Generalized permutative representation of Cuntz algebra. II
—Irreducible decomposition of periodic cycle—

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Abstract

We show irreducible decomposition formula for a certain class of representations of Cuntz algebra by succeeding our previous work. The complete reducibility of this class of representations is automatically proved by their decomposition formulae.

1 Introduction

In ordinary theory of operator algebra, an irreducible decomposition of representation is not studied so much because that decomposition is not unique in general. In spite of such common sense, we can construct non trivial decomposition theory of a class of representations of Cuntz algebra with complete reducibility and uniqueness of irreducible decomposition.

In [8], we introduce a class of representations of Cuntz algebra with parameter in a subset of the tensor space over a finite dimensional complex vector space. It is a kind of generalization of permutative representation with cycle by [5, 6, 7]. We have already shown uniqueness, irreducibility and equivalence for this class of representations for "non periodic" case. We review these results in Theorem 2.3. In this article, we show the uniqueness of decomposition (Lemma 3.4) and the irreducible decomposition formula about this class (Theorem 5.4). By this decomposition, the complete reducibility of this class is automatically proved (Corollary 5.5).

2 Preliminaries

We review the symbol and known results in this section.

Put $N \geq 2$. Let $S(\mathbf{C}^N) \equiv \{z \in \mathbf{C}^N : \|z\| = 1\}$ be the unit complex sphere in $\mathbf{C}^N$ and

$$TS(\mathbf{C}^N) \equiv \bigcup_{k\geq 1} S(\mathbf{C}^N)^{\otimes k},$$

$$S(\mathbf{C}^N)^{\otimes k} \equiv \left\{ w^{(1)}\otimes\cdots\otimes w^{(k)} \in (\mathbf{C}^N)^{\otimes k} : \begin{array}{l} w^{(i)} \in S(\mathbf{C}^N) \\ i=1,\ldots,k \end{array} \right\}.$$

Let $\mathcal{O}_N$ be the Cuntz algebra with generators $s_1,\ldots,s_N$ which satisfy $s_i^*s_j = \delta_{ij}I$ and $\sum_{i=1}^N s_is_i^* = I$. For $z=(z_1,\ldots,z_N)\in S(\mathbf{C}^N)$, we denote

$$s(z) \equiv z_1s_1+\cdots+z_Ns_N.$$

For $w = w^{(1)}\otimes\cdots\otimes w^{(k)} \in S(\mathbf{C}^N)^{\otimes k}$, define

$$s(w) \equiv s(w^{(1)})\cdots s(w^{(k)}).$$

In this paper, a representation always means a unital *-representation.

Definition 2.1 *$(\mathcal{H},\pi,\Omega)$ is the GP(= generalized permutative) representation of $\mathcal{O}_N$ with cycle by $w\in S(\mathbf{C}^N)^{\otimes k}$ if $(\mathcal{H},\pi)$ is a cyclic representation of $\mathcal{O}_N$ with the cyclic unit vector Ω which satisfies the following equation:*

$$\pi(s(w))\Omega = \Omega. \tag{2.1}$$

We denote $GP(w)\equiv(\mathcal{H},\pi,\Omega)$. A unit vector $\Omega\in\mathcal{H}$ which satisfies the equation (2.1) is called the GP vector of $(\mathcal{H},\pi)$ with respect to w. k is called the length of cycle of $(\mathcal{H},\pi,\Omega)$.

A naive meaning of "cycle" is explained in a paragraph after Definition 3.3 in [8]. Correspondence with ordinary permutative representation in [5] is explained in subsection 3.3 in [8] and subsection 7.1 in this article. The generalization of "chain type" is treated in [9]. Examples are shown in subsection 4.4, 6.2 in [8] and section 6 in this article. The relation between states and GP representations are shown in subsection 6.1 in [8] and subsection 7.5 in this article. In our paper, a symbol $GP(w)$ means a property of representation or a representation itself as the case may be.

In order to review our results in [8], we prepare some notions about parameters in $TS(\mathbf{C}^N)$ and representations.

Definition 2.2 (i) *$w\in S(\mathbf{C}^N)^{\otimes k}$ is periodic if there is $\sigma\in\mathbf{Z}_k\setminus\{id\}$ such that $\hat{\sigma}(w)=w$ where $\hat{}$ is an action of the cyclic group $\mathbf{Z}_k$ on $(\mathbf{C}^N)^{\otimes k}$ by transposition of tensor factors.*

(ii) *$w\in S(\mathbf{C}^N)^{\otimes k}$ is non periodic if w is not periodic.*

(iii) *For $w,w^{'}\in TS(\mathbf{C}^N)$, $w\sim w^{'}$ if there are $k\geq 1$ and $\sigma\in\mathbf{Z}_k$ such that $w,w^{'}\in S(\mathbf{C}^N)^{\otimes k}$ and $\hat{\sigma}(w)=w^{'}$.*

(iv) *For two representations* $(\mathcal{H}_1, \pi_1)$ *and* $(\mathcal{H}_2, \pi_2)$ *of* $\mathcal{O}_N$, $(\mathcal{H}_1, \pi_1) \sim (\mathcal{H}_2, \pi_2)$ *means the unitary equivalence between* $(\mathcal{H}_1, \pi_1)$ *and* $(\mathcal{H}_2, \pi_2)$.

(v) *A data* $(\mathcal{H}, \{s_1, \ldots, s_N\})$ *means a representation of* $\mathcal{O}_N$ *as a family* $\{s_1, \ldots, s_N\}$ *of operators on* $\mathcal{H}$ *which satisfies the condition of generators of* $\mathcal{O}_N$.

