HIGHER DIRECT IMAGES OF LOG CANONICAL DIVISORS AND POSITIVITY THEOREMS

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Abstract. In this paper, we investigate higher direct images of log canonical divisors. After we reformulate Kollár's torsion-free theorem, we treat the relationship between higher direct images of log canonical divisors and the canonical extensions of Hodge filtration of gradedly polarized variations of mixed Hodge structures. As a corollary, we obtain a logarithmic version of Fujita-Kawamata's semi-positivity theorem. By this semi-positivity theorem, we generalize Kawamata's positivity theorem and apply it to the study of a log canonical bundle formula. The final section is an appendix, which is a result of Morihiko Saito.

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1. Introduction

In this paper, we investigate higher direct images of log canonical divisors.

First, we reformulate Kollár's torsion-free theorem and vanishing theorem. This part is more or less known to experts. See [EV] and [Am1, Section 3]. However, we explain the details since there are no appropriate references for our purposes and torsion-freeness will play important roles in this paper.

Next, we treat the relationship between higher direct images of log canonical divisors and the canonical extensions of Hodge filtration of gradedly polarized variations of mixed Hodge structures.

Let $f : X \longrightarrow Y$ be a surjective morphism between non-singular projective varieties and D a simple normal crossing divisor on X. We assume that D is *strongly horizontal* (see Definition 1.2.9) with respect to f. Then, under some suitable assumptions, $R^if_*\omega_{X/Y}(D)$ is characterized as the (upper) canonical extension of the bottom Hodge filtration of the suitable polarized variation of mixed Hodge structures. When $D = 0$, it is the theorem of Kollár and Nakayama (see [Ko2, Theorem 2.6] and [N1, Theorem 1]). If Y is a curve, then the above theorem immediately follows from the study of the gradedly polarized variation of mixed Hodge structures by Steenbrink and Zucker (see [SZ, §5 The geometric case]). By this characterization, it is not difficult to see that $R^if_*\omega_{X/Y}(D)$ is semi-positive on some monodromy conditions. It is a logarithmic version of Fujita-Kawamata's semi-positivity theorem.

Finally, by using this semi-positivity theorem, we generalize Kawamata's positivity theorem. This is one of the main purposes of this paper. As a corollary, we obtain a log canonical bundle formula for log canonical pairs, which is a slight generalization of [FM, Section 4].

We don't pursue further applications or generalizations to make this paper readable.

The final section is an appendix, which is a result of Morihiko Saito.

1.1. Main Results.

Let us explain the results of this paper more precisely. We will work over $\mathbb{C}$, the complex number field, throughout this paper.

1.1.1. In Section 2, we reformulate Kollár's torsion-free theorem.

Theorem (cf. Theorems 2.1.1, 2.1.2). *Let $f : X \longrightarrow Y$ be a surjective morphism between projective varieties. Assume that X is non-singular and $D := \sum_{i\in I} D_i$ a simple normal crossing divisor on X. We assume that D is* strongly horizontal *with respect to f, that is, every irreducible*

component of $D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k}$, *where* $\{i_1, \cdots, i_k\} \subset I$, *is dominant onto* Y (*see* Definition 1.2.9). *Then* $R^i f_* \omega_{X/Y}(D)$ *is torsion-free.*

It is a special case of [Am1, Theorem 3.2 (i)]. His theorem is much more general than ours. We explain the details and give a precise proof. Our proof is a modification of Arapura's argument [Ar, Theorem 1] and relies on the theory of (geometric) variation of mixed Hodge structures over curves. So, it is a warm-up to the next section.

In subsection 2.2, we treat a slight generalization of Kollár's vanishing theorem (see Theorem 2.2.1). We note that we don't use it later. Thus, we omit it here.

1.1.2. Section 3 is one of the main parts of this paper. It is a logarithmic generalization of the theorem of Kollár and Nakayama. As a corollary, we obtain a logarithmic generalization of Fujita-Kawamata's semi-positivity theorem.

Theorem (cf. Theorems 3.1.3, 3.1.6). *Let* $f : X \longrightarrow Y$ *be a surjective morphism between non-singular projective varieties and* D *a simple normal crossing divisor on* X, *which is strongly horizontal with respect to* f. *Let* Σ *be a simple normal crossing divisor on* Y. *We put* $Y_0 := Y \setminus \Sigma$. *If* f *is smooth and* D *is relatively normal crossing over* Y_0, *then* $R^i f_* \omega_{X/Y}(D)$ *is the upper canonical extension of the bottom Hodge filtration. In particular, it is locally free.*

Note that on the above assumptions we have a (geometric) variation of mixed Hodge structures on Y_0. Our theorem is a direct consequence of [SZ, §5] when Y is a curve. If $D = 0$, then it is the theorem of Kollár and Nakayama (see [Ko2, Theorem 2.6] and [N1, Theorem 1]). A key point of our proof is the torsion-freeness of $R^i f_* \omega_{X/Y}(D)$ that is obtained in Section 2.

We put $X_0 := f^{-1}(Y_0)$, $D_0 := D \cap X_0$, $f_0 := f|_{X_0}$, and $d := \dim X - \dim Y$.

Theorem (cf. Theorem 3.2.5). *We further assume that all the local monodromies on the local system* $R^{d+i} f_{0*} \mathbb{C}_{X_0 - D_0}$ *around every irreducible component of* Σ *are unipotent, then* $R^i f_* \omega_{X/Y}(D)$ *is a semi-positive vector bundle.*

As stated above, it is a logarithmic version of Fujita-Kawamata's semi-positivity theorem. This theorem will play crucial roles in Section 4.

1.1.3. Section 4 deals with a generalization of Kawamata's positivity theorem [Ka5, Theorem 2]. The statement is too technical to state here. So, please see Theorem 4.1.1. Our proof is essentially the same as

Kawamata's. In his proof, he used Fujita-Kawamata's semi-positivity theorem. On the other hand, we use the semi-positivity of $f_*\omega_{X/Y}(D)$, which is obtained in Section 3. Roughly speaking, Kawamata's original positivity theorem holds for (sub) klt (Kawamata log terminal) pairs and our theorem (Theorem 4.1.1) does for (sub) lc (log canonical) pairs. We believe that this difference is big for some applications. We note that Kawamata's original positivity theorem already played important roles in various situations.

In subsection 4.2, we treat only one application of our positivity theorem. It is a slight generalization of [F1, Theorem 0.2]. See Theorem 4.2.1. We don't pursue other applications here.

1.1.4. In Section 5, we formulate and prove a log canonical bundle formula for lc pairs. For the precise formula, see Theorem 5.11. This section is essentially the same as [FM, Section 4], where we formulate and prove it for klt pairs. The only one nontrivial point is the semi-positivity (nefness) of the *log-semistable part* $L^{\log,ss}_{X/Y}$ (see Theorem 5.15). It is a direct consequence of the positivity theorem obtained in Section 4.

When we wrote [FM, Section 4], Kawamata's positivity theorem was proved only for (sub) klt pairs. So we formulated a log canonical bundle formula for klt pairs. Since we generalized Kawamata's positivity theorem in Section 4, there are no difficulties to formulate and prove a log canonical bundle formula for lc pairs. We repeat the formulation in details for the readers' convenience. We note that it is conjectured that the log-semistable part $L^{\log,ss}_{X/Y}$ is semi-ample. It is proved only for elliptic fibrations, Abelian fibrations, $K3$ fibrations and so on. We recommend the readers to see [F2, Section 6] for the details.

Note that F. Ambro treated a log canonical bundle formula in a slightly different formulation. We don't pursue this formulation in this paper. For the details, see his preprint [Am2]. Related topics are [Am1] and [Sh].

1.1.5. Section 6 is an appendix, which is a remark on Section 3. After I finished the preliminary version of this paper, I asked Professor Morihiko Saito about the topic in Section 3. I received an e-mail [SE] from him, where he gave a different proof (Proposition 2 in 6.1) to Theorems 3.1.3 and 3.1.6. It depends on the theory of mixed Hodge Modules [Sa1], [Sa2]. I insert it into this paper as an appendix. Note that I made no contribution to Section 6.

1.1.6. Subsection 1.2 collects some basic definitions and fix our notation. We also recall some vanishing theorems. After checking subsection 1.2 quickly, the readers can read any section independently with referring results obtained in other sections.

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1.2. **Preliminaries.** Let us recall the basic definitions and fix our notation (cf. [KMM], [KM], and [K$^+$]). We also recall some vanishing theorems. We will work over $\mathbb{C}$, the complex number field, throughout this paper.

Definition 1.2.1 ($\mathbb{Q}$-divisors)**.** Let X be a normal variety and B, B' $\mathbb{Q}$-divisors on X. If $B - B'$ is effective, we write $B \succ B'$ or $B' \prec B$. We write $B \sim B'$ if $B - B'$ is a principal divisor on X (linear equivalence of $\mathbb{Q}$-divisors). Let B_+, B_- be the effective $\mathbb{Q}$-divisors on X without common irreducible components such that $B_+ - B_- = B$. They are called the *positive* and the *negative* parts of B.

Definition 1.2.2 (Operations of $\mathbb{Q}$-divisors)**.** Let $B = \sum b_i B_i$ be a $\mathbb{Q}$-divisor, where b_i are rational numbers and B_i are mutually prime Weil divisors. We define

$$\begin{aligned} \llcorner B \lrcorner &:= \sum \llcorner b_i \lrcorner B_i, \text{ the round down of } B, \\ \ulcorner B \urcorner &:= \sum \ulcorner b_i \urcorner B_i = -\llcorner -B \lrcorner, \text{ the round up of } B, \\ B^{<1} &:= \sum_{b_i<1} b_i B_i, \end{aligned}$$

where for $r \in \mathbb{R}$, we define $\llcorner r \lrcorner := \max\{t \in \mathbb{Z}; t \leq r\}$.

Definition 1.2.3 (Vertical and horizontal)**.** Let $f : X \longrightarrow S$ be a surjective morphism between varieties. Let B^h, B^v be the $\mathbb{Q}$-divisors on X with $B^h + B^v = B$ such that an irreducible component of Supp$B$ is contained in SuppB^h if and only if it is mapped onto S. They are called the *horizontal* and the *vertical* parts of B over S. A $\mathbb{Q}$-divisor B is said to be *horizontal* (resp. *vertical*) over S if $B = B^h$ (resp. $B = B^v$). The phrase "over S" might be suppressed if there is no danger of confusion.

Definition 1.2.4 (Canonical divisor)**.** Let X be a normal variety. The *canonical divisor* K_X is defined so that its restriction to the regular part of X is a divisor of a regular n-form. The reflexive sheaf of rank one $\omega_X := \mathcal{O}_X(K_X)$ corresponding to K_X is called the *canonical sheaf*.

The following is the definition of singularities of pairs. Note that the definitions in [KMM] or [KM] are slightly different from ours.

Definition 1.2.5 (Discrepancies and singularities for pairs)**.** Let X be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \longrightarrow X$ be a proper birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + D) + \sum a(E, X, D)E,$$

where the sum runs over all the distinct prime divisors $E \subset Y$, and $a(E, X, D) \in \mathbb{Q}$. This $a(E, X, D)$ is called the *discrepancy* of E with respect to (X, D). We define

$$\mathrm{discrep}(X, D) := \inf_E \{a(E, X, D) \mid E \text{ is exceptional over } X\}.$$

On the assumption that $d_i \leq 1$ for every i, we say that (X, D) is

$$\begin{cases} \text{sub klt} \\ \text{sub lc} \end{cases} \quad \text{if} \quad \mathrm{discrep}(X, D) \quad \begin{cases} > -1 \quad \text{and} \quad \lfloor D \rfloor \leq 0, \\ \geq -1. \end{cases}$$

If (X, D) is sub klt (resp. sub lc) and D is effective, then we say that (X, D) is *klt* (resp. *lc*). Here klt (resp. lc) is short for *Kawamata log terminal* (resp. *log canonical*).

Definition 1.2.6 (Center of lc singularities)**.** Let X be a normal variety and D a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. A subvariety W of X is said to be a *center of log canonical singularities* for the pair (X, D), if there exists a proper birational morphism from a normal variety $\mu : Y \longrightarrow X$ and a prime divisor E on Y with the discrepancy coefficient $a(E, X, D) \leq -1$ such that $\mu(E) = W$.

Remark 1.2.7 (cf. [KM, Lemmas 2.29, 2.30, and 2.45])**.** Let X be a non-singular variety and D a simple normal crossing divisor on X. Then (X, D) is lc. More precisely, (X, D) is a typical example of dlt pairs (see Remark 1.2.16 below). Let $D = \sum_{i \in I} D_i$ be the irreducible decomposition of D. Then, W is a center of log canonical singularities for the pair (X, D) if and only if W is an irreducible component of $D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k}$ for some $\{i_1, i_2, \cdots, i_k\} \subset I$.

Definition 1.2.8 (Log canonical threshold)**.** Let (X, D) be a sub lc pair and $\Delta \neq 0$ a $\mathbb{Q}$-Cartier divisor on X. We put

$$c_0 := \max\{c \in \mathbb{R} \mid (X, D + c\Delta) \text{ is sub lc}\}.$$

We call c_0 *the log canonical threshold for the pair* (X, D) *with respect to* Δ. We note that $c_0 \in \mathbb{Q}$.

Furthermore, we assume that (X, D) is lc (resp. klt) and Δ is an effective Weil divisor. Then $0 \leq c_0 \leq 1$ (resp. $0 < c_0 \leq 1$).

We introduce the following new notion, which will play important roles in this paper.

Definition 1.2.9 (Strongly horizontal)**.** Let (X, D) be a sub lc pair and $f : X \longrightarrow Y$ a surjective morphism. If all the center of log canonical singularities for the pair (X, D) are dominant onto Y, then we call D *strongly horizontal with respect to* f.

Remark 1.2.10. Let $f : X \longrightarrow Y$ and D be as in Definition 1.2.9. If (X, D) is sub klt, then D is strongly horizontal with respect to f. It is obvious by the definition. We note that $D = 0$ is strongly horizontal when X is non-singular.

We note the following easy fact.

Lemma 1.2.11. *Let $f : X \longrightarrow Y$ and D be as in* Definition 1.2.9*, that is, D is strongly horizontal with respect to f. Let Λ be a free linear system on Y and $V \in \Lambda$ a general member. We put $W := f^{-1}(V)$. Then $D|_W$ is strongly horizontal with respect to $f_W := f|_W : W \longrightarrow V$.*

The following notion was introduced by M. Reid.

Definition 1.2.12 (Nef and log big divisors)**.** Let (X, D) be lc and L a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. The divisor L is called *nef and log big* on (X, D) if L is nef and big, and $(L^{\dim W} \cdot W) > 0$ for every center of log canonical singularities W for the pair (X, D). We note that an ample divisor is nef and log big.

We prepare some vanishing theorems. The following lemma is a special case of [Fk, Lemma], which is a variant of the Kawamata-Viehweg vanishing theorem (cf. [No]).

Lemma 1.2.13. *Let X be a non-singular complete variety and D a simple normal crossing divisor on X. Let H be a nef and log big divisor on X. Then $H^i(X, K_X + D + H) = 0$ for every $i > 0$.*

Proof. If $D = 0$, then $H^i(X, K_X + H) = 0$ for every $i > 0$ by the Kawamata-Viehweg vanishing theorem. So, we can assume that $D \neq 0$.

Let D_0 be an irreducible component of D. We consider the following exact sequence:

$$\begin{aligned}\cdots &\to H^i(X, K_X + D - D_0 + H) \to H^i(X, K_X + D + H)\\ &\to H^i(D_0, K_{D_0} + (D - D_0)|_{D_0} + H|_{D_0}) \to \cdots .\end{aligned}$$

By the inductions on the number of the irreducible components of D and on $\dim X$, the first and the last terms are zero. Therefore, we obtain that $H^i(X, K_X + D + H) = 0$ for every $i > 0$. □

The following proposition is a slight generalization of the Grauert-Riemenschneider vanishing theorem.

Proposition 1.2.14. *Let $f : X \longrightarrow Y$ be a proper birational morphism from a non-singular variety X. Let D be a simple normal crossing divisor on X, where D may be zero. Assume that f is an isomorphism at every generic point of center of log canonical singularities for the pair (X, D). Then $R^i f_*\omega_X(D) = 0$ for $i > 0$.*

Proof. If $D = 0$, then it is nothing but the Grauert-Riemenschneider vanishing theorem. So, we can assume that $D \neq 0$. Let D_0 be an irreducible component of D. We consider the following exact sequence:

$$\cdots \to R^i f_*\omega_X(D - D_0) \to R^i f_*\omega_X(D) \to R^i f_*\omega_{D_0}((D - D_0)|_{D_0}) \to \cdots .$$

Then $R^i f_*\omega_X(D) = 0$ for every positive i by the same argument as in the proof of Lemma 1.2.13. □

The following corollary is an easy consequence of Proposition 1.2.14, which will be used in Sections 2 and 3.

Corollary 1.2.15. *Let $f : Z \longrightarrow X$ be a proper birational morphism from a non-singular variety Z. Let D be a reduced Weil divisor on X such that (X, D) is lc. Assume that D' is the strict transform of D and f is an isomorphism over a Zariski open set U of X such that U contains every generic point of center of log canonical singularities for the pair (X, D). Then $f_*\omega_Z(D') \simeq \omega_X(D)$.*

We further assume that the f-exceptional locus $\mathrm{Exc}(f)$ *(see* 1.2.17 (e) *below) and $D' \cup \mathrm{Exc}(f)$ are both simple normal crossing divisors on Z. Then f is an isomorphism at every generic point of center of log canonical singularities for the pair (Z, D') and $R^i f_*\omega_Z(D') = 0$ for $i > 0$. In particular, f induces a one to one correspondence between the generic points of center of log canonical singularities for the pair (Z, D') and those for the pair (X, D).*

Proof. First, we write

$$K_Z + D' = f^*(K_X + D) + \sum_i a_i E_i,$$

where E_i is an f-exceptional irreducible Cartier divisor on Z for every i. Since f is an isomorphism over U that contains every generic point of log canonical singularities for the pair (X, D), we have that $a_i > -1$ for every i. Thus, we obtain that $f_*\omega_Z(D') \simeq \omega_X(D)$.

Next, let W be a center of log canonical singularities for the pair (Z, D'). Then $W \not\subset \mathrm{Exc}(f)$ since $\mathrm{Exc}(f)$ and $D' \cup \mathrm{Exc}(f)$ are both simple normal crossing divisors. Therefore, f is an isomorphism at every generic point of center of log canonical singularities for the pair (Z, D'). So, we can apply Proposition 1.2.14. Thus, we obtain that $R^i f_*\omega_Z(D') = 0$ for $i > 0$. The final statement is obvious by the above arguments. □

Remark 1.2.16 (Divisorial log terminal)**.** The notion of *dlt* pairs may help the readers to understand Corollary 1.2.15. Here, dlt is short for *divisorial log terminal*. It is one of the most useful variants of *log terminal singularities*. In this paper, however, we don't use it explicitly. So, we omit the details. For the precise definition and the basic properties of dlt pairs, see [Sb], [KM, §2.3], and [F3].

Finally, we fix the following notation and convention.

1.2.17 (Notation and Convention). Let $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\geq 0}$) be the set of positive (resp. non-negative) integers.

(a) An *algebraic fiber space* $f : X \longrightarrow Y$ is a proper surjective morphism between non-singular projective varieties X and Y with connected fibers.

(b) Let $f : X \longrightarrow Y$ be a dominant morphism between varieties. We put $\dim f := \dim X - \dim Y$.

(c) The words *locally free sheaf* and *vector bundle* are used interchangeably.

(d) A Cartier (resp. Weil) divisor D on a normal variety X and the associated line bundle (resp. rank one reflexive sheaf) $\mathcal{O}_X(D)$ are used interchangeably if there is no danger of confusion.

(e) Let $f : X \longrightarrow Y$ be a proper birational morphism between normal varieties. By the *exceptional locus* of f, we mean the subset $\{x \in X \mid \dim f^{-1}f(x) \geq 1\}$ of X, and denote it by $\mathrm{Exc}(f)$. We note that $\mathrm{Exc}(f)$ is of pure codimension one in X if f is birational and Y is $\mathbb{Q}$-factorial.

(f) When we use the desingularization theorem, we often forbid unnecessary blow-ups implicitly, that is, we don't use the weak Hironaka theorem but the original Hironaka theorem. Unnecessary blow-ups sometimes make the proof more difficult. We recommend the readers to see [Ma, Remark 6-2 (iii)], [Sb, Resolution

Lemma], and [BEV, Corollary 7.9], in particular, [BEV, 7.12 The motivation]. See also Remark 2.1.3 below. My private note [F3] may help the readers to understand the subtleties of the desingularization theorem and various kinds of log terminal singularities.

(g) Let X be a normal variety and D a $\mathbb{Q}$-divisor on X. A *log resolution* of (X, D) is a proper birational morphism $g : Y \longrightarrow X$ such that Y is non-singular, $\mathrm{Exc}(g)$ and $\mathrm{Exc}(g) \cup g^{-1}(\mathrm{Supp} D)$ are both simple normal crossing divisors. See [KM, Notation 0.4 (10)].

2. Torsion-freeness and Vanishing theorem

In this section, we generalize Kollár's torsion-free theorem and vanishing theorem: [Ko1, Theorem 2.1]. The following is the main theorem of this section (see also Theorem 2.1.2 and Theorem 2.2.1).

Theorem. *Let X be a non-singular projective variety, D a simple normal crossing divisor on X, Y an arbitrary (reduced) projective variety, and $f : X \longrightarrow Y$ a surjective morphism. Then*

(i) *If H is an ample divisor on Y, then $H^j(Y, H \otimes R^i f_* \omega_{X/Y}(D)) = 0$ for $j > 0$.*

Assume furthermore that D is strongly horizontal with respect to f. Then

(ii) *$R^i f_* \omega_X(D)$ is torsion-free for $i \geq 0$.*

(iii) *$R^i f_* \omega_X(D) = 0$ if $i > \dim f$. Furthermore, if $D \neq 0$, then $R^i f_* \omega_X(D) = 0$ for $i \geq \dim f$.*

The statement (iii) is obvious by (ii). So, it is sufficient to prove (i) and (ii). In Theorem 2.1.1, we treat (ii) in a more general setting. We will prove (i), which is not used later, in the subsection 2.2.

2.1. Torsion-free theorem.

The following is the main theorem of this subsection. It is a special case of [Am1, Theorem 3.2 (i)]. We adopt Arapura's proof of torsion-freeness (see the proof of Theorem 1 in [Ar]). This proof is suitable for our paper.

Theorem 2.1.1 (Torsion-freeness). *Let $f : X \longrightarrow Y$ be a surjective morphism from a non-singular projective variety X to a (possibly singular) projective variety Y. Let D be a simple normal crossing divisor on X. Assume that D is strongly horizontal with respect to f. Let L be a semi-ample line bundle on X. Then, for all i, the sheaves $R^i f_*(\omega_X(D) \otimes L)$ are torsion-free. In particular, $R^i f_* \omega_X(D)$ is torsion-free for every i.*

Proof. In order to explain the plan of the proof, let us introduce the following notation, where $f : X \longrightarrow Y$ and the divisor D are as in the statement.

$$P_n^{\log} : \begin{cases} \text{If } \dim \mathrm{Supp}(R^i f_* \omega_X(D)_{\mathrm{tor}}) \leq n \text{ for all } i, \text{ then the} \\ \text{sheaves } R^i f_* \omega_X(D) \text{ are torsion-free for all } i, \text{ where} \\ R^i f_* \omega_X(D)_{\mathrm{tor}} \text{ is the torsion part of } R^i f_* \omega_X(D). \end{cases}$$

$$P^{\log}(Y) : \begin{cases} \text{The sheaf } R^i f_* \omega_X(D) \text{ is torsion-free for every } i. \end{cases}$$

$$Q^{\log}(Y) : \begin{cases} \text{Then the sheaf } R^i f_*(\omega_X(D) \otimes L) \text{ is torsion-free for} \\ \text{every } i. \end{cases}$$

It is enough to prove the following four claims:

- $P^{\log}(Y)$ implies $Q^{\log}(Y)$.
- $P^{\log}(Y)$ when Y is a curve.
- $Q^{\log}(\mathbb{P}^1)$ implies $P_0^{\log}$.
- $P_{n-1}^{\log}$ implies $P_n^{\log}$.

Step 1 ($P^{\log}(Y)$ implies $Q^{\log}(Y)$)**.** Since L is semi-ample, there exists an $m > 0$ for which L^m is generated by global sections. Hence, by Bertini's theorem, we can find $B \in |L^m|$ such that B is smooth and $B+D$ is a simple normal crossing divisor on X. Let $\pi : Z \longrightarrow X$ be the m-fold cyclic covering branched along B. Then $\pi_* \omega_Z \simeq \bigoplus_{j=0}^{m-1} \omega_X \otimes L^j$. Therefore, $\omega_X(D) \otimes L$ is a direct summand of $\pi_* \omega_Z(\pi^* D)$. Since π is finite, we have

$$R^i(f \circ \pi)_* \omega_Z(\pi^* D) = R^i f_*(\pi_* \omega_Z(\pi^* D)) = \bigoplus_{j=0}^{m-1} R^i f_*(\omega_X(D) \otimes L^j).$$

By $P^{\log}(Y)$, the left hand side is torsion-free. We note that $\pi^* D$ is a simple normal crossing divisor on Z and strongly horizontal with respect to $f \circ \pi$. Then, so is $R^i f_*(\omega_X(D) \otimes L)$.

Step 2 ($P^{\log}(Y)$ when Y is a curve)**.** Now suppose that Y is a curve. By the Stein factorization, we can assume that Y is smooth. Let Y_0 be a non-empty Zariski open set of Y such that f is smooth and D is relatively normal crossing over Y_0. By blowing up X, we can assume that $\mathrm{Supp}(f^{-1}P \cup D)$ is simple normal crossing for $P \in Y \setminus Y_0$ (cf. Corollary 1.2.15). If $f : X \longrightarrow Y$ is semi-stable, then the theorem follows from [SZ, (5.7)] (see also Theorem 3.1.4 and Step 1 in the proof of Theorem 3.1.3 below). If $f : X \longrightarrow Y$ is not semi-stable, then we apply the semi-stable reduction theorem (cf. [KMMS, Chapter II] and [SSU2,

I.9]). We consider the following commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\alpha} & X' & \xleftarrow{\beta} & \widetilde{X} \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ Y & \xleftarrow[\tau]{} & Y' & = & Y', \end{array}$$

where $\tau : Y' \to Y$ is a finite morphism from a non-singular projective curve and X' is the normalization of $X \times_Y Y'$, and β is a birational morphism such that $\widetilde{f} : \widetilde{X} \to Y'$ is semi-stable. We note that we can assume that β is an isomorphism over a non-empty Zariski open set of Y' (cf. 1.2.17 (f)). Then $R^if_*\omega_X(D)$ is a direct summand of $\tau_*R^if'_*\omega_{X'}(\alpha^*D)$. So, it is sufficient to check the local freeness of $R^if'_*\omega_{X'}(\alpha^*D)$. We note that $R^if'_*\omega_{X'/Y'}(\alpha^*D) \simeq R^i\widetilde{f}_*\omega_{\widetilde{X}/Y'}(D')$, where D' is the proper transform of α^*D (cf. Corollary 1.2.15). We can assume that $\mathrm{Supp}(\widetilde{f}^{-1}P \cup D')$ is simple normal crossing for every $P \in Y'$ (cf. [SSU2, I.9]). Since $\widetilde{f}$ is semi-stable, $R^i\widetilde{f}_*\omega_{\widetilde{X}/Y'}(D')$ is locally free by the above argument. Thus, we obtain that $R^if_*\omega_X(D)$ is locally free for every i.

Step 3 ($Q^{\log}(\mathbb{P}^1)$ implies $P_0^{\log}$)**.** We assume that the sheaf $R^if_*\omega_X(D)$ has torsion supported on a finite set of points $S := \{p_1, \cdots, p_r\}$ for some i. Now, take a pencil of hyperplane sections of Y. We can assume that the base locus is disjoint from S and that the preimage of the base locus in X is smooth and meets all the centers of log canonical singularities of (X, D) transversally. Blow up the base locus and its preimage in X to get a diagram.

$$\begin{array}{rccc} f' : & X' & \longrightarrow & Y' \\ & \searrow & & \swarrow g \\ & & \mathbb{P}^1 & \end{array}$$

Let $\mathcal{H}$ be an ample line bundle on Y'. Replacing $\mathcal{H}$ by $\mathcal{H}^k$ for some $k \gg 0$, we can assume that $R^pg_*(\mathcal{H} \otimes R^qf'_*\omega_{X'}(D')) = 0$ for all $p > 0$ and for all q, where D' is the strict transform of D on X'. Therefore, the spectral sequence collapses to give isomorphisms $g_*(\mathcal{H} \otimes R^qf'_*\omega_{X'}(D')) = R^q(g \circ f')_*(f'^*\mathcal{H} \otimes \omega_{X'}(D'))$. By $Q^{\log}(\mathbb{P}^1)$, the right hand side is torsion-free. However, by the assumption, the sheaf $\mathcal{H} \otimes R^if'_*\omega_{X'}(D')$ has torsion supported at the points $p_j \in Y'$. Therefore, $g_*(\mathcal{H} \otimes R^if'_*\omega_{X'}(D'))$ has torsion at the points $g(p_j)$. This is a contradiction. Thus, the sheaf $R^if_*\omega_X(D)$ must be torsion-free.

Step 4 ($P_{n-1}^{\log}$ implies $P_n^{\log}$)**.** Assume that $\dim \mathrm{Supp}(R^if_*\omega_X(D)_{\mathrm{tor}}) \leq n$ for all i. We suppose that for some i the sheaf $R^if_*\omega_X(D)$ is not

torsion-free. Then there must be a positive dimensional component of $\mathrm{Supp}(R^if_*\omega_X(D)_{\mathrm{tor}})$ by $P^{\log}_{n-1}$.

Let $\mathcal{H}$ be a very ample line bundle on Y and let $B \in |\mathcal{H}|$ be a general member such that f^*B is smooth and $f^*B + D$ is a simple normal crossing divisor. Then $R^if_*\omega_{f^*B}(D|_{f^*B}) \simeq R^if_*\omega_X(D) \otimes \mathcal{O}_B(B)$. Applying $P^{\log}_{n-1}$ to $(f^*B, D|_{f^*B}) \longrightarrow B$, we obtain that the left hand side is torsion-free. This contradicts the assumption that $R^if_*\omega_X(D)$ has torsion. So, we obtain the required result.

Therefore, we complete the proof. □

We can omit the assumption that X and Y are projective in Theorem 2.1.1. We will use Theorem 2.1.2 in the proof of Theorem 3.1.3.

Theorem 2.1.2. *Let $f : X \longrightarrow Y$ be a projective surjective morphism from a non-singular variety to a (possibly singular) variety. Let D be a simple normal crossing divisor on X. Assume that D is strongly horizontal with respect to f. Let L be a semi-ample line bundle on X. Then, for all i, the sheaves $R^if_*(\omega_X(D) \otimes L)$ are torsion-free. In particular, $R^if_*\omega_X(D)$ is torsion-free for every i.*

Proof. By Step 1 in the proof of Theorem 2.1.1, it is enough to prove that $R^if_*\omega_X(D)$ is torsion-free for every i. Since the statement is local, we can shrink Y and assume that Y is quasi-projective. We can take a suitable compactification and assume that X and Y are both projective (see Remark 2.1.3 below). Thus, by Theorem 2.1.1, we obtain the required result. □

Remark 2.1.3. Here, we had better use Szabó's resolution lemma: [Sb, Resolution Lemma]. See also [Ma, Remark 6-2 (iii)] or [BEV, Corollary 7.9]. We recommend the readers to see [BEV, 7.12 The motivation].

The following example implies that if D is not strongly horizontal, then the torsion-freeness is not necessarily true.

Example 2.1.4. Let Y be a non-singular projective surface and $f : X \longrightarrow Y$ be a blow-up at a point $P \in Y$. We put $D := f^{-1}(P)$. We consider the following exact sequence:

$$0 \longrightarrow \omega_X \longrightarrow \omega_X(D) \longrightarrow \omega_D \longrightarrow 0.$$

Then we obtain that $R^1f_*\omega_X(D) \simeq R^1f_*\omega_D \simeq H^1(D, \omega_D) \simeq \mathbb{C}_P$ by the Grauert-Riemenschneider vanishing theorem. So, $R^1f_*\omega_X(D)$ is not torsion-free.

Corollary 2.1.5. *Let $f : X \longrightarrow Y$ and D be as in* Theorem 2.1.2. *Assume that L is a relatively ample line bundle on X. Then $R^if_*(\omega_X(D)\otimes L) = 0$ for $i > 0$.*

Proof. It is essentially the same as [Ar, Corollary 2 (i)]. It is sufficient to use Lemma 1.2.13, instead of the Kodaira vanishing theorem. □

2.2. **Vanishing theorem.** The following is a slight generalization of Kollár's vanishing theorem: [Ko1, Theorem 2.1 (iii)]. It is also a special case of [Am1, Theorem 3.2 (ii)]. We adopt the proof of [Ko3, 9.14 Theorem]. We will not use this vanishing theorem later. So, the readers can skip this subsection.

Theorem 2.2.1 (Vanishing Theorem)**.** *Let $f : X \longrightarrow Y$ be a morphism from a non-singular projective variety X onto a variety Y. Let D be a simple normal crossing divisor on X. Let H be an ample line bundle on Y. Then*

$$H^i(Y, H \otimes R^jf_*\omega_X(D)) = 0$$

for $i > 0$ and $j \geq 0$.

Proof. Let n be a positive integer such that $n \geq 2$ and the linear system $|H^n|$ is base point free. Take a general member $E \in |H^n|$ such that $Z := f^{-1}(E)$ is smooth and $Z \cup D$ is a simple normal crossing divisor. By [EV, 5.1 b)],

$$H^i(X,\omega_X(D) \otimes f^*H) \longrightarrow H^i(X,\omega_X(D) \otimes f^*H^{1+kn}) \tag{1}$$

is injective for $i \geq 0$. We prove the theorem by induction on $\dim Y$. The assertion is evident if $\dim Y = 0$. We have an exact sequence:

$$0 \to \omega_X(D) \otimes f^*H^t \to \omega_X(D) \otimes f^*H^{t+n} \to \omega_Z(D|_Z) \otimes (f^*H^t|_Z) \to 0.$$

Using induction and the corresponding long exact sequence, we obtain that

$$H^i(Y, H^t \otimes R^jf_*\omega_X(D)) \simeq H^i(Y, H^{t+n} \otimes R^jf_*\omega_X(D))$$

for $i \geq 2$. By the Serre vanishing theorem,

$$H^i(Y, H^{t+kn} \otimes R^jf_*\omega_X(D)) = 0$$

for $k \gg 0$. Thus,

$$H^i(Y, H^t \otimes R^jf_*\omega_X(D)) = 0$$

for $t \geq 1$ and $i \geq 2$. Once this much of the theorem is established, the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, H^t \otimes R^qf_*\omega_X(D)) \Longrightarrow E^{p+q} = H^{p+q}(X,\omega_X(D) \otimes f^*H^t)$$

has only two columns, and therefore it degenerates. This means that

$$0 \longrightarrow E_2^{1,j} \longrightarrow E^{j+1} \longrightarrow E_2^{0,j+1} \longrightarrow 0.$$

Then we have

$$\begin{array}{ccc} 0 \longrightarrow & H^1(Y, H \otimes R^i f_* \omega_X(D)) & \longrightarrow \quad H^{j+1}(X, \omega_X(D) \otimes f^*H) \\ & \downarrow & \downarrow \\ 0 \longrightarrow & H^1(Y, H^{1+kn} \otimes R^i f_* \omega_X(D)) & \longrightarrow H^{j+1}(X, \omega_X(D) \otimes f^* H^{1+kn}). \end{array}$$

Using (1), this implies that

$$H^1(Y, H \otimes R^j f_* \omega_X(D)) \longrightarrow H^1(Y, H^{1+kn} \otimes R^j f_* \omega_X(D))$$

is injective for every k. As before, by the Serre vanishing theorem, this implies that $H^1(Y, H \otimes R^j f_* \omega_X(D)) = 0$. We complete the proof. □

3. Variation of mixed Hodge structures

To investigate $R^i f_* \omega_X(D)$, we use the notion of gradedly polarized variation of mixed Hodge structures. We note that all the variations of mixed Hodge structures which we treat in this section are geometric.

3.1. **Canonical Extension.** In this subsection, we generalize [Ko2, Theorem 2.6] and [N1, Theorem 1].

3.1.1. Let $f : X \longrightarrow Y$ be a projective surjective morphism between non-singular varieties over $\mathbb{C}$. Let D be a simple normal crossing divisor on X such that D is strongly horizontal. Assume that there is a non-empty Zariski open set Y_0 of Y such that $\Sigma := Y \setminus Y_0$ is a simple normal crossing divisor on Y and that $f_0 : X_0 \longrightarrow Y_0$ is smooth and D_0 is relatively normal crossing over Y_0, where $X_0 := f^{-1}(Y_0)$, $f_0 := f|_{X_0}$ and $D_0 := D \cap X_0$. The local system $R^i f_{0*} \mathbb{C}_{X_0 - D_0}$ on Y_0 forms a gradedly polarized variation of mixed Hodge structure (see [SSU1]).

3.1.2. Assume that all the local monodromies of the local system $R^k f_{0*} \mathbb{C}_{X_0 - D_0}$ around Σ are unipotent. Put $\mathcal{H}_0^k := (R^k f_{0*} \mathbb{C}_{X_0 - D_0}) \otimes \mathcal{O}_{Y_0}$ and let $F^p(\mathcal{H}_0^k)$ be the p-th Hodge filtration of $\mathcal{H}_0^k$. Let $\mathcal{H}_Y^k$ be the canonical extension of $\mathcal{H}_0^k$ to Y. Then there exists an extension $F^p(\mathcal{H}_Y^k)$ of $F^p(\mathcal{H}_0^k)$, which is locally free. See [SZ, §5 The geometric case], [SSU2, I.10], and [Ks, Lemma 1.11.2]. We note that $F^p(\mathcal{H}_Y^k) = j_* F^p(\mathcal{H}_0^k) \cap \mathcal{H}_Y^k$, where $j : Y_0 \longrightarrow Y$ is the natural inclusion. As stated above, in this paper, we only treat *geometric* gradedly polarized variation of mixed Hodge structures.

The following is the main theorem of this subsection (see also Theorem 3.1.6). The proof is essentially the same as the proof of [N1, Theorem 1].

Theorem 3.1.3. *Under the same notation as in* 3.1.1, *let* $\omega_{X/Y} := \omega_X \otimes f^*\omega_Y^{-1}$ *and* $d := \dim f$. *Assume that all the local monodromies of the local system* $R^{d+i}f_{0*}\mathbb{C}_{X_0-D_0}$ *around* Σ *are unipotent. Then there exists an isomorphism*

$$R^if_*\omega_{X/Y}(D) \simeq F^d(\mathcal{H}_Y^{d+i}).$$

In particular, $R^if_*\omega_{X/Y}(D)$ *is locally free.*

Our proof of Theorem 3.1.3 relies on the following theorem. We can take it out from ([SZ, §5 The geometric case]) with a little effort. See also [SSU2, I.10].

Theorem 3.1.4 ([SZ, §5])**.** *Let* $f : X \longrightarrow Y$ *be a projective surjective morphism from a non-singular variety* X *onto a non-singular curve* Y. *Let* D *be a simple normal crossing divisor on* X. *Assume that there is a divisor* Σ *on* Y *such that* f *is smooth over* $Y_0 := Y \setminus \Sigma$ *and* D *is relatively normal crossing over* Y_0 *and that* $C \cup D$ *is a simple normal crossing divisor, where* $C := (f^*\Sigma)_{red}$. *Assume that all the local monodromies on* $R^if_{0*}\mathbb{C}_{X_0-D_0}$ *around* Σ *are unipotent. Then*

$$\mathcal{H}_Y^i \simeq \mathbf{R}^if_*\Omega_{X/Y}^\bullet(\log(C \cup D))$$

and

$$F^p(\mathcal{H}_Y^i) \simeq \mathbf{R}^if_*F^p(\Omega_{X/Y}^\bullet(\log(C \cup D)))$$

for all p.

Here, $\Omega_{X/Y}^\bullet(\log(C \cup D))$ *is the* relative log complex:

$$\Omega_{X/Y}^\bullet(\log(C\cup D)) := \Omega_X^\bullet(\log(C\cup D))/f^*\Omega_Y^1(\log \Sigma)\wedge\Omega_X^\bullet(\log(C\cup D))[-1],$$

and $K^\bullet = F^p(\Omega_{X/Y}^\bullet(\log(C \cup D)))$ *is a complex such that*

$$K^q = \begin{cases} 0 & \text{if} \quad q < p \\ \Omega_{X/Y}^q(\log(C \cup D)) & \text{otherwise.} \end{cases}$$

Proof of Theorem 3.1.3. By Corollary 1.2.15 and 1.2.17 (f), we can assume that $C \cup D$ is a simple normal crossing divisor on X without loss of generality, where $C := (f^*\Sigma)_{red}$.

Step 1 (The case when $\dim Y = 1$)**.** By Theorem 3.1.4, we have

$$F^d(\mathcal{H}_Y^{d+i}) \simeq \mathbf{R}^if_*\Omega_{X/Y}^d(\log(C \cup D)).$$

On the other hand, $\Omega_{X/Y}^d(\log(C\cup D)) \simeq \omega_{X/Y}(C-f^*\Sigma+D)$. Therefore, if f is semi-stable, then $R^if_*\omega_{X/Y}(D) \simeq F^d(\mathcal{H}_Y^{d+i})$. If f is not semi-stable, then we use the covering arguments in [Ko2, Lemma 2.11] and [Ks, Lemma 1.9.1]. Thus, we obtain $F^d(\mathcal{H}_Y^{d+i}) \simeq R^if_*(\Omega_{X/Y}^d(\log(C \cup D)) \otimes \mathcal{O}_X(\sum(a_i-1)C_i)) \simeq R^if_*\omega_{X/Y}(D)$, where $f^*\Sigma := \sum a_iC_i$. Note

that the middle term is the *upper canonical extension* (cf. [Ko2, Definition 2.3]) and there is no difference between the *canonical extension* and the upper canonical extension (the *right canonical extension* in the proof of [Ks, Lemma 1.9.1]), since all the local monodromies are unipotent. See Remark 6.2 below.

Step 2 (The case when $l := \dim Y \geq 2$). We shall prove by induction on l.

By Step1, there is an open subset Y_1 of Y such that $\mathrm{codim}(Y \setminus Y_1) \geq 2$ and that

$$R^i f_* \omega_{X/Y}(D)|_{Y_1} \simeq F^d(\mathcal{H}_Y^{d+i})|_{Y_1}.$$

Since $F^d(\mathcal{H}_Y)$ is locally free, we obtain a homomorphism

$$\varphi_Y^i : R^i f_* \omega_{X/Y}(D) \longrightarrow F^d(\mathcal{H}_Y^{d+i}).$$

By Theorem 2.1.2, $\mathrm{Ker}\varphi_Y^i = 0$. We put $G_Y^i := \mathrm{Coker}\varphi_Y^i$. Taking a general hyperplane cut, we see that $\mathrm{Supp} G_Y^i$ is a finite set by the induction hypothesis. Assume that $G_Y^i \neq 0$. Take a point $P \in G_Y^i$. Let $\mu : W \longrightarrow Y$ be the blowing up at P and set $E = \mu^{-1}(P)$. Then $E \simeq \mathbb{P}^{l-1}$. By 1.2.17 (f), we can take a projective birational morphism $p : X' \longrightarrow X$ from a non-singular variety X' with the following properties:

(i) the composition $X' \longrightarrow X \longrightarrow Y \dashrightarrow W$ is a morphism.
(ii) p is an isomorphism over X_0.
(iii) $\mathrm{Exc}(p)$ and $\mathrm{Exc}(p) \cup D'$ are simple normal crossing divisors on X', where D' is the strict transform of D.

By Corollary 1.2.15, we obtain that $R^i f_* \omega_{X/Y}(D) \simeq R^i (f \circ p)_* \omega_{X'/Y}(D')$ for every i. We note that D' is strongly horizontal with respect to $f \circ p$. By replacing (X, D) with (X', D'), we can assume that there is a morphism $g : X \longrightarrow W$ such that $f = \mu \circ g$. Since $g : X \longrightarrow W$ is in the same situation as f, we obtain the exact sequence:

$$0 \to R^i g_* \omega_{X/W}(D) \to F^d(\mathcal{H}_W^{d+i}) \to G_W^i \to 0.$$

Tensoring $\mathcal{O}_W(\nu E)$ for $0 \leq \nu \leq l-1$ and applying $R^j \mu_*$ for $j \geq 0$ to each ν, we have a exact sequence

$$\begin{aligned} 0 \quad &\to \quad \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \to \mu_*(F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W(\nu E)) \\ &\to \quad \mu_*(G_W^i \otimes \mathcal{O}_W(\nu E)) \to R^1 \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \\ &\to \quad R^1 \mu_*(F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W(\nu E)) \to 0 \end{aligned}$$

and $R^q \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \simeq R^q \mu_*(F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W(\nu E))$ for $q \geq 2$.

By Lemma 3.1.5 below, $F^d(\mathcal{H}_W^{d+i}) \simeq \mu^* F^d(\mathcal{H}_Y^{d+i})$. We have

$$\mu_*(F^d(\mathcal{H}_W^{d+i}) \otimes \mathcal{O}_W(\nu E)) \simeq F^d(\mathcal{H}_Y^{d+i})$$

and

$$R^q\mu_*(F^d(\mathcal{H}^{d+i}_W)\otimes\mathcal{O}_W(\nu E))=0$$

for $q\geq 1$. Therefore, $R^q\mu_*(R^ig_*\omega_{X/W}(D)\otimes\mathcal{O}_W(\nu E))=0$ for $q\geq 2$ and

$$\begin{aligned}0 &\to \mu_*(R^ig_*\omega_{X/W}(D)\otimes\mathcal{O}_W(\nu E))\to\mu_*(F^d(\mathcal{H}^{d+i}_W)\otimes\mathcal{O}_W(\nu E))\\ &\to \mu_*(G^i_W\otimes\mathcal{O}_W(\nu E))\to R^1\mu_*(R^ig_*\omega_{X/W}(D)\otimes\mathcal{O}_W(\nu E))\to 0\end{aligned}$$

is exact. Since $\omega_W=\mu^*\omega_Y\otimes\mathcal{O}_W((l-1)E)$, we have a spectral sequence

$$E_2^{p,q}=R^p\mu_*(R^qg_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-1)E))\Longrightarrow R^{p+q}f_*\omega_{X/Y}(D).$$

However, $E_2^{p,q}=0$ for $p\geq 2$ by the above argument; thus

$$\begin{aligned}0 &\to R^1\mu_*R^{i-1}g_*\omega_{X/Y}(D)\to R^if_*\omega_{X/Y}(D)\\ &\to \mu_*(R^ig_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-1)E))\to 0.\end{aligned}$$

By Theorem 2.1.2, $R^1\mu_*R^{i-1}g_*\omega_{X/Y}(D)=0$. Therefore, for $q\geq 1$, we obtain

(a) $R^if_*\omega_{X/Y}(D)\simeq\mu_*(R^ig_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-1)E))$ and
(b) $R^q\mu_*(R^ig_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-1)E))=0$.

Next, we shall consider the following commutative diagram:

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
R^ig_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-2)E) & \longrightarrow & R^ig_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-1)E)\\
\downarrow & & \downarrow\\
F^d(\mathcal{H}^{d+i}_W)\otimes\mathcal{O}_W((l-2)E) & \longrightarrow & F^d(\mathcal{H}^{d+i}_W)\otimes\mathcal{O}_W((l-1)E)\\
\downarrow & & \downarrow\\
G^i_W\otimes\mathcal{O}_W((l-2)E) & \longrightarrow & G^i_W\otimes\mathcal{O}_W((l-1)E)\\
\downarrow & & \downarrow\\
0 & & 0
\end{array}$$

By applying μ_*, we have

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
\mu_*(R^ig_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-2)E)) & \longrightarrow & \mu_*(R^ig_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-1)E))\\
\downarrow & & \downarrow\\
F^d(\mathcal{H}^{d+i}_Y) & \simeq & F^d(\mathcal{H}^{d+i}_Y)\\
\downarrow & & \downarrow\\
\mu_*(G^i_W\otimes\mathcal{O}_W((l-2)E)) & \longrightarrow & \mu_*(G^i_W\otimes\mathcal{O}_W((l-1)E))\\
 & & \downarrow\\
 & & 0
\end{array}$$

By (a) and (b), $G^i_Y \simeq \mu_*(G^i_W \otimes \mathcal{O}_W((l-1)E))$ and

$$\mu_*(G^i_W \otimes \mathcal{O}_W((l-2)E)) \to \mu_*(G^i_W \otimes \mathcal{O}_W((l-1)E))$$

is surjective. Since $\dim \mathrm{Supp} G^i_W = 0$ and $E \cap \mathrm{Supp} G^i_W \neq \emptyset$, it follows that $G^i_W = 0$ by Nakayama's lemma. Therefore, $G^i_Y = 0$. This prove the theorem.

□

The following lemma played an essential role in the proof of Theorem 3.1.3.

Lemma 3.1.5. *Let $f : X \longrightarrow Y$ be and D be as in* 3.1.1. *Let $\pi : V \longrightarrow Y$ be a morphism from a non-singular variety such that* $\mathrm{Supp}\pi^{-1}(\Sigma)$ *is a simple normal crossing divisor on V. Then we obtain the gradedly polarized variation of mixed Hodge structures on $V_0 := V \setminus \pi^{-1}(\Sigma)$ by the base change. Assume that all the local monodromies of the local system $R^k f_{0*}\mathbb{C}_{X_0 - D_0}$ around Σ are unipotent. Then $\pi^* F^p(\mathcal{H}^k_Y) \simeq F^p(\mathcal{H}^k_V)$, where $\mathcal{H}^k_V$ is the canonical extension of $\mathcal{H}_{V_0} := \pi^* \mathcal{H}^k_0$ to V.*

Proof. Note that $F^p(\mathcal{H}^k_Y) = j_* F^p(\mathcal{H}^k_0) \cap \mathcal{H}^k_Y$, where $j : Y_0 \longrightarrow Y$ is the natural inclusion. See, for example, [SSU2, I.10] and [Ks, Lemma 1.11.2]. Then we have $\pi^* F^p(\mathcal{H}^k_Y) \simeq F^p(\mathcal{H}^k_V)$. See [Ka3, Proposition 1]. □

By using the unipotent reduction theorem, we obtain the following theorem.

Theorem 3.1.6. *We use the same notation and assumptions as in* 3.1.1. *We put $\omega_{X/Y} := \omega_X \otimes \omega_Y^{-1}$ and $d := \dim f$. Then $R^i f_* \omega_{X/Y}(D)$ is locally free. More precisely, $R^i f_* \omega_{X/Y}(D)$ is the upper canonical extension of $R^i f_{0*} \omega_{X_0/Y_0}(D_0)$ (see* [Ko2, Definition 2.3] *and Remark* 6.2 *below).*

Proof. (cf. [Ko2, Reduction 2.12]) It is sufficient to prove the local freeness of $R^i f_* \omega_{X/Y}(D)$. We already checked that $R^i f_* \omega_{X/Y}(D)$ is the upper canonical extension in codimension one (see Step 1 in the proof of Theorem 3.1.3). We use the unipotent reduction theorem with respect to the local system $R^{d+i} f_{0*}\mathbb{C}_{X_0 - D_0}$. First, we can assume that $\mathrm{Supp}(D \cup f^{-1}(\Sigma))$ is a simple normal crossing divisor (cf. Corollary 1.2.15 and 1.2.17 (f)). We consider the following commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\alpha} & X' & \xleftarrow{\beta} & \widetilde{X} \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ Y & \xleftarrow[\tau]{} & Y' & = & Y', \end{array}$$

where $\tau : Y' \to Y$ is a finite morphism from a non-singular variety obtained by Kawamata's covering trick, X' is the normalization of $X \times_Y Y'$, β is a projective birational morphism from a non-singular variety $\widetilde{X}$, and D' is a simple normal crossing divisor on $\widetilde{X}$ that is the strict transform of α^*D. We can assume that $\widetilde{f} : \widetilde{X} \longrightarrow Y'$ and D' satisfy the conditions and the assumptions in 3.1.1 and Theorem 3.1.3 for a suitable simple normal crossing divisor Σ' on Y'. By Proposition 1.2.14 and Corollary 1.2.15, we can check that $R^i\widetilde{f}_*\omega_{\widetilde{X}}(D') \simeq R^if'_*\omega_{X'}(\alpha^*D)$ for $i \geq 0$. We note that (X', α^*D) is lc and every center of log canonical singularities for the pair (X', α^*D) is dominant onto Y'. Thus, $R^if'_*\omega_{X'}(\alpha^*D)$ is locally free for $i \geq 0$. Since $R^if_*\omega_X(D)$ is a direct summand of $\tau_*R^if'_*\omega_{X'}(\alpha^*D)$, we obtain that $R^if_*\omega_X(D)$ is locally free for $i \geq 0$. So, we complete the proof. □

3.2. **Semi-positivity theorem.** In this subsection, we prove the semi-positivity of $R^if_*\omega_{X/Y}(D)$ on suitable assumptions. It is a generalization of Fujita-Kawamata's semi-positivity theorem and related to [Ka1, Theorem 32]. See Caution 3.2.6 below.

First, let us recall the definition of semi-positive vector bundles.

Definition 3.2.1 (Semi-positivity)**.** Let V be a complete variety and $\mathcal{E}$ a locally free sheaf on V. We say that $\mathcal{E}$ is *semi-positive* if and only if the tautological line bundle $\mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1)$ is nef on $\mathbb{P}_V(\mathcal{E})$. We note that $\mathcal{E}$ is semi-positive if and only if for every complete curve C and morphism $g : C \longrightarrow V$ every quotient line bundle of $g^*\mathcal{E}$ has non-negative degree.

We collect the basic properties of semi-positive vector bundles for the readers' convenience. We omit the proof here. Details are left to the readers. See the corresponding part of [L].

Proposition 3.2.2 (Properties of semi-positive vector bundles)**.** *Let V be a complete variety. Then we have the following properties*:

(i) *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be vector bundles on V. Then the direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ is semi-positive if and only if both $\mathcal{E}_1$ and $\mathcal{E}_2$ are semi-positive.*

(ii) *A vector bundle $\mathcal{E}$ on V is semi-positive if and only if so is $S^k\mathcal{E}$ for every k, where $S^k\mathcal{E}$ is the k-th symmetric product of $\mathcal{E}$.*

(iii) *If $\mathcal{E}$ is semi-positive and $\mathcal{F}$ is a semi-positive (resp. an ample) vector bundle on V, then $\mathcal{E} \otimes \mathcal{F}$ is semi-positive (resp. ample).*

(iv) *Any tensor product or exterior product of semi-positive vector bundles is semi-positive.*

Remark 3.2.3. It is obvious that a line bundle $\mathcal{L}$ is semi-positive if and only if $\mathcal{L}$ is nef. We note that, in some literatures (for example, [L]), semi-positive vector bundles are called *nef vector bundles*.

The following lemma, which is not difficult to prove, will be used in the proof of Theorem 3.2.5. We leave the details to the readers.

Lemma 3.2.4 (Extension of semi-positive vector bundles). *Let Y be a complete variety. Assume that there exists a short exact sequence on Y:*

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

such that both $\mathcal{E}$ and $\mathcal{E}''$ are semi-positive vector bundles. Then so is $\mathcal{E}$.

The next theorem is the main theorem of this subsection. We will use it in Section 4 to generalize Kawamata's positivity theorem.

Theorem 3.2.5 (Semi-positivity theorem). *Let $f : X \longrightarrow Y$ be a projective surjective morphism between non-singular varieties over $\mathbb{C}$. Let D be a simple normal crossing divisor on X such that D is strongly horizontal. Assume that there is a non-empty Zariski open set Y_0 of Y such that $\Sigma := Y \setminus Y_0$ is a simple normal crossing divisor on Y and that $f_0 : X_0 \longrightarrow Y_0$ is smooth and D_0 is relatively normal crossing over Y_0, where $X_0 := f^{-1}(Y_0)$, $f_0 := f|_{X_0}$ and $D_0 := D \cap X_0$. Let $\omega_{X/Y} := \omega_X \otimes f^*\omega_Y^{-1}$ and $d := \dim f$. Assume that all the local monodromies of the local system $R^{d+i}f_{0*}\mathbb{C}_{X_0-D_0}$ around Σ are unipotent. Let Z be a complete subvariety of Y. Then the restriction $R^if_*\omega_{X/Y}(D)|_Z$ is semi-positive. In particular, if Y is complete, then $R^if_*\omega_{X/Y}(D)$ is semi-positive.*

Proof. Let

$$0 \subset \cdots \subset W_k \subset W_{k+1} \subset \cdots \subset \mathcal{H}_0^{d+i} := R^{d+i}f_{0*}\mathbb{C}_{X_0-D_0}$$

be the weight filtration of the gradedly polarized variation of mixed Hodge structures and

$$0 \subset \cdots \subset \widetilde{W}_k \subset \widetilde{W}_{k+1} \subset \cdots \subset \mathcal{H}_Y^{d+i}$$

be the canonical extension of the above weight filtration. Then, the vector bundle $F^d(\mathcal{H}_Y^{d+i})$ induces the canonical extension of the bottom Hodge filtration on each $\mathrm{Gr}_k^{\widetilde{W}}(\mathcal{H}_Y^{d+i})$. Therefore, Lemma 3.2.4, [Ka3, Theorem 2] and [Ka1, §4] [1] imply that $F^d(\mathcal{H}_Y^{d+i})|_Z$ is semi-positive. On the other hand, by Theorem 3.1.3, we have the following isomorphism

$$R^if_*\omega_{X/Y}(D) \simeq F^d(\mathcal{H}_Y^{d+i}).$$

Thus, we obtain that $R^if_*\omega_{X/Y}(D)|_Z$ is semi-positive. □

[1] Kawamata's proof of semi-positivity theorem heavily relies on the asymptotic behavior of the Hodge metric near a puncture. It is not so easy for the non-expert to take it out from [Sc, §6]. We recommend the readers to see [P, Sections 2, 3] or [Z]. Section 4 of [F4] is an exposition of Fujita-Kawamata's semi-positivity theorem.

Caution 3.2.6. The semi-positivity of $f_*\omega_{X/Y}(D)$ in Theorem 3.2.5 is very similar to [Ka1, Theorem 32]. Unfortunately, our theorem: Theorem 3.2.5, does not contain Theorem 32 in [Ka1]. Note that we assume that D is strongly horizontal, in particular, D has no vertical components. This assumption is a little stronger than Kawamata's.

4. A generalization of Kawamata's positivity theorem

4.1. **Positivity theorem.** The following theorem is one of the main results in this paper. It is a slight but important generalization of Kawamata's positivity theorem (see [Ka5, Theorem 2]). See also [Ka4] for the case where the fibers are curves.

Theorem 4.1.1 (A generalization of Kawamata's positivity theorem). *Let $f : X \longrightarrow B$ be a projective surjective morphism between non-singular quasi-projective varieties with connected fibers. Let $P = \sum P_j$ and $Q = \sum_l Q_l$ be simple normal crossing divisors on X and B, respectively, such that $f^{-1}(Q) \subset P$ and f is smooth over $B \setminus Q$. Let $D = \sum_j d_j P_j$ be a $\mathbb{Q}$-divisor on X, where d_j may be positive, zero or negative, which satisfies the following conditions*:

(1) *$f : \mathrm{Supp}(D^h) \longrightarrow B$ is relatively normal crossing over $B \setminus Q$, and $f(\mathrm{Supp}(D^v)) \subset Q$.*
(2) *$d_j \leq 1$ if P_j is horizontal.*
(3) *$\dim_{k(\eta)} f_*\mathcal{O}_X(\ulcorner -D^{<1} \urcorner) \otimes_{\mathcal{O}_B} k(\eta) = 1$ for the generic point η of B.*
(4) *$K_X + D \sim_{\mathbb{Q}} f^*(K_B + L)$ for some $\mathbb{Q}$-divisor L on B.*

We put

$$\gamma_l := \max\{\gamma \in \mathbb{R} | (X, D+\gamma f^*Q_l) \text{is sub lc over the generic point of} Q_l\},$$

that is, γ_l is the log canonical threshold for the pair (X, D) with respect to f^*Q_l over the generic point of Q_l. We put

$$\delta_l := 1 - \gamma_l$$

and define

$$\Delta_0 := \sum_l \delta_l Q_l$$
$$M := L - \Delta_0.$$

Then $M \cdot C \geq 0$ for every projective curve C on B. In particular, if B is projective, then M is nef in the usual sense.

Remark 4.1.2. We use the same notation as in Theorem 4.1.1. Let

$$\begin{aligned} f^*Q_l &= \sum_j w_{lj}P_j, \\ \bar{d}_j &= \frac{d_j + w_{lj} - 1}{w_{lj}} \text{ if } f(P_j) = Q_l. \end{aligned}$$

Then we can check easily that

$$\delta_l = \max\{\bar{d}_j \mid f(P_j) = Q_l\}$$

by the definition of the log canonical threshold. So, our definition of M coincides with Kawamata's.

Remark 4.1.3. In [Ka5, Theorem 2], it is assumed that $d_j < 1$ for all j. On this assumption, it is obvious that $D^{<1} = D$. So, our theorem contains original Kawamata's positivity theorem.

Before we give the proof of Theorem 4.1.1, we recall the following well-known lemma. It may help the readers to understand the proof of Theorem 4.1.1, which is related to the toroidal geometry. We learned it from [AK]. Proposition 3.1 and Lemma 6.2 of [AK] are useful for us.

Lemma 4.1.4. *Let X be a toric variety and D the complement of the big torus. Then (X, D) is lc.*

The following proof is essentially the same as Kawamata's. We repeat his arguments in details for the readers' convenience. In his proof, he used Fujita-Kawamata's semi-positivity theorem. On the other hand, we apply Theorem 3.2.5.

Proof of Theorem 4.1.1. By replacing D by $D - f^*\Delta_0$, we can assume that $\Delta_0 = 0$. Then we have an inequality $d_j \leq 1 - w_{lj}$ for $f(P_j) = Q_l$, and the equality holds for some j for each l.

By the semi-stable reduction theorem and Kawamata's covering trick, we obtain a semi-stable reduction in codimension one in the following sense: there exists a finite morphism $h : B' \longrightarrow B$ from a non-singular quasi-projective variety B' with a simple normal crossing divisor $Q' := \mathrm{Supp}(h^*Q) = \sum_{l'} Q'_{l'}$ such that the induced morphism $f' : X' \longrightarrow B'$ from a desingularization X' of $X \times_B B'$ is semi-stable over the generic points of Q'. Let $g : X' \longrightarrow X$ be the induced morphism. We may assume that $P' = \mathrm{Supp}(g^*P) = \sum_{j'} P'_{j'}$ is a simple normal crossing divisor again:

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xleftarrow{h} & B'. \end{array}$$

Let $Z \subset B'$ be a closed subset of codimension two or larger that is contained in Q' and such that $Q' \setminus Z$ is smooth and f' is semi-stable over $B' \setminus Z$. We can define naturally a $\mathbb{Q}$-divisor $D' = \sum_{j'} d'_{j'} P'_{j'}$ on X' such that $K_{X'} + D' \sim_{\mathbb{Q}} f'^*(K_{B'} + h^*L)$. We calculate the coefficients $d'_{j'}$. If $P'_{j'}$ is horizontal and $g(P'_{j'}) = P_j$, then $d'_{j'} = d_j$. We are not concerned with those $P'_{j'}$ such that $f'(P'_{j'}) \subset Z$.

We consider the case where $f'(P'_{j'}) = Q'_{l'}$. Then, we have an inequality $d'_{j'} \leq 0$, and the equality holds for some j' for each l'. The inequality $d'_{j'} \leq 0$ holds for any j' such that $f'(P'_{j'}) = Q'_{l'}$, because it is stable under the blow-ups of X. We note that $\gamma_l = 1 - \delta_l$ is the log canonical threshold for the pair (X, D) with respect to the divisor f^*Q_l over the generic point of Q_l.

Let m be a positive integer such that mL is a Weil divisor, $X'_m := X' \times_{B'} \cdots \times_{B'} X'$ the m-tuple fiber product of X' over B', and $f'_m : X'_m \longrightarrow B'$ the projection. Then X'_m has only Gorenstein toric singularities over $B' \setminus Z$. Note that f'_m is weakly semi-stable over $B' \setminus Z$ by [AK, Definition 0.1, Corollary 1.6, Lemma 6.2]. Let $D'_m = \sum_{i=1}^m p_i^* D'$ be the sum over the i-th projections $p_i : X'_m \longrightarrow X'$. Then we have $K_{X'_m} + D'_m \sim_{\mathbb{Q}} f'^*_m(K_{B'} + mh^*L)$ over $B' \setminus Z$.

Let r be the smallest positive integer such that $r(K_{X'_m} + D'_m - f'^*_m(K_{B'} + mh^*L)) \sim 0$ over $B' \setminus Z$, and θ a non-zero rational function such that

$$\mathrm{div}(\theta) = r(K_{X'_m} + D'_m - f'^*_m(K_{B'} + mh^*L))$$

over $B' \setminus Z$. Let $\pi : \widetilde{X} \longrightarrow X'_m$ be the normalization of the main irreducible component X'^o_m of X'_m in the field $\mathbb{C}(X'^o_m)(\sqrt[r]{\theta})$, and $\widetilde{f} : \widetilde{X} \longrightarrow B'$ the induced morphism. We note that π may ramify along the support of D'_m over $B' \setminus Z$, and $\widetilde{f}$ may have non-connected fibers. Then we obtain

$$\pi_* \mathcal{O}_{\widetilde{X}} \simeq \bigoplus_{k=0}^{r-1} \mathcal{O}_{X'_m}(kK_{X'_m} + \llcorner kD'_m \lrcorner - kf'^*_m(K_{B'} + mh^*L))$$

over $B' \setminus Z$. By the duality,

$$\pi_* \omega_{\widetilde{X}} \simeq \bigoplus_{k=0}^{r-1} \mathcal{O}_{X'_m}((1-k)K_{X'_m} + \ulcorner -kD'_m \urcorner + kf'^*_m(K_{B'} + mh^*L))$$

over $B' \setminus Z$. Let $\rho : W \longrightarrow \widetilde{X}$ be a resolution of singularities such that $\lambda := f'_m \circ \pi \circ \rho : W \longrightarrow B'$ is smooth over $B' \setminus Q'$. We note that since $\widetilde{X} \longrightarrow B'$ is equisingular over $B' \setminus Q'$ by the construction, there is a simultaneous resolution over $B' \setminus Q'$ by the canonical desingularization

theorem. We can further assume that there exists a simple normal crossing divisor F on W such that F is relatively normal crossing over $B' \setminus Q'$, strongly horizontal over B', and $\rho_*\omega_W(F) \simeq \omega_{\widetilde{X}}(\pi^*E)$ over $B' \setminus Z'$, where $E := D'_m - D'^{<1}_m$. We note that $(\widetilde{X}, \pi^*E)$ is log canonical such that all the discrepancies of the divisors whose centers are not dominant onto B' are non-negative over $B' \setminus Z$.

We consider the local system $R^d\lambda_{0*}\mathbb{C}_{W_0 - F_0}$, where $B'_0 := B' \setminus Q'$, $W_0 := \lambda^{-1}(B'_0)$, $\lambda_0 := \lambda|_{W_0}$, $F_0 := F \cap W_0$, and $d := \dim \lambda$. By the covering trick again, we construct a finite morphism $h' : B'' \longrightarrow B'$ from a non-singular quasi-projective variety B'' with a simple normal crossing divisor $Q'' := \mathrm{Supp}(h'^*Q') = \sum_{l''} Q''_{l''}$ such that the induced morphism $\lambda' : W' \longrightarrow B''$ from a desingularization $\mu : W' \longrightarrow W \times_{B'} B''$ has unipotent local monodromies on the local system $(h')^{-1}R^d\lambda_{0*}\mathbb{C}_{W_0 - F_0}$ around the irreducible components of Q''. It is a unipotent reduction with respect to the local system $R^d\lambda_{0*}\mathbb{C}_{W_0 - F_0}$.

Let X'' be the normalization of the main irreducible components of the fiber product $X'_m \times_{B'} B''$. Let $f'' : X'' \longrightarrow B''$ and $g' : X'' \longrightarrow X'_m$ be the induced morphisms. Since f' is semi-stable over $B' \setminus Z$, we have $g'^*K_{X'_m/B'} = K_{X''/B''}$ over $B'' \setminus Z'$ for $Z' = h'^{-1}(Z)$. Thus $\theta' = g'^*\theta$ is a rational function such that

$$\mathrm{div}(\theta') = r(K_{X''/B''} + g'^*D'_m - f''^*mh'^*h^*L)$$

over $B'' \setminus Z'$. Let $\widetilde{X}'$ be the normalization of X'' in the field $\mathbb{C}(X'')(\sqrt[r]{\theta'})$. We note that f'' is weakly semi-stable over $B'' \setminus Z'$ by [AK, Lemma 6.2]. Let $\pi' : \widetilde{X}' \longrightarrow X''$ and $\widetilde{f}' : \widetilde{X}' \longrightarrow B''$ be the induced morphisms:

$$\begin{array}{ccc}
W & \longleftarrow & W' \\
{\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho'} \\
\widetilde{X} & \xleftarrow{\widetilde{g}} & \widetilde{X}' \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\
X'_m & \xleftarrow{g'} & X'' \\
{\scriptstyle f'_m}\downarrow & & \downarrow{\scriptstyle f''} \\
B' & \xleftarrow{h'} & B''.
\end{array}$$

Since π' ramifies only along the support of $D'' = g'^*D'_m$ over $B'' \setminus Z'$, $\widetilde{X}'$ has only toric singularities there. We have

$$\pi'_*\omega_{\widetilde{X}'} \simeq \bigoplus_{k=0}^{r-1} \mathcal{O}_{X''}((1-k)K_{X''} + \ulcorner -kD'' \urcorner + kf''^*(K_{B''} + mh'^*h^*L))$$

over $B''\setminus Z'$ by the same arguments as above. We put $E':=D''-D''^{<1}$. We note that $(\widetilde{X}', \pi'^*E')$ is lc such that all the discrepancies of the divisors whose centers are not dominant onto B'' are non-negative over $B''\setminus Z'$. So, we can assume that there exists a simple normal crossing divisor F' on W' such that F' is strongly horizontal, relatively normal crossing over $B''\setminus Q''$, and that $\rho'_*\omega_{W'}(F')\simeq \omega_{\widetilde{X}'}(\pi'^*E')$. We note that since $(\widetilde{X}', \pi'^*E')$ is equisingular over $B''\setminus Q'$, there is a simultaneous resolution over $B''\setminus Q'$ by the canonical desingularization theorem. We can assume that $W'=W\times_{B'}B''$ and $F'=F\times_{B'}B''$ over $B''\setminus Q''$. We put $\lambda'=f''\circ\pi'\circ\rho'$, $W'_0:=(\lambda')^{-1}(B''\setminus Q'')$, $\lambda'_0:=\lambda'|_{W'_0}$, and $F'_0:=F'\cap W'_0$. Then, $\lambda'_*\omega_{W'}(F')\simeq f''_*\pi'_*\omega_{\widetilde{X}'}(\pi'^*E')$ is semi-positive when it is restricted to a projective subvariety by Theorem 3.2.5 since all the local monodromies on $R^d\lambda'_{0*}\mathbb{C}_{W'_0-F'_0}$ around Q'' are unipotent.

By the above argument, we have that

$$f''_*\mathcal{O}_{X''}(\ulcorner -D''^{<1}\urcorner)\otimes\mathcal{O}_{B''}(mh'^*h^*L)$$

is a direct summand of $\lambda'_*\omega_{W'}(F')$. Since we have $d'_{j'}\leq 0$ with the equality for some j' for each l', we have $f''_*\mathcal{O}_{X''}(\ulcorner -D''^{<1}\urcorner)\simeq\mathcal{O}_{B''}$. So, $\mathcal{O}_{B''}(mh'^*h^*L)$ is a direct summand of $\lambda'_*\omega_{W'}(F')$. Thus we obtain that $M\cdot C=L\cdot C\geq 0$ for every projective curve C on B. □

4.2. **Applications.** The following is a slight generalization of [F1, Theorem 0.2]. We explain the proof in details for the readers' convenience.

Theorem 4.2.1. *Let (X,Δ) be a proper sub lc pair and $f:X\longrightarrow S$ a proper surjective morphism onto a normal variety S with connected fibers such that Δ is strongly horizontal with respect to f. Assume that $\dim_{k(\eta)} f_*\mathcal{O}_X(\ulcorner -\Delta^{<1}\urcorner)\otimes_{\mathcal{O}_S}k(\eta)=1$, where η is the generic point of S. We further assume that there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A on S such that $K_X+\Delta\sim_{\mathbb{Q}} f^*A$. Let H be an ample Cartier divisor on S, and ϵ a positive rational number. Then there exists a $\mathbb{Q}$-divisor B on S such that*

$$K_S+B\sim_{\mathbb{Q}} A+\epsilon H,$$

$$K_X+\Delta+\epsilon f^*H\sim_{\mathbb{Q}} f^*(K_S+B),$$

and that the pair (S,B) is sub klt.

Furthermore, if $f_*\mathcal{O}_X(\ulcorner -\Delta^{<1}\urcorner)\simeq\mathcal{O}_S$, then we can make B effective, that is, (S,B) is klt. In particular, S has only rational singularities.

Proof. (cf. Proof of [N2, Theorem 2]). By using the desingularization theorem, we have the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\nu} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\mu} & S, \end{array}$$

where

(i) Y and T are non-singular projective varieties,
(ii) ν and μ are projective birational morphisms,
(iii) we define $\mathbb{Q}$-divisors D and L on Y and T by the following relations:

$$K_Y + D = \nu^*(K_X + \Delta),$$
$$K_T + L \sim_{\mathbb{Q}} \mu^* A,$$

(iv) there are simple normal crossing divisors P and Q on Y and T such that they satisfy the conditions of Theorem 4.1.1 and there exists a set of positive rational numbers $\{s_l\}$ such that $\mu^*H - \sum_l s_l Q_l$ is ample.

By the construction, the conditions (1) and (4) of Theorem 4.1.1 are satisfied. Since (X, Δ) is sub lc, the condition (2) of Theorem 4.1.1 is satisfied. The condition (3) of Theorem 4.1.1 can be checked by the following claim. Note that μ is birational. We put $h := f \circ \nu$.

Claim (A). $\mathcal{O}_S \subset h_*\mathcal{O}_Y(\ulcorner -D^{<1} \urcorner) \subset f_*\mathcal{O}_X(\ulcorner -\Delta^{<1} \urcorner)$.

Proof of Claim (A). First, we have $\mathcal{O}_S \simeq h_*\mathcal{O}_Y \subset h_*\mathcal{O}_Y(\ulcorner -D^{<1} \urcorner)$, since $\mathcal{O}_Y \subset \mathcal{O}_Y(\ulcorner -D^{<1} \urcorner)$. Next, we have

$$\begin{array}{rcccl} & \Gamma(U, \nu_*\mathcal{O}_Y(\ulcorner -D^{<1} \urcorner)) & \subset & \Gamma(U \setminus Z, \nu_*\mathcal{O}_Y(\ulcorner -D^{<1} \urcorner)) \\ = & \Gamma(U \setminus Z, \mathcal{O}_X(\ulcorner -\Delta^{<1} \urcorner)) & = & \Gamma(U, \mathcal{O}_X(\ulcorner -\Delta^{<1} \urcorner)), \end{array}$$

where U is a Zariski open set of X and $Z := \nu(\mathrm{Exc}(\nu))$. So we have $\nu_*\mathcal{O}_Y(\ulcorner -D^{<1} \urcorner) \subset \mathcal{O}_X(\ulcorner -\Delta^{<1} \urcorner)$. Then we obtain $h_*\mathcal{O}_Y(\ulcorner -D^{<1} \urcorner) \subset f_*\mathcal{O}_X(\ulcorner -\Delta^{<1} \urcorner)$. We complete the proof of Claim (A). □

So we can apply Theorem 4.1.1 to $g : Y \longrightarrow T$. The divisors Δ_0 and M are as in Theorem 4.1.1. Then M is nef. Since M is nef, we have that

$$M + \epsilon\mu^*H - \epsilon' \sum_l s_l Q_l$$

is ample for $0 < \epsilon' \leq \epsilon$. We take a general Cartier divisor

$$F_0 \in |m(M + \epsilon\mu^*H - \epsilon' \sum_l s_l Q_l)|$$

for a sufficiently large and divisible integer m. We can assume that $\mathrm{Supp}(F_0 \cup \sum_l Q_l)$ is a simple normal crossing divisor. And we define $F := (1/m)F_0$. Then

$$L + \epsilon\mu^*H \sim_{\mathbb{Q}} F + \Delta_0 + \epsilon' \sum_l s_l Q_l.$$

Let $B_0 := F + \Delta_0 + \epsilon' \sum_l s_l Q_l$ and $\mu_* B_0 = B$. We have $K_T + B_0 = \mu^*(K_S + B)$. By the definition of Δ_0 and the assumption that Δ is strongly horizontal with respect to f, $\llcorner \Delta_0 \lrcorner \leq 0$. So $\llcorner F + \Delta_0 + \epsilon' \sum_l s_l Q_l \lrcorner \leq 0$ when ϵ' is small enough. Then (S, B) is sub klt. By the construction we have

$$K_S + B \sim_{\mathbb{Q}} A + \epsilon H,$$

$$K_X + \Delta + \epsilon f^*H \sim_{\mathbb{Q}} f^*(K_S + B).$$

If we assume furthermore that $f_*\mathcal{O}_X(\ulcorner -\Delta^{<1} \urcorner) \simeq \mathcal{O}_S$, we can prove the following claim. For the notation: δ_l, w_{lj}, and $\bar{d}_j$, see Theorem 4.1.1.

Claim (B). If $\mu_* Q_l \neq 0$, then $\delta_l \geq 0$.

Proof of Claim (B). If $\ulcorner -d_j \urcorner \geq w_{lj}$ for every j, then $\ulcorner -D^{<1} \urcorner \geq g^*Q_l$. So $g_*\mathcal{O}_Y(\ulcorner -D^{<1} \urcorner) \supset \mathcal{O}_T(Q_l)$. Then $\mathcal{O}_S \simeq h_*\mathcal{O}_Y(\ulcorner -D^{<1} \urcorner) \supset \mu_*\mathcal{O}_T(Q_l)$ by Claim (A). It is a contradiction. So we have that $\ulcorner -d_j \urcorner < w_{lj}$ for some j. Since w_{lj} is an integer, we have that $-d_j + 1 \leq w_{lj}$. Then $\bar{d}_j \geq 0$. We get $\delta_l \geq 0$. □

Therefore, B is effective if $f_*\mathcal{O}_X(\ulcorner -\Delta^{<1} \urcorner) \simeq \mathcal{O}_S$. This completes the proof. □

Remark 4.2.2. Under the same notation and assumptions as in Theorem 4.2.1, we further assume that S is projective and $f_*\mathcal{O}_X(\ulcorner -\Delta^{<1} \urcorner) \simeq \mathcal{O}_S$. Then the (generalized) cone theorem holds on S with respect to A. For the details, see [F1, Section 4]. We note that [F1, Section 4] was completely generalized in [Am1]. In his notation, (S, A) is a projective *quasi-log variety*.

5. Log canonical bundle formula

In [FM, Section 4], we formulated and proved a log canonical bundle formula for klt pairs. In this section, we give a log canonical bundle formula for log canonical pairs. The main theorem of this section is Theorem 5.11.

5.1. Let $f : X \longrightarrow S$ be a proper surjective morphism of a normal variety X of dimension $n = m + l$ to a non-singular l-fold S such that

(i) (X, Δ) is a sub lc pair (assumed lc from 5.9 and on), and

(ii) the generic fiber F of f is a geometrically irreducible variety with $\kappa(F, (K_X + \Delta)|_F) = 0$. We fix the smallest positive integer b such that the $f_*\mathcal{O}_X(b(K_X + \Delta)) \neq 0$.

Proposition 5.2 ([FM, Propositions 2.2, 4.2])**.** There exists one and only one $\mathbb{Q}$-divisor D modulo linear equivalence on S with a graded $\mathcal{O}_S$-algebra isomorphism

$$\bigoplus_{i\geq 0} \mathcal{O}_S(\llcorner iD \lrcorner) \cong \bigoplus_{i\geq 0} (f_*\mathcal{O}_X(\llcorner ib(K_X + \Delta) \lrcorner - ibf^*K_S))^{**}.$$

Furthermore, the above isomorphism induces the equality*:*

$$b(K_X + \Delta) = f^*(bK_S + D) + B^{\Delta},$$

where B^{Δ} is a $\mathbb{Q}$-divisor on X such that $f_*\mathcal{O}_X(\llcorner iB^{\Delta}_{+} \lrcorner) \simeq \mathcal{O}_S$ for every $i > 0$ and $\operatorname{codim}_S(f(\operatorname{Supp} B^{\Delta}_{-})) \geq 2$.

Remark 5.3. In Proposition 5.2, we note that for an arbitrary open set $U \subset S$, $D|_U$ and $B^{\Delta}|_{f^{-1}(U)}$ depend only on $f|_{f^{-1}(U)}$ and $\Delta|_{f^{-1}(U)}$.

Definition 5.4 ($L^{\log}_{X/Y}$)**.** We denote the $\mathbb{Q}$-divisor D given in Proposition 5.2 by $L_{(X,\Delta)/S}$ or simply by $L^{\log}_{X/S}$ if there is no danger of confusion.

Definition 5.5 (s_P and t_P)**.** Let P be a prime divisor on S. We set $s^{\Delta}_P := b(1 - t^{\Delta}_P)$, where t^{Δ}_P is the log canonical threshold of f^*P with respect to $(X, \Delta - (1/b)B^{\Delta})$ over the generic point η_P of P:

$$t^{\Delta}_P := \max\{t \in \mathbb{R} \mid (X, \Delta - (1/b)B^{\Delta} + tf^*P) \text{ is sub lc over } \eta_P\}.$$

Note that $s_P \in \mathbb{Q}$ and $s^{\Delta}_P \neq 0$ only for a finite number of codimension one points P because there exists a nonempty Zariski open set $U \subset S$ such that $s^{\Delta}_P = 0$ for every prime divisor P with $P \cap U \neq \emptyset$. We may simply write s_P rather than s^{Δ}_P if there is no danger of confusion. We note that s^{Δ}_P depends only on $f|_{f^{-1}(U)}$ and $\Delta|_{f^{-1}(U)}$ where U is an open set containing P.

Definition 5.6 (Log-semistable part $L^{\log,ss}_{X/S}$)**.** We set

$$L^{ss}_{(X,\Delta)/S} := L_{(X,\Delta)/S} - \sum_P s^{\Delta}_P P$$

and call it the *log-semistable part* of f. We may simply denote it by $L^{\log,ss}_{X/S}$ if there is no danger of confusion.

Remark 5.7. We note that D, $L_{(X,\Delta)/S}$, s^{Δ}_P, t^{Δ}_P and $L^{ss}_{(X,\Delta)/S}$ are birational invariants of (X, Δ) over S in the following sense. Let (X', Δ') be a projective sub lc pair and $\sigma : X' \longrightarrow X$ a birational morphism such that $K_{X'} + \Delta' - \sigma^*(K_X + \Delta)$ is an effective σ-exceptional $\mathbb{Q}$-divisor.

Then the above invariants for $f \circ \sigma$ and (X', Δ') are equal to those for f and (X, Δ).

5.8. Putting the above symbols together, we have *the log canonical bundle formula* for (X, Δ) over S:

$$b(K_X + \Delta) = f^*(bK_S + L^{\log, ss}_{X/S}) + \sum_P s^{\Delta}_P f^*P + B^{\Delta}, \tag{2}$$

where B^{Δ} is a $\mathbb{Q}$-divisor on X such that $f_*\mathcal{O}_X(\llcorner iB^{\Delta}_{+} \lrcorner) \simeq \mathcal{O}_S$ $(\forall i > 0)$ and $\mathrm{codim}_S(f(\mathrm{Supp} B^{\Delta}_{-})) \geq 2$.

We need to pass to a certain birational model $f' : X' \longrightarrow S'$ to understand the log-semistable part more clearly and to make the log canonical bundle formula more useful.

5.9. From now on, we assume that (X, Δ) is lc.

By Proposition 5.2, we have

$$K_X + \Delta - \frac{1}{b}B^{\Delta} = f^*(K_S + \frac{1}{b}L^{\log}_{X/S}).$$

Let $g : Y \longrightarrow X$ be a log resolution of $(X, \Delta - (1/b)B^{\Delta})$ with G a $\mathbb{Q}$-divisor on Y such that

$$K_Y + G = g^*(K_X + \Delta - \frac{1}{b}B^{\Delta}).$$

We put $K_Y + \Theta = g^*(K_X + \Delta)$. Let $\Sigma \subset S$ be an effective divisor satisfying the following conditions;

1. $h := f \circ g$ is projective,
2. h is smooth and $\mathrm{Supp} G^h$ is relatively normal crossing over $S \setminus \Sigma$,
3. $h(\mathrm{Supp} G^v) \subset \Sigma$, and
4. f is flat over $S \setminus \Sigma$.

Let $\pi : S' \longrightarrow S$ be a proper birational morphism from a non-singular quasi-projective variety such that

(i) $\Sigma' := \pi^{-1}(\Sigma)$ is a simple normal crossing divisor,
(ii) π induces an isomorphism $S' \setminus \Sigma' \cong S \setminus \Sigma$, and
(iii) the irreducible component X_1 of $X \times_S S'$ dominating S' is flat over S'.

Let X' be the normalization of X_1, and $f' : X' \longrightarrow S'$ the induced morphism. Let $g' : Y' \longrightarrow X'$ be a log resolution such that $Y' \setminus h'^{-1}(\Sigma') \cong Y \times_X X_1 \setminus \alpha^{-1}(\Sigma')$, where $h' := f' \circ g'$ and $\alpha : Y \times_X X_1 \longrightarrow S'$. Let Θ' be the $\mathbb{Q}$-divisor on Y' such that $K_{Y'} + \Theta' = (\tau \circ g')^*(K_X + \Delta)$, where $\tau : X' \longrightarrow X$ is the induced morphism. We put

$$K_{Y'} + G' = (\tau \circ g)^*(K_X + \Delta - \frac{1}{b}B^{\Delta}).$$

Furthermore, we can assume that $\mathrm{Supp}(h'^{-1}(\Sigma')\cup G')$ is a simple normal crossing divisor, and $h'(\mathrm{Supp}G'^{v}) \subset \Sigma'$. We note that $\mathrm{Supp}G'^{h}$ is relatively normal crossing over $S' \setminus \Sigma'$ by the construction.

Later we treat horizontal or vertical divisors on X, X' or Y' over S without referring to S. Note that a $\mathbb{Q}$-divisor on X' or Y' is horizontal (resp. vertical) over S if and only if it is horizontal (resp. vertical) over S'.

We note that the horizontal part $(\Theta')^h_-$ of the negative part Θ'_- of Θ' is g'-exceptional.

$$\begin{array}{ccccc} Y & \longleftarrow & Y \times_X X_1 & \longleftarrow & Y' \\ {\scriptstyle g}\downarrow & & \downarrow & & \downarrow{\scriptstyle g'} \\ X & \longleftarrow & X_1 & \longleftarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle f'} \\ S & \underset{\pi}{\longleftarrow} & S' & = & S' \end{array}$$

Remark 5.10. The definition of g in the above 5.9 is slightly different from that in [FM]. In [FM, 4.4], $g : Y \longrightarrow X$ is a log resolution of (X, Δ). However, it is better to assume that g is a log resolution of $(X, \Delta - (1/b)B^{\Delta})$ for the proof of Theorem 5.15. See the conditions in Theorem 4.1.1.

The following formula is the main theorem of this section. It is a slight generalization of [FM, Theorem 4.5].

Theorem 5.11 (Log canonical bundle formula)**.** Under the above notation and assumptions, let Ξ be a $\mathbb{Q}$-divisor on Y' such that (Y', Ξ) is sub lc and $\Xi - \Theta'$ is effective and exceptional over X. (Note that Ξ exists since (X, Δ) is lc.) Then the log canonical bundle formula

$$b(K_{Y'} + \Xi) = (h')^*(bK_{S'} + L^{ss}_{(Y',\Xi)/S'}) + \sum_P s^{\Xi}_P (h')^*(P) + B^{\Xi}$$

for (Y', Ξ) over S' has the following properties:

(i) $h'_*\mathcal{O}_{Y'}(\llcorner iB^{\Xi}_+ \lrcorner) \simeq \mathcal{O}_{S'}$ for all $i > 0$,

(ii) B^{Ξ}_- is g'-exceptional and $\mathrm{codim}_{S'}(h'(\mathrm{Supp}B^{\Xi}_-)) \geq 2$,

(iii) the following holds for every $i > 0$:

$$\begin{aligned} H^0(X, \llcorner ib(K_X + \Delta) \lrcorner) &\simeq H^0(Y', \llcorner ib(K_{Y'} + \Xi) \lrcorner) \\ &\simeq H^0(S', \llcorner ibK_{S'} + iL^{\log,ss}_{Y'/S'} + \sum i\, s^{\Xi}_P P \lrcorner), \end{aligned}$$

(iv) $L^{\log,ss}_{Y'/S'} \cdot C \geq 0$ for every projective curve C on S', in particular, $L^{\log,ss}_{Y'/S'}$ is nef if S is complete, and

(v) under the assumption that (Y', Ξ) is lc, let N be a positive integer such that $Nh'^*(L^{\log,ss}_{Y'/S'})$ and $bN\Xi^v$ are Weil divisors. Then for each P, there exist $u_P \in \mathbb{Z}_{>0}$ and $v_P \in \mathbb{Z}_{\geq 0}$ such that $0 \leq v_P \leq bN$ and

$$s_P = \frac{bNu_P - v_P}{Nu_P}.$$

Before we give the proof, we note the following remark, which is obvious by the definition of s_P and t_P.

Remark 5.12. Let (Y, Θ') be as in Theorem 5.11. If Θ' is strongly horizontal with respect to h', then $s_P < 1$, equivalently, $v_P > 0$. We note that Θ' is strongly horizontal with respect to h' if and only if so is Δ with respect to f.

Proof of Theorem 5.11. First, (i) is obvious by the formula (2) in 5.8. Similarly, (ii) follows because $(g')_*(B^{\Xi}_{-}) = 0$ by the equidimensionality of f'.

By (ii) and the conditions on Ξ, the following holds for all $i > 0$:

$$\begin{aligned} H^0(X, \llcorner ib(K_X + \Delta)\lrcorner) &\simeq H^0(Y', \llcorner ib(K_{Y'} + \Xi)\lrcorner) \\ &\simeq H^0(Y', \llcorner ib(K_{Y'} + \Xi) + iB^{\Xi}_{-}\lrcorner). \end{aligned}$$

By the log canonical bundle formula and then by (i), we have

$$\begin{aligned} &H^0(Y', \llcorner ib(K_{Y'} + \Xi) + iB^{\Xi}_{-}\lrcorner) \\ &\quad \simeq H^0(Y', \llcorner i(h')^*(bK_{S'} + L^{ss}_{(Y',\Xi)/S'} + \sum s^{\Xi}_P P) + iB^{\Xi}_{+}\lrcorner) \\ &\quad \simeq H^0(S', \llcorner ibK_{S'} + iL^{\log,ss}_{Y'/S'} + \sum i\, s^{\Xi}_P P\lrcorner). \end{aligned}$$

Thus (iii) is settled. The property (iv) will be settled by Theorem 5.15 below, and (v) at the end of this section. □

The following proposition is [FM, Proposition 4.6]. For the proof, see [FM].

Proposition 5.13. Under the notation and the assumptions of *Theorem 5.11*, $L^{ss}_{(Y',\Xi)/S'}$ does not depend on the choice of Ξ. In particular, $L^{ss}_{(Y',\Theta')/S'} = L^{ss}_{(Y',\Theta'_+)/S'} = L^{ss}_{(Y',\Xi)/S'}$.

Remark 5.14. The log canonical bundle formula: Theorem 5.11, coincides with [FM, Thoeorem 4.5] if (Y', Ξ) is sub klt. So, [FM, Proposition 4.7] holds without any changes.

The next theorem is Theorem 5.11 (iv).

Theorem 5.15. The log-semistable part $L^{\log,ss}_{Y'/S'}$ is nef when it is restricted to a complete subvariety of S', that is, $L^{\log,ss}_{Y'/S'} \cdot C \geq 0$ for every projective curve C on S'.

Proof. By the definition of B^{Δ}, $f_*\mathcal{O}_X(\ulcorner (1/b)B^{\Delta}_{+} \urcorner) \simeq \mathcal{O}_S$ holds. This implies that ${h'}_*\mathcal{O}_{Y'}(-G'^{<1})$ satisfies the condition (3) in Theorem 4.1.1. See also Claim (A) in the proof of Theorem 4.2.1. By the construction 5.9, G' and Σ' satisfy the conditions (1) and (2) in Theorem 4.1.1. We note that $\mathrm{Supp}({h'}^{-1}(\Sigma') \cup G')$ is a simple normal crossing divisor on Y'. So, we can apply Theorem 4.1.1 to $h' : (Y', \Theta' - (1/b)B^{\Theta'}) = (Y', G') \longrightarrow S'$. Thus, we obtain that $L^{\log,ss}_{Y'/S'} \cdot C \geq 0$ for every projective curve C on S'. □

We recall the following lemma to prove Theorem 5.11.(v).

Lemma 5.16 ([FM, Lemma 4.12])**.** Under the notation and the assumptions of *Theorem 5.11 (v)*, assume that S' is a curve. Then the following holds.

$$b(K_{Y'/S'} + \Xi + ((h')^{-1}\Sigma')_{red}) \succ (h')^*(L^{\log,ss}_{Y'/S'} + b\Sigma').$$

Finally, we give the proof of Theorem 5.11 (v) for the readers' convenience. It is essentially the same as [FM, Proof of 4.5 (i)].

Proof of Theorem 5.11 (v). Replacing S' with a general hyperplane-section H and Y' by $(h')^*(H)$, we can immediately reduce to the case where S' is a curve. For simplicity, Ξ in B^{Ξ}, s^{Ξ}_P and t^{Ξ}_P will be suppressed during the proof. We note that B is effective.

By the hypothesis, the vertical part D of the Weil divisor

$$bN(K_{Y'/S'} + \Xi^v) - (h')^*NL^{\log,ss}_{Y'/S'} = N\sum_P s_P(h')^*P + NB - bN\Xi^h$$

is a Weil divisor. We note that

$$D = N\sum_P s_P(h')^*P + NB^v = bN(K_{Y'/S'} + \Xi) - (h')^*NL^{\log,ss}_{Y'/S'} - NB^h.$$

By Lemma 5.16, we have

$$bN(K_{Y'/S'} + \Xi) - (h')^*NL^{\log,ss}_{Y'/S'} + bN\,((h')^{-1}\Sigma')_{red} \succ (h')^*bN\Sigma'.$$

Whence

$$D + NB^h + bN\,((h')^{-1}\Sigma')_{red} \succ (h')^*bN\Sigma'. \tag{3}$$

Let D_P and B^v_P be the parts of D and B^v lying over P. Let $(h')^*P = \sum_k a_k F_k$ be the irreducible decomposition. Then $D_P - NB^v_P = Ns_P(h')^*P$ and $\mathrm{Supp}(D_P - Ns_P(h')^*P) \not\supset F_c$ for some c by the

definition of B_P^v. In particular $Ns_Pa_c \in \mathbb{Z}$. Furthermore, comparing the coefficients of F_c in the formula (3), we obtain $Ns_Pa_c + bN \geq bNa_c$, that is, $Na_cs_P \geq bN(a_c - 1)$. Since (Y', Ξ) is lc, we have $t_P \geq 0$ and hence $s_P \leq b$. Hence $u_P := a_c$ works. □

6. Appendix: A remark on Section 3 by M. Saito

In this section, we give a different proof to Theorems 3.1.3, 3.1.6. It is based on the theory of mixed Hodge Modules [Sa1], [Sa2]. As I explained in 1.1.5, the following 6.1 is [SE]. I made no contribution in this section.

6.1 ([SE]). Let X be a smooth complex algebraic variety, and D a divisor with normal crossings whose irreducible components D_i are smooth. Let $U = X \setminus D$ with the inclusion $j : U \to X$. Let $(M; F, W)$ be a bifiltered (left) $\mathcal{O}_X$-Module underlying a mixed Hodge Module. Assume that $L := M|_U$ is a locally free $\mathcal{O}_U$-Module, i.e. it underlies an admissible variation of mixed Hodge structure on U.

By the definition of pure Hodge Modules, we have the strict support decomposition

$$\mathrm{Gr}_k^W(M, F) = \bigoplus_Z (M_{k,Z}, F),$$

where Z is either X or a closed irreducible variety of D (by the assumption on $M|_U$), and the $M_{k,Z}$ have no nontrivial subobject or quotient object with strictly smaller support.

Proposition 1. Let $p_0 = \min\{p : F_pM \neq 0\}$, and assume

$$F_{p_0}M_{k,Z} = 0 \quad \text{if } Z \subset D. \tag{1}$$

Then we have the canonical isomorphism

$$F_{p_0}M = j_*F_{p_0}L \cap L^{>-1} \tag{2}$$

where $L^{>a}$ is the Deligne extension of L such that the eigenvalues of the residue of the connection are contained in $(a, a+1]$.

Proof. We first consider the case $M = j_!L$, where $j_!$ is defined to be the composition $\mathbf{D}j_*\mathbf{D}$. Here $\mathbf{D}$ denotes the functor assigning the dual, and j_* coincides with the usual direct image as $\mathcal{O}$-Modules. In this case the filtration F on M is given by

$$F_pM = \sum_i F_i\mathcal{D}_X(F_{p-i}L^{>-1}), \tag{3}$$

(see e.g. [Sa1, (3.10.8)]), where F on $\mathcal{D}_X$ is the filtration by the order of operator, and $F_pL^{>-1}$ is given by $j_*F_pL \cap L^{>-1}$ as usual. So the isomorphism (2) is clear.

In general we use the canonical morphism $u : j_!L \to M$, see [Sa1, (4.2.11)]. By the above result, it is enough to show the vanishing of F_{p_0} for $\mathrm{Ker}\, u$ and $\mathrm{Coker}\, u$, because the functor assigning F_p is an exact functor for mixed Hodge Modules. Furthermore the functor assigning $F_p\mathrm{Gr}_k^W$ is also exact. So we may replace u with $\mathrm{Gr}_k^W u : \mathrm{Gr}_k^W j_!L \to \mathrm{Gr}_k^W M$. This morphism is compatible with the decomposition by strict support, and condition (1) is also satisfied for $j_!L$ (using (3)). So the assertion follows from the fact that $\mathrm{Gr}_k^W u$ induces an isomorphism between the direct factors with strict support X (this follows from the definition of the Hodge filtration on pure Hodge Modules, see e.g. [Sa1, (3.10.12)]. □

We apply this to the direct image of $\mathcal{D}$-Modules. Here it is easier to use right $\mathcal{D}$-Modules (because it simplifies many definitions) and we use the transformation between right and left $\mathcal{D}$-Modules, which is defined by assigning $\Omega_X^{\dim X} \otimes_{\mathcal{O}_X} M$ to a left $\mathcal{D}$-Module M, where $\Omega_X^{\dim X}$ has the filtration F such that $\mathrm{Gr}_p^F = 0$ for $p \neq -\dim X$. We define the Hodge filtration F on the right $\mathcal{D}$-Module ω_X by $F_p\omega_X = \omega_X$ for $p \geq 0$ and 0 otherwise. Then (ω_X, F) is pure of weight $-\dim X$ (and $(\Omega_X^{\dim X}, F)$ has weight $\dim X$). We can verify that $\mathrm{Gr}_{-k}^W(j_*\omega_U, F)$ is the direct sum of $(\iota_I)_*(\omega_{D_I}, F)$ with $\dim D_I = k$, where $D_I = \cap_{i\in I} D_i$ with the inclusion $\iota_I : D_I \to X$, see [Sa1, (3.10.8) and (3.16.12)]. (Here the direct image $(\iota_I)_*$ is defined by tensoring the polynomial ring $\mathbf{C}[\partial_1, \dots, \partial_r]$ over $\mathbf{C}$ if $I = \{1, \dots, r\}$, where $\partial_i = \partial/\partial x_i$ with $(x_1, \dots, x_n)$ a local coordinate system such that $D_i = x_i^{-1}(0)$.) We also see that $F_0 j_*\omega_U = \omega_X(D)$, and

$$F_0 H^i f_*(j_*\omega_U) = R^i f_*\omega_X(D)$$

by the definition of the direct image of filtered right $\mathcal{D}$-Modules, using the strictness of the Hodge filtration F on the direct image.

Proposition 2. Let X, U, D, j be as above. Let $f : X \to Y$ be a proper morphism of smooth complex algebraic varieties, and D' be a divisor with normal crossings on Y. Assume that every irreducible component of any intersections of D_i is dominant to Y and smooth over $Y \setminus D'$. Then condition (1) with $p_0 = 0$ is satisfied for the direct image of a filtered (right) $\mathcal{D}$-Module $H^i f_*(j_*\omega_U, F)$.

Proof. Consider the weight spectral sequence of filtered (right) $\mathcal{D}$-Modules

$$E_1^{-k,i+k} = H^i f_* \mathrm{Gr}_k^W(j_*\omega_U, F) \Rightarrow H^i f_*(j_*\omega_U, F),$$

which underlies a spectral sequence of mixed Hodge Modules and degenerates at E_2. Since $\mathrm{Gr}_k^W(j_*\omega_U, F)$ is calculated as above and the direct image of a pure Hodge Module by a proper morphism is pure,

the assertion is reduced to the proper case, where it is well known. (Indeed, it is reduced to the torsion-freeness using the decomposition by strict support as above.) □

6.2. Finally, we add one remark for the readers' convenience.

Remark 6.3 (Deligne's extension)**.** In the above Proposition 1 and [Sa2, p.513], $L^{>a}$ (resp. $L^{\geq a}$) is *Deligne's extension* of L such that the eigenvalues of the residue of the connection are contained in $(a, a+1]$ (resp. $[a, a+1)$). In our notation: Kollár's notation [Ko2, Definition 2.3], $L^{>-1}$ (resp. $L^{\geq 0}$) is called the *upper* (resp. *lower*) *canonical extension* of L. In [Ks, Lemma 1.9.1], $L^{>-1}$ (resp. $L^{\geq 0}$) is called the *right* (resp. *left*) *canonical extension* of L.

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