Period map of hyperelliptic curves.
KOMATSU Makoto (e-mail: komatu@kurims.kyoto-u.ac.jp)

Abstract

Monodromy covering of complement of discriminant of parameter space of versal deformation of curve singularity of type A_{2n}, is regarded as total space of $\mathbf{C}^*$-bundle. For n=2, we had that Rosenhain's normal form gives trivialization of the bundle. Moreover, under our trivialization, we gave factor of automorphy which expresses monodromy group action.

Contents

1 Introduction.

For any positive integer n, we define that

$$F_{A_{2n}}(x,y,t) := -y^2 + x^{2n+1} + t_2 x^{2n-1} + \cdots + t_{2n}x + t_{2n+1} \ .$$

$F_{A_{2n}}$ is universal unfolding of the polynomial $-y^2 + x^{2n+1}$. Moreover, we define that

$$\Xi_{A_{2n}} := \{(x,y,t) \in \mathbf{C}^2 \times \mathbf{C}^{2n} | F_{A_{2n}}(x,y,t) = 0\} \ .$$

It is called versal deformation of curve singularity of type A_{2n}. The parameter space $\mathbf{C}^{2n}$ is denoted by $S_{A_{2n}}$. That is, $S_{A_{2n}} := \mathbf{C}^{2n}(\ni t = (t_2, \ldots, t_{2n+1}))$. Moreover, π denotes the natural projection $\Xi_{A_{2n}} \ni (x,y,t) \mapsto t \in S_{A_{2n}}$. We write $X_t := \pi^{-1}(t)$. On $\Xi_{A_{2n}}$ and $S_{A_{2n}}$ we define $\mathbf{C}^*$-action as

$$\begin{aligned} \lambda \cdot (x,y,t) &:= (\lambda^2 x, \quad \lambda^{2n+1} y, \quad \lambda \cdot t) \\ \lambda \cdot t &:= (\lambda^4 t_2, \ \ldots \ , \ \lambda^{4n+2} t_{2n+1}) \end{aligned}$$

This action has fixed points on $S_{A_{2n}}$. But we can lift this action to one on $(S_{A_{2n}} - D_{A_{2n}})^\wedge$, and there the action is fixed point free. So $(S_{A_{2n}} - D_{A_{2n}})^\wedge$ is regarded as total space of $\mathbf{C}^*$-bundle. Here we think of the following problem.

Problem 1 *Clarify the structure of the above* $\mathbf{C}^*$*-bundle* $(S_{A_{2n}} - D_{A_{2n}})^\wedge \to \mathbf{C}^* \backslash (S_{A_{2n}} - D_{A_{2n}})^\wedge$.

At the present time, only for $n = 1$, answer to the problem is already known. For $n = 1$, the answer is a classical result, which we will see later (in subsection 2.4). In the following, we think of the problem for $n = 2$.

Remark. For any integer $g > 1$, $\mathbf{C}^*\backslash(S_{A_{2g}} - D_{A_{2g}})$ is regarded as a moduli space of hyperelliptic curve of genus g with one Weierstrass point on it. That is, suppose that

$$MH'_g := \left\{ (R, W) \,\middle|\, \begin{array}{l} R \text{ is hyperelliptic compact Riemann surface of genus } g\ , \\ W \text{ is one of Weierstrass points on } R\ . \end{array} \right\} .$$

Moreover, for $(R, W), (R', W') \in MH'_g$,

$$(R, W) \sim (R', W') :\Longleftrightarrow \exists \phi : R \xrightarrow{\sim} R' \text{ (biholomorphic) such that } \phi(W) = W' .$$

And we write $MH_g := MH'_g/\sim$. Furthermore, $\overline{X_t}$ denotes compact Riemann surface given by doing resolution of singularities of $X_t \sqcup \{\infty\}$. Then the map

$$S_{A_{2g}} - D_{A_{2g}} \ni t \longmapsto (\overline{X_t}, \infty) \in MH'_g$$

gives bijection $\mathbf{C}^*\backslash(S_{A_{2g}} - D_{A_{2g}}) \xrightarrow{\sim} MH_g$. Therefore, $S_{A_{2g}} - D_{A_{2g}}$ is total space of a $\mathbf{C}^*$-bundle with MH_g as its base space. $\square$

Here we avoid the problem for A_{2n+1}. The case A_{2n+1} with $n \geq 1$, is rather different from that of A_{2n}. Therefore we cannot apply the way of A_2 to A_{2n+1}. As for the problem for A_{2n+1}, we have no idea now. In the case A_{2n}, using a period mapping and applying a well-known framework of automorphic forms, we can see that the transition functions of the bundle S_{A_2} are given as a factor of automorphy. In the following section we review the framework of automorphic forms.

2 Framework of automorphic forms.

In this section we review a well-known framework of automorphic forms.

2.1 Equivariant group action on a trivial bundle and a factor of automorphy.

Suppose X be a complex manifold, and G be a group acting on X discontinuously. Then the following (2-1-1), (2-1-2) are equivalent.

(2-1-1) To give a factor of automorphy $j : G \times X \to \mathbf{C}^*$.

(2-1-2) To give a G-action on $\mathbf{C}^* \times X$ which satisfies the following (i), (ii).

- **(i)** The G-action is commutative to the natural $\mathbf{C}^*$-action on $\mathbf{C}^* \times X$.
- **(ii)** The G-action is equivariant to the natural projection $\mathbf{C}^* \times X \to X$.

In fact, if a factor of automorphy j is given, we can give a G-action on $\mathbf{C}^* \times X$ using j as follows:

$$\mathbf{C}^* \times X \ni (\lambda, x) \overset{\sigma}{\longmapsto} (j(\sigma, x)^{-1}\lambda, \sigma(x)) \in \mathbf{C}^* \times X \qquad (\sigma \in G)\ . \tag{1}$$

It can be easily seen that this G-action satisfy the above (i) and (ii). On the other hand, suppose that a G-action on $\mathbf{C}^* \times X$ satisfying (i) and (ii) is given. Then we define a map $j : G \times X \to \mathbf{C}^*$ by the following relation:

$$(1, x) \overset{\sigma}{\longmapsto} (j(\sigma, x)^{-1}, \sigma(x)) \qquad (\sigma \in G, x \in X)\ . \tag{2}$$

Then this j is a factor of automorphy. Those two procedures now explained are inverse to each other.

2.2 Invariant ring and ring of automorphic forms.

In general, when a group G is acting on a ring R, we denote by R^G the G-invariant subring of R. And for any complex analytic space X, we denote by $\mathcal{O}(X)$ the ring of all of holomorphic functions on X. Moreover, if a group G is acting on X and a factor of automorphy $j : G \times X \to \mathbf{C}^*$ is given, then for any integer k, we define that

$$A_k(X, G, j) := \{ f \in \Gamma(X, \mathcal{O}_X) \mid f(\sigma(x)) = j(\sigma, x)^k f(x) \text{ for any } x \in X,\ \sigma \in G \} \,. \tag{3}$$

In this article, only the case that $\sum_{k \in \mathbf{Z}} A_k(X, G, j)$ is direct sum, is appear. Note that the following relations hold:

$$\bigoplus_{k \in \mathbf{Z}} A_k(X, G, j) \cong \Gamma(X, \mathcal{O}_X)[\lambda, \lambda^{-1}]^G \subseteq \Gamma(\mathbf{C}^* \times X, \mathcal{O}_{\mathbf{C}^* \times X})^G \cong \Gamma((\mathbf{C}^* \times X)/G, \mathcal{O}_{(\mathbf{C}^* \times X)/G}) \,. \tag{4}$$

In (4), only the first isomorphism may be unfamiliar (at least, to the author). Therefore we explain it. Suppose f be an element of $\mathcal{O}(X)[\lambda, \lambda^{-1}]^G$. We express f as Laurent polynomial in λ:

$$f(\lambda, x) = \sum_{k \in \mathbf{Z}} \lambda^k f_k(x) \qquad (\text{ finite sum}) \tag{5}$$

where $f_k \in \mathcal{O}(X)$. From the expansion, f satisfies the equality

$$f(j(\sigma, x)^{-1} \lambda, \sigma(x)) = \sum_k j(\sigma, x)^{-k} \lambda^k f_k(\sigma(x)) \tag{6}$$

for any $\sigma \in G$. Because f is G-invariant, (1), (5) and (6) imply that

$$f_k(\sigma(x)) = j(\sigma, x)^k f_k(x) \quad (\forall \sigma \in G,\ \forall x \in X,\ \forall k \in \mathbf{Z}) \,.$$

That is, f_k is a (G, j)-automorphic form of weight k. On the oter hand, for given finite set $\{f_k\}$ (where $f_k \in A_k(X, G, j)$ for any k), if we define f by (5), we can easily see that f is an element of $\mathcal{O}(X)[\lambda, \lambda^{-1}]^G$.