Specially, for $w, w' \in TS(\mathbf{C}^N)$, $GP(w) \sim GP(w')$ means that two cyclic representations of $\mathcal{O}_N$ which have cyclic vectors satisfying condition (2.1) with respect to w, w' are unitarily equivalent.

Theorem 2.3 *([8])*

(i) *(Existence) For any* $w \in TS(\mathbf{C}^N)$, *there exists* $GP(w)$, *that is, there exists a cyclic representation* $(\mathcal{H}, \pi, \Omega)$ *of* $\mathcal{O}_N$ *which satisfies condition (2.1).*

(ii) *(Uniqueness and irreducibility) If* $w \in TS(\mathbf{C}^N)$ *is non periodic, then* $GP(w)$ *is unique up to unitary equivalence, and irreducible.*

(iii) *(Equivalence) For non periodic elements* $w, w' \in TS(\mathbf{C}^N)$, $GP(w) \sim GP(w')$ *if and only if* $w \sim w'$.

Proof. (i) Proposition 3.4 in [8]. (ii) The uniqueness is in Proposition 5.4 in [8]. The irreducibility is in Proposition 5.5 in [8]. (iii) Proposition 5.11 in [8]. ∎

By Theorem 2.3 (ii), we can regard a symbol $GP(w)$ as the representative element of an equivalence class of irreducible representations of $\mathcal{O}_N$ which satisfies condition (2.1) with respect to non periodic $w \in TS(\mathbf{C}^N)$.

Note that our results in Theorem 2.3 (ii), (iii) are assumed the non-periodicity with respect to parameter $w \in TS(\mathbf{C}^N)$. Our aims in this article are the classification and the computation of GP representation $GP(w)$ of $\mathcal{O}_N$ in the periodic case.

Lemma 2.4 *Let* $(\mathcal{H}, \pi, \Omega)$ *be the GP representation of* $\mathcal{O}_N$ *by non periodic* $w \in TS(\mathbf{C}^N)$. *If* $x \in \mathcal{H}$ *satisfies* $< x|\Omega >= 0$, *then the following holds:*

$$\lim_{m \to \infty} (\pi(s(w))^*)^m x = 0.$$

Proof. See Lemma 5.2 in [8]. ∎

3 Disjointness and uniqueness of decomposition of GP representations

Before we show the decomposition formulae, we mention about general properties of GP representations of Cuntz algebra.

Lemma 3.1 *Let* $(\mathcal{H},\pi)$ *be a representation of* $\mathcal{O}_N$. *If there is the GP vector* Ω *of* $(\mathcal{H},\pi)$ *with respect to* $w \in TS(\mathbf{C}^N)$, *then there is a subrepresentation* $V \subset \mathcal{H}$ *of* $\mathcal{O}_N$ *such that* $(V,\pi|_V,\Omega)$ *is* $GP(w)$.

Proof. The subrepresentation $V \equiv \pi(\mathcal{O}_N)\Omega$ is cyclic and it satisfies the condition (2.1) in Definition 2.1. Hence it is $GP(w)$. ∎

By Lemma 3.1, the existence of GP vector in a given representation of Cuntz algebra always assures the existence of subrepresentation which is a GP representation. From here, we consider the relation among GP representations which are defined on a common Hilbert space.

Lemma 3.2 *Let* $(\mathcal{H},\pi)$ *be a representation of* $\mathcal{O}_N$. *Assume that there is the GP vector* $\Omega \in \mathcal{H}$ *with respect to non periodic* $w \in TS(\mathbf{C}^N)$. *If a vector* $x \in \mathcal{H}$ *satisfies* $< x|\Omega > \neq 0$, *then* $\Omega \in \pi(\mathcal{O}_N)x$.

Proof. By assumption and Lemma 3.1, if put $V \equiv \pi(\mathcal{O}_N)\Omega$, then $(V,\pi|_V,\Omega)$ is the GP representation of $\mathcal{O}_N$ by w. By assumption, we can denote

$$x = a\Omega + y$$

where $a \in \mathbf{C}$, and $y \in \mathcal{H}$ such that $a \neq 0$, $< y|\Omega >= 0$. By Lemma 2.4,

$$\lim_{n\to\infty} \{\pi(s(w))^*\}^n x = a\Omega.$$

Hence $\Omega \in \pi(\mathcal{O}_N)x$. ∎

Lemma 3.3 *(Disjointness) Let* $(\mathcal{H},\pi)$ *be a representation of* $\mathcal{O}_N$. *Assume that* $M \geq 1$ *and there are GP vectors* $\Omega_1,\ldots,\Omega_M$ *of* $(\mathcal{H},\pi)$ *with respect to non periodic parameters* $w_1,\ldots,w_M \in TS(\mathbf{C}^N)$, *respectively. If* $w_1,\ldots,w_M$ *are mutually inequivalent each other, then there is a subrepresentation* $(V,\pi|_V)$ *of* $(\mathcal{H},\pi)$ *such that*

$$(V,\pi|_V) \sim \bigoplus_{i=1}^{M} GP(w_i).$$

Proof. If $M = 1$, then it holds by Lemma 3.1. Assume that $M \geq 2$. By Lemma 3.1, there is a subrepresentation $(V_i, \pi|_{V_i}, \Omega_i)$ in $\mathcal{H}$ which is $GP(w_i)$ for each $i = 1, \ldots, M$. Since w_i is non periodic, $GP(w_i)$ is irreducible by Theorem 2.3 (ii). By assumption, $w_i \not\sim w_j$, hence $\pi|_{V_i} \not\sim \pi|_{V_j}$ by Theorem 2.3 (iii) when $i \neq j$. If $V_i \cap V_j \neq \{0\}$ for $i \neq j$, then its intersection generates both V_i and V_j by $\pi(\mathcal{O}_N)$. Hence $V_i = V_j$ and $(V_i, \pi|_{V_i}) = (V_j, \pi|_{V_j})$. This is contradiction. Hence $V_i \cap V_j = \{0\}$ when $i \neq j$. Therefore a subspace $V \equiv \bigoplus_{i=1}^{M} V_i$ satisfies the condition of the statement. ∎

By Lemma 3.3, the inequivalence of parameters induces the disjointness of associated GP representations automatically.

