CLASSIFICATION OF PRIMARY $\mathbb{Q}$-FANO 3-FOLDS WITH ANTI-CANONICAL DU VAL K3 SURFACES. I

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Abstract. Let X be a non-Gorenstein $\mathbb{Q}$-Fano 3-fold with only cyclic quotient terminal singularities such that the class of $-K_X$ generates the group of numerical equivalence classes of divisors, and $|-K_X|$ contains Du Val K3 surfaces. We prove that $g(X) := h^0(-K_X) - 2 \leq 8$ and give the classification of X with $g(X) \geq 6$.

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Notation, terminology and convention.

$\mathbb{N}$: The set of positive integers.
$\sim$: Linear equivalence.
$\equiv$: Numerical equivalence (only when it is used for two $\mathbb{Q}$-Cartier divisors).
$\mathbb{F}_n$: $\mathbb{P}(\mathscr{O} \oplus \mathscr{O}(-n))$ over $\mathbb{P}^1$.
$\overline{\mathbb{F}}_n$: Surface obtained by contracting the minimal section of $\mathbb{F}_n$.
Q^3: Smooth quadric 3-fold.
ODP: Ordinary double point, i.e., singularity analytically isomorphic to $\{xy+z^2+u^2=0 \subset \mathbb{C}^4\}$.
$\sigma_{a_1,a_2,\dots,a_d}(V_i)$: The Schubert cycle $\{[\mathbb{C}^k] \mid \dim \mathbb{C}^k \cap V_{n-k+i-a_i} \geq i\}$ in $G(k,n)$ for a complete flag $0 \in V_1 \subset V_2 \subset \cdots \subset V_n$.
r-plane: An r-dimensional linear subspace of a projective space.
$\langle X, Y\rangle$: The linear hull of subvarieties X and Y in a projective space.
$\mathscr{I}_{S/X}$: The ideal sheaf of a closed subvariety S in a variety X.

$|D \otimes \mathscr{I}_{S/X}|$: The sub-linear system of $|D|$ consisting the members containing S.
$\mu_{X,S} : \widetilde{X}_S \to X$: The blow-up of a variety X along a sub-variety S.
$E_{X,S}$: The closed subscheme of $\tilde{X}_S$ defined by ${\mu_{X,S}}^{-1}\mathscr{I}_{S/X}$.
$\nu_{X,S} : \widetilde{X}_S \to X_S$: The rational map defined by $|{\mu_{X,S}}^* L_X - E_{X,S}|$, where L_X is a naturally defined very ample divisor on X (for example, X is embedded in a projective space by L_X).
Abuse of notation: We use the same notation for the strict transforms of curves as original ones. We indicate the variety where curves are located so there is no confusion.
Singular curve (resp. line) of X: A curve (resp. line) contained in $\operatorname{Sing} X$.

Definitions

$\mathbb{Q}$-Fano variety: Definition 1.1.
Decomposable and indecomposable: Definition 1.2.
Primary $\mathbb{Q}$-Fano variety: Definition 1.3.
Genus $g(X)$ of a $\mathbb{Q}$-Fano 3-fold X: $g(X) := h^0(-K_X) - 2$.
Standard weighted blow-up: Definition 2.1.
Economic exceptional divisor: Definition 2.4.
Basic set-up: 2.12.
Exceptional plane: Definition 2.9.
Good plane: Definition 2.9.
Rules of the game: 2.14.
Notation and terminology for weighted dual graphs: See the beginning of the subsection 3.1.
Good line: Definition 3.9.
$F_{W,\Delta}$ and C_Δ: Definition 4.3.

1. Introduction

We will work over $\mathbb{C}$, the complex number field.

1.1. Definition and history of $\mathbb{Q}$-Fano 3-folds.

Definition 1.1. A projective variety X is called a $\mathbb{Q}$-*Fano variety* if X has only terminal singularities and $-K_X$ is ample. If X is also Gorenstein, i.e, $-K_X$ is Cartier, we just say that X is a *Fano variety*.

Among $\mathbb{Q}$-Fano varieties, $\mathbb{Q}$-factorial ones with Picard number 1 form an important class since the minimal model conjecture asserts that every projective variety is birational to a terminal $\mathbb{Q}$-factorial variety which is a $\mathbb{Q}$-Fano fiber space with relative Picard number 1 (so-called *Mori fiber space*) or a minimal variety. In dimension 3, the minimal model conjecture was solved affirmatively by efforts of many birational geometers, especially by Y. Kawamata, J. Kollár, S. Mori, M. Reid and V. V. Shokurov. So the classification of $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-folds with Picard number 1 is a natural problem. In this introduction, let X be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with Picard number 1 henceforth.

In case X is smooth, the classification is classical. This was considered by G. Fano [Fan37] for the first time, and modernized and completed mainly by T. Fujita [Fuj80], [Fuj81], [Fuj84], V. A. Iskovskih [Isk77], [Isk78], [Isk79], [Isk90] and V. V. Shokurov [Sho79b], [Sho79a]. Moreover the work of K. Takeuchi [Take89] simplified and amplified the classification in case $\operatorname{Pic} X \simeq \mathbb{Z}(-K_X)$ (i.e., X is a smooth primary Fano 3-fold as defined below) based on the theory of extremal ray.

1.2. **Mukai's classification of indecomposable Fano 3-folds.**

S. Mukai [Muk95b] found a complete different method for the classification of smooth primary Fano 3-folds. To explain this, we need some definitions.

Definition 1.2. We say that a Weil divisor D on a variety is *decomposable* if there exists a decomposition as $D \sim A + B$, where A and B are Weil divisors with $h^0(A) \geq 2$ and $h^0(B) \geq 2$. Otherwise we say that D is *indecomposable*. A $\mathbb{Q}$-Fano 3-fold W, where we allow W to have canonical singularities, is said to be *indecomposable* if $-K_W$ is indecomposable. In particular, smooth primary Fano 3-folds are indecomposable.

Mukai determined the defining equation of an indecomposable (canonical Gorenstein) Fano 3-fold W in $\mathbb{P}(H^0(-K_W))$. More precisely for $7 \leq g(W) \leq 10$ for example, he proved that W is a linear section of some Grassmannian, where $g(W) := h^0(-K_W) - 2$, which is called *genus of* W. His method uses the theory of vector bundle. Let $S \in |-K_W|$ be a Du Val K3 surface (the existence of such an S was proved by Shokurov [Sho79b] and Reid [Reid83]). Mukai constructed the vector bundle on S which defines the embedding of S into some Grassmannian and then he extended the embedding of S to that of W.

The reason why he considered indecomposable Fano 3-folds is that it is suitable for inductive treatment; start with a smooth primary Fano 3-fold, blow it up at a general point and take the anti-canonical model. Then the anti-canonical model, which is again a Fano 3-fold, is no longer smooth but still indecomposable. By using this inductive structure of indecomposable Fano 3-folds and the above descriptions of them with genera ≤ 12, he gave a new (and generalized) proof of the famous genus bound $g(X) \leq 12$ and $g(X) \neq 11$. We remark here that his descriptions of indecomposable Fano 3-folds play crucial roles in this paper.

In case X is singular but Gorenstein, Mukai's and Takeuchi's methods work if X is primary, and Fujita's method works for the other cases (see [Fuj85], [Fuj90]).

1.3. **Non-Gorenstein $\mathbb{Q}$-Fano 3-fold.**
So we are interested in the case that X is non-Gorenstein. Kawamata [Kaw92] proved that such X's form a bound family and hence in particular, there are universal bounds (in fact, effective ones) of $(-K_X)^3$, the number of non-Gorenstein points and the index of singularities. We expect that we can improve those bounds by studying X more explicitly. Similarly to the smooth case, we make a case division by

$$qF(X) := \max\{t \in \mathbb{N} | -K_X \equiv tD \text{ for an ample Weil divisor} D\},$$

which is called $\mathbb{Q}$-*Fano index of* X.

First let us mention the case that $qF(X) \geq 2$. In the Gorenstein case, Fujita [Fuj80]–[Fuj90] studied the linear system $|D|$ and proved that $|D|$ contains a smooth member S. Then he succeeded in classifying X by using properties of S. But in the non-Gorenstein case, $|D|$ may be empty or have only bad members. So the strategy in the Gorenstein case does not work immediately. We should find out some new method.

1.4. **Primary $\mathbb{Q}$-Fano 3-fold X.**

Definition 1.3. X is called a *primary* $\mathbb{Q}$-*Fano* 3-*fold* if $qF(X) = 1$.

Now let us focus on primary $\mathbb{Q}$-Fano 3-folds X. So far there are mainly two methods to aim at the classification of primary $\mathbb{Q}$-Fano 3-folds.

One is based on the theory of extremal rays, which was developed by Takeuchi [Take89] for the smooth case as we stated above and was generalized in [Taka02a] and [Taka02b]. In [loc.cit], we treated the case that $\operatorname{Pic} X \simeq \mathbb{Z}(-2K_X)$ and almost completed the classification. Though this method need many numerical calculations, the list of the possibilities of X made by it is very precise, i.e., almost all the possibilities really exist.

The other is based on the theory of the Hilbert function and the unprojection. Let $R := \bigoplus_{i\geq 0} H^0(\mathscr{O}_X(-iK_X))$. Then R is known to be Gorenstein (see for example [Wat81]). R is written as the quotient of the polynomial ring $R' := k[x_1, \ldots, x_r]$, where x_i are lifts of generators of R. R' has the natural grading which comes from that of R. Let $a_i := \mathrm{wt}\ x_i$. When we apply this method, we always embed X in the weighted projective space $\mathbb{P}(a_1, \ldots, a_r) := \operatorname{Proj} k[x_1, \ldots, x_r]$. By the Riemann-Roch theorem for a projective terminal 3-fold [Kaw86], [Reid87] and the Kodaira-Kawamata-Viewheg vanishing theorem, the Hilbert function $P(t) := \sum_{i=0}^{\infty} h^0(\mathscr{O}_X(-iK_X))t^i$ of R is determined by $(-K_X)^3$ and the data of non-Gorenstein points (so-called *baskets*), whose possibilities are finite by [Kaw92]. We also note that $P(t)$ has the Gorenstein symmetry. So we can make a list of the possibilities of X and $P(t)$. Actually by this method, S. Altınok [Alt], I. Fletcher [Fle00] and Reid gave the classification of primary $\mathbb{Q}$-Fano 3-folds with codimension not greater than 3 in $\mathbb{P}(a_1, \dots, a_r)$. Since in these cases, the type of equations of X is known a priori by commutative algebra, the existence of X is easy to check. Unfortunately it is difficult to check whether they really exist in case codimension ≥ 4. The unprojection will be useful for this problem. The unprojection was defined and studied by S. Papadakis and Reid in [PRe02], [Pap01a], [Pap01b], which is roughly speaking the tool to produce a variety with bigger codimension in some weighted projective space from that with smaller codimension by contracting a divisor. The simplest example is the inverse of a usual projection.

1.5. **Main theorem.**

In this paper, we propose a new method and apply this for primary $\mathbb{Q}$-Fano 3-folds with the following properties:

1.4. *X is a primary $\mathbb{Q}$-Fano 3-fold such that*

(1) *X is non-Gorenstein.*
(2) *There exists a Du Val $K3$ surface in $|-K_X|$.*
(3) *X has only cyclic quotient terminal singularities.*

We conjecture that (2) holds if $g(X)$ is appropriately big. This is a modified version of the general elephant conjecture by Reid. In [Taka02b, §1], this conjecture was treated in case $-2K_X$ is Cartier and under some extra assumptions, we proved that it is affirmative if $g(X) \geq 2$.

The condition (3) can be considered to be that of generality. We conjecture that there is a small deformation of $\mathbb{Q}$-Fano 3-folds X such that nearby fibers have only quotient terminal singularities. This was proved to be affirmative in case $-2K_X$ is Cartier by T. Minagawa [Min99a, Theorem 2.4] and the author [Taka02b, §2].

The following is the main result:

Theorem 1.5. *Let X be as in* 1.4. *Then $g(X) \leq 8$. Assume that $g(X) \geq 6$. Then any singularity of X is a $1/r\,(1, -1, 1)$-singularity for some r. Moreover the following hold.*

(A) *Assume that* $g(X) = 8$. *Then* X *has at most two singular points and they are* $1/2\,(1,1,1)$*-singularities.*

(B) *Assume that* $g(X) = 7$. *Then* X *has only one singular point and it is a* $1/r\,(1,-1,1)$*-singularity with* $r = 2, 3, 4$.

(C) *Assume that* $g(X) = 6$. *Then one of the following holds:*

$(C1)$ X *has two singular points and they are* $1/r_i\,(1,-1,1)$*-singularities* $(r_1 \le r_2)$ *with*

$$(r_1, r_2) = (2,2), (2,3), (2,4) \text{ or } (3,3).$$

$(C2)$ X *has only one singular point and it is a* $1/r\,(1,-1,1)$*-singularity with* $r = 2, 3, 4, 5$.

Actually we can also obtain information of $\mathrm{Bs}\,|-K_X|$ and birational properties of X in the course of the proof up to deformation of X. See for summaries the subsections 5.4, 6.4 and 7.6 and the section 8.

As is known by the result, there are very few possibilities of X. So we expect that we can find nice biregular characterizations of such $\mathbb{Q}$-Fano 3-folds and consequently we can remove extra assumptions in 1.4.

We will study the existence problem of X elsewhere because it needs more calculations like [Taka02b, §4, 5].

1.6. **Structure of the paper.**
Hereafter let X be as in 1.4. This paper is organized as follows:

In the section 2, we prove that if $g(X) \ge 2$, then by an explicit birational map, X is transformed to an indecomposable Fano 3-fold W with $g(W) = g(X)$ and with at least one plane (see 2.12). This is a basic result for our treatment of X in the following sections. For this construction, we need the assumptions (2) and (3) in 1.4. As a corollary, we obtain $g(X) \le 8$ (see Corollary 2.11). For proving this fact, Mukai's classification of indecomposable Fano 3-folds is indispensable since his theorem claims in particular that they do not contain planes if their genus are greater than 8. Moreover in the section 2, we investigate singularities and exceptional planes on W (see Definition 2.9) and its relation with singularities of X (see Corollary 2.8 and Proposition 2.13). In particular, if we deform X necessarily, then $\mathrm{Sing}\,W$ is contained in exceptional planes on W and is the union of lines and a finite number of points. Moreover any two exceptional planes on W can intersect only at one point (see 2.14 (a)–(c)). These properties of W turn out to restrict the geometry of W more than expected. In fact, the classification of X is almost reduced to that of indecomposable Fano 3-folds satisfying 2.14 (a)–(c).

In the section 3, we continue the study of the relation of the geometries of X and W by restricting all to the strict transforms of a general member of $|-K_X|$ and introducing weighted dual graphs. We derive more delicate properties of W which does not follow from 2.14 (a)–(c).

In the section 4, we study the projection of $W \dashrightarrow W_\Delta$ from its linear subspace Δ especially in case W_Δ is a del Pezzo 3-fold. In the sections 5, 6 and 7, we see that this situation often occurs.

Based on the results in the previous sections, we classify X (or W) with $g(X) = 8, 7, 6$ in the sections 5, 6, 7 respectively.

In the section 8, we summarize the results and re-describe them from the view point of the minimal model program.

1.7. **Outline of the proof.**
Roughly speaking, it suffices to classify indecomposable Fano 3-folds W satisfying 2.14 (a)–(c).

The method is very simple and classical; we study the projections of W from planes, singular lines and singular points very closely. We explain the strategy in case $g(X) = 7$. Though the case that $g(X) = 8$ is simpler and the case that $g(X) = 6$ is harder, the basic process is similar to this case.

First we consider the projection $W \dashrightarrow W_P$ from a plane P on W. By the geometry of $\mathrm{OG}(5, 10)$, we see that W_P is a $(2, 2)$-complete intersection in $\mathbb{P}^5$ (see Proposition 6.5). Also for X with other genera, we determine W_P or W_l for a singular line l by using the geometry of the ambient variety like this. Moreover we can prove that W_P has only terminal singularities (see Proposition 4.8).

Next we calculate the degree of the 'weighted center' C_P of $\nu_{W,P}$ (see Definition 4.3 for the precise definition) rather formally (see Propositions 4.4 and 4.6). Since W_P is a del Pezzo 3-fold, we have that $\mu_{W,P}(F_{W,P}) \in |L_W - P|$ (Proposition 4.4 (1)). Together with 2.14 (c), this fact restricts the possibilities of C_P. In fact, we can prove that C_P is irreducible and reduced (Proposition 6.5 (2)). This implies that P is the unique plane on W and moreover a good plane as defined in Definition 2.9 (see Proposition 4.9). In particular, we see that:

$\operatorname{Sing} X$ consists of one $1/r\,(1, -1, 1)$-singularity for some r; if $r = 2$, then W has no singular line; if $r \geq 3$, then W has a unique singular line l on P and has a cA_{r-2}-singularity generically along l.

Now whether W_P is factorial or not becomes essentially related to $\operatorname{Sing} X$. W_P is factorial if and only if W has no singular line, i.e., $r = 2$ (Proposition 4.9). So if W_P is not factorial, then there is a unique singular line l on P. In this case, we consider the projection $W \dashrightarrow W_l$ from l and can prove that W_l is a smooth quintic del Pezzo 3-fold (Proposition 6.7). Similarly to C_P, we can define C_l for $\nu_{W,l}$. C_l is reduced but not necessarily irreducible. If C_l is irreducible, then we see that $E_{W,l}$

is irreducible. This implies that W has a cA_1-singularity generically along l, i.e., $r = 3$ (Theorem 6.10 (2) (a)). If C_l is reducible, then we see that $E_{W,l}$ has two components and $\widetilde{W}_l$ has no curve singularity dominating l. This implies that W has a cA_2-singularity generically along l, i.e., $r = 4$ (Theorem 6.10 (2) (b)).

Like this, we can explain why $\operatorname{Sing} X$ is bounded effectively by studying the geometry of W_P and W_l.

1.8. **Case X has lower genus.**
The technique in the sections 2 and 3 works also for the case that $2 \le g(X) \le 5$. But that in the section 4 is only for the case where W_Δ is a del Pezzo 3-fold, where Δ is a plane $P \subset W$ or a singular line $l \subset W$. It is sufficient for the case $g(X) \ge 6$ except the one case where $g(X) = 6$ and $W_P \simeq Q^3$. As we know in the subsection 7.5, the case where W_P is not a del Pezzo 3-fold is hard to treat with. Unfortunately the possibilities of W_Δ have wider ranges in the cases where $g(X) \le 5$. But we believe that we can also classify these cases after some efforts in the future.

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2. Indecomposable Fano model W

2.1. Construction of an indecomposable Fano model.

Definition 2.1. For a $1/r\,(a, -a, 1)$-singularity with $(r, a) = 1$, the weighted blow-up with weight $1/r\,(a, r-a, 1)$ is called *the standard weighted blow-up.*

Lemma 2.2. *Let X be a projective 3-fold with only cyclic singularities and $1/r_i\,(a_i, -a_i, 1)$ the types of singularities of X. Let*

$$X_m \xrightarrow{g_m} X_{m-1} \xrightarrow{g_{m-1}} \cdots \xrightarrow{g_2} X_1 := X$$

be the composite of the standard weighted blow-ups of cyclic quotient singularities such that X_m is smooth (X_m is so-called the economic resolution). *Then*

$$(-K_{X_m})^3 = (-K_X)^3 - \sum \frac{b_i(r_i - b_i)}{r_i},$$

where b_i *is the integer such that* $1 \le b_i \le r_i$ *and* $a_i b_i \equiv 1 \pmod{r_i}$.

Proof. Let $x \in \operatorname{Sing} X$ and $1/r\,(a, -a, 1)$ the type of x. Let b be the integer such that $1 \le b \le r$ and $ab \equiv 1 \pmod r$. Define c by $ab = rc+1$. We may assume that

$$X_k \xrightarrow{g_k} X_{k-1} \xrightarrow{g_{k-1}} \cdots \xrightarrow{g_k} X_1 := X$$

be the economic resolution of x. It suffices to prove that

$$(-K_{X_k})^3 = (-K_X)^3 - \frac{b(r-b)}{r}.$$

By the standard calculation of the weighted blow-up, we see that

$$(-K_{X_2})^3 = (-K_X)^3 - \frac{1}{ra(r-a)}. \tag{2.1}$$

On X_2 over x, there are $1/a\,(r, -r, 1)$-singularity and $1/(r-a)\,(r, -r, 1)$-singularity (the argument below works if $a = 1$ or $r - a = 1$). Let b' (resp. b'') be the integer such that $1 \le b' \le a$ (resp. $1 \le b'' \le r - a$) and $rb' \equiv 1 \pmod a$ (resp. $rb'' \equiv 1 \pmod{r-a}$). Then it is easy to see that

$$\begin{aligned} b' &= a - c. \\ b'' &= b - c. \end{aligned} \tag{2.2}$$

By induction, we have

$$(-K_{X_k})^3 = (-K_{X_2})^3 - \frac{b'(a-b')}{a} - \frac{b''(r-a-b'')}{r-a}. \tag{2.3}$$

By (2.1), (2.2) and (2.3), we obtain the assertion.

□

The following proposition is a key to our treatment below.

Proposition 2.3. *Let* X *be a* $\mathbb{Q}$*-Fano 3-fold with only cyclic quotient singularities. Let* $g' : X_m \to X$ *be the economic resolution of* X *as in Lemma 2.2. Then* $(-K_{X_m})^3 = 2g(X) - 2$. *Moreover assume that* $|-K_X|$ *contains a Du Val* $K3$ *surface and* $h^0(-K_X) \ge 2$. *Then there exists a composite of a finite number of blow-ups*

$$X_n \xrightarrow{g_n} X_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_{m+1}} X_m$$

along $\gamma_i \simeq \mathbb{P}^1$ *with* $-K_{X_i} \cdot \gamma_i < 0$ *such that* $-K_{X_n}$ *is nef.*

Proof. By the Riemann-Roch theorem for terminal 3-folds [Kaw86], [Reid87] and the Kodaira-Kawamata-Viewheg vanishing theorem, we have

$$(-K_X)^3 = 2h^0(-K_X) - 6 + \sum \frac{b_i(r_i - b_i)}{r_i}.$$

Hence by Lemma 2.2, we obtain

$$(-K_{X_m})^3 = 2h^0(-K_X) - 6 = 2g(X) - 2.$$

Define $g_{i+1} : X_{i+1} \to X_i$ $(i \geq m)$ inductively as the blow-up along a curve γ_i with $-K_{X_i} \cdot \gamma_i < 0$. Let $D \in |-K_X|$ be a Du Val K3 surface and D_i the strict transform of D on X_i $(i \geq m)$. Then we can show inductively the following:

(A) X_i is smooth.
(B) $D_i \in |-K_{X_i}|$.
(C) D_i is a Du Val K3 surface.
(D) $\gamma_i \simeq \mathbb{P}^1$.

For $i = m$, (A) is clear. Since the discrepancy of every step of the economic resolution is minimal and D is a Du Val K3 surface, $D_m \in |-K_{X_m}|$ and D_m is also a Du Val K3 surface. So (B) and (C) hold for $i = m$. Since $\gamma_m \subset \mathrm{Bs}\,|-K_{X_m}|_{D_m}|$, we have $\gamma_m \simeq \mathbb{P}^1$ by [Ale91, Corollary 1.5]. So (D) is checked for $i = m$.

Assume that we have the assertion for i-th step. By (D) for i-th step, (A) holds for $(i+1)$-st step. Since D_i is generically smooth along γ_i, (B) holds for $(i+1)$-st step. By (C) for i-th step and $D_{i+1} \to D_i$ is crepant, (C) holds for $(i+1)$-st step. Since $\gamma_{i+1} \subset \mathrm{Bs}\,|-K_{X_{i+1}}|_{D_{i+1}}|$, (D) holds for $(i+1)$-st step by [ibid.].

Note that $h^0(-K_{X_i}) = h^0(-K_X) \geq 2$. Hence γ_i is contained in the intersection of two general members of $|-K_{X_i}|$. So the sequence of the above blow-ups terminate, i.e., for some $n \geq m$, $-K_{X_n}$ is nef. $\square$

Definition 2.4. We say that a g-exceptional divisor F is *economic* (resp. *non-economic*) if F is extracted by $g_i : X_i \to X_{i-1}$ with $i \leq m$ (resp. $i > m$).

Definition 2.5. Let W be an indecomposable Fano 3-fold. Then an irreducible surface S (resp. an irreducible curve C) on W is called a *plane* (resp. a *line*, a *conic*) if and only if $S \simeq \mathbb{P}^2$ and $(-K_W)^2 S = 1$ (resp. $C \simeq \mathbb{P}^1$ and $-K_W \cdot C = 1, 2$). Note that if $-K_W$ is very ample, then a plane, a line and a conic are usual ones in the projective space $\mathbb{P}^*(H^0(-K_W))$.

A kind of surprise is the following:

Corollary 2.6. *Let the assumptions be as in Proposition* 2.3. *We use the notation there. Moreover assume that X is indecomposable and $g(X) \geq 2$. Then*

(1) $-K_{X_i} \cdot \gamma_i = -1$ *for any* $i \geq m$ *and* $\mathscr{N}_{\gamma_i/X_i} \simeq \mathscr{O}_{\mathbb{P}_1}(-1) \oplus \mathscr{O}_{\mathbb{P}_1}(-2)$ *or* $\mathscr{O}_{\mathbb{P}_1} \oplus \mathscr{O}_{\mathbb{P}_1}(-3)$.

(2) X_n *is an indecomposable weak Fano* 3*-fold with* $(-K_{X_n})^3 = (-K_{X_m})^3$. *In particular,* $|-K_{X_n}|$ *is free by* [Isk79, Theorem 6.3] *and* [Muk95b, Theorem 6.5].

(3) *Let* $h : Z \to W$ *be the anti-canonical model. Then by deforming* X *if necessarily, we may assume that the image of an* h*-exceptional divisor is a point or a line on* W.

Proof. By $g(X) \geq 2$, we have

$$(-K_{X_m})^3 = 2g(X) - 2 \geq 2.$$

By the formula

$$(-K_{X_{i+1}})^3 = (-K_{X_i})^3 - 2\{(-K_{X_i} \cdot \gamma_i) + 1\},$$

we have $(-K_{X_{i+1}})^3 \geq (-K_{X_i})^3$. Hence X_n is a weak Fano 3-fold, and indecomposability of X_n follows from primarity of X. By the Riemann-Roch theorem, $(-K_{X_n})^3 = 2h^0(-K_{X_n}) - 6$. Since $h^0(-K_{X_n}) = h^0(-K_X)$, we have $(-K_{X_m})^3 = (-K_{X_n})^3$. So $-K_{X_i} \cdot \gamma_i = -1$ for any $i \geq m$. Since $\deg \mathscr{N}_{\gamma_i/X_i} = -3$, we can write $\mathscr{N}_{\gamma_i/X_i} \simeq \mathscr{O}_{\mathbb{P}_1}(a) \oplus \mathscr{O}_{\mathbb{P}_1}(-a-3)$ with $a \geq -1$. Let F_i be the g_i-exceptional divisor, C_0 its minimal section, and r a ruling. Then $-K_{X_{i+1}}|_{F_{i+1}} = C_0 + (a+2)r$ whence $-K_{X_{i+1}} \cdot C_0 = -a - 1$. If $-K_{X_{i+1}} \cdot C_0 \geq 0$, then $a = -1$ and we have the former case. If $-K_{X_{i+1}} \cdot C_0 < 0$, then C_0 becomes the center of g_j for some $j > i + 1$. Since $-K_{X_j} \cdot C_0 = -1$, we have $-K_{X_{i+1}} \cdot C_0 = -1$ and so $a = 0$.

Let E be an h-exceptional divisor which is contracted to a curve on W which is not a line. After a sequence of flops $Z \dashrightarrow Z'$, E can be contracted primitively. Then by [Min99b, Theorem 0.9], E disappears after a suitable small deformation of Z'. By the existence of smooth simultaneous flops [Pin83, Theorem 3], this implies that E disappears after a suitable small deformation of Z, which induces that of X. So we obtain (3). □

Remark. (i) For Corollary 2.6, we are inspired by [Ale94, Theorem 4.5]. In contrast to [ibid.], however, we need to construct a Gorenstein model very explicitly for our purpose.

(ii) For No. 3.1' in [Taka02a], $h : Z \to X$ is the blow-up at the unique $\frac{1}{2}(1,1,1)$-singularity and the image of the h-exceptional divisor on W is a curve of degree 5. However No. 3.1' can be deformed to No. 3.1, for which h is a small contraction.

Corollary 2.7. *Let the assumptions be as in Corollary* 2.6. *Let F_i be the g_i-exceptional divisor and F_i' the strict transform of F_i on X_n. We denote by C_0 the minimal section of F_i $(i \geq m+1)$, which is $\mathbb{F}_1$ or $\mathbb{F}_3$, and by r a ruling. Then one of the following holds:*

(a) (*economic case*)

- $(a1)$ $F_i \simeq \mathbb{P}^2$ *and* $D_{X_i}|_{F_i}$ *is a line* l. $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ *is an empty set, a simple point or* l. *Moreover if* $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ *is an empty set, then* $g(F_i')$ *is a* $1/r\,(1,-1,1)$*-singularity for some* r *and* $|-K_{X_i}|$ *has no base point over* $g(F_i')$.
- $(a2)$ $F_i \simeq \overline{\mathbb{F}}_k$ $(k \geq 2)$ *and* $D_{X_i}|_{F_i}$ *is a ruling* r. $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ *is the vertex of* F_i *or* r. *Moreover if* $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ *is the vertex of* F_i, *then* $g(F_i')$ *is a* $1/r\,(1,-1,1)$*-singularity for some* r *and* $\mathrm{Bs}\,|-K_{X_n}| \cap g^{-1}(g(F_i'))$ *is at most one simple point on the exceptional divisor resolving the* $\frac{1}{2}(1,1,1)$*-singularity over* $g(F_i')$.
- $(a3)$ $F_i \simeq \mathbb{P} := \mathbb{P}(k,m,1)$ $(k \geq 2$ *and* $m \geq 2)$ *and* $D_{X_i}|_{F_i}$ *is the unique member* C *of* $|\mathscr{O}_{\mathbb{P}}(1)|$. $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ *is* C.

