Representations of the Cuntz algebra $\mathcal{O}_2$ arising from real quadratic transformations

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Abstract

We construct representations of the Cuntz algebra $\mathcal{O}_2$ from real quadratic transformations of 1-dimensional dynamical systems. By intertwining relations among transformations, we derive relations among operators associated with representations of $\mathcal{O}_2$. From these relations, we show properties of these representations.

1 Introduction

In the theory of dynamical system, 1-dimensional dynamical systems of real quadratic functions on closed intervals are well known. In [3], such system is introduced as a typical example of chaotic dynamical system. The key point of this orbit analysis in [3] is some *intertwining relation* between a quadratic transformation and some piecewise linear transformation *but not conjugation between them.*

On the other hand, our interests are the construction of representations of Cuntz algebras and their analysis([1, 4, 8]). For these intentions, we construct a representation from a 1-dimensional dynamical system by a quadratic transformation as follows: Consider a real quadratic transformation Q on a closed interval $[-2,2]$ by

$$Q:[-2,2]\to[-2,2];\quad Q(x)\equiv x^2-2. \tag{1.1}$$

We define a representation $(L_2[-2,2],\pi_q)$ of the Cuntz algebra $\mathcal{O}_2$ arising from Q by

$$(\pi_q(s_i)\phi)(x)=m_i(x)\phi(Q(x))\quad (i=1,2,\ \phi\in L_2[-2,2]) \tag{1.2}$$

where $m_i(x)=\chi_{D_i}(x)\sqrt{2|x|}$, χ_{D_i} is the characteristic function on D_i, $i=1,2$, $D_1=[0,2]$, $D_2=[-2,0]$ and s_1,s_2 are generators of $\mathcal{O}_2$.

Theorem 1.1 *Let $(L_2[-2,2],\pi_q)$ be the representation of $\mathcal{O}_2$ defined in (1.2).*

(i) *Put a function*

$$\Omega(x) \equiv \frac{1}{\sqrt{\pi}} \frac{1}{(4-x^2)^{1/4}} \quad (x \in [-2,2]).$$

Then Ω is unique eigen vector in $L_2[-2,2]$ of an operator $\pi_q(s_1) + \pi_q(s_2)$ up to scalar multiples and its eigen value is $\sqrt{2}$.

(ii) *$(L_2[-2,2], \pi_q)$ is unitarily equivalent to one of the GP representation of $\mathcal{O}_2$ in [4].*

(iii) *$(L_2[-2,2], \pi_q)$ is irreducible.*

We introduce several representations of Cuntz algebras in section 2. In section 3, we show intertwining relations of transformations of dynamical systems and derive relations of operators from them. By using these relations, we show Theorem 1.1.

2 Representations of the Cuntz algebra

In this section we review some basic facts on the Cuntz algebras and their representations on measure spaces. For $N \geq 2$, a C^*-algebra with generators $s_1, \ldots, s_N$ which satisfy the following relations([2])

$$s_i^* s_j = \delta_{ij} I \quad (i, j = 1, \ldots, N), \qquad \sum_{i=1}^{N} s_i s_i^* = I \tag{2.1}$$

is called *the Cuntz algebra* and is denoted by $\mathcal{O}_N$. $\mathcal{O}_N$ is non commutative, infinite dimensional, separable, simple and unique up to isomorphisms. Hence there is no finite dimensional representation of $\mathcal{O}_N$ except 0-representation.

2.1 GP representations of $\mathcal{O}_N$ with 1-cycle

We show only the 1-cycle case about GP representations of Cuntz algebras with cycle in [4]. The GP representation is used to characterize representation introduced in § 2.3. In this paper, a representation always means a unital *-representation on a complex Hilbert space.

Let $S(\mathbf{C}^N) \equiv \{z \in \mathbf{C}^N : \|z\| = 1\}$ be the complex sphere in a complex vector space $\mathbf{C}^N$.

Definition 2.1 *$(\mathcal{H},\pi,\Omega)$ is the GP(= generalized permutative) representation of $\mathcal{O}_N$ by $z=(z_1,\ldots,z_N)\in S(\mathbf{C}^N)$ if $(\mathcal{H},\pi)$ is a cyclic representation of $\mathcal{O}_N$ and Ω is the unit cyclic vector which satisfies the following equation:*

$$\pi(z_1s_1+\cdots+z_Ns_N)\Omega=\Omega. \tag{2.2}$$

We denote $GP(z)\equiv(\mathcal{H},\pi,\Omega)$.

For two representations $(\mathcal{H}_1,\pi_1)$ and $(\mathcal{H}_2,\pi_2)$ of $\mathcal{O}_N$, $(\mathcal{H}_1,\pi_1)\sim(\mathcal{H}_2,\pi_2)$ means the unitary equivalence between $(\mathcal{H}_1,\pi_1)$ and $(\mathcal{H}_2,\pi_2)$.

Theorem 2.2 *(Properties of GP representation with 1-cycle)*

(i) *(Existence and Uniqueness) For any $z\in S(\mathbf{C}^N)$, $GP(z)$ exists uniquely up to unitary equivalence.*

(ii) *(Irreducibility) For any $z\in S(\mathbf{C}^N)$, $GP(z)$ is irreducible.*

(iii) *(Equivalence) For any $z,z'\in S(\mathbf{C}^N)$, $GP(z)\sim GP(z')$ if and only if $z=z'$.*

(iv) *(Uniqueness of eigen vector) For $z\in S(\mathbf{C}^N)$, Ω in (2.2) is unique up to scalar multiples.*

Proof. (i) Proposition 3.4 and Proposition 5.4 in [4]. (ii), (iii) Example 6.3 (ii) in [4]. (iv) Corollary 5.3 in [4]. ∎

We show examples of GP representation. Let $l_2(\mathbf{N})$ be a Hilbert space with the canonical basis $\{e_n : n\in\mathbf{N}\}$ where $\mathbf{N}=\{1,2,3,\ldots\}$. We make the following representation $(l_2(\mathbf{N}),\pi_S)$ of the Cuntz algebra $\mathcal{O}_N$ which is called *the standard representation of* $\mathcal{O}_N$:

$$\pi_S(s_i)e_n\equiv e_{N(n-1)+i}\quad(i=1,\ldots,N,\,n\in\mathbf{N}). \tag{2.3}$$

From this definition, we have $\pi_S(s_i)^*e_{N(n-1)+j}=\delta_{ij}e_n$ for $i,j=1,\ldots,N$ and $n\in\mathbf{N}$. Note that $(l_2(\mathbf{N},\pi_S)$ is one of permutative representations of $\mathcal{O}_N$ by [1]. Since $\pi_S(s_1)e_1=e_1$, $(l_2(\mathbf{N}),\pi_S,e_1)$ is $GP(z)$ for $z=(1,0,\ldots,0)\in S(\mathbf{C}^N)$ and irreducible.