Lemma 3.4 *(Uniqueness of irreducible decomposition) Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$. If there is a family $\{w_i\}_{i=1}^{M}$ of non periodic elements in $TS(\mathbf{C}^N)$ such that $(\mathcal{H}, \pi) \sim \bigoplus_{i=1}^{M} GP(w_i)$, then this decomposition is unique up to unitary equivalence.*

Proof. Let $\{V_i : i = 1, \ldots, M\}$ be an orthogonal family of subspaces of $\mathcal{H}$ such that $(V_i, \pi|_{V_i})$ is a subrepresentation of $\mathcal{O}_N$ which is unitarily equivalent to $GP(w_i)$ for $i = 1, \ldots, M$, respectively. Then there is an orthonormal family $\{\Omega_i : i = 1, \ldots, M\}$ of vectors in $\mathcal{H}$ such that $\Omega_i \in V_i$ and $\pi(s(w_i))\Omega_i = \Omega_i$ for $i = 1, \ldots, M$.

Assume that $\mathcal{H} = \bigoplus_{\lambda \in \Lambda} V_\lambda^{'}$ is another irreducible decomposition. Then there is $\lambda \in \Lambda$ such that there is $x_\lambda \in V_\lambda^{'}$ and $< \Omega_i | x_\lambda > \neq 0$. By Lemma 3.2, $\Omega_i \in \pi(\mathcal{O}_N) x_\lambda = V_\lambda^{'}$. Hence $V_\lambda^{'} = V_i$. In this way, we have $\lambda_i \in \Lambda$ such that $V_{\lambda_i}^{'} = V_i$ for each $i = 1, \ldots, M$. Therefore $(V_{\lambda_i}^{'}, \pi|_{V_i^{'}}) \sim (V_i, \pi|_{V_i})$. Automatically, we have a numbering $\Lambda = \{\lambda_i : i = 1, \ldots, M\}$ and $V_{\lambda_1}^{'} \oplus \cdots \oplus V_{\lambda_M}^{'} = V_1 \oplus \cdots \oplus V_M$. Hence we finish to show the uniqueness of decomposition. ∎

Corollary 3.5 *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$. If there are two families $\{w_j\}_{j=1}^{M}$ and $\{v_l\}_{l=1}^{M^{'}}$ of non periodic elements in $TS(\mathbf{C}^N)$ such that $(\mathcal{H}, \pi) \sim \bigoplus_{j=1}^{M} GP(w_j)$ and $(\mathcal{H}, \pi) \sim \bigoplus_{l=1}^{M^{'}} GP(v_l)$, then $M = M^{'}$ and there is $\sigma \in \mathfrak{S}_M$ such that $w_{\sigma(i)} \sim v_i$ for each $i = 1, \ldots, M$.*

Proof. Note that $\{w_j\}_{j=1}^{M}$ and $\{v_l\}_{l=1}^{M^{'}}$ arise two irreducible decompositions by Theorem 2.3 (ii). By Lemma 3.4, $M = M^{'}$ and there is $\sigma \in \mathfrak{S}_M$ such that $GP(v_i) \sim GP(w_{\sigma(i)})$ for each $i = 1, \ldots, M$. By Theorem 2.3 (iii), $w_{\sigma(i)} \sim v_i$ for each $i = 1, \ldots, M$. ∎

4 Degeneracy of periodic cycle

For a given $w \in TS(\mathbf{C}^N)$, we always have a representation $GP(w)$ by Theorem 2.3 (i). Although, the uniqueness of $GP(w)$ does not always hold when w is periodic. In order to classify $GP(w)$ for periodic w, we introduce a new parameter here.

Denote

$$\begin{aligned} TS_P(\mathbf{C}^N) \equiv &\ \{w \in TS(\mathbf{C}^N) : w \text{ is periodic }\}, \\ TS_{NP}(\mathbf{C}^N) \equiv &\ \{w \in TS(\mathbf{C}^N) : w \text{ is non periodic }\}. \end{aligned}$$

Then

$$TS(\mathbf{C}^N) = \quad TS_P(\mathbf{C}^N) \sqcup TS_{NP}(\mathbf{C}^N), \tag{4.1}$$

$$\begin{aligned} TS_P(\mathbf{C}^N) = &\ \{w^{\otimes k} : w \in TS(\mathbf{C}^N),\, k \geq 2\} \\ = &\ \{w^{\otimes k} : w \in TS_{NP}(\mathbf{C}^N),\, k \geq 2\} \end{aligned} \tag{4.2}$$

where $v^{\otimes k} = \underbrace{v \otimes \cdots \otimes v}_{k}$. Particularly, $S(\mathbf{C}^N) \subset TS_{NP}(\mathbf{C}^N)$. The ambiguity of GP representation of periodic case occurs because the period number of parameter *degenerates* in the level of representation in general. We show this meaning in the following.

Definition 4.1 *Let $(\mathcal{H}, \pi, \Omega)$ be $GP(v^{\otimes p})$ for $v \in TS_{NP}(\mathbf{C}^N)$ and $p \geq 1$.*

(i) *A complex subspace $W(\mathcal{H}, \pi, \Omega) \equiv \mathrm{Lin} < \{\pi(s(v))^l \Omega \in \mathcal{H} : l = 0, \ldots, p-1\} >$ is called the period subspace of $\mathcal{H}$.*

(ii) *$(\mathcal{H}, \pi, \Omega)$ is degenerate if $\dim W(\mathcal{H}, \pi, \Omega) < p$.*

(iii) *$\dim W(\mathcal{H}, \pi, \Omega)$ is called the proper period of $(\mathcal{H}, \pi, \Omega)$.*

Lemma 4.2 *Under assumption and notation in Definition 4.1, the followings hold:*

(i) *$1 \leq \dim W(\mathcal{H}, \pi, \Omega) \leq p$.*

(ii) *$\pi(s(v))|_{W(\mathcal{H},\pi,\Omega)}$ is a unitary from $W(\mathcal{H}, \pi, \Omega)$ to $W(\mathcal{H}, \pi, \Omega)$.*

(iii) *$\pi(s(v))^p|_{W(\mathcal{H},\pi,\Omega)}$ is the identity operator on $W(\mathcal{H}, \pi, \Omega)$.*

(iv) *If $p = 1$, then $(\mathcal{H}, \pi, \Omega)$ is always non degenerate.*

Proof. (i) By definition of $W(\mathcal{H}, \pi, \Omega)$, it holds. (ii) $\pi(s(v))$ is an isometry on $\mathcal{H}$ for any $v \in TS(\mathbf{C}^N)$. By definition of $W(\mathcal{H}, \pi, \Omega)$, $\pi(s(v))W(\mathcal{H}, \pi, \Omega) \subset W(\mathcal{H}, \pi, \Omega)$. Hence $\pi(s(v))$ is an isometry on $W(\mathcal{H}, \pi, \Omega)$, too. Because of (i), $\pi(s(v))$ is a unitary automatically. (iii) Since $\pi(s(v))^p\Omega = \pi(s(v^{\otimes p}))\Omega = \Omega$, $\pi(s(v))^p(\pi(s(v))^l\Omega) = \pi(s(v))^{p+l}\Omega = \pi(s(v))^l\Omega$ for each $l = 0, \ldots, p-1$. Hence $\pi(s(v))^p$ is the identity on $W(\mathcal{H}, \pi, \Omega)$. (iv) By definition of $W(\mathcal{H}, \pi, \Omega)$, $\dim W(\mathcal{H}, \pi, \Omega) = 1$ when $p = 1$. ∎

Lemma 4.3 *Let $(\mathcal{H}, \pi, \Omega)$ be $GP(v^{\otimes p})$ for $p \geq 2$. If the proper period of $(\mathcal{H}, \pi, \Omega)$ is q, then the period subspace of $(\mathcal{H}, \pi, \Omega)$ is spanned by eigen vectors $\Omega_1, \ldots, \Omega_q$ of $\pi(s(v))$ with mutually different eigen values $\xi_1, \ldots, \xi_q$ in $\{e^{2\pi\sqrt{-1}l/p} : l = 0, \ldots, p-1\}$.*

Proof. Let W be the period subspace of $(\mathcal{H}, \pi, \Omega)$. We identify $\pi(s_i)$ and s_i for $i = 1, \ldots, N$ here. Assume $\dim W = q$. Then a family $\{s(v)^l\Omega\}_{l=0}^{q-1}$ of vectors is linearly independent in W. By Lemma 4.2, an operator $A \equiv s(v)|_W$ is an action of $\mathbf{Z}_p$ on W. Therefore W is decomposed into (one-dimensional) irreducible representations $W_1, \ldots, W_q$ of $\mathbf{Z}_p$. Choose $h_i \in W_i$, $h_i \neq 0$ for $i = 1, \ldots, q$. Then there is $\xi_i \in \mathbf{C}$ such that $Ah_i = \xi_i h_i$ for each $i = 1, \ldots, q$. A representation W of $\mathbf{Z}_p$ is cyclic if and only if any component in the irreducible decomposition of W has multiplicity 1. Hence $\xi_1, \ldots, \xi_q$ are mutually different. By definition of A,

$$s(v)h_i = Ah_i = \xi_i h_i \quad (i = 1, \ldots, q).$$

By Lemma 4.2 (ii), $|\xi_i| = 1$. By Lemma 4.2 (iii), $\xi_i \in \{e^{2\pi\sqrt{-1}l/p} : l = 0, \ldots, p-1\}$ for each $i = 1, \ldots, q$. ∎

Lemma 4.4 *If $p \geq 2$ and $v \in TS_{NP}(\mathbf{C}^N)$, then for each $1 \leq q \leq p$, there exists $(\mathcal{H}, \pi, \Omega)$ which is $GP(v^{\otimes p})$ with the proper period q.*

Proof. We construct them concretely. If $q = 1$, then $GP(\xi v)$ satisfies the condition of $GP(v^{\otimes p})$ for each number $\xi \in \mathbf{C}$ such that $\xi^p = 1$.

Assume that $q \geq 2$. Choose a subset $\{\xi_i\}_{i=1}^q \subset \{e^{2\pi\sqrt{-1}l/p} : l = 0, \ldots, p-1\}$ which consists of q-mutually different elements. Prepare a family

$\{(V_i,\pi_i,\Omega_i)\}_{i=1}^q$ of GP representations with respect to a family $\{\xi_i v\}_{i=1}^q$ of parameters. Put

$$\mathcal{H}\equiv V_1\oplus\cdots\oplus V_q,\quad \Omega\equiv\frac{1}{\sqrt{q}}\sum_{i=1}^q\Omega_i\in\mathcal{H},\quad \pi\equiv\pi_1\oplus\cdots\oplus\pi_q.$$

Then $\pi(s(v^{\otimes p}))\Omega=\pi(s(v))^p\Omega=\Omega$ and $\dim W=q$ for $W\equiv \mathrm{Lin}<\{\pi(s(v))^l\Omega\}_{l=0}^{p-1}>$. Note that vectors $((\xi_i)^k)_{k=1}^q$ and $((\xi_j)^k)_{k=1}^q$ in $\mathbf{C}^q$ are linearly independent when $i\neq j$. Hence $\Omega,\pi(s(v))\Omega,\ \ldots,\ \pi(s(v))^{q-1}\Omega\in \mathrm{Lin}<\{\Omega_i : i=1,\ldots,q\}>$ are linearly independent in $\mathcal{H}$. From this, $\Omega_1,\ldots,\Omega_q\in \mathrm{Lin}<\{\pi(s(v))^{i-1}\Omega : i=1,\ldots,q\}>$. Therefore $\Omega_i\in\pi(\mathcal{O}_N)\Omega$ for each $i=1,\ldots,q$. Hence

$$V_i=\pi_i(\mathcal{O}_N)\Omega_i\subset\pi(\mathcal{O}_N)\Omega$$

for each $i=1,\ldots,q$. Therefore $\pi(\mathcal{O}_N)\Omega=\mathcal{H}$ and $(\mathcal{H},\pi,\Omega)$ is cyclic. Since $(\mathcal{H},\pi,\Omega)$ satisfies condition (2.1) with respect to $v^{\otimes p}$ and the proper period of $(\mathcal{H},\pi,\Omega)$ is q, we obtain $(\mathcal{H},\pi,\Omega)$ in the statement. ∎

By Lemma 4.4, $GP(v^{\otimes p})$ is not unique when $p\geq 2$. Hence we classify representations which satisfy the condition of $GP(v^{\otimes p})$ in the next section.

5 Irreducible decomposition of GP representation

For $p\geq 2$ and $v\in TS_{NP}(\mathbf{C}^N)$, we classify $GP(v^{\otimes p})$ and show decomposition formulae. In order to classify periodic case, we introduce new symbols.

Definition 5.1 *A representation $(\mathcal{H},\pi,\Omega)$ of $\mathcal{O}_N$ is $GP(v^{\otimes p};q)$ if $(\mathcal{H},\pi,\Omega)$ is $GP(v^{\otimes p})$ which has the proper period q.*

Lemma 5.2 *For each $1\leq q\leq p$, there exists $GP(v^{\otimes p};q)$.*

Proof. By Lemma 4.4, it holds. ∎

Theorem 5.3 *Let $p\geq 2$ and $1\leq q\leq p$. If $(\mathcal{H},\pi,\Omega)$ is $GP(v^{\otimes p};q)$ for $v\in TS_{NP}(\mathbf{C}^N)$, then there is a subset $\{\xi_i\}_{i=1}^q\subset\{e^{2\pi\sqrt{-1}l/p}: l=0,\ldots,p-1\}$ such that $\xi_i\neq\xi_j$ when $i\neq j$ and the following equivalence holds:*

$$(\mathcal{H},\pi)\sim\bigoplus_{i=1}^q GP(\xi_i v).$$

Proof. By Lemma 4.3, there are eigen vectors $\Omega_1,\ldots,\Omega_q$ of $\pi(s(v))$ in $\mathcal{H}$ with mutually different eigen values $\eta_1,\ldots,\eta_q$, respectively where we normalize them as $\|\Omega_i\| = 1$ for $i = 1,\ldots,q$. Hence if we let $w_i \equiv \overline{\eta}_i v$ for $i = 1,\ldots,q$, then the following equations hold:

$$\pi(s(w_i))\Omega_i = \overline{\eta}_i\pi(s(v))\Omega_i = \overline{\eta}_i(\eta_i\Omega_i) = \Omega_i \quad (i = 1,\ldots,q).$$

By assumption, w_i is non periodic for each $i = 1,\ldots,q$. Furthermore $\{w_i\}_{i=1}^q$ are mutually inequivalent in $TS(\mathbf{C}^N)$. By Lemma 3.3, there is a subrepresentation $(V,\pi|_V)$ of $(\mathcal{H},\pi)$ such that

$$(V,\pi|_V) \sim \bigoplus_{i=1}^{q} GP(w_i) \sim \bigoplus_{i=1}^{q} GP(\overline{\eta}_i v) \sim \bigoplus_{i=1}^{q} GP(\xi_i v) \tag{5.1}$$

where $\xi_i \equiv \overline{\eta}_i \in \{e^{2\pi\sqrt{-1}l/p} : l = 0,\ldots,p-1\}$ for $i = 1,\ldots,q$. On the other hand, $\Omega \in \mathrm{Lin} < \{\Omega_1,\ldots,\Omega_q\} > \subset V$ by definition of Ω_i in Lemma 4.3. By assumption, Ω is a cyclic vector of a representation $(\mathcal{H},\pi)$. Therefore

$$\mathcal{H} = \pi(\mathcal{O}_N)\Omega \subset V \subset \mathcal{H}. \tag{5.2}$$

By (5.1) and (5.2),

$$\mathcal{H} = V \sim \bigoplus_{i=1}^{q} GP(\xi_i v).$$

Hence the statement holds. ∎

Theorem 5.4 *(Decomposition formula) Assume that $(\mathcal{H},\pi,\Omega)$ is $GP(v^{\otimes p})$ for $v \in TS_{NP}(\mathbf{C}^N)$ and $p \geq 2$. Then there exists $1 \leq q \leq p$ and a subset $\{\xi_i\}_{i=1}^q \subset \{e^{2\pi\sqrt{-1}l/p} : l = 0,\ldots,p-1\}$ uniquely such that the following irreducible decomposition holds and this decomposition is unique up to unitary equivalence:*

$$(\mathcal{H},\pi) \sim \bigoplus_{i=1}^{q} GP(\xi_i v).$$

Proof. Let q be the proper period of $(\mathcal{H},\pi,\Omega)$. Then $(\mathcal{H},\pi,\Omega)$ is $GP(v^{\otimes p};q)$. By Theorem 5.3, the irreducible decomposition holds. The uniqueness holds by Lemma 3.4. ∎

Corollary 5.5 (i) *(Complete reducibility) The GP representation of $\mathcal{O}_N$ with cycle which is defined in Definition 2.1 is completely reducible.*

(ii) *Let* $v \in TS_{NP}(\mathbf{C}^N)$ *and* $p \geq 2$. *Assume that* $(\mathcal{H}_i, \pi_i, \Omega_i)$ *is* $GP(v^{\otimes p}; q_i)$ *for* $i = 1, 2$, *respectively. If* $q_1 \neq q_2$, *then* $(\mathcal{H}_1, \pi_1) \not\sim (\mathcal{H}_2, \pi_2)$.

(iii) *Irreducible decomposition of GP representation with cycle is closed in GP representations with cycle, too.*

Proof.
(i) By (4.1), it is sufficient to consider two cases $w \in TS_{NP}(\mathbf{C}^N)$ and $w \in TS_P(\mathbf{C}^N)$ about $GP(w)$. When $w \in TS_{NP}(\mathbf{C}^N)$, $GP(w)$ is irreducible by Theorem 2.3 (ii). Hence the statement holds in this case. Assume that $w \in TS_P(\mathbf{C}^N)$. By (4.2), we can write $w = v^{\otimes p}$ for $v \in TS_{NP}(\mathbf{C}^N)$ and $p \geq 2$. By Theorem 5.4, this case is completely reducible, too.
(ii) By Theorem 5.4 and Corollary 3.5, the statement holds by comparing the components of decompositions.
(iii) By Theorem 5.4, it holds. ∎

Corollary 5.6 *Assume that* $v \in TS_{NP}(\mathbf{C}^N)$ *and* $p \geq 2$.

(i) *Let* $(\mathcal{H}, \pi, \Omega)$ *be* $GP(v^{\otimes p}; 1)$. *Then there is* $\xi \in \mathbf{C}$ *such that* $\xi^p = 1$ *and* $(\mathcal{H}, \pi, \Omega)$ *is* $GP(\xi v)$.

(ii) $GP(v^{\otimes p}; 1)$ *is irreducible.*

(iii) *There are just* ${}_pC_q = \frac{p!}{q!(p-q)!}$ *number of inequivalent representations of* $\mathcal{O}_N$ *which are* $GP(v^{\otimes p}; q)$ *for* $1 \leq q \leq p$.

(iv) *If* $GP(v^{\otimes p})$ *does not degenerate, then*

$$GP(v^{\otimes p}) = GP(v) \oplus GP(\xi v) \oplus GP(\xi^2 v) \oplus \cdots \oplus GP(\xi^{p-1} v)$$

where $\xi \equiv e^{2\pi\sqrt{-1}/p}$.

(v) $GP(v^{\otimes p}; p)$ *is unique up to unitary equivalence.*

6 Example

Let $\{\varepsilon_i : i = 1, \ldots, N\}$ be the canonical basis of $\mathbf{C}^N$ and $\{e_n : n \in \mathbf{N}\}$ that of $l_2(\mathbf{N})$, too where $\mathbf{N} = \{1, 2, 3, \ldots, \}$.

Example 6.1 Let $\xi \equiv e^{2\pi\sqrt{-1}/p}$ and $\eta \equiv e^{2\pi\sqrt{-1}/q}$ for positive integers p and q. If p and q are prime each other, then $GP(\xi\varepsilon_1) \oplus GP(\eta\varepsilon_1) \sim GP(\varepsilon_1^{\otimes pq}; 2)$. This equivalence holds on $\mathcal{O}_N$ for any $N \geq 2$.

Example 6.2 Let $(l_2(\mathbf{N}), \pi_S)$ be *the standard representation of* $\mathcal{O}_N$ which is defined by $\pi_S(s_i)e_n \equiv e_{N(n-1)+i}$ for $n \geq 1$ and $i = 1, \ldots, N$. Let $\gamma_z : \mathcal{O}_N \to \mathcal{O}_N$ be the so-called gauge action for $z \in U(1)$ on $\mathcal{O}_N$ which is defined by $\gamma_z(s_i) \equiv z s_i$ for $i = 1, \ldots, N$. Then the following holds:

$$\bigoplus_{i=1}^{p} \left(l_2(\mathbf{N}),\, \pi_S \circ \left(\gamma_{\xi_p}\right)^{i-1} \right) \sim GP(\varepsilon_1^{\otimes p}; p)$$

where $\xi_p \equiv e^{2\pi\sqrt{-1}/p}$ for $p \geq 1$. This shows that suitable combination of GP representations is cyclic, too. In this way, GP representations are related to the group action on $\mathcal{O}_N$. We treat this study in [10].

Example 6.3 For a unitary matrix $g \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$, a representation $(l_2(\mathbf{N}), \{s_1, s_2\})$ of $\mathcal{O}_2$ which is defined by

$$s_1 e_1 \equiv a e_2 + c e_3, \quad s_1 e_2 \equiv e_1, \quad s_2 e_1 \equiv b e_2 + d e_3, \quad s_2 e_2 \equiv e_4,$$

$$s_1 e_n \equiv e_{2n-1}, \quad s_2 e_n \equiv e_{2n} \quad (n \geq 3)$$

satisfies

$$s_1(\alpha s_1 + \beta s_2)e_1 = e_1$$

for $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \equiv g^{-1} \in U(2)$. Hence this representation is $GP(w)$ for $w \equiv \varepsilon_1 \otimes (\alpha\varepsilon_1 + \beta\varepsilon_2)$. If $\alpha\beta \neq 0$, then this representation does not belong to the class of representation by [5, 6, 7]. $\beta \neq 0$ if and only if w is non periodic if and only if $GP(w)$ is irreducible. If $\beta = 0$, then $|\alpha| = 1$ and the following equivalence holds:

$$GP((\alpha^{1/2}\varepsilon_1)^{\otimes 2}; 2) \sim GP(\alpha^{1/2}\varepsilon_1) \oplus GP(-\alpha^{1/2}\varepsilon_1).$$

Note that this representation is determined by only $(\alpha, \beta) \in \mathbf{C}^2$, $|\alpha|^2 + |\beta|^2 = 1$ up to unitary equivalence. Furthermore for each two elements in $S(\mathbf{C}^2)$, associated representations are inequivalent each other. In this sense, $S(\mathbf{C}^2)$ is a parameter space of unitary equivalence classes of representations of $\mathcal{O}_2$. This structure is shown in [10] in more detail.

7 Application

7.1 Permutative representation with cycle

A class of representations in [5, 6, 7] with cycle is a subclass of GP representations with cycle (see subsection 3.3 in [8], too). It is corresponded to a family

$$\{z\varepsilon_I \in TS(\mathbf{C}^N) : z \in U(1),\, I \in \{1,\ldots,N\}^k,\, k \geq 1\}$$

of parameters where $\varepsilon_I \equiv \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_k}$ for $I = (i_1,\ldots,i_k)$. In [1, 4], the symbol $GP(z\varepsilon_I)$ is denoted by $\mathrm{Rep}(I;\bar{z})$. Hence our decomposition formula in Theorem 5.4 holds about them, too. For example, when $p \geq 2$, in the case of non degenerate proper period does not degenerate decomposed as follows:

$$GP(z\varepsilon_I^{\otimes p};p) = GP((z^{1/p}\varepsilon_I)^{\otimes p};p) \sim \bigoplus_{j=1}^{p} GP(\xi_p^{j-1}\cdot z^{1/p}\varepsilon_I)$$

where $I \in \{1,\ldots,N\}^k$ is a non periodic multi index. Note that a term in the right hand side $GP(\xi_p^{j-1}\cdot z^{1/p}\varepsilon_I)$ means an irreducible representation $(\mathcal{H},\{s_1,\ldots,s_N\})$ which satisfies

$$s_I\Omega = \overline{\xi_p^{j-1}\cdot z^{1/p}}\Omega$$

for suitable non zero vector Ω where $s_I \equiv s(\varepsilon_I)$.

7.2 Spectrum of Cuntz algebra

Let $\mathrm{Spec}\mathcal{O}_N$ be the set of all unitary equivalence classes of irreducible representations of $\mathcal{O}_N$. We treat relation between non periodic case of GP representation and $\mathrm{Spec}\mathcal{O}_N$ in subsection 6.3 in [8].

If $GP(w)$ is irreducible for $w \in TS_P(\mathbf{C}^N)$, $GP(w)$ is equivalent to $GP(v)$ for some $v \in TS_{NP}(\mathbf{C}^N)$. Hence the following equality holds:

$$\{GP(w) : w \in TS(\mathbf{C}^N),\, GP(w) \text{ is irreducible }\}/\sim$$
$$= \{GP(w) : w \in TS_{NP}(\mathbf{C}^N)\}/\sim .$$

Therefore the subset of $\mathrm{Spec}\mathcal{O}_N$ associated with GP representation with cycle is arisen from only $TS_{NP}(\mathbf{C}^N)$. Chain case is treated in [9].

7.3 Classification of endomorphisms of Cuntz algebra

In [11], we classify a class of unital *-endomorphisms of Cuntz algebra by computing the branching rule of them on permutative representations. For example an endomorphism ρ of $\mathcal{O}_2$ which is defined by

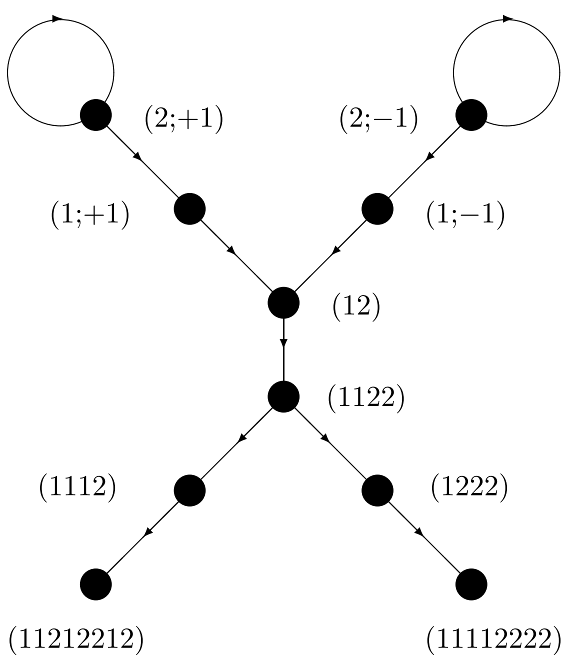
$$\begin{cases} \rho(s_1) \equiv & s_1 s_2 s_1^* + s_1 s_1 s_2^*, \\ \rho(s_2) \equiv & s_2 \end{cases} \tag{7.1}$$

arises a transformation of representations of $\mathcal{O}_2$ as

$$(\mathcal{H}, \pi) \mapsto (\mathcal{H}, \pi \circ \rho). \tag{7.2}$$

By this transformation, we get information about ρ. We denote ρ in (7.1) by ψ_{12}. If $\rho^{'}$ is an endomorphism of $\mathcal{O}_2$ which is unitarily equivalent to ρ in $\mathcal{O}_2$, then its branching rule equals to that of ρ. Hence the branching rule is an invariant of equivalence class of endomorphisms. We show the branching rule of ψ_{12} without proof.

<u>Branching rule of $\rho = \psi_{12}$</u>



In this figure, each vertex means a permutative representation of $\mathcal{O}_2$ and a directed edge means an action of ψ_{12}. For example, $(1;\pm1)\circ\rho \sim (12)$ where $(1;\pm1) \equiv GP(\pm\varepsilon_1)$ and $(12) \equiv GP(\varepsilon_1 \otimes \varepsilon_2)$. In general, the right hand side in (7.2) is decomposed into direct sum of irreducible representations. For example, a vertex with label $(2;\pm1)$ means a representation which satisfies GP condition $s_2\Omega_\pm = \pm\Omega_\pm$, and the figure shows the following branching of ψ_{12} on $(2;\pm1)$:

$$(2;\pm1)\psi_{12} \sim (2;\pm1) \oplus (1;\pm1).$$

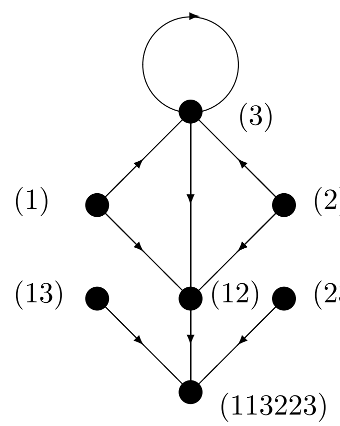
ψ_{12} is irreducible, that is, $(\psi_{12}(\mathcal{O}_2))^{'} \cap \mathcal{O}_2 = \mathbf{C}I$, and not automorphism of $\mathcal{O}_2$. In this way, we have computed branchings of many endomorphisms in

[11]. The origin of this class of endomorphisms of Cuntz algebra was brought by N.Nakanishi. He found an endomorphism of $\mathcal{O}_3$ by only combinatrix method for sake of pure mathematical interest. It is the following:

$$\begin{cases} t_1 \equiv & s_1s_2s_3^* + s_2s_3s_1^* + s_3s_1s_2^*, \\ t_2 \equiv & s_2s_1s_3^* + s_3s_2s_1^* + s_1s_3s_2^*, \\ t_3 \equiv & s_1s_1s_1^* + s_2s_2s_2^* + s_3s_3s_3^*. \end{cases} \tag{7.3}$$

It is easy to show that t_1, t_2, t_3 satisfy relations of generators of $\mathcal{O}_3$. We illustrate the figure of the branching rule of an endomorphism by Nakanishi (7.3) on permutative representations of $\mathcal{O}_3$ without proof:

Branching rule of an endomorphism by Nakanishi



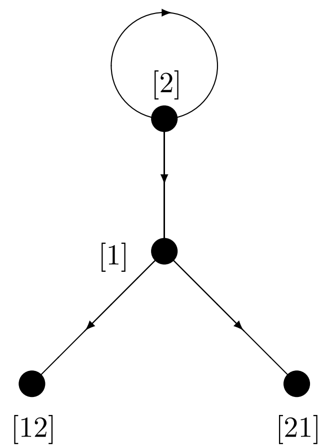
This endomorphism (7.3) is irreducible and not automorphism of $\mathcal{O}_3$, too.

7.4 Representation of Fermion algebra

In [1, 2, 3, 4], we apply our decomposition theory of representations of Cuntz algebra on its $U(1)$-fixed point subalgebra $\mathcal{O}_N^{U(1)}$. Fortunately, permutative representations are completely reducible on $\mathcal{O}_N^{U(1)}$, too. Specially, $\mathcal{O}_2^{U(1)}$ is isomorphic to algebra of fermions, so-called CAR algebra. We have many decomposition formulae for representations of Fermion algebra.

For example, the restriction of ψ_{12} in (7.1) on $CAR \equiv \mathcal{O}_2^{U(1)}$ is an endomorphism of CAR, too. The branching rule of $\psi_{12}|_{CAR}$ is given as follows:

Branching rule of $\psi_{12}|_{CAR}$



In the above figure, each vertex means an equivalence class of irreducible representations of CAR. For example a symbol [1] means $GP(\varepsilon_1)|_{CAR}$. By $U(1)$-invariance of CAR algebra in $\mathcal{O}_2$, $GP(\varepsilon_1)|_{CAR} = GP(-\varepsilon_1)|_{CAR}$. Furthermore the cyclic symmetry of permutative representation of $\mathcal{O}_2$ breaks on CAR algebra and the following branching happens:

$$GP(\varepsilon_1 \otimes \varepsilon_2)|_{CAR} = [12] + [21]$$

where $[12] \neq [21]$ as equivalence classes of irreducible representations of CAR.

7.5 State

In [8], we show the relation between pure states and GP representations with cycle for non periodic parameters in the sense of GNS representation. On the other hand, in [4], we consider a state on direct sum of two irreducible GP representations. This case brings a representation of $\mathcal{O}_2^{U(1)}$ which is not type I in general. It includes Araki-Woods factor as special case. In this point of view, there are problems to determine the type of GNS representation of a state on reducible GP representation.

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