(b) (*non-economic case*)

- $(b1)$ $F_i \simeq \mathbb{F}_1$ *and* $D_{X_i}|_{F_i}$ *is an irreducible member* C *of* $|C_0 + r|$. $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ *is an empty set, a simple point or* C.
- $(b2)$ $F_i \simeq \mathbb{F}_1$ *and* $D_{X_i}|_{F_i}$ *is the union of* C_0 *and a ruling* r. $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ *is* C_0 *or* $C_0 + r$.
- $(b3)$ $F_i \simeq \mathbb{F}_3$ *and* $D_{X_i}|_{F_i} = C_0 + r_1 + r_2$, *where* r_i *are two different rulings.* $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ *is* C_0, $C_0 + r_i$ *or* $C_0 + r_1 + r_2$.
- $(b4)$ $F_i \simeq \mathbb{F}_3$ *and* $D_{X_i}|_{F_i} = C_0 + 2r$, *where* r *is a ruling.* $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ *is* $C_0 + 2r$.

In particular

- (i) $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ *is connected.*
- (ii) *For the case* (b), γ_{i-1} $(i \geq m+1)$ *is a non-reduced component of* $\mathrm{Bs}\,|-K_{X_{i-1}}|$ *if and only if and* $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ *contains a curve.*

Proof. The proof is an easy exercise. Here we only prove the 'Moreover' parts of (a1) and (a2) by using the other descriptions.

First assume that $F_i \simeq \mathbb{P}^2$ and $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ is an empty set. We may assume that g_j $(j < i)$ is the standard weighted blow-up of a singularity over $g(F_i)$. Let F_j'' $(j < i)$ be the strict transform of F_j on X_i. Then by induction, we see that $\mathrm{Bs}\,|-K_{X_i}| \cap F_j''$ is an empty set for any $j < i$. So $|-K_{X_i}|$ has no base point over $g(F_i')$. Moreover we know that (a3) does not occurs for any $j < i$ and in particular this shows that $g(F_i')$ is a $1/r\,(1,-1,1)$-singularity for some r.

Next assume that $F_i \simeq \overline{\mathbb{F}}_k$ $(k \geq 2)$ and $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ is the vertex v of F_i. We may assume that g_j $(j \leq l)$ is the standard weighted blow-up of a singularity over $g(F_i)$ and g_{i+1} is the standard weighted blow-up at v. Let F_i'' be the strict transform of F_i on X_{i+1}. Then $\operatorname{Bs}|-K_{X_{i+1}}| \cap F_i''$ is an empty set. Moreover similarly to the previous case, we can prove that $|-K_{X_{i+1}}|$ has no base point over $g(F_i')$ and $g(F_i')$ is a $1/r\,(1,-1,1)$-singularity for some r. So by the descriptions in (a1) (resp. (a2)) (except the 'Moreover' part), we see that for $l > j > i$, $\operatorname{Bs}|-K_{X_n}| \cap F_j'$ is an empty set (resp. $\operatorname{Bs}|-K_{X_n}| \cap F_l'$ is at most a simple point). $\square$

Corollary 2.8. *Let the assumptions be as in Corollary 2.7. We use the notation there. Moreover denote by F_i' the strict transform of F_i on X_n. Then $h(F_i')$ is a plane, a conic, a line, or a point. More precisely,*

(i) *$h(F_i')$ is a plane if and only if*
  (i1) *the case (a1) in Corollary 2.7 holds and $\operatorname{Bs}|-K_{X_i}| \cap F_i = \emptyset$, or*
  (i2) *the case (b1) in Corollary 2.7 holds and $\operatorname{Bs}|-K_{X_i}| \cap F_i = \emptyset$.*

(ii) *$h(F_i')$ is a conic if and only if the case (b3) in Corollary 2.7 holds and $\operatorname{Bs}|-K_{X_i}| \cap F_i = C_0$.*

(iii) *$h(F_i')$ is a line if and only if*
  (iii1) *$\operatorname{Bs}|-K_{X_i}| \cap F_i$ is a simple point and the case (a1), (a2) or (b1) in Corollary 2.7 holds,*
  (iii2) *the case (b2) in Corollary 2.7 holds and $\operatorname{Bs}|-K_{X_i}| \cap F_i = C_0$, or*
  (iii3) *the case (b3) in Corollary 2.7 holds and $\operatorname{Bs}|-K_{X_i}| \cap F_i = C_0 + r_1$ or $C_0 + r_2$.*

(iv) *$h(F_i')$ is a point if and only if $\operatorname{Bs}|-K_{X_i}| \cap F_i = D_{X_i}|_{F_i}$.*

Proof. Since $h^0(-K_Z) = h^0(-K_{X_i})$, we have

$$h^0(-K_Z - F_i') = h^0(-K_{X_i} - F_i).$$

On the other hand, we have

$$h^0(-K_Z - F_i') = h^0(-K_W \otimes \mathscr{I}_{h(F_i')/W}).$$

Let $h' : Z \to W'$ be the morphism defined by $|-K_Z|$. We have a natural morphism $\pi : W \to W'$. Let $s := \dim\langle h'(F_i')\rangle$. If $\pi|_{h(F_i')}$ is birational (resp. not birational), then

$$h^0(-K_W \otimes \mathscr{I}_{h(F_i')/W}) = h^0(-K_{W'} \otimes \mathscr{I}_{h'(F_i')/W'})$$

$$(\text{resp. } h^0(-K_W \otimes \mathscr{I}_{h(F_i')/W}) \leq h^0(-K_{W'} \otimes \mathscr{I}_{h'(F_i')/W'})).$$

Hence we have the formula:

$$\dim \operatorname{Im}(H^0(-K_{X_i}) \to H^0(-K_{X_i}|_{F_i})) = s+1$$

$$(\text{resp.} \dim \operatorname{Im}(H^0(-K_{X_i}) \to H^0(-K_{X_i}|_{F_i})) \geq s+1).$$

Thus by Corollary 2.7 (a) and (b), we have $s \leq 2$.

Assume that $s = 2$. Then by Corollary 2.7,

(A1) $\mathrm{Bs}\,|-K_{X_i}| \cap F_i = \emptyset$ and the case (a1) or (b1) in Corollary 2.7 holds, or

(A2) the case (b3) in Corollary 2.7 holds and $\mathrm{Bs}\,|-K_{X_i}| \cap F_i = C_0$.

It is easy to see that if the case (A1) occurs, then $h(F_i)$ is a plane. Assume that the case (A2) occurs. Then we may assume that g_{i+1} is the blow-up along C_0. Let F_i'' be the strict transform of F_i on X_{i+1}. Then $\mathrm{Bs}\,|-K_{X_{i+1}}| \cap F_i'' = \emptyset$. So $h(F_i')$ is a conic.

Assume that $s = 1$. Then clearly $h(F_i)$ is a line and it is easy to see that one of (iii1)–(iii3) holds.

Assume that $s = 0$. Then clearly $h(F_i)$ is a point and it is easy to see that $\mathrm{Bs}\,|-K_{X_i}| \cap F_i = D_{X_i}|_{F_i}$. □

Definition 2.9. We say that a plane on W is an *exceptional plane* if its strict transform on Z is g-exceptional. An exceptional plane is called a *good plane* if it satisfies Corollary 2.8 (i1).

The following is a refinement of indecomposability of W:

Corollary 2.10. *Let the assumptions be as in Corollary* 2.7. *Then any member of* $|-K_W|$ *is a prime divisor, or a union of a prime divisor and exceptional planes. In particular* W *contains no surface of degree between* 2 *and* $g(X) - 2$ *if* $g(X) \geq 4$.

Proof. By Corollary 2.8, any irreducible component of a member $\in |-K_W|$ which is not an exceptional plane is not contracted by $W \dashrightarrow X$. Hence by primarity of X, the former half of the assertion follows.

If $g(X) \geq 4$ (and hence $-K_W$ is very ample), and there exists a surface S of degree between 2 and $g(X) - 2$ on W, then $\dim\langle S\rangle \leq g(X) - 1$ whence $|L_W \otimes \mathscr{I}_{S/W}|$ has a movable part. This contradicts the former half of the assertion. □

Corollary 2.11. *Let the assumptions and the notation be as in Corollary* 2.6. *Moreover assume that* X *is non-Gorenstein. Then*

(1) W *contains at least one plane.*

(2) $-K_W$ *is not very ample if and only if* $g(X) = 2$.

(3) (*The genus bound*) $g(X) \leq 8$.

Proof. (1) This follows from Corollary 2.8 (i).

(2) By [Isk79, Theorem 6.3] and [Muk95b, Theorem 6.5], it suffices to deny the case that $g(X) = 3$ and $|-K_W|$ defines a double cover over a quadric 3-fold W'. Assume that this case occurs. Then by (1), W' contains a plane and hence W' is singular. Thus $\mathscr{O}_{W'}(1)$ is decomposable and hence so is $-K_W$, a contradiction.

(3) We are inspired by the proof of [Muk95b, Theorem 6.5]. By [ibid.], we have that $g(X) = g(W) \leq 10$ or $= 12$.

If $g(W) = 12$, then W is smooth by [ibid.]. Hence by Lefschetz's theorem, W cannot contain a plane, a contradiction to (1). If $g(W) = 10$ (resp. $g(W) = 9$), W is a linear section of the 5-dimensional G_2-variety G in $\mathbb{P}^{13}$ (resp. the 6-dimensional Symplectic Grassmannian G in $\mathbb{P}^{12}$) by [ibid.]. By [LM, Corollary 4.10], G contains no plane, a contradiction to (1).

□

2.2. **Refinement of the results for a primary X.**

2.12 (Basic set-up)**.** *Let X be as in* 1.4. *In the previous subsection* 2.1, *we obtained the following for this X:*

(1) *A sequence of birational morphisms*

$$X_n \xrightarrow{g_n} X_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_2} X_1 := X,$$

where g_i is the standard weighted blow-up of a cyclic quotient singularity for $i \leq m$ or the blow-up along $\gamma_{i-1} \simeq \mathbb{P}^1$ with $-K_{X_{i-1}} \cdot \gamma_{i-1} = -1$ for $i > m$. We denote the composite of these birational morphisms by g. X_n is a weak Fano 3*-fold, which we denote by Z.*

(2) *Denote the anti-canonical model of Z by $h : Z \to W$. W is an indecomposable Fano* 3*-fold with $g(W) = g(X)$. W contains at least one plane.*

(3) *We may assume that the image of an h-exceptional divisor is a point or a line on W.*

We fix this set-up from here on till the end of the paper.

Proposition 2.13. (1) *An h-exceptional divisor is g-exceptional. Assume that $-K_W$ is very ample. Then the strict transform of a plane contained in W is g-exceptional, i.e., any plane on W is exceptional. In particular by Corollary* 2.8 (*i*)*, a plane on W satisfies* (*i*1) *or* (*i*2) *in loc. cit. Assume that $-K_W$ is not very ample. Let $h' : Z \to W'$ be the morphism defined by $|-K_Z|$. Note that there is a natural morphism $\pi : W \to W'$. Let P be a plane on W and P' the plane on W such that $\pi(P) = \pi(P')$. Then exactly one of the strict transforms of P and P' is g-exceptional.*

(2) $\operatorname{Sing} W$ *is contained in the unions of exceptional planes.*

(3) *Let $G_k \subset Z$ be the strict transforms of exceptional planes, and $G_{k,j}$ their images on X_j. Then any connected component of* $\operatorname{Bs}|-K_{X_j}|$ *intersects exactly one $G_{k,j}$ and contains it. In particular, any two exceptional planes do not intersect along a curve by* (1).

Proof. (1) Let H be an h-exceptional divisor. By 2.12 (3), there is a member L of $|-K_W|$ containing $h(H)$ and $|-K_W \otimes \mathscr{I}_{h(H)/W}|$ is movable. Then h^*L is a member of $|-K_Z|$ and is reducible. On the other hand, g_*h^*L is a member of $|-K_X|$ and hence it must be irreducible by primarity of X. Hence H is contracted by g. The rest can be similarly proved.

(2) Let w be a terminal singularity of W. Note that $h^{-1}(w)$ is 1-dimensional. By Corollary 2.8, a g-exceptional divisor is the strict transform of a plane or an h-exceptional divisor. So if w is not contained in any exceptional plane, then $h^{-1}(w)$ is disjoint from g-exceptional divisors whence $g(h^{-1}(w))$ is numerically trivial for $-K_X$, a contradiction.

So it suffices to show that $h(F)$ is contained in an exceptional plane for an h-exceptional divisor F. Since F is g-exceptional by (1) and is crepant, general fibers of $h|_F$ intersect a g-exceptional divisor F' which is extracted after the extraction of the transform of F. Hence $h(F) \subset h(F')$. So the assertion follows by induction.

(3) By the construction of g and Corollary 2.8 (i), any connected component of $\mathrm{Bs}\,|-K_{X_j}|$ contains one of $G_{k,j}$. Assume by contradiction that there exists a connected component of $\mathrm{Bs}\,|-K_{X_j}|$ intersecting both $G_{k,j}$ and $G_{k',j}$. By Corollary 2.8 (i), we see then that $G_{k,j}, G_{k',j} \subset \mathrm{Bs}\,|-K_{X_j}|$ and $G_k \cap G_{k'} = \emptyset$. Hence there exists $i \geq j+1$ such that
   (i) the connected components l_k and $l_{k'}$ of $\mathrm{Bs}\,|-K_{X_i}|$ containing $G_{k,i}$ and $G_{k',i}$ respectively are distinct, and
   (ii) there exists a connected component of $\mathrm{Bs}\,|-K_{X_{i-1}}|$ containing $G_{k,i-1}$ and $G_{k',i-1}$.

Hence the g_i-exceptional divisor F_i intersects l_k and $l_{k'}$. This implies that $\mathrm{Bs}\,|-K_{X_i}| \cap F_i$ is not connected, a contradiction to Corollary 2.7.

□

2.14 (Rules of the game). *We collect the properties of W which we use frequently afterward:*

(*a*) Sing W *is the union of lines and a finite number of points contained in exceptional planes on* W (2.12 (3) *and Proposition* 2.13 (2)).

(*b*) *The intersection of two exceptional planes does not contain a curve* (*Proposition* 2.13 (3)).

(*c*) *Any member of* $|-K_W|$ *is a prime divisor, or a union of a prime divisor and exceptional planes. In particular* W *is indecomposable and if* $g(X) \geq 4$*, then* W *contains no surface of degree between* 2 *and* $g(X)-2$ (*Corollary* 2.10).

The following corollary is not needed in this paper but it should be useful for the classification of X with $g(X) \leq 5$.

Corollary 2.15 (Existence of index 2 model). *Let P_i be the strict transforms of planes on W. Let $Z \dashrightarrow Z'$ be the flop of $(-1,-1)$-curves on the strict transforms of planes which are not good. Then the strict transforms of all the P_i on Z' are copies of $\mathbb{P}^2$ and can be contracted to $1/2\,(1,1,1)$-singularities at a time. Let $Z' \to Z''$ be the contraction and $Z'' \to W'$ the anti-canonical model. Then W' is a canonical indecomposable $\mathbb{Q}$-Fano 3-fold such that all the non-Gorenstein points are $1/2\,(1,1,1)$-singularities.*

Proof. The assertion is clear except that all the non-Gorenstein points on W' are $1/2\,(1,1,1)$-singularities. Let z be a $1/2\,(1,1,1)$-singularity on Z''. Since $-K_{Z'}$ is nef, there is no curve through z which is numerically trivial for $K_{Z''}$. This implies the assertion. $\square$

3. Study of the relation between X and W by using weighted dual graphs

3.1. **Basic properties of weighted dual graphs.**

3.1. *In this section, we use the notation as in* 2.12. *Let D be a general member of $|-K_X|$ and D_* the strict transform of D on $*$. Note that D has only $1/r\,(a,-a)$-singularity at a $1/r\,(a,-a,1)$-singularity of X by weight-reason* (*see* [Reid87, (4.10)] *for example*)*. Hence $D_{X_m} \to D$ is just the composite of the minimal resolutions of singular points contained in* $\operatorname{Sing} X$.

Notation and terminology for weighted dual graphs

Γ_{X_i}: The dual graph with

vertices: curves in $\operatorname{Bs}|-K_{X_i}|_{D_i}|$ and curves in the intersections between the exceptional divisors for $X_i \to X$ and D_{X_i}, and

weights of vertices: the intersection numbers of the corresponding curves with $-K_{X_i}$.

α_v: The curve on D_i corresponding to a vertex $v \in \Gamma_{X_i}$.

v_α: The vertex $v \in \Gamma_{X_i}$ corresponding to a curve α.

$\Gamma_{X_i,P}$: The connected component of Γ_{X_i} having a vertex $v \in \Gamma_{X_i,P}$ such that α_v contains the image of the strict transform of an exceptional plane P. This makes sense by Proposition 2.13 (3).

Fixed vertex: A vertex corresponding to a curve in $\operatorname{Bs}|-K_{X_i}|_{D_{X_i}}|$.

Economic vertex: A vertex in Γ_{X_m} corresponding to an economic exceptional divisor.

Economic chain: A chain in Γ_{X_m} consisting of economic vertices.

Claim 3.2. *Γ_{X_i} is a disjoint union of trees and the curve corresponding to a vertex is a copy of $\mathbb{P}^1$. In particular the sum of g-exceptional divisors is a simple normal crossing divisor.*

Proof. By [Ale91, Corollary 1.5], the assertion is true for Γ_X. Since $D_i \to D$ is a composite of blow-ups at points and along smooth curves and the vertices in Γ_X correspond to copies of $\mathbb{P}^1$'s, the former half of the assertion follows. Since the economic resolution is locally toric, the sum of exceptional divisors for $X_m \to X$ is a simple normal crossing divisor. Hence the latter half of the assertion follows from the former one. □

Proposition 3.3 (Properties of graphs)**.** *Γ_{X_i} satisfy the following:*

(1) *No vertex in Γ_Z is not fixed.*
(2) *If $\Gamma_{X_i,P}$ $(i \geq m)$ has a fixed vertex, then at least one fixed vertex has weight -1. In particular if P is not a good plane, then at least one fixed vertex in $\Gamma_{X_m,P}$ has weight -1.*
(3) *An exceptional vertex in Γ_{X_i} $(i \geq m)$ has the weight $-1, 0$ or 1.*
(4) *Two exceptional vertices in Γ_{X_i} $(i \geq m)$ with the weights 1 do not intersect.*
(5) *Two vertices in Γ_{X_i} $(i \geq m)$ with the weights -1 do not intersect.*

Proof. (1) follows since $\operatorname{Bs}|-K_Z| \cap D_Z = \operatorname{Bs}|-K_Z|_{D_Z}|$ by $h^1(\mathscr{O}_Z) = 0$, and $|-K_Z|$ is free. (2) follows from the construction of g. (3) and (4) follow by the construction of g and Corollary 2.8. (5) follows by Corollary 2.6 (1). □

We investigate how these weighted dual graphs are changed by $g_i|_{D_i}$ $(i \geq m+1)$.

3.4 (Operations of graphs)**.** *In all cases, $g_i|_{D_i}$ reduces the weights of vertices intersecting $v_{\gamma_{i-1}} \in \Gamma_{X_{i-1}}$ by 1 because of Claim 3.2. Moreover the graph changes at $v_{\gamma_{i-1}}$ according as the cases $(b1)$–$(b4)$ in Corollary 2.7 as follows:*

$(b1)$ *The shape of graph is unchanged. $v_{\gamma_{i-1}}$ changes to* 1.
$(b2)$ *$v_{\gamma_{i-1}}$ changes to* 0 —— 1 *or* 1 —— 0, *where* 1 *(resp.* 0*) corresponds to the fiber-component of $F_i|_{D_{X_i}}$ (resp. the minimal section of F_i).*
$(b3)$ *$v_{\gamma_{i-1}}$ changes to* 1 —— -1 —— 1, *where* 1 *(resp.* -1*) corresponds to a fiber-component of $F_i|_{D_{X_i}}$ (resp. the minimal section of F_i).*
$(b4)$ *$v_{\gamma_{i-1}}$ changes to* -1 —— 1 *or* 1 —— -1, *where* 1 *(resp.* -1*) corresponds to the fiber-component of $F_i|_{D_{X_i}}$ (resp. the minimal section of F_i).*

In the cases $(b2)$–$(b4)$*, we cannot specify the way how the new vertices in* Γ_{X_i} *intersect the vertices which intersected* $v_{\gamma_{i-1}}$ *in* $\Gamma_{X_{i-1}}$*.*

Claim 3.5. *Assume that* X *has only* $1/r\,(1,-1,1)$*-singularities. Then* -1 *in* Γ_{X_m} *intersects at least two economic vertices (clearly such vertices belong to different economic chains). Moreover if* -1 *intersects only two economic vertices, then one of the economic chains to which the two vertices belong corresponds to a* $1/s\,(1,-1,1)$*-singularity with* $s \geq 3$*.*

Proof. Clearly -1 intersects at least one economic vertex. Assume that -1 intersects only one economic vertex v. Let γ be the curve corresponding to -1 and F the exceptional divisor corresponding to v. Let u be the index of the singularity corresponding to the economic chain having v. Since X has only $1/r\,(1,-1,1)$-singularities, F is isomorphic to $\mathbb{P}^2$ (then we set $t=1$) or $\mathbb{F}_t$ for some $t \geq 2$. Then it is easy to see that $-K_X \cdot \gamma = -t/u$, a contradiction. The latter half can be similarly proved. □

Note that the number of vertices of the dual graph never decreases. Since $|-K_Z|$ is free and D_Z is a general member of $|-K_Z|$, vertices with weight 0 (resp. 1) in Γ_Z correspond to h-exceptional divisors which are contracted to lines by h (resp. the strict transforms of exceptional planes on W). Hence we have

Corollary 3.6. *Let* P *be an exceptional plane on* W*. Set*

$$d_1 := \#\{\textit{curves } \alpha \textit{ in } \mathrm{Bs}\,|-K_X|_D| \textit{ such that } v_\alpha \in \Gamma_{X,P}\}.$$

$$d_2 := \#\{h\textit{-exceptional divisors contracted to lines in } P\}.$$

Then

$$\sum (r_i - 1) + d_1 \leq 1 + d_2,$$

where the summation $\sum (r_i - 1)$ *is taken over the singularities contained in the curves corresponding to vertices of* $\Gamma_{X,P}$*.*

3.2. Applications of weighted dual graphs.

Proposition 3.7 (A characterization of a good plane)**.** *Let* P *be an exceptional plane on* W *and assume that* $\Gamma_{Z,P}$ *is a chain*

$$1 \;\text{——}\; 0 \;\text{——}\; \cdots\cdots \;\text{——}\; 0$$

with $r-1$ *vertices, where* 1 *corresponds to* P *and other vertices have weights* 0*. Then it is the resolution graph of a* $1/r\,(-1,1)$*-singularity of* D *which is a* $1/r\,(-1,1,1)$*-singularity on* X*. In particular* P *is a good plane.*

Proof. Assume the contrary. Then $\textcircled{1}$ corresponds to a non-economic exceptional divisor, which we denote by F. By 3.4 and the weights of $\Gamma_{Z,P}$, the contraction of F corresponds to the inverse of the operation (b1) or (b2) and hence by the contraction of F, $\Gamma_{Z,P}$ changes to

$$\textcircled{-1} —— \textcircled{1} —— \textcircled{0} —— \cdots —— \textcircled{0}$$

with $r-1$ or $r-2$ vertices according as the case (b1) or (b2). By considering inductively, only the inverse of (b1) or (b2) appears before the inverse of (b4) does and the graph is of the form

$$\textcircled{0} —— \cdots —— \textcircled{0} —— \textcircled{-1} —— \textcircled{1} —— \textcircled{0} —— \cdots\cdots —— \textcircled{0}\,.$$

Note that the intersection of the curves corresponding to $\textcircled{1}$ and its right-hand $\textcircled{0}$ is a smooth point of the strict transform of D since so is the intersection of the corresponding curves on D_Z and the contraction corresponding to the inverse of (b1) or (b2) is an isomorphism near the intersection.

Assume that the inverse of (b4) appears at this step. We denote this graph by $\Gamma_{X_i,P}$. Then $\textcircled{-1} —— \textcircled{1}$ corresponds to $F_i|_{D_{X_i}}$, where the multiplicity of the curve corresponding to $\textcircled{1}$ is two in $F_i|_{D_{X_i}}$. So since the intersection number between F_i and the curve corresponding to $\textcircled{0}$ on the right of $\textcircled{1}$ is 1 by Claim 3.2, the intersection of the curves corresponding to these $\textcircled{1}$ and $\textcircled{0}$ is a singular point of D_{X_i}, a contradiction.

Hence only the inverse of (b1) or (b2) appears and $\Gamma_{X_m,P}$ is of the form

$$\textcircled{0} —— \cdots —— \textcircled{0} —— \textcircled{-1} —— \textcircled{1} —— \textcircled{0} —— \cdots\cdots —— \textcircled{0}\,.$$

Note that any economic chain in $\Gamma_{X_m,P}$ contains at least one $\textcircled{1}$ and at least two $\textcircled{1}$'s for a $1/s\,(c, s-c, 1)$-singularity with $c>1$ and $s-c>1$, we conclude that

$$\textcircled{1} —— \textcircled{0} —— \cdots\cdots —— \textcircled{0}$$

is the economic chain for a $1/s\,(1,-1,1)$-singularity with some s. Then, however, this contradicts Claim 3.5 and we finish the proof. $\square$

The following three propositions are used only in the subsections 7.3 and 7.5.

Proposition 3.8. *Let P be an exceptional plane on W and assume that there are at most A_1-singular lines on P. Then one of the following holds:*

(1) *P is a good plane corresponding to a $1/r\,(1,-1,1)$-singularity with $r=2,3$.*

(2) *P is not a good plane and any singularity contained in the curves corresponding to the vertices of $\Gamma_{X,P}$ is a $1/2\,(1,1,1)$-singularity. $\Gamma_{X_m,P}$ is not a chain.*

Moreover if the assumption of this proposition is satisfied for any exceptional plane P on W, then (2) does not occur for any P.

Proof. Since the length of a chain in $\Gamma_{X_m,P}$ is at most 3, an economic chain in $\Gamma_{X_m,P}$ corresponds to a $1/s\,(1,-1,1)$-singularity with $s = 2,3,4$.

Assume by contradiction that $\Gamma_{X_m,P}$ has an economic chain corresponding to a $1/4\,(1,-1,1)$-singularity. By Proposition 3.7 and the assumption of this proposition, $\Gamma_{X_m,P}$ is not a chain. By the shape of $\Gamma_{X_m,P}$, other vertices in $\Gamma_{X_m,P}$ are not economic. By Proposition 3.3 (2), at least one of fixed vertices in $\Gamma_{X_m,P}$ has weight -1. This contradicts Claim 3.5.

Assume that there exists an economic chain L corresponding to a $1/3\,(1,-1,1)$-singularity in $\Gamma_{X_m,P}$. We prove that $L = \Gamma_{X_m,P}$. Assume the contrary. Then by the shape of $\Gamma_{X_m,P}$, other vertices in $\Gamma_{X_m,P}$ are not economic. So we can obtain a contradiction as the above case.

Finally we treat the case that any economic chain in $\Gamma_{X_m,P}$ corresponds to a $1/2\,(1,1,1)$-singularity. By Claim 3.5, $\Gamma_{X_m,P}$ is not a chain if P is not good. It remains to show the last assertion. If the assumption of this proposition is satisfied for any exceptional plane P on W, then X has only $1/2\,(1,1,1)$-singularities. So by [Taka02b, Theorem 1.0], $Z = X_m$. Hence $\Gamma_{X_m,P}$ is the disjoint union of economic chains corresponding to $1/2\,(1,1,1)$-singularities. $\square$

Definition 3.9. Let l be a singular line on W and P the exceptional plane on W containing l. l is called a *good line* if the dual graph of the strict transform of P and h-exceptional divisors over l is a chain.

Proposition 3.10. *Assume that there is at most one h-exceptional divisor contracted to a point on W. Then X has only $1/r\,(1,-1,1)$-singularities. Let*

$$\textcircled{1} \text{ --- } \textcircled{0} \text{ --- } \cdots\cdots \text{ --- } \textcircled{0}$$

be the economic chain in Γ_{X_m} for a $1/r\,(1,-1,1)$-singularity x. Then one of (1), (2) or (3) holds:

(1) (1-1) *no vertex in*

$$\textcircled{1} \text{ --- } \textcircled{0} \text{ --- } \cdots\cdots \text{ --- } \textcircled{0}$$

is a fixed one.

(1-2) *any* $\textcircled{0}$ *does not intersect a fixed vertex.*

(1-3) ① *does not intersect a fixed vertex, or intersects only one fixed vertex* v*. If the former case occurs, then* ① *corresponds to a good plane. If the latter case occurs, then* $\mathrm{Bs}\,|-K_{X_m}|$ *is reduced along* α_v*.*

(2) $r = 3$.

(2-1) $v :=$ ⓪ *is a fixed vertex and* $\mathrm{Bs}\,|-K_{X_m}|$ *is reduced along* α_v*.*

(2-2) ① *does not intersect a fixed vertex except* ⓪*.*

Hence in this case, the economic exceptional divisor over x *corresponding to* ⓪ *becomes the* h*-exceptional divisor contracted to a point on* W *by Corollary* 2.8*.*

(3) $r = 2$ *and* ① *is a fixed vertex. Hence in this case, the economic exceptional divisor over* x *becomes the* h*-exceptional divisor contracted to a point on* W *by Corollary* 2.8*.*

In particular for any exceptional plane P*,* $\Gamma_{X_m,P}$ *contains at most one economic chain which is not of type* (1) *and if* W *has only cDV singularities, then* Γ_{X_m} *contains only economic chains of type* (1)*.*

Proof. If there exists a $1/r\,(a, r-a, 1)$-singularity with $a, r-a > 1$, then at least two economic exceptional divisors becomes h-exceptional divisors contracted to a point by Corollary 2.8, a contradiction.

Assume that there is no divisor in the economic chain

$$① — ⓪ — \cdots\cdots — ⓪$$

which becomes an h-exceptional divisor contracted to a point on W. Then by a similar reason to the above, (1-1), (1-2) follows. We show (1-3). If ① does not intersect a fixed vertex, then ① corresponds to a good plane by Proposition 3.7. Assume that ① intersects at least one fixed vertex. Let F be the exceptional divisor corresponding to ①. Then by $(-K_{X_m})^2F = 1$, ① intersects only one fixed vertex v and $\mathrm{Bs}\,|-K_{X_m}|$ is reduced along α_v.

Assume that there is a (unique) vertex v in the economic chain such that the divisor corresponding to v is an h-exceptional divisor contracted to a point on W.

If F corresponds to one of ⓪, then any exceptional divisor corresponding to another ⓪ becomes an h-divisor contracted to a point on W by Corollary 2.8. Hence $r = 3$ and the former half of (2-1) holds. Moreover (2-2) and the latter half of (2-1) can be proved similarly to (1-3).

If F corresponds to ①, then $\mathrm{Bs}\,|-K_{X_m}| \cap F$ is a line. Hence if $r \geq 3$, then any exceptional divisor corresponding to one of ⓪ would become an h-divisor contracted to a point on W by Corollary 2.8, a contradiction. Hence $r = 2$ and (3) holds. $\square$

Proposition 3.11. *Assume that there is at most one h-exceptional divisor contracted to a point on W. Let P be an exceptional plane. Assume that $\Gamma_{Z,P}$ is a chain and P is not a good plane. Then one of the following holds:*

(1) $\Gamma_{X_m,P}$ *is*

$$0 \text{ — } \cdots \text{ — } 0 \text{ — } 1 \text{ — } {-1} \text{ — } 1 \text{ — } 0 \text{ — } \cdots \text{ — } 0\ ,$$

where

$$0 \text{ — } \cdots \text{ — } 0 \text{ — } 1$$

(resp.

$$1 \text{ — } 0 \text{ — } \cdots \text{ — } 0)$$

corresponds to a $1/r\,(1,-1,1)$-singularity for some r (resp. a $1/s\,(1,-1,1)$-singularity for some s).

In this case, $\Gamma_{Z,P}$ is

$$0 \text{ — } \cdots \text{ — } 0 \text{ — } 1 \text{ — } 0 \text{ — } \cdots \text{ — } 0\ ,$$

where the length of the left (resp. right)

$$0 \text{ — } \cdots \text{ — } 0$$

is $r-1$ (resp. $s-1$). If P is the unique exceptional plane, then W has only cDV singularities on P.

(2) $\Gamma_{X_m,P}$ *is*

$$1 \text{ — } 0 \text{ — } {-1} \text{ — } 1 \text{ — } 0 \text{ — } \cdots \text{ — } 0\ ,$$

where

$$1 \text{ — } 0$$

(resp.

$$1 \text{ — } 0 \text{ — } \cdots \text{ — } 0)$$

corresponds to a $1/3\,(1,-1,1)$-singularity and (resp. a $1/s\,(1,-1,1)$-singularity for some s).

In this case, $\Gamma_{Z,P}$ is

$$0 \text{ — } 1 \text{ — } 0 \text{ — } \cdots \text{ — } 0\ ,$$

where the length of

$$0 \text{ — } \cdots \text{ — } 0$$

is s. $\textcircled{0}$ on the right-side in $\Gamma_{X_m,P}$ corresponds to an h-exceptional divisor F contracted to a point on W. Hence P has two good lines and F is contracted to their intersection.

(3) $\Gamma_{X_m,P}$ *is*

$$\textcircled{0} - \cdots - \textcircled{0} - \textcircled{1} - \textcircled{-1} - \textcircled{1} - \textcircled{-1} - \textcircled{1} - \textcircled{0} - \cdots - \textcircled{0},$$

where

$$\textcircled{0} - \cdots - \textcircled{0} - \textcircled{1}$$

corresponds to a $1/r\,(1,-1,1)$-singularity for some r,

$$\textcircled{1} - \textcircled{0} - \cdots - \textcircled{0}$$

corresponds to a $1/s\,(1,-1,1)$-singularity for some s, and $\textcircled{1}$ between two $\textcircled{-1}$'s corresponds to a $1/2\,(1,1,1)$-singularity. In this case, $\Gamma_{Z,P}$ is

$$\textcircled{0} - \cdots - \textcircled{0} - \textcircled{1} - \textcircled{0} - \cdots - \textcircled{0},$$

where the length of the left (resp. right)

$$\textcircled{0} - \cdots - \textcircled{0}$$

is r (resp. s). $\textcircled{1}$ between two $\textcircled{-1}$'s in $\Gamma_{X_m,P}$ corresponds to an h-exceptional divisor F contracted to a point on W. Hence P has two good lines and F is contracted to their intersection.

Proof. First we give the descriptions of $\Gamma_{X_m,P}$. Since P is not a good plane, $\Gamma_{X_m,P}$ contains at least one $\textcircled{-1}$ by Proposition 3.3 (2). Hence by Claim 3.5 and Proposition 3.10, $\Gamma_{X_m,P}$ contains one of the following:

(a)

$$\textcircled{0} - \cdots - \textcircled{0} - \textcircled{1} - \textcircled{-1} - \textcircled{1} - \textcircled{0} - \cdots - \textcircled{0}.$$

This is the case (1).

(b)

$$\textcircled{1} - \textcircled{0} - \textcircled{-1} - \textcircled{1} - \textcircled{0} - \cdots - \textcircled{0}.$$

This is the case (2).

(c)

$$
\begin{array}{l}
0 — \cdots — 0 — 1 — \\
\qquad — -1 — 1 — \bigcirc — \cdots — \bigcirc — \\
\qquad\qquad\qquad — 1 — 0 — \cdots — 0 .
\end{array}
$$

We prove that this is the case (3). We denote

$$\bigcirc — \cdots — \bigcirc$$

by Γ'. If Γ' does not contain -1, then the operations to obtain Γ_Z terminate without resolving the base curves corresponding to fixed vertices of Γ', a contradiction to Proposition 3.3 (1). Hence Γ' contains -1 and by Claim 3.5, $\Gamma' = -1$. Hence we have (3).

Next we give the descriptions of $\Gamma_{Z,P}$. In any case, we can see that the base curves are reduced in $\Gamma_{X_m,P}$ by Proposition 3.10. So the blow-ups along the base curves induce the operations of type (b1) in 3.4. Hence we obtain the desired descriptions.

□

4. Properties of various linear projections of W

In this section, Δ is a linear subspace Δ of W.

4.1. **Study of the center of $\nu_{W,\Delta}$ for a linear subspace Δ of W.**

Lemma 4.1. *Let Δ be a linear subspace of W satisfying one of the following:*

(*a*) (*only if $g(X)=6$*) *Δ is the point v as in Proposition* 7.1 (2),

(*b*) *Δ is a singular line l.*

  (*b*1) *W has only hypersurface singularities at any point on l, or*

  (*b*2) (*only if $g(X)=6$*) *V is the cone and $v \in l$.*

(*c*) *Δ is a plane P. W has only local complete intersection singularities at any point on P.*

Then $\mu_{W,\Delta}$ is crepant and hence $\widetilde{W}_\Delta$ has only canonical singularities.

Proof. Assume that Δ satisfies (a). Since $\operatorname{mult}_v W = 5$ by Proposition 7.1 (2), the assertion holds by [Rei80, Theorem 2.11].

Assume that Δ satisfies (b1). We can check the assertion locally around a point w. So W (resp. l) is a hypersurface (resp. a copy of $\mathbb{C}$) in $\mathbb{C}^4$ locally analytically near w. By explicit computations, we can see that $\mu_{W,l}$ is crepant.

Assume that Δ satisfies (b2). Let $\mu' : W' \to \widetilde{W}_v$ be the blow-up along the strict transform l' of l on $\widetilde{W}_v$ and E' the μ'-exceptional divisor. Since $E_{W,v}$ has only hypersurface singularities by Proposition 7.8 (a), so is $\widetilde{W}_v$. Hence by the computation in the case (b1), μ' is crepant and by the conclusion in the case (a), so is $\mu'' := \mu' \circ \mu_{W,v}$. Let $\pi : W' \to W''$ be the canonical model for $\mu''^* L_W - \mu'^* E_{W,v} - E'$ over W and $\pi' : W'' \to W$ the natural morphism. Let $E'_{W,v}$ and E'' be the images of $\mu'^* E_{W,v}$ and E' on W'' respectively. Then

$$H^0(\pi'^* L_W - E'_{W,v} - E'') \simeq H^0({\mu_{W,l}}^* L_W - E_{W,l}).$$

Since $\pi'^* L_W - E'_{W,v} - E''$ is π''-ample, this implies that $W'' \simeq \widetilde{W}_l$ over W. Consequently $\mu_{W,l}$ is crepant since so is π'.

Assume that Δ satisfies (c). We can check the assertion locally around a point w. By [Lau77, Theorem 3.13], the embedded dimension of W at $w \leq 5$. So locally analytically near w, W can be embedded in $\mathbb{C}^5$. Let $x_1, \ldots, x_5$ be the coordinates of $\mathbb{C}^5$. Then we can write

$$W = \{f = g = 0\}.$$
$$P = \{x_1 = x_2 = x_3 = 0\}.$$

Since $P \not\subset \operatorname{Sing} W$ and W has only hypersurface singularities except a finite set of points, we can see that $\mu_{W,P}$ is crepant by explicit computations. $\square$

Lemma 4.2. *Let $\pi : S \to T$ be a birational morphism between normal surfaces. Assume that*

(1) *S has only canonical singularities,*
(2) *$-K_S$ is π-ample, and*
(3) *$C := \operatorname{exc} \pi$ is connected.*

Then $C \simeq \mathbb{P}^1$ and $-K_S \cdot C = 1$. Moreover C contains at most one singularity of S and its type is A_m for some $m \in \mathbb{N}$.

Proof. This easily follows from [LS85, Theorem 0.1]. $\square$

Definition 4.3. Assume that $\mu_{W,\Delta}$ is crepant and W_Δ is a normal 3-fold such that K_{W_Δ} is $\mathbb{Q}$-Cartier. Define

$$F_{W,\Delta} := {\nu_{W,\Delta}}^*(-K_{W_\Delta}) - (-K_{\widetilde{W}_\Delta}).$$
$$C_\Delta := {\nu_{W,\Delta}}_*(-K_{\widetilde{W}_\Delta} \cdot F_{W,\Delta}).$$

Proposition 4.4. *Let the assumptions be as in Definition* 4.3 *and assume that W_Δ is a del Pezzo* 3*-fold of degree* ≥ 3. *Set* $s := \dim \Delta$ (*note that $W_\Delta \subset \mathbb{P}^{g(X)-s}$*). *Then*

(1) *$E_{W,\Delta} = {\nu_{W,\Delta}}^* L_{W_\Delta} - F_{W,\Delta}$ and $F_{W,\Delta} = {\mu_{W,\Delta}}^* L_W - 2E_{W,\Delta}$.*

(2) *Assume moreover that $\nu_{W,\Delta}$ is birational. Then*

$$\deg C_\Delta = \begin{cases} 2g(X)-2 \text{ if } s=0. \\ 4\deg W_\Delta - 2g(X)+2 \text{ if } s=1. \\ 4\deg W_\Delta - 2g(X)+3 \text{ if } s=2. \end{cases}$$

(3) *In addition to the assumption of (2), assume that $g(X) \geq 4$. Then*

$$C_\Delta = \delta C'_\Delta + \sum d_i m_i,$$

where $\delta, d_i \in \mathbb{N}$, and C'_Δ and m_i are irreducible curves such that

(3-1) *$C'_\Delta \neq \emptyset$ and $\langle C'_\Delta \rangle$ is a hyperplane of $\mathbb{P}^{g(X)-s}$,*

(3-2) *$\deg C'_\Delta \geq p_a(C'_\Delta) + g(X) - s - 1$, and*

(3-3) *if $\dim\langle m_i \rangle$ is smaller than $g(X)-s-1$, then m_i is the image of the strict transform of a plane, or that of a $\mu_{W,\Delta}$-exceptional divisor.*

Remark. In the below sections, we see that $C_\Delta = C'_\Delta$ or $C_\Delta = C'_\Delta + dm$, where $d \in \mathbb{N}$ and m is a line.

Proof. (1) This follows from the definition of $F_{W,\Delta}$ and

$$\nu_{W,\Delta}{}^* L_{W_\Delta} = \mu_{W,\Delta}{}^* L_W - E_{W,\Delta}.$$

(2) We have

$$\begin{aligned} \deg C_\Delta &= (L_{W_\Delta} \cdot C_\Delta) = \nu_{W,\Delta}{}^* L_{W_\Delta}(-K_{\widetilde{W_\Delta}}) F_{W,\Delta} \\ &= (L_W)^3 - 3(\mu_{W,\Delta}{}^* L_W)^2 E_{W,\Delta} + 2\mu_{W,\Delta}{}^* L_W (E_{W,\Delta})^2. \end{aligned}$$

The second equality follows from Lemma 4.2.

If $s=0$, then $\deg C_\Delta = 2g(X)-2$. We treat the other cases. By $(\mu_{W,\Delta}{}^* L_W - E_{W,\Delta})^3 = \deg W_\Delta$ and $(\nu_{W,\Delta}{}^* L_{W_\Delta})^2 F_{W,\Delta} = 0$, we have

$$\mu_{W,\Delta}{}^* L_W (E_{W,\Delta})^2 = 2(\mu_{W,\Delta}{}^* L_W)^2 E_{W,\Delta} + 2\deg W_\Delta - (L_W)^3.$$

This gives

$$\deg C_\Delta = 4d - (2g(X)-2) + (\mu_{W,\Delta}{}^* L_W)^2 E_{W,\Delta}.$$

If $s=1$ (resp. 2), then $(\mu_{W,\Delta}{}^* L_W)^2 E_{W,\Delta} = 0$ (resp. 1). So we have the assertion.

(3) Assume that C_Δ contains an irreducible and reduced curve n contained in an r-plane with $r \leq g(X)-s-2$. Let F' be the irreducible component of $F_{W,\Delta}$ such that $\nu_{W,\Delta}(F') = n$. Then $\mu_{W,\Delta}(F')$ is contained in an $(r+s+1)$-plane with $r+s+1 \leq g(X)-1$. Thus $|L_W \otimes \mathscr{I}_{\mu_{W,\Delta}(F')/W}|$ has the movable part. This implies that if $\mu_{W,\Delta}(F')$ is a divisor, then it is a plane by 2.14 (c).

Note that by (1), $\mu_{W,\Delta}(F_{W,\Delta}) \in |L_W|$ if Δ is not a plane, or $\mu_{W,\Delta}(F_{W,\Delta}) \in |L_W - \Delta|$ if Δ is a plane. Hence $F_{W,\Delta}$ contains an irreducible component F'' such that $\mu_{W,\Delta}(F'')$ is a divisor and is not a plane. So $C'_\Delta := \nu_{W,\Delta}(F'')$ is not contained in an r-plane with $r \le g(X) - s - 2$. On the other hand, C'_Δ is contained in a hyperplane section of W_Δ by (1). Hence $\langle C'_\Delta \rangle$ is a hyperplane of $\mathbb{P}^{g(X)-s}$. This implies that

$$p_a(C'_\Delta) \le \deg C'_\Delta - (g(X) - s - 1).$$

Thus we obtain (3).

□

Lemma 4.5. *Let C be an irreducible projective curve and L be a Cartier divisor on C such that $d := \deg L \ge 2p_a(C)+2$ (note that in particular, L is very ample). Let $\phi : C \to \mathbb{P}^{d-p_a(C)}$ be the embedding defined by $|L|$. Then $\phi(C)$ is the intersections of quadrics.*

Proof. See [ACGH, p.142, F-2]. The result is stated only for a smooth curve but the proof works also for a singular curve since the general position theorem [ibid. p.109] has nothing to do with singularities of a curve. □

Proposition 4.6. *Let the assumptions be as in Proposition* 4.4 (3) *and assume that C_Δ is reducible. Then C'_Δ is not the intersection of quadrics. Assume moreover that*

$$\deg C'_\Delta = p_a(C'_\Delta) + g(X) - s - 1$$

and for any i, $\dim\langle m_i \rangle$ is smaller than $g(X) - s - 1$. Then

$$\deg C'_\Delta \ge 2(g(X) - s) - 3.$$

Proof. Define $\overline{C}_\Delta$ to be the closed subscheme associated to the ideal $\nu_{W,\Delta_*}\mathscr{O}_{\widetilde{W}_\Delta}(-F_{W,\Delta})$. Note that $\overline{C}_\Delta = C_\Delta$ as a cycle.

Assume that C'_Δ is the intersection of quadrics. Then since $\overline{C}_\Delta \neq C'_\Delta$ and $-K_{W_\Delta} = 2L_{W_\Delta}$, there is a member $D \in |-K_{W_\Delta}|$ such that $C'_\Delta \subset D$ but $\overline{C}_\Delta \not\subset D$. Let F'' be the irreducible component of $F_{W,\Delta}$ such that $\nu_{W,P}(F'') = C'_\Delta$. Set $D' := \nu^*_{W,\Delta}D - F''$. By the assumption and Proposition 4.4 (3-3), m_i is the image of the strict transform of a plane or that of a $\mu_{W,\Delta}$-exceptional divisor. Hence $-K_W - \mu_{W,\Delta}(D')$ is linearly equivalent to at most a sum of planes. By Proposition 2.13 (1), this implies that the strict transform of $\mu_{W,\Delta}(D')$ is contained in $|-K_X|$. This means that

$$h^0(-K_X) \ge h^0(-K_{W_\Delta} \otimes \mathscr{I}_{C'_\Delta/W_\Delta}) > h^0(-K_{W_\Delta} \otimes \mathscr{I}_{\overline{C}_\Delta/W_\Delta}) = h^0(-K_W),$$

a contradiction.

If $\deg C'_\Delta = p_a(C'_\Delta) + g(X) - s - 1$, then C'_Δ is embedded in $\mathbb{P}^{g(X)-s-1}$ by $|L_{W_\Delta}|_{C'_\Delta}|$. Hence the assertion follows from (1) and Lemma 4.5. □

Proposition 4.7. *Let l be a singular line on W and the assumptions as in Proposition 4.4 (3) for $\Delta = l$. Then any plane on W intersects l.*

Proof. Assume that there exists a plane Q such that $Q \cap l = \emptyset$. Then $\nu_{W,l}(Q')$ is a plane on W_l, where Q' is the strict transform of Q on $\widetilde{W_l}$. By Proposition 4.4 (1), $\nu_{W,l}(E_{W,l})$ is a hyperplane section of W_l. Hence $m := \nu_{W,l}(Q') \cap \nu_{W,l}(E_{W,l})$ is a line. Since $Q' \cap E_{W,l} = \emptyset$, there is an irreducible component F of $F_{W,l}$ dominating m and intersecting both Q' and $E_{W,l}$ along curves (in particular $F \not\subset E_{W,l}$). Then $\mu_{W,l}(F)$ is a surface contained in $W \cap L$, where L is a 3-plane containing l. So since W is the intersection of quadrics, we have $\deg \mu_{W,l}(F) \leq 2$. Thus by $g(X) \geq 4$ and 2.14 (c), $\mu_{W,l}(F)$ is a plane. On the other hand, $Q' \cap F$ contains a curve and $Q' \simeq Q$. Hence $Q \cap \mu_{W,l}(F)$ is also a curve, a contradiction to 2.14 (b). □

4.2. **Classification of del Pezzo 3-folds appearing as W_Δ for the projection from a linear subspace Δ of W.**

Proposition 4.8. *Assume that W_Δ is a (possibly non-normal) del Pezzo 3-fold of degree ≤ 5. Then W_Δ has only canonical singularities. Moreover the following hold:*

(1) *If $\deg W_\Delta = 5$, then W_Δ is smooth.*
(2) *If $\deg W_\Delta = 4$, then W_Δ has only terminal singularities, and one of the following holds:*
  (2-1) *W_Δ is factorial, or*
  (2-2) *W_Δ is not factorial. A factorization $\widehat{W} \to W_\Delta$ is a small resolution and $\rho(\widehat{W}) = 2$ (and hence there are two factorizations). Moreover one of two factorizations has a contraction of $E \simeq \mathbb{P}^2$ to a smooth point on a smooth quintic del Pezzo 3-fold, and the other has a $\mathbb{P}^1$-bundle structure over $\mathbb{P}^2$. W_Δ contains a unique plane E_1, which is the image of E. If the assumptions is as in Proposition 4.4 (3), then $\nu_{W,\Delta}(E_{W,\Delta})$ is a hyperplane section containing E_1.*

Proof.
Step 1. W_Δ is not covered by planes (resp. irreducible quadrics if $\deg W_\Delta \geq 4$).

Assume by contradiction that W_Δ is covered by planes (resp. irreducible quadrics if $\deg W_\Delta \geq 4$). We can choose a plane (resp. an irreducible quadric) $S \subset W_\Delta$ which is neither the strict transform of Δ (only if $\dim \Delta = 2$), that of a plane on W, nor the image of a $\mu_{W,\Delta}$-exceptional divisor. Then the member of $|L_W|$ corresponding to that of $|L_{W_\Delta} \otimes \mathscr{I}_{S/W_\Delta}|$ is reducible and contains at least two irreducible components neither of which is a plane since $|L_{W_\Delta} \otimes \mathscr{I}_{S/W_\Delta}|$ has the movable part. This contradicts 2.14 (c).

Step 2. We classify an extremal ray of a $\mathbb{Q}$-factorial terminalization of W_Δ roughly.

Let $\pi' : W'_\Delta \to W_\Delta$ be the normalization and $\pi'' : W^1_\Delta \to W'_\Delta$ be a $\mathbb{Q}$-factorial terminalization such that $K_{W^1_\Delta}$ is π''-nef. Set $\pi := \pi' \circ \pi''$. Then we can write $K_{W^1_\Delta} = \pi^* K_{W_\Delta} - Z$, where Z is an effective divisor. If $Z \neq 0$, then we can find an extremal ray R of W^1_Δ such that $Z \cdot R > 0$ (this follows from the cone theorem. See for example [Reid94, the proof of Theorem 1.1]). If $Z = 0$, then let R be any extremal ray of W^1_Δ. Note that if $Z = 0$, then W^1_Δ is Gorenstein. Hence we can use the classification of [Mor82] and [Cut88]. Let p be the extremal contraction associated to R and $L_{W^1_\Delta} := \pi^* L_{W_\Delta}$. By $-K_{W^1_\Delta} = 2L_{W^1_\Delta} + Z$, p satisfies one of the following:

(1) p is $(3,0)$-type. This case occurs if and only if $W^1_\Delta \simeq W_\Delta$, i.e., W_Δ is terminal and factorial.
(2) $Z = 0$ and p is of $(3,1)$-type. A general fiber D is a quadric in $\mathbb{P}^3$ and $L_{W^1_\Delta}|_D$ is a hyperplane section.
(3) $Z = 0$ and p is a $\mathbb{P}^1$-bundle and $L_{W^1_\Delta}|_r$ is a hyperplane section for a fiber r of p.
(4) $Z = 0$ and p is of $(2,0)$-type which contracts $E \simeq \mathbb{P}^2$ to a smooth point. $L_{W^1_\Delta}|_E$ is a hyperplane section.
(5) $Z > 0$ and p is of $(3,1)$-type. A general fiber D is $\mathbb{P}^2$ and $L_{W^1_\Delta}|_D$ is a hyperplane section.
(6) $Z > 0$ and p is of $(2,0)$-type.

First we prove that $Z = 0$. In particular, we see then that W_Δ has only canonical singularities.

If (5) occurs, then $\pi(D)$ is a plane. Thus W_Δ is covered by planes, a contradiction to Step 1.

If (6) occurs, then by $-K_{W^1_\Delta} = 2L_{W^1_\Delta} + Z$ and [AW93, Theorem], $|L_{W^1_\Delta}|$ is p-free. Hence we can choose a smooth member D of $|L_{W^1_\Delta}|$ locally near $\operatorname{exc} p$. By restricting p to D, we have a K_D-negative contraction of $(1,0)$-type, i.e., $p|_D$ is the blow-down of a (-1)-curve $F := \operatorname{exc} p|_D$. But we have $-K_D \cdot F > 1$ by $-K_{W^1_\Delta} = 2L_{W^1_\Delta} + Z$, a contradiction.

Step 3.

We prove that if (2) occurs, or (3) occurs and the base S of p is not $\mathbb{P}^2$, then $\deg W_\Delta \leq 3$.

If (2) occurs, then $\pi(D)$ is a quadric surface. So by Step 1, this case occurs only if $\deg W_\Delta \leq 3$. Assume that (3) occurs and S is not $\mathbb{P}^2$. Note that S is a smooth weak del Pezzo surface by [Cut88] and [MM85, the proof of Proposition 4.16]. So if S is not $\mathbb{P}^2$, then S is covered by smooth rational curves n such that $n^2 = 0$. Let $M_n := p^*n$. Then

$$(-K_{W^1_\Delta})^2 M_n = (-K_{W^1_\Delta} - M_n)^2 M_n = (-K_{M_n})^2 = 8.$$

This implies that $\pi(M_n)$ is a quadric surface. Hence $\deg W_\Delta \leq 3$ by Step 1.

Step 4. We treat the case that (3) of Step 2 occurs and $S \simeq \mathbb{P}^2$.

By the assumption, $\rho(W^1_\Delta) = 2$.

Assume that W^1_Δ has a crepant divisor T. First we treat the case that $\pi(T)$ is a point. If $\deg W_\Delta \geq 3$, then it intersects T at one point since a fiber of p is mapped to a line on W_Δ. Thus T is a section of p. Then it is easy to see that $W^1_\Delta = \mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(-3))$ and hence $(L_{W^1_\Delta})^3 = 9$. So $\deg W_\Delta \leq 2$.

Next we treat the case that $\pi(T)$ is a curve. We can write

$$T \equiv z(-K_{W^1_\Delta}) - uD,$$

where D is the pull back of a line by p. By

$$(-K_{W^1_\Delta})^2 T = 0 \text{ and } (-K_{W^1_\Delta})^2 D = 12,$$

we have $3u = 2z \deg W_\Delta$. Moreover $T \cdot m = 2z$ for a fiber m of p so $2z \in \mathbb{Z}$. Let n be a general fiber of T. Then by $T \cdot n = -2$, we have $u(D \cdot n) = 2$. We can easily derive the following solutions, where we set $z' = 2z$:

(a) $\deg W_\Delta = 1$. $z' = 1, 2, 3, 6$ and $u = z'/3$.
(b) $\deg W_\Delta = 2$. $z' = 1, 3$ and $u = 2z'/3$.
(c) $\deg W_\Delta = 3$. $z' = 1, 2$ and $u = z'$.

So if $\deg W_\Delta \geq 4$, then $W^1_\Delta \to W_\Delta$ is a small resolution and in particular W_Δ is terminal. Moreover, by the method of [Take89], we see that $\deg W_\Delta \leq 4$, i.e., $\deg W_\Delta = 4$.

Step 5. We treat the case that (4) of Step 2 occurs.

Let W^2_Δ be the target of p and W^3_Δ be the anti-canonical model of W^2_Δ. It is easy to see that W^3_Δ is a del Pezzo 3-fold with only canonical singularities and $\deg W^3_\Delta = \deg W_\Delta + 1$.

Case 1.

Assume that W^3_Δ is smooth. Then we show that $\deg W_\Delta \leq 4$. We use the classification of smooth del Pezzo 3-folds [Fuj80]–[Fuj84]. Note that $W^3_\Delta = W^2_\Delta$ in this case. If $\deg W^3_\Delta = 6$, then

$$W^3_\Delta \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \text{ or a divisor of } (1,1)\text{-type on } \mathbb{P}^2 \times \mathbb{P}^2.$$

Moreover $L_{W^3_\Delta}$ is a divisor of $(1,1,1)$-type (resp. the restriction of a divisor of $(1,1)$-type) if $W^3_\Delta \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (resp. a divisor of $(1,1)$-type on $\mathbb{P}^2 \times \mathbb{P}^2$). Then $L_{W^1_\Delta} = p^*(L_{W^2_\Delta}) - E$ is decomposable, a contradiction. Hence $\deg W_\Delta \leq 4$.

Case 2.

Assume that W^3_Δ is a cone over a del Pezzo surface of degree ≥ 3. Then W^3_Δ contains families of quadrics, which are cones over conics in the del Pezzo surface. Since $p(E)$ is a smooth point of W^2_Δ and is disjoint from the exceptional locus of $W^2_\Delta \to W^3_\Delta$, a general quadric in W^3_Δ does not pass through the image of $p(E)$. Hence a general quadric in W^3_Δ is transformed to that in W_Δ. Thus by Step 1, we have $\deg W_\Delta = 3$.

Case 3. Assume that $\deg W_\Delta \geq 4$, W^3_Δ is singular and is not a cone. We prove that this case does not occur. By [Fuj85], $\deg W^3_\Delta \leq 6$ and W^3_Δ (resp. W_Δ) is (si21) or (si111) in [ibid.] if $\deg W^3_\Delta = 5$ (resp. $\deg W^3_\Delta = 6$). We use the notation in [ibid.]. Note that W^3_Δ (resp. W_Δ) is denoted by V in [ibid.].

Assume that $\deg W^3_\Delta = 5$. If W^2_Δ is (si21), then

$$(H_\alpha)^2(H_\alpha + H_\tau)F = 2,$$

where F is the pull back of a ruling of $M \simeq \mathbb{F}_1$ on $\widetilde{W}$, where $\widetilde{W}$ is in the notation in [ibid.]. Hence a surface which is the intersection of general members of $|(H_\alpha + H_\tau)|_{\widetilde{V}}|$ and $|F|_{\widetilde{V}}|$ is transformed to a quadric in W_Δ, a contradiction. If W^2_Δ is (si111), then

$$(H_\alpha)^2(H_\alpha + H_\tau)H_\eta = 2.$$

Hence a surface which is the intersection of general members of $|(H_\alpha + H_\tau)|_{\widetilde{V}}|$ and $|H_\eta|_{\widetilde{V}}|$ is transformed to a quadric in W_Δ, a contradiction.

Assume that $\deg W^3_\Delta = 6$. Then we prove that L_{W_Δ} is decomposable. In this case, $\widetilde{V}$ of Fujita's notation is contained in a $\mathbb{P}^1 \times \mathbb{P}^2$-bundle over $\mathbb{P}^1$ such that the pull-back of L_{W_Δ} is the restriction of an ample decomposable tautological divisor of $\tilde{V}$. Hence L_{W_Δ} is decomposable. This implies that W is decomposable, a contradiction.

Consequently we see that $\deg W_\Delta \leq 4$. Moreover if $\deg W_\Delta = 4$, then W^3_Δ is smooth and in particular, we have $\rho(W^1_\Delta) = 2$.

We study the case that $\rho(W^1_\Delta) = 2$. Assume that W^1_Δ has a crepant divisor T. Then $\pi(T)$ cannot be a point. Indeed, otherwise the curve $E \cap T (\neq \emptyset)$ is K-negative and K-trivial at the same time, a contradiction. Hence $\pi(T)$ is a curve. We can write $T \equiv z(-K_{W^1_\Delta}) - uE$. By $(-K_{W^1_\Delta})^2 T = 0$ and $(-K_{W^1_\Delta})^2 E = 4$, we have $u = 2z \deg W^1_\Delta$. Since W^3_Δ is a del Pezzo 3-fold, we have $2z \in \mathbb{Z}$. Let n be a general fiber of

T. Then by $T \cdot n = -2$, we have $u(E \cdot n) = 2$. Thus we can easily derive the following solutions:

(a) $\deg W^1_\Delta = 1$, $E \cdot n = 1$ and $z = 1$.
(b) $\deg W^1_\Delta = 1$, $E \cdot n = 2$ and $z = 1/2$.
(c) $\deg W^1_\Delta = 2$, $E \cdot n = 1$ and $z = 1/2$.

Hence if $\deg W_\Delta = 4$, then $W^1_\Delta \to W_\Delta$ is a small resolution and in particular W_Δ is terminal.

Step 6. We complete the proof.

By Steps 4 and 5, if $\deg W_\Delta = 5$, then W_Δ is smooth. Assume that $\deg W_\Delta = 4$. Since $\rho(W^1_\Delta) = 2$, the method of [Take89] applies. We can easily show that after the flop $W^1_\Delta \dashrightarrow W^{1+}_\Delta$, the cases (3) and (4) interchange.

We may assume that W^1_Δ satisfies (4) of Step 2. Assume that there exists a plane P on W_Δ which is not the image of E. Let P' be the strict transform of P on W^3_Δ of Step 5. Then P' is contained in a 3-plane, a contradiction to $\operatorname{Pic} W^3_\Delta = \mathbb{Z}[L_{W^3_\Delta}]$. Hence the uniqueness of the plane on W_Δ follows.

Assume that $\mu_{W,\Delta}$ is crepant. Considering the total transform on W of a hyperplane section containing E_1, we see by 2.14 (c) and Proposition 4.7 that E_1 is the image of a $\mu_{W,\Delta}$-exceptional divisor, or the strict transform of Δ (only if Δ is a plane). Thus by Proposition 4.4 (1), $\nu_{W,\Delta}(E_{W,\Delta})$ is a hyperplane section containing E_1. $\square$

Remark. In this paper, we do not need the case that $\deg W_\Delta \leq 3$ and so we did not study these case in detail.

4.3. Relation between C_Δ and singularities of X.

We gather the results which hold when W_Δ is a del Pezzo 3-fold and C_Δ is irreducible.

Proposition 4.9. *Let P be a plane on W and the assumption as in Proposition 4.4 (3) for $\Delta = P$. Assume that $\deg W_P \leq 5$ and C_P is irreducible. Then P is unique and is a good plane.*

Assume moreover that $\nu_{W,P}(E_{W,P})$ is irreducible (this holds for example if W_P is factorial by Proposition 4.4 (1)). Then $\operatorname{Sing} X$ consists of one $1/2\,(1,1,1)$-singularity.

Proof.

Step 1. We prove the uniqueness of P.

If there were a plane $P' \neq P$ such that $P \cap P' \neq \emptyset$, then it would be contracted to a line on W_P. This contradicts irreducibility of C_P. Assume that there is a plane $P' \neq P$ such that $P \cap P' = \emptyset$. Then it is mapped to a plane P'' on W_P. Since $\nu_{W,P}(E_{W,P})$ is a hyperplane section by Proposition 4.4 (1), P'' intersects $\nu_{W,P}(E_{W,P})$ along a line

m. So there is a $\nu_{W,P}$-exceptional divisor over m. By Proposition 4.4 (3-1), $m \neq C'_\Delta$. But this contradicts irreducibility of C_P. So P is a unique plane.

Step 2. We prove that P is good.

By irreducibility of C_P, no $\mu_{W,P}$-exceptional divisor is $\nu_{W,P}$-exceptional and the strict transform of P is not contracted to a curve by $\nu_{W,P}$.

Case 1. $\nu_{W,P}(E_{W,P})$ **is reducible.**

By Proposition 4.8, $\deg W_P = 3, 4$. Assume that $\deg W_P = 4$. Then by Proposition 4.8 (2-2), we have $\nu_{W,P}(E_{W,P}) = E_1 + E_2$, where E_1 is a plane, and E_2 is a cubic surface such that $\langle E_2 \rangle$ is a 4-plane. Assume that $\deg W_P = 3$. Then since C_P is a twisted cubic curve and hence it is not a plane curve, we have $\nu_{W,P}(E_{W,P}) = E_1 + E_2$, where E_1 is a plane, and E_2 is a quadric surface.

We prove that E_1 on $\widetilde{W}_P$ is not a $\mu_{W,P}$-exceptional divisor. Take a general line n in E_1. Then

$$\mu^*_{W,P} L_W \cdot n = 2 \tag{4.1}$$

since $C_P \not\subset E_1$. So $\mu_{W,P}(E'_1)$ is not a point. If $\mu_{W,P}(E'_1)$ is a line, then a $\mu_{W,P}$-fiber δ in E'_1 is the strict transform of a line on E_1. Thus $\delta \cap n$ is one point and hence by (4.1), the image of E'_1 on W is a conic, a contradiction. So E'_1 is not a $\mu_{W,P}$-exceptional divisor whence it is the strict transform of P.

Consequently we see that $E_{W,P}$ is reduced and has 2 components, which implies that P is a good plane by Proposition 3.7.

Case 2. $\nu_{W,P}(E_{W,P})$ **is irreducible.**

It suffices to prove that $E_{W,P}$ is irreducible since this implies that W has only isolated singularities along P and by Proposition 3.7, P is a good plane corresponding to a $1/2\,(1,1,1)$-singularity.

Assume the contrary. Then the strict transform P' of P is contracted to a point v by $\nu_{W,P}$. Thus $\nu_{W,P}(E_{W,P})$ is a cone since the images of $\mu_{W,P}$-fibers are lines on W_P and they pass through v. Besides $\nu_{W,P}(E_{W,P})$ is a non-normal del Pezzo surface.

Let E' be the irreducible component of $E_{W,P}$ different from P'. We show that $E' \to l$ is a $\mathbb{P}^1$-bundle and in particular E' is smooth. Let $D \in |{\nu_{W,P}}^* L_{W_\Delta}|$ be a general member. Since $|{\nu_{W,P}}^* L_{W_\Delta}|$ is free, $D|_{E'}$ is irreducible and reduced (as a scheme). Moreover since

$${\nu_{W,P}}^* L_{W_\Delta} \cdot f = 1 \tag{4.2}$$

for a general fiber f of $\mu_{W,P}|_{E'}$, $D|_{E'} \to l$ is birational and hence $D|_{E'} \simeq \mathbb{P}^1$. This implies that no fiber of $\mu_{W,P}|_{E'}$ is contained in $\operatorname{Sing} E'$. By (4.2) for a general fiber f of $\mu_{W,P}|_{E'}$, we have $f \simeq \mathbb{P}^1$ and hence there is no horizontal singular locus over l. Thus we know that E' is regular in codimension 1. Let $\pi : \widetilde{E}' \to E'$ be the normalization. Since $\widetilde{E}' \to l$ is flat and its general fiber r is $\mathbb{P}^1$ such that $((\nu_{W,P} \circ \pi)^* L_{W_\Delta}|_{E'}) \cdot r = 1$, we see that $E' \to l$ is a $\mathbb{P}^1$-bundle by a standard argument. Since E' is regular in codimension 1, any fiber f' of $\mu_{W,P}|_{E'}$ is generically reduced and ${\nu_{W,P}}^* L_{W_\Delta} \cdot f' = 1$. So by flatness of $E' \to l$, $f' \simeq \mathbb{P}^1$, i.e., $E' \to l$ is a $\mathbb{P}^1$-bundle.

Since C_P is irreducible, $\nu_{W,P}$ is isomorphic over the generic point of a ruling of $\nu_{W,P}(E_{W,P})$. This means that $\nu_{W,P}(E_{W,P})$ is normal, a contradiction. □

Proposition 4.10. *Let P be a plane and assume that there is at least one singular line on P. Moreover assume that*

(1) *for any singular line l on P, the assumptions as in Proposition* 4.4 (3) *hold for $\Delta = l$,*

(2) *W_l is factorial, and*

(3) *C_l is irreducible.*

Then $\operatorname{Sing} X$ *consists of one* $1/3\,(1,-1,1)$*-singularity and P is a good plane corresponding to this singularity.*

Proof. By irreducibility of C_l and Proposition 4.7, P is the unique plane on W and any irreducible component of $E_{W,l}$ is not $\nu_{W,l}$-exceptional. On the other hand, $\nu_{W,l}(E_{W,l})$ is a hyperplane section of W_l by Proposition 4.4 (1), which is irreducible and reduced by factoriality of W_l. Thus $E_{W,l}$ is irreducible and reduced, which implies that W_l has a cA_1-singularity generically along l. Hence by Proposition 3.8, P is a good plane corresponding to a $1/3\,(1,-1,1)$-singularity. □

5. $g(X) = 8$

The main results in this section are Theorems 5.2 and 5.5.

5.1. **Preliminaries.**

By [Gus83] and [Muk95b, Theorem 6.5],

$$W \simeq G(2,6) \cap \mathbb{P}^9.$$

From here on in this section, U_i and U_i^j mean i-dimensional vector subspaces of $\mathbb{C}^6$ and U_x means the 2-dimensional vector subspace of $\mathbb{C}^6$ corresponding to $x \in G(2,6)$.

By the cellular decomposition of $G(2,6)$, we know that there are the following two types of a plane P on $G(2,6)$:

(A) $P = \sigma_{4,2}(U_1, U_4)$ (we call such a plane a $\sigma_{4,2}$*-plane*).

(B) $P = \sigma_{3,3}(U_3)$ (we call such a plane a $\sigma_{3,3}$*-plane*).

Similarly a line is of the form $\sigma_{4,3}(U_1, U_3)$.

5.2. **Case W has a $\sigma_{3,3}$-plane.**
In this subsection, we assume there exists a $\sigma_{3,3}$-plane $P := \sigma_{3,3}(U_3)$ on W.

Lemma 5.1. *There is no line l in W such that $l \not\subset P$ and $l \cap P \neq \emptyset$.*

Proof. Assume that there is a line l in W such that $l \not\subset P$ and $l \cap P \neq \emptyset$. We can write $l = \sigma_{4,3}(U_1, U_3')$ for some U_1 and U_3'. By the assumption on l, we have $U_1 \subset U_3$ and $\dim U_3 \cap U_3' = 2$. Let $P' = \sigma_{4,2}(U_1, U_3 + U_3')$. Then $l \subset P'$ and $P \cap P'$ is a line. Thus P' is contained in $\langle P, l\rangle$ whence P' is contained in W, a contradiction to 2.14 (b). □

Theorem 5.2. *W has only isolated singularities. Hence P is a good plane and corresponds to a* $1/2\,(1,1,1)$*-singularity of X by Proposition* 3.7*. Moreover X is No.* 4.8 *in the tables of* [Taka02a]*. In particular P is the unique plane on W.*

Proof.
Step 1. We consider the projection $G(2,6) \dashrightarrow G(2,6)_{\sigma_2(U_3)}$.

This coincides with the rational map

$$G(2,6) \dashrightarrow G(2,3) \simeq \mathbb{P}^2$$

defined by

$$[\mathbb{C}^2] \mapsto [\mathrm{Im}\,(\mathbb{C}^2 \to \mathbb{C}^6/U_3)].$$

By this description, for $x, y \in G(2,6)$, x and y is in the same fiber of the projection if and only if $U_x + U_3 = U_y + U_3$. Hence a fiber of the projection is isomorphic to $G(2,5)$.

Step 2. Let Γ be a fiber of $G(2,6) \dashrightarrow G(2,6)_{\sigma_2(U_3)}$ and $\gamma = \Gamma|_W$. Then we see that γ is a subscheme supported on P, or the union of an irreducible conic and a subscheme supported on P.

Indeed, since W is not contained in $G(2,5)$ and $P \subset \Gamma$, we know that γ is 2-dimensional. Assume by contradiction that γ has a 2-dimensional component $\gamma' \neq P$. Then by $\deg G(2,5) = 5$, 2.14 (c) and Lemma 5.1, γ' is a plane disjoint from P. So $\gamma = G(2,5) \cap \mathbb{P}^5$ and hence γ is a linear complete intersection in $G(2,5)$. In particular γ is purely 2-dimensional and connected. This is impossible. Thus the 2-dimensional part of γ is supported on P.

Since γ is 2-dimensional and is a linear section of $G(2,5)$, there is a 2-dimensional linear complete intersection Γ' of $G(2,5)$ containing γ. Let $\Gamma' = dP + Q$ be the decomposition as a 2-cycle, where Q does not contain P as a component. Since $\deg \Gamma' = 5$, we have $\deg Q \leq 4$. Assume that $\deg Q = 4$. Let $\iota : P \to \Gamma'$ be the natural inclusion. Then $K_P = \mathscr{O}_{\mathbb{P}_2}(-3)$ and $\iota^* K_{\Gamma'} = \mathscr{O}_{\mathbb{P}^2}(-1)$, we have that $P \cap Q$ is a conic and hence $Q|_W$ is the union of $P \cap Q$ and a curve q such that $\deg q \leq 2$.

Thus by the connectedness of γ and Lemma 5.1, q is an irreducible conic intersecting P. Assume that $\deg Q \leq 3$. Then since $P \cap Q$ is a curve, we obtain the same conclusion as above.

Step 3. Let $S := \sigma_2(U_3)|_W$. We see that $S_{\mathrm{red}} = P$ and S is generically non-reduced.

Indeed, for $x \in \sigma_2(U_3) \cap W$, there exists $y \in \sigma_{3,3}(U_3)$ such that $\dim U_x \cap U_y = 1$. Then we have $\langle x, y \rangle \subset G(2,6)$ whence $\langle x, y \rangle \subset W$. So by Lemma 5.1, we have $\langle x, y \rangle \subset P$, i.e., $S_{\mathrm{red}} \subset P$. The inclusion in the other direction is clear.

Take ${U_4}^i$ $(i = 1, 2, 3)$ such that

$$
\begin{gathered}
{U_4}^1 \cap {U_4}^2 \cap {U_4}^3 = U_3 \\
{U_4}^1 + {U_4}^2 + {U_4}^3 = \mathbb{C}^6.
\end{gathered}
$$

Then we have

$$\sigma_2(U_3) = \sigma_1({U_4}^1) \cap \sigma_1({U_4}^2) \cap \sigma_1({U_4}^3).$$

Since $\sigma_1({U_4}^i)$ is singular along $G(2, {U_4}^i)$, S is generically non-reduced.

Step 4. Let $\pi : \overline{W}_S \to \widetilde{W}_S$ be the normalization. Let $\mu'_{W,S} := \mu_{W,S} \circ \pi$, $\nu'_{W,S} := \nu_{W,S} \circ \pi$ and $E'_{W,S} := \pi^* E_{W,S}$. Note that γ in Step 2 is a fiber of $W \dashrightarrow W_S$. So by Steps 1 and 2, we see that $W_S \simeq \mathbb{P}^2$ and a general fiber of $\nu'_{W,S}$ is the strict transform of an irreducible conic intersecting P. Moreover since $\nu'_{W,S}$ is generically a $\mathbb{P}^1$-bundle over a smooth surface, the degree of a 1-dimensional fiber with respect to $-K_{\overline{W}_S}$ is two.

Step 5. We show that neither a $\mu'_{W,S}$-exceptional divisor nor the strict transform P' of P on $\overline{W}_S$ is $\nu'_{W,S}$-exceptional. Moreover we prove that $\mu'_{W,S}$ is crepant.

Case 1. Assume that a $\mu'_{W,S}$-exceptional divisor E' is $\nu'_{W,S}$-exceptional, or P' is contracted to a curve on W_S.

Note that if the former case occurs, then $\nu'_{W,S}(E')$ is a curve since a $\mu'_{W,S}$-exceptional curve is not $\nu'_{W,S}$-exceptional. Let $\delta := \nu'_{W,S}(E')$ if the former case occurs (resp. $\delta := \nu'_{W,S}(P')$ if the latter case occurs). Then since

$${\nu'_{W,S}}^* \delta \sim \deg \delta ({\mu'_{W,S}}^* L_W - E'_{W,S}),$$

we know by considering ${\mu'_{W,S}}_* ({\nu'_{W,S}}^* \delta)$ that ${\nu'_{W,S}}^* \delta$ contains a component Δ which is neither P' nor a $\mu'_{W,S}$-exceptional divisor. By Step 2, $\nu'_{W,S}(\Delta)$ is not a point. Hence a general fiber l of $\nu'_{W,S}|_{{\nu'_{W,S}}^{-1}(\delta)}$ is reducible. Since ${\mu'_{W,S}}^* L_W \cdot l = 2$ and ${\mu'_{W,S}}^* L_W$ is $\nu'_{W,S}$-ample, an irreducible component of $\mu'_{W,S}(l)$ is a line intersecting P. This contradicts Lemma 5.1.

Since

$$-K_{\overline{W}_S} \cdot l' = {\mu'_{W,S}}^* L_W \cdot l' = 2$$

for a general $\nu'_{W,S}$-fiber l', any $\mu'_{W,S}$-exceptional divisor is crepant by the conclusion of Case 1. In particular, $\overline{W}_S$ has only canonical singularities and any $\mu'_{W,S}$-exceptional divisor is mapped to a line or a point on W.

Case 2. Assume that $\nu'_{W,S}(P')$ is a point.

Then by

$$({\mu'_{W,S}}^* L_W - E'_{W,S}) \cdot l = 0 \tag{5.1}$$

for a 1-dimensional $\nu'_{W,S}$-fiber l, there is a $\mu'_{W,S}$-exceptional divisor. Hence by Step 3, we can write $E'_{W,S} = dP' + F$, where $d \geq 2$ and F is a non-zero $\mu'_{W,S}$-exceptional divisor. Thus by (5.1) and the assumption of Case 2, we have $F \cdot l = 2$ for a 1-dimensional $\nu'_{W,S}$-fiber l. This implies that if F contains at least two components, then there is a conic intersecting P at two points, a contradiction to that W is the intersection of quadrics. Hence $F = 2F'$, where F' is irreducible. In particular, $\nu'_{W,S}|_{F'}$ is birational. Let ζ be a general line on W_S. We may assume that $\nu'_{W,S}|_{F'}$ is an isomorphism over ζ. Let ζ' be the strict transform of ζ on F'. Then

$$({\mu'_{W,S}}^* L_W - E'_{W,S}) \cdot \zeta' = 1. \tag{5.2}$$

Note that by [Kol96, II, Theorem 2.8], $\overline{W}_S$ is smooth outside P'.

Subcase 2.1. Assume that $\mu'_{W,S}(F')$ is a line.

Since a fiber m of $\mu'_{W,S}|_{F'}$ is mapped isomorphically onto $\nu'_{W,S}(m)$, we see that $\deg \nu'_{W,S}(m) = 1, 2$. Thus we have

$${\mu'_{W,S}}^* L_W \cdot \zeta' = 1, 2$$

respectively. Then by $-K_{F'} \cdot \zeta' = 3$, we have $F' \cdot \zeta' = -2, -1$. On the other hand, by (5.2), we have $F' \cdot \zeta' = 0, 1/2$, a contradiction.

Subcase 2.2. Assume that $\mu'_{W,S}(F')$ is a point.

Then by $-K_{F'} \cdot \zeta' = 3$, we have $F' \cdot \zeta' = -3$. Hence by (5.2), we have ${\mu'_{W,S}}^* L_W \cdot \zeta' = -5$, a contradiction.

Step 6. We complete the proof.

By Step 5 and (5.1), $\nu'_{W,P}$ is equi-dimensional and $E'_{W,S} = 2P'$. In particular this implies that $\mu'_{W,S}$ is small by Step 5. Since $E'_{W,S}$ is $\nu'_{W,S}$-ample and $P' \cdot l = 1$ for a $\nu'_{W,P}$-fiber l, $\overline{W}_S$ is a $\mathbb{P}^1$-bundle over W_S. Since $\overline{W}_S$ is smooth and $\mu'_{W,S}$ is small, W has no singular line whence X has only $1/2\,(1,1,1)$-singularities. It is easy to see that $\pi : \widetilde{W}_P \to S$

is $f' : Y' \to X'$ in [Taka02a]. Hence X is No. 4.8 and we finished the proof.

$\square$

5.3. **Case W has a $\sigma_{4,2}$-plane.**
Assume that W has a $\sigma_{4,2}$-plane $P := \sigma_{4,2}(U_1, U_4)$. Then by the results of the previous subsection, W has no $\sigma_{3,3}$-plane. We study the projection $W \dashrightarrow W_P$ in detail.

Proposition 5.3. (1) *The projection $W \dashrightarrow W_P$ is the restriction of the projection $G(2,6) \dashrightarrow G(2,6)_{\sigma_4(U_1)}$.*
(2) *$G(2,6)_{\sigma_4(U_1)} \simeq G(2,5)$.*

Proof. (1) Note that

$$\sigma_4(U_1) = \{[\mathbb{C}^2] | U_1 \subset \mathbb{C}^2\} \simeq \mathbb{P}^4.$$

So $\sigma_4(U_1) \cap \langle W \rangle$ is a linear subspace containing $P = \sigma_{4,2}(U_1, U_4)$. Hence $\sigma_4(U_1) \cap \langle W \rangle = P$. This proves the assertion.

(2) Let $x, y \in G(2,6) \setminus \sigma_4(U_1)$. The images of x and y by the projection coincide if and only if $\langle \sigma_4(U_1), x \rangle = \langle \sigma_4(U_1), y \rangle$. Moreover this is equivalent to that there exists $z \in \sigma_4(U_1)$ such that $y \in \langle x, z \rangle$. Assume that such a z exists. Then $\langle x, z \rangle \subset G(2,6)$ since $G(2,6)$ is an intersection of quadrics and $\langle x, z \rangle \cap G(2,6)$ contains at least three points x, y, z. Hence $\dim U_x \cap U_z = 1$. Let u_1 be a basis of $U_x \cap U_z$ and u_2 (resp. $u_2{}'$) a basis of U_1 (a vector $\in U_x$ such that $u_1, u_2{}'$ form a basis of U_x). Note that u_1, u_2 form a basis of U_z. Then $y \in \langle x, z \rangle$ means that for some $a, b \in \mathbb{C}$, $u_1, au_2 + bu_2{}'$ form a basis of U_y. Since $y \notin \sigma_4(U_1)$, $b \neq 0$. Thus $U_x \equiv U_y (\operatorname{mod} U_1)$. By reversing this argument, we can see that there exists $z \in \sigma_4(U_1)$ such that $y \in \langle x, z \rangle$ if and only if $U_x \equiv U_y (\operatorname{mod} U_1)$. Hence

$$G(2,6) \dashrightarrow G(2,6)_{\sigma_4(U_1)}$$

is nothing but the rational map defined by

$$[\mathbb{C}^2] \mapsto [\operatorname{Im}(\mathbb{C}^2 \to \mathbb{C}^6/U_1)]$$

whence $G(2,6)_{\sigma_4(U_1)} \simeq G(2,5)$.

$\square$

Since $\sigma_4(U_1) \cap \langle W \rangle = P$, we have $\langle W \rangle_P \simeq \mathbb{P}^6$.

Proposition 5.4. (1) *$W_P = G(2,5) \cap \langle W \rangle_P$.*
(2) *W_P is smooth, i.e., W_P is a smooth quintic del Pezzo 3-fold.*

Proof. (1) Let x be a point of $G(2,6) \setminus \sigma_4(U_1)$. By the proof of Proposition 5.3 (2), the fiber of

$$G(2,6) \dashrightarrow G(2,6)_{\sigma_4(U_1)}$$

containing x is

$$\{[\mathbb{C}^2] | \mathbb{C}^2 \subset U_1 \oplus U_x\} \simeq \mathbb{P}^2.$$

Note that a fiber of the projection $\mathbb{P}^{14} \dashrightarrow \mathbb{P}^9$ from $\sigma_4(U_1)$ (resp. $\langle W \rangle \dashrightarrow \langle W \rangle_P$) is a $\mathbb{P}^5$ (resp. $\mathbb{P}^3$). Hence in a fiber of $\mathbb{P}^{14} \dashrightarrow \mathbb{P}^9$, the fibers of $W \dashrightarrow W_P$ and $\langle W \rangle \dashrightarrow \langle W \rangle_P$ intersect. Hence we have $W_P = G(2,5) \cap \langle W \rangle_P$.

(2) This follows from (1) and Proposition 4.8.

□

Theorem 5.5. *Let C_P be as in Definition* 4.3. *Then* $\deg C_P = 7$ *and one of the following occurs:*

(1) *C_P is irreducible. In this case,* $\operatorname{Sing} X$ *consists of one* $1/2\,(1,1,1)$-*singularity. Moreover X is No.* 1.13 *in the tables of* [Taka02a].

(2) *$C_P = C'_P + m$, where m is a line. $p_a(C'_P) = 0$. In this case,* $\operatorname{Sing} X$ *consists of two* $1/2\,(1,1,1)$-*singularities. Moreover X is No.* 1.14 *in the tables of* [Taka02a].

Proof. Since W has only local complete intersection singularities, we can apply Propositions 4.4 and 4.6 by Lemma 4.1 and Proposition 5.4 (2). Then we have $C_P = C'_P$, or $\deg C'_P = 6$ and $p_a(C'_P) = 0$.

Case 1. C_P is irreducible.

Since W_P is smooth and $\deg W_P = 5$, we can apply Proposition 4.9 and then we obtain (1).

Case 2. $C_P = C'_P + m$, where m is a line.

Let

$$\nu'_{W,P} : \widetilde{W}'_P \to W_P$$

be the blow-up along C_P. Since C_P is the union of two smooth curves, it has only planar singularities. So $\widetilde{W}'_P$ has only Gorenstein terminal singularities. We can prove that $\nu_{W,P}(P')$ is not a point, where P' is the strict transform of P on $\widetilde{W}_P$. Indeed, if $\nu_{W,P}(P')$ were a point, then $\nu_{W,P}(E_{W,P})$ would be a cone and the embedded dimension of $\nu_{W,P}(E_{W,P})$ at the vertex is 5, a contradiction since W_P is smooth. So $\nu_{W,P}$ is equi-dimensional and hence $\widetilde{W}_P$ and $\tilde{W}'_P$ are isomorphic in codimension 1. On the other hand, $-K_{\widetilde{W}_P}$ and $-K_{\widetilde{W}'_P}$ are relatively ample over W_P. So $\widetilde{W}_P$ and $\widetilde{W}'_P$ are actually isomorphic by the negativity lemma.

By Proposition 5.4 (2), any two planes intersect mutually. Hence W has at most two planes since planes different from P are contracted to lines on W_P, and C_P contains only one line.

Subcase 2.1. W contains two planes.

In this case, any component of $E_{W,P}$ is not $\nu_{W,P}$-exceptional. Thus $E_{W,P}$ is irreducible since so is $\nu_{W,P}(E_{W,P})$. This implies that P is a good plane corresponding to a $1/2\,(1,1,1)$-singularity. So we obtain the assertion in this case.

Subcase 2.2. W **contains one plane.**

We deny this case. By Proposition 4.4 (3-3), m is the image of an irreducible component of $E_{W,P}$. Recall that P' is not contracted to a point by $\nu_{W,P}$. Hence $E_{W,P} = P' + E'$, where P' is the strict transform of P and E' is the other irreducible component, which dominates a singular line l on W. Similarly to the last part of the proof of Proposition 4.9, we can prove that E' is a $\mathbb{P}^1$-bundle over l.

Assume that $\nu_{W,P}(P') = m$. Then $\nu_{W,P}(E')$ is a non-normal del Pezzo surface since it is covered by lines. Since W_P is smooth, $\nu_{W,P}(E')$ is not a cone. So $E' \to \nu_{W,P}(E')$ is finite and birational. Moreover since E' is smooth, $E' \to \nu_{W,P}(E')$ is the normalization. Let D be the non-normal locus of $\nu_{W,P}(E')$ and D' the pull back of D on E'. By [Fuj85] or [Reid94], $D' \to D$ has degree 2. This means that a general fiber f of $P' \to m$ intersects E' at two points. Then, however, $\mu_{W,P}(f)$ is a line intersecting l at two points, a contradiction.

Assume that $\nu_{W,P}(E') = m$. Let δ be a general fiber of $E' \to m$. Since C_P is reduced, we have $E' \cdot \delta = -1$. So by Proposition 4.4 (1), we have $P' \cdot \delta = 2$ whence $\nu_{W,P}(P')$ is non-normal. On the other hand, we can prove that $\nu_{W,P}(P')$ is covered by irreducible conics as below. Thus by [Fuj85] or [Reid94], $\nu_{W,P}(P')$ must be normal. This is the final contradiction.

Since W_P is smooth and C_P has only planar singularities, $\widetilde{W}_P$ has only isolated singularities (recall that $\nu_{W,P}$ is the blow-up along C_P). Let $\gamma \subset \widetilde{W}_P$ be the strict transform of a general line on P. We may assume that $\gamma \subset \operatorname{Reg} \widetilde{W}_P$ and $P' \to P$ is isomorphic near γ (hence in particular $\gamma \subset \operatorname{Reg} P'$). By $-K_{P'} \cdot \gamma = 3$ and $-K_{\widetilde{W}_P} \cdot \gamma = 1$, we have $P' \cdot \gamma = -2$. Moreover by $E' \cdot \gamma = 1$, we have $E_{W,P} \cdot \gamma = -1$. So by ${\nu_{W,P}}^* L_{W_P} = {\mu_{W,P}}^* L_W - E_{W,P}$, we have ${\nu_{W,P}}^* L_{W_P} \cdot \gamma = 2$, i.e., $\nu_{W,P}(\gamma)$ is a conic. So $\nu_{W,P}(P')$ is covered by irreducible conics. $\square$

5.4. **Summary of the results in the case $g(X) = 8$.**
Assume that $g(X) = 8$. Then X has only $1/2\,(1,1,1)$-singularities and any plane on W is good. Hence X is No. 1.13, 1.14 or 4.8 . More precisely, X is No. 1.13 or 1.14 (resp. No. 4.8) if and only if any plane on W is a $\sigma_{4,2}$-plane (resp. a $\sigma_{3,3}$-plane).

6. $g(X) = 7$

The main result in this section is Theorem 6.10.

6.1. **Preliminaries.**
By [Muk95b, Theorem 6.5],

$$W \simeq \mathrm{OG}(5,10) \cap \mathbb{P}^8,$$

where $\mathrm{OG}(5,10)$ is the orthogonal Grassmannian embedded in $\mathbb{P}^{15}$ by the spinor embedding.

Recall the following:

Proposition 6.1. $\mathrm{OG}(5,10)$ *is defined in* $\mathbb{P}^{15}$ *by the following* 10 *quadratic forms:*

$$\begin{aligned}
N_1 &:= x_\emptyset x_{2345} - x_{23}x_{45} + x_{24}x_{35} - x_{25}x_{34}\\
N_{-1} &:= x_{12}x_{1345} - x_{13}x_{1245} + x_{14}x_{1235} - x_{15}x_{1234}\\
N_2 &:= x_\emptyset x_{1345} - x_{13}x_{45} + x_{14}x_{35} - x_{15}x_{34}\\
N_{-2} &:= x_{12}x_{2345} - x_{23}x_{1245} + x_{24}x_{1235} - x_{25}x_{1234}\\
N_3 &:= x_\emptyset x_{1245} - x_{12}x_{45} + x_{14}x_{25} - x_{15}x_{24}\\
N_{-3} &:= x_{13}x_{2345} - x_{23}x_{1345} + x_{34}x_{1235} - x_{35}x_{1234}\\
N_4 &:= x_\emptyset x_{1235} - x_{12}x_{35} + x_{13}x_{25} - x_{15}x_{23}\\
N_{-4} &:= x_{14}x_{2345} - x_{24}x_{1345} + x_{34}x_{1245} - x_{45}x_{1234}\\
N_5 &:= x_\emptyset x_{1234} - x_{12}x_{34} + x_{13}x_{24} - x_{14}x_{23}\\
N_{-5} &:= x_{15}x_{2345} - x_{25}x_{1345} + x_{35}x_{1245} - x_{45}x_{1235}
\end{aligned}$$

Proof. See [Muk95a, Proposition 1.9] □

Proposition 6.2. *Let* x *be any point of* $\mathrm{OG}(5,10)$*. Then*

$$T_x\mathrm{OG}(5,10) \cap \mathrm{OG}(5,10)$$

is the cone over $G(2,5)$ *with the vertex* x*, where* $T_x\mathrm{OG}(5,10)$ *is the projectivized tangent cone of* $\mathrm{OG}(5,10)$ *at* x*.*

Proof. Since $\mathrm{OG}(5,10)$ is homogeneous, we may assume that x is the $x_\emptyset$-point regarding the coordinate as in Proposition 6.1. Then

$$T_x\mathrm{OG}(5,10) = \{x_{1234} = x_{1235} = x_{1245} = x_{1345} = x_{2345} = 0\}.$$

Hence $T_x\mathrm{OG}(5,10)\cap\mathrm{OG}(5,10)$ is defined by 5×5 Pfaffians with entries x_{ij} with $1 \le i < j \le 5$. So we have the assertion. □

Proposition 6.3. *Let* $(x \in l \subset P)$ *be a triplet consisting of a point, a line and a plane in* $\mathrm{OG}(5,10)$*. Then up to the projective equivalence in* $\mathbb{P}^{15}$*, this is unique.*

Proof. By Proposition 6.2, l (resp. P) corresponds to a point l' (resp. a line P') in $G(2,5)$. So it suffices to prove the uniqueness of a pair $(l' \in P')$. This can be easily verified by noting that a line in $G(2,5)$ is of the form $\sigma_{3,2}(U_1, U_3)$, where U_i is an i-dimensional vector sub-space of $\mathbb{C}^5$. □

Proposition 6.4. *Assume that W is a 3-dimensional linear complete intersection of a Grassmannian G and has only canonical singularities. Then there is a 5-dimensional (resp. 4-dimensional) linear complete intersection A (resp. B) of G such that A is smooth (resp. B has only isolated singularities and is smooth outside singular curves of W and a finite number of points $w \in W$ where emb-dim$_w W = 5$).*

Proof. Let H be a d-dimensional linear complete intersection of G containing W. Then

$$\begin{aligned}\dim |L_H \otimes \mathscr{I}_{W/H}| &= \dim\langle H\rangle - (\dim\langle W\rangle + 1)\\ &= \dim H - \dim W - 1 = d - 4.\end{aligned}$$

On the other hand, we can prove that members $\in |L_H \otimes \mathscr{I}_{W/H}|$ singular at a point of $\operatorname{Sing} W$ form at most one dimensional family. Indeed, since W is a complete intersection in G and has only canonical singularities, the embedded dimension of W at any singular point (resp. any singular point except a finite number of points) ≤ 5 (resp. $= 4$) by [Lau77, Theorem 3.13]. So by a simple dimension count, we can prove the assertion.

Hence by induction, we can take a desired A. As for B, just note that we can take a 4-dimensional linear complete intersection which is smooth at the generic points on singular curves by the estimates of the embedded dimensions of W at singular points. □

6.2. **Projection of W from a plane.**

Proposition 6.5. *Let P be a plane on W. Then*

(1) *W_P is a $(2,2)$-complete intersection in $\mathbb{P}^5$.*
(2) *C_P is irreducible.*
(3) *P is unique and is a good plane.*

Proof. As in Proposition 6.3, we may assume that P is the $(x_\emptyset, x_{12}, x_{13})$-plane. Project $\mathrm{OG}(5,10)$ from $H :=$ the $(x_\emptyset, x_{12}, x_{13}, x_{23})$-plane. Then $\mathrm{OG}(5,10)_H$ is $\{N_{-4} = N_{-5} = 0\}$ in $\mathbb{P}^{11}$. Since W is a linear section of $\mathrm{OG}(5,10)$, we have that $P = H \cap W$. Hence W_P is a $(2,2)$-complete intersection in $\mathbb{P}^5$.

Since W has only local complete intersection singularities, we can apply Propositions 4.4 and 4.6 by (1) and Lemma 4.1. Then we have (2). Hence by Proposition 4.9, we have (3). □

From here on, we denote the unique plane on W by P.

6.3. **Projection of W from a singular line.**

Proposition 6.6. *Assume that W_P is factorial. Then* $\operatorname{Sing} X$ *consists of one* $1/2\,(1,1,1)$*-singularity.*

Proof. This directly follows from Propositions 4.9 and 6.5. □

From now on, we may assume that W_P is not factorial. Then by Proposition 4.8 (2-2), $\nu_{W,P}(E_{W,P})$ is a reducible hyperplane section on W_P. So there is a unique singular line l on P. We study the projection $W \dashrightarrow W_l$.

Proposition 6.7. (1) *W_l is a smooth quintic del Pezzo 3-fold.*
(2) *$p_a(C_P) = 1$.*

Proof. (1) We use the notation as in Proposition 4.8 (2-2). In the proof of Proposition 4.9, we saw that E_1 is the strict transform of P. Let E_2 be as in that proof. Clearly E_2 is the image of the $\mu_{W,P}$-exceptional divisor contracted to l. Let ϱ be the image of P on W_l. Then $W \dashrightarrow W_P$ is the composite of $W \dashrightarrow W_l$ and $W_l \dashrightarrow (W_l)_\varrho$. Hence E_1 is the image of the strict transform of $E_{W_l,\varrho}$. The linear system Γ consisting of images of members $\in |L_{W_l}|$ is

$$|L_{W_P} + E_1| = |2L_{W_P} - E_2|.$$

Recall that we can choose the factorization $\overline{W}_P$ of W_P such that the strict transform of E_1 can be contracted to a smooth point. Let $\overline{W}_P \to \widehat{W}_P$ be the contraction morphism. Then $\widehat{W}_P$ is a smooth quintic del Pezzo 3-folds and this is embedded in $\mathbb{P}^6$ by the strict transforms of members of Γ. So $W_l = \widehat{W}_P$.

(2) If $E_2 \simeq \overline{\mathbb{F}}_3$, then clearly $p_a(C_P) = 1$. Assume that $E_2 \simeq \mathbb{F}_1$. Since a $\mu_{W,P}$-fiber is mapped to a line, it is a ruling of E_2. Since $\nu_{W,P}$ is the blow-up along C_P at the generic point of C_P, a ruling of E_2 intersects C_P at two points. Hence $p_a(C_P) = 1$. □

Lemma 6.8. *There is a smooth* 4*-dimensional linear complete intersection B of* $\mathrm{OG}(5,10)$ *containing W.*

Proof. Since P is a good plane and P is unique, W has only cDV singularities. Hence by Proposition 6.4, there is a 5-dimensional (resp. 4-dimensional) linear complete intersection A (resp. B) of $\mathrm{OG}(5,10)$ such that A is smooth (resp. B has only isolated singularities and is smooth outside l). Assume that $E_{B,l}$ is reducible. Then $E_{B,l}$ contains a fiber F of $\mathbb{P}^3$-bundle $E_{A,l} \to l$. Since $E_{B,l}$ is a Cartier divisor on $\widetilde{B}_l$, so is $E_{B,l} \cap \widetilde{W}_l$ on $\widetilde{W}_l$. Hence $F \cap \widetilde{W}_l \neq \emptyset$ is a divisor on $\widetilde{W}_l$. Then, however, $\nu_{W,l}(F \cap \widetilde{W_l})$ is a surface in $\mathbb{P}^3$, a contradiction to the fact that $\operatorname{Pic} W_l = \mathbb{Z}[L_{W_l}]$. Thus $E_{B,l}$ is irreducible and hence $E_{B,l}$ is a $\mathbb{P}^2$-bundle. This implies that $\widetilde{B}_l$ is smooth along $E_{B,l}$ and so is B along l, i.e., B is smooth. $\square$

Proposition 6.9. *Let C_l and C'_l be as in Definition* 4.3. *Then* $\deg C_l = 8$. *Moreover one of the following holds:*

(1) *C_l is irreducible.*

(2) *$C_l = C'_l + m$, where m is a line.* $\deg(C'_l) = 7$ *and* $p_a(C'_l) = 1$.

Moreover ϱ is the unique singularity of C'_l and $\operatorname{mult}_\varrho C'_l = \deg C'_l - 5$, *where ϱ is the image of P on W_l.*

Proof. Since W has only hypersurface singularities by Lemma 6.8, we can apply Propositions 4.4 and 4.6 by Lemma 4.1 and Proposition 6.7 (1). Let $d := \deg C'_l$ and $p := p_a(C'_l)$. If $p = d - 5$ and C_l is reducible, then $d \geq 9$ by Proposition 4.6, a contradiction. Hence we have the following possibilities:

(i) $d = 6$ and $p = 0$.
(ii) $d = 7$ and $p = 0, 1$.
(iii) $d = 8$ and $0 \leq p \leq 3$.

If $d = 6$ and $p = 0$, then by the Riemann-Roch theorem,

$$h^0(\mathscr{O}_{C'_l}(2L_{W_l})) = 2d + 1 - p = 13.$$

Hence by the proof of Proposition 4.6, we obtain

$$h^0(-K_X) \geq h^0(-K_{W_l} \otimes \mathscr{I}_{C'_l/W_l}) \geq h^0(-K_{W_l}) - 13 = 10,$$

a contradiction. Since C_P is the transform of C'_l and $\deg C_P = 5$, the case that $d = 7$ and $p = 0$, or $d = 8$ and $p = 0, 1$ is excluded. Moreover we see that ϱ is a singular point of C'_l and $\operatorname{mult}_\varrho C'_l = d - 5$. Since C_P is smooth, ϱ is the unique singularity of C'_l. So we have the assertion. $\square$

By the following theorem, we know the possibilities of the generic type of singularity of W along l and hence those of $\operatorname{Sing} X$.

Theorem 6.10. *Let B be as in Proposition* 6.4.

(1) *Let γ be a line on B. Then*

$$\mathscr{N}_{\gamma/B} \simeq \mathscr{O}_{\mathbb{P}^1}(-2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}, \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1), \text{ or } \mathscr{O}_{\mathbb{P}^1}^{\oplus 3}.$$

(2) *Let l be a singular line on W. Then any irreducible component E of $E_{W,l}$ dominates l and a general fiber of $\mu_{W,l}|_E$ is irreducible. Moreover one of the following holds:*

(a) *$E_{W,l}$ is irreducible. In this case, W has a cA_1-singularity generically along l. Hence* $\operatorname{Sing} X$ *consists of one* $1/3\,(1,-1,1)$-*singularity.*

(b) *$E_{W,l}$ is reducible and the following hold:*
   (b-1) *$E_{W,l}$ has an irreducible component E' which is $\nu_{W,l}$-exceptional (clearly $\mu_{W,l}(E') = l$).*
   (b-2) *$\mathscr{N}_{l/B} \simeq \mathscr{O}_{\mathbb{P}^1}(-2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}$.*
   (b-3) *$E_{W,l} = E' + E''$, where E'' is irreducible and $E' \neq E''$.*
   (b-4) *$\nu_{W,l}(E'')$ is a non-normal del Pezzo surface and $\nu_{W,l}(E')$ coincides with $\operatorname{Sing} \nu_{W,l}(E'')$.*
   (b-5) *$E'' \to \nu_{W,l}(E'')$ is finite. Moreover W has a cA_2-singularity generically along l. Hence $\operatorname{Sing} X$ consists of one $1/4\,(1,-1,1)$-singularity.*

Proof. (1) Note that there is a 2-dimensional complete intersection S of $\mathrm{OG}(5, 10)$ containing γ and contained in B. Since $\mathscr{N}_{\gamma/S} \simeq \mathscr{O}_{\mathbb{P}^1}(-2)$, the assertion follows from standard computations by using the normal bundle sequences.

(2) If $E_{W,l}$ has an irreducible component D mapped to a point on l, then $D \simeq \mathbb{P}^2$ since B is smooth. Then, however, $\nu_{W,l}(D)$ is a plane on W_l, a contradiction to Proposition 6.7. Hence $E_{W,l}$ has at most two components since W has a cDV singularity generically along l. So if $E_{W,l}$ is non-reduced, then $E_{W,l} = 2E'$ for some E'. Thus $\nu_{W,l}(E_{W,l})$ is also non-reduced since $\nu_{W,l}(E_{W,l}) \neq 0$ by Proposition 4.4 (1). This contradicts Proposition 6.7. So $E_{W,l}$ is reduced and this implies that W has a cA-singularity generically along l.

Let E be an irreducible component of $E_{W,l}$. Recall that $\operatorname{exc} h$ is a simple normal crossing divisor by Claim 3.2 and hence the restriction of h to the strict transform of E on Z has an irreducible general fiber. Since $\mu_{W,l}$ factors through Z or its birational transform by a composite of flops, this is also the case for E.

Assume that $E_{W,l}$ is irreducible. Then the assertion follows from irreducibility of a general fiber of $\mu_{W,l}|_{E_{W,l}}$.

Assume that $E_{W,l}$ is reducible. Then by Proposition 4.4 (1) and Proposition 6.7, one of irreducible components of $E_{W,l}$ (we denote this by E') is $\nu_{W,l}$-exceptional.

Let $H := -E_{B,l}|_{E_{B,l}}$ and F is a fiber of $E_{B,l} \to l$. Then we have

$$(\mu_{B,l}^* L_B - E_{B,l})|_{E_{B,l}} = H + F.$$

By the assumption, the exceptional locus of the morphism defined by $|H + F|$ contains a two dimensional subset. So

$$\mathscr{N}_{l/B} \simeq \mathscr{O}_{\mathbb{P}^1}(-2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}$$

and E' is the unique member of $|H - 2F|$. Since

$$\widetilde{W_l} \in |{\mu_{W,l}}^* L_B - 2E_{B,l}|,$$

we have $E_{W,l} \in |2H+F|$. Hence $E_{W,l} = E' + E''$, where E'' is an irreducible member of $|H+3F|$. Let C_0 be the minimal section of E'' and r a ruling. Since

$$(H+F)^2(H+3F) = 5$$

and $|H+F|$ is free, we have

(i) $E'' \simeq \mathbb{F}_1$ and $(H+F)|_{E''} \sim C_0 + 3r$,
(ii) $E'' \simeq \mathbb{F}_3$ and $(H+F)|_{E''} \sim C_0 + 4r$, or
(iii) $E'' \simeq \mathbb{F}_5$ and $(H+F)|_{E''} \sim C_0 + 5r$.

But the last case does not occur since otherwise $\nu_{W,P}(E'')$ is a cone over a quintic curve, which cannot lie on a smooth 3-fold. In particular, $(H+F)|_{E''}$ is ample, and so $E'' \to \nu_{W,l}(E'')$ is finite. Let $\delta := E' \cap E''$, which is a generically section of $E_{B,l} \to l$. By

$$(\delta^2)_{E''} = (H-2F)^2(H+3F) = -1,$$

we have $\delta = C_0$ if $E'' \simeq \mathbb{F}_1$ (resp. $\delta = C_0 + r'$ for some ruling r' if $E'' \simeq \mathbb{F}_3$). This implies that the image of E' is $\operatorname{Sing} \nu_{W,l}(E_{W,l})$.

Finally we prove that W has a cA_2-singularity generically along l. By irreducibility of general fibers of $\mu_{W,l}|_{E'}$ and $\mu_{W,l}|_{E''}$, we have only to prove that there is no singular curve of $\widetilde{W_l}$ dominating l. This condition is equivalent to that the irreducible component l' of δ dominating l is not a singular curve of $\widetilde{W_l}$. Since $E'' \to \nu_{W,l}(E'')$ is finite, $\nu_{W,l}(l')$ is a curve. On the other hand, by Proposition 6.9, C_l is reduced so l' cannot be a singular curve of $\widetilde{W_l}$.

□

6.4. **Summary of the results in the case $g(X) = 7$.**
W has a unique plane P and P is a good plane. In particular $\operatorname{Sing} X$ consists of one $1/r\,(1,-1,1)$-singularity for some r.

(1) Assume that W_P is factorial. Then $r = 2$.
(2) Assume that W_P is not factorial. Then W has a singular line l and one of the following holds:
  (2-1) C_l is irreducible. In this case, W has a cA_1-singularity generically along l. Hence $r = 3$.
  (2-2) C_l is reducible and $\deg C'_l = 7$. In this case, W has a cA_2-singularity generically along l. Hence $r = 4$.

7. $g(X) = 6$

In this section, we prove Theorem 1.5 in case $g(X) = 6$. See the subsection 7.6 for more precise statements.

In the subsection 7.1, we prove that W is a quadric section of V, where V is a smooth quintic del Pezzo 4-fold or the cone over a smooth quintic del Pezzo 3-fold (Propositions 7.1). Unlike the other cases, W is not a linear section of the key variety and there are two possibilities of the key variety. These fact make the situation more complicated than the other cases.

In the subsection 7.2, we study the projection of both W and V from a singular line l on W (if W has no singular line, then we see that X has only $1/2\,(1,1,1)$-singularities. So we may assume that W has at least one singular line). In the subsections 7.1 and 7.2, we see that there are essentially three cases (we denote by P the plane on W containing l):

(a) V is smooth and W_P is a cubic in $\mathbb{P}^4$.
(b) V is smooth and $W_P \simeq Q^3$.
(c) V is the cone.

We would like to mention here that it is interesting that the difference of two possibilities of V can be explained by the property of W_l (see Proposition 7.6). In Proposition 7.7, we study the relation between the singularity of W along l and the geometry of W_l if V is smooth. There is a similar statement in case $g(X) = 7$ (Theorem 6.10) but the existence of the singularity of W_l produces a new phenomena (see (b-5-2) in this proposition). We study this situation in Proposition 7.11 if l is contained in the plane $P \subset W$ such that $W_P \simeq Q^3$.

In the subsection 7.3, we prove that the case (c) does not occur. The proof is very long. First of all we determine the configuration of singular lines and the possibilities of $\operatorname{Sing} W$ along them. Then we deny them by using the techniques in the section 3.

In the subsection 7.4, we prove the main result for the case (a). This case can be treated like the other cases for $g(X) = 7, 8$ since both W_P and W_l are del Pezzo 3-folds and we can use the technique in the section 4 for them.

Lastly we treat the case (b) in the subsection 7.5 after long but elementary calculations in Proposition 7.11. In this case, the study of $W \dashrightarrow W_P$ is harder than the case that W_P is a del Pezzo 3-fold. One reason for this is the following: it is easy to see that even if we define $F_{W,P}$ and C_P for this case as in Definition 4.3, $\mu_{W,P}(F_{W,P})$ is not a hyperplane section of W. Hence we cannot restrict the possibilities of C_P as in the proof of Proposition 4.4 (3).

7.1. **Preliminaries.**

Proposition 7.1. *W is a quadric section of one of the following:*

(1) *a smooth quintic del Pezzo 4-fold V.*

(2) *a cone V over a smooth quintic del Pezzo 3-fold V'. In this case, W contains the vertex v of V and any two planes intersect only at v. Moreover $W_v \simeq V'$.*

Proof. By [Gus82] and [Muk95b, Theorem 6.5], W is a quadric section of a quintic del Pezzo 4-fold V. By [Fuj85, (2.9) Theorem], V is one of the following:

(i) V is smooth.
(ii) V is a cone over a quintic del Pezzo n-fold V' with the $(3-n)$-dimensional vertex, where $n = 1, 2$.
(iii) V is a cone over a singular quintic del Pezzo 3-fold V' which is not a cone.
(iv) V is singular but is not a cone.
(v) V is a cone over a smooth quintic del Pezzo 3-fold V'.

Assume that V satisfies (ii) or (iii). Then $V_v \simeq V'$. Since W is not a cone, $W \dashrightarrow W_v$ is generically finite. Hence $W_v \simeq V'$ and by Proposition 4.8, V' is smooth. If V satisfies (iv), then we can prove that $|L_V|$ is decomposable similarly to Step 5 in the proof of Proposition 4.8. This implies that $|L_W|$ is also decomposable, a contradiction.

Assume that V satisfies (v). If a plane P on W does not pass the vertex v of V, then V contains a 3-plane $\langle P, v\rangle$ whence a general hyperplane section V' of V which does not pass v contains a plane, a contradiction (V' is a smooth quintic del Pezzo 3-fold). Hence any plane on W passes v so in particular, $v \in W$. Moreover by 2.14 (b), any two planes intersect only at v.

□

7.2. **Projection of W from a singular line.**

Let l be a singular line and V as in Proposition 7.1. In this subsection, we study the projections $V \dashrightarrow V_l$ and $W \dashrightarrow W_l$.

Proposition 7.2. (1) *Assume that V is smooth, or V is the cone and $v \notin l$. Then V_l is a 4-dimensional quadric.*

(2) *Assume that V is smooth. Then $\nu_{V,l}$ is a birational divisorial contraction which contracts an irreducible divisor $F_{V,l}$ to a cubic surface S spanning a 4-plane and hence $S \simeq \mathbb{F}_1$ or $\overline{\mathbb{F}}_3$. Any fiber is a copy of $\mathbb{P}^1$ or $\mathbb{P}^2$. Moreover the following hold:*

(2-1) *Let ϱ be a smooth point of S which is the image of a plane containing l. Then V_l has an ODP at ϱ, and locally analytically,*

$$(\varrho \in S \subset V_l) \simeq (o \in \{x = z = t = 0\} \subset \{xy + zw + t^2 = 0\}),$$

where the latter objects are located in $\mathbb{C}^5$ *whose coordinates are* $x, \dots, t$.

(2-2) *If* $S \simeq \overline{\mathbb{F}}_3$, *then the vertex is the image of a plane containing* l *and is a smooth point of* V_l.

(2-3) $\nu_{V,l}$ *is the blow-up along* S *outside* $\operatorname{Sing} S$.

Proof. (1) By $-K_V = 3L_V$, we have $\deg \mathscr{N}_{l/V} = 1$ and hence ${E_{V,l}}^4 = 1$. Thus $({\mu_{V,l}}^* L_V - E_{V,l})^4 = 2$ whence V_l is a 4-dimensional quadric.

(2) By
$$ {\nu_{V,l}}^* L_{V_l} = {\mu_{V,l}}^* L_V - E_{V,l} $$
and
$$ -K_{\widetilde{V}_l} = {\mu_{V,l}}^*(-K_V) - 2E_{V,l}, $$
we have
$$ dF_{V,l} = {\nu_{V,l}}^*(-K_{V_l}) - (-K_{\widetilde{V}_l}) = {\mu_{V,l}}^* L_V - 2E_{V,l}, $$
where d is the codimension of $S := \nu_{V,l}(F_{V,l})$ in V_l. Thus $d = 1$, and by ${E_{V,l}}^4 = 1$, we have
$$ \deg S = -({\nu_{V,l}}^* L_{V_l})^2 {F_{V,l}}^2 = 3. $$
Moreover since $E_{V,l} = {\nu_{V,l}}^* L_{V_l} - F_{V,l}$ and $E_{V,l}$ does not move, there is only one hyperplane in $\mathbb{P}^5$ containing S, i.e., $\langle S \rangle$ is $\mathbb{P}^4$. Thus $S \simeq \mathbb{F}_1$ or $\overline{\mathbb{F}}_3$. Recall that a fiber of $\nu_{V,l}$ is the strict transform of the intersection of V and a plane containing l. So a fiber is the strict transform of a line or a plane since V is an intersection of quadrics. Hence by [AW98, Theorem], we see that (2-2) holds and V_l has an ODP at ϱ in (2-1).

We show the local analytic description in (2-1). We start with
$$ (\varrho \in S \subset V_l) \simeq (o \in \{x = z = t = 0\} \subset \{xa_1 + za_2 + ta_3 = 0\}), $$
where the latter objects are located in $\mathbb{C}^5$ whose coordinates are $x, \dots, t$ and
$$ f := xa_1 + za_2 + ta_3 $$
is the equation of the ODP. Let l_i be the linear part of a_i. If one of $l_i \equiv 0$, then the rank of the quadric part of f is not greater than 4, a contradiction. By the rank condition of the quadric term of f, one of l_i contains y. We may assume that $a_1 = y$. Similarly we may assume that $a_2 = w$. Let
$$ a_3 = \alpha x + \beta y + \gamma z + \delta w + \epsilon t. $$
By replacing y with $y + \alpha t$ and w with $w + \gamma t$, we may assume that $\alpha = \gamma = 0$. Moreover by replacing x with $x + \beta t$ and z with $z + \delta t$, we may assume that $\beta = \delta = 0$. Hence by replacing t with $\sqrt{\epsilon} t$, we have the desired expression.

Lastly we prove (2-3). Let $U_l := V_l \setminus \operatorname{Sing} S$. Let $\nu' : U'_l \to U_l$ be the blow-up along $S \setminus \operatorname{Sing} S$. Then $\operatorname{exc}\nu'$ is an irreducible divisor and hence ${\nu_{W,l}}^{-1}(U_l)$ and U'_l are isomorphic in codimension 1. Moreover U'_l is smooth and $-K_{U'_l}$ is ν'-ample. Recall that $-K_{{\nu_{W,l}}^{-1}(U_l)}$ is $\nu_{V,l}$-ample. Hence ${\nu_{W,l}}^{-1}(U_l)$ and U'_l are isomorphic over U_l.

□

7.3. *Let P be the plane on W containing l and ϱ the image of P by $W \dashrightarrow W_l$. Let $W_l \dashrightarrow (W_l)_\varrho$ (resp. $V_l \dashrightarrow (V_l)_\varrho$) be the projection of W_l (resp. V_l) from ϱ. Note that the composite*

$$W \dashrightarrow W_l \dashrightarrow (W_l)_\varrho$$

$$(\textit{resp. } V \dashrightarrow V_l \dashrightarrow (V_l)_\varrho)$$

is the projection of W (resp. V) from P.

Proposition 7.4. *W_l is a $(2,2)$-complete intersection in $\mathbb{P}^5$. If $\varrho \in \operatorname{Sing} W_l$ (resp. $\notin \operatorname{Sing} W_l$), then $W_P \simeq Q^3$ (resp. W_P is a cubic in $\mathbb{P}^4$).*

Assume that W_l is factorial and $W_P \simeq Q^3$. Then $\nu_{W_l,\varrho}(E_{W_l,\varrho})$ is a quadric and $\nu_{W_l,\varrho}$ is the blow-up along an irreducible curve C_ϱ which is a $(2,2)$-complete intersection in $\mathbb{P}^3$.

If C_ϱ passes through the singularity of $\nu_{W_l,\varrho}(E_{W_l,\varrho})$, then W_l has one singularity, which is analytically isomorphic to

$$\{x^2 + y^2 + z^2 + w^k = 0\} \text{ with } k = 4, 5.$$

If C_ϱ does not pass through the singularity of $\nu_{W_l,\varrho}(E_{W_l,\varrho})$, then W_l has two singularities, which are analytically isomorphic to

$$\{x^2 + y^2 + z^2 + w^k = 0\} \text{ with } k = 2, 3.$$

Proof. Assume that V is smooth, or V is the cone and $v \notin l$. Then since

$$\widetilde{W_l} \in |{\mu_{V,l}}^*(2L_V) - 2E_{V,l}|,$$

we have $W_l \in |2L_{V_l}|$. Hence W_l is a $(2,2)$-complete intersection in $\mathbb{P}^5$ by Proposition 7.2. Assume that V is the cone and $v \in l$. Then since W_l is a projection of W_v from a (smooth) point, W_l is a $(2,2)$-complete intersection in $\mathbb{P}^5$.

Since

$$({\mu_{W_l,\varrho}}^* L_{W_l} - E_{W_l,\varrho})^3 = 4 - \operatorname{mult}_\varrho W_l$$

and W_l has only terminal singularities by Proposition 4.8, the second assertion follows.

The description of $\nu_{W_l,\varrho}$ is easy to verify so we only prove irreducibility of C_ϱ. It is easy to see that the image of the sum of $\nu_{W_l,\varrho}$-exceptional divisors on W_l is a hyperplane section, and any $\nu_{W_l,\varrho}$-exceptional divisor is not $\mu_{W_l,\varrho}$-exceptional. Hence by factoriality of W_l, the sum of $\nu_{W_l,\varrho}$-exceptional divisors is irreducible and so is C_ϱ. □

Proposition 7.5. *Let C_l and C'_l be as in Definition 4.3. Then $\deg C_l = 6$. Moreover one of the following holds:*

(1) *C_l is irreducible.*
(2) *$C_l = C'_l + m$, where m is a line. $\deg C'_l = 5$ and $p_a(C'_l) = 0$.*

Proof. We can apply Propositions 4.4 and 4.6 by Lemma 4.1 and Proposition 7.4. Hence we can easily obtain the assertion. □

Proposition 7.6. *W_l is not factorial if and only if V is the cone and $v \in l$. If this is the case, then the strict transform of $E_{W,v}$ on W_l is a cubic surface.*

Proof. Assume that V is the cone and $v \in l$. Since $W_v \dashrightarrow W_l$ is the projection from a singular point of $\nu_{W,v}(E_{W,v})$ and the multiplicity of any singular point of $\nu_{W,v}(E_{W,v})$ is 2, the image of $\nu_{W,v}(E_{W,v})$ on W_l is a cubic surface. This implies that W_l is not factorial.

Conversely assume that W_l is not factorial. Then W_l contains a unique plane E_1 by Proposition 4.8 (2-2). By Proposition 4.7, no plane on W is mapped to a divisor on W_l. Consider the total transform on W of a hyperplane section containing E_1. Then we see that E_1 is the image of a $\mu_{W,l}$-exceptional divisor by 2.14 (c). By Proposition 4.4 (1), $\nu_{W,l}(E_{W,l})$ is a hyperplane section hence it is the sum of E_1 and a cubic surface E_2 such that $\langle E_2 \rangle$ is a 4-plane.

We prove that E_2 is contracted to a point by $W_l \dashrightarrow W$. If C_l is irreducible, then clearly $C_l \subset E_2$. Assume that $C_l = C'_l + m$, where m is a line. Then $C'_l \subset E_2$. Since $\overline{\mathbb{F}}_3$ contains no $\mathbb{P}^1$ of degree 5, $E_2 \simeq \mathbb{F}_1$. Since

$$C_l = \mathrm{Bs}\,|-K_{W_l} \otimes \mathscr{I}_{C_l/W_l}|$$

and the minimal section C_0 of $\mathbb{F}_1$ is contained in $\mathrm{Bs}\,|-K_{W_l} \otimes \mathscr{I}_{C_l/W_l}|$, we have that $m = C_0$. So in any case, $C_l \subset E_2$ and $C_l \in |2L_{E_2}|$. This implies that E_2 is contracted to a point by $W_l \dashrightarrow W$ since for a general curve γ in E_2, $-K_{\widetilde{W}_l} \cdot \gamma = 0$. Assume that V is smooth, or V is the cone and $v \notin l$. Then the embedded dimension of W at any point $\in l$ is at most 4. So a $\mu_{W,l}$-exceptional divisor contracted to a point on l is a copy of $\mathbb{P}^2$, a contradiction. Thus V is the cone and $v \in l$. □

Proposition 7.7. *Assume that V is smooth.*

(1) *Let γ be a line on V. Then*

$$\mathscr{N}_{\gamma/V} \simeq \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \text{ or } \mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1).$$

(2) *Let l be a singular line. Then any irreducible component E of $E_{W,l}$ dominates l and a general fiber of $\mu_{W,l}|_E$ is irreducible. Moreover one of the following holds:*
  - (a) *$E_{W,l}$ is irreducible. In this case, W has a cA_1-singularity generically along l.*
  - (b) *$E_{W,l}$ is reducible. In this case, W has a cA_i-singularity $(i \geq 2)$ generically along l. Moreover the following hold:*
    - $(b$-1$)$ *$E_{W,l}$ has an irreducible component E' which is $\nu_{W,l}$-exceptional (clearly $\mu_{W,l}(E') = l$).*
    - $(b$-2$)$ *$\mathscr{N}_{l/V} \simeq \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}$.*
    - $(b$-3$)$ *$E_{W,l} = E' + E''$, where E'' is irreducible and $E' \neq E''$.*
    - $(b$-4$)$ *$\nu_{W,l}(E'')$ is a non-normal del Pezzo surface and $\nu_{W,l}(E')$ coincides with $\operatorname{Sing} \nu_{W,l}(E'')$.*
    - $(b$-5$)$ *One of the following holds:*
      - $(b$-5-1$)$ *$\nu_{W,l}(E'')$ is not a cone. In this case, $E'' \to \nu_{W,l}(E'')$ is finite. Moreover W has a cA_2-singularity generically along l. In particular, l is a good line.*
      - $(b$-5-2$)$ *$\nu_{W,l}(E'')$ is a cone. In this case, the vertex of the cone is a singular point of W_l.*
    - $(b$-6$)$ *The plane containing l is the unique plane on W.*

Proof. (1) This can be shown similarly to Theorem 6.10 (1).

(2) Since V is smooth, W_l is factorial by Proposition 7.6. Hence by the same way as the first part of the proof of Theorem 6.10 (2), we can see that $E_{W,l}$ is irreducible, or $E_{W,l} = E' + E''$, where $E' \neq E''$, E' and E'' are irreducible, and

$$\mu_{W,l}(E') = \mu_{W,l}(E'') = l.$$

So in particular, W has a cA-singularity generically along l. Irreducibility of a general fiber of $\mu_{W,l}|_E$ also follows as in the proof of Theorem 6.10 (2).

If $E_{W,l}$ is irreducible, then the assertion follows from irreducibility of a general fiber of $\mu_{W,l}|_{E_{W,l}}$.

Assume that $E_{W,l}$ is reducible. Then since $\nu_{W,l}(E_{W,l}) \in |L_{W_l}|$ by Proposition 4.4 (1) and W_l is factorial, one of irreducible components of $E_{W,l}$, say E', is $\nu_{W,l}$-exceptional. Let $H := -E_{V,l}|_{E_{V,l}}$ and F is a fiber of $E_{V,l} \to l$. Then

$$(\mu_{V,l}^* L_V - E_{V,l})|_{E_{V,l}} = H + F.$$

By the assumption, the exceptional locus of the morphism defined by $|H + F|$ contains a two dimensional subset. So

$$\mathscr{N}_{l/V} \simeq \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}$$

and E' is the unique member of $|H-F|$. Since

$$\widetilde{W_l} \in |\mu_{W,l}{}^*(2L_V) - 2E_{V,l}|,$$

we have $E_{W,l} \in |2H+2F|$. Hence $E'' \in |H+3F|$. Let C_0 be the minimal section of E'' and r a ruling. Since

$$(H+F)^2(H+3F) = 4$$

and $|H+F|$ is free, one of the following three cases occurs:
(i) $E'' \simeq \mathbb{F}_0$ and $(H+F)|_{E''} \sim C_0 + 2r$,
(ii) $E'' \simeq \mathbb{F}_2$ and $(H+F)|_{E''} \sim C_0 + 3r$, or
(iii) $E'' \simeq \mathbb{F}_4$ and $(H+F)|_{E''} \sim C_0 + 4r$.

Let $\delta := E' \cap E''$, which is a generically section of $E_{B,l} \to l$. By

$$(\delta^2)_{E''} = (H-F)^2(H+3F) = 0,$$

we have

$$\begin{cases} \delta \sim C_0 & \text{if } E'' \simeq \mathbb{F}_0, \\ \delta \sim C_0 + r & \text{if } E'' \simeq \mathbb{F}_2, \\ \delta \sim C_0 + 2r & \text{if } E'' \simeq \mathbb{F}_4. \end{cases}$$

This implies that $\nu_{W,l}(E') = \operatorname{Sing} \nu_{W,l}(E'')$.

Assume that $E'' \simeq \mathbb{F}_0$ or $\mathbb{F}_2$. Then $(H+F)|_{E''}$ is ample, and so $E'' \to \nu_{W,l}(E'')$ is finite. By the almost same way as in the proof of Theorem 6.10 (b-5), we can prove that W has a cA_2-singularity generically along l. This gives (b-5-1).

Assume that $E'' \simeq \mathbb{F}_4$. Then

$$(H+F)|_{E''} \cdot C_0 = 0$$

so C_0 is $\nu_{W,l}$-exceptional. Since the embedded dimension of the vertex of $\nu_{W,l}(E'')$ is 4, the vertex is a singular point of W_l. This gives (b-5-2).

(b-6) follows from Proposition 4.7.

□

Remark. For (b-5-2), we cannot obtain the bound of the type of singularity along l here. Proposition 7.11 is devoted to the study of the singularity along l if (b-5-2) occurs for l and $W_P \simeq Q^3$ for the plane $P \subset W$ containing l.

7.3. **Case V is the cone.**

Assume that V is the cone. We use the notation as in Proposition 7.1. This subsection is devoted to prove that this does not occur.

Let $W \dashrightarrow W_v$ be the projection from the vertex v of V. By Lemma 4.1 and Proposition 7.1 (2), we can apply the results in the section

4. We begin by stating basic properties of $W \dashrightarrow W_v$ in the next proposition.

Proposition 7.8. (*a*) *$E_{W,v}$ is a quintic del Pezzo surface. If $E_{W,v}$ is non-normal, then $E_{W,v}$ is* (*c*) *or* (*d*) *in* [Reid94, Theorem 1.1]. *If $E_{W,v}$ is normal, then $\nu_{W,v}(E_{W,v})$ has only Du Val singularities and its type are A_i $(i = 1, 2, 3, 4)$, $A_1 + A_1$ or $A_1 + A_2$.*

(*b*) *C_v is the intersection between $\nu_{W,v}(E_{W,v})$ and a member D of $|-K_{W_v}|$, where D has only Du Val singularities.*

(*c*) *$\nu_{W,v}$ is the blow-up along C_v.*

(*d*) *Assume that $\nu_{W,v}(E_{W,v})$ is normal. Then $\widetilde{W}_v$ has only cDV singularities and $E_{W,v}$ is the unique h-exceptional divisor contracted to a point on W.*

(*e*) *Let l be a singular line on W. Then one of the following holds:*

(*e*-1) *$(v \in l)$ l is the image of the fiber of $\nu_{W,v}$ over a point w of C_v such that emb-dim$_w C_v = 3$. In particular, w is contained in* $\operatorname{Sing} \nu_{W,v}(E_{W,v})$.

(*e*-2) *$(v \notin l)$ l is the image of a curve which is isomorphically mapped to a multiple line m in C_v. l is a good line and the singularity of W along l is generically of type cA_{k-1}, where k is the multiplicity of m in C_v.*
In particular, we see by this description that there is at most one singular line $\not\ni v$ on one plane.

(*f*) *At least one plane on W is not good.*

Proof. (a) By Proposition 4.4 (1), $\nu_{W,v}(E_{W,v}) \in |L_{W_v}|$. Thus by $E_{W,v} \simeq \nu_{W,v}(E_{W,v})$, $E_{W,v}$ is a quintic del Pezzo surface. Moreover since W_v is smooth, $E_{W,v}$ cannot be a cone. Hence we have the last two assertions by [Fuj85] and [Reid94, Theorem 1.1].

(b) This can be easily checked at the generic points of irreducible components.

(c) Let $\nu' : W' \to W_v$ be the blow-up along C_v. Since C_v is a complete intersection in W_v, W' has only hypersurface singularities and ν' is equi-dimensional. Thus W' and $\widetilde{W}_v$ are isomorphic in codimension 1. On the other hand, both $-K_{W'}$ and $-K_{\widetilde{W}_v}$ are relatively ample over W_v. Hence W' and $\widetilde{W}_v$ are isomorphic.

(d) Based on (b), we calculate the blow up $\nu_{W,v}$ analytically locally as follows:

Let w be a point of W_v and f (resp. g) is the local equation of D (resp. $\nu_{W,v}(E_{W,v})$) in $\mathbb{C}^3$ with coordinates x, y, z. Then

$$\widetilde{W}_v = \{(x, y, z; p : q) | pf - qg = 0\} \subset \mathbb{C}^3 \times \mathbb{P}^1.$$

So $\widetilde{W}_v$ has only cDV singularities over w except $(0,0,0;0:1)$ and if $\nu_{W,v}(E_{W,v})$ has at most a Du Val singularity at w, then $\widetilde{W}_v$ has only cDV singularities all over w. So if $\nu_{W,v}(E_{W,v})$ is normal, then $\widetilde{W}_v$ has only cDV singularities. Moreover if $\nu_{W,v}(E_{W,v})$ is normal, then there is no singular curve of $\widetilde{W}_v$ on $E_{W,v}$ and hence $E_{W,v}$ is the unique h-exceptional divisor contracted to a point on W.

(e) This easily follows from the local description of $\nu_{W,v}$ and Lemma 4.2.

(f) If any plane on W is good, then W has only cDV singularities by Corollary 2.8 and Proposition 2.13 (1). This is absurd.

$\square$

Proposition 7.9. *V is not the cone.*

Proof. Assume that V is the cone. Then by Proposition 7.8 (f), there exists at least one singular line on W.

Case 1. W contains at least two planes.

Let l be a singular line. By Proposition 7.4, W_l is a del Pezzo 3-fold and so by Proposition 4.7, l intersects any plane on W. Thus $v \in l$ since any two planes intersect only at v by Proposition 7.1 (2). Moreover C_l is reducible since the image of a plane which does not contain l is a line in C_l. Hence by Proposition 7.5, $\deg C'_l = 5$. This implies that W_l has only two planes and any component of $E_{W,l}$ is not $\nu_{E,l}$-exceptional. Since $\nu_{W,l}(E_{W,l}) \in |L_{W_l}|$ by Proposition 4.4 (1), and the strict transform of $E_{W,v}$ is contained in $\nu_{W,l}(E_{W,l})$ as a cubic surface by Proposition 7.6, the part of $E_{W,l}$ dominating l is irreducible and reduced. This implies that W has only a cA_1-singularity generically along l. Moreover we can show that there are at most two singular lines in one plane and this leads to a contradiction by Proposition 3.8. Indeed, by Proposition 7.8 (a), the assertion is true if $E_{W,v}$ is normal. Assume that $E_{W,v}$ is non-normal. Then since two planes are mapped to disjoint lines on W_v, the non-normal locus of $E_{W,v}$ is not the image of a plane. This implies that the image of a plane contains only one singular point of $E_{W,v}$ whence a plane contains at most one singular line.

Case 2. W contains only one plane.

Denote this plane by P.

Step 1. We restrict the possibilities of C_v.

Since P is contracted to a line on W_v, C_v is reducible. So

$$d := \deg C'_v \leq 9.$$

Let $p := p_a(C'_v)$. By Proposition 4.4, we have that $p \leq d-5$. Moreover, if $p = d-5$, then by Proposition 4.6, $d \geq 9$. By these facts, we have the following possibilities:

(i) $d = 6$ and $p = 0$.
(ii) $d = 7$ and $p = 0, 1$.
(iii) $d = 8$ and $0 \leq p \leq 2$.
(iv) $d = 9$ and $0 \leq p \leq 4$.

Let m be the image of P on W_v. By the above list of the possibilities and Proposition 4.4 (3-3), we have $C_v = C'_v + (10-d)m$ since $E_{W,v}$ is irreducible and is not $\nu_{W,v}$-exceptional.

If $d = 6$ and $p = 0$, or $d = 7$ and $p = 1$, then by the Riemann-Roch theorem,

$$h^0(\mathscr{O}_{C'_v}(2L_v)) = 2d + 1 - p \leq 14.$$

Hence by the proof of Proposition 4.6, we obtain

$$h^0(-K_X) \geq h^0(-K_{W_v} \otimes \mathscr{I}_{C'_v/W_v}) \geq h^0(-K_{W_v}) - 14 = 9,$$

a contradiction. Since P is not a good plane by Proposition 7.8 (f), P contains at least one singular line $l \ni v$. Since $\deg C'_l = 5$ or 6 by Proposition 7.5, and C'_l is the transform of C_v, the case that $d = 8$ and $p = 0$, or $d = 9$ and $p = 0, 1$ is excluded. By this observation, we know that if $d \geq 8$, then the image of l on C'_v is a singularity of C'_v. Now the possibilities of C'_v are:

(i) $d = 7$ and $p = 0$.
(ii) $d = 8$ and $p = 1, 2$.
(iii) $d = 9$ and $p = 2, 3, 4$.

Step 2. We treat the case that $\deg C'_l = 5$.

In this case, the strict transform $E'_{W,v}$ of $E_{W,v}$ on W_l is $\mathbb{F}_1$ since $\overline{\mathbb{F}}_3$ cannot contain a smooth rational curve of degree 5. Assume that $E_{W,v}$ is non-normal. Then since $E_{W,v} \dashrightarrow E'_{W,v}$ is the projection from a singular point, $E'_{W,v}$ is a cone, a contradiction. So $E_{W,v}$ is normal.

Since C'_l is the strict transform of C'_v and $\deg C'_l = 5$, we see that the image of l on C'_v is a singularity of C'_v and in particular $d \geq 8$. Moreover since C'_l is smooth, C'_v has only one singularity. On the other hand, as we saw in Step 1, the image of a singular line $\ni v$ on C'_v is a singularity of C'_v since $d \geq 8$. Thus l is the unique singular line $\ni v$. Hence by Proposition 7.8 (a), Γ_Z is one of the following:

(1) $(d = 8)$

$$\textcircled{0} —— \textcircled{1} —— \textcircled{0} —— \cdots —— \textcircled{0}\ ,$$

where the length of the chain

$$\begin{tikzcd} 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \end{tikzcd}$$

is at most 4. The left 0 corresponds to the singular line $\not\ni v$.

(2) $(d = 8)$

$$\begin{tikzcd} 0 \arrow[r, dash] & 0 \arrow[r, dash] \arrow[d, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \\ & 1 \arrow[d, dash] & & & \\ & 0 & & & \end{tikzcd},$$

where the length of the chain

$$\begin{tikzcd} 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \end{tikzcd}$$

is at most 4. The 0 at the bottom corresponds to the singular line $\not\ni v$.

(3) $(d = 9)$

$$\begin{tikzcd} 0 \arrow[r, dash] & 0 \arrow[r, dash] \arrow[d, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \\ & 1 & & & \end{tikzcd},$$

where the length of the chain

$$\begin{tikzcd} 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \end{tikzcd}$$

is at most 4.

We deny this case in Step 4 together with the case that $\deg C'_l = 6$ and $E_{W,v}$ is normal.

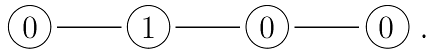
Step 3. We treat the case that $\deg C'_l = 6$.

The assumption implies that any component of $E_{W,l}$ is not $\nu_{W,l}$-exceptional. Moreover since the image of $E_{W,v}$ on W_l is a cubic surface by Proposition 7.6, the part of $E_{W,l}$ dominating l is irreducible and reduced. So W has only a cA_1-singularity generically along l. By Step 2, $\deg C'_{l'} = 6$ for any singular line $l' \ni v$. Hence W has only a cA_1-singularity generically along singular lines $\ni v$.

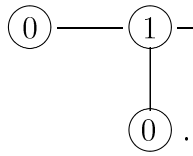
Assume that $d = 8, 9$. Then W has only cA_1-singular lines. This contradicts Proposition 3.8 since P is not good plane by Proposition 7.8 (f) and P is the unique plane. Thus we may assume that $d = 7$.

Assume that $E_{W,v}$ is normal. Recall that by Proposition 7.8 (a), the number of singular lines $\ni v$ is at most 2. Hence Γ_Z is one of the following:

(i)



(ii)



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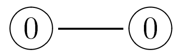


corresponds to the singular line $\not\ni v$.

We deny this case in Step 4.

Assume that $E_{W,v}$ is non-normal. Then by Proposition 7.8 (a), $E_{W,v}$ is (c) or (d) in [Reid94, Theorem 1.1]. Let $\pi : \widetilde{E} \to E_{W,v}$ be the normalization and $\widetilde{C}$ (resp. $\widetilde{C}'$) the strict transform of C_v (resp. C'_v) on $\widetilde{E}$. Let C_0 be the minimal section of $\widetilde{E}$ and r a ruling.

Then one of the following holds:

(a)
$$\widetilde{E} \simeq \mathbb{F}_3, \widetilde{C}' \sim C_0 + 6r, \text{ and } \widetilde{C} - \widetilde{C}' \sim C_0 + 2r.$$
By this description, we see that the image of P on W is the non-normal locus of $E_{W,v}$ and $\widetilde{C} - \widetilde{C}' = C_0 + 2r'$ for some fiber r'. Hence a general member D of $|-K_{W_v}|$ intersects one of the branches of $\nu_{W,v}(E_{W,v})$ along $\operatorname{Sing} \nu_{W,v}(E_{W,v})$ simply. So by the explicit calculation of the blow-up, we see that the singular curve on $\widetilde{W}_v$ dominating m is contained in $E_{W,v}$ whence it is contracted to v on W. Hence there is no singular line $\not\ni v$ on W. This contradicts Proposition 3.8.

(b)
$$\widetilde{E} \simeq \mathbb{F}_3, \widetilde{C}' \sim 2C_0 + 5r, \text{ and } \widetilde{C} - \widetilde{C}' \sim 3r.$$
Then, however, $\widetilde{C}'$ is reducible, a contradiction.

(c)
$$\widetilde{E} \simeq \mathbb{F}_1, \widetilde{C}' \sim C_0 + 5r, \text{ and } \widetilde{C} - \widetilde{C}' \sim C_0 + r.$$
Then C_v contains two different lines, a contradiction.

(d)

$$\widetilde{E} \simeq \mathbb{F}_1, \widetilde{C}' \sim 2C_0 + 3r, \text{ and } \widetilde{C} - \widetilde{C}' \sim 3r.$$

Then $p_a(\widetilde{C}') = 1$, a contradiction to that $p = 0$.

Now we have denied the case that $E_{W,v}$ is non-normal.

Step 4. We deny the case that $E_{W,v}$ is normal.

In this case, $E_{W,v}$ is the unique h-exceptional divisor contracted to a point on W by Proposition 7.8 (d). So we can apply Proposition 3.10 and Proposition 3.11.

Assume that Γ_Z is a chain. Then we obtain a contradiction to Proposition 3.11 since two good lines does not intersect at v.

In the other cases, Γ_Z has one vertex with three edges. The length of a branch is at most 2 and at least one branch has length 1. Note that at least one of fixed vertices in Γ_{X_m} has weight -1 by Proposition 3.3 (2).

If there is a vertex 1 as in Proposition 3.10 (3) which intersects three fixed vertices, then any economic chain in Γ_Z corresponds to a $1/2\,(1,1,1)$-singularity. This contradicts [Taka02b, Theorem 1.0].

Hence by Proposition 3.10, Γ_{X_m} contains one of the following:

(a)

$$0 \text{ — } \cdots \text{ — } 0 \text{ — } 1 \text{ — } -1 \text{ — } 1 \text{ — } 0 \text{ — } \cdots \text{ — } 0\ .$$

In this case, another branch intersects -1. Since the length of this branch is at most 2, it contains at most one economic chain. Hence by Claim 3.5, all the vertices in this branch have weights ≥ 0. Note that the curve corresponding to -1 is reduced in $\mathrm{Bs}\,|-K_{X_m}|$ by Proposition 3.10. Hence the blow-up along the curve corresponding to -1 induces the operation (b1) in 3.4. So if the vertex of this branch intersecting -1 has weight ≥ 1, then the operations to obtain Γ_Z terminate after resolving the base curve corresponding to -1 in the above graph. Thus Γ_{X_m} is one of the following:

(a-1)

$$\begin{array}{ccccccccccccccccc} 0 & \text{—} & \cdots & \text{—} & 0 & \text{—} & 1 & \text{—} & -1 & \text{—} & 1 & \text{—} & 0 & \text{—} & \cdots & \text{—} & 0 \\ & & & & & & & & | & & & & & & & & \\ & & & & & & & & 1\ , & & & & & & & & \end{array}$$

where the bottom 1 is the economic chain for a $\frac{1}{2}(1,1,1)$-singularity.

(a-2)

$$
\begin{array}{ccccccccccccccccc}
0 & — & \cdots & — & 0 & — & 1 & — & -1 & — & 1 & — & 0 & — & \cdots & — & 0 \\
 & & & & & & & & | & & & & & & & & \\
 & & & & & & & & 0 & & & & & & & & \\
 & & & & & & & & | & & & & & & & & \\
 & & & & & & & & 1\,, & & & & & & & &
\end{array}
$$

where in the bottom $0 — 1$, 1 is the economic chain for a $\frac{1}{2}(1,1,1)$-singularity and 0 is a fixed vertex.

(a-3)

$$
\begin{array}{ccccccccccccccccc}
0 & — & \cdots & — & 0 & — & 1 & — & -1 & — & 1 & — & 0 & — & \cdots & — & 0 \\
 & & & & & & & & | & & & & & & & & \\
 & & & & & & & & 0 & & & & & & & & \\
 & & & & & & & & | & & & & & & & & \\
 & & & & & & & & 1\,, & & & & & & & &
\end{array}
$$

where the bottom $0 — 1$ is the economic chain for a $\frac{1}{3}(1,-1,1)$-singularity.

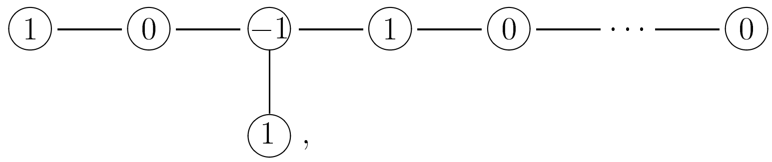
Similarly to the above, we see that the blow-ups along the base curves of $|-K_{X_m}|$ induce the operations of type (b1) in 3.4. So we can easily obtain Γ_Z. Then, however, W has only cDV singularities for the cases (a-1) and (a-2) and any singular line passes through v for the case (a-3), a contradiction.

(b)

$$1 — 0 — -1 — 1 — 0 — \cdots — 0\,.$$

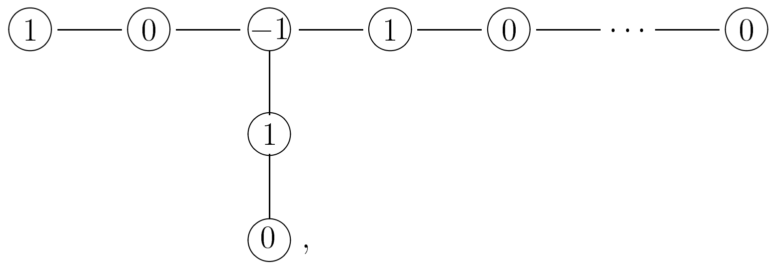
In this case, another branch intersects -1 or the left 0. Since the length of this branch is at most 2, it contains at most one economic chain. Hence by Claim 3.5, all the vertices is branch have weights ≥ 0. Assume that the vertex of this branch intersecting -1 or 0 has weight ≥ 1. Then similarly to the case (a), the operations to obtain Γ_Z terminate after resolving the base curves corresponding to $0 — -1$ in the above graph. Hence Γ_{X_m} is one of the following:

(b-1)



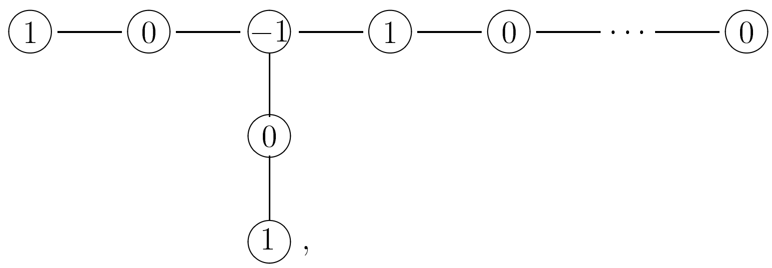
where the bottom 1 is the economic chain for a $\frac{1}{2}(1,1,1)$-singularity

(b-2)



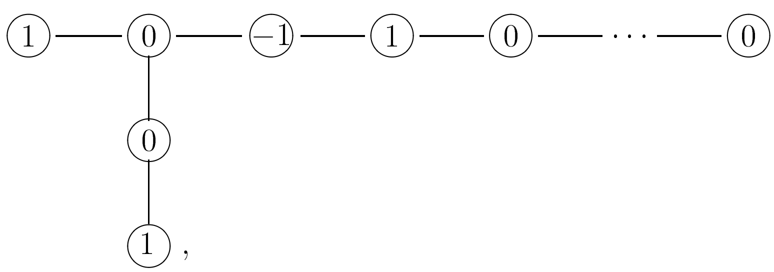
where the bottom 1 —— 0 is the economic chain for a $\frac{1}{3}(1,-1,1)$-singularity.

(b-3)



where in the bottom 0 —— 1, 1 is the economic chain for a $\frac{1}{2}(1,1,1)$-singularity and 0 is a fixed vertex.

(b-4)



where in the bottom 0 —— 1, 1 is the economic chain for a $\frac{1}{2}(1,1,1)$-singularity and 0 is a fixed vertex.

Similarly to the case (a), we see that the blow-ups along the base curves of $|-K_{X_m}|$ induce the operations of type (b1) in 3.4. So we can easily obtain Γ_Z. Then, however, any singular line passes through v for the cases (b-1) and (b-2), and W does not have a cA-singularity generically along a singular line for the case (b-3), a contradiction. Assume that the case (b-4) occurs. Let γ be the curve corresponding to -1 and $r-1$ the length of

$$1 \text{ --- } 0 \text{ --- } \cdots \text{ --- } 0\,.$$

Then it is easy to see that $-K_X \cdot \gamma = 1/3 - 1/r$. Hence we have $r \geq 4$. Then, however, Γ_Z has a branch with length ≥ 4, a contradiction.

(c)

$$0 \text{ --- } \cdots \text{ --- } 0 \text{ --- } 1 \text{ --- } -1 \text{ --- } 1 \text{ --- }$$
$$\text{ --- } \bigcirc \text{ --- } \cdots \text{ --- } \bigcirc \text{ --- }$$
$$\text{ --- } 1 \text{ --- } 0 \text{ --- } \cdots \text{ --- } 0\,.$$

There is another branch intersecting -1 or $\bigcirc \text{ --- } \cdots \text{ --- } \bigcirc$. Then, however, there is a branch with length ≥ 3, a contradiction.

$\square$

7.4. Case V is smooth and W_P is a cubic in $\mathbb{P}^4$ for some plane P.

Theorem 7.10. *Assume that V is smooth and W_P is a cubic in $\mathbb{P}^4$ for some plane P. Then*

(1) *C_P is a twisted cubic curve in $\mathbb{P}^3$.*
(2) *P is the unique plane on W and a good plane corresponding to a $1/r\,(1,-1,1)$-singularity for $r = 2, 3$.*

Proof. Since W_P is a del Pezzo 3-fold of degree 3, we can apply Proposition 4.4 and we see that C_P is a twisted cubic curve. Hence by Proposition 4.9, P is the unique plane on W and a good plane. We may assume that P contains a singular line l. By Proposition 7.4, the image of P on W_l is a smooth point of W_l. So by Proposition 7.2, $S \simeq \overline{\mathbb{F}}_3$ (see Proposition 7.2 for the definition of S). Since $\overline{\mathbb{F}}_3$ does not contain a smooth rational curve of degree 5, C_l is irreducible by Proposition 7.5. By Proposition 7.6, W_l is factorial. Hence by Proposition 4.10, P corresponds to a $1/3\,(1,-1,1)$-singularity. $\square$

7.5. Case V is smooth and $W_P \simeq Q^3$ for some plane P.

Assume that V is smooth and $W_P \simeq Q^3$ for some plane P in this subsection. Then by Theorem 7.10, $W_P \simeq Q^3$ for any plane P.

First we give descriptions of $\nu_{W,l}$ locally analytically near a fiber contained in $\operatorname{Sing} \widetilde{W}_l$ for a singular line l.

Proposition 7.11. *Let l be a singular line and P the plane on W containing l. Assume that $W_P \simeq Q^3$. Then*

(1) *$\widetilde{W}_l$ has only cDV singularities and any singular curve is contained in a fiber of $\nu_{W,l}$. Hence W also has only cDV singularities.*

(2) *Let ϱ be the image of P by $W \dashrightarrow W_l$. There is a singular curve on ${\nu_{W,l}}^{-1}(\varrho)$ if and only if $\varrho \in \operatorname{Sing} C_l$. If a singular curve exists on ${\nu_{W,l}}^{-1}(\varrho)$, then it is a line (denote it by δ) and $\widetilde{W}_l$ has a cA-singularity generically along δ. Moreover if $k = 2, 3$ for ϱ (see Proposition 7.4 for the definition of k), then $\widetilde{W}_l$ has a cA_1-singularity generically along δ.*

(3) *Let w be a point $\neq \varrho$ on C_l (recall that ${\nu_{W,l}}^{-1}(w) \simeq \mathbb{P}^1$).*
   - (3-1) *If w is a smooth point of W_l, then ${\nu_{W,l}}^{-1}(w) \not\subset \operatorname{Sing} \widetilde{W}_l$.*
   - (3-2) *Assume that w is a singular point. Then $w \in \operatorname{Sing} C_l$ and $\widetilde{W}_l$ has a cA_1-singularity generically along ${\nu_{W,l}}^{-1}(w)$. Moreover $\nu_{W,l}(E_{W,l})$ is a non-normal cone and w is its vertex.*

Proof. Since W_l has only isolated singularities by Proposition 4.8 and C_l is reduced by Proposition 7.5, any singular curve on $\widetilde{W}_l$ is contained in a fiber of $\nu_{W,l}$.

Step 1. We study the singularities of $\widetilde{W}_l$ on ${\nu_{W,l}}^{-1}(\varrho)$.

Analytically locally near ϱ, we use the description in Proposition 7.2 (2-1). W_l is the intersection of V_l and a smooth hypersurface M in $\mathbb{C}^5$. By the symmetry of variables, we have only to consider the following three cases:

(A)
$$M = \{x = f(y, z, w, t)\}.$$

Then in $\mathbb{C}^4$,
$$W_l = \{fy + zw + t^2 = 0\}.$$
$$C_l = \{z = t = f(y, 0, z, 0) = 0\}.$$

We denote the i-th part of f by f_i. Let

$$f_1 := az + bt + cy + dw.$$
$$f_2 := Q(z, t) + l_1(y, w)z + l_2(y, w)t + L_1(y, w)L_2(y, w),$$

where $a, \dots, d \in \mathbb{C}$, Q is a quadric form, and l_i and L_i are linear forms.

Since C_l is smooth or the union of two smooth curves at ϱ, we have that

$$cy + dw \not\equiv 0 \text{ or } L_1 L_2 \not\equiv 0.$$

(B)

$$M = \{y = f(x, z, w, t)\}.$$

Then in $\mathbb{C}^4$,

$$W_l = \{xf + zw + t^2 = 0\}.$$
$$C_l = \{x = z = t = 0\}.$$

We denote the i-th part of f by f_i. Let

$$f_1 := ax + bz + ct + dw.$$
$$f_2 := Q(x, z, t) + \alpha wx + \beta wz + \gamma w^2,$$

where

$$a, \dots, d, \alpha, \dots, \gamma \in \mathbb{C},$$

and Q is a quadric form.

(C)

$$M = \{t = f(x, y, z, w)\}.$$

Then in $\mathbb{C}^4$,

$$W_l = \{xy + zw + f^2 = 0\}.$$
$$C_l = \{x = z = f(0, y, 0, w) = 0\}.$$

We denote the i-th part of f by f_i. Let

$$f_1 := ax + bz + cy + dw.$$
$$f_2 := Q(x, z) + l_1(y, w)x + l_2(y, w)z + L_1(y, w)L_2(y, w),$$

where $a, \dots, d \in \mathbb{C}$, Q is a quadric form, and l_i and L_i are linear forms. Since C_l is smooth or the union of two smooth curves at ϱ, we have that

$$cy + dw \not\equiv 0 \text{ and } L_1 L_2 \not\equiv 0.$$

Case (A)

$$\widetilde{W_l} = \{(y, z, w, t; p : q : r) | z : t : f = p : q : r\} \subset \mathbb{C}^4 \times \mathbb{P}^2.$$

We check the singularities of $\widetilde{W_l}$ on

$$\{y = z = w = t = 0\}.$$

By blowing up at the origin, we can easily show that if C_l is singular at ϱ, then $k = 2, 3$ for ϱ if and only if

$$b \neq 0, \text{ or}$$
$$a \neq 0, b = 0 \text{ and } L_1(1, -a)L_2(1, -a) \neq 0.$$

The chart $p \neq 0$.
Set $q' := q/p$ and $r' := r/p$. Then

$$\widetilde{W_l} = \{(y, z, w, t, q', r') | t = q'z, f = r'z, w = -r'y - (q')^2 z\} \subset \mathbb{C}^6.$$

By erasing t and w, we have

$$\widetilde{W_l} = \{(y, z, q', r') | f(y, z, -r'y - (q')^2 z, q'z) - r'z = 0\} \subset \mathbb{C}^4.$$

It is easy to see that

$$\operatorname{Sing} \widetilde{W_l} \cap \{y = z = 0\} =$$
$$\{y = z = c - dr' = a - d(q')^2 + bq' - r' = 0\}.$$

Assume that C_l is smooth at ϱ. Then we can see that $\widetilde{W_l}$ has at most isolated singularities on this chart (see Step 3).

Assume that C_l is singular at ϱ. Then

$$\operatorname{Sing} \widetilde{W_l} \cap \{y = z = 0\} = \{y = z = a + bq' - r' = 0\}.$$

Set

$$r'' := r' - (a + bq').$$

Then

$$\widetilde{W_l} = \{(y, z, q', r'') |$$
$$F := f(y, z, -(r'' + a + bq')y - (q')^2 z, q'z) - (r'' + a + bq')z = 0\} \subset \mathbb{C}^4.$$

So we can see that the quadric part of F is $Lz + \eta y^2$, where L is a linear form $\ni r''$ and $\eta \in \mathbb{C}$. This implies that $\widetilde{W_l}$ has only cA-singularity at any point of

$$\operatorname{Sing} \widetilde{W_l} \cap \{y = z = 0\}.$$

Assume that $k = 2, 3$. We prove that $\widetilde{W_l}$ has a cA_1-singularity generically along

$$\{y = z = r'' = 0\}.$$

If we regard q' as a constant, then the quadric part of F is

$$Mz + L_1(y, -(a + bq')y)L_2(y, -(a + bq')y),$$

where M is a linear form $\ni r''$. It is easy to see that

$$L_1(y, -(a + bq')y)L_2(y, -(a + bq')y) \not\equiv 0$$

if and only if $b \neq 0$, or

$$L_1(1,-a)L_2(1,-a) \neq 0.$$

As we saw above, this holds if $k = 2, 3$.

The chart $q \neq 0$.

Set $p' := p/q$ and $r' := r/q$. Then

$$\widetilde{W_l} = \{(y,z,w,t,p',r') | t = -r'y - p'w, z = -p'(r'y + p'w), f = r't\} \subset \mathbb{C}^6.$$

By erasing t and z, we have

$$\begin{aligned} \widetilde{W_l} = \{(y,w,p',r') | G := & f(y, -p'(r'y+p'w), w, -r'y-p'w) \\ +r'(r'y+p'w) = 0\} \subset \mathbb{C}^4. \end{aligned}$$

It is easy to see that

$$\begin{aligned} &\operatorname{Sing} \widetilde{W_l} \cap \{y = w = 0\} = \\ &\{y = w = c - r'(ap' + b - r') = d - p'(ap' + b - r') = 0\}. \end{aligned}$$

Assume that C_l is smooth at ϱ (see Step 3). Then we can see that $\widetilde{W_l}$ has at most isolated singularities on this chart.

Assume that C_l is singular at ϱ. Then

$$\begin{aligned} &\operatorname{Sing} \widetilde{W_l} \cap \{y = w = 0\} = \\ &\{y = w = ap' + b - r' = 0\} \cup \{y = w = p' = r' = 0\}. \end{aligned}$$

The quadric part of G is

$$-b(r'y + p'w) + L_1L_2$$

so $\widetilde{W_l}$ has an ODP at

$$\{y = w = p' = r' = 0\}$$

if $b \neq 0$. Set

$$r'' := r' - b.$$

Then

$$\begin{aligned} \widetilde{W_l} = \{(y,w,p',r'') | H := f(y, -p'\{(r''+b)y + p'w\}, w, -(r''+b)y - p'w) \\ +(r''+b)\{(r''+b)y + p'w\} = 0\} \subset \mathbb{C}^4. \end{aligned}$$

Since we checked the singularities on the chart $p \neq 0$, we have only to check the singularity at

$$\{y = w = p' = r'' = 0\}.$$

We can see that the quadric part of H is $bLy + L_1L_2$, where L is a linear form $\ni r''$. If $b \neq 0$, or $b = 0$ and L_1 is not a multiple of L_2, then $\widetilde{W_l}$ has only cA-singularity at

$$\{y = w = p' = r'' = 0\}.$$

Assume that $b = 0$ and L_1 is a multiple of L_2. Then we see that the cubic part of H is

$$H_3 := (r'' - ap' - l_2)(r''y + p'w) + f_3(y, 0, w, 0),$$

where f_3 is the cubic part of f. Let $y' := L_1$ and we regard it as a new coordinate. If $y = \zeta y' + \eta w$ with $\zeta \neq 0$, then after completing the square with respect to y', the cubic part of H contains $r''p'w$ but not $(r'')^3$ and hence is not a cube. If $w = \zeta' y'$ with $\zeta' \neq 0$, then after completing the square with respect to y', the cubic part of H contains $(r'')^2 y$ but not $(r'')^3$ and hence is not a cube. So $\widetilde{W}_l$ has at worst cD-singularity at

$$\{y = z = p' = r'' = 0\}.$$

<u>The chart $r \neq 0$.</u>
Set $p' := p/r$ and $q' := q/r$. Then

$$\widetilde{W}_l = \{(y, z, w, t, p', q') | z = p'f, t = q'f, y = -p'w - (q')^2 f\} \subset \mathbb{C}^6.$$

We have only to check the singularity at the origin but clearly $\widetilde{W}_l$ is smooth at the origin.

Case (B)

$$\widetilde{W}_l = \{(x, z, w, t; p : q : r) | x : z : t = p : q : r\} \subset \mathbb{C}^4 \times \mathbb{P}^2.$$

We check the singularities of $\widetilde{W}_l$ on

$$\{x = z = w = t = 0\}.$$

<u>The chart $p \neq 0$.</u>
Set $q' := q/p$ and $r' := r/p$. Then

$$\widetilde{W}_l = \{(x, z, w, t, q', r') | z = q'x, t = r'x, f + q'w + (r')^2 x = 0\} \subset \mathbb{C}^6.$$

By erasing z and t, we have

$$\widetilde{W}_l = \{(x, w, q', r') | f(x, q'x, w, r'x) + q'w + (r')^2 x = 0\} \subset \mathbb{C}^4.$$

It is easy to see that

$$\begin{aligned} &\operatorname{Sing} \widetilde{W}_l \cap \{x = w = 0\} = \\ &\{x = w = a + bq' + cr' + (r')^2 = d + q' = 0\}. \end{aligned}$$

Then we can see that $\widetilde{W}_l$ has at most isolated singularities on this chart (see Step 3).

<u>The chart $q \neq 0$.</u>
Set $p' := p/q$ and $r' := r/q$. Then

$$\widetilde{W}_l = \{(x, z, w, t, p', r') | x = p'z, t = r'z, p'f + w + (r')^2 z = 0\} \subset \mathbb{C}^6.$$

By erasing x and t, we have

$$\widetilde{W_l} = \{(z, w, p', r') | p' f(p'z, z, w, r'z) + w + (r')^2 z = 0\} \subset \mathbb{C}^4.$$

Since we checked the singularities on the chart $p \neq 0$, we have only to check the singularity on

$$\{z = w = p' = 0\}.$$

But $\widetilde{W_l}$ is smooth on

$$\{z = w = p' = 0\}.$$

The chart $r \neq 0$.
Set $p' := p/r$ and $q' := q/r$. Then

$$\widetilde{W_l} = \{(x, z, w, t, p', q') | x = p't, z = q't, p'f + q'w + t = 0\} \subset \mathbb{C}^6.$$

By erasing x and z, we have

$$\widetilde{W_l} = \{(w, t, p', q') | p' f(p't, q't, w, t) + q'w + t = 0\} \subset \mathbb{C}^4.$$

Since we checked the singularities on the charts $p \neq 0$ and $q \neq 0$, we have only to check the singularity at

$$\{w = t = p' = q' = 0\}.$$

But $\widetilde{W_l}$ is smooth at

$$\{w = t = p' = q' = 0\}.$$

Case (C)

$$\widetilde{W_l} = \{(x, y, z, w; p : q : r) | x : z : f = p : q : r\} \subset \mathbb{C}^4 \times \mathbb{P}^2.$$

We check the singularities of $\widetilde{W_l}$ on

$$\{x = y = z = w = 0\}.$$

By the symmetry of variables, we have only to consider the charts $p \neq 0$ and $r \neq 0$.

The chart $p \neq 0$.
Set $q' := q/p$ and $r' := r/p$. Then

$$\widetilde{W_l} = \{(x, y, z, w, q', r') | z = q'x, f = r'x, y = -q'w - (r')^2 x\} \subset \mathbb{C}^6.$$

By erasing z and y, we have

$$\widetilde{W_l} = \{(x, w, q', r') | f(x, -q'w - (r')^2 x, q'x, w) - r'x = 0\} \subset \mathbb{C}^4.$$

It is easy to see that

$$\operatorname{Sing} \widetilde{W_l} \cap \{x = w = 0\} = \quad \{x = w = cq' - d = a - c(r')^2 + bq' - r' = 0\}.$$

Assume that C_l is smooth at ϱ. Then we can see that $\widetilde{W_l}$ has at most isolated singularities on this chart (see Step 3).

Assume that C_l is singular at ϱ. Then

$$\operatorname{Sing}\widetilde{W_l} \cap \{x = w = 0\} = \{x = w = a + bq' - r' = 0\}.$$

Set

$$r'' := r' - (a + bq').$$

Then

$$\widetilde{W_l} = \{(x, w, q', r'') | F := f(x, -q'w - (r'' + a + bq')^2 x, q'x, w) \\ -(r'' + a + bq')x = 0\} \subset \mathbb{C}^4.$$

So we can see that the quadric part of F is $Lx + \eta w^2$, where L is a linear form $\ni r''$ and $\eta \in \mathbb{C}$. This implies that $\widetilde{W_l}$ has only cA-singularity at any point of

$$\operatorname{Sing}\widetilde{W_l} \cap \{x = w = 0\}.$$

Moreover if we regard q' as a constant, then the quadric part of F is

$$Lx + L_1(-q'w, w)L_2(-q'w, w),$$

where M is a linear form $\ni r''$. Since

$$L_1(-q'w, w)L_2(-q'w, w) \not\equiv 0,$$

$\widetilde{W_l}$ has a cA_1-singularity generically along

$$\{x = w = r'' = 0\}.$$

<u>The chart $r \neq 0$.</u>

Set $p' := p/r$ and $q' := q/r$. Then

$$\widetilde{W_l} = \{(y, z, w, t, p', q') | x = p'f, z = q'f, p'y + q'w + f = 0\} \subset \mathbb{C}^6.$$

By erasing x and z, we have

$$\widetilde{W_l} = \{(y, w, p', q') | f(-p'(p'y + q'w), y, -q'(p'y + q'w), w) \\ +p'y + q'w = 0\} \subset \mathbb{C}^4.$$

We have only to check the singularity at the origin but clearly $\widetilde{W_l}$ has an ODP at the origin.

Step 2. We study singularities of $\widetilde{W_l}$ on ${\nu_{W,l}}^{-1}(w)$ for a point $w \neq \varrho$.

If w is a smooth point, then $\widetilde{W_l}$ has at most isolated cA_1-singularities on ${\nu_{W,l}}^{-1}(w)$ since C_l is reduced and has only planar singularities.

Assume that w is a singular point. If $w \notin \operatorname{Sing} C_l$, then we can easily see that ${\nu_{W,l}}^{-1}(w) \simeq \mathbb{P}^2$ by taking a local coordinate. But by Proposition 4.7, ${\nu_{W,l}}^{-1}(\varrho)$ is the unique 2-dimensional fiber, a contradiction. Hence $w \in \operatorname{Sing} C_l$.

We prove that C_l is a local complete intersection in W_l. Indeed, let

$$C_l := \{x_1 = x_2 = k(x_2, x_3) = 0\}$$

locally, where x_1–x_4 are coordinates of $\mathbb{C}^4$. Then W_l is of the form

$$\{x_1 l + x_2 m + nk = 0\},$$

where $l, k, n \in \mathbb{C}[x_1, \dots, x_4]$. We have only to prove that n is a unit. If n is not a unit, then ${\nu_{W,l}}^{-1}(w) \simeq \mathbb{P}^2$ so we obtain a contradiction as above.

Since ϱ is also a singular point on W_l, $k = 2, 3$ for w by Proposition 7.4. We use the local equation of W_l at w as in Proposition 7.4. As we proved above, C_l is a local complete intersection in W_l. So by the symmetry of the equation, we may assume that C_l is the intersection in $\mathbb{C}^4$ between W_l and the following:

(i) $\{x = f(z, w)\} \cap \{y = g(z, w)\}$.
(ii) (only for $k = 3$) $\{x = f(y, z)\} \cap \{w = g(y, z)\}$.

Hence in $\mathbb{C}^2$ with coordinates z, w for (i) (resp. with coordinates y, z for (ii)),

$$C_l = \{f^2 + g^2 + z^2 + w^k = 0\}$$

(resp.

$$C_l = \{f^2 + y^2 + z^2 + g^3 = 0\}).$$

We denote the i-th part of f and g by f_i and g_i respectively. Moreover we set

$$f_1 := \alpha z + \beta w$$

and

$$g_1 = \gamma z + \delta w.$$

Since C_l is the union of two smooth curves at w, we see that

$$f_1{}^2 + g_1{}^2 + z^2 + \delta_{2,k} w^k \not\equiv 0$$

for (i) (resp.

$$f_1{}^2 + y^2 + z^2 \not\equiv 0$$

for (ii), which always holds).

<u>The case (i).</u>

$$\widetilde{W}_l = \{(x, y, z, w; p : q) | y - g : x - f = p : q\} \subset \mathbb{C}^4 \times \mathbb{P}^1.$$

By the symmetry of the equation, it is sufficient to check the singularity of $\widetilde{W}_l$ on the chart $p \neq 0$. On this chart, by setting $q' = q/p$,

$$\widetilde{W}_l = \{F := \{q'(y - g) + f\}^2 + y^2 + z^2 + w^k = 0\}.$$

First we see that $\widetilde{W}_l$ has a cA_1-singularity generically along ${\nu_{W,l}}^{-1}(w)$. By considering q' as a constant, the quadric part F_2 of F is

$$\{q'(y-g_1)+f_1\}^2+y^2+z^2+\delta_{2,k}w^k.$$

By completing the square with respect to y,

$$F_2=(1+(q')^2)\{y+\frac{q'(f_1-q'g_1)}{1+(q')^2}\}^2+\frac{(f_1-q'g_1)^2}{1+(q')^2}+z^2+\delta_{2,k}w^k.$$

It suffices to prove that for a general q',

$$\operatorname{rank}(\frac{(f_1-q'g_1)^2}{1+(q')^2}+z^2+\delta_{2,k}w^k)=2.$$

Assume that $k=2$ and

$$\operatorname{rank}((f_1-q'g_1)^2+(1+(q')^2)(z^2+w^2))\leq 1$$

for all q'. Then we have

$$\alpha^2+\beta^2+1=\gamma^2+\delta^2+1=\alpha\gamma+\beta\delta=0.$$

This implies that

$$\alpha-\delta=\beta+\gamma=0$$

or

$$\alpha+\delta=\beta-\gamma=0.$$

Hence

$${f_1}^2+{g_1}^2+z^2+w^2\equiv 0,$$

a contradiction.

Assume that $k=3$ and

$$\operatorname{rank}((f_1-q'g_1)^2+(1+(q')^2)z^2)\leq 1$$

for all q'. This implies that $\beta=\delta=0$. Then we can easily see that C_l has an ordinary cusp at w, a contradiction.

Thus $\widetilde{W}_l$ has a cA_1-singularity generically along ${\nu_{W,l}}^{-1}(w)$.

Next we see that $\widetilde{W}_l$ has only a cDV singularity at any point on ${\nu_{W,l}}^{-1}(w)$. Let c be an arbitrary complex number and

$$q'':=q'-c.$$

Then the quadric part $F_{c,2}$ of $F(y,z,w,q''+c)$ is

$$\{c(y-g_1)+f_1\}^2+y^2+z^2+\delta_{2,k}w^k.$$

It is easy to see that $F_{c,2}\not\equiv 0$. If $\operatorname{rank}F_{c,2}\geq 2$, then $\widetilde{W}_l$ has only a cA-singularity at $(0,0,0,c)$. So we may assume that

$$\{c(y-g_1)+f_1\}^2+y^2+z^2+\delta_{2,k}w^k=(y')^2 \tag{7.1}$$

for some linear form y'. The cubic part $F_{c,3}$ of $F(y, z, w, q'' + c)$ is

$$2\{c(y - g_1) + f_1\}\{q''(y - g_1) - cg_2 + f_2\} + \delta_{3,k}w^3.$$

We regard y' as a new coordinate.

Assume that we can write

$$y = \zeta y' + \eta z + \theta w \text{ with } \zeta \neq 0.$$

Then after completing the square with respect to y', the cubic part becomes

$$2\{c(\eta z + \theta w - g_1) + f_1\}\{q''(\eta z + \theta w - g_1) - cg_2 + f_2\} + \delta_{3,k}w^3.$$

By (7.1), $c(y - g_1) + f_1$ is not a multiple of y' whence

$$c(\eta z + \theta w - g_1) + f_1 \neq 0.$$

Assume that

$$\eta z + \theta w - g_1 \neq 0.$$

Then the cubic part contains lq'', where l is a linear form which is not divisible by q''. But it does not contain q'' so it is not a cube. Assume that

$$\eta z + \theta w - g_1 = 0.$$

Then by (7.1),

$$\{c\zeta' y' + f_1\}^2 + (\zeta' y' + g_1)^2 + z^2 + \delta_{2,k}w^k = (y')^2.$$

Hence we have

$$\begin{aligned} {f_1}^2 + {g_1}^2 + z^2 + \delta_{2,k}w^k &\equiv 0. \\ g_1 + cf_1 &\equiv 0. \end{aligned}$$

Then $k = 3$, and f_1 and g_1 are multiples of z. This implies that C_l has a simple cusp at w, a contradiction.

Assume that we can write

$$y' = \eta' z + \theta' w.$$

Then by (7.1),

$$c^2 + 1 = 0 \text{ and } f_1 - cg_1 = 0.$$

So

$$F_{c,2} = z^2 + \delta_{2,k}w^k$$

whence $k = 3$ and $y' = z$. Then after completing the square with respect to z, the cubic part contains y^2q'' but not $(q'')^3$ and so is not a cube.

Consequently in any case $\widetilde{W_l}$ has only a cDV singularity at $(0, 0, 0, c)$.

The case (ii).

$$\widetilde{W_l} = \{(x, y, z, w; p : q) | w - g : x - f = p : q\} \subset \mathbb{C}^4 \times \mathbb{P}^1.$$

First we check the singularity of $\widetilde{W_l}$ on the chart $p \neq 0$. On this chart, by setting $q' = q/p$, we have

$$\widetilde{W_l} = \{F := \{q'(w - g) + f\}^2 + y^2 + z^2 + w^3 = 0\}.$$

We see that $\widetilde{W_l}$ has a cA_1-singularity at any point on ${\nu_{W,l}}^{-1}(w)$ with $q' \neq 0$. By considering q' as a constant, the quadric part of F is

$$\{q'(w - g_1) + f_1\}^2 + y^2 + z^2.$$

For a $q' \neq 0$, we may regard $q'(w - g_1) + f_1$ as a coordinate. Hence for a $q' \neq 0$,

$$\operatorname{rank}\left(\{q'(w - g_1) + f_1\}^2 + y^2 + z^2\right) = 3$$

and we are done. We check the singularity at the origin. Note that

$$F_2 = {f_1}^2 + y^2 + z^2,$$

which is $\not\equiv 0$. If $\operatorname{rank} F_2 \geq 2$, $\widetilde{W_l}$ has a cA-singularity at the origin. So we may assume that $F_2 = (y')^2$ for some linear form y'. Then we have $f_1 \not\equiv 0$ and f_1 is not a multiple of y'. Since

$$F_3 = 2f_1 q'(w - g_1) + w^3,$$

after completing the square with respect to y', the cubic part contains $q'wl$ with some linear form l but not $(q')^3$. Thus it is not a cube. Hence $\widetilde{W_l}$ has at worst cD-singularity at the origin.

Next we check the singularity of $\widetilde{W_l}$ on the chart $q \neq 0$. On this chart, by setting $p' = p/q$, we have

$$\widetilde{W_l} = \{G := \{x^2 + y^2 + z^2 + \{p'(x - f) + g\}^3 = 0\}.$$

It is enough to check the singularity at the origin. Since

$$G_2 = x^2 + y^2 + z^2,$$

$\widetilde{W_l}$ has a cA_1-singularity at the origin and we are done.

Step 3. We finish the proof.

(2) was proved in Step 1 and (3) was proved in Step 2. We finish the proof of (1) now. In Step 2, we proved that $\widetilde{W_l}$ has only cDV singularities on ${\nu_{W,l}}^{-1}(w)$ for any $w \neq \varrho$. Assume that $\varrho \in \operatorname{Sing} C_l$. Then we proved in Step 1 that $\widetilde{W_l}$ has only cDV singularities on ${\nu_{W,l}}^{-1}(\varrho)$. Thus (1) holds in this case. So we may assume that $\varrho \notin \operatorname{Sing} C_l$ for any l and P. Then we proved that $\widetilde{W_l}$ has only isolated singularities on ${\nu_{W,l}}^{-1}(\varrho)$. Hence by Proposition 7.7, P is a good plane and so W has only cDV singularities. $\square$

Theorem 7.12. *Let P be a plane on W and assume that there is a singular line on P. Then one of the following occurs:*

(1) *P is a good plane corresponding to a $1/r\,(1,-1,1)$-singularity with $r = 3, 4, 5$.*

(2) *$\Gamma_{X_m,P}$ is*

$$\textcircled{1} \;\text{---}\; \textcircled{-1} \;\text{---}\; \textcircled{1} \;\text{---}\; \textcircled{0}\ .$$

The left $\textcircled{1}$ corresponds to a $1/2\,(1,1,1)$-singularity and the right

$$\textcircled{1} \;\text{---}\; \textcircled{0}$$

corresponds to a $1/3\,(1,-1,1)$-singularity.

(3) *$\Gamma_{X_m,P}$ is*

$$\textcircled{0} \;\text{---}\; \textcircled{1} \;\text{---}\; \textcircled{-1} \;\text{---}\; \textcircled{1} \;\text{---}\; \textcircled{0}\ .$$

The left $\textcircled{0} \;\text{---}\; \textcircled{1}$ and the right $\textcircled{1} \;\text{---}\; \textcircled{0}$ correspond to $1/3\,(1,-1,1)$-singularities.

(4) *$\Gamma_{X_m,P}$ is*

$$\textcircled{1} \;\text{---}\; \textcircled{-1} \;\text{---}\; \textcircled{1} \;\text{---}\; \textcircled{0} \;\text{---}\; \textcircled{0}\ .$$

The left $\textcircled{1}$ corresponds to a $1/2\,(1,1,1)$-singularity and the right

$$\textcircled{1} \;\text{---}\; \textcircled{0} \;\text{---}\; \textcircled{0}$$

corresponds to a $1/4\,(1,-1,1)$-singularity.

Except the case (1) *with $r = 3$, P is the unique plane on W by Proposition* 7.7 (*b*-6).

Proof.

Step 1. We give a rough classification of singular lines on P.

By Proposition 7.7 and Proposition 7.11 (2), one of the following holds:

(1) P is a good plane.

(2) P contains two good lines.

(3) P contains the unique singular line l and l is of type A but not good.

Assume that (2) occurs. Then by Proposition 3.11 and Proposition 7.11 (1), Γ_{X_m} is

$$\textcircled{0} \;\text{---}\; \cdots \;\text{---}\; \textcircled{0} \;\text{---}\; \textcircled{1} \;\text{---}\; \textcircled{-1} \;\text{---}\; \textcircled{1} \;\text{---}\; \textcircled{0} \;\text{---}\; \cdots \;\text{---}\; \textcircled{0},$$

where

$$\textcircled{0} \;\text{---}\; \cdots \;\text{---}\; \textcircled{0} \;\text{---}\; \textcircled{1}$$

corresponds to a $1/r\,(1,-1,1)$-singularity and

$$\begin{tikzcd} 1 \arrow[r, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \end{tikzcd}$$

corresponds to a $1/s\,(1,-1,1)$-singularity for some r, s.

We show that (3) does not occurs. Assume the contrary. Since P is not good, Γ_{X_m} contains at least one -1 by Proposition 3.3 (2). Hence by Claim 3.5 and Proposition 3.10, Γ_{X_m} contains

$$\begin{tikzcd} 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \arrow[r, dash] & 1 \arrow[r, dash] & -1 \arrow[r, dash] & 1 \arrow[r, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \end{tikzcd}.$$

By the shape of Γ_Z, Proposition 3.3 (2) and Claim 3.5, the possibilities of Γ_{X_m} are as follows:

(i)

$$\begin{tikzcd} 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \arrow[r, dash] & 1 \arrow[r, dash] & -1 \arrow[r, dash] \arrow[d, dash] & 1 \arrow[r, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \\ & & & & 1, & & & & \end{tikzcd}$$

where the bottom 1 is the economic chain for a $\frac{1}{2}(1,1,1)$-singularity

(ii)

$$\begin{tikzcd} 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \arrow[r, dash] & 1 \arrow[r, dash] & -1 \arrow[r, dash] \arrow[d, dash] & 0 \arrow[r, dash] & 1 \arrow[r, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \\ & & & & 1, & & & & & \end{tikzcd}$$

where the bottom 1 is the economic chain for a $\frac{1}{2}(1,1,1)$-singularity, the right

$$\begin{tikzcd} 1 \arrow[r, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \end{tikzcd}$$

is the economic chain for a $\frac{1}{r}(1,-1,1)$-singularity, and 0 between -1 and 1 is a fixed vertex.

Similarly to the proof of Proposition 3.11, we see that the blow-ups along the base curves of $|-K_{X_m}|$ induce the operations of type (b1) in 3.4. So we can easily obtain Γ_Z. Then, however, W has three (resp. two) singular lines on P for (i) (resp. (ii)), a contradiction.

Step 2.

Case 1. $\#\operatorname{Sing} W_l = 1$.

In this case, $\operatorname{Sing} W_l = \{\varrho\}$. So by Proposition 7.11 (3), $\widetilde{W_l}$ has no singular curve outside $\nu_{W,l}^{-1}(\varrho)$ and hence W has a cA_1- or cA_2-singularity generically along l by Proposition 7.7 (1).

There are two cases:

(a) $\varrho \in \operatorname{Sing} C_l$.

In this case, P contains two good lines. If $\#\operatorname{Sing} W_{l'} = 1$ for the other line l', then Γ_Z is one of the following (we use the notation as in Step 1):

(a-1)

$$0 \text{ --- } 1 \text{ --- } 0$$

($r = s = 2$). By Claim 3.5, this case does not occur.

(a-2)

$$0 \text{ --- } 1 \text{ --- } 0 \text{ --- } 0$$

($r = 2, s = 3$).

(a-3)

$$0 \text{ --- } 0 \text{ --- } 1 \text{ --- } 0 \text{ --- } 0$$

($r = s = 3$).

We treat the case that $\#\operatorname{Sing} W_{l'} = 2$ for the other line l' in Case 2 below.

(b) $\varrho \notin \operatorname{Sing} C_l$.

In this case, there is no singular curve of $\widetilde{W_l}$ on $\nu_{W,l}^{-1}(\varrho)$ by Proposition 7.11 (2). Hence P is a good plane. Hence P corresponds to a $1/r\,(1, -1, 1)$-singularity for $r = 3, 4$.

Case 2. $\#\operatorname{Sing} W_l = 2$.

Let $\operatorname{Sing} W_l = \{\varrho, \varrho'\}$. By Proposition 7.11 (3), $\widetilde{W_l}$ has a cA_1-singularity along $\nu_{W,l}^{-1}(\varrho') \simeq \mathbb{P}^1$. Hence W has a cA_3-singularity generically along l by Proposition 7.7 (1).

Moreover by Proposition 7.11 (2), one of the following holds:

(i) $\varrho \in \operatorname{Sing} C_l$.

In this case, $\nu_{W,l}^{-1}(\varrho)$ contains one singular curve and the type is generically cA_1. Hence Γ_Z is

$$0 \text{ --- } 1 \text{ --- } 0 \text{ --- } 0 \text{ --- } 0$$

($r = 2, s = 4$).

(ii) $\varrho \notin \operatorname{Sing} C_l$.

In this case, $\nu_{W,l}^{-1}(\varrho)$ contains no singular curve. Hence Γ_Z is

$$1 \text{ --- } 0 \text{ --- } 0 \text{ --- } 0\,.$$

So P is a good plane corresponding to a $1/5\,(1, -1, 1)$-singularity.

□

If W has no singular line, then W contains at most two planes by [Taka02a]. If there exists at least one singular line on W, then W

contains at most two planes by Propositions 4.7 and 7.5. Hence by Proposition 7.7 (b-6) and Theorem 7.12, we obtain the possibilities of $\operatorname{Sing} X$ as summarized in the next subsection.

7.6. **Summary of the results in the case $g(X) = 6$.**
We summarize the results when X has a singularity with index ≥ 3. We emphasize the relation among the geometries of X, W and W_l for a singular line l.

Note that W has at least one singular line and W_l is factorial for any singular line l. Moreover one of the following holds:

If W_P is a cubic in $\mathbb{P}^4$ for a plane P, then (1-1) always holds. Hence for the other cases, $W_P \simeq Q^3$ for any plane P on W.

(1) $\operatorname{Sing} X$ consists of one $1/r\,(1,-1,1)$-singularity. In this case, W has the unique plane P and P is a good plane. Hence W has the unique singular line l on P.
 - (1-1) $(r = 3)$
   C_l is irreducible.
 - (1-2) $(r \geq 4)$
   - $C_l = C'_l + m$, where m is a line.
   - m is the image of one component of $E_{W,l}$.
   - $\nu_{E,l}(E_{W,l})$ is non-normal.
   - The image of P on $W_l \in \operatorname{Reg} C_l \cap \operatorname{Sing} W_l$.
   - (1-2-1) $(r = 4)$
     - $\widetilde{W_l}$ has no singular curve.
     - $\nu_{E,l}(E_{W,l})$ is not a cone.
   - (1-2-2) $(r = 5)$
     - $\widetilde{W_l}$ has only one singular curve, which is the component l' of the intersection between two components of $E_{W,l}$ dominating l. The generic type of the singularity along l' is cA_1.
     - $\nu_{W,l}(E_{W,l})$ is a cone.

(2) $\operatorname{Sing} X$ consists of a $1/r_1\,(1,-1,1)$-singularity x_1 and a $1/r_2\,(1,-1,1)$-singularity x_2 with

$$(r_1, r_2) = (2,3), (2,4) \text{ or } (3,3).$$

 - (2-1) (Two singularities x_i correspond to two good planes P_i on W.)
   - $(r_1, r_2) = (2,3)$ or $(3,3)$.
   - There is the unique singular line l_2 on P_2 and if $r_1 = 2$, then there is no singular line on P_1 (resp. if $r_1 = 3$, then there is the unique singular line l_1 on P_1).
   - $C_{l_i} = C'_{l_i} + m_i$, where m_i is a line.

- m_i is the image of P_{3-i}.
- The image of P_i on $W_{l_i} \in \operatorname{Reg} C_{l_i} \cap \operatorname{Sing} W_{l_i}$.

(2-2) (W contains only one plane P.)

- There are two good lines l_i on P and l_i corresponds to x_i. $\Gamma_{Z,P}$ is one of the following:
  $(r_1, r_2) = (2, 3)$
  $$0 — 1 — 0 — 0$$
  $(r_1, r_2) = (2, 4)$
  $$0 — 1 — 0 — 0 — 0$$
  $(r_1, r_2) = (3, 3)$
  $$0 — 0 — 1 — 0 — 0$$
- If $r_1 = 2$ (resp. $r_1 = 3$), then C_{l_1} is irreducible (resp. $C_{l_1} = C'_{l_1} + m_1$, where m_1 is a line). $C_{l_2} = C'_{l_2} + m_2$, where m_2 is a line.
- m_i is the image of one component of E_{W,l_i}.
- If $C_{l_i} = C'_{l_i} + m_i$, then $\nu_{W,l_i}(E_{W,l_i})$ is non-normal and is a cone if $(r_1, r_2) = (2, 4)$ and $i = 2$ (resp. is not a cone otherwise).
- The image of P_i on $W_{l_i} \in \operatorname{Sing} C_{l_i} \cap \operatorname{Sing} W_{l_i}$.
- Let γ be the set theoretic base locus of $|-K_X|$. Then $\gamma \simeq \mathbb{P}^1$ and γ contains x_1 and x_2.
  $$-K_X \cdot \gamma = 1/6, 1/4, 1/3$$
  if
  $$(r_1, r_2) = (2, 3), (2, 4) \text{ or } (3, 3)$$
  respectively.

8. Birational maps from X to other Mori fiber spaces

8.1. Correction of the classification of extremal contractions of $(2, 1)$-type from singular 3-folds with only $1/2\,(1, 1, 1)$-singularities. In [Taka02a, Proposition 2.2], we gave the classification of extremal contractions as in the title. But one case is missing so we give the correct statement and proof here.

Proposition 8.1. *Let X be a singular analytic 3-fold with only $1/2\,(1, 1, 1)$-singularities and $f : X \to (Y, Q)$ an extremal contraction of $(2, 1)$-type to a germ (Y, Q) (see* [Taka02a, Definition 2.0] *for the definition). Let $E := \operatorname{exc} f$ and $C := f(E)$. Assume that E is irreducible and X contains at least one singularity over Q. Then one of the following holds:*

(*A*) *C is singular at Q and* $\operatorname{mult}_Q C = 3$. *$Y$ is smooth at Q.*
(*B*) *C is smooth at Q. In this case, the description of f is exactly as in* [Taka02a, Proposition 2.2 (4)] *except* (4*e*).

Assume that X is projective. Then

$$(-K_E)^2 = 8(1 - g(\overline{C})) - 2m - 18m',$$

where $\overline{C}$ is the normalization of C and m' (*resp. m*) *is the number of fibers of type* (*A*) (*resp.* (*B*)).

Proof. [(2), ibid.] holds without any change. Hence there is a unique $1/2\,(1,1,1)$-singularity on X over Q and we denote it by P. Let $g : Z \to X$ be the blow-up at P and $F := \operatorname{exc} g$. Then by [(2), ibid.] again, $-K_Z$ is nef over Y and the flopping curves are the strict transforms of components of the f-fiber over Q. Since Z is smooth and a sequence of $(g_*^{-1}E)$-flops terminates, we see that after a finite sequence of $(g_*^{-1}E)$-flop $Z \dashrightarrow Z'$, Z' is also smooth, and there exists neither a $(g_*^{-1}E)$-flopping contraction or a flipping contraction from Z' over Y. Let E' be the strict transform of E on Z'. Since $-K_{Z'}$ is nef over Y and a general fiber of E' over Y is E'-negative by f-ampleness of $-E$, there exists an extremal ray R in $\overline{\mathrm{NE}}(Z'/Y)$ with respect to $K_{Z'}$ such that $E' \cdot R < 0$. By the choice of Z', the contraction $g' : Z' \to X'$ of R is the divisorial contraction of E'. By [Mor82, Theorem 3.3], X' is smooth. Moreover a general curve in the strict transform F' of F on X' is $K_{X'}$-negative. Hence there exists the divisorial contraction $f' : X' \to Y'$ of F'.

First we prove that $Y' = Y$. Assume the contrary. Then $Y' \to Y$ is a small contraction and the exceptional curves are the strict transforms of $(g_*^{-1}E)$-flopped curves. So there exists a flipping contraction $Y' \to Y''$ over Y since $-K_{X'}$ is f'-ample, and at least one $(g_*^{-1}E)$-flopped curve intersects E' and is not contained in E'. This is a contradiction.

Next we prove that the following case in [Mor82, Theorem 3.3] does not occur:

$$(F', -K_{X'}|_{F'}) \simeq (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)).$$

Assume that this case occurs. Let l' be the strict transform of a line on F' such that $E' \cap l' \neq \emptyset$. Since $Y' = Y$, there exist such l''s and they cover $(g')^{-1}F'$. Then we have $-K_{Z'} \cdot l' = 0$. This implies that $-K_Z \cdot l' = 0$ for general l''s and they cover F, a contradiction.

Hence we have the following possibilities:

(i) $(F', -K_{X'}|_{F'}) \simeq (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2))$.
(ii) F' is a quadric in $\mathbb{P}^3$ and $-K_{X'}|_{F'} = \mathscr{O}_{\mathbb{P}^3}(1)|_{F'}$.

Let F'' be the strict transform of F' on Z'. Now we calculate

$$(-K_{Z'} - F'')^2 F''$$

in two ways. First by

$$-K_{Z'} = (g')^*(-K_{X'}) - E' \text{ and } (E')^2(g')^*F' = -\operatorname{mult}_Q C,$$

we have

$$(-K_{Z'} - F'')^2 F'' = (-K_{F'})^2 - \operatorname{mult}_Q C.$$

On the other hand, by

$$(F'')^3 = (F')^3,\ (-K_{Z'})^2 F'' = (-K_Z)^2 F \text{ and } (-K_{Z'})(F'')^2 = (-K_Z)F^2,$$

we have

$$(-K_{Z'} - F'')^2 F'' = (-K_F)^2 - \{F^3 - (F')^3\} = 5 + (F')^3.$$

Hence for (i), $\operatorname{mult}_Q C = 3$ and for (ii), $\operatorname{mult}_Q C = 1$. This gives the descriptions (A) and (B). The calculation of $(-K_E)^2$ is easy so we omit it. $\square$

Remark. In [Taka02a], the contraction of this type appears as f' in the case that $h^0(-K_X) = 4$ and $z = u = 1$ (see [p.32, ibid.]). The formulas on this page are correct except (5-1-4$'$). (5-1-4$'$) should be replaced by

$$(-K_{X'})^3 - 2 - 16n = 8g(\overline{C}) + 2m + 18m'.$$

However by (5-1-2$'$), we have $(-K_{X'})^3 \leq 16$ so $m' = 0$, i.e., there is no fiber of type (A). By this consideration, we see that the tables in [Taka02a] are correct.

8.2. **Description of birational maps.**

During the proof of the main theorem, we obtained rational maps from X to another Mori fiber spaces. We re-describe them from the view point of the minimal model program. We use the notation in previous sections (for example, in 2.12).

Let x be a $1/r\,(1,-1,1)$-singularity on X with the highest index. Let $f : Y \to X$ be the standard weighted blow-up at x and $E := \operatorname{exc} f$. Let P_x be the plane such that the strict transforms of P_x and E belong to the same connected component of $\operatorname{exc} g$ (P_x is well-defined by Proposition 2.13 (3)). Then we have the following diagram:

$$\begin{tikzcd} & Y \arrow[r, dashed] \arrow[dl, "f"'] & Y' \arrow[dr, "f'"] & \\ X & & & X', \end{tikzcd}$$

satisfying

(1) $Y \dashrightarrow Y'$ is one flop or a composite of one flop $Y \dashrightarrow Y'_1$ and one flip $Y'_1 \dashrightarrow Y'$. See the below more precise descriptions.

(2) For the case that $g(X) = 8$ and W contains a $\sigma_{3,3}$-plane, $X' \simeq \mathbb{P}^2$ and f' is a $\mathbb{P}^1$-bundle. For the other cases, f' is of $(2,1)$-type, and one of the following holds:
  (2-1) $(r = 2)$ W_P is a factorial del Pezzo 3-fold of degree $g(X) - 3$, or $g(X) = 6$ and $W_P \simeq Q^3$. In this case, $X' \simeq W_P$ and the center of f' is C'_P, or
  (2-2) $(r \geq 3)$. There is a singular line l_x corresponding to x, and W_{l_x} is a factorial del Pezzo 3-fold of degree $g(X) - 2$. In this case, $X' \simeq W_{l_x}$ and the center of f' is C'_{l_x}. We denote by ϱ the image of P_x on W_{l_x}.

  See the below for the descriptions of f'.

(3) Assume that $X' \simeq W_\Delta$ ($\Delta = P_x$ or l_x). Then the following hold:
  (3-1) $Y \dashrightarrow Y'$ contains a flip if and only if $C_\Delta = C'_\Delta + m$, where m is a line. Moreover if this is the case, then m is the image of the flipped curve.
  (3-2) Let $E'_{W,\Delta}$ be $E_{W,\Delta}$ if $r = 2$ (resp. the irreducible component of $E_{W,\Delta}$ such that $E'_{W,\Delta} \cap P_x$ does not contain a curve dominating l_x if $r \geq 3$). Then $E'_{W,\Delta}$ is the strict transform of E.

Descriptions of $Y \dashrightarrow Y'$ and f'.

The statements below with $(*)$ are verified by playing 2-ray game as in [Taka02a].

X has one $1/2\,(1,1,1)$-singularity.

(a) $(*)$ W_P is smooth.
(b) $Y = Z$ and $Y' = \widetilde{W}_P$. $h : Z \to W$ and $\mu_{W,P} : \widetilde{W}_P \to W$ are flopping contractions, and hence $Y \dashrightarrow Y'$ is a flop.
(c) $f' = \nu_{W,P}$. $\nu_{W,P}$ is the blow-up of W_P along C_P, where

$$(*)\ C_P \text{ is smooth, and}$$

$$\begin{cases} \deg C_P = 2g(X) - 9, \text{ and} \\ g(C_P) = g(X) - 6 \text{ (with } (*) \text{ if } g(X) = 8) \end{cases}$$

if W_P is a del Pezzo 3-fold (resp.

$$\begin{cases} \deg C_P = 9, \text{ and} \\ (*)\ g(C_P) = 6 \end{cases}$$

if $g(X) = 6$ and $W_P \simeq Q^3$). In particular, Y' is smooth.

X has two $1/2\,(1,1,1)$-singularities.

(a) $g(X) = 6, 8$.
(b) $Y \dashrightarrow Y'$ is a composite of a flop $Y \dashrightarrow Y'_1$ and a flip $Y'_1 \dashrightarrow Y'$.

(c) f' is the blow-up of W_P along C'_P. If $g(X)=8$, then

$$\begin{cases} \deg C'_P = 6 \text{ and} \\ C'_P \simeq \mathbb{P}^1. \end{cases}$$

If $g(X)=6$ and $W_P \simeq Q^3$, then

$$\begin{cases} (*)\ C'_P \text{ is smooth,} \\ \deg C'_P = 8, \text{ and} \\ (*)\ p_a(C_P) = 3. \end{cases}$$

In particular, Y' is smooth.

We have $g(X) \le 7$ below.

<u>X has one $1/3\,(1,-1,1)$-singularity.</u>

(a) $Y \dashrightarrow Y'$ is a flop.

(b) (b-1) If $g(X)=7$, then W_{l_x} is smooth, and f' is an extremal contraction of type (A) as in Proposition 8.1.

$$\begin{cases} C_{l_x} \text{ is irreducible,} \\ \deg C_{l_x} = 8, \text{ and} \\ (*)\ p_a(C_{l_x}) = 2. \end{cases}$$

ϱ is the unique singularity of C_{l_x} and $\operatorname{mult}_{\varrho} C_{l_x} = 3$.

(b-2) If $g(X)=6$, then W_{l_x} is singular, and f' is an extremal contraction of type (B) as in Proposition 8.1.

$$\begin{cases} (*)\ C_{l_x} \text{ is smooth,} \\ \deg C_{l_x} = 6 \text{ and} \\ (*)\ p_a(C_{l_x}) = 2. \end{cases}$$

$\operatorname{Sing} W_{l_x} = \{\varrho\}$ and $\varrho \in C_{l_x}$.

<u>X has one $1/4\,(1,-1,1)$-singularity.</u>

(a) $Y \dashrightarrow Y'$ is a composite of a flop $Y \dashrightarrow Y'_1$ and a flip $Y'_1 \dashrightarrow Y'$. By the flip, the $1/3\,(1,-1,1)$-singularity on Y'_1 disappears and a Gorenstein terminal singularity y' appears on Y' such that $f(y') = \varrho$.

(b) f' is the blow-up of W_{l_x} along C'_{l_x} with

$$\begin{cases} \deg C'_{l_x} = 2g(X) - 7 \text{ and} \\ p_a(C'_{l_x}) = g(X) - 6. \end{cases}$$

(b-1) If $g(X)=7$, then W_{l_x} is smooth. ϱ is the unique singularity of C'_{l_x} and $\operatorname{mult}_{\varrho} C'_{l_x} = 2$.

(b-2) If $g(X)=6$, then $\varrho \in \operatorname{Sing} W_{l_x}$ and C'_{l_x} is smooth.

(c) $\nu_{W,l_x}(E_{W,l_x})$ is non-normal and is not a cone.

The following hold below:

- $g(X) = 6$.
- f' is an extremal contraction of type (B) as in Proposition 8.1.
- $$\begin{cases} \deg C'_{l_x} = 5 \text{ and} \\ C'_{l_x} \simeq \mathbb{P}^1. \end{cases}$$

<u>X has one $1/5\,(1,-1,1)$-singularity.</u>

(a) $Y \dashrightarrow Y'$ is a composite of a flop $Y \dashrightarrow Y'_1$ and a flip $Y'_1 \dashrightarrow Y'$. By the flip, the $1/4\,(1,-1,1)$-singularity on Y'_1 disappears and a $1/2\,(1,1,1)$-singularity y' appears on Y' such that $f(y') = \varrho$.

(b) $\operatorname{Sing} W_{l_x} = \{\varrho\}$ and $\varrho \in C'_{l_x}$.

(c) $\nu_{W,l_x}(E_{W,l_x})$ is a non-normal cone.

<u>X has two singularities, one of which is not a $1/2\,(1,1,1)$-singularity.</u>

(a) one of the following holds:
  (a-1) X has one $1/2\,(1,1,1)$-singularity and one $1/3\,(1,-1,1)$-singularity,
  (a-2) X has one $1/2\,(1,1,1)$-singularity and one $1/4\,(1,-1,1)$-singularity, or
  (a-3) X has two $1/3\,(1,-1,1)$-singularities.

(b) $Y \dashrightarrow Y'$ is a composite of a flop $Y \dashrightarrow Y'_1$ and a flip $Y'_1 \dashrightarrow Y'$. By the flip, a $1/2\,(1,1,1)$-singularity on Y'_1 disappears for (a-1) and the $1/3\,(1,-1,1)$-singularity on Y'_1 disappears and a $1/2\,(1,1,1)$-singularity appears on Y' for (a-2) and (a-3).

(c) $\varrho \in \operatorname{Sing} W_{l_x}$ and $\operatorname{Sing} W_{l_x} \subset C'_{l_x}$. W_{l_x} has one singularity for (a-1) and two singularities for (a-2) and (a-3).

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