More about details for GP representations, see [4, 5, 6, 7].

2.2 Isometries arising from transformations on measure spaces

By simplicity and uniqueness of $\mathcal{O}_N$, it is sufficient to define operators $S_1, \ldots, S_N$ on an infinite dimensional Hilbert space which satisfy (2.1) in order to construct a representation of $\mathcal{O}_N$.

We introduce an easy method to construct partial isometries from maps on a measure space.

Let (X, μ) be a measure space and $Y \subset X$ a measurable subset of X.

Definition 2.3 (i) *$RN(Y, X)$ is the set of measurable maps on Y defined by*

$$RN(Y, X) \equiv \{f : Y \to X \mid \textit{there exists } \Phi_f \textit{ and } \Phi_f > 0 \textit{ a.e. } Y\}$$

where Φ_f is the Radon-Nikodým derivative of $\mu \circ f$ with respect to μ on Y.

(ii) *$RN_{loc}(X) \equiv \bigcup_{Y \subset X} RN(Y, X)$ where Y is taken from all measurable subsets of X.*

For $f, g \in RN_{loc}(X)$, we denote the domain and the range of f by $D(f)$ and $R(f)$, respectively and $(f \circ g)(x) \equiv f(g(x))$ for $x \in D(g)$ when $D(f) \supset R(g)$.

Lemma 2.4 (i) *If $f \in RN_{loc}(X)$, then $f^{-1} \in RN_{loc}(X)$.*

(ii) *For $f, g \in RN_{loc}(X)$, $f \circ g \in RN_{loc}(X)$ when $D(f) \supset R(g)$.*

(iii) *$\Phi_{f \circ g} = \Phi_g \cdot ((\Phi_f) \circ g)$.*

Proof. These are easily checked by direct computation and property of Radon-Nikodým derivative. ∎

Note that $RN_{loc}(X)$ is a groupoid by Lemma 2.4 (ii).

Definition 2.5 *For $f \in RN_{loc}(X)$ define an operator $S(f) : L_2(X, \mu) \to L_2(X, \mu)$ by*

$$(S(f)\phi)(x) \equiv \begin{cases} \left\{\Phi_f\left(f^{-1}(x)\right)\right\}^{-1/2} \phi(f^{-1}(x)) & (\, x \in R(f)\,), \\ 0 & (\textit{ otherwise }) \end{cases} \tag{2.4}$$

for $\phi \in L_2(X, \mu)$ and $x \in X$.

We simply denote

$$S(f) = \mathcal{L}_f M_{\Phi_f^{-1/2}} \tag{2.5}$$

where M_g is the multiplication operator of $g \in L_\infty(X,\mu)$ defined by

$$(\mathcal{L}_f\phi)(x) \equiv \chi_{R(f)}(x)\phi(f^{-1}(x)) \quad (x \in X)$$

and χ_Y is the characteristic function on $Y \subset X$.

Lemma 2.6 (i) *For $f \in RN_{loc}(X)$, $S(f)$ is a partial isometry on $L_2(X,\mu)$ with the initial projection $M_{\chi_{D(f)}}$ and the range projection $M_{\chi_{R(f)}}$.*

(ii) *For $f \in RN_{loc}(X)$, $S(f)^* = S(f^{-1})$.*

(iii) $S(id_Y) = M_{\chi_Y}$.

(iv) *For $f, g \in RN_{loc}(X)$, $\mathcal{L}_f\mathcal{L}_g = \mathcal{L}_{f\circ g}$ when $D(f) \supset R(g)$.*

(v) *For $f \in RN_{loc}(X)$ and $g \in L_\infty(X)$, $\mathcal{L}_f M_g = M_{\chi_{R(f)}} M_{g\circ f^{-1}}\mathcal{L}_f$.*

Proof. (i), (iv) and (v) follow by simple computation. (ii) Since $R(f^{-1}) = D(f)$, $D(f^{-1}) = R(f)$ and the property of Radon-Nikodým derivative, it follows. (iii) Since $\Phi_{id_Y} = \chi_Y$, it follows. ∎

Let $\mathrm{PIso}(L_2(X,\mu))$ be the groupoid of partial isometries on $L_2(X,\mu)$ by usual product of operators.

Lemma 2.7 *A map $S : RN_{loc}(X) \to \mathrm{PIso}(L_2(X,\mu))$ is a groupoid homomorphism, that is,*

$$S(f)S(g) = S(f\circ g) \tag{2.6}$$

when $f, g \in RN_{loc}(X)$ and $D(f) \supset R(g)$.

Proof. By Lemma 2.4 (iii) and Lemma 2.6 (iv),(v),

$$S(f)S(g) = \mathcal{L}_f M_{\Phi_f^{-1/2}}\mathcal{L}_g M_{\Phi_g^{-1/2}} = \mathcal{L}_{f\circ g} M_{((\Phi_f)\circ g)^{-1/2}} M_{\Phi_g^{-1/2}} = S(f\circ g).$$

∎

Remark that $f\circ g$ in rhs of (2.6) is only the composition of two maps f and g but not special product of them.

2.3 Representations of Cuntz algebra on a measure space

The notion of branching function system was introduced in [1] in order to construct a representation of $\mathcal{O}_N$ from a family of maps. We introduce a branching function system to define the representation from a dynamical system in section 3.

Let $N \geq 2$ and (X, μ) a measure space.

Definition 2.8 *$f = \{f_i\}_{i=1}^N$ is a branching function system over (X, μ) if $f_i \in RN_{loc}(X)$ and $D(f_i) = X$, put $R_i \equiv R(f_i)$, then $\mu(R_i \cap R_j) = 0$, $i \neq j$ and $\mu\left(X \setminus \bigcup_{i=1}^N R_i\right) = 0$.*

Proposition 2.9 *For a branching function system $f = \{f_i\}_{i=1}^N$ on (X, μ),*

$$\pi_f(s_i) \equiv S(f_i) \quad (i = 1, \ldots, N),$$

defines a representation $(L_2(X, \mu), \pi_f)$ of $\mathcal{O}_N$ where S is the map in Definition 2.5.

Proof. It is straightforward to show that $S(f_1), \ldots, S(f_N)$ satisfy the relations (2.1) by Lemma 2.6, Lemma 2.7 and Definition 2.8. ∎

We show several examples of branching function systems associated with 1-dimensional dynamical systems in [8].

Lemma 2.10 *Let (Y, ν) be another measure space. Assume that $\varphi : X \to Y$ is a measurable bijection a.e. X and Y and its Radon-Nikodým derivative Φ_φ is positive a.e. X.*

(i) *An operator $U_\varphi : L_2(X, \mu) \to L_2(Y, \nu)$ defined by $U_\varphi \equiv \mathcal{L}_\varphi M_{\Phi_\varphi^{1/2}}$ is a unitary.*

(ii) *For $f \in RN_{loc}(X)$, $\varphi \circ f \circ \varphi^{-1} \in RN_{loc}(Y)$ and $U_\varphi S(f) U_\varphi^* = S(\varphi \circ f \circ \varphi^{-1})$.*

(iii) *If $f = \{f_i\}_{i=1}^N$ is a branching function system over (X, μ), then $\{\varphi \circ f_i \circ \varphi^{-1}\}_{i=1}^N$ is a branching function system over (Y, ν), too.*

Proof. These are easily proved by direct computation. ∎

Proposition 2.11 *Let $f = \{f_i\}_{i=1}^N$ and $g = \{g_i\}_{i=1}^N$ be branching function systems over measure spaces (X, μ) and (Y, ν), respectively. Assume that there is a map $\varphi : X \to Y$ which satisfies the assumption in Lemma 2.10 and a map identity $g_i = \varphi \circ f_i \circ \varphi^{-1}$ holds for $i = 1, \ldots, N$. Then $(L_2(X, \mu), \pi_f)$ and $(L_2(Y, \nu), \pi_g)$ are unitarily equivalent.*

Proof. By Lemma 2.10 (ii), we can show $S(g_i) = U_\varphi S(f_i) U_\varphi^*$ for $i = 1, \ldots, N$. These relations induce a unitary equivalence between π_f and π_g in Proposition 2.9 immediately. ∎

3 Operator relations arising from intertwiner among transformations in dynamical systems

Under the preparation in section 2, we show the property of representation of $\mathcal{O}_2$ arising from a quadratic transformation. The first, we consider intertwining relations among transformations of dynamical systems. Next we derive relations of operator on a Hilbert space which are associated with representations of $\mathcal{O}_2$. The properties (2.6) of the map S plays important role in this procedure.

3.1 Intertwining relations of transformations

For study of dynamical system, the strongest equivalence relation between two systems is the conjugation([3]). For two dynamical systems (X, F) and (Y, G) are *conjugate* if there is a bijective map $\varphi : X \to Y$ such that $G = \varphi \circ F \circ \varphi^{-1}$ with some conditions (continuity, measurability, conservation of measure, differentiability and so on) of φ depending on the aim of study and class of dynamical systems. φ is called *a conjugacy.*

In spite that we do not know the existence of conjugation map between two dynamical systems, if we have an intertwiner $\varphi : X \to Y$ between F and G, that is, $G \circ \varphi = \varphi \circ F$, (where φ is not always injective), then it is useful to consider relations between (X, F) and (Y, G) in some situation. φ is called *a semiconjugacy*([3]).

In order to introduce intertwining relations of transformations associated with Q, we prepare two transformations except Q in (1.1) according to § 10.2 in [3]. Put two transformations

$$V, C : [-2, 2] \to [-2, 2]; \quad V(x) \equiv 2|x| - 2, \quad C(x) \equiv -2\cos\frac{\pi}{2}x, \tag{3.1}$$

and $D_1 \equiv [0, 2]$, $D_2 \equiv [-2, 0]$. Then every restriction $V|_{D_i}, C|_{D_i}, Q|_{D_i} : D_i \to [-2, 2]$ is invertible for each $i = 1, 2$. Hence we can take inverse transformations $v_i, c_i, q_i : [-2, 2] \to D_i$ by

$$v_i \equiv (V|_{D_i})^{-1}, \quad c_i \equiv (C|_{D_i})^{-1}, \quad q_i \equiv (Q|_{D_i})^{-1} \tag{3.2}$$

for each $i = 1, 2$. Concretely we have the followings:

$$\begin{cases} v_1(x) = & \frac{1}{2}x + 1, \\ v_2(x) = & -\frac{1}{2}x - 1, \end{cases} \quad \begin{cases} c_1(x) = & \frac{2}{\pi}\arccos(-\frac{1}{2}x), \\ c_2(x) = & -\frac{2}{\pi}\arccos(-\frac{1}{2}x), \end{cases} \quad \begin{cases} q_1(x) = & \sqrt{x+2}, \\ q_2(x) = & -\sqrt{x+2}. \end{cases}$$

Lemma 3.1 $\{v_1, v_2\}, \{c_1, c_2\}, \{q_1, q_2\}$ *are branching function systems on* $[-2, 2]$ *for the case* $N = 2$ *with respect to Lebesgue measure in Definition 2.8.*

Proof. Note

$$\Phi_{v_i}(x) = \frac{1}{2}, \quad \Phi_{c_i}(x) = \frac{2}{\pi}\frac{1}{\sqrt{4-x^2}}, \quad \Phi_{q_i}(x) = \frac{1}{2}\frac{1}{\sqrt{x+2}} \tag{3.3}$$

for $i = 1, 2$ and $x \in [-2, 2]$. Hence every Radon-Nikodým derivative of v_i, c_i, q_i is positive a.e. $[-2, 2]$. Other conditions in Definition 2.8 are easily checked. ∎

Remark that the following intertwining relation holds on $[-2, 2]$:

$$C \circ V = Q \circ C. \tag{3.4}$$

(3.4) follows immediately from the double angle's formula and periodicity of cosine function. C is a semiconjugacy between V and Q but not a conjugacy because C is not injective on $[-2, 2]$. We derive several identities among v_i, c_i, q_i by dividing the domain of (3.4).

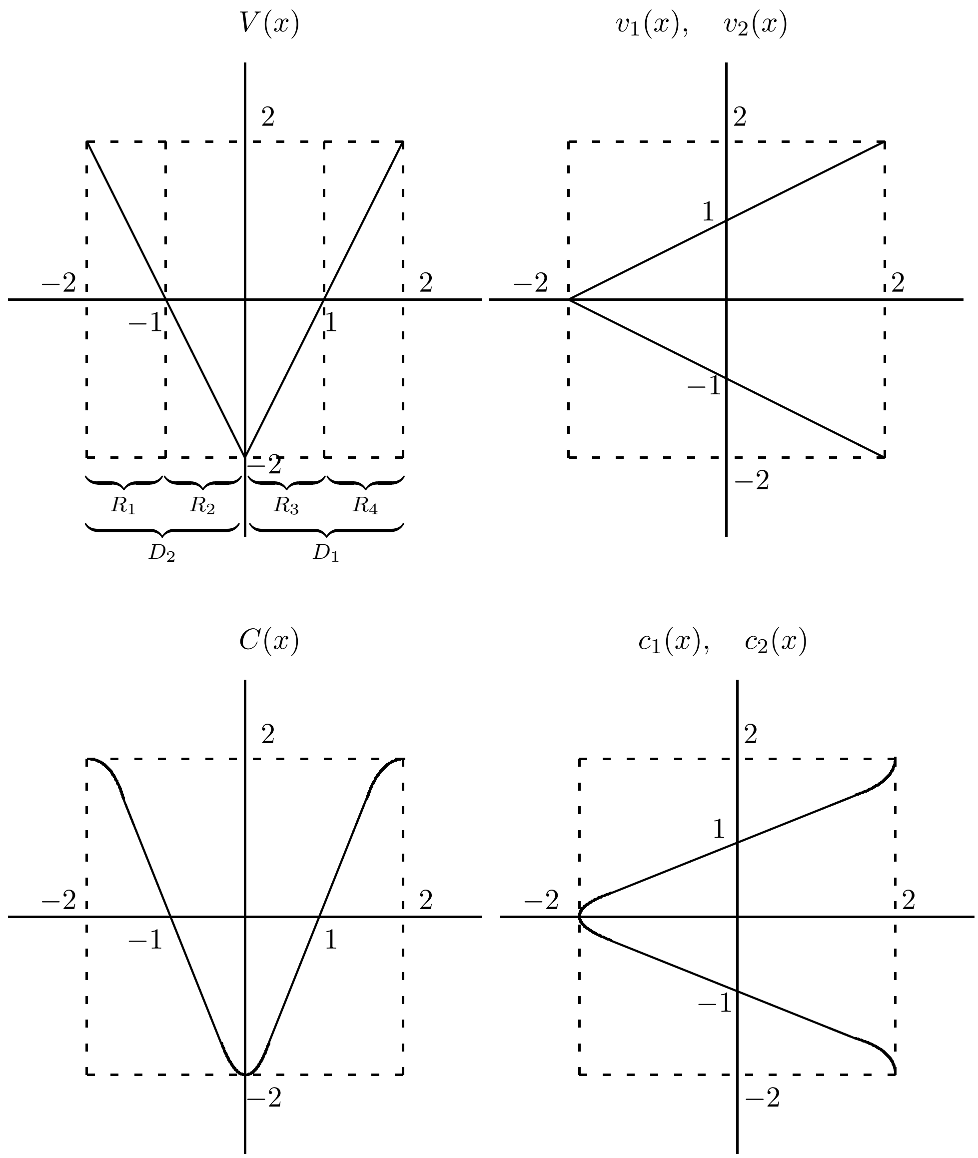
Let R_i be a subinterval of $[-2, 2]$ defined by

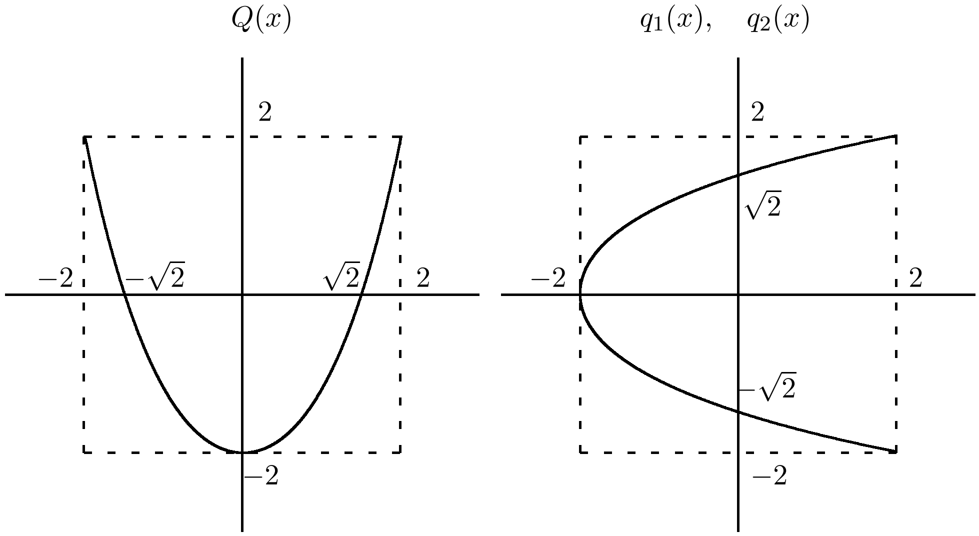
$$R_i \equiv [-2 + (i-1),\ -2 + i] \quad (i = 1, 2, 3, 4).$$

Then both sides of (3.4) are injective on R_i. V and C map R_i's as follows:

$$V(R_i) = C(R_i) = D_1, \quad V(R_j) = C(R_j) = D_2 \quad (i = 1, 4,\ j = 2, 3)$$

where we note $D_1 = R_3 \cup R_4, \quad D_2 = R_1 \cup R_2$.


$V(x)$
$v_1(x), \quad v_2(x)$
$C(x)$
$c_1(x), \quad c_2(x)$
2
1
−1
−2
R_1
R_2
R_3
R_4
D_2
D_1



Lemma 3.2 *The following relations hold for v_i, c_i, q_i in (3.2):*

$$v_2 \circ c_1 = c_2 \circ q_1, \quad v_1 \circ c_2 = c_1 \circ q_2, \quad v_2 \circ c_2 = c_2 \circ q_2, \quad v_1 \circ c_1 = c_1 \circ q_1. \tag{3.5}$$

Proof. By (3.4), we have

$$(C \circ V)|_{R_i} = (Q \circ C)|_{R_i} \tag{3.6}$$

for each $i = 1, 2, 3, 4$. When $i = 1$, $(C \circ V)|_{R_1} = (C \circ v_2^{-1})|_{R_1} = (C)|_{D_2} \circ (v_2^{-1})|_{R_1} = c_2^{-1} \circ (v_2^{-1})|_{R_1}$. In the same way, $(Q \circ C)|_{R_1} = q_2^{-1} \circ (c_2^{-1})|_{R_1}$. Therefore $c_2^{-1} \circ (v_2^{-1})|_{R_1} = q_2^{-1} \circ (c_2^{-1})|_{R_1}$. By taking inverse of both sides, we get the third identity in (3.5). In the same way, we get others by checking in cases $i = 2, 3, 4$ for (3.6). ∎

By Lemma 3.2, we have the following relations:

$$v_1 = \begin{cases} c_1 \circ q_1 \circ c_1^{-1} & (\text{ on } D_1), \\ c_1 \circ q_2 \circ c_2^{-1} & (\text{ on } D_2), \end{cases} \quad v_2 = \begin{cases} c_2 \circ q_1 \circ c_1^{-1} & (\text{ on } D_1), \\ c_2 \circ q_2 \circ c_2^{-1} & (\text{ on } D_2). \end{cases} \tag{3.7}$$

3.2 Relations of operators arising from a semiconjugacy

By Lemma 3.1 and Proposition 2.9, we have three representations π_v, π_c, π_q of $\mathcal{O}_2$ on $L_2[-2, 2]$ associated with three branching function systems $v =$

$\{v_1, v_2\}$, $c = \{c_1, c_2\}$, $q = \{q_1, q_2\}$. By definition of π_v, π_c, π_q, they are given as couples of operators $\{S(v_1), S(v_2)\}$, $\{S(c_1), S(c_2)\}$, $\{S(q_1), S(q_2)\}$ on $L_2[-2, 2]$, and they are concretely obtained as follows:

$$\begin{cases} (S(v_1)\phi)(x) = & 2^{1/2}\chi_{[0,2]}(x)\phi(2x-2), \\ (S(v_2)\phi)(x) = & 2^{1/2}\chi_{[-2,0]}(x)\phi(-2x-2), \end{cases}$$

$$\begin{cases} (S(c_1)\phi)(x) = & \sqrt{\frac{\pi}{2}\sin\frac{\pi}{2}|x|}\,\chi_{[0,2]}(x)\phi(-2\cos(\pi/2)x), \\ (S(c_2)\phi)(x) = & \sqrt{\frac{\pi}{2}\sin\frac{\pi}{2}|x|}\,\chi_{[-2,0]}(x)\phi(-2\cos(\pi/2)x), \end{cases} \tag{3.8}$$

$$\begin{cases} (S(q_1)\phi)(x) = & 2^{1/2}|x|^{1/2}\chi_{[0,2]}(x)\phi(x^2-2), \\ (S(q_2)\phi)(x) = & 2^{1/2}|x|^{1/2}\chi_{[-2,0]}(x)\phi(x^2-2) \end{cases}$$

for $\phi \in L_2[-2, 2]$ and $x \in [-2, 2]$ where we use simple notation like (2.5) instead of (2.4).

By Lemma 2.7, (3.7) and Lemma 2.6 (ii), we have the followings:

$$\begin{cases} S(v_1) = & S(c_1)S(q_1)S(c_1)^* + S(c_1)S(q_2)S(c_2)^*, \\ S(v_2) = & S(c_2)S(q_1)S(c_1)^* + S(c_2)S(q_2)S(c_2)^*. \end{cases} \tag{3.9}$$

Note that (3.9) is essentially derived from (3.4). Let

$$A_1 \equiv S(v_1) + S(v_2), \quad A_2 \equiv S(q_1) + S(q_2), \quad B \equiv S(c_1) + S(c_2)$$

and $\Omega_0 \equiv B^*\mathbf{1}$ where $\mathbf{1}$ is the constant function on $[-2, 2]$ with value 1.

Lemma 3.3 (i) $A_1\mathbf{1} = \sqrt{2}\mathbf{1}$. (ii) $A_2\Omega_0 = \sqrt{2}\Omega_0$.

Proof. (i) By definition of $S(v_i)$, $S(v_i)\mathbf{1} = \sqrt{2}\chi_{D_i}$ for $i = 1, 2$. Hence the statement holds. (ii) Since $S(c_1)^*\mathbf{1} = S(c_2)^*\mathbf{1}$, $B^*\mathbf{1} = 2S(c_i)^*\mathbf{1}$ for $i = 1, 2$,

$$A_2\Omega_0 = (S(q_1)B^* + S(q_2)B^*)\mathbf{1} = 2(S(q_1)S(c_1)^* + S(q_2)S(c_2)^*)\mathbf{1}.$$

Note $B^*A_1 = 2(S(q_1)S(c_1)^* + S(q_2)S(c_2)^*)$ by (2.1) with respect to $S(c_1), S(c_2)$. Hence $A_2\Omega_0 = B^*A_1\mathbf{1} = \sqrt{2}B^*\mathbf{1} = \sqrt{2}\Omega_0$. ∎

Let Ω be the normalization of Ω_0 and denote $q_I = q_{i_1} \circ \cdots \circ q_{i_k}$ for a multiindex $I = (i_1, \ldots, i_k) \in \{1, 2\}^k$, $k \geq 1$.

Lemma 3.4 *For* $I \in \{1,2\}^k$, $k \ge 1$, $(S(q_I)\Omega)(x) = \chi_{D_I}(x) 2^{k/2}\Omega(x)$ *where* $D_I \equiv q_I([-2,2])$.

Proof. Since $q_i \circ c_i^{-1} = c_i^{-1} \circ v_i$, $S(q_i)\Omega = \chi_{D_i}\sqrt{2}\Omega$ for $i = 1,2$. For any measurable subset $Y \subset [-2,2]$, $S(q_i) M_{\chi_Y} = M_{\chi_{q_i(Y)}} S(q_i)$ for $i = 1,2$ by Lemma 2.6 (v). Hence

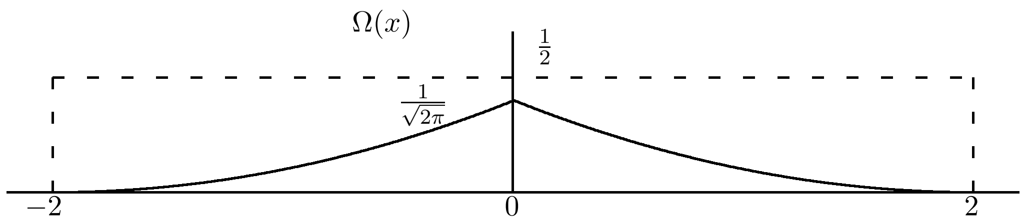
$$S(q_I)\Omega = 2^{1/2} S(q_{I'}) M_{\chi_{D_{i_k}}} \Omega = (2^{1/2})^2 S(q_{I''}) M_{\chi_{D_{(i_{k-1},i_k)}}} \Omega = \cdots = 2^{k/2} M_{\chi_{D_I}} \Omega$$

when $I = (i_1, \ldots, i_k)$ where $I' = (i_1, \ldots, i_{k-1})$ and $I'' = (i_1, \ldots, i_{k-2})$. ∎

Since $S(c_i)^* \mathbf{1} = \Phi_{c_i}^{1/2}$, $i = 1,2$, we have

$$\Omega(x) = \frac{1}{\sqrt{\pi}} \frac{1}{(4-x^2)^{1/4}}. \tag{3.10}$$

Note that Ω is the following function:



$\Omega(x)$ is positive on $-2 < x < 2$. Furthermore the followings hold:

$$\sum_{I \in \{1,2\}^k} \chi_{D_I} \cdot \Omega = \Omega, \quad (\chi_{D_I} \cdot \Omega)(x) = \begin{cases} \Omega(x) & (x \in D_I), \\ 0 & (\text{otherwise}), \end{cases} \quad (I \in \{1,2\}^k,\ k \ge 1).$$

Theorem 3.5 *Let* $(L_2[-2,2], \pi_q)$ *be the representation of* $\mathcal{O}_2$ *by* $q = \{q_1, q_2\}$ *in Proposition 2.9 and* Ω *in (3.10). Then* $(L_2[-2,2], \pi_q, \Omega)$ *is GP*(z) *for* $z \equiv \frac{1}{\sqrt{2}}(1,1)$ *in Definition 2.1.*

Proof. By definition, $\|\Omega\| = 1$. By Lemma 3.3,

$$\pi_q \left(\frac{1}{\sqrt{2}} (s_1 + s_2) \right) \Omega = \frac{1}{\sqrt{2}} (S(q_1) + S(q_2)) \Omega = \Omega.$$

Hence $(L_2[-2,2], \pi_q)$ satisfies (2.2) for $z = \frac{1}{\sqrt{2}}(1,1)$. By Lemma 3.4, $K \equiv \{\chi_{D_I} \cdot \Omega : I \in \{1,2\}^k,\ k \ge 1\} \subset \pi_q(\mathcal{O}_2)\Omega$. Since $\Omega(x) > 0$ for $-2 < x < 2$,

$L_2[-2,2] = \overline{\mathrm{Lin}K} \subset \pi_q(\mathcal{O}_2)\Omega$. Therefore $(L_2[-2,2], \pi_q)$ is a cyclic representation of $\mathcal{O}_2$ with a unit cyclic vector Ω. Hence $(L_2[-2,2], \pi_q, \Omega)$ is $GP(z)$ by Definition 2.1 for $z = \frac{1}{\sqrt{2}}(1,1)$. ∎

Proof of Theorem 1.1. Note that the definition of π_q in (1.2) is same in (3.8). The uniqueness of eigen vector Ω of $\pi_q(s_1) + \pi_q(s_2)$ in Theorem 3.5 follows from Theorem 2.2 (iv). Hence (i) is proved. (ii) follows from Theorem 3.5. (iii) follows from Theorem 2.2 (ii). ∎

We check the eigen equation (2.2) for Theorem 3.5 by direct computation according to definitions of operators in (1.2) and the eigen function in (3.10). Note

$$(\Phi_{c_i} \circ Q)(x) = \frac{1}{|x|}\Phi_{c_i}(x), \quad (\Phi_{q_i} \circ Q)(x) = \frac{1}{2|x|}, \tag{3.11}$$

$$\Omega(q_i^{-1}(x)) = \Omega(Q(x)) = \frac{1}{\sqrt{\pi}} \frac{1}{(4-(x^2-2)^2)^{1/4}} = \frac{1}{\sqrt{|x|}}\Omega(x) \tag{3.12}$$

for $i = 1, 2$ and $x \in [-2,2]$. From these relations,

$$(S(q_i)\Omega)(x) = \chi_{D_i}(x)\left\{\Phi_{q_i}(Q(x))\right\}^{-1/2}\Omega(Q(x)) = \chi_{D_i}(x)\sqrt{2}\Omega(x).$$

Hence $\frac{1}{\sqrt{2}}(S(q_1)+S(q_2))\Omega = \Omega$ and Ω is the eigen vector of $\frac{1}{\sqrt{2}}(S(q_1)+S(q_2))$ with eigen value 1 certainly.

4 Generalization

We generalize Theorem 3.5 slightly.

By Proposition 2.11, if we have a conjugacy between two branching function systems, then we have a unitary equivalence between two representations of $\mathcal{O}_N$. Therefore we construct a new branching function system from the system $([-2,2], Q)$ in (1.1) by the following conjugacy $\varphi_{(a,b)}$: Let $a, b \in \mathbf{R}$, $a < b$ and $\varphi_{(a,b)} : [-2,2] \to [a,b]$ defined by

$$\varphi_{(a,b)}(x) = \frac{b-a}{4}x + \frac{a+b}{2}, \quad \varphi_{(a,b)}^{-1}(x) = \frac{4}{b-a}x - 2\frac{a+b}{b-a}.$$

Note $\varphi_{(a,b)}$ satisfies the assumption in Lemma 2.10. Put $Q^{(a,b)} \equiv \varphi_{(a,b)} \circ Q \circ \varphi_{(a,b)}^{-1}$, $q_{\pm}^{(a,b)} \equiv \left(Q^{(a,b)}|_{D_{\pm}^{(a,b)}}\right)^{-1}$ for $D_{-}^{(a,b)} \equiv [a, \frac{b+a}{2}]$, $D_{+}^{(a,b)} \equiv [\frac{b+a}{2}, b]$. Then

$$Q^{(a,b)}(x) = \frac{4}{b-a}\left(x - \frac{a+b}{2}\right)^2 + a, \quad q_{\pm}^{(a,b)}(x) = \frac{a+b \pm \sqrt{(b-a)(x-a)}}{2}.$$

The 1-dimensional dynamical system $\left([a,b],Q^{(a,b)}\right)$ is conjugate with $\left([-2,2],\,Q=Q^{(-2,2)}\right)$ by construction. The representation $(L_2[a,b],\pi_{q^{(a,b)}})$ of $\mathcal{O}_2$ associated with the branching function system $q^{(a,b)}=\{q_{\pm}^{(a,b)}\}$ is given by

$$\pi_{q^{(a,b)}}(s_1)\equiv S(q_{+}^{(a,b)}),\quad \pi_{q^{(a,b)}}(s_2)\equiv S(q_{-}^{(a,b)}),$$

$$\left(S\left(q_{\pm}^{(a,b)}\right)\phi\right)(x)=2\sqrt{\frac{|2x-b-a|}{b-a}}\chi_{D_{\pm}}(x)\phi\left(Q^{(a,b)}(x)\right) \tag{4.1}$$

for $\phi\in L_2[a,b]$ and $x\in[a,b]$. Put $\Omega^{(a,b)}\equiv U_{\varphi_{(a,b)}}\Omega$ where $U_{\varphi_{(a,b)}}$ is in Lemma 2.10 (i). Then

$$\Omega^{(a,b)}(x)=\frac{1}{\sqrt{\pi}}\frac{1}{\{(3b-a-2x)(b-3a+2x)\}^{1/4}}.$$

For example, $Q^{(-1,1)}(x)=2x^2-1$, $Q^{(-b,b)}(x)=\frac{4}{b}(x-\frac{b}{2})^2$, $Q^{(0,b)}(x)=\frac{2}{b}x^2-b$ for $b>0$.

By Theorem 3.5 and Proposition 2.11, we have the following result immediately.

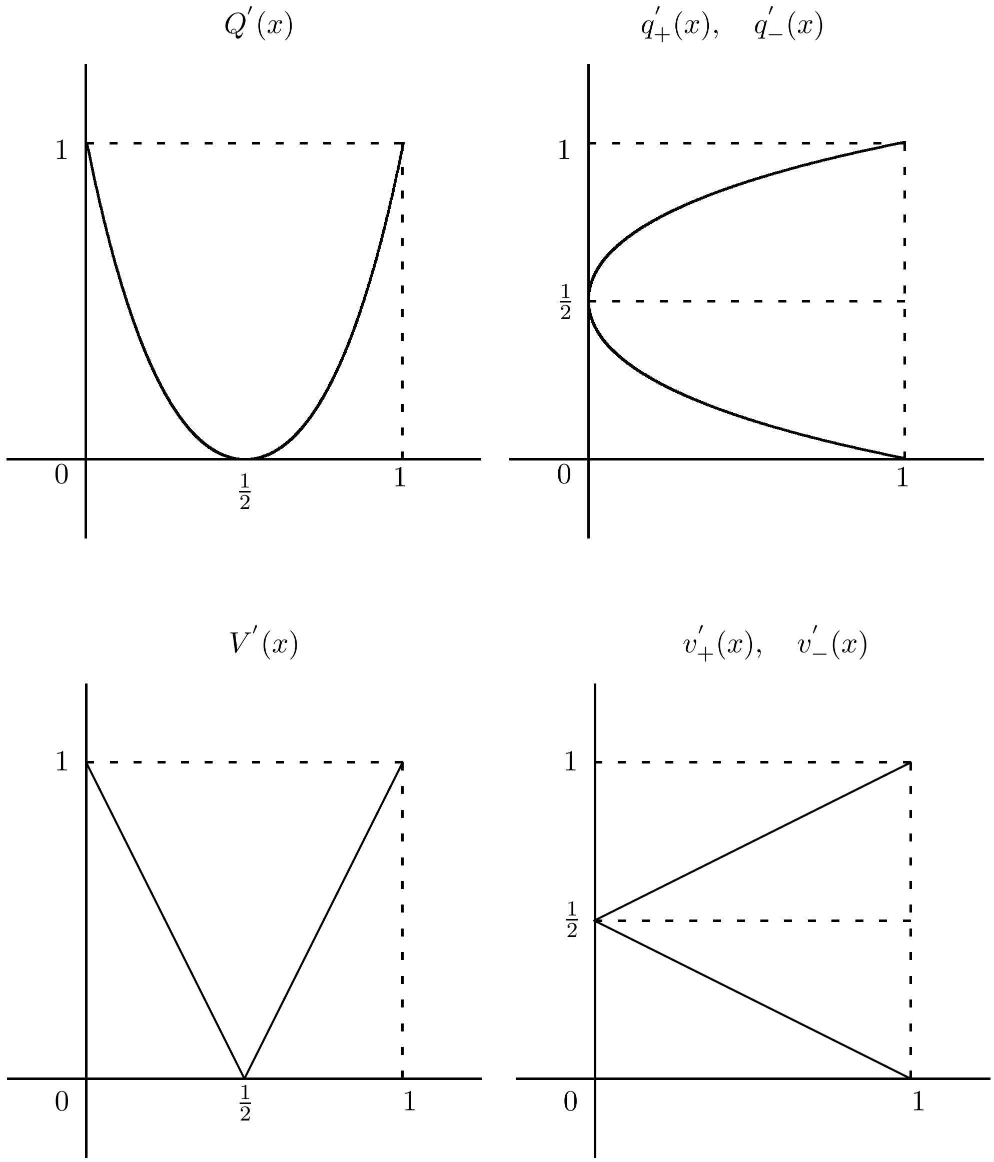
Corollary 4.1 *The representation $(L_2[a,b],\pi_{q^{(a,b)}})$ of $\mathcal{O}_2$ is $GP(z)$ for $z=\frac{1}{\sqrt{2}}(1,1)$. Specially, $(L_2[a,b],\pi_{q^{(a,b)}})$ is irreducible. $\Omega^{(a,b)}$ is unique eigen vector of $\frac{1}{\sqrt{2}}(\pi_{q^{(a,b)}}(s_1)+\pi_{q^{(a,b)}}(s_2))$ with eigen value 1.*

Remark that a given quadratic function does not always appears as $Q^{(a,b)}$. For example a transformation $P(x)=x^2$ on $[0,1]$ is impossible to describe as $Q^{(a,b)}$.

Example 4.2 Denote $Q'(x)\equiv Q^{(0,1)}$, $q'_{\pm}\equiv q_{\pm}^{(0,1)}$, $V'\equiv\varphi_{(0,1)}\circ V\circ\varphi_{(0,1)}^{-1}$ and $v'_{\pm}\equiv\varphi_{(0,1)}\circ v_{\pm}\circ\varphi_{(0,1)}^{-1}$ where $v_+\equiv v_1$ and $v_-\equiv v_2$ in (3.2). Then

$$Q'(x)=(2x-1)^2,\quad q'_{\pm}(x)=\frac{1\pm\sqrt{x}}{2},\quad V'(x)=|2x-1|,\quad v'_{\pm}(x)=\frac{1\pm x}{2}$$

for $x\in[0,1]$.



For a branching function system $q' = \{q'_\pm\}$ on $[0,1]$, we have a representation $\{S(q'_\pm)\}$ of $\mathcal{O}_2$ on $L_2[0,1]$ as follows:

$$(S(q'_\pm)\phi)(x) = \chi_{D'_\pm}(x)\sqrt{2|2x-1|}\phi\left((2x-1)^2\right)$$

for $\phi \in L_2[0,1]$ and $x \in [0,1]$ where $D'_+ \equiv [\frac{1}{2},1]$ and $D'_- \equiv [0,\frac{1}{2}]$. The eigen vector is given by

$$\Omega^{'}(x) \equiv \Omega^{(0,1)}(x) = \sqrt{\frac{2}{\pi}} \frac{1}{(x-x^2)^{1/4}}.$$

We can check directly that $\Omega^{'}$ is the eigen vector of $\frac{1}{\sqrt{2}}(\pi_{q'}(s_1)+\pi_{q'}(s_2))$ with eigen vector 1.

5 Discussion

C*-algebras associated with dynamical systems are often studied([9]). However there are few studies about representations but not algebras themselves. It is not clear that relations between the properties of dynamical systems and those of representations of $\mathcal{O}_N$ by comparison with those between algebras themselves and dynamical systems. Our interest is the construction of representation of $\mathcal{O}_N$ and its characterization. Hence we list up forward problems in this point of view.

(1) In the theory of dynamical system, a transformation

$$Q_c(x) \equiv x^2 + c$$

is more general. We give a representation of $\mathcal{O}_2$ arising from a quadratic map Q_{-2} and characterize it. Can we construct similar representation from Q_c when $c \neq -2$? If it is possible, then how can we characterize it ?

For example, $f_1(x) = \frac{1}{2}x^2$ and $f_2(x) = \frac{1}{2}x^2 + \frac{1}{2}$ gives a branching function system $f = \{f_1, f_2\}$ on $[0,1]$. However we have no idea to characterize the representation $(L_2[0,1], \pi_f)$ of $\mathcal{O}_2$ yet.

Can we treat the dynamical system of polynomial (transformation) $P(x) = a_0 + a_1 x + \cdots + a_n x^n$?

(2) Equations (3.9) is derived from (3.4). That is, equations of transformations(or real functions) change operator relations by the map S. We are interest in other equation which is similar in this case.

(3) Equations (3.9) of six operators means a relation of three representations π_v, π_c, π_q of $\mathcal{O}_2$ on $L_2[-2,2]$. This relation seems unique. How should they be studied ? We expect that similar relations will be derived from other semiconjugacies of dynamical systems.

(4) We can show easily that $(L_2[-2,2], \pi_v, \mathbf{\frac{1}{2}})$ in (3.2) is $GP(z)$ for $z = \frac{1}{\sqrt{2}}(1,1)$ where $\mathbf{\frac{1}{2}}$ is the constant function on $[-2,2]$ with value $\frac{1}{2}$. But

$(L_2[-2,2],\pi_c)$ is not known yet. Is $(L_2[-2,2],\pi_c)$ unitarily equivalent to $(L_2[-2,2],\pi_v)$, too ?

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