2.3 Our plan.

We denote by D_{A_n} the discriminant set of S_{A_n}:

$$D_{A_n} := \{ t \in S_{A_n} | F_{A_n}(x, 0, t) \text{ has multiple roots.} \} \tag{7}$$

We treat $S_{A_n} - D_{A_n}$ rather than S_{A_n} itself. Suppose that there exist X and G which make the left hand side of the following diagram

$$\begin{tikzcd}
S_{A_n} - D_{A_n} \arrow[d] & (\mathbf{C}^* \times X)/G \arrow[l, "\sim"'] \arrow[d] & \mathbf{C}^* \times X \arrow[l, "u"'] \arrow[d, shift right] & \ni & (1, x) \\
\mathbf{C}^* \backslash (S_{A_n} - D_{A_n}) & X/G \arrow[l, "\sim"'] & X \arrow[l] \arrow[u, shift right, "s"'] \arrow[ul, "u \circ s"] & \ni & x \arrow[u, mapsto, "s"']
\end{tikzcd}$$

Diagram-1

commutative, where u is a natural projection, and s is a global section of the trivial bundle $\mathbf{C}^* \times X \to X$ defined as in the above diagram. Then by (4), the ring $\mathbf{C}[t_2, \ldots, t_{n+1}]$ is regarded as a subring of $\mathcal{O}(X)[\lambda, \lambda^{-1}]^G$, and hence it is regarded as a subring of the ring of (G, j)-automorphic forms. Moreover, transition functions of the bundle $S_{A_n} - D_{A_n}$ is given as a factor of automorphy j. By the way, the G-actions on the total space and on the base space of the bundle $\mathbf{C}^* \times X \to X$ are equivariant to the projection. Hence, by the relation (2) the section s satisfies

$$s(\sigma(x)) = j(\sigma, x) \cdot \sigma(s(x)) \qquad (\forall \sigma \in G,\ \forall x \in X) \,.$$

Moreover, the $\mathbf{C}^*$-actions on $(\mathbf{C}^* \times X)/G$ and on $\mathbf{C}^* \times X$ are equivariant to the map u. And, in addition, u is G-invariant. Therefore, we have

$$(u \circ s)(\sigma(x)) = j(\sigma, x) \cdot (u \circ s)(x) \qquad (\forall \sigma \in G,\ \forall x \in X) \,.$$

Keeping the above framework in mind, we consider Problem 1 for $n = 4$ as follows.

(2-3-1) We take an open dense subset of Siegel upper half space of degree two, say $\mathbf{H}_2^*$, as X in **Diagram-1**.

(2-3-2) Next we investigate the effect of G-action on the map $u \circ s$ to obtain a factor of automorphy j explicitly.

2.4 Example. (A_2-type curve singularity.)

As an example, we review the answer to the problem 1 for $n = 2$ (cf. [Sai]). In order to adapt the problem to the theory of Weierstrass' $\wp$ function, we modify the definition of F_{A_2} as follows:

$$F_{A_2}(x, y, g) := -y^2 + 4x^3 - g_2 x - g_3 \ .$$

Then $S_{A_2} = \mathbf{C}^2$ and $D_{A_2} = \{g \in S_{A_2} | g_2^3 - 27g_3^2 = 0\}$. In this case, using the following multi-valued holomorphic mapping:

$$S_{A_2} - D_{A_2} \ni g \mapsto \left(\int_{A(g)} \frac{dx}{y}, \quad \int_{B(g)} \frac{dx}{y} \Big/ \int_{A(g)} \frac{dx}{y} \right) \in \mathbf{C}^* \times \mathbf{H} \ , \tag{8}$$

we can apply the above framework to $S_{A_2} - D_{A_2}$, where $G = SL(2, \mathbf{Z})$ and $X = \mathbf{H}$. As a consequence, we obtain that $S_{A_2} - D_{A_2} \cong \mathbf{C}^* \times \mathbf{H}/SL(2, \mathbf{Z})$. Moreover, we have $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) = c\tau + d$, and obtain the expression of g_i $(i = 2, 3)$ as (G, j)-automorphic forms, which coincide to the well-known expressions as Eisenstein series.

3 Definition of period mapping.

We denote that $S := S_{A_4}$, $\Xi := \Xi_{A_4}$, and $D := D_{A_4}$. Discriminant of the polynomial $F_{A_4}(x, 0, t) \in (\mathbf{C}[t])[x]$ is as follows:

$$\begin{aligned} \Delta(t) : = \ & 3125t_5^4 - 3750t_2t_3t_5^3 + 2000t_2t_4^2t_5^2 + 2250t_3^2t_4t_5^2 - 900t_2^3t_4t_5^2 + 825t_2^2t_3^2t_5^2 \\ & + 108t_2^5t_5^2 - 1600t_3t_4^3t_5 + 560t_2^2t_3t_4^2t_5 - 630t_2t_3^3t_4t_5 - 72t_2^4t_3t_4t_5 + 108t_3^5t_5 \\ & + 16t_2^3t_3^3t_5 + 256t_4^5 - 128t_2^2t_4^4 + 144t_2t_3^2t_4^3 + 16t_2^4t_4^3 - 27t_3^4t_4^2 - 4t_2^3t_3^2t_4^2 \ . \end{aligned}$$

By (7), we have $D = \{t \in S \mid \Delta(t) = 0\}$. We take a point $t_0 \in S - D$. t_0 is used as a base point of the fundamental group of $S - D$. Projection $\pi : \Xi - \pi^{-1}(D) \to S - D$ has the property of local triviality. Hence $\pi_1(S - D, t_0)$ acts on $H_1(X_{t_0}, \mathbf{Z})$, and then we have what is called monodromy representation of $\pi_1(S - D, t_0)$ and monodromy covering of $S - D$. Here we define them. Suppose C be an element of $H_1(X_{t_0}, \mathbf{Z})$ and γ be an element of $\pi_1(S - D, t_0)$. Then we denote by $\gamma(C)$ an element of $H_1(X_{t_0}, \mathbf{Z})$ given by modifying C continuously along the path γ. Thus γ is regarded as an automorphism of $H_1(X_{t_0}, \mathbf{Z})$. Moreover, this action preserves the intersection form $\langle \ , \ \rangle$ on $H_1(X_{t_0}, \mathbf{Z})$. Therefore we have the following anti-homomorphism:

$$\rho^* : \pi_1(S - D, t_0) \longrightarrow \mathrm{Aut}(H_1(X_{t_0}, \mathbf{Z}), \langle \ , \ \rangle) \qquad \text{(monodromy representation)} \ , \tag{9}$$

where $\mathrm{Aut}(H_1(X_{t_0}, \mathbf{Z}), \langle \ , \ \rangle)$ denotes all of automorphisms of $H_1(X_{t_0}, \mathbf{Z})$ which preserve the intersection form $\langle \ , \ \rangle$. Note that for any $\gamma, \gamma' \in \pi_1(S - D, t_0)$, we define the product $\gamma\gamma'$ by joining the end point of γ to the initial point of γ'. $\Gamma := \rho^*(H_1(X_{t_0}, \mathbf{Z}))$ is called as monodromy group. We take a symplectic basis of $H_1(X_{t_0}, \mathbf{Z})$ as in **Figure-1**. Then by the basis, the following group isomorphism holds:

$$\begin{aligned} & E : \mathrm{Aut}(H_1(X_t, \mathbf{Z}), \langle \ , \ \rangle) \xrightarrow{\sim} Sp(4, \mathbf{Z}) \qquad \gamma \longmapsto M \\ & \text{where} \qquad (\gamma(A_1) \ \gamma(A_2) \ \gamma(B_1) \ \gamma(B_2)) = (A_1 \ A_2 \ B_1 \ B_2)M \ , \end{aligned} \tag{10}$$

is obtained, where $t = t_0$. By the isomorphism, Γ is regarded as a subgroup of $Sp(4, \mathbf{Z})$. Now we define a covering space of $S - D$ as follows:

$$(S-D)^\wedge := (\text{universal covering space of } S - D)/\,\mathrm{Kernel}(\rho^*).$$

$(S–D)^\wedge$ is called as monodromy covering. Natural projection $(S–D)^\wedge \to S-D$ is denoted by σ. Here we can define a period mapping.

$$P : (S–D)^\wedge \ni h \longmapsto \begin{pmatrix} \omega_{11}(h) & \omega_{12}(h) & \omega_{13}(h) & \omega_{14}(h) \\ \omega_{21}(h) & \omega_{22}(h) & \omega_{23}(h) & \omega_{24}(h) \end{pmatrix} \in M_{2,4}(\mathbf{C}) , \qquad \omega_{ij}(h) := \int_{A_j(h)} \frac{x^{i-1}dx}{y} ,$$

where $A_1(h),\ A_2(h),\ A_3(h) = B_1(h),\ A_4(h) = B_2(h)$ are symplectic basis of $H_1(X_{\sigma(h)}, \mathbf{Z})$ and depend on h "continuously". That is, each $A_j(h)$ is a local system. We choose one element $h_0 \in \sigma^{-1}(t_0)$, and on the h_0, take $A_j(h_0)$ $(j = 1, 2, 3, 4)$ as in the **Figure-1**.

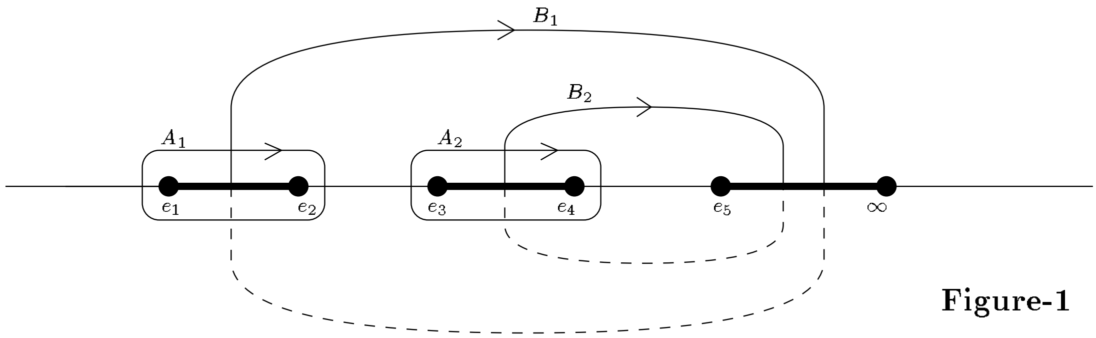


Figure-1

Remark. Each $A_j(t)$ is multi-valued on $S-D$. But, on $(S–D)^\wedge$, each $A_j(t)$ is single-valued. In fact $(S–D)^\wedge$ is the minimal covering on which each $A_j(t)$ is single-valued. Therefore the above period map P is single-valued.

By the definition of P, each $P(h)$ $(h \in (S–D)^\wedge)$ is a 2×4 matrix. We define a map φ as

$$\varphi : \mathrm{Image}(P) \ni (\Omega_A\ \Omega_B) \longmapsto (\Omega_A^{-1}\Omega_B) \in \mathbf{H}_2 ,$$

where Ω_A, Ω_B denote the left 2×2 part, the right 2×2 part of the 2×4 matrix $P(h)$, respectively.

4 Monodromy covering and configuration space of ramified points.

We denote the n-th symmetric group by S_n. The aim of this section is to give a well-known homomorphism $Sp(4, \mathbf{Z}) \to S_6$ explicitly, to review a result of A'Campo about monodromy group of the deformation of curve singularity of type A_4, with more precise consideration, and to show that the monodromy covering $\sigma : (S–D)^\wedge \to S–D$ is factored by a configuration space of five roots of $F(x, 0, t_0)$.

4.1 $Sp(4, \mathbf{Z})$-action on $H_1(X_{t_0}, \mathbf{Z})/2H_1(X_{t_0}, \mathbf{Z})$.

Suppose that i, j are elements of $\{1, \ldots, 6\}$. Now we take a path on $\overline{X_{t_0}}$ which has e_i as its initial point and e_j as its end point, where e_6 means ∞. Then the path and its image under the hyperelliptic involution of X_{t_0} make a closed path on $\overline{X_{t_0}}$, which determine an element of $H_1(X_{t_0}, \mathbf{Z})$. we denote it by $[e_i, e_j]$. $[e_i, e_j]$ is uniquely determined by e_i, e_j up to $\mathrm{mod}\,2H_1(X_{t_0}, \mathbf{Z})$. Under the assumption that the basis of $H_1(X_{t_0}, \mathbf{Z})$ is given as in the **Figure-1**, the six cycles $[e_i, e_6]$ are written as follows:

$$\begin{array}{l} [e_1, e_6] \equiv B_1, \quad [e_2, e_6] \equiv A_1 + B_1, \quad [e_3, e_6] \equiv A_1 + B_2, \\ [e_4, e_6] \equiv A_1 + A_2 + B_2, \quad [e_5, e_6] \equiv A_1 + A_2, \quad [e_6, e_6] \equiv 0, \quad \mathrm{mod}\,2H_1(X_{t_0}, \mathbf{Z}) . \end{array} \tag{11}$$

Here we write

$$[e_i, e_j] \equiv \varepsilon_1''(ij)A_1 + \varepsilon_2''(ij)A_2 + \varepsilon_1'(ij)B_1 + \varepsilon_2'(ij)B_2 \quad \mathrm{mod}\ 2H_1(X_{t_0}, \mathbf{Z})$$

where $\varepsilon_k''(ij), \varepsilon_k'(ij) \in \{0, 1\}$ for $i, j \in \{1, \ldots, 6\}$, $k \in \{1, 2\}$. Here we note that the six $(\varepsilon'(i6)\varepsilon''(i6)) + (1101)$ $(i \in \{1, \ldots, 6\})$ coincide $\mathrm{mod}(2\mathbf{Z})^4$ with the elements of **OTC** in (67):

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
$(\varepsilon'(i6)\varepsilon''(i6)) + (1101)$	(0101)	(0111)	(1011)	(1010)	(1110)	(1101)

.

Remark. (1101) corresponds to the Riemann constant. That is, (1101) corresponds to $B_1+B_2+A_2$, and $\frac{1}{2}(\int_{B_1+B_2+A_2}\omega_1, \int_{B_1+B_2+A_2}\omega_2)$ is what is called Riemann constant, where ω_1,ω_2 are the basis of $\mathbf{C}$-vector space of holomorphic 1-forms on X_{t_0} satisfying $\int_{A_j}\omega_i=\delta_{ij}$ (Kronecker's delta). ∎

By Appendix A.3, any element of $Sp(4,\mathbf{Z})$ is regarded as an element of S_6 using the homomorphism b in (68). Here we note that, for any $M\in Sp(4,\mathbf{Z})$ and for any $i\in\{1,\dots,6\}$ the relation

$$M\circ((\varepsilon'(i6)\ \varepsilon''(i6))+(1101))=(\varepsilon'(M(i)6)\ \varepsilon''(M(i)6))+(1101) \qquad \bmod (2\mathbf{Z})^4$$

holds. The aim of this subsection is to prove the following lemma.

Lemma 2 *For any* $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(4,\mathbf{Z})$ *and for any* $i,j\in\{1,\dots,6\}$, *the following equality* $(\bmod(2\mathbf{Z})^4)$ *holds.*

$$M\begin{pmatrix} {}^t\varepsilon''(ij)\\ {}^t\varepsilon'(ij)\end{pmatrix}\equiv\begin{pmatrix} {}^t\varepsilon''(M(i)M(j))\\ {}^t\varepsilon'(M(i)M(j))\end{pmatrix} \bmod (2\mathbf{Z})^4 .$$

Proof. We have only to prove the case $i\neq j$. By the definition of $M\circ\varepsilon$, it satisfies that

$$M\circ\varepsilon - M\circ\delta=\varepsilon M^{-1}-\delta M^{-1}$$

for any $M\in Sp(2g,\mathbf{Z})$, $\varepsilon,\delta\in\mathbf{Z}^{2g}$. Therefore, for any $i,j\in\{1,\dots,6\}$ satisfying $i\neq j$, and for any $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(4,\mathbf{Z})$, we have

$$\begin{aligned}
\varepsilon(ij)M^{-1} &\equiv (\varepsilon(i6)+(1101)-\varepsilon(j6)-(1101))M^{-1}\\
&\equiv (\varepsilon(i6)+(1101))M^{-1}-(\varepsilon(j6)+(1101))M^{-1}\\
&\equiv M\circ(\varepsilon(i6)+(1101))-M\circ(\varepsilon(j6)+(1101))\\
&\equiv \varepsilon(M(i)6)+(1101)-\varepsilon(M(j)6)-(1101)\\
&\equiv \varepsilon(M(i)M(j)),
\end{aligned}$$

where $\varepsilon(ij)=(\varepsilon_1'(ij)\ \varepsilon_2'(ij)\ \varepsilon_1''(ij)\ \varepsilon_2''(ij))\in\{0,1\}^4$, and "$\equiv$" means $\bmod(2\mathbf{Z})^4$. By the way, for $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(2g,\mathbf{Z})$, $M^{-1}=\begin{pmatrix} {}^tD & -{}^tB\\ -{}^tC & {}^tA\end{pmatrix}$. Therefore,

$$\begin{pmatrix} {}^t\varepsilon'(M(i)M(j))\\ {}^t\varepsilon''(M(i)M(j))\end{pmatrix}\equiv{}^tM^{-1}\begin{pmatrix} {}^t\varepsilon'(ij)\\ {}^t\varepsilon''(ij)\end{pmatrix}=\begin{pmatrix} D & -C\\ -B & A\end{pmatrix}\begin{pmatrix} {}^t\varepsilon'(ij)\\ {}^t\varepsilon''(ij)\end{pmatrix}.$$

In other words,

$$\begin{pmatrix} {}^t\varepsilon''(M(i)M(j))\\ {}^t\varepsilon'(M(i)M(j))\end{pmatrix}\equiv\begin{pmatrix} A & -B\\ -C & D\end{pmatrix}\begin{pmatrix} {}^t\varepsilon''(ij)\\ {}^t\varepsilon'(ij)\end{pmatrix}\equiv\begin{pmatrix} A & B\\ C & D\end{pmatrix}\begin{pmatrix} {}^t\varepsilon''(ij)\\ {}^t\varepsilon'(ij)\end{pmatrix} \quad \bmod (2\mathbf{Z})^4 .$$

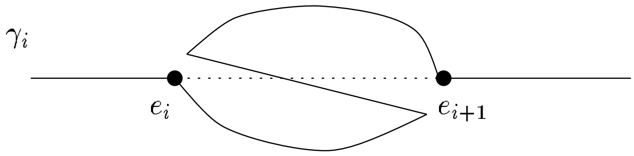
This completes the lemma. ∎

4.2 Monodromy group.

The aim of this subsection is to investigate a result of A'Campo precisely. It is convenient that the monodromy representation is modified to be homomorphism. So now we define ρ.

$$\rho(\gamma):=K(E\circ\rho^*(\gamma))^{-1}K^{-1} \qquad \text{where} \quad K:=\begin{pmatrix} I_2 & 0\\ 0 & -I_2\end{pmatrix}. \tag{12}$$

Note that, $K\notin Sp(4,\mathbf{Z})$ but $KMK^{-1}\in Sp(4,\mathbf{Z})$ for any $M\in Sp(4,\mathbf{Z})$. Since $\pi_1(S-D,t_0)$ is isomorphic to the Artin braid group of five strings, it has canonical generators $\gamma_1,\dots,\gamma_4$ where each γ_i is given by the exchange of e_i and e_{i+1} counterclockwisely as in the following figure.



Then it can be easily seen that

$$E\circ\rho^*(\gamma_1)=\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\qquad E\circ\rho^*(\gamma_2)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\-1&1&1&0\\1&-1&0&1\end{pmatrix},$$
$$E\circ\rho^*(\gamma_3)=\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix},\qquad E\circ\rho^*(\gamma_4)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-1&0&1\end{pmatrix}. \tag{13}$$

Obviously, as a subset of $Sp(4,\mathbf{Z})$, monodromy group $\Gamma=\rho(\pi_1(S-D,t_0))$ is generated by the above four matrices. In [ACa], A'Campo gave the following

Lemma 3 (A'Campo [ACa]) $\Gamma_2(2)\subset\Gamma\subset Sp(4,\mathbf{Z})$, *and* $\Gamma/\Gamma_2(2)\cong S_5$.

Using the homomorphism (68), let us obtain a more precise characterization of Γ. Here we define that $\Gamma' := \{M\in Sp(4,\mathbf{Z})|M(6)=6\}$, where each M is regarded as an element of S_6 by the map (68). Then obviously, we obtain that

$$\Gamma_2(2)\subset\Gamma'\subset Sp(4,\mathbf{Z}),\ \text{and}\ \Gamma'/\Gamma_2(2)\cong S_5\ . \tag{14}$$

Moreover, it can be easily seen that $\rho(\gamma_i)\in\Gamma'$ for any $i\in\{1,2,3,4\}$, which implies that $\Gamma\subset\Gamma'$. Therefore, by (14) and Lemma 3, we obtain that $\Gamma=\Gamma'$.

4.3 Monodromy covering and a configuration space of five roots of $F(x,0,t_0)$.

First we define a natural homomorphism $\pi_1(S-D,t_0)\to S_5$. Here we use an element $(e_1,\dots,e_5)\in\mathbf{C}^5$ where $e_1,\dots,e_5$ are roots of $F(x,0,t_0)$ as in **Figure-1**. Note that $\pi_1(S-D,t_0)$ is isomorphic to the Artin braid group of five strings. Now we take an element $\gamma\in\pi_1(S-D,t_0)$. Then five roots of $F(x,0,t)$, say $(e_1(t),\dots,e_5(t))$, starting from $(e_1,\dots,e_5)$, move "along γ" to arrive a point. We denote the end point by $(e_{\gamma^{-1}(1)},\dots,e_{\gamma^{-1}(5)})$. Thus we have a group homomorphism $\pi_1(S-D,t_0)\to S_5$.

Here we obtain the following lemma.

Lemma 4 *The diagram:*

$$\begin{array}{ccc}\pi_1(S-D,t_0) & \longrightarrow\!\!\!\!\!\rightarrow & S_5\\ \rho\downarrow & & \downarrow\\ \Gamma_2(1) & \longrightarrow\!\!\!\!\!\rightarrow & S_6\end{array}$$

Diagram-2

is commutative, where $\pi_1(S-D,t_0)\to S_5$ *is the homomorphism given above,* $\Gamma_2(1)\to S_6$ *is given in* (68), $\pi_1(S-D,t_0)\to\Gamma_2(1)$ *is given in* (12), *and* $S_5\hookrightarrow S_6$ *is natural embedding.*

Proof. We take a $\gamma\in\pi_1(S-D,t_0)$ arbitrarily. Then by the definition of ρ^*, any cycle $[e_i,e_j]\in H_1(X_{t_0},\mathbf{Z})$ is mapped by $\rho^*(\gamma)$ to $[e_{\gamma^{-1}(i)},e_{\gamma^{-1}(j)}]\in H_1(X_{t_0},\mathbf{Z})$, up to $2H_1(X_{t_0},\mathbf{Z})$. On the other hand, Lemma 2 implies that $[e_i,e_j]$ is mapped by $E\circ\rho^*(\gamma)$ to $[e_{(E\circ\rho^*(\gamma))(i)},e_{(E\circ\rho^*(\gamma))(j)}] \bmod 2H_1(X_{t_0},\mathbf{Z})$. Therefore, for any $i,j\in\{1,\dots,6\}$,

$$[e_{\gamma^{-1}(i)},e_{\gamma^{-1}(j)}]\equiv[e_{(E\circ\rho^*(\gamma))(i)},e_{(E\circ\rho^*(\gamma))(j)}]\qquad \bmod 2H_1(X_{t_0},\mathbf{Z})$$

is valid. As a result, we have that $(E\circ\rho^*(\gamma))(i)=\gamma^{-1}(i)$ for any $i\in\{1,\dots,6\}$. Therefore, $E\circ\rho^*(\gamma)=\gamma^{-1}$ in S_6. Hence, it can be easily seen that $\rho(\gamma)=\gamma$ in S_6. This completes the proof. ∎

Definition 5

$$
\begin{aligned}
V &:= \{(e_1,\dots,e_5) \in \mathbf{C}^5 | e_1 + \cdots + e_5 = 0\}\ , \\
D' &:= \{e \in V | e_i = e_j \ \textit{for some distinct}\ i,j \in \{1,\dots,5\}\}\ .
\end{aligned}
$$

From the above lemma, we have the following corollary.

Corollary 6 *The monodromy covering $\sigma : (S\text{–}D)^\wedge \to S-D$ is factored by $V-D'$. That is, there exists a covering map $e : (S\text{–}D)^\wedge \to V-D'$ such that the following diagram is commutative.*

$$
\begin{tikzcd}
 & & \widetilde{S-D}\ \text{:universal covering space} \arrow[dll] \arrow[dl] \arrow[dd] \\
V-D' \arrow[drr] & \widehat{S-D} \arrow[l, dashed] \arrow[dr, "\sigma"] & \\
 & & S-D
\end{tikzcd}
$$

5 Rosenhain's formula.

In this section, first we define root functions by modifying Rosenhain's expression arising from a theory of periods on curves of genus two. Then we obtain some automorphic property of the functions under the action of the monodromy group Γ.

5.1 Rosenhain's formula and root functions.

Suppose t be any point of $S-D$. We write $F(x,y,t)$ as

$$
F(x,y,t) = -y^2 + (x-e_1)\cdots(x-e_5)\ .
$$

Then X_t with a basis of $H_1(X_t,\mathbf{Z})$ taken as in the **Figure-1** gives a period matrix $\tau \in \mathbf{H}_2$. Rosenhain [Ros] gave expressions of anharmonic ratios of four of six ramified points of X_t by theta constants:

$$
\frac{e_k - e_1}{e_2 - e_1} = \lambda_k(\tau) \qquad (k = 3,4,5)\ , \tag{15}
$$

where

$$
\lambda_3(\tau) = \frac{\vartheta_{134}^2(\tau)\vartheta_{135}^2(\tau)}{\vartheta_{124}^2(\tau)\vartheta_{125}^2(\tau)}, \qquad \lambda_4(\tau) = \frac{\vartheta_{143}^2(\tau)\vartheta_{145}^2(\tau)}{\vartheta_{123}^2(\tau)\vartheta_{125}^2(\tau)}, \qquad \lambda_5(\tau) = \frac{\vartheta_{153}^2(\tau)\vartheta_{154}^2(\tau)}{\vartheta_{123}^2(\tau)\vartheta_{124}^2(\tau)}\ . \tag{16}
$$

For the sake of convenience, we define $\lambda_1(\tau) := 0$, $\lambda_2(\tau) := 1$. The invariableness of each λ_i under the action of $\Gamma_2(2)$, is almost trivial by the transformation formula of theta constants. For any $i \in \{1,\dots,5\}$, we define functions β_i as the product of λ_i with the least common multiple of denominators of λ_3, λ_4, and λ_5, that is,

$$
\beta_i := \vartheta_{123}^2\vartheta_{124}^2\vartheta_{125}^2\lambda_i \qquad (1 \le i \le 5)\ . \tag{17}
$$

Then we can write β_i as follows:

$$
\beta_1 = 0, \quad \beta_2 = \vartheta_{213}^2\vartheta_{214}^2\vartheta_{215}^2, \quad \beta_3 = \vartheta_{312}^2\vartheta_{314}^2\vartheta_{315}^2, \quad \beta_4 = \vartheta_{412}^2\vartheta_{413}^2\vartheta_{415}^2, \quad \beta_5 = \vartheta_{512}^2\vartheta_{513}^2\vartheta_{514}^2.
$$

Moreover, we define functions α_i as follows.

$$\alpha_i := \beta_i - \frac{1}{5}\sum_{j=1}^{5}\beta_j = \frac{1}{5}\sum_{j=1}^{5}(\beta_i - \beta_j) \ . \tag{18}$$

Using those α_i ($i \in \{1,\ldots,5\}$), we define a map $F : \mathbf{H}_2 \to S$ as follows:

$$F : \mathbf{H}_2 \ni \tau \longmapsto t \in S \ , \tag{19}$$
$$\text{where} \qquad t_i = (-1)^i \sum_{1\le\nu_1<\cdots<\nu_i\le 5} \alpha_{\nu_1}\cdots\alpha_{\nu_i} \qquad (i \in \{2,3,4,5\}) \ . \tag{20}$$

Since each θ_{ijk} ($1 \le i < j < k \le 5$), and hence each $\beta_i - \beta_j$ ($i < j$) has no zeros on $\mathbf{H}_2^*$ we conclude that $F(\mathbf{H}_2^*) \subset S - D$. Moreover, by (15), (18), for any $h \in (S{-}D)^\wedge$, there exists $\lambda \in \mathbf{C}^*$ such that $\lambda \cdot \sigma(h) = F \circ \varphi \circ P(h)$, and hence the equality

$$\sigma(\lambda \cdot h) = F \circ \varphi \circ P(\lambda \cdot h)$$

holds. As a result, we have the following lemma.

Lemma 7 *For any $\tau \in \mathbf{H}_2^*$, we have*

$$\varphi \circ P(\sigma^{-1}(F(\tau))) = \{M \circ \tau | M \in \Gamma\} \ .$$

5.2 Five functions as modular forms.

The functions $\alpha_1, \ldots, \alpha_5$, which was defined in the previous subsection, have modular property under the action of Γ over $\mathbf{H}_2$. In this subsection we obtain the modular property and investigate the factor of automorphy. To begin with, we show the following easy lemma.

Lemma 8 $\Gamma \ni M \mapsto \chi(M) := \kappa(M)^2 \exp[2\pi\sqrt{-1}\phi(M,(1101))] \in \mathbf{C}^*$ *is group homomorphism.*

Proof. First note that a formula of ϕ defined in (62). For any $M_1, M_2 \in Sp(2g,\mathbf{Z})$ and $\varepsilon \in \mathbf{Z}^{2g}$, simple computation gives

$$\begin{aligned}\phi(M_2M_1,\varepsilon) &= \phi(M_2, M_1\circ\varepsilon) + \phi(M_1,\varepsilon) - \phi(M_2, M_1\circ 0) + \\ &+\frac{1}{2}(\varepsilon'^tD_3 - \varepsilon''^tC_3)^t[(A_3{}^tB_3)_0 - (-(C_1{}^tD_1)_0{}^tB_2 + (A_1{}^tB_1)_0{}^tA_2 + (A_2{}^tB_2)_0)] \ ,\end{aligned}$$

where $M_3 := M_2M_1$, $M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ ($i = 1,2,3$). Note that $[\cdots]$ is an element of $(2\mathbf{Z})^{2g}$. Hence, with the equality given in (64), we have

$$\begin{aligned}&\kappa(M_2M_1)^2 \exp[2\pi\sqrt{-1}\phi(M_2M_1,\varepsilon)] \\ &= \kappa(M_2)^2 \exp[2\pi\sqrt{-1}\phi(M_2, M_1\circ\varepsilon)]\kappa(M_1)^2 \exp[2\pi\sqrt{-1}\phi(M_1,\varepsilon)] \ .\end{aligned} \tag{21}$$

Therefore, if $g = 2$, $M_1, M_2 \in \Gamma$ and $\varepsilon = (1101)$, we obtain a result that we wanted. ∎

Therefore, the map

$$\Gamma \times \mathbf{H}_2 \ni (M,\tau) \longmapsto \kappa(M,\tau)^2 \exp[2\pi\sqrt{-1}\phi(M,(1101))] \in \mathbf{C}^* \tag{22}$$

is a factor of automorphy. The following lemma gives a square root of (22).

Lemma 9 *Suppose that $\varepsilon = (1101)$. Then*

$$\Gamma \times \mathbf{H}_2 \ni (M,\tau) \longmapsto \kappa(M,\tau)\exp[\pi\sqrt{-1}(\phi(M,\varepsilon) - \frac{1}{2}\varepsilon'^t(-\varepsilon'^tB + \varepsilon''^tA + (A^tB)_0 - \varepsilon''))] \in \mathbf{C}^*$$

is a factor of automorphy.

Proof. For any $M_1, M_2 \in Sp(2g, \mathbf{Z})$, $\varepsilon \in \mathbf{Z}^{2g}$, we have

$$\begin{aligned}
& (A_3{}^tB_3)_0 - (-(C_1{}^tD_1)_0{}^tB_2 + (A_1{}^tB_1)_0{}^tA_2 + (A_2{}^tB_2)_0) \\
= \; & [-\varepsilon'^tB_3 + \varepsilon''^tA_3 + (A_3{}^tB_3)_0 - \varepsilon''] - [-\varepsilon'^tB_2 + \varepsilon''^tA_2 + (A_2{}^tB_2)_0 - \varepsilon''] \\
- \; & [-\varepsilon'^tB_1 + \varepsilon''^tA_1 + (A_1{}^tB_1)_0 - \varepsilon''] + 2\delta'^tB_2 - 2\delta''^tA_2 + 2\delta'' , \\
& \phi(M_2, M_1 \circ \varepsilon) + \varepsilon'^t(\delta'^tB_2 - \delta''^tA_2 + \delta'') \\
= \; & \phi(M_2, \varepsilon) + (\varepsilon'^tD_2 - \varepsilon''^tC_2 + (C_2{}^tD_2)_0 - \varepsilon')(-B_2{}^t\delta' + A_2{}^t\delta'') \\
+ \; & (\delta'^tD_2 - \delta''^tC_2 + (C_2{}^tD_2)_0)^t(-\delta'^tB_2 + \delta''^tA_2 + (A_2{}^tB_2)_0) - \delta'^t\delta'' \\
- \; & 2(C_2{}^tD_2)_0(-B_2{}^t\delta' + A_2{}^t\delta'') - (C_2{}^tD_2)_0{}^t(A_2{}^tB_2)_0 ,
\end{aligned}$$

where $M_3 := M_2M_1$ and $\delta := \frac{1}{2}(M_1 \circ \varepsilon - \varepsilon)$. Using the above equalities, we can prove the lemma by simple computation. ∎

Here we denote that, for any $M \in \Gamma$, $\tau \in \mathbf{H}_2$,

$$j_{1101}(M, \tau) := \kappa(M, \tau) \exp[\pi\sqrt{-1}(\phi(M, \varepsilon) - \frac{1}{2}\varepsilon'^t(-\varepsilon'^tB + \varepsilon''^tA + (A^tB)_0 - \varepsilon''))] \det(C\tau + D) ,$$

where $\varepsilon = (1101)$. Note that $j_{1101}(M, \tau)^2 = \kappa(M)^2(\det(C\tau + D))^3 \exp[2\pi\sqrt{-1}\phi(M, (1101))]$. The factor of automorphy j_{1101} is important by the following lemma.

Lemma 10 *For each* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, $i \in \{1, 2, 3, 4, 5\}$ *and* $\tau \in \mathbf{H}_2$, *it satisfies that*

$$\alpha_{M(i)}(M \circ \tau) = j_{1101}(M, \tau)^2 \alpha_i(\tau) .$$

Proof. By the definition of α_i, we can write as

$$\alpha_{M(i)}(M \circ \tau) = \frac{1}{5} \sum_{j=1}^{5} (\beta_{M(i)}(M \circ \tau) - \beta_{M(j)}(M \circ \tau)) .$$

Hence now let us investigate the factor of automorphy of $\beta_i - \beta_j$ under the action of Γ. By the way, with the aid of formulas:

$$\begin{aligned}
\vartheta_{135}^2\vartheta_{145}^2 &= \vartheta_{136}^2\vartheta_{146}^2 + \vartheta_{132}^2\vartheta_{142}^2 , & \vartheta_{134}^2\vartheta_{154}^2 &= \vartheta_{136}^2\vartheta_{156}^2 + \vartheta_{132}^2\vartheta_{152}^2 , \\
\vartheta_{143}^2\vartheta_{153}^2 &= \vartheta_{146}^2\vartheta_{156}^2 + \vartheta_{142}^2\vartheta_{152}^2 , & \vartheta_{125}^2\vartheta_{135}^2 &= \vartheta_{126}^2\vartheta_{136}^2 + \vartheta_{124}^2\vartheta_{134}^2 , \\
\vartheta_{125}^2\vartheta_{145}^2 &= \vartheta_{126}^2\vartheta_{146}^2 + \vartheta_{123}^2\vartheta_{143}^2 , & \vartheta_{124}^2\vartheta_{154}^2 &= \vartheta_{126}^2\vartheta_{156}^2 + \vartheta_{123}^2\vartheta_{153}^2 ,
\end{aligned}$$

the differences $\beta_i - \beta_j$ are written as follows:

$$\beta_i - \beta_j = \mathrm{sign}(i - j) \cdot \vartheta_{ijk}^2\vartheta_{ijl}^2\vartheta_{ijm}^2 , \tag{23}$$

where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$, and $\mathrm{sign}(x) := \pm 1$ if $\pm x > 0$.

Remark. By (23), we have

$$\prod_{1 \leq i < j \leq 5} (\beta_j - \beta_i) = \Theta^6, \qquad \text{where} \qquad \Theta := \prod_{1 \leq i < j < k \leq 5} \vartheta_{ijk} . \qquad \square$$

Note that, for any $M \in Sp(4, \mathbf{Z})$, (71) holds. Then, for any $M \in \Gamma$, $\tau \in \mathbf{H}_2$ and $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$,

$$\begin{aligned}
& \beta_{M(i)}(M \circ \tau) - \beta_{M(j)}(M \circ \tau) \\
= \; & \mathrm{sign}(M(i) - M(j))\vartheta_{M(i)M(j)M(k)}^2(M \circ \tau)\vartheta_{M(i)M(j)M(l)}^2(M \circ \tau)\vartheta_{M(i)M(j)M(m)}^2(M \circ \tau) \\
= \; & \mathrm{sign}(M(i) - M(j))\vartheta_{M(ijk)}^2(M \circ \tau)\vartheta_{M(ijl)}^2(M \circ \tau)\vartheta_{M(ijm)}^2(M \circ \tau) \\
= \; & \mathrm{sign}(M(i) - M(j))\kappa(M)^6 \exp 2\pi\sqrt{-1}\phi(M, ijk) \exp 2\pi\sqrt{-1}\phi(M, ijl) \exp 2\pi\sqrt{-1}\phi(M, ijm) \\
& \times \det(C\tau + D)^3\vartheta_{ijk}^2(\tau)\vartheta_{ijl}^2(\tau)\vartheta_{ijm}^2(\tau) \\
= \; & \mathrm{sign}(M(i) - M(j)) \, \mathrm{sign}(i - j)\kappa(M)^6 \exp 2\pi\sqrt{-1}(\phi(M, ijk) + \phi(M, ijl) + \phi(M, ijm)) \\
& \times \det(C\tau + D)^3(\beta_i(\tau) - \beta_j(\tau)) .
\end{aligned}$$

Hence, the proof of Lemma 10 is reduced to the proof of the following lemma.

Lemma 11 *Suppose that i,j,k,l,m are permutation of $1,2,3,4,5$, Then for any $M \in \Gamma$, the following equality holds:*

$$\begin{aligned} & \operatorname{sign}(M(i)-M(j))\operatorname{sign}(i-j)\kappa(M)^6 \exp 2\pi\sqrt{-1}(\phi(M,ijk)+\phi(M,ijl)+\phi(M,ijm)) \\ = \;& \kappa(M)^2 \exp 2\pi\sqrt{-1}\phi(M,6)\ . \end{aligned}$$

Proof. First we define two sets:

$$\begin{aligned} S &:= \{A \mid A \subset \{1,2,3,4,5\}, \#A = 2\} = \{\{i,j\} \mid i,j \in \{1,2,3,4,5\}, i \neq j\}, \\ T &:= \{\text{even theta characteristics of genus two}\} \ \ (\subset \{0,1\}^4)\ . \end{aligned}$$

Note that $\#S = \#T = 10$. It can be easily checked that the map

$$S \ni \{i,j\} \longmapsto ij6 \in T \tag{24}$$

gives bijection from S to T. Here we define Γ-action on S as follows: for any $M \in \Gamma$ and $\{i,j\} \in S$, we define

$$M : \{i,j\} \longmapsto \{M(i), M(j)\}\ .$$

Then the Γ-actions on S and on T are compatible to the map (24), that is, the map (24) satisfies

$$\{M(i), M(j)\} \longmapsto M(ij6)$$

for any $\{i,j\} \in S$ and $M \in \Gamma$. Now we denote that

$$\begin{aligned} \chi_1(M,\{i,j\}) &:= \operatorname{sign}(M(i)-M(j))\operatorname{sign}(i-j)\ , \\ \chi_2(M,\{i,j\}) &:= \kappa(M)^6 \exp 2\pi\sqrt{-1}(\phi(M,ijk)+\phi(M,ijl)+\phi(M,ijm)) \\ &\qquad \times \kappa(M)^{-2}\exp(-2\pi\sqrt{-1}\phi(M,6))\ . \end{aligned}$$

What we want to prove is the equality

$$\chi_1(M,\{i,j\}) = \chi_2(M,\{i,j\}) \qquad (\forall M \in \Gamma, \forall \{i,j\} \in S)\ .$$

The proof is decomposed into two steps.
Step 1 For any $M, M' \in \Gamma$, $\{i,j\} \in S$ and $\mu \in \{1,2\}$, the equality

$$\chi_\mu(M'M,\{i,j\}) = \chi_\mu(M',\{M(i),M(j)\})\chi_\mu(M,\{i,j\})$$

holds. In fact, if $\mu = 1$, this equality is trivial. On the oter hand, if $\mu = 2$, this equality holds by (21) and (71).
Step 2 Suppose $M = M_\nu^{\pm 1}$ $(\nu = 1,2,3,4)$, where $M_\nu := \rho(\gamma_\nu)$. Then we have

$$\chi_1(M,\{i,j\}) = \chi_2(M,\{i,j\}) \qquad \text{for } \forall \{i,j\} \in S\ .$$

Now we check this fact by giving the values $\chi_\mu(M_\nu^{\pm 1},\{i,j\})$ explicitly. First let us give $\chi_2(M_\nu^{\pm 1},\{i,j\})$. Suppose n be any integer. Then

$$\begin{aligned} 4\phi(M_1^n,\varepsilon) &= n(-(\varepsilon_1')^2 + 2\varepsilon_1')\ , \qquad & 4\phi(M_2^n,\varepsilon) &= n(\varepsilon_1'' - \varepsilon_2'')^2\ , \\ 4\phi(M_3^n,\varepsilon) &= n(-(\varepsilon_2')^2 + 2\varepsilon_2')\ , \qquad & 4\phi(M_4^n,\varepsilon) &= n(\varepsilon_2'')^2\ . \end{aligned}$$

Especially, $4\phi(M_\nu^n,(1101)) = n$ $(\forall \nu = 1,2,3,4)$. Thus, it can be easily seen that, in the meaning of mod.4,

$$\begin{aligned} & 4\phi(M_\nu^n,ijk) + 4\phi(M_\nu^n,ijl) + 4\phi(M_\nu^n,ijm) - 4\phi(M_\nu^n,(1101)) \\ & = \begin{cases} 2n & (\text{when } \{i,j\} = \{\nu,\nu+1\}) \\ 0 & (\text{otherwise}) \end{cases} \ . \end{aligned} \tag{25}$$

On the other hand, it is well known (cf.[R-F] p90) that, for any $\nu \in \{1,3,4\}$, the equalities

$$\kappa(M_\nu)^2 = \kappa(M_\nu^{-1})^2 = 1 \tag{26}$$

hold. Moreover, by the decomposition $M_2 = {}^-C_2{}^+B_2{}^-C_2{}^+A_{12}{}^+C_2{}^-B_2{}^-C_1$ and relation (64), the equality (26) also holds for $\nu = 2$. Consequently, if $\{i,j,k,l,m\} = \{1,2,3,4,5\}$, then for any $\nu \in \{1,2,3,4\}$ we have

$$\chi_\mu(M_\nu^{\pm 1}, \{i,j\}) = \begin{cases} -1 & (\text{when } \{i,j\} = \{\nu, \nu+1\}) \\ 1 & (\text{otherwise}) \end{cases} \tag{27}$$

with $\mu = 2$. On the other hand, for any $\nu \in \{1,2,3,4\}$, the action of $M_\nu^{\pm 1}$ over $\{1,\ldots,5\}$ coincides with that of $(\nu, \nu+1)$. Therefore, (27) with $\mu = 1$ holds. Hence the claim of **Step 2** is verified. Thus the proof of Lemma 11 is completed. ∎

Thus the proof of Lemma 10 is completed. ∎

6 Monodromy covering as $\mathbf{C}^*$-bundle.

By lifting the $\mathbf{C}^*$-action on $S-D$ to $(S–D)^\wedge$, we can define a $\mathbf{C}^*$-action on $(S–D)^\wedge$. The aim of this section is to show that, with the $\mathbf{C}^*$-action, $(S–D)^\wedge$ becomes the total space of a $\mathbf{C}^*$-bundle in the strict sense.

6.1 Injectivity of P.

In proving the injectivity of P, we use the following well known fact, which is a part of the Torelli's theorem.

Fact 12 (Torelli) *Suppose that:*

- X_1, X_2 *are compact Riemann surfaces of genus two.*
- *For each* $k(=1,2)$, $A_{k1}, A_{k2}, B_{k1}, B_{k2}$ *are* $\mathbf{Z}$*-basis of* $H_1(X_k, \mathbf{Z})$ *such that* $\langle A_{ki}, B_{kj}\rangle = \delta_{ij}$, $\langle A_{ki}, A_{kj}\rangle = \langle B_{ki}, B_{kj}\rangle = 0$, *where* $\langle\ \ ,\ \ \rangle$ *is the intersection form on* X_k *and* δ_{ij} *is Kronecker's delta.*
- ω_{k1}, ω_{k2} *are holomorphic 1-forms on* X_k, *linearly independent over* $\mathbf{C}$ *and satisfying* $\int_{A_{kj}} \omega_{ki} = \delta_{ij}$.
- *We denote that* $\tau_k := \begin{bmatrix} \int_{B_{k1}} \omega_{k1} & \int_{B_{k2}} \omega_{k1} \\ \int_{B_{k1}} \omega_{k2} & \int_{B_{k2}} \omega_{k2} \end{bmatrix}$.

Then

$$[\, \tau_1 = \tau_2 \,] \Longrightarrow \begin{bmatrix} {}^\exists f : X_1 \xrightarrow{\sim} X_2 \quad (\textit{biregular}) \quad \textit{such that} \\ f^*\omega_{2i} = \omega_{1i}\ (i=1,2), \quad f_*A_{1j} = A_{2j},\ f_*B_{1j} = B_{2j}\ (j=1,2) \end{bmatrix}.$$

Proof. See, for example, [Mar] or [Mum1]. ∎

Lemma 13 *P is injective.*

Proof. Suppose that h, h' are elements of $(S-D)^\wedge$. To avoid confusion, only in the proof, we use new letters to write defining equations of $X_{\sigma(h)}$ and $X_{\sigma(h')}$ as follows:

$$X_{\sigma(h)} : y^2 = (x-e_1)\cdots(x-e_5) , \qquad X_{\sigma(h')} : w^2 = (z-e_1')\cdots(z-e_5') .$$

We denote by $\overline{X_t}$ a compact Riemann surface given by the resolution of singularity of $X_t \cup \{\infty\}$. Suppose that $P(h) = P(h')$. Then by the Torelli's theorem there exists a biholomorphic bijective map $f : \overline{X_{\sigma(h)}} \to \overline{X_{\sigma(h')}}$ satisfying

$$f^*\left(\frac{z^{i-1}dz}{w}\right) = \frac{x^{i-1}dx}{y} \qquad (i = 1, 2) , \tag{28}$$

$$f_* A_j(h) = A_j(h') \qquad (j = 1, 2, 3, 4) . \tag{29}$$

The divisor on $\overline{X_{\sigma(h)}}$ given by $\frac{dx}{y}$ is $2\cdot\infty$. On the other hand, the divisor on $\overline{X_{\sigma(h')}}$ given by $\frac{dz}{w}$ is also $2\cdot\infty$. Therefore, by (28) with $i = 1$, we obtain that $f(\infty) = \infty$. Hence there exists a constant $\lambda \in \mathbf{C}^*$ such that

$$\{e_1', \ldots, e_5'\} = \{\lambda^2 e_1, \ldots, \lambda^2 e_5\} , \tag{30}$$

$$f \text{ coincides with the map } \quad (x, y) \longmapsto (z, w) = (\lambda^2 x, \lambda^5 y) . \tag{31}$$

By (31) we obtain $f^*\left(\frac{zdz}{w}\right) = \frac{1}{\lambda}\frac{xdx}{y}$. Therefore, together with the relation (28) we have that $\lambda = 1$. Hence $\{e_1', \ldots, e_5'\} = \{e_1, \ldots, e_5\}$ and f is the trivial isomorphism (that is, (31) with $\lambda = 1$). Moreover, this map satisfy the condition (29) and hence we conclude that $h = h'$. ∎

6.2 Injectivity of $(dP)_h$.

First we review a well known fact.

Fact 14 (Saito, K.) *On the above situation and notations, we have*

$${}^t\left(\frac{\partial\omega_{1k}}{\partial t_2}\ \frac{\partial\omega_{1k}}{\partial t_3}\ \frac{\partial\omega_{1k}}{\partial t_4}\ \frac{\partial\omega_{1k}}{\partial t_5}\right){}^tT = {}^t\left(-\frac{3}{2}\omega_{1k}\ \frac{5}{2}\omega_{2k}\ \frac{15}{2}\omega_{3k}\ \frac{15}{2}\omega_{4k}\right)$$

for each $k \in \{1, 2, 3, 4\}$, where

$$T = \begin{pmatrix} 2t_2 & 3t_3 & 4t_4 & 5t_5 \\ -15t_3 & 6t_2^2 - 20t_4 & 4t_2t_3 - 25t_5 & 2t_2t_4 \\ 60t_4 - 10t_2^2 & 75t_5 - 27t_2t_3 & 10t_2t_4 - 18t_3^2 & 20t_2t_5 - 9t_3t_4 \\ 25t_5 + 15t_2t_3 & 18t_2t_4 - 6t_2^3 & 40t_2t_5 - 3t_3t_4 - 4t_2^2t_3 & 10t_3t_5 - 4t_4^2 - 2t_2^2t_4 \end{pmatrix}$$

and $\det T = -75\Delta$.

Proof. First we note that

$$\frac{\partial}{\partial t_k}\int_{A_k(t)}\frac{dx}{y} = -\int_{A_k(t)}\frac{x^{5-k}dx}{2yf} , \qquad (k = 2, 3, 4, 5) , \tag{32}$$

and, for any fixed $t \in S - D$,

$$d\left(\frac{x^n}{y}\right) = \frac{nx^{n-1}}{y}dx - \frac{x^n\dfrac{\partial f}{\partial x}}{2yf}dx = \frac{2nx^{n-1}f - x^n\dfrac{\partial f}{\partial x}}{2yf}dx . \tag{33}$$

Here we denote, for any integer n,

$$W_n := 2nx^{n-1}f - x^n\frac{\partial f}{\partial x} .$$

Then we can write (33) as

$$d\left(\frac{x^n}{y}\right) = \frac{W_n}{2yf}dx \qquad (n \in \mathbf{Z})\,. \tag{34}$$

For simplisity, we write $\mathbf{Q}[t] := \mathbf{Q}[t_2,t_3,t_4,t_5]$ and $\mathbf{Q}[t,x] := \mathbf{Q}[t_2,t_3,t_4,t_5,x]$. For any nonnegative integer n, W_n is a polynomial of x with coefficients in $\mathbf{Q}[t]$, whose leading term is $(2n-5)x^{n+4}$. Therefore, it can be easily seen that, as $\mathbf{Q}[t]$-modules,

$$\mathbf{Q}[t,x] = \bigoplus_{k=0}^{\infty}\mathbf{Q}[t]x^k = \left(\bigoplus_{k=0}^{3}\mathbf{Q}[t]x^k\right)\bigoplus\left(\bigoplus_{n=0}^{\infty}\mathbf{Q}[t]W_n\right)\,.$$

Hence, each $P \in \mathbf{Q}[t,x]$ has unique expression as

$$\begin{aligned} &P = \sum_{k=0}^{3}\varphi_k x^k + \sum_{n=0}^{deg_x P-4}\psi_n W_n\,, \\ &\text{where} \quad \varphi_k,\psi_n \in \mathbf{Q}[t] \quad (k \in \{0,1,2,3\}\,,\ n \in \{0,1,\ldots,deg_x P-4\})\,. \end{aligned} \tag{35}$$

By (32), (34) and (35), we have

$$\begin{aligned} \int_{A_k(t)}\frac{P}{2yf}dx &= \sum_{k=0}^{3}\varphi_k(t)\int_{A_k(t)}\frac{x^k}{2yf}dx + \sum_{n=0}^{deg_x P-4}\psi_n(t)\int_{A_k(t)}\frac{W_n}{2yf}dx \\ &= -\sum_{k=0}^{3}\varphi_k(t)\frac{\partial}{\partial t_{5-k}}\int_{A_k(t)}\frac{dx}{y}\,. \end{aligned}$$

What we need for our purpose is to get $\{\varphi_k\}$ satisfying (35) when $P = x^i f$ (i=0,1,2,3).

First, $W_n (n \in \{0,1,2,3,4\})$ are as follows.

$$\begin{aligned} W_0(x) &:= -5x^4 - 3t_2x^2 - 2t_3x - t_4\,, \\ W_1(x) &:= -3x^5 - t_2x^3 + t_4x + 2t_5\,, \\ W_2(x) &:= -x^6 + t_2x^4 + 2t_3x^3 + 3t_4x^2 + 4t_5x\,, \\ W_3(x) &:= x^7 + 3t_2x^5 + 4t_3x^4 + 5t_4x^3 + 6t_5x^2\,, \\ W_4(x) &:= 3x^8 + 5t_2x^6 + 6t_3x^5 + 7t_4x^4 + 8t_5x^3\,. \end{aligned}$$

From these formulae, we have expression of $x^i f$ into the form like (35) for each i(=0,1,2,3) as follows.

$$\begin{aligned} 3f &= 2t_2x^3 + 3t_3x^2 + 4t_4x + 5t_5 - W_1\,, \\ 5xf &= 15t_3x^3 + (20t_4 - 6t_2^2)x^2 + (25t_5 - 4t_2t_3)x - 2t_2t_4 - 5W_2 + 2t_2W_0\,, \\ -15x^2f &= 10(6t_4 - t_2^2)x^3 + (75t_5 - 27t_2t_3)x^2 + (10t_2t_4 - 18t_3^2)x + 20t_2t_5 - 9t_3t_4 \\ &\quad -15W_3 - 10t_2W_1 - 9t_3W_0\,, \\ -15x^3f &= 5(5t_5 + 3t_2t_3)x^3 + (18t_2t_4 - 6t_2^3)x^2 + (40t_2t_5 - 3t_3t_4 - 4t_2^2t_3)x \\ &\quad +10t_3t_5 - 4t_4^2 - 2t_2^2t_4 - 5W_4 - 10t_2W_2 - 5t_3W_1 - (4t_4 + 2t_2^2)W_0\,. \end{aligned}$$

Finally, we have

$$\begin{aligned} -\frac{3}{2}\int_{A_k(t)}\frac{dx}{y} &= \left[2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + 4t_4\frac{\partial}{\partial t_4} + 5t_5\frac{\partial}{\partial t_5}\right]\int_{A_k(t)}\frac{dx}{y}\,, \\ -\frac{5}{2}\int_{A_k(t)}\frac{dx}{y} &= \left[15t_3\frac{\partial}{\partial t_2} + (20t_4 - 6t_2^2)\frac{\partial}{\partial t_3} + (25t_5 - 4t_2t_3)\frac{\partial}{\partial t_4} - 2t_2t_4\frac{\partial}{\partial t_5}\right]\int_{A_k(t)}\frac{dx}{y}\,, \\ \frac{15}{2}\int_{A_k(t)}\frac{x^2dx}{y} &= \Bigg[10(6t_4 - t_2^2)\frac{\partial}{\partial t_2} + (75t_5 - 27t_2t_3)\frac{\partial}{\partial t_3} \\ &\quad +(10t_2t_4 - 18t_3^2)\frac{\partial}{\partial t_4} + (20t_2t_5 - 9t_3t_4)\frac{\partial}{\partial t_5}\Bigg]\int_{A_k(t)}\frac{dx}{y}\,, \\ \frac{15}{2}\int_{A_k(t)}\frac{x^3dx}{y} &= \Bigg[5(5t_5 + 3t_2t_3)\frac{\partial}{\partial t_2} + (18t_2t_4 - 6t_2^3)\frac{\partial}{\partial t_3} \\ &\quad +(40t_2t_5 - 3t_3t_4 - 4t_2^2t_3)\frac{\partial}{\partial t_4} + (10t_3t_5 - 4t_4^2 - 2t_2^2t_4)\frac{\partial}{\partial t_5}\Bigg]\int_{A_k(t)}\frac{dx}{y}\,. \end{aligned}$$

Therefore, the matrix T is given. We can check the equality $\det T = -75\Delta$ by computer. ∎

Lemma 15 (Saito, K.) *The differential of P at any point $h \in (S\text{–}D)^\wedge$, that is, $(dP)_h : T_h((S\text{–}D)^\wedge) \to T_{P(h)}(M_{2,4}(\mathbf{C}))$ is injective.*

Proof. It is sufficient to prove that $\det \dfrac{\partial(\omega_{11}, \ldots, \omega_{14})}{\partial(t_2, \ldots, t_5)} \neq 0$ for any $h \in (S\text{–}D)^\wedge$. Then, by Fact 14 it is sufficient to see that $\det(\omega_{ij})_{i,j=1,\ldots,4} \neq 0$ for any $h \in (S\text{–}D)^\wedge$. In general, suppose X be a compact Riemann surface of genus $g(\geq 1)$ and $\mathcal{M}_X$ be the sheaf of germs of meromorphic functions on X, then there exists the following canonical isomorphism as $\mathbf{C}$ vector spaces (of $2g$-dimensional):

$$H^1(X, \mathbf{C}) \cong Hom_{\mathