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# A General Two-Sided Matching Market with Discrete Concave Utility Functions †

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**Abstract**

In the theory of two-sided matching markets there are two standard models: (i) the marriage model due to Gale and Shapley and (ii) the assignment model due to Shapley and Shubik. Recently, Eriksson and Karlander introduced a hybrid model, which was further generalized by Sotomayor. In this paper, we propose a common generalization of these models by utilizing the framework of discrete convex analysis introduced by Murota, and verify the existence of a pairwise-stable outcome in our general model.

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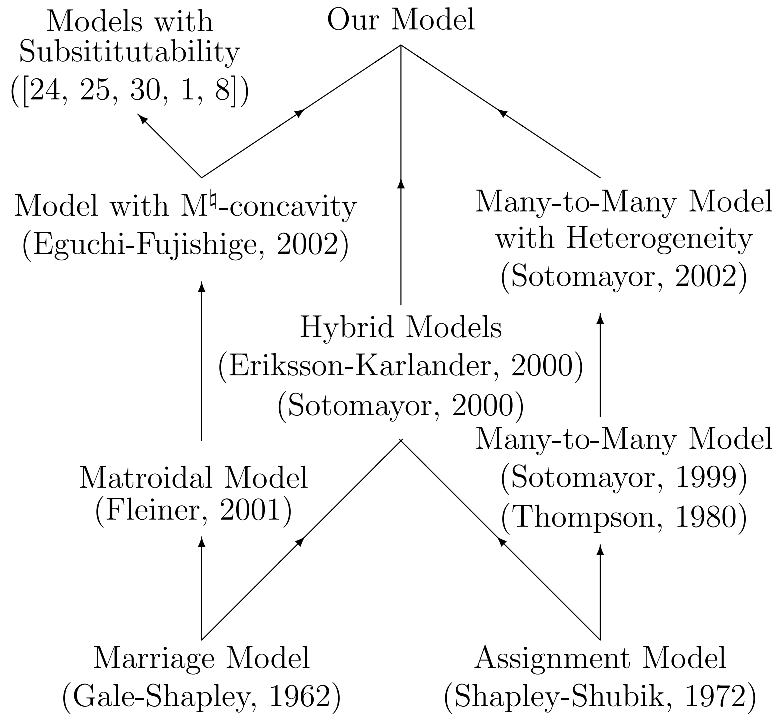


Figure 1: Hierarchy of two-sided matching market models: an arrow from model A to model B means that B is a generalization of A.

# 1. Introduction

In the theory of two-sided matching markets there are two standard models: (i) the marriage model due to Gale and Shapley [11] and (ii) the assignment model due to Shapley and Shubik [28]. The former does not allow money or transferable utilities whereas the latter does (see Roth and Sotomayor [26]). Our goal is to propose a common generalization of these models, and to verify the existence of a pairwise-stable outcome[1] in our model. Our model includes several well known special cases as shown in Fig. 1.

In the marriage model there are sets of men and women of the same size, and each person has a strict preference order on persons of the opposite gender. A matching is a set of disjoint man-woman pairs. Gale and Shapley [11] introduced the concept of pairwise-stability[2] of a matching, and gave a constructive proof

[1]Several concepts of stability such as pairwise-stability, corewise-stability, and setwise-stability have been discussed for the marriage and assignment models, and their extensions. This paper concentrates on the most elementary concept, pairwise-stability, since a special case of our model, the many-to-many marriage model, may not have a setwise-stable matching [30].

[2]In the marriage model the three concepts of pairwise-stability, corewise-stability, and setwise-stability are equivalent.

of the existence of such a matching (also see Sotomayor [29]). Since Gale and Shapley's paper a large number of variations and extensions have been proposed, and algorithmic progress has also been made (see e.g., Gusfield and Irving [13] and Baïou and Balinski [2]). Recently, Fleiner [7] extended the marriage model to the framework of matroids, and showed the existence of a pairwise-stable outcome. The preference of each person in his model can be described by a linear utility function on a matroidal domain. Eguchi and Fujishige [5] extended this formulation to a more general one in terms of discrete convex analysis, which was developed by Murota [17, 18, 20] as a unified framework in discrete optimization. In the Eguchi-Fujishige model, the preference of each agent is described by a discrete concave function, called an $\mathrm{M}^\natural$-concave function[3]. The Eguchi-Fujishige model is also a concrete example (in terms of utility functions) of the generalized models (in terms of choice functions with substitutability) by Roth [24, 25], Sotomayor [30], Alkan and Gale [1], and Fleiner [8], because an $\mathrm{M}^\natural$-concave function defines a choice function with substitutability (see Lemma 5.2).

In the other standard model, the assignment model, if a man and a woman form a partnership, then they obtain a total profit that may be divided into payoffs. An outcome consists of payoff vectors for men and women, and a matching. Shapley and Shubik [28] showed that the core[4] of this model is nonempty. Various extensions of this model have also been proposed. Sotomayor [31] showed the existence of a pairwise-stable outcome in a many-to-many model in which each agent is permitted to form multiple partnerships with agents of the opposite set without multiple partnerships of each pair[5]. Thompson [34] verified the nonemptiness of the core in a many-to-many model with heterogeneous agents, in which multiple partnerships of each pair are allowed. Sotomayor [33] also proved the nonemptiness of the core for a generalization of Thompson's model[6].

Progress has also been made toward unifying the marriage model and the assignment model. Kaneko [14] formulated a general model that includes both, and proved the nonemptiness of the core. Roth and Sotomayor [27] proposed a general model that also encompasses both, and investigated the lattice property for payoffs in the core, though the existence of a pairwise-stable outcome is not guaranteed in their model[7].

Our model is independent of the above two models. It is motivated by the

[3]The symbol $\natural$ in $\mathrm{M}^\natural$ is read "natural."

[4]The three concepts of stability are equivalent in this model.

[5]In this model, pairwise-stability is equivalent to setwise-stability, but is not equivalent to corewise-stability.

[6]In Thompson's model and Sotomayor's generalization, corewise-stability and setwise-stability are equivalent, but are not equivalent to pairwise-stability.

[7]The three concepts of stability are equivalent in this model.

model due to Eriksson and Karlander [6] and Sotomayor's generalization [32][8]. In the Eriksson-Karlander model, the set of agents is partitioned into two categories, one of "rigid" agents and the other of "flexible" agents. Rigid agents do not receive side payments, as in the marriage model, while flexible agents do, as in the assignment model. Sotomayor [32] investigated this hybrid model and gave a non-constructive proof of the existence of a pairwise-stable outcome.

We propose a common further generalization that preserves the existence of a pairwise-stable outcome. Our model has the following features:

- the preference of agents on each side is expressed by a discrete concave utility function, called an M$^\natural$-concave function,
- each agent is permitted to form partnerships with more than one agent on the opposite side,
- each pair is permitted to form multiple partnerships,
- the set of pairs is arbitrarily partitioned into a set of flexible pairs and a set of rigid pairs.

An M$^\natural$-concave function has nice features as a utility function, as we shall discuss in the next section. Our main result is that our model always has a pairwise-stable outcome. Corollaries are the existence of pairwise-stable outcomes in the above-mentioned special cases of our models (see Fig. 1). Although the present work is motivated by theoretical considerations, we believe it will contribute toward reinforcing the applicability of the two-sided matching market models.

This paper is organized as follows. Section 2 explains M$^\natural$-concavity together with its properties and gives characterizations of pairwise-stability of the marriage model and the assignment model in terms of utility functions. Section 3 describes our model based on discrete convex analysis. Section 4 shows that several existing models are special cases of our model. Section 5 proposes an algorithm for finding a pairwise-stable outcome and prove its correctness, which shows our main theorem.

# 2. Preliminaries

## 2.1. M$^\natural$-Concavity

Let $E$ be a nonempty finite set, and let $\mathbf{Z}$ and $\mathbf{R}$ be the sets of integers and reals, respectively. Let $\mathbf{Z}^E$ be the set of integral vectors $x = (x(e) : e \in E)$. Also, let

[8]The three concepts of stability are equivalent in their models.

$\mathbf{R}^E$ denote the set of real vectors indexed by $E$. For each $x \in \mathbf{Z}^E$, we define its positive support $\mathrm{supp}^+(x)$ and its negative support $\mathrm{supp}^-(x)$ by

$$\mathrm{supp}^+(x) = \{e \in E \mid x(e) > 0\}, \quad \mathrm{supp}^-(x) = \{e \in E \mid x(e) < 0\}.$$

For any $x, y \in \mathbf{Z}^E$, $x \wedge y$ and $x \vee y$ are the vectors whose $e$th components $(x \wedge y)(e)$ and $(x \vee y)(e)$ are, respectively, $\min\{x(e), y(e)\}$ and $\max\{x(e), y(e)\}$ for all $e \in E$. For each $S \subseteq E$, we denote by $\chi_S$ the characteristic vector of $S$, defined by $\chi_S(e) = 1$ if $e \in S$ and $\chi_S(e) = 0$ otherwise. We simply write $\chi_e$ instead of $\chi_{\{e\}}$ for all $e \in E$, while we denote by $\chi_0$ the zero vector in $\mathbf{Z}^E$, where $0 \notin E$. For a vector $p \in \mathbf{R}^E$ and a function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$, we define functions $\langle p, x\rangle$ and $f[p](x)$ in $x \in \mathbf{Z}^E$ by

$$\langle p, x\rangle = \sum_{e \in E} p(e)x(e), \quad f[p](x) = f(x) + \langle p, x\rangle.$$

We also denote the set of maximizers of $f$ on $U \subseteq \mathbf{Z}^E$ and the *effective domain* of $f$ by

$$\begin{aligned} &\arg\max\{f(y) \mid y \in U\} = \{x \in U \mid f(x) \geq f(y) \text{ for all } y \in U\}, \\ &\mathrm{dom}\, f = \{x \in \mathbf{Z}^E \mid f(x) > -\infty\}. \end{aligned}$$

A function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M$^\natural$-concave* [20, 21] if it satisfies[9]

(M$^\natural$) for any $x, y \in \mathrm{dom}\, f$ and any $e \in \mathrm{supp}^+(x - y)$, there exists $e' \in \mathrm{supp}^-(x - y) \cup \{0\}$ such that

$$f(x) + f(y) \leq f(x - \chi_e + \chi_{e'}) + f(y + \chi_e - \chi_{e'}).$$

Here are two simple examples of M$^\natural$-concave functions.

**Example 1**: *A set $\mathcal{I}$ of subsets of $E$ is called the* family of independent sets *of a matroid on $E$ if it satisfies the following three conditions: (a) $\emptyset \in \mathcal{I}$, (b) if $X \subseteq Y \in \mathcal{I}$ then $X \in \mathcal{I}$, and (c) for any $X, Y \in \mathcal{I}$ with $|X| < |Y|$, there exists $e \in Y \setminus X$ such that $X \cup \{e\} \in \mathcal{I}$, where $|X|$ denotes the cardinality of $X$. For any family $\mathcal{I} \subseteq 2^E$ of independent sets of a matroid and any $w \in \mathbf{R}^E$, the function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ defined by*

$$f(x) = \begin{cases} \sum_{e \in X} w(e) & \text{if } x = \chi_X \text{ for some } X \in \mathcal{I} \\ -\infty & \text{otherwise} \end{cases}$$

*for all $x \in \mathbf{Z}^E$ is M$^\natural$-concave (see Murota [20]).* ∎

[9] Condition (M$^\natural$) is denoted by ($-$M$^\natural$-EXC) in [20].

**Example 2**: *A nonempty family $\mathcal{T}$ of subsets of $E$ is called a* laminar family *if $X \cap Y = \emptyset$, $X \subseteq Y$ or $Y \subseteq X$ holds for all $X, Y \in \mathcal{T}$. For a laminar family $\mathcal{T}$ and a family of univariate concave functions $f_X : \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$ indexed by $X \in \mathcal{T}$, the function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ defined by*

$$f(x) = \sum_{X \in \mathcal{T}} f_X \left( \sum_{e \in X} x(e) \right)$$

*for all $x \in \mathbf{Z}^E$ is M$^\natural$-concave if* $\mathrm{dom}\, f \neq \emptyset$ *(see Murota [20]).* ∎

An M$^\natural$-concave function has nice features as a utility function. In mathematical economics, a utility function is usually assumed to be concave. For any M$^\natural$-concave function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$, there exists an ordinary concave function $\bar{f} : \mathbf{R}^E \to \mathbf{R} \cup \{-\infty\}$ such that $\bar{f}(x) = f(x)$ for all $x \in \mathbf{Z}^E$ [17]. That is, any M$^\natural$-concave function on $\mathbf{Z}^E$ has a concave extension on $\mathbf{R}^E$. An M$^\natural$-concave function $f$ also satisfies submodularity [22]: $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$ for all $x, y \in \mathrm{dom}\, f$.

Next we consider natural generalizations of the gross substitutability and single improvement condition that were originally proposed for set functions by Kelso and Crawford [15] and Gul and Stacchetti [12], respectively.

(GS) For any $p, q \in \mathbf{R}^E$ and any $x \in \arg\max f[-p]$ such that $p \leq q$ and $\arg\max f[-q] \neq \emptyset$, there exists $y \in \arg\max f[-q]$ such that $y(e) \geq x(e)$ for all $e \in E$ with $p(e) = q(e)$.

(SI) For any $p \in \mathbf{R}^E$ and any $x, y \in \mathrm{dom}\, f$ with $f[-p](x) < f[-p](y)$,

$$f[-p](x) < \max_{e \in \mathrm{supp}^+(x-y) \cup \{0\}} \max_{e' \in \mathrm{supp}^-(x-y) \cup \{0\}} f[-p](x - \chi_e + \chi_{e'}).$$

Here we assume that $E$ denotes the set of indivisible commodities, $p \in \mathbf{R}^E$ a price vector of commodities, $x \in \mathbf{Z}^E$ a consumption of commodities, and $f(x)$ a monetary valuation for $x$. The above conditions are interpreted as follows. Condition (GS) says that when each price increases or remains the same, the consumer wants a consumption such that the numbers of the commodities whose prices remain the same do not decrease. Condition (SI) guarantees that the consumer can bring consumption $x$ nearer to any better consumption $y$ by changing the consumption of at most two commodities. The equivalence between gross substitutability and the single improvement condition for set functions was first pointed out by Gul and Stacchetti [12], and the equivalence between the single improvement condition and M$^\natural$-concavity for set functions was by Fujishige and Yang [10]. Murota and Tamura [23] showed that an M$^\natural$-concave function satisfies (GS) and (SI), and conversely, M$^\natural$-concavity is characterized by (SI), and by a stronger version of (GS) under a certain natural assumption. Danilov, Koshevoy and Lang [3] characterized M$^\natural$-concavity by another strengthened property of (GS) under a natural assumption. M$^\natural$-concavity also implies substitutability (see Lemma 5.2).

## 2.2. The Marriage Model and the Assignment Model

In this subsection we characterize pairwise-stability of the marriage model and the assignment model in terms of utility functions. These characterizations will be useful to understand pairwise-stability in our model.

Let $M$ and $W$ denote two disjoint sets of agents and $E = M \times W$. Agents in $M$ and $W$ are interpreted as men and women, respectively. To each pair $(i, j) \in E$, a pair $(a_{ij}, b_{ij}) \in \mathbf{R}^2$ is associated. Here $a_{ij}$ and $b_{ij}$ are interpreted as profits of $i$ and $j$ in the assignment model. And they define preferences in the marriage model: man $i \in M$ *prefers* $j_1$ to $j_2$ if $a_{ij_1} > a_{ij_2}$, and $i$ is *indifferent* between $j_1$ and $j_2$ if $a_{ij_1} = a_{ij_2}$ (similarly, the preferences of woman $j \in W$ are defined by $b_{ij}$'s). We assume that $a_{ij} > 0$ if $j$ is *acceptable* to $i$, and $a_{ij} = -\infty$ otherwise, and $b_{ij} > 0$ if $i$ is *acceptable* to $j$, and $b_{ij} = -\infty$ otherwise. Let $\{0, 1\}^E$ denote the set of all 0-1 vectors $x$ on $E$, i.e., $x_{ij} = 0$ or 1 for all $(i, j) \in E$. Define two aggregated utility functions $f_M$ for $M$ and $f_W$ for $W$ as follows: for all $x \in \mathbf{Z}^E$,

$$f_M(x) = \begin{cases} \sum_{(i,j)\in E} a_{ij}x_{ij} & \text{if } x \in \{0,1\}^E \text{ and } \sum_{j\in W} x_{ij} \le 1 \text{ for all } i \in M \\ -\infty & \text{otherwise,} \end{cases} \tag{2.1}$$

$$f_W(x) = \begin{cases} \sum_{(i,j)\in E} b_{ij}x_{ij} & \text{if } x \in \{0,1\}^E \text{ and } \sum_{i\in M} x_{ij} \le 1 \text{ for all } j \in W \\ -\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

As shown in Example 1, $f_M$ and $f_W$ are M$^\natural$-concave.

We now consider the marriage model. A *matching* is a subset of $E$ such that every agent appears at most once. Given a matching $X$, $i \in M$ (respectively $j \in W$) is called *unmatched in* $X$ if there exists no $j \in W$ (resp. $i \in M$) with $(i, j) \in X$. A matching $X$ is called *pairwise-stable*[10] if there exist $q \in \mathbf{R}^M$ and $r \in \mathbf{R}^W$ such that

(m1) $q_i = a_{ij} > -\infty$ and $r_j = b_{ij} > -\infty$ for all $(i, j) \in X$,

(m2) $q_i = 0$ (resp. $r_j = 0$) if $i$ (resp. $j$) is unmatched in $X$,

(m3) $q_i \ge a_{ij}$ or $r_j \ge b_{ij}$ for all $(i, j) \in E$.

The above-defined pairwise-stability in the marriage model can also be characterized in terms of utility functions $f_M$ and $f_W$ given by (2.1) and (2.2). Let $\mathbf{1}$ denote

[10] Here we consider weak pairwise-stability for a variation in which indifference is allowed. Note that strong pairwise-stability is defined by the conditions (m1), (m2), and (m3′) $[q_i > a_{ij}$ or $r_j \ge b_{ij}]$ and $[q_i \ge a_{ij}$ or $r_j > b_{ij}]$ for all $(i, j) \in E$. A strongly pairwise-stable matching does not always exist, and hence, we restrict our attention to weak pairwise-stability.

the vector of all ones on $E$. A 0-1 vector $x$ on $E$ is pairwise-stable in the marriage model[11] if and only if there exist 0-1 vectors $z_M$ and $z_W$ such that

$$\mathbf{1} = z_M \vee z_W, \tag{2.3}$$

$$x \text{ maximizes } f_M \text{ in } \{y \in \mathbf{Z}^E \mid y \leq z_M\}, \tag{2.4}$$

$$x \text{ maximizes } f_W \text{ in } \{y \in \mathbf{Z}^E \mid y \leq z_W\}. \tag{2.5}$$

This characterization can be interpreted as follows. We note that a 0-1 vector $x$ satisfying (2.4) and (2.5) must be a matching since $x \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$. For a matching $x$, condition (2.4) (resp. (2.5)) says that each man (resp. woman) selects one of the best partners among partners in $z_M$ (resp. $z_W$). Therefore, (2.3) guarantees that there is no pair whose members prefer each other to their partners in $x$ or to being alone in $x$. Conversely, for a pairwise-stable matching $x$, $z_M$ can be constructed as follows. Set $z_M(i,j) = 0$ for all pairs $(i,j) \in E$ such that $i$ prefers $j$ to his partner or to being alone in $x$ (note that by pairwise-stability of $x$, $j$ does not prefer $i$ to her partner or to being alone in $x$), and set $z_M(i,j) = 1$ otherwise. Similarly, $z_W$ can be constructed from $x$. Then, (2.3)$\sim$(2.5) hold.

The assignment model allows side payments, which is not the case for the marriage model. An *outcome* is a triple consisting of payoff vectors $q = (q_i : i \in M) \in \mathbf{R}^M$, $r = (r_j : j \in W) \in \mathbf{R}^W$, and a subset $X \subseteq E$, denoted by $(q, r; X)$. An outcome $(q, r; X)$ is called *pairwise-stable* if

(a1) $X$ is a matching,

(a2) $q_i + r_j = a_{ij} + b_{ij}$ for all $(i,j) \in X$,

(a3) $q_i = 0$ (resp. $r_j = 0$) if $i$ (resp. $j$) is unmatched in $X$,

(a4) $q \geq \mathbf{0}$, $r \geq \mathbf{0}$, and $q_i + r_j \geq a_{ij} + b_{ij}$ for all $(i,j) \in E$,

where $\mathbf{0}$ denotes a zero vector of an appropriate dimension and $p_{ij} (= b_{ij} - r_j = q_i - a_{ij})$ means a side payment from $j$ to $i$ for each $(i,j) \in X$. The pairwise-stability says that no pair $(i,j) \notin X$ will be better off by forming a partnership. Shapley and Shubik [28] proved the existence of a pairwise-stable outcome by linear programming duality and integrality. The maximum weight bipartite matching problem with weights $(a_{ij} + b_{ij})$ and its dual problem are formulated as linear

[11] We identify a subset $X$ with its characteristic vector $\chi_X$.

programs:

$$
\begin{array}{ll}
\text{Maximize} & \displaystyle\sum_{(i,j)\in E} (a_{ij}+b_{ij})x_{ij} \\
\text{subject to} & \displaystyle\sum_{j\in W} x_{ij} \le 1 \quad \text{for all } i \in M \\
& \displaystyle\sum_{i\in M} x_{ij} \le 1 \quad \text{for all } j \in W \\
& x_{ij} \ge 0 \quad \text{for all } (i,j) \in E, \\
\text{Minimize} & \displaystyle\sum_{i\in M} q_i + \sum_{j\in W} r_j \\
\text{subject to} & q_i + r_j \ge a_{ij} + b_{ij} \quad \text{for all } (i,j) \in E \\
& q_i \ge 0 \quad \text{for all } i \in M \\
& r_j \ge 0 \quad \text{for all } j \in W.
\end{array}
$$

Recall that the primal problem has an integral optimal solution, a matching. Thus, $(q, r; X)$ is pairwise-stable if and only if $x = \chi_X$, $q$ and $r$ are optimal solutions of the above dual problems, because (a1) and (a4) require primal and dual feasibility and because (a2) and (a3) mean complementary slackness. Furthermore, pairwise-stability in the assignment model can be characterized by using utility functions in (2.1) and (2.2). A 0-1 vector $x$ on $E$ is pairwise-stable[12] if and only if there exists $p \in \mathbf{R}^E$ such that

$$x \text{ maximizes } f_M[+p], \tag{2.6}$$

$$x \text{ maximizes } f_W[-p]. \tag{2.7}$$

A pairwise-stable outcome $(q, r; X)$ gives $x = \chi_X$ together with $p$ satisfying (2.6) and (2.7) by putting $p_{ij} = b_{ij} - r_j$ for all $(i,j) \in E$. Conversely, $x = \chi_X$ and $p$ satisfying (2.6) and (2.7) lead us to a pairwise-stable outcome $(q, r; X)$ such that $q_i = a_{ij} + p_{ij}$ and $r_j = b_{ij} - p_{ij}$ for all $(i,j) \in X$ and $q_i = 0$ (resp. $r_j = 0$) for all $i$ (resp. $j$) unmatched in $X$.

# 3. Model Description and the Main Theorem

Let $M$ and $W$ denote two disjoint sets of agents and $E$ be a finite set. In our model, utilities (in monetary terms) of $M$ and $W$ over $E$ are, respectively, described by M$^\natural$-concave functions $f_M, f_W : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$. In the exemplary models described in Sections 2.2 and 4, $E = M \times W$, and $f_M$ and $f_W$ can be regarded as aggregations of the utilities of $M$-agents and $W$-agents (see Remark 1 given below). We assume

[12] A 0-1 vector $x$ is pairwise-stable in the assignment model if and only if there exists a pairwise-stable outcome $(q, r; X)$ with $x = \chi_X$.

that $E$ is arbitrarily partitioned into two subsets $F$ (the set of *flexible elements*) and $R$ (the set of *rigid elements*)[13]. We also assume that $f_M$ and $f_W$ satisfy:

**(A)** Effective domains $\mathrm{dom}\, f_M$ and $\mathrm{dom}\, f_W$ are bounded and hereditary, and have $\mathbf{0}$ as a common minimum point,

where heredity means that $\mathbf{0} \leq x_1 \leq x_2 \in \mathrm{dom}\, f_M$ (resp. $\mathrm{dom}\, f_W$) implies $x_1 \in \mathrm{dom}\, f_M$ (resp. $\mathrm{dom}\, f_W$). Heredity implies that each agent can arbitrarily decrease the multiplicity of partnerships he is in part of without permission from his partners, similarly as in other two-sided matching market models.

Let $z$ be an integral vector such that

$$\mathrm{dom}\, f_M \cup \mathrm{dom}\, f_W \subseteq \{y \in \mathbf{Z}^E \mid \mathbf{0} \leq y \leq z\}.$$

For a vector $d$ on $E$ and $S \subseteq E$, let $d|_S$ denote the restriction of $d$ on $S$. Taking conditions (2.3)$\sim$(2.7) into account, we say that $x \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ is an *$f_M f_W$-pairwise-stable solution with respect to* $(F, R)$, or simply *$f_M f_W$-pairwise-stable solution*, if there exist $p \in \mathbf{R}^E$ and $z_M, z_W \in \mathbf{Z}^R$ such that

$$p|_R = \mathbf{0}, \tag{3.1}$$

$$z|_R = z_M \vee z_W, \tag{3.2}$$

$$x \in \arg\max\{f_M[+p](y) \mid y|_R \leq z_M\}, \tag{3.3}$$

$$x \in \arg\max\{f_W[-p](y) \mid y|_R \leq z_W\}. \tag{3.4}$$

Condition (3.1) states that there are no side payments for rigid elements. Condition (3.2) replaces the upper bound vector $\mathbf{1}$ in (2.4) by $z$. Obviously, if $E = R$ then our model includes the marriage model, and if $E = F$ then it includes the assignment model.

Before giving our main result, we give two illustrations of our model.

**Example 3**: *We consider the problem of allocating dance partners between set $M = \{m_1, m_2\}$ of two men and set $W = \{w_1, w_2\}$ of two women. Here, $E = M \times W$. We assume that they have the following preferences:*

- *everyone wants to dance as many times as possible, up to four times,*
- *$m_1$ prefers $w_1$ to $w_2$,*
- *$w_1$ and $w_2$ are indifferent for $m_2$,*
- *every woman wants to dance with $m_1$ and $m_2$ as equally as possible.*

[13] In the Eriksson-Karlander model [6], $M$ and $W$ are, respectively, partitioned into $\{M_F, M_R\}$ and $\{W_F, W_R\}$, and we have $F = M_F \times W_F$ and $R = E \setminus F$, where $E = M \times W$.

*Denoting by $x_{ij}$ the number of times $m_i$ and $w_j$ dance together, we can describe their preferences by the following four utility functions:*

$$
\begin{aligned}
f_{m_1}(x_{11}, x_{12}) &= \begin{cases} 10x_{11} + 8x_{12} & \text{if } 0 \le x_{11} + x_{12} \le 4 \\ -\infty & \text{otherwise,} \end{cases} \\
f_{m_2}(x_{21}, x_{22}) &= \begin{cases} 10x_{21} + 10x_{22} & \text{if } 0 \le x_{21} + x_{22} \le 4 \\ -\infty & \text{otherwise,} \end{cases} \\
f_{w_1}(x_{11}, x_{21}) &= \begin{cases} 10x_{11} + 10x_{21} - (x_{11}^2 + x_{21}^2)/2 & \text{if } 0 \le x_{11} + x_{21} \le 4 \\ -\infty & \text{otherwise,} \end{cases} \\
f_{w_2}(x_{12}, x_{22}) &= \begin{cases} 10x_{12} + 10x_{22} - (x_{12}^2 + x_{22}^2)/2 & \text{if } 0 \le x_{12} + x_{22} \le 4 \\ -\infty & \text{otherwise.} \end{cases}
\end{aligned}
$$

*Then, the two aggregated utility functions $f_M = f_{m_1} + f_{m_2}$ and $f_W = f_{w_1} + f_{w_2}$ are concave functions defined by the laminar families*

$$
\begin{aligned}
&\{\, \{(1,1)\}, \{(1,2)\}, \{(2,1)\}, \{(2,2)\}, \{(1,1),(1,2)\}, \{(2,1),(2,2)\} \,\} \quad \text{and} \\
&\{\, \{(1,1)\}, \{(1,2)\}, \{(2,1)\}, \{(2,2)\}, \{(1,1),(2,1)\}, \{(1,2),(2,2)\} \,\},
\end{aligned}
$$

*respectively (see Example 2), namely $M^\natural$-concave functions, where the pairs $(m_i, w_j)$ are abbreviated by $(i, j)$. Thus, the problem can be formulated by our model with $F = \emptyset$. It is enough to set $z = (4, 4, 4, 4)$. We have three $f_M f_W$-pairwise-stable solutions $x$ together with $z_M$ and $z_W$ as follows.*

| $x = (x_{11}, x_{12}, x_{21}, x_{22})$ | $z_M$ | $z_W$ |
|---|---|---|
| $(4, 0, 0, 4)$ | $(4, 4, 4, 4)$ | $(4, 0, 0, 4)$ |
| $(3, 1, 1, 3)$ | $(3, 4, 4, 4)$ | $(4, 1, 1, 4)$ |
| $(2, 2, 2, 2)$ | $(2, 4, 4, 4)$ | $(4, 4, 4, 4)$ |

∎

**Example 4**: *We consider a problem similar to Example 3, except that pair $(1, 2)$ is flexible. Think of a situation when $m_1$ and $w_1$ are professional and the others are amateur, and $m_2$ and $w_1$ are a married couple. A lesson fee is allowed between a professional and an amateur, and a lesson fee between husband and wife is meaningless. Since there exists a flexible pair, utility functions must be represented precisely in monetary terms. We adopt the utility functions of Example 3. Then, we have the following $f_M f_W$-pairwise-stable solutions $x$, together with $p$, $z_M$, and $z_W$:*

| $x = (x_{11}, x_{12}, x_{21}, x_{22})$ | $p$ | $z_M$ | $z_W$ |
|---|---|---|---|
| $(3, 1, 1, 3)$ | $(0, \alpha, 0, 0)$ | $(3, -, 4, 4)$ | $(4, -, 1, 4)$ |
| $(2, 2, 2, 2)$ | $(0, \beta, 0, 0)$ | $(2, -, 4, 4)$ | $(4, -, 4, 2)$ |
| $(1, 3, 3, 1)$ | $(0, 2, 0, 0)$ | $(4, -, 4, 4)$ | $(1, -, 4, 1)$ |
| $(0, 4, 4, 0)$ | $(0, \gamma, 0, 0)$ | $(4, -, 4, 4)$ | $(0, -, 4, 0)$ |

*where* $1 \leq \alpha \leq 2$, $-1 \leq \beta \leq 2$ *and* $2 \leq \gamma \leq 6.5$. ∎

Our main result is the following theorem about the existence of an $f_M f_W$-pairwise-stable solution in our model.

**Theorem 3.1** (Main Theorem): *For any* $\mathrm{M}^\natural$*-concave functions* $f_M, f_W : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ *satisfying* (A) *and for any partition* $(F, R)$ *of* $E$*, there exists an* $f_M f_W$*-pairwise-stable solution with respect to* $(F, R)$.

A proof of the main theorem will be given in Section 5.

**Remark 1**: *In our model, each of* $M$ *and* $W$ *is regarded as a single aggregate agent but can be interpreted as a set of agents. Let* $M = \{1, \cdots, m\}$, $W = \{1, \cdots, w\}$, *and* $E = M \times W$. *Also, define* $E_i = \{i\} \times W$ *for all* $i \in M$, *and* $E_j = M \times \{j\}$ *for all* $j \in W$. *Suppose that each agent* $i \in M$ *has an* $\mathrm{M}^\natural$*-concave utility function* $f_i : \mathbf{Z}^{E_i} \to \mathbf{R} \cup \{-\infty\}$ *on* $E_i$, *and that each agent* $j \in W$ *has an* $\mathrm{M}^\natural$*-concave utility function* $f_j : \mathbf{Z}^{E_j} \to \mathbf{R} \cup \{-\infty\}$ *on* $E_j$. *Aggregations* $f_M(x) = \sum_{i \in M} f_i(x|_{E_i})$ *and* $f_W(x) = \sum_{j \in W} f_j(x|_{E_j})$ *in* $x \in \mathbf{Z}^E$ *are also* $\mathrm{M}^\natural$*-concave. Moreover,* $E$ *can arbitrarily be partitioned into a set of flexible pairs and a set of rigid pairs. It should be noted that this model is mathematically equivalent to our model.* ∎

**Remark 2**: *Our definition of an* $f_M f_W$*-pairwise-stable solution apparently depends on* $z$. *But it does not essentially depend on* $z$ *as long as* $z$ *is large enough so that* $\mathrm{dom}\, f_M \cup \mathrm{dom}\, f_W \subseteq \{y \in \mathbf{Z}^E \mid \mathbf{0} \leq y \leq z\}$. *One can see that* $x \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ *is an* $f_M f_W$*-pairwise-stable solution with respect to* $(F, R)$ *if and only if there exist* $p \in \mathbf{R}^E$, *disjoint subsets* $R_M$ *and* $R_W$ *of* $R$, $z_M \in \mathbf{Z}^{R_M}$, *and* $z_W \in \mathbf{Z}^{R_W}$ *satisfying (3.1) and*

$$
\begin{aligned}
x &\in \arg\max\{f_M[+p](y) \mid y|_{R_M} \leq z_M\}, \\
x &\in \arg\max\{f_W[-p](y) \mid y|_{R_W} \leq z_W\}.
\end{aligned}
$$

∎

**Remark 3**: *When* $M$ *and* $W$ *are, respectively, a set of workers and a set of firms, the coordinates of* $p$ *can be interpreted as salaries and hence* $p$ *should be nonnegative. Although our model does not impose such a condition, the nonnegativity of* $p$ *can be derived as follows. Suppose that* $f_W(x)$ *denotes the total profit of the firms obtained by allocation* $x$ *between workers and firms, and that* $\mathrm{dom}\, f_M$ *is the set of allocations acceptable for workers and* $f_M$ *is identically zero on* $\mathrm{dom}\, f_M$. *Then, for an* $f_M f_W$*-pairwise-stable solution* $x$ *and for a flexible element* $e$ *with* $x(e) > 0$, *we have* $p(e) \geq 0$ *because* $f_M[+p](x) \geq f_M[+p](x - \chi_e)$ *and* $f_M(x) = f_M(x - \chi_e) = 0$.
∎

**Remark 4**: *When* $E = F$, $x \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ *is an* $f_M f_W$*-pairwise-stable solution if and only if there exists* $p \in \mathbf{R}^E$ *such that*

$$x \in \arg\max f_M[+p], \tag{3.5}$$

$$x \in \arg\max f_W[-p]. \tag{3.6}$$

*It is a direct consequence of the following theorem that the set of all* $f_M f_W$*-pairwise-stable solutions coincides with the set of all maximizers of* $f_M + f_W$ *(see also Murota [20]).*

**Theorem 3.2** ([17]): *For* $M^\natural$*-concave functions* $f_1, f_2 : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ *and a point* $x^* \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$*, we have* $x^* \in \arg\max(f_1 + f_2)$ *if and only if there exists* $p^* \in \mathbf{R}^E$ *such that* $x^* \in \arg\max f_1[+p^*]$ *and* $x^* \in \arg\max f_2[-p^*]$*. Furthermore, for such a* $p^*$*, we have*

$$\arg\max(f_1 + f_2) = \arg\max(f_1[+p^*]) \cap \arg\max(f_2[-p^*]).$$

*Since (A) guarantees that* $\mathrm{dom}\,(f_M + f_W)$ *is nonempty and bounded,* $f_M + f_W$ *has a maximizer, which implies the existence of an* $f_M f_W$*-pairwise-stable solution with respect to* $(E, \emptyset)$*. We also give an algorithm for finding an* $f_M f_W$*-pairwise-stable solution with respect to* $(E, \emptyset)$ *in Section 5.2.* ∎

# 4. Existing Special Models

The marriage model and the assignment model are special cases of our model as described in Section 2.2. In this section we show that several extensions of these models are also special cases of ours.

## 4.1. Extensions of the Marriage Model

Fleiner [7] has generalized the marriage model to matroids. A triple $\mathcal{M} = (E, \mathcal{I}, >)$ is called an *ordered matroid*, if $\mathcal{I}$ is the family of independent sets of a matroid on $E$ and $>$ is a linear order on $E$. An element $e \in E$ is *dominated* by $X \subseteq E$ if $e \in X$ or there exists $Y \in \mathcal{I}$ such that $Y \subseteq X$, $\{e\} \cup Y \notin \mathcal{I}$ and $e' > e$ for all $e' \in Y$. The set of elements dominated by $X$ is denoted by $D_{\mathcal{M}}(X)$. Given two ordered matroids $\mathcal{M}_M = (E, \mathcal{I}_M, >_M)$ and $\mathcal{M}_W = (E, \mathcal{I}_W, >_W)$, $X \subseteq E$ is called an $\mathcal{M}_M\mathcal{M}_W$*-kernel* if

(m4) $X \in \mathcal{I}_M \cap \mathcal{I}_W$ and $D_{\mathcal{M}_M}(X) \cup D_{\mathcal{M}_W}(X) = E$.

The marriage model $(M, W, \{a_{ij}\}, \{b_{ij}\})$ without indifference can be formulated as the matroidal model. Let $E$ be the set of pairs $(i, j)$ with $a_{ij}, b_{ij} > 0$. Also, define

$E_i = \{(i,j) \in E \mid j \in W\}$ for all $i \in M$, and $E_j = \{(i,j) \in E \mid i \in M\}$ for all $j \in W$. It is known that

$$\begin{aligned} \mathcal{I}_M &= \{X \subseteq E \mid |X \cap E_i| \leq 1 \text{ for all } i \in M\}, \text{ and} \\ \mathcal{I}_W &= \{X \subseteq E \mid |X \cap E_j| \leq 1 \text{ for all } j \in W\} \end{aligned}$$

are the families of independent sets of matroids. Then, $X$ is a matching if and only if $X \in \mathcal{I}_M \cap \mathcal{I}_W$. By defining linear orders $>_M$ and $>_W$ on $E$ so that $(i,j_1) >_M (i,j_2)$ whenever $a_{ij_1} > a_{ij_2}$, and $(i_1,j) >_W (i_2,j)$ whenever $b_{i_1j} > b_{i_2j}$, a matching $X$ is an $\mathcal{M}_M\mathcal{M}_W$-kernel if and only if for any pair $(i,j) \in E \setminus X$ there exists either $(i,j')$ or $(i',j)$ in $X$ such that either $(i,j') >_M (i,j)$ or $(i',j) >_W (i,j)$. Hence, the set of $\mathcal{M}_M\mathcal{M}_W$-kernels coincides with the set of pairwise-stable matchings. The matroidal model can easily be modified so that indifference is allowed.

Eguchi and Fujishige [5] proposed a model based on M$^\natural$-concavity, which is a restriction of our model in which $E = R$ and $\mathrm{dom}\, f_M$, $\mathrm{dom}\, f_W \subseteq \{0,1\}^E$. We identify a subset of $E$ with its characteristic vector. The above matroidal model can be recognized as a special case of this model in which utility functions are linear. Let $\mathcal{M}_M = (E, \mathcal{I}_M, >_M)$ and $\mathcal{M}_W = (E, \mathcal{I}_W, >_W)$ be an instance of the matroidal model. We describe linear orders $>_M$ and $>_W$ by positive numbers $\{a_e\}$ and $\{b_e\}$ such that $a_{e'} > a_e \iff e' >_M e$ and $b_{e'} > b_e \iff e' >_W e$, and define functions $f_M$ and $f_W$ by

$$f_M(X) = \begin{cases} \sum_{e \in X} a_e & \text{if } X \in \mathcal{I}_M \\ -\infty & \text{otherwise,} \end{cases} \qquad f_W(X) = \begin{cases} \sum_{e \in X} b_e & \text{if } X \in \mathcal{I}_W \\ -\infty & \text{otherwise,} \end{cases} \tag{4.1}$$

which are M$^\natural$-concave by Example 1. From basic theorems in matroid theory, we can show that a subset $X$ of $E$ is an $\mathcal{M}_M\mathcal{M}_W$-kernel if and only if it is $f_Mf_W$-pairwise-stable for the M$^\natural$-concave functions specified by (4.1).

Our model with $E = R$ includes all of the above-mentioned models.

## 4.2. Extensions of the Assignment Model

Sotomayor [33] proposed an extension of the assignment model in which $M$ and $W$ denote sets of firms and workers, respectively, and each firm $i \in M$ has a quota of $\alpha_i (> 0)$ units of labor-time for hiring workers, and each worker $j \in W$ can supply at most $\beta_j (> 0)$ units of time. Pair $(i,j)$ can earn $c_{ij} (= a_{ij} + b_{ij})$ per unit time. Instead of considering matchings, let $x_{ij}$ be the number of time units for which $i$ hires $j$, and let $x$ be called a *labor allocation*. A labor allocation $x \in \mathbf{Z}^{M \times W}$ is called *feasible* if $x \geq \mathbf{0}$ and the following two inequalities hold:

$$\sum_{j \in W} x_{ij} \leq \alpha_i \qquad \text{for all } i \in M, \tag{4.2}$$

$$\sum_{i \in M} x_{ij} \leq \beta_j \qquad \text{for all } j \in W. \tag{4.3}$$

For any subsets $M' \subseteq M$ and $W' \subseteq W$, let $P(M', W')$ denote the maximum of $\sum_{i \in M'} \sum_{j \in W'} c_{ij} x_{ij}$ over all feasible labor allocations $x$. We call a pair $(q, r) \in \mathbf{R}^M \times \mathbf{R}^W$ a *money allocation*. Let $q(M) = \sum_{i \in M} q_i$ and $r(W) = \sum_{j \in W} r_j$. A money allocation $(q, r)$ is *feasible* if $q \geq \mathbf{0}$, $r \geq \mathbf{0}$, and $q(M) + r(W) \leq P(M, W)$. It is in the *core* if it is feasible and $q(M') + r(W') \geq P(M', W')$ for all coalitions $M' \subseteq M$ and $W' \subseteq W$. She showed that an element of the core is derived from a dual optimal solution of the transportation problem:

$$\text{Maximize} \sum_{(i,j) \in E} c_{ij} x_{ij} \quad \text{subject to} \quad (4.2),\ (4.3),\ x \geq \mathbf{0},$$

which implies the nonemptiness of the core. Therefore, in our context, by defining M$^\natural$-concave functions $f_M$ and $f_W$ as

$$f_M(x) = \begin{cases} \displaystyle\sum_{(i,j) \in E} c_{ij} x_{ij} & \text{if } x \in \mathbf{Z}^E \text{ satisfies (4.2) and } x \geq \mathbf{0} \\ -\infty & \text{otherwise,} \end{cases}$$

$$f_W(x) = \begin{cases} 0 & \text{if } x \in \mathbf{Z}^E \text{ satisfies (4.3) and } x \geq \mathbf{0} \\ -\infty & \text{otherwise,} \end{cases}$$

an $f_M f_W$-pairwise-stable solution $x$, together with $p$, gives a money allocation $(q, r)$ in the core. Such an allocation $(q, r)$ is defined by

$$q_i = \sum_{j : x_{ij} > 0} (c_{ij} + p_{ij}) x_{ij} \quad \text{for all } i \in M,$$
$$r_j = \sum_{i : x_{ij} > 0} (-p_{ij}) x_{ij} \quad \text{for all } j \in W.$$

However, the converse does not necessarily hold, as the core may strictly contain the set of dual optimal solutions (see [33, Example 2]).

Kelso and Crawford [15] introduced a many-to-one labor market model in which the profit function of each firm satisfies gross substitutability and the utility function of each worker is strictly increasing (not necessarily linear) in salary. Danilov, Koshevoy, and Murota [4] provided, for the first time, a model that is based on discrete convex analysis. Our model is closely related to these models.

## 4.3. A Hybrid Model

Eriksson and Karlander [6] proposed a hybrid model of the marriage model and the assignment model. In this model, agents are partitioned into two categories, called *flexible agents* and *rigid agents*, that is, $M$ and $W$ are partitioned into $(M_F, M_R)$

and $(W_F, W_R)$, and $F$ and $R$ are defined by

$$\begin{aligned} F &= \{(i,j) \in E \mid i \in M_F \text{ and } j \in W_F\}, \\ R &= \{(i,j) \in E \mid i \in M_R \text{ or } j \in W_R\}. \end{aligned}$$

A generalization of the hybrid model was also given by Sotomayor [32]. Here, we adopt the notion of pairwise-stability of her generalized version. An outcome $(q, r; X)$ is called *pairwise-stable* if

(h1) $X$ is a matching,

(h2) $q_i + r_j = a_{ij} + b_{ij}$ for all $(i,j) \in X$,

(h3) $q_i = a_{ij} > -\infty$ and $r_j = b_{ij} > -\infty$ for all $(i,j) \in X \cap R$,

(h4) $q_i = 0$ (resp. $r_j = 0$) if $i$ (resp. $j$) is unmatched in $X$,

(h5) $q \geq \mathbf{0}$, $r \geq \mathbf{0}$, and $q_i + r_j \geq a_{ij} + b_{ij}$ for all $(i,j) \in F$,

(h6) $q_i \geq a_{ij}$ or $r_j \geq b_{ij}$ for all $(i,j) \in R$.

When $E = R$ (resp. $E = F$), Conditions (h1)$\sim$(h6) are obviously equivalent to (m1)$\sim$(m3) (resp. (a1)$\sim$(a4)). As is seen from the discussion in Section 2.2, our model includes this hybrid model as a special case.

# 5. Proof

In this section we prove our main theorem, Theorem 3.1. We give a constructive proof by combining two algorithms, one for the marriage case and the other for the assignment case. We divide our arguments into three parts that deal with

(i) a variant of the marriage model,

(ii) a variant of the assignment model, and

(iii) a combination of the two.

Readers will easily understand the argument for the general model (iii) by first understanding the algorithms for (i) and (ii). The algorithm for (i) is interesting in its own right as it is a natural generalization of the Gale-Shapley algorithm [11]. On the other hand, the other parts of our constructive proof are rather technical; for example, the algorithm for (ii), called a successive shortest path algorithm, is a generalization of an algorithm for a network flow problem.

## 5.1. The Marriage Case

For a given partition $(F, R)$ we give an algorithm for finding $x_M, x_W \in \mathbf{Z}^E$ and $z_M, z_W \in \mathbf{Z}^R$ satisfying (3.2) and

$$\begin{aligned} x_M &\in \arg\max\{f_M(y) \mid y|_R \leq z_M\}, & (5.1)\\ x_W &\in \arg\max\{f_W(y) \mid y|_R \leq z_W\}, & (5.2)\\ x_M|_R &= x_W|_R. & (5.3) \end{aligned}$$

Here it should be noted that $(F, R)$ can be any partition of $E$ and that if $E = R$ and if there exist $z_M, z_W \in \mathbf{Z}^E$ satisfying (3.2), (5.1), and (5.2) with $x_M = x_W = x$, then $x \in \mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ is an $f_M f_W$-pairwise-stable solution. Hence the algorithm proposed below can find an $f_M f_W$-pairwise-stable solution with respect to $(\emptyset, E)$.

We first state three fundamental lemmas, which hold without Assumption (A).

**Lemma 5.1** ([19]): *Let $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ be an M$^\natural$-concave function and $U$ be a nonempty subset of $E$. Define the function $f^U : \mathbf{Z}^U \to \mathbf{R} \cup \{\pm\infty\}$ by*

$$f^U(x) = \sup\{f(y) \mid y \in \mathbf{Z}^E,\ y|_U = x\}$$

*for each $x \in \mathbf{Z}^U$. If $f^U(x) < +\infty$ for all $x \in \mathbf{Z}^U$, then $f^U$ is an M$^\natural$-concave function. In particular, if $\mathrm{dom}\, f$ is bounded, then $f^U$ is M$^\natural$-concave.*

**Lemma 5.2**: [14] *Let $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ be an M$^\natural$-concave function and $z_1, z_2 \in \mathbf{Z}^E$ be such that $z_1 \geq z_2$, $\arg\max\{f(y) \mid y \leq z_1\} \neq \emptyset$, and $\arg\max\{f(y) \mid y \leq z_2\} \neq \emptyset$.*

(a) *For any $x_1 \in \arg\max\{f(y) \mid y \leq z_1\}$, there exists $x_2$ such that*

$$x_2 \in \arg\max\{f(y) \mid y \leq z_2\} \quad \textit{and} \quad z_2 \wedge x_1 \leq x_2.$$

(b) *For any $x_2 \in \arg\max\{f(y) \mid y \leq z_2\}$, there exists $x_1$ such that*

$$x_1 \in \arg\max\{f(y) \mid y \leq z_1\} \quad \textit{and} \quad z_2 \wedge x_1 \leq x_2.$$

**Proof.** (a): Let $x_2$ be a minimizer of $\sum\{x_1(e) - x_2(e) \mid e \in \mathrm{supp}^+((z_2 \wedge x_1) - x_2)\}$ on $\arg\max\{f(y) \mid y \leq z_2\}$. We show $z_2 \wedge x_1 \leq x_2$. Suppose, to the contrary, that

[14] This lemma says that $C : \mathrm{dom}\, f \to 2^{\mathrm{dom}\, f}$ defined by $C(z) = \arg\max\{f(y) \mid y \leq z\}$ satisfies "substitutability," where $2^{\mathrm{dom}\, f}$ denotes the set of all subsets of $\mathrm{dom}\, f$. In fact, if $\mathrm{dom}\, f \subseteq \{0,1\}^E$ then statements (a) and (b) are equivalent to conditions of substitutability in Sotomayor [30, Definition 4], and if $C$ always gives a singleton (in this case (a) and (b) are equivalent) then (a) and (b) are equivalent to persistence (substitutability) in Alkan and Gale [1].

there exists $e \in E$ with $\min\{z_2(e), x_1(e)\} > x_2(e)$. Then $e \in \mathrm{supp}^+(x_1 - x_2)$. By ($\mathrm{M}^\natural$), there exists $e' \in \mathrm{supp}^-(x_1 - x_2) \cup \{0\}$ with

$$f(x_1) + f(x_2) \leq f(x_1 - \chi_e + \chi_{e'}) + f(x_2 + \chi_e - \chi_{e'}). \tag{5.4}$$

If $e' \neq 0$, then $x_1(e') < x_2(e') \leq z_2(e') \leq z_1(e')$. Hence $x_1 - \chi_e + \chi_{e'} \leq z_1$, which implies $f(x_1) \geq f(x_1 - \chi_e + \chi_{e'})$. This, together with (5.4), yields $f(x_2) \leq f(x_2 + \chi_e - \chi_{e'})$. Moreover, since $z_2(e) > x_2(e)$, we have $x'_2 = x_2 + \chi_e - \chi_{e'} \leq z_2$. It follows that $x'_2 \in \arg\max\{f(y) \mid y \leq z_2\}$ and $x'_2(e') \geq \min\{z_2(e'), x_1(e')\}$ if $e' \neq 0$, which contradicts the minimality of $x_2$.

(b): Let $x_1$ be a minimizer of $\sum\{x_1(e) - x_2(e) \mid e \in \mathrm{supp}^+((z_2 \wedge x_1) - x_2)\}$ on $\arg\max\{f(y) \mid y \leq z_1\}$. We show $z_2 \wedge x_1 \leq x_2$. Suppose, to the contrary, that there exists $e \in E$ with $\min\{z_2(e), x_1(e)\} > x_2(e)$. Then $e \in \mathrm{supp}^+(x_1 - x_2)$. By ($\mathrm{M}^\natural$), there exists $e' \in \mathrm{supp}^-(x_1 - x_2) \cup \{0\}$ with

$$f(x_1) + f(x_2) \leq f(x_1 - \chi_e + \chi_{e'}) + f(x_2 + \chi_e - \chi_{e'}). \tag{5.5}$$

Since $x_2(e) < z_2(e)$, we have $x_2 + \chi_e - \chi_{e'} \leq z_2$, which implies $f(x_2) \geq f(x_2 + \chi_e - \chi_{e'})$. This, together with (5.5), yields $f(x_1) \leq f(x_1 - \chi_e + \chi_{e'})$. Obviously $x'_1 = x_1 - \chi_e + \chi_{e'} \leq z_1$. However, this contradicts the minimality of $x_1$ because $x_2(e') \geq \min\{z_2(e'), x'_1(e')\}$ if $e' \neq 0$. ∎

**Lemma 5.3**: *For an $\mathrm{M}^\natural$-concave function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ and a vector $z_2 \in \mathbf{Z}^E$ suppose that $\arg\max\{f(y) \mid y \leq z_2\} \neq \emptyset$. For any $x \in \arg\max\{f(y) \mid y \leq z_2\}$ and any $z_1 \in \mathbf{Z}^E$ such that* (i) *$z_1 \geq z_2$ and* (ii) *if $x(e) = z_2(e)$ then $z_1(e) = z_2(e)$, we have $x \in \arg\max\{f(y) \mid y \leq z_1\}$.*

**Proof.** Assume to the contrary that the assertion is not satisfied. Let $x'$ be a point minimizing $\sum\{y(e) - z_2(e) \mid e \in \mathrm{supp}^+(y - z_2)\}$ in $y$ subject to $y \leq z_1$ and $f(y) > f(x)$. By the assumption, there exists $e \in E$ with $x'(e) > z_2(e) > x(e)$. By ($\mathrm{M}^\natural$) for $x'$, $x$, and $e$, there exists $e' \in \mathrm{supp}^-(x' - x) \cup \{0\}$ such that

$$f(x') + f(x) \leq f(x' - \chi_e + \chi_{e'}) + f(x + \chi_e - \chi_{e'}).$$

Since $x + \chi_e - \chi_{e'} \leq z_2$, we have $f(x) \geq f(x + \chi_e - \chi_{e'})$, which implies $f(x') \leq f(x' - \chi_e + \chi_{e'})$. Obviously, $x' - \chi_e + \chi_{e'} \leq z_1$, However, this contradicts the minimality of $x'$ because if $e' \neq 0$, then $z_2(e') \geq x(e') > x'(e')$. ∎

Our algorithm for finding $x_M, x_W \in \mathbf{Z}^E$ and $z_M, z_W \in \mathbf{Z}^R$ satisfying (3.2), (5.1), (5.2), and (5.3) is a natural generalization of the Gale-Shapley algorithm [11], which consists of proposal and rejection steps. Although we can deal with more general cases, we illustrate our algorithm by considering a labor allocation model in which $M$ and $W$ are sets of firms and workers. At each iteration, firms first offer a labor allocation $x_M$ maximizing their aggregated utility $f_M$ under the constraint $x_M \leq z_M$, where $z_M((i,j))$ represents firm $i$'s quota of time units for hiring worker $j$. This is a proposal step. Next is a rejection step. Given the offer of firms, workers maximize the aggregated utility $f_W$ among possible allocations less than or equal to $x_M$. For workers' choice $x_W$ and $e = (i,j)$, if $x_W(e) < x_M(e)$ then $z_M(e)$ is updated as $x_W(e)$ because worker $j$ does not supply more than $x_W(e)$ time units to firm $i$. Our algorithm iterates the above two steps until $x_M = x_W$.

To describe our algorithm in more detail, we assume that we are initially given $x_M, x_W \in \mathbf{Z}^E$ and $z_M, z_W \in \mathbf{Z}^R$ satisfying (3.2), (5.1) and the following:

$$\begin{aligned} x_W &\in \arg\max\{f_W(y) \mid\ y|_R \leq z_W \vee x_M|_R\}, && (5.6)\\ x_W|_R &\leq x_M|_R. && (5.7) \end{aligned}$$

We can easily compute these initial vectors by setting $z_M = z|_R$, $z_W = \mathbf{0}$, and finding $x_M$ and $x_W$ such that

$$\begin{aligned} x_M &\in \arg\max\{f_M(y) \mid\ y|_R \leq z_M\},\\ x_W &\in \arg\max\{f_W(y) \mid\ y|_R \leq x_M|_R\}. \end{aligned}$$

Here is our algorithm.

**Algorithm G_GS** $(f_M, f_W, x_M, x_W, z_M, z_W)$

**Input:** M$^\natural$-concave functions $f_M, f_W$ and $x_M$, $x_W$, $z_M$, $z_W$ satisfying (3.2), (5.1), (5.6), (5.7).

**Step 1.** Find $x_M \in \arg\max\{f_M(y) \mid\ x_W|_R \leq y|_R \leq z_M\}$.

**Step 2.** Find $x_W \in \arg\max\{f_W(y) \mid\ y|_R \leq x_M|_R\}$.

**Step 3.** For each $e \in R$ with $x_M(e) > x_W(e)$, set $z_M(e) := x_W(e)$, $z_W(e) := z(e)$[15].

**Step 4.** If $x_M|_R = x_W|_R$ then output $(x_M, x_W, z_M, z_W \vee x_M|_R)$. Else go to Step 1.

From (A), $x_M$ and $x_W$ are well-defined within the effective domains and G_GS terminates after at most $\sum_{e \in R} z(e)$ iterations, because $\sum_{e \in R} z_M(e)$ strictly decreases at each iteration. In order to show that the outputs of G_GS satisfy (3.2), (5.1), (5.2), and (5.3), we establish two lemmas.

Let us assume that $x_M^{(0)}$, $x_W^{(0)}$, $z_M^{(0)}$, and $z_W^{(0)}$ are the input vectors. Also let $x_M^{(i)}$, $x_W^{(i)}$, $z_M^{(i)}$, and $z_W^{(i)}$ be obtained from $x_M^{(i-1)}$, $x_W^{(i-1)}$, $z_M^{(i-1)}$, and $z_W^{(i-1)}$ at the $i$th iteration in G_GS for $i = 1, 2, \cdots, n$, where the algorithm terminates at the $n$th iteration.

[15] $c := d$ means that we assign the (current) value of $d$ to $c$.

**Lemma 5.4**: *For all* $i = 0, 1, \cdots, n$, *we have*

$$x_M^{(i+1)} \in \arg\max \left\{ f_M(y) \;\middle|\; y|_R \leq z_M^{(i)} \right\}. \tag{5.8}$$

**Proof.** We prove (5.8) by induction on $i$. For $i = 0$, (5.8) holds by (5.1) and (5.7). We assume that for some $l$ with $0 \leq l < t$, (5.8) holds for each $i$ with $0 \leq i \leq l$, and we show (5.8) for $i = l + 1$. Since $x_M^{(l+1)} \in \arg\max\{f_M(y) \mid y|_R \leq z_M^{(l)}\}$ and $z_M^{(l)} \geq z_M^{(l+1)}$, Lemma 5.1 and Lemma 5.2 (a) guarantee the existence of $x \in \arg\max\{f_M(y) \mid y|_R \leq z_M^{(l+1)}\}$ with $z_M^{(l+1)} \wedge x_M^{(l+1)}|_R \leq x|_R$. By the modification of $z_M$, this implies (5.8) for $i = l + 1$ because $z_M^{(l+1)} \wedge x_M^{(l+1)}|_R = x_W^{(l+1)}|_R$. ∎

**Lemma 5.5**: *For all* $i = 0, 1, \cdots, n$, *we have*

$$x_W^{(i)} \in \arg\max \left\{ f_W(y) \;\middle|\; y|_R \leq z_W^{(i)} \vee x_M^{(i)}|_R \right\}. \tag{5.9}$$

**Proof.** We show (5.9) by induction on $i$. For $i = 0$, (5.9) holds by (5.6). We assume that for some $l$ with $0 \leq l < t$, (5.9) holds for each $i$ with $0 \leq i \leq l$, and we show (5.9) for $i = l + 1$. By Lemma 5.1, Lemma 5.2 (b), and (5.9) for $i = l$, there exists $x$ such that

$$x \in \arg\max \left\{ f_W(y) \;\middle|\; y|_R \leq z_W^{(l)} \vee (x_M^{(l)}|_R) \vee (x_M^{(l+1)}|_R) \right\}, \tag{5.10}$$

$$\left( z_W^{(l)} \vee x_M^{(l)}|_R \right) \wedge x|_R \leq x_W^{(l)}|_R. \tag{5.11}$$

On the other hand, by the definition of $x_M$ we have

$$x_W^{(l)}|_R \leq x_M^{(l+1)}|_R. \tag{5.12}$$

From (5.10), (5.11), and (5.12), we have $x|_R \leq x_M^{(l+1)}|_R$ and hence $f_W(x) = f_W(x_W^{(l+1)})$. If $z_W^{(l+1)} = z_W^{(l)}$, then we immediately obtain (5.9) for $i = l + 1$. So, we assume that $z_W^{(l+1)} \neq z_W^{(l)}$. By the modification of $z_W$, we have $x_W^{(l+1)}(e) < x_M^{(l+1)}(e)$ if $z_W^{(l)}(e) < z_W^{(l+1)}(e)$. Hence it follows from Lemma 5.3 that (5.9) holds for $i = l+1$. ∎

The correctness of G_GS follows from Lemmas 5.4 and 5.5.

**Theorem 5.6**: *The outputs of* G_GS *satisfy* (3.2), (5.1), (5.2), *and* (5.3).

**Proof.** From Lemmas 5.4 and 5.5 we have for $i = n$

$$\begin{aligned} x_M^{(n)} &\in \arg\max \left\{ f_M(y) \;\middle|\; y|_R \leq z_M^{(n)} \right\}, \\ x_W^{(n)} &\in \arg\max \left\{ f_W(y) \;\middle|\; y|_R \leq z_W^{(n)} \vee x_M^{(n)}|_R \right\}, \\ x_M^{(n)}|_R &= x_W^{(n)}|_R. \end{aligned}$$

Because of the way in which we modified $z_M$, $z_W$, and $x_M$, we have

$$z_M^{(n)} \vee \left( z_W^{(n)} \vee x_M^{(n)}|_R \right) = z|_R.$$

∎

The following is a by-product of Theorem 5.6.

**Corollary 5.7**: *For any* $M^\natural$*-concave functions* $f_M, f_W : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ *satisfying* (A)*, there exists an* $f_M f_W$*-pairwise-stable solution with respect to* $(\emptyset, E)$.

## 5.2. The Assignment Case

In this subsection we explain a successive shortest path algorithm (SSP) for finding a maximizer of $f_M + f_W$. It is a modified version of an algorithm given by Moriguchi and Murota [16]. As discussed in Remark 4, SSP finds an $f_M f_W$-pairwise-stable solution with respect to $(E, \emptyset)$. We recall that SSP is used as a basic procedure for finding an $f_M f_W$-pairwise-stable solution for our general case.

Before describing SSP, we state several known results on M$^\natural$-concave functions. Let $y(E) = \sum_{e \in E} y(e)$ for all $y \in \mathbf{Z}^E$. For an M$^\natural$-concave function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$, we define $\hat{f} : \mathbf{Z}^{\{0\} \cup E} \to \mathbf{R} \cup \{-\infty\}$ by

$$\hat{f}(y_0, y) = \begin{cases} f(y) & \text{if } y_0 = -y(E) \\ -\infty & \text{otherwise} \end{cases}$$

for all $(y_0, y) \in \mathbf{Z}^{\{0\} \cup E}$. Function $\hat{f}$ is called an *M-concave function* and can be characterized by the following exchange property[16] [17, 18]:

(M) for all $x, y \in \mathrm{dom}\, \hat{f}$ and all $e \in \mathrm{supp}^+(x - y)$, there exists $e' \in \mathrm{supp}^-(x - y)$ such that

$$\hat{f}(x) + \hat{f}(y) \le \hat{f}(x - \chi_e + \chi_{e'}) + \hat{f}(y + \chi_e - \chi_{e'}).$$

In particular, an M-concave function is also M$^\natural$-concave. We denote $\{0\} \cup E$ by $\hat{E}$. For a vector $x \in \mathbf{R}^E$ we denote by $\hat{x}$ the vector $(-x(E), x) \in \mathbf{R}^{\hat{E}}$. For a vector $\tilde{p} = (p_0, p) \in \mathbf{R}^{\hat{E}}$,

$$x \in \arg\max f[p - p_0 \mathbf{1}] \iff \hat{x} \in \arg\max \hat{f}[\tilde{p}]. \tag{5.13}$$

Thus, the problem of finding an $f_M f_W$-pairwise-stable solution with respect to $(E, \emptyset)$ is equivalent to that of finding a maximizer of $\hat{f}_M + \hat{f}_W$.

The maximizers of an M-concave function have a useful characterization[17].

[16] (M) is written as (−M-EXC) in [20].

[17] The sum of two M-concave functions is not M-concave in general. So we need a sophisticated characterization for the maximizers of the sum of two M-concave functions (see Theorem 3.2).

**Theorem 5.8** ([17, 18]): *For any M-concave function* $\hat{f} : \mathbf{Z}^{\hat{E}} \to \mathbf{R} \cup \{-\infty\}$ *and* $x \in \mathrm{dom}\, \hat{f}$, $x \in \arg\max \hat{f}$ *if and only if* $\hat{f}(x) \geq \hat{f}(x - \chi_e + \chi_{e'})$ *for all* $e, e' \in \hat{E}$.

The following property is a direct consequence of property (M).

**Lemma 5.9**: *For any M-concave function* $\hat{f}$, $\arg\max \hat{f}$ *satisfies: for any* $x, y \in \arg\max \hat{f}$ *and any* $e \in \mathrm{supp}^+(x - y)$, *there exists* $e' \in \mathrm{supp}^-(x - y)$ *such that* $x - \chi_e + \chi_{e'}$, $y + \chi_e - \chi_{e'} \in \arg\max \hat{f}$.

A set $B$ of integral vectors satisfying the property in Lemma 5.9 is called an *M-convex set*. M-convex sets have the following property:

**Lemma 5.10** ([9, Lemma 4.5]): *Let* $B$ *be an M-convex set. For any* $x \in B$ *and any distinct* $e_1, e'_1, e_2, e'_2, \cdots, e_r, e'_r \in \hat{E}$, *if* $x - \chi_{e_i} + \chi_{e'_i} \in B$ *for all* $i = 1, \cdots, r$ *and* $x - \chi_{e_i} + \chi_{e'_j} \notin B$ *for all* $i, j$ *with* $i < j$, *then* $y = x - \sum_{i=1}^{r} (\chi_{e_i} - \chi_{e'_i}) \in B$.

Now, we return to explaining SSP. Let $\hat{x}_M$ and $\hat{x}_W$ be arbitrary maximizers of $\hat{f}_M$ and $\hat{f}_W$, respectively. We construct a directed graph $G = (\hat{E}, A)$ and an arc length $\ell \in \mathbf{R}^A$ as follows. Arc set $A$ has two disjoint parts:

$$\begin{array}{rcl} A_M & = & \{(e, e') \mid e, e' \in \hat{E},\ e \neq e',\ \hat{x}_M - \chi_e + \chi_{e'} \in \mathrm{dom}\, \hat{f}_M\}, \\ A_W & = & \{(e', e) \mid e, e' \in \hat{E},\ e \neq e',\ \hat{x}_W - \chi_e + \chi_{e'} \in \mathrm{dom}\, \hat{f}_W\}, \end{array} \tag{5.14}$$

and $\ell \in \mathbf{R}^A$ is defined by

$$\ell(a) = \begin{cases} \hat{f}_M(\hat{x}_M) - \hat{f}_M(\hat{x}_M - \chi_e + \chi_{e'}) & \text{if } a = (e, e') \in A_M \\ \hat{f}_W(\hat{x}_W) - \hat{f}_W(\hat{x}_W - \chi_e + \chi_{e'}) & \text{if } a = (e', e) \in A_W. \end{cases} \tag{5.15}$$

The length function $\ell$ is nonnegative due to Theorem 5.8.

For a set $S$ of specified source vertices of $\hat{E}$, let $d : \hat{E} \to \mathbf{R} \cup \{+\infty\}$ denote the shortest distances from $S$ to all vertices in $G$ with respect to $\ell$. Then, for all arcs $a = (e, e') \in A$

$$\ell(a) + d(e) - d(e') \geq 0.$$

Let $t$ be an arbitrary vertex of $\hat{E}$ reachable from $S$, and define $\tilde{p} \in \mathbf{R}^{\hat{E}}$ by $\tilde{p}(e) = \min\{d(e), d(t)\}$ for all $e \in \hat{E}$. It follows from the nonnegativity of $\ell$ that for all arcs $a = (e, e') \in A$

$$\ell(a) + \tilde{p}(e) - \tilde{p}(e') \geq 0.$$

The above system of inequalities is equivalent to

$$\begin{array}{l} \hat{f}_M(\hat{x}_M) - \hat{f}_M(\hat{x}_M - \chi_e + \chi_{e'}) + \tilde{p}(e) - \tilde{p}(e') \geq 0 \\ \hat{f}_W(\hat{x}_W) - \hat{f}_W(\hat{x}_W - \chi_e + \chi_{e'}) - \tilde{p}(e) + \tilde{p}(e') \geq 0 \end{array}$$

for all $e, e' \in \hat{E}$, which is further equivalent to

$$\hat{x}_M \in \arg\max \hat{f}_M[+\tilde{p}], \quad \hat{x}_W \in \arg\max \hat{f}_W[-\tilde{p}],$$

by Theorem 5.8. Note that for each arc $a = (e, e') \in A$, $\ell_{\tilde{p}}(a) = \ell(a) + \tilde{p}(e) - \tilde{p}(e')$ is the length of $a$ in the directed graph defined in the same way as above for $\hat{f}_M[+\tilde{p}]$, $\hat{f}_W[-\tilde{p}]$, $\hat{x}_M$, and $\hat{x}_W$. Also note that $\ell_{\tilde{p}}(a) = 0$ for all arcs $a$ in a shortest path from $S$ to $t$.

Let $P$ be a shortest path from $S$ to $t$ in $G$ with the minimum number of arcs. Since $\ell_{\tilde{p}}(a) = 0$ for all $a \in P$,

$$\begin{aligned} &\hat{x}_M - \chi_e + \chi_{e'} \in \arg\max \hat{f}_M[+\tilde{p}] && \text{for all } (e, e') \in P \cap A_M, \\ &\hat{x}_W - \chi_e + \chi_{e'} \in \arg\max \hat{f}_W[-\tilde{p}] && \text{for all } (e', e) \in P \cap A_W. \end{aligned} \tag{5.16}$$

Since $P$ has the minimum number of arcs,

$$\hat{x}_M - \chi_e + \chi_{e''} \notin \arg\max \hat{f}_M[+\tilde{p}], \quad \hat{x}_W - \chi_{e''} + \chi_e \notin \arg\max \hat{f}_W[-\tilde{p}] \tag{5.17}$$

for all vertices $e$ and $e''$ of $P$ such that $(e, e'') \notin P$ and $e$ appears earlier than $e''$ in $P$. Furthermore, arcs of $A_M$ and $A_W$ appear alternately in $P$. For otherwise, assume that two consecutive arcs $(e, e'), (e', e'') \in P$ belong to $A_M$. Then, by (M)

$$\hat{f}_M(\hat{x}_M + \chi_e - \chi_{e'}) + \hat{f}_M(\hat{x}_M + \chi_{e'} - \chi_{e''}) \leq \hat{f}_M(\hat{x}_M) + \hat{f}_M(\hat{x}_M + \chi_e - \chi_{e''}),$$

which yields

$$\ell(e, e') + \ell(e', e'') \geq \ell(e, e''),$$

a contradiction to the minimality (with respect to the number of arcs) of $P$. Consequently,

$$\begin{aligned} a_1 = (e_1, e'_1), a_2 = (e_2, e'_2) \in P \cap A_M, a_1 \neq a_2 &\implies \{e_1, e'_1\} \cap \{e_2, e'_2\} = \emptyset, \\ a_1 = (e_1, e'_1), a_2 = (e_2, e'_2) \in P \cap A_W, a_1 \neq a_2 &\implies \{e_1, e'_1\} \cap \{e_2, e'_2\} = \emptyset. \end{aligned} \tag{5.18}$$

From Lemmas 5.9 and 5.10 together with (5.16), (5.17), and (5.18), we have

$$\hat{x}'_M \equiv \hat{x}_M - \sum_{(e, e') \in P \cap A_M} (\chi_e - \chi_{e'}) \in \arg\max \hat{f}_M[+\tilde{p}], \tag{5.19}$$

$$\hat{x}'_W \equiv \hat{x}_W - \sum_{(e', e) \in P \cap A_W} (\chi_e - \chi_{e'}) \in \arg\max \hat{f}_W[-\tilde{p}]. \tag{5.20}$$

Conditions (5.19) and (5.20) guarantee that if $S = \mathrm{supp}^+(\hat{x}_M - \hat{x}_W)$ and $t \in \mathrm{supp}^-(\hat{x}_M - \hat{x}_W)$, then we can decrease the distance between new maximizers $\hat{x}'_M$ and $\hat{x}'_W$ by modifying $\tilde{p}$ as above.

The above discussion leads us to Algorithm SSP for finding a maximizer of $\hat{f}_M + \hat{f}_W$ described as follows.

**Algorithm SSP**

**Step 0.** Find $\hat{x}_M \in \arg\max \hat{f}_M$ and $\hat{x}_W \in \arg\max \hat{f}_W$. Set $\tilde{p} := \mathbf{0}$.

**Step 1.** If $\hat{x}_M = \hat{x}_W$ then stop.

**Step 2.** Construct $G$ and compute $\ell$ for $\hat{f}_M[+\tilde{p}]$, $\hat{f}_W[-\tilde{p}]$, $\hat{x}_M$ and $\hat{x}_W$ by (5.14) and (5.15). Set $S := \mathrm{supp}^+(\hat{x}_M - \hat{x}_W)$, $T := \mathrm{supp}^-(\hat{x}_M - \hat{x}_W)$.

**Step 3.** Compute the shortest distances $d(e)$ from $S$ to all $e \in \hat{E}$ in $G$ with respect to $\ell$. Find a shortest path $P$ from $S$ to $T$ with the minimum number of arcs.

**Step 4.** For each $e \in \hat{E}$, set $\tilde{p}(e) := \tilde{p}(e) + \min\{d(e), \sum_{a\in P} \ell(a)\}$. Update $\hat{x}_M$ and $\hat{x}_W$ by (5.19) and (5.20). Go to Step 1.

Under (A), a shortest path $P$ in Step 3 always exists because if there is no path from $\mathrm{supp}^+(\hat{x}_M - \hat{x}_W)$ to $\mathrm{supp}^-(\hat{x}_M - \hat{x}_W)$, then $\mathrm{dom}\, \hat{f}_M \cap \mathrm{dom}\, \hat{f}_W$ must be empty (see [20]). By (5.19) and (5.20), the algorithm preserves

$$\hat{x}_M \in \arg\max \hat{f}_M[+\tilde{p}], \quad \hat{x}_W \in \arg\max \hat{f}_W[-\tilde{p}].$$

Hence, if SSP terminates, then it finds a maximizer of $\hat{f}_M + \hat{f}_W$. Since $P$ is a path from $\mathrm{supp}^+(\hat{x}_M - \hat{x}_W)$ to $\mathrm{supp}^-(\hat{x}_M - \hat{x}_W)$, and arcs of $A_M$ and $A_W$ appear alternately in $P$, $\sum_{e\in\hat{E}} |\hat{x}_M(e) - \hat{x}_W(e)|$ decreases by two after each execution of Step 4, which guarantees the termination of SSP.

When $E = F$, as we see from the above discussion, we can relax (A) to the requirement that $\mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ is nonempty and bounded.

**Corollary 5.11**: *For any $M^\natural$-concave functions $f_M, f_W : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ such that* $\mathrm{dom}\, f_M \cap \mathrm{dom}\, f_W$ *is nonempty and bounded, there exists an $f_M f_W$-pairwise-stable solution with respect to $(E, \emptyset)$.*

**Remark 5**: *In Step 4 of SSP, we can update $\tilde{p}$ as: for each $e \in \hat{E}$,*

$$\tilde{p}(e) := \tilde{p}(e) + \min\{d(e), \sum_{a\in P} \ell(a)\} - \sum_{a\in P} \ell(a)$$

*while preserving $\hat{x}_M \in \arg\max \hat{f}_M[+\tilde{p}]$ and $\hat{x}_W \in \arg\max \hat{f}_W[-\tilde{p}]$. The subtraction of a constant does not affect the correctness of SSP since $\hat{x}_M(\hat{E}) = \hat{x}_W(\hat{E}) = 0$. This modified version will be used in our algorithm in Section 5.3.* ∎

## 5.3. The General Case

In this subsection, we give an algorithm for finding an $f_M f_W$-pairwise-stable solution for the general case, that is, an algorithm for finding $x_M, x_W \in \mathbf{Z}^E$, $p \in \mathbf{R}^E$ and $z_M, z_W \in \mathbf{Z}^R$ satisfying

$$
\begin{aligned}
p|_R &= \mathbf{0}, && (5.21)\\
z|_R &= z_M \vee z_W, && (5.22)\\
x_M &\in \arg\max\{f_M[+p](y) \mid \ y|_R \le z_M\}, && (5.23)\\
x_W &\in \arg\max\{f_W[-p](y) \mid \ y|_R \le z_W\}, && (5.24)\\
x_M &= x_W. && (5.25)
\end{aligned}
$$

The algorithm has the following two phases:

**(i)** Phase 1 finds $x_M, x_W \in \mathbf{Z}^E$, $p \in \mathbf{R}^E$, and $z_M, z_W \in \mathbf{Z}^R$ satisfying (5.21), (5.22), (5.23), (5.24), and the following conditions:

$$
\begin{aligned}
x_M|_R &= x_W|_R, && (5.26)\\
x_M &\le x_W. && (5.27)
\end{aligned}
$$

Note that if we further get (5.25), then $x(= x_M = x_W)$ is an $f_M f_W$-pairwise-stable solution. Phase 1 relies on two algorithms, G_GS and SSP described in the previous subsections (see Fig. 2).

**(ii)** Phase 2 executes part of Phase 1 with the inputs obtained from the outputs of Phase 1 by interchanging the roles of $M$ and $W$. The outputs of Phase 2 satisfy (5.21)∼(5.25).

Before giving the details of Phase 1, we show a basic property of an M$^\natural$-concave function.

**Lemma 5.12**: *Let* $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ *be an M*$^\natural$*-concave function. For an element* $e \in E$*, let* $z_1, z_2 \in \mathbf{Z}^E$ *be such that* $z_1 = z_2 + \chi_e$, $\arg\max\{f(y) \mid y \le z_1\} \neq \emptyset$*, and* $\arg\max\{f(y) \mid y \le z_2\} \neq \emptyset$*. Then, the following two statements hold:*

(a) *For any* $x \in \arg\max\{f(y) \mid y \le z_1\}$*, there exists* $e' \in \{0\} \cup E$ *such that*

$$x - \chi_e + \chi_{e'} \in \arg\max\{f(y) \mid y \le z_2\}.$$

(b) *For any* $x \in \arg\max\{f(y) \mid y \le z_2\}$*, there exists* $e' \in \{0\} \cup E$ *such that*

$$x + \chi_e - \chi_{e'} \in \arg\max\{f(y) \mid y \le z_1\}.$$

**Phase 1**

Set $z_M := z|_R$, $z_W := \mathbf{0}$ and $p := \mathbf{0}$. Find $x_M \in \arg\max\{f_M(y) \mid y|_R \le z_M\}$ and $x_W \in \arg\max\{f_W(y) \mid y|_R \le x_M|_R\}$.

(G_GS): Modify $x_M, x_W \in \mathbf{Z}^E$ and $z_M, z_W \in \mathbf{Z}^R$ so that

$$z|_R = z_M \vee z_W, \tag{5.22}$$

$$x_M \in \arg\max\{f_M[+p](y) \mid y|_R \le z_M\}, \tag{5.23}$$

$$x_W \in \arg\max\{f_W[-p](y) \mid y|_R \le z_W\}, \tag{5.24}$$

$$x_M|_R = x_W|_R. \tag{5.26}$$

If $x_M \le x_W$ (5.27), then Phase 1 terminates.

(SSP): Modify $x_M, x_W \in \mathbf{Z}^E$ and $p \in \mathbf{R}^E$ preserving (5.23), (5.24) and

$$p|_R = \mathbf{0} \tag{5.21}$$

to decrease $\sum\{x_M(e) - x_W(e) \mid e \in \mathrm{supp}^+(x_M - x_W)\}$.

Modify $x_W$ slightly if necessary, and go to G_GS.

Figure 2: An outline of Phase 1 consisting of G_GS and SSP: the equation numbers correspond to those in the text.

**Proof.** (a) If $x \le z_2$, then it suffices to set $e' = e$. Hence, we assume that $x(e) = z_1(e) = z_2(e) + 1$. Let $x'$ be any element of $\arg\max\{f(y) \mid y \le z_2\}$. By (M$^\natural$) for $x$, $x'$, and $e$, there exists $e' \in \{0\} \cup \mathrm{supp}^-(x - x')$ such that

$$f(x) + f(x') \le f(x - \chi_e + \chi_{e'}) + f(x' + \chi_e - \chi_{e'}).$$

Since $x' + \chi_e - \chi_{e'} \le z_1$ and $x \in \arg\max\{f(y) \mid y \le z_1\}$, the above inequality implies $f(x') \le f(x - \chi_e + \chi_{e'})$, that is, $x - \chi_e + \chi_{e'} \in \arg\max\{f(y) \mid y \le z_2\}$.

(b) If $x \in \arg\max\{f(y) \mid y \le z_1\}$, then it suffices to set $e' = e$. Hence we assume that $x \notin \arg\max\{f(y) \mid y \le z_1\}$. Then there exists $x' \in \arg\max\{f(y) \mid y \le z_1\}$ with $x'(e) = z_1(e)$, and by (M$^\natural$) for $x'$, $x$, and $e$, there exists $e' \in \{0\} \cup \mathrm{supp}^-(x' - x)$ such that

$$f(x') + f(x) \le f(x' - \chi_e + \chi_{e'}) + f(x + \chi_e - \chi_{e'}).$$

Since $x' - \chi_e + \chi_{e'} \le z_2$ and $x \in \arg\max\{f(y) \mid y \le z_2\}$, we have $f(x') \le f(x + \chi_e - \chi_{e'})$, that is, $x + \chi_e - \chi_{e'} \in \arg\max\{f(y) \mid y \le z_1\}$. ∎

We next give the procedure for Phase 1, where $f_M^{z_M}$ and $f_W^{z_W}$ are M$^\natural$-concave functions defined by

$$f_M^{z_M}(x) = \begin{cases} f_M(x) & \text{if } x|_R \le z_M \\ -\infty & \text{otherwise,} \end{cases}$$

$$f_W^{z_W}(x) = \begin{cases} f_W(x) & \text{if } x|_R \leq z_W \\ -\infty & \text{otherwise.} \end{cases}$$

**Phase 1**

**Step 0.** Set $z_M := z|_R$, $z_W := \mathbf{0}$, $p := \mathbf{0}$. Find $x_M \in \arg\max\{f_M(y) \mid y|_R \leq z_M\}$ and $x_W \in \arg\max\{f_W(y) \mid y|_R \leq x_M|_R\}$.

**Step 1.** Set $(x_M, x_W, z_M, z_W) := \mathrm{G\_GS}(f_M[+p], f_W[-p], x_M, x_W, z_M, z_W)$.

**Step 2.** If $x_M \leq x_W$, then output $(x_M, x_W, p, z_M, z_W)$ and Phase 1 terminates.

**Step 3.** Construct $G$ and compute $\ell$ for $\hat{f}_M^{z_M}[+(0,p)]$, $\hat{f}_W^{z_W}[-(0,p)]$, $\hat{x}_M$, $\hat{x}_W$ by (5.14) and (5.15). Set $S := \mathrm{supp}^+(x_M - x_W)$, $T := \{0\} \cup R \cup \mathrm{supp}^-(x_M - x_W)$.

**Step 4.** Compute the shortest distances $d(e)$ from $S$ to all $e \in \hat{E}$ in $G$ with respect to $\ell$. Find a shortest path $P$ from $S$ to $T$ with a minimum number of arcs.

**Step 5.** For all $e \in \hat{E}$, set $\tilde{p}(e) := \tilde{p}(e) + \min\{d(e), \ell(P)\} - \ell(P)$, where $\tilde{p} = (p_0, p)$ and $\ell(P) = \sum_{a \in P} \ell(a)$. Update $\hat{x}_M$ and $\hat{x}_W$ by (5.19) and (5.20).

**Step 6.** If the terminal vertex $e$ of $P$ is in $R$ and the last arc $a$ of $P$ is in $A_M$, then choose $e' \in \{0\} \cup E$ such that $x_W + \chi_e - \chi_{e'} \in \arg\max\{f_W(y) \mid y|_R \leq z_W \vee x_M|_R\}$, and set $x_W := x_W + \chi_e - \chi_{e'}$. Go to Step 1.

Before analyzing Phase 1 in detail, we give several remarks. Steps 3$\sim$5 are the same as Algorithm SSP except for the definitions of $S$, $T$, and $\tilde{p}$. In order to achieve $p|_R = \mathbf{0}$ and $p_0 = 0$ as required by (5.21) and (5.13), we have modified the way of updating $S$, $T$, and $\tilde{p}$ in original SSP. As we will show later, (5.26) is satisfied just before Step 3, and hence $S$ and $T$ are disjoint. By (A), there exist arcs from all $e \in \mathrm{supp}^+(x_M - x_W)$ to 0 in $G$, which guarantees the existence of a path from $S$ to $T$. In Step 6 we can choose $e' \in \{0\} \cup E$ such that $x_W + \chi_e - \chi_{e'} \in \arg\max\{f_W(y) \mid y|_R \leq z_W \vee x_M|_R\}$, due to Lemma 5.12 (b), as will be shown later.

It is a direct consequence of the next lemma that (5.21)$\sim$(5.24), (5.26), and (5.27) hold at the termination of Phase 1.

**Lemma 5.13**: *The following three statements hold for* Phase 1.

(a) *Just after* Step 0*,*

$$x_W \in \arg\max\{f_W[-p](y) \mid y|_R \leq z_W \vee x_M|_R\}, \tag{5.28}$$

$$x_W|_R \leq x_M|_R, \tag{5.29}$$

*and* (5.21)$\sim$(5.23) *hold.*

(b) *Just after* Step 1*,* (5.21)$\sim$(5.24) *and* (5.26) *hold.*

(c) *Just after* Step 6*,* (5.21)$\sim$(5.23)*,* (5.28) *and* (5.29) *hold.*

**Proof.** Assertion (a) holds trivially. We will prove (b) and (c) by induction on the number of iterations from Step 1 through Step 6. We assume that (b) and (c) hold at the $l$th iteration, and we show that these statements also hold at the $(l+1)$st iteration.

(b): By either (a) or (c) for the $l$th iteration, (5.21)$\sim$(5.23), (5.28) and (5.29) hold just before Step 1. From the argument in Section 5.1, G_GS outputs $x_M$, $x_W$, $z_M$ and $z_W$ satisfying (5.22)$\sim$(5.24) and (5.26). On the other hand, (5.21) is satisfied because $p$ is not changed by G_GS.

(c): By (b), we have (5.21)$\sim$(5.24) and (5.26) just before Step 3. As Steps 3$\sim$6 change neither $z_M$ nor $z_W$, (5.22) holds just after Step 6. By the argument in Section 5.2 (see also Remark 5) and the definitions of $f_M^{z_M}$ and $f_W^{z_W}$, Lemmas 5.9 and 5.10 imply that (5.23) and (5.24) hold just after Step 5. Since $\{0\} \cup R \subseteq T$, the shortest distances from $S$ to vertices in $\{0\} \cup R$ are greater than or equal to $\ell(P)$, and hence $(p_0, p)$ updated in Step 5 satisfies (5.21) and $p_0 = 0$. To show (5.28) and (5.29), we consider three cases: (i) the terminal vertex $e$ of $P$ is in $T \setminus R$, (ii) $e \in R$ and the last arc $a$ of $P$ is in $A_W$, and (iii) $e \in R$ and the last arc $a$ of $P$ is in $A_M$. In case (i), (5.26) holds just after Step 5, and hence $x_M|_R \le z_W$ holds. This and (5.24) imply (5.28). In cases (ii) and (iii), just after Step 5 we have $x_M(e) = x_W(e) + 1$ and $x_M(e'') = x_W(e'')$ for all $e'' \in R \setminus \{e\}$, and hence (5.29). Furthermore, in case (ii), because $x_M|_R$ was not changed in Step 5, (b) implies that $x_M|_R \le z_W$ just after Step 6. This, together with (5.24), implies (5.28). In case (iii), because $x_M(e)$ was increased by one in Step 5, (5.28) may not hold just before Step 6. However, Lemma 5.12 (b) guarantees that $x_W$ updated in Step 6 satisfies (5.28) and (5.29). ∎

We next show that Phase 1 terminates in a finite number of iterations. To show this, we state a lemma.

**Lemma 5.14**: *If* G_GS *has inputs satisfying* (5.26) *in* Phase 1*, then it can terminate by simply setting* $z_W := z_W \vee x_M|_R$.

**Proof.** By (a) and (c) of Lemma 5.13, the inputs $x_M$, $x_W$, $z_M$ of G_GS, and $z_W$ modified as above satisfy (5.22)$\sim$(5.24) and (5.26). ∎

In the sequel we assume that G_GS in Phase 1 is executed as shown in Lemma 5.14. Then we have the following lemma.

**Lemma 5.15**: Phase 1 *terminates in a finite number of iterations.*

**Proof.** We first observe that Step 1 with $x_M|_R \neq x_W|_R$ is executed finitely many times. Let $e$ and $e'$ be the elements defined in the previous Step 6. By the discussion in Section 5.1, if G_GS has inputs with $x_M|_R \neq x_W|_R$, then either $z_M(e)$ or $z_M(e')$ is decreased by at least one. Since Steps 2∼6 preserve $z_M$ and $z_W$, and since Step 1 does not increase $z_M$, Step 1 with $x_M|_R \neq x_W|_R$ is executed finitely many times, due to (A).

We next observe that the cycle of Steps 1∼6 in which (5.26) (i.e., $x_M|_R = x_W|_R$) is retained is executed consecutively finitely many times, by showing that either $\alpha = \sum\{x_M(e'') - x_W(e'') \mid e'' \in \mathrm{supp}^+(x_M - x_W)\}$ is decreased by one or $\beta = \sum\{x_M(e'') \mid e'' \in R\}$ is increased by one at each iteration. Let $e$, $e'$ and $a$ be the elements defined in Step 6. If (5.26) holds just after Step 6, then we have one of the following two cases: (i) $e \in T \setminus R$, (ii) $e \in R$, $a \in A_M$ and $e' \in \hat{E} \setminus R$. We first assume that either case (i) or the subcase of (ii) where $e' \in T \setminus R$ occurs. As we assumed on the basis of Lemma 5.14, Step 1 with (5.26) changes neither $x_M$ nor $x_W$. Lemma 5.13 (b) yields that $S \cap T = \emptyset$. Since there always exists a path from $S$ to $T$, an execution of Steps 3∼5 reduces $\alpha$ by one. Furthermore, Step 6 does not increase $\alpha$. We next assume that the other subcase of (ii) where $e' \in E \setminus T$ occurs. In this case, an execution of Steps 3∼5 decreases $\alpha$ by one and Step 6 increases $\alpha$ by one, and hence $\alpha$ remains the same. However $\beta$ is increased by one. Thus, the cycle of Steps 1∼6 preserving (5.26) is executed consecutively finitely many times, due to (A).

Hence, Phase 1 terminates in a finite number of iterations. ∎

Before explaining Phase 2, we state a lemma.

**Lemma 5.16**: Phase 1 *can be executed so that it preserves*

$$x_M \geq x_W \tag{5.30}$$

*once this relation holds.*

**Proof.** We assume that (5.30) holds in some iteration of Phase 1. If (5.30) holds just before Step 3, the inequality is preserved by the execution of Steps 3∼5 because $T = \{0\} \cup R$. Obviously, the modification of $x_W$ in Step 6 does not destroy (5.30). It remains to show that (5.30) can be kept while executing G_GS. Without loss of generality, we assume that $x_M|_R \neq x_W|_R$ at the beginning of G_GS. Then, $x_M(e_1) = x_W(e_1) + 1$ for some $e_1 \in R$ and $x_M(e_2) = x_W(e_2)$ for all $e_2 \in R \setminus \{e_1\}$. In G_GS, we can apply Lemma 5.12 (a) to update $x_M$ and Lemma 5.12 (b) to update $x_W$, respectively. Hence, at the end of the $l$th iteration (but not the final one) of G_GS, we have for some $e \in R$ and for all $e'' \in R \setminus \{e\}$

$$x_M^{(l)}(e) = x_W^{(l)}(e) + 1, \quad x_M^{(l)}(e'') = x_W^{(l)}(e'').$$

Moreover, G_GS terminates when $e' \in \{0\} \cup F$ for $e'$ in Lemma 5.12. If $e' \in F$, then either $x_M(e')$ is increased by one or $x_W(e')$ is decreased by one. Hence, relation (5.30) is preserved by G_GS. ∎

Let $(x'_M, x'_W, p', z'_M, z'_W)$ be the outputs of Phase 1. Phase 2 is the same as Phase 1 except that it starts from Step 2 with the inputs $(x'_W, x'_M, p', z'_W, z'_M)$, namely, the roles of $M$ and $W$ are interchanged in Phase 2. Since $x'_W \geq x'_M$ holds, Lemma 5.16 says that Phase 2 preserves this relation. Lemmas 5.13 and 5.15 guarantee that Phase 2 terminates in a finite number of iterations and outputs $x_M$, $x_W$, $p$, $z_M$, and $z_W$ satisfying (5.21)$\sim$(5.25). We have thus shown our main result, Theorem 3.1.

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