REMARKS ON ALGEBRAIC FIBER SPACES

OSAMU FUJINO

Abstract. In this paper, we collect basic properties of the Albanese dimension and explain how to generalize the main theorem of [F2]. This paper is a supplement and a generalization of [F2]. We also prove an inequality of irregularities for algebraic fiber spaces in the appendix, which is an exposition of Fujita-Kawamata's semipositivity theorem.

1. Introduction

The following is the main theorem of [F2]. In this paper, we recall the Albanese dimension and collect its basic properties. Then we explain how to generalize Theorem 1.1 in Section 3. We note that Section 3 is not self-contained. Finally, we explain Fujita-Kawamata's semipositivity theorem and prove an inequality of irregularities for algebraic fiber spaces in Section 4.

Theorem 1.1 (Main theorem of [F2]). *Let $f : X \to Y$ be a surjective morphism between non-singular projective varieties with connected fibers. Let F be a sufficiently general fiber of f. Assume that F has maximal Albanese dimension. Then $\kappa(X) \geq \kappa(Y) + \kappa(F)$.*

The main theorem of this paper is Theorem 1.3 below. See also Theorem 3.4. Before we state the main theorem, we recall *Albanese dimension* and define *Albanese fiber dimension* for non-singular projective varieties. The notion of Albanese dimension was already defined by many mathematicians.

Definition 1.2 (Albanese dimension and Albanese fiber dimension). Let X be a non-singular projective variety. Let $\mathrm{Alb}(X)$ be the *Albanese variety* of X and $\alpha_X : X \to \mathrm{Alb}(X)$ the *Albanese map*. We define the *Albanese dimension* $\mathrm{d}_{\mathrm{Alb}}(X)$ as follows;

$$\mathrm{d}_{\mathrm{Alb}}(X) := \dim(\alpha_X(X)).$$

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We call $\mathrm{fd}_{\mathrm{Alb}}(X) := \dim X - \mathrm{d}_{\mathrm{Alb}}(X)$ the *Albanese fiber dimension* of X. By the definition, it is obvious that $0 \leq \mathrm{d}_{\mathrm{Alb}}(X) \leq \dim X$ and $0 \leq \mathrm{fd}_{\mathrm{Alb}}(X) \leq \dim X$. We note that $\mathrm{d}_{\mathrm{Alb}}(X) = \mathrm{rank}_{\mathcal{O}_X}(\mathrm{Im}(H^0(X, \Omega^1_X) \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega^1_X))$.

We say that X *has maximal Albanese dimension* when $\dim X = \mathrm{d}_{\mathrm{Alb}}(X)$, equivalently, $\mathrm{fd}_{\mathrm{Alb}}(X) = 0$.

The following is the main theorem of this paper, which is a generalization of Theorem 1.1. See also Theorem 3.4 below.

Theorem 1.3. *Let $f : X \to Y$ be a surjective morphism between non-singular projective varieties with connected fibers. Let F be a sufficiently general fiber of f. Assume that $\mathrm{fd}_{\mathrm{Alb}}(F) \leq 3$. Then $\kappa(X) \geq \kappa(Y) + \kappa(F)$.*

The following corollary is obvious by Theorem 1.3 and Proposition 2.14 (2) below.

Corollary 1.4. *Let $f : X \to Y$ be a surjective morphism between non-singular projective varieties with connected fibers and F a sufficiently general fiber of f. Assume that $\dim F = 4$ and the irregularity $q(F) > 0$. Then $\kappa(X) \geq \kappa(Y) + \kappa(F)$.*

The idea of the proof of Theorem 1.3 is to combine Theorem 1.1 with Theorem 1.5 below. It is a special case of [Ka3, Corollary 1.2].

Theorem 1.5 (c.f. [Ka3, Corollary 1.2])**.** *Let $f : X \to Y$ be a surjective morphism between non-singular projective varieties. Assume that $\dim X - \dim Y \leq 3$ or sufficiently general fiber of f are birationally equivalent to Abelian varieties. If $\kappa(Y) \geq 0$, then*

$$\kappa(X) \geq \kappa(F) + \max\{\kappa(Y), \mathrm{Var}(f)\},$$

where F is a sufficiently general fiber of f.

For $\mathrm{Var}(f)$ and other positive answers to Iitaka's conjecture, see [M, Sections 6, 7].

We summarize the contents of this paper: In section 2, we define Albanese dimension and Albanese fiber dimension for complete (not necessarily non-singular) varieties and collect their several basic properties. Section 3 sketches the proof of the main theorem: Theorem 1.3. The proof depends on Theorem 1.1 and the arguments in [F2]. In section 4, which is an appendix, we explain Fujita-Kawamata's semipositivity theorem and prove an inequality about *irregularities* for algebraic fiber spaces (Theorem 4.13). It may be useful for the study of the relative Albanese map. The statement is as follows;

Theorem 1.6 (Inequality of irregularities). *Let $f : X \to Y$ be a surjective morphism between non-singular projective varieties with connected fibers. Then*

$$q(Y) \leq q(X) \leq q(Y) + q(F),$$

where F is a general fiber of f.

We recommend the readers who are only interested in Theorem 1.6 to read Section 4 directly after checking the notation below. Section 4 is independent of the results in Sections 2 and 3. Section 4 is intended to the readers who are not familiar with the technique of the higher dimensional algebraic geometry. It is an expository section.

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We fix the notation used in this paper.

Notation. We will work over the complex number field $\mathbb{C}$ throughout this paper.

(i) A *sufficiently general point* z (resp. *subvariety* Γ) of the variety Z means that z (resp. Γ) is not contained in the countable union of certain proper Zariski closed subsets. We say that a subvariety Γ (resp. point z) is *general* in Z if it is not contained in a certain proper Zariski closed subsets.

Let $f : X \to Y$ be a morphism between varieties. A *sufficiently general fiber* (resp. *general fiber*) $X_y = f^{-1}(y)$ of f means that y is a sufficiently general point (resp. general) in Y.

(ii) An *algebraic fiber space* $f : X \to Y$ is a proper surjective morphism between non-singular projective varieties X and Y with connected fibers.

(iii) Let $f : X \to Y$ be a surjective morphism between varieties. We put $\dim f := \dim X - \dim Y$. Let X be a variety and $\mathcal{F}$ a coherent sheaf on X. We write $h^i(X, \mathcal{F}) = \dim_{\mathbb{C}} H^i(X, \mathcal{F})$. If X is non-singular projective, then $q(X) := h^1(X, \mathcal{O}_X)$ denotes the *irregularity* of X.

(iv) The words *locally free sheaf* and *vector bundle* are used interchangeably.

(v) Since most questions we are interested in are birational ones, we usually make birational modifications freely whenever it is necessary. If no confusion is likely, we denote the new objects with the old symbols.

(vi) Let X be a non-singular projective variety. If the *Kodaira dimension* $\kappa(X) > 0$, then we have the *Iitaka fibration* $f : X \to Y$, where X and Y are non-singular projective varieties and Y is of dimension $\kappa(X)$, such that the sufficiently general fiber of f is non-singular, irreducible with $\kappa = 0$. Since the Iitaka fibration is determined only up to birational equivalence, we used the above abuses in (v). For the basic properties of the Kodaira dimension and the Iitaka fibration, see [U, Chapter III] or [M, Sections 1,2].

2. Albanese dimension

In this section, we collect several basic properties of the Albanese dimension and the Albanese fiber dimension. The next lemma is easy to check.

Lemma 2.1. *Let $f : W \to V$ be a birational morphism between non-singular projective varieties. Then $(\mathrm{Alb}(V), \alpha_V \circ f)$ is the Albanese variety of W. In particular, $\mathrm{d}_{\mathrm{Alb}}(V) = \mathrm{d}_{\mathrm{Alb}}(W)$.*

Proof. This is obvious. See, for example, [U, Proposition 9.12]. □

By the above lemma, we can define the Albanese dimension and the Albanese fiber dimension for singular varieties.

Definition 2.2. Let X be a complete variety. We define $\mathrm{d}_{\mathrm{Alb}}(X) := \mathrm{d}_{\mathrm{Alb}}(\widetilde{X})$, where $\widetilde{X}$ is a non-singular projective variety that is birationally equivalent to X. We put $\mathrm{fd}_{\mathrm{Alb}}(X) := \dim X - \mathrm{d}_{\mathrm{Alb}}(X)$.

By Lemma 2.1, the *Albanese dimension* $\mathrm{d}_{\mathrm{Alb}}(X)$ and the *Albanese fiber dimension* $\mathrm{fd}_{\mathrm{Alb}}(X)$ are well-defined birational invariants of X.

Definition 2.3 (Varieties of maximal Albanese dimension)**.** Let X be a complete variety. If $\mathrm{fd}_{\mathrm{Alb}}(X) = 0$, then we say that X is *of maximal Albanese dimension* or *has maximal Albanese dimension*. We note that $\kappa(X) \geq 0$ if $\mathrm{fd}_{\mathrm{Alb}}(X) = 0$. By [Ka1, Theorem 1], $\kappa(X) = 0$ and $\mathrm{fd}_{\mathrm{Alb}}(X) = 0$ implies that $\alpha_X : X \to \mathrm{Alb}(X)$ is birational.

Definition 2.4 (Albanese dimension of fibers)**.** Let $f : X \to Y$ be a surjective morphism between non-singular projective varieties. By taking the Stein factorization of f, we obtain;

$$\begin{array}{ccc} X & \longrightarrow & Z \\ & \searrow \quad \swarrow & \\ & Y & \end{array} .$$

We shrink Z suitably and take a finite étale cover $\widetilde{Z} \to Z$ such that $\widetilde{X} := X \times_Z \widetilde{Z} \to \widetilde{Z}$ has a section. Then, we obtain a relative Albanese map $\widetilde{X} \to \mathrm{Alb}(\widetilde{X}/\widetilde{Z}) \to \widetilde{Z}$ (see also 3.1 below). By this relative

Albanese map, it is easy to see that the Albanese dimension $\mathrm{d}_{\mathrm{Alb}}(X_z)$ is independent of $z \in U$, where U is a suitable non-empty Zariski open set of Z and X_z is a fiber of $X \to Z$. We note that we can assume that the irregularity $q(X_z)$ is independent of $z \in U$.

We put $\mathrm{d}_{\mathrm{Alb}}(f) := \mathrm{d}_{\mathrm{Alb}}(X_z)$, where z is a general point of Z and $\mathrm{fd}_{\mathrm{Alb}}(f) := \dim f - \mathrm{d}_{\mathrm{Alb}}(f)$. We call $\mathrm{d}_{\mathrm{Alb}}(f)$ (resp. $\mathrm{fd}_{\mathrm{Alb}}(f)$) the *Albanese dimension* (resp. *Albanese fiber dimension*) of f.

The following proposition will play important roles in the proof of the main theorem (c.f. [F2, Proposition 2.3 (3)]).

Proposition 2.5. *Let V be a non-singular projective variety and X a closed subvariety of V. If X is general in V, then*

$$\mathrm{fd}_{\mathrm{Alb}}(X) \leq \dim X - \dim \alpha_V(X) \leq \mathrm{fd}_{\mathrm{Alb}}(V).$$

Proof. Let $\widetilde{X} \to X$ be a resolution. We consider the following commutative diagram;

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \mathrm{Alb}(\widetilde{X}) \\ \downarrow & & \downarrow \\ V & \xrightarrow{\alpha_V} & \mathrm{Alb}(V). \end{array}$$

Since X is general in V, $\dim V - \dim X \geq \mathrm{d}_{\mathrm{Alb}}(V) - \dim \alpha_V(X)$. Therefore, $\mathrm{fd}_{\mathrm{Alb}}(V) \geq \dim X - \dim \alpha_V(X) \geq \mathrm{fd}_{\mathrm{Alb}}(X)$. □

The next example shows that the equality does not necessarily holds in Proposition 2.5.

Example 2.6. Let $X := \mathbb{P}^1 \times E$, where E is an elliptic curve. Let $p : X \to E$ be the second projection and C an irreducible curve on X. Then $\mathrm{fd}_{\mathrm{Alb}}(C) = 0$ unless $p(C)$ is a point. If C is a fiber of p, then it is obvious that $\mathrm{fd}_{\mathrm{Alb}}(C) = 1$. We note that $\mathrm{fd}_{\mathrm{Alb}}(X) = 1$.

Example 2.7. Let A be an Abelian surface and X a one point blow-up of A. Let E be the (-1)-curve on X. Then $\mathrm{fd}_{\mathrm{Alb}}(E) = 1$ since $E \simeq \mathbb{P}^1$. Therefore, $1 = \mathrm{fd}_{\mathrm{Alb}}(E) > \mathrm{fd}_{\mathrm{Alb}}(X) = \mathrm{fd}_{\mathrm{Alb}}(A) = 0$.

Corollary 2.8 (Easy addition of the Albanese dimension)**.** *Let $f : X \to Y$ be a surjective morphism between non-singular projective varieties. Then*

$$\mathrm{d}_{\mathrm{Alb}}(X) \leq \dim \alpha_X(F) + \dim Y \leq \mathrm{d}_{\mathrm{Alb}}(f) + \dim Y.$$

Proof. This easily follows from Proposition 2.5. We note that Definition 2.4. □

Example 2.9. Let A be an Abelian surface and H be a non-singular very ample divisor on A. We take a general member $H' \in |H|$. We can write $H' = H + (h)$, where $h \in \mathbb{C}(A)$. Consider the rational map;

$$h : A \dashrightarrow \mathbb{P}^1.$$

By blowing up the points $H \cap H'$, we obtain an algebraic fiber space;

$$f : X \longrightarrow Y,$$

which is birationally equivalent to h. In this case, $\mathrm{d}_{\mathrm{Alb}}(X) = 2$, $\mathrm{d}_{\mathrm{Alb}}(Y) = 0$, and $\mathrm{d}_{\mathrm{Alb}}(f) = 1$. Therefore, we can not replace $\dim Y$ with $\mathrm{d}_{\mathrm{Alb}}(Y)$ in Corollary 2.8.

The following claim is a variant of [F2, Proposition 2.4].

Proposition 2.10. *Let V be a non-singular projective variety and $f : V \to W$ be the Iitaka fibration. Then $\mathrm{fd}_{\mathrm{Alb}}(W) \leq \mathrm{fd}_{\mathrm{Alb}}(V)$. In particular, if V is of maximal Albanese dimension, then so is W.*

Proof. We put $m := \mathrm{fd}_{\mathrm{Alb}}(V)$ and $n := \dim V$. Let F be a sufficiently general fiber of f. Then $\kappa(F) = 0$ and $\mathrm{fd}_{\mathrm{Alb}}(F) \leq m$ by Proposition 2.5. By [Ka1, Theorem 1], $\alpha_F : F \to \mathrm{Alb}(F)$ is an algebraic fiber space. By the following diagram;

$$\begin{array}{ccc} F & \longrightarrow & \mathrm{Alb}(F) \\ \downarrow & & \downarrow \\ V & \xrightarrow{\alpha_V} & \mathrm{Alb}(V), \end{array}$$

$\alpha_V(F)$ is an Abelian variety. Since there exists at most countably many Abelian subvarieties in $\mathrm{Alb}(V)$, there is an Abelian subvariety A of $\mathrm{Alb}(V)$ such that $\alpha_V(F)$ is a translation of A for general fibers F. We note that $\dim A \leq \dim F = n - \dim W = n - \kappa(V)$.

Let $\psi : \mathrm{Alb}(V) \to \mathrm{Alb}(V)/A$ be the quotient map. By the definition of A, $\psi \circ \alpha_V$ induces a rational map $\varphi : W \dashrightarrow \mathrm{Alb}(V)/A$. Since $\mathrm{Alb}(V)/A$ is Abelian, φ is a morphism. By the universality of $(\mathrm{Alb}(W), \alpha_W)$, φ factors through $\mathrm{Alb}(W)$. Therefore,

$$\begin{aligned} \mathrm{d}_{\mathrm{Alb}}(W) &\geq \dim \varphi(W) \\ &= \dim \psi(\alpha_V(V)) \\ &\geq n - m - \dim F \\ &= \kappa(V) - m. \end{aligned}$$

Note that $\dim W = \kappa(V)$. Thus, we have the required inequality $\mathrm{fd}_{\mathrm{Alb}}(W) \leq m$. □

Remark 2.11. We can generalize Proposition 2.10 as follows without difficulties. Details are left to the readers.

Let $V \to W$ be a surjective morphism between non-singular projective varieties. If $q(f) = \mathrm{d}_{\mathrm{Alb}}(f)$, then $\mathrm{fd}_{\mathrm{Alb}}(W) \leq \mathrm{fd}_{\mathrm{Alb}}(V)$.

The following example says that the equality doesn't necessarily hold in Proposition 2.10.

Example 2.12. Let S be a $K3$ surface and C is a non-singular projective curve with the genus $g(C) \geq 2$. We put $X := S \times C$. Then the second projection $X \to C$ is the Iitaka fibration. It is easy to check that $\mathrm{fd}_{\mathrm{Alb}}(X) = 2$ and $\mathrm{fd}_{\mathrm{Alb}}(C) = 0$.

We collect several basic properties of the Albanese dimension and Albanese fiber dimension for the reader's convenience. First, we recall the following obvious fact.

Lemma 2.13. *Let X be a non-singular projective variety. Then the irregularity $q(X) = 0$ if and only if $\mathrm{d}_{\mathrm{Alb}}(X) = 0$.*

Proposition 2.14. (1) *Let X be a complete variety with $\dim X = n$. Then $\mathrm{fd}_{\mathrm{Alb}}(X) \leq n$.*

(2) *Let X be an $(n+1)$-dimensional non-singular projective variety with the irregularity $q(X) > 0$. Then $\mathrm{fd}_{\mathrm{Alb}}(X) \leq n$.*

(3) *Let $f : X \to Y$ be a generically finite morphism between non-singular projective varieties. Then $\mathrm{d}_{\mathrm{Alb}}(X) \geq \mathrm{d}_{\mathrm{Alb}}(Y)$. Equivalently, $\mathrm{fd}_{\mathrm{Alb}}(X) \leq \mathrm{fd}_{\mathrm{Alb}}(Y)$.*

(4) *Let $f : X \to Y$ be a surjective morphism between non-singular projective varieties. Then $\mathrm{fd}_{\mathrm{Alb}}(X) \leq \mathrm{fd}_{\mathrm{Alb}}(Y) + \dim f$.*

(5) *Let X and Y be non-singular projective varieties. Then $\mathrm{d}_{\mathrm{Alb}}(X \times Y) = \mathrm{d}_{\mathrm{Alb}}(X) + \mathrm{d}_{\mathrm{Alb}}(Y)$. Equivalently, $\mathrm{fd}_{\mathrm{Alb}}(X \times Y) = \mathrm{fd}_{\mathrm{Alb}}(X) + \mathrm{fd}_{\mathrm{Alb}}(Y)$.*

Proof. The claims (1), (3), and (4) are obvious by the definition. For (2), we note that $\alpha_X(X)$ is not a point by Lemma 2.13. For (5), note that $(\mathrm{Alb}(X) \times \mathrm{Alb}(Y), \alpha_X \times \alpha_Y)$ is the Albanese variety of $X \times Y$ by the Künneth formula. □

Lemma 2.15. *Let V be a non-singular projective variety and H a non-singular ample divisor on V. Then $\mathrm{Alb}(H) \to \mathrm{Alb}(V)$ is surjective (resp. an isomorphism) if $\dim V \geq 2$ (resp. $\dim V \geq 3$).*

Proof. Apply the Kodaira vanishing theorem to $H^1(V, \mathcal{O}_V(-H))$ and $H^2(V, \mathcal{O}_V(-H))$. □

Proposition 2.16. *Let V be a non-singular projective variety and H a non-singular ample divisor on V. Assume that H is general in V. Then $\mathrm{fd}_{\mathrm{Alb}}(H) = \mathrm{fd}_{\mathrm{Alb}}(V) - 1$ if $\dim V \geq 3$.*

Proof. It is obvious by Lemma 2.15. □

The following lemma is a key lemma in Section 3 (see 3.1 below).

Lemma 2.17. *Let X be a non-singular projective variety and $\alpha_X : X \to \mathrm{Alb}(X)$ the Albanese mapping. We consider the Stein factorization;*

$$X \longrightarrow W \longrightarrow \alpha_X(X).$$

Then $\mathrm{fd}_{\mathrm{Alb}}(W) = 0$, that is, W has maximal Albanese dimension.

Proof. It is obvious by the definition. See Proposition 2.14 (3). □

3. Sketch of the proof of the main theorem

In this section, we sketch the proof of the main theorem: Theorem 1.3. We explain how to modify the arguments in [F2, Section 4]. For the details, see [F2].

3.1 (Algebraic fiber space associated to the relative Albanese map)**.** Let $f : X \to Y$ be an algebraic fiber space. From now on, we often replace Y with its non-empty Zariski open set and X with its inverse image. We denote the new objects with the old symbols. We can assume that f is smooth. We can assume that there exists a finite étale cover $\pi : \widetilde{Y} \to Y$ such that π is Galois and $\widetilde{f} : \widetilde{X} := X \times_Y \widetilde{Y} \to \widetilde{Y}$ has a section. Then there exists the relative Albanese map $\alpha_{\widetilde{X}/\widetilde{Y}} : \widetilde{X} \to \mathrm{Alb}(\widetilde{X}/\widetilde{Y})$ over $\widetilde{Y}$. We note that $\mathrm{Alb}(\widetilde{X}/\widetilde{Y}) \simeq \mathrm{Pic}^0(\mathrm{Pic}^0(\widetilde{X}/\widetilde{Y})/\widetilde{Y})$. Put $k := \deg \pi$ and $G := \mathrm{Gal}(\widetilde{Y}/Y)$ the Galois group of π. Then G acts on $\widetilde{X} = X \times_Y \widetilde{Y}$ and $\mathrm{Alb}(\widetilde{X}/\widetilde{Y}) = \mathrm{Alb}(X/Y) \times_Y \widetilde{Y}$. Thus G acts on $\alpha_{\widetilde{X}/\widetilde{Y}}$ as follows;

$$(g \cdot \alpha_{\widetilde{X}/\widetilde{Y}})(x) = g^{-1}(\alpha_{\widetilde{X}/\widetilde{Y}}(gx)),$$

where $g \in G$ and $x \in \widetilde{X}$. We put $\beta_{\widetilde{X}/\widetilde{Y}} := \sum_{g \in G} g \cdot \alpha_{\widetilde{X}/\widetilde{Y}} : \widetilde{X} \to \mathrm{Alb}(\widetilde{X}/\widetilde{Y})$. Let $\widetilde{\Gamma} \subset \widetilde{X} \times_{\widetilde{Y}} \mathrm{Alb}(\widetilde{X}/\widetilde{Y})$ be the graph of $\beta_{\widetilde{X}/\widetilde{Y}}$ and $\Gamma \subset X \times_Y \mathrm{Alb}(X/Y)$ the image of $\widetilde{\Gamma}$. By the construction of $\beta_{\widetilde{X}/\widetilde{Y}}$, Γ induces a morphism $\beta_{X/Y} : X \to \mathrm{Alb}(X/Y)$ over Y. Let us see $\beta_{X/Y}$ fiberwise. Then it is the composition of the Albanese map and the multiplication by k of the Albanese variety up to a translation. By taking the Stein factorization of $\beta_{X/Y}$, we obtain $X \to Z \to Y$. Compactify X, Y, and Z. Then, after taking resolutions, we obtain;

$$f : X \xrightarrow{g} Z \xrightarrow{h} Y$$

such that

(i) it is birationally equivalent to the given fiber space $f : X \to Y$.
(ii) X, Y, and Z are non-singular projective varieties.

(iii) g and h are algebraic fiber spaces.
(iv) the general fibers of h have maximal Albanese dimension.
(v) if $\kappa(X_y) \geq 0$ for sufficiently general fibers X_y of f, then $\kappa(X_z) \geq 0$ for sufficiently general points $z \in Z$. This is an easy consequence of the easy addition of the Kodaira dimension.

Proof of Theorem 1.3. Let $f : X \to Y$ be the given fiber space.

Step 1. If $\kappa(Y) = -\infty$ or $\kappa(F) = -\infty$, then the inequality is obviously true. From now on, we assume that $\kappa(Y) \geq 0$ and $\kappa(F) \geq 0$.

Step 2. If $\kappa(F) = 0$, then we construct the fiber space associated to the relative Albanese map 3.1, we obtain

$$f : X \xrightarrow{g} Z \xrightarrow{h} Y,$$

such that $\dim X - \dim Z \leq 3$ by the assumption $\mathrm{fd}_{\mathrm{Alb}}(F) \leq 3$ and the sufficiently general fiber of h are of maximal Albanese dimension by 3.1 (iv). By 3.1 (v), the Kodaira dimension of the sufficiently general fiber of g is non-negative. Therefore, by Theorem 1.1 and Theorem 1.5,

$$\begin{aligned} \kappa(X) &\geq \kappa(Z) + \kappa(X_z) \\ &\geq \kappa(Z) \\ &\geq \kappa(Y) + \kappa(Z_y) \\ &\geq \kappa(Y), \end{aligned}$$

where z (resp. y) is a sufficiently general point of Z (resp. Y). Note that $\kappa(Z_y) \geq 0$ since $\mathrm{fd}_{\mathrm{Alb}}(Z_y) = 0$ (see Definition 2.3). Therefore, we obtained the required inequality when $\kappa(F) = 0$. From now on, we assume that $\kappa(F) > 0$.

Step 3. We recall the following useful lemma ([F2, Lemma 4.2]).

Lemma 3.2 (Induction Lemma). *Under the same notation as in* Theorem 1.3*, it is sufficient to prove that* $\kappa(X) > 0$ *on the assumption that* $\kappa(Y) \geq 0$ *and* $\kappa(F) > 0$.

Remark 3.3. We prove this lemma in [F2] on the assumption that $\mathrm{fd}_{\mathrm{Alb}}(F) = 0$. The same proof works on the weaker assumption that $\mathrm{fd}_{\mathrm{Alb}}(F) \leq 3$. We only have to replace the words "maximal Albanese dimension" with "$\mathrm{fd}_{\mathrm{Alb}}(\cdot) \leq 3$" in the proof of [F2, Lemma 4.2]. The key point is Proposition 2.5. By this, the inductive arguments work. Details are left to the reader.

Step 4. By taking the fiber space associated to the relative Albanese map 3.1, we obtain

$$f : X \xrightarrow{g} Z \xrightarrow{h} Y.$$

Since we want to prove $\kappa(X) > 0$, we can assume that the sufficiently general fibers of g and h have zero Kodaira dimension by Theorem 1.1 and Theorem 1.5 (see Step 2 above). On this assumption, the general fibers of h are birationally equivalent to Abelian varieties. We can further assume that $\mathrm{Var}(g) = \mathrm{Var}(h) = 0$ by Theorem 1.5. Therefore, we can apply the same proof as in [F2]. Then we obtain $\kappa(X) > 0$. For the details, see the latter part of [F2, Proof of the theorem]. We note that $\kappa(F) \geq 1$.

Therefore, we complete the proof. □

The following theorem is obvious by the proof of Theorem 1.3. For the conjecture $C^+_{n,m}$, see [M, Section 7] and Theorem 1.5.

Theorem 3.4. *Suppose that $C^+_{n,m}$ holds for every algebraic fiber spaces on the assumption that $n - m \leq k$. Let $f : X \to Y$ be an algebraic fiber space. If $\mathrm{fd}_{\mathrm{Alb}}(F) \leq k$, where F is a sufficiently general fiber of f, then $\kappa(X) \geq \kappa(Y) + \kappa(F)$.*

4. Appendix: Semi-positivity and Inequality of Irregularities

The aim of this section is to explain Fujita-Kawamata's semi-positivity theorem in the geometric situation, that is, we prove the semi-positivity of $R^i f_* \omega_{X/Y}$ on suitable assumptions, where $f : X \to Y$ is a surjective morphism between non-singular projective varieties. Then we prove an inequality of irregularities for algebraic fiber spaces (see Theorem 4.13), which is an easy consequence of the semi-positivity theorem. This inequality seems to be useful when we treat (relative) Albanese variety.

In spite of its importance, the results and the statements about the semi-positivity theorem are scattered over various papers (see [M, §5]). This is one of the reason why I decided to write down this section [1]. It is surprising that there are no good references about the semi-positivity of $R^i f_* \omega_{X/Y}$. Kawamata's proof of the semi-positivity theorem (c.f. [Ka2, Theorem 2] and [Ka1, §4]) heavily relies on the asymptotic behavior of the Hodge metric near a puncture. It is not so easy for the non-expert to take it out from [S, §6]. We recommend the reader to see [P, Sections 2, 3] or [Z][2]. Our proof depends on [Ka2, Proposition 1], [Ko1], [Ko2], and Viehweg's technique. It is essentially the same as

[1] It took long time to find the statement [Ka2, Theorem 2]. I hope that this section will contribute to distribute Fujita-Kawamata's semi-positivity package.

[2] For the proof of the semi-positivity theorem [Ka2, §4 (2)], it is sufficient to know the asymptotic behavior of the Hodge metric of the VHS on a *curve*.

[Ko1, Corollary 3.7]. It is much simpler than the original proof. We note that Kawamata's proof can be applied to non-geometric situations. So, his theorem is much stronger than the results explained in this section. See the original article [Ka1, §4].

Remark 4.1. This section is a supplement of [M, §5 Part I], especially, [M, (5.3) Theorem]. The joint paper with S. Mori [FM] generalized [M, §5 Part II] and treated several applications. The paper [F1] gave a precise proof of [M, (5.15.9)(ii)] from the Hodge theoretic viewpoint.

Let us recall the definition of *semi-positive vector bundles*.

Definition 4.2 (Semi-positive vector bundles)**.** Let V be a complete variety and $\mathcal{E}$ a locally free sheaf on V. We say that $\mathcal{E}$ is *semi-positive* if and only if the tautological line bundle $\mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1)$ is nef on $\mathbb{P}_V(\mathcal{E})$. We note that $\mathcal{E}$ is semi-positive if and only if for every complete curve C and morphism $g : C \to V$ every quotient line bundle of $g^*\mathcal{E}$ has non-negative degree.

The following result is obtained by Kollár and Nakayama. For the details, see, for example, the original articles [Ko2, Theorem 2.6], [N, Theorem 1], or [M, (5.3), (5.4)].

Theorem 4.3. *Let $f : V \to W$ be a projective surjective morphism between non-singular varieties. We assume that there exists a simple normal crossing divisor Σ on W such that f is smooth over $W \setminus \Sigma$. Then $R^i f_* \mathcal{O}_V$ and $R^i f_* \omega_{V/W}$ are locally free for every i. Put $W_0 := W \setminus \Sigma$, $V_0 := f^{-1}(W_0)$, $f_0 := f|_{V_0}$, and $k := \dim f$. Moreover, if all the local monodromies on $R^{k+i} f_{0*} \mathbb{C}_{V_0}$ around Σ are unipotent, then $R^i f_* \omega_{V/W}$ is characterized by the* canonical extension *of $R^{k+i} f_{0*} \mathbb{C}_{V_0}$.*

The following is the main theorem of this section (see [M, (5.3) Theorem]).

Theorem 4.4 (Semi-positivity theorem)**.** *Let $f : V \to W$ be a surjective morphism between non-singular projective varieties with $k := \dim f$. Let Σ be a simple normal crossing divisor on W such that f is smooth over $W_0 := W \setminus \Sigma$. Put $V_0 := f^{-1}(W_0)$ and $f_0 := f|_{V_0}$. We assume that all the local monodromies on $R^{k+i} f_{0*} \mathbb{C}_{V_0}$ around Σ are unipotent. Then $R^i f_* \omega_{V/W}$ is a semi-positive vector bundle on W.*

Before we prove Theorem 4.4, we fix the notation and convention used below.

4.5. Let $f : V \to W$ be a surjective morphism between varieties. Let

$$f^s : V^s := V \times_W V \times_W \cdots \times_W V \to W \quad (\text{product taken } s \text{ times})$$

$$V^{(s)} = \text{desingularization of } \ V^s, f^{(s)} : V^{(s)} \to W.$$

Lemma 4.6. *On the same assumption as in* Theorem 4.4*, we have that* $(R^if_*\omega_{V/W})^{\otimes s}$ *is a direct summand of* $R^{si}f^{(s)}_*\omega_{V^{(s)}/W}$ *for every positive integer* s*. We note that* $(f_*\omega_{V/W})^{\otimes s} \simeq f^{(s)}_*\omega_{V^{(s)}/W}$ *for every positive integer* s*.*

Proof. We use the induction on s. First, when $s = 1$, the claim is obvious. Next, we assume that $(R^if_*\omega_{V/W})^{\otimes(s-1)}$ is a direct summand of $R^{(s-1)i}f^{(s-1)}_*\omega_{V^{(s-1)}/W}$. We consider the following commutative diagram;

$$\begin{array}{ccccc} V & \longleftarrow & V\times_W V^{(s-1)} & \longleftarrow & V^{(s)} \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ W & \underset{f^{(s-1)}}{\longleftarrow} & V^{(s-1)} & \underset{id}{\longleftarrow} & V^{(s-1)}. \end{array}$$

We can assume that $(f^{(s-1)})^{-1}(\Sigma)$ is simple normal crossing without loss of generality. Since $R^if_*\omega_{V/W}$ is the canonical extension of $R^if_{0*}\omega_{V_0/W_0}$ by Theorem 4.3 (see [Ko2, Theorem 2.6], [N, Theorem 1]), we have the following isomorphism: $f^{(s-1)*}R^if_*\omega_{V/W} \simeq R^ig_*\omega_{V^{(s)}/V^{(s-1)}}$ by [Ka2, Proposition]. Therefore,

$$\begin{aligned} R^jf^{(s-1)}_*R^ig_*\omega_{V^{(s)}/W} &\simeq R^jf^{(s-1)}_*R^ig_*(\omega_{V^{(s)}/V^{(s-1)}}\otimes g^*\omega_{V^{(s-1)}/W}) \\ &\simeq R^jf^{(s-1)}_*(R^ig_*\omega_{V^{(s)}/V^{(s-1)}}\otimes\omega_{V^{(s-1)}/W}) \\ &\simeq R^jf^{(s-1)}_*(f^{(s-1)*}R^if_*\omega_{V/W}\otimes\omega_{V^{(s-1)}/W}) \\ &\simeq R^if_*\omega_{V/W}\otimes R^jf^{(s-1)}_*\omega_{V^{(s-1)}/W}, \end{aligned}$$

for every j. By [Ko2, Theorem 3.4], $R^{(s-1)i}f^{(s-1)}_*R^ig_*\omega_{V^{(s)}/W}$ is a direct summand of $R^{si}f^{(s)}_*\omega_{V^{(s)}/W}$. We assumed that $(R^if_*\omega_{V/W})^{\otimes(s-1)}$ is a direct summand of $R^{(s-1)i}f^{(s-1)}_*\omega_{V^{(s-1)}/W}$. Therefore, $(R^if_*\omega_{V/W})^{\otimes s}$ is a direct summand of $R^{si}f^{(s)}_*\omega_{V^{(s)}/W}$. □

The following lemma is obvious by Kollár's vanishing theorem [Ko1, Theorem 2.1]. For the *regularities*, see [Kl, p.307 Definitions 1, 2 and Proposition 1].

Lemma 4.7. *Let* $f : X \to Y$ *be a surjective morphism between projective varieties,* X *non-singular,* $m := \dim Y + 1$*. Let* L *be an ample line bundle on* Y *which is generated by its global sections. Then* $R^if_*\omega_X$ *is* m*-regular (with respect to* L*) for every* i*. That is,* $H^j(Y, R^if_*\omega_X \otimes L^{\otimes(m-j)}) = 0$ *for every* $j > 0$*. In particular,* $R^if_*\omega_X \otimes L^{\otimes l}$ *is generated by its global sections for* $l \geq m$*.*

The next lemma is not difficult to prove. For the proof, see, for example, [Ko1, Proof of Corollary 3.7].

Lemma 4.8. *Let $\mathcal{E}$ be a vector bundle on a complete variety Y. Assume that there exists a line bundle $\mathcal{L}$ on Y and $\mathcal{L} \otimes \mathcal{E}^{\otimes s}$ is generated by its global sections for every s. Then $\mathcal{E}$ is semi-positive.*

Sketch of the proof. It is not difficult to see that $\mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(s) \otimes \pi^*\mathcal{L}$ is nef for every s, where $\pi : \mathbb{P}_Y(\mathcal{E}) \to Y$. Therefore, $\mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$ is nef. □

Proof of Theorem 4.4. Let L be an ample line bundle on W which is generated by its global sections. We put $\mathcal{E} := R^if_*\omega_{V/W}$ and $\mathcal{L} := L^{\otimes m} \otimes \omega_W$, where $m = \dim Y + 1$. Then $R^{si}f^{(s)}_*\omega_{V^{(s)}/W} \otimes \mathcal{L}$ is generated by its global sections for every s by Lemma 4.7. By Lemma 4.6, $(R^if_*\omega_{V/W})^{\otimes s} \otimes \mathcal{L}$ is generated by its global sections. Therefore, $R^if_*\omega_{V/W}$ is semi-positive by Lemma 4.8. □

The following theorem is well-known. We write it for the reader's convenience. We will use it in the proof of Theorem 4.13. We reduce it to Theorem 4.4 by the semi-stable reduction theorem.

Theorem 4.9. *In* Theorem 4.4*, if W is a curve, then the assumption on the monodoromies is not necessary, that is, $R^if_*\omega_{V/W}$ is always semi-positive for every i.*

Proof. Without loss of generality, we can assume that $\mathrm{Supp} f^*(P)$ is simple normal crossing for every point $P \in W$. By the semi-stable reduction theorem, we consider the following commutative diagram;

$$\begin{array}{ccccccc}
V & \longleftarrow & V \times_W \widetilde{W} & \xleftarrow{\nu} & V' & \longleftarrow & \widetilde{V} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \widetilde{f}} \\
W & \underset{\pi}{\longleftarrow} & \widetilde{W} & \underset{id}{\longleftarrow} & \widetilde{W} & \underset{id}{\longleftarrow} & \widetilde{W},
\end{array}$$

where $\pi : \widetilde{W} \to W$ is a finite cover, ν is the normalization, and $\widetilde{f} : \widetilde{V} \to \widetilde{W}$ is a semi-stable reduction of f. By the flat base change theorem, we have $\pi^*R^if_*\omega_{V/W} \simeq R^ig_*\omega_{V \times_W \widetilde{W}/\widetilde{W}}$ for every i. We consider the following exact sequence, which is induced by the trace map;

$$0 \longrightarrow \nu_*\omega_{V'/\widetilde{W}} \xrightarrow{tr} \omega_{V \times_W \widetilde{W}/\widetilde{W}} \longrightarrow \delta \longrightarrow 0,$$

where δ is the cokernel of tr. Since $\mathrm{Supp}\, R^{i-1}g_*\delta \subsetneq \widetilde{W}$, we obtain a generically isomorphic inclusion;

$$0 \to R^if'_*\omega_{V'/\widetilde{W}} \simeq R^i\widetilde{f}_*\omega_{\widetilde{V}/\widetilde{W}} \to R^ig_*\omega_{V \times_W \widetilde{W}/\widetilde{W}} \simeq \pi^*R^if_*\omega_{V/W}.$$

We note that ν is finite and V' has at worst rational Gorenstein singularities. Since $\widetilde{f}$ is semi-stable, all the local monodromies are unipotent (see, for example, [M, (4.6.1)]). Thus $R^if_*\omega_{\widetilde{V}/\widetilde{W}}$ is semi-positive by Theorem 4.4. By the above inclusion, we can check easily that $R^if_*\omega_{V/W}$ is semi-positive. We note that W is a curve. □

The following theorem is a slight generalization of Theorem 4.4. It seems to be well-known to specialists. It is buried in Kawamata's proof of the semi-positivity theorem (see [Ka1, §4]). We note that W is not necessarily complete in Theorem 4.10 below.

Theorem 4.10. *Let $f : V \to W$ be a proper surjective morphism between non-singular varieties with connected fibers. Assume that there exists a simple normal crossing divisor Σ on W such that f is smooth over $W_0 := W \setminus \Sigma$. We put $V_0 := f^{-1}(W_0)$, $f_0 := f|_{V_0}$, and $k := \dim f$. We assume that all the local monodromies on $R^{k+i}f_{0*}\mathbb{C}_{V_0}$ around Σ are unipotent. Let C be a complete curve on W. Then the restriction $(R^if_*\omega_{V/W})|_C$ is semi-positive.*

Proof. By Chow's lemma, desingularization theorem, and [Ka2, Proposition 1], we can assume that V and W are quasi-projective and Σ is a simple normal crossing divisor. We note that if $\psi : W' \to W$ is a proper birational morphism from a quasi-projective variety, then there exists a complete curve C' on W' such that $\psi(C') = C$. Let $g : \widetilde{C} \to C \subset W$ be the normalization of C. If $C \cap W_0 \neq \emptyset$, then $g^*R^if_*\omega_{X/Y} \simeq R^ih_*\omega_{D/\widetilde{C}}$ is semi-positive by Theorem 4.4, where D is a desingularization of the main component of $V \times_W \widetilde{C}$ and $h : D \to \widetilde{C}$. So, we have to treat the case when $C \subset \Sigma$. By cutting W, we can take an irreducible surface S on W such that $S \not\subset \Sigma$ and $C \subset S$. Take a desingularization $\pi : \widetilde{S} \to S \subset W$ of S such that $\pi^{-1}(\Sigma)$ is simple normal crossing on $\widetilde{S}$ and there is a smooth projective irreducible curve $\widetilde{C}$ on $\widetilde{S}$ with $\pi(\widetilde{C}) = C$. By [Ka2, Proposition 1], we have $\pi^*R^if_*\omega_{V/W} \simeq R^ih_*\omega_{T/\widetilde{S}}$, where T is a desingularization of the main component of $V \times_W \widetilde{S}$ and $h : T \to \widetilde{S}$. So, it is sufficient to check that $\pi^*R^if_*\omega_{V/W} \simeq R^ih_*\omega_{T/\widetilde{S}}$ is semi-positive on $\widetilde{C}$. We compactify $h : T \to \widetilde{S}$. Then we obtain an algebraic fiber space $\overline{h} : \overline{T} \to \overline{S}$. After modifying $\overline{h}$ birationally, we can assume that $\Delta := (\overline{S} \setminus \widetilde{S}) \cup \pi^{-1}(\Sigma)$ is a simple normal crossing divisor on $\overline{S}$. We can assume that $\overline{h}$ is smooth over $\overline{S} \setminus \Delta$. We note that $\widetilde{C}$ is an irreducible component of Δ. By Kawamata's covering trick [Ka1, Theorem 17], we can take a finite cover $\varphi : S' \to \overline{S}$ which induces a unipotent reduction $h' : T' \to S'$ (see [Ka1, Corollary 18] or [M, (4.5)]). We put $U := \varphi^{-1}(\widetilde{S})$. By Theorem 4.4, $R^ih'_*\omega_{T'/S'}$ is semi-positive and

$(R^ih'_*\omega_{T'/S'})|_U \simeq \varphi^*((R^i\overline{h}_*\omega_{\overline{T}/\overline{S}})|_{\widetilde{S}}) \simeq \varphi^*(R^ih_*\omega_{T/\widetilde{S}})$. Therefore, it is easy to check that $(R^ih_*\omega_{T/\widetilde{S}})|_{\widetilde{C}}$ is semi-positive since $\widetilde{C} \subset \widetilde{S}$. So, we obtain the required result. □

The following corollary is obvious by Theorem 4.10. It will be useful when we study algebraic fiber spaces in the relative setting.

Corollary 4.11. *In the same notation and assumptions as in* Theorem 4.10*, we further assume that W is proper over a variety B. Then $R^if_*\omega_{V/W}$ is semi-positive over B, that is, $(R^if_*\omega_{V/W})|_F$ is semi-positive for every fiber F of $W \to B$.*

Remark 4.12. In the proof of Theorem 4.10, we only use the following fact that $R^if_*\omega_{V/W}$ is characterized as the canonical extension of the bottom Hodge filtration.

The next theorem is an easy consequence of the semi-positivity theorem. For the proof, we use Theorem 4.9.

Theorem 4.13. *Let $f : X \to Y$ be an algebraic fiber space. Then*

$$q(Y) \leq q(X) \leq q(Y) + q(F),$$

where F is a general fiber. Moreover, if $q(F) = 0$, then $\mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ is an isomorphism.

Proof. By modifying f birationally, we can assume that there exists a simple normal crossing divisor Σ on Y such that f is smooth over $Y \setminus \Sigma$ and $\mathrm{Supp} f^*\Sigma$ is simple normal crossing. By Leray's spectral sequence, we have

$$0 \to H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X) \to H^0(Y, R^1f_*\mathcal{O}_X) \to 0.$$

We note that $H^2(Y, \mathcal{O}_Y) \to H^2(X, \mathcal{O}_X)$ is injective. Therefore, $q(Y) \leq q(X) = q(Y) + h^0(Y, R^1f_*\mathcal{O}_X)$. Let H be a very ample divisor on Y such that $h^0(Y, R^1f_*\mathcal{O}_X \otimes \mathcal{O}_Y(-H)) = 0$. We take H general. We have;

$$0 \to R^1f_*\mathcal{O}_X \otimes \mathcal{O}_Y(-H) \to R^1f_*\mathcal{O}_X \to R^1f_*\mathcal{O}_{f^{-1}H} \to 0.$$

Thus, we obtain that $h^0(Y, R^1f_*\mathcal{O}_X) \leq h^0(Y, R^1f_*\mathcal{O}_{f^{-1}H})$. By repeating this argument, we can assume that Y is a curve. By Theorem 4.9, $(R^if_*\mathcal{O}_X)^\vee \simeq R^{\dim f - i}\omega_{X/Y}$ is semi-positive. Then $h^0(Y, R^1f_*\mathcal{O}_X) \leq q(F)$ by Lemma 4.14 below. We note that the rank of $R^1f_*\mathcal{O}_X$ is $q(F)$. Thus, we get the required inequality. The latter part is obvious. □

Lemma 4.14. *Let $\mathcal{E}$ be a vector bundle on a non-singular projective curve V. Assume that the dual vector bundle $\mathcal{E}^\vee$ is semi-positive. Then $h^0(V, \mathcal{E}) \leq r$. In particular, if $h^0(V, \mathcal{E}) = r$, then $\mathcal{E}$ is trivial.*

Proof. We take a basis $\{\varphi_1, \cdots, \varphi_l\}$ of $H^0(V, \mathcal{E})$. We have to prove that $l \leq k$. We define $\psi_i := (\varphi_1, \cdots, \varphi_i) : \mathcal{O}_V^{\oplus i} \to \mathcal{E}$ for every i. We put $\psi_0 = 0$ for inductive arguments.

Claim. $\mathrm{Im}\psi_i \simeq \mathcal{O}_V^{\oplus i}$ is a subbundle of $\mathcal{E}$ for every i, where $\mathrm{Im}\psi_i$ denotes the image of ψ_i.

Proof of Claim. We use the induction on i. We assume that $\mathrm{Im}\psi_{i-1} \simeq \mathcal{O}_V^{\oplus(i-1)}$ is a subbundle of $\mathcal{E}$. Thus, we have $h^0(V, \mathrm{Im}\psi_{i-1}) = i - 1$. Therefore, $\mathrm{Im}\psi_i$ has rank i. Let $\mathcal{F}$ be the double dual of $\mathrm{Im}\psi_i$. Then $\mathcal{F}$ is a rank i subbundle of $\mathcal{E}$ such that ψ_i factors through $\mathcal{F}$. Since $\mathcal{F}^\vee$ is semi-positive by the semi-positivity of $\mathcal{E}^\vee$ and $\mathcal{F}$ is semi-positive by the definition of $\mathcal{F}$, we obtain that $\psi_i : \mathcal{O}_V^{\oplus i} \simeq \mathcal{F}$. □

Thus we obtain that $l \leq r$, that is, $h^0(V, \mathcal{E}) \leq r$. The latter part is obvious. □

The following corollary is a slight generalization of [Ka1, Corollary 2]. The proof is obvious by Theorem 4.13.

Corollary 4.15. *Let $f : X \to Y$ be an algebraic fiber space. Assume that sufficiently general fibers have zero Kodaira dimension. Then $0 \leq q(X) - q(Y) \leq q(F) \leq \dim f$, where F is a general fiber of f. Furthermore, if $q(X) - q(Y) = \dim f$, then general fibers are birationally equivalent to Abelian varieties.*

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 Japan

E-mail address: fujino@kurims.kyoto-u.ac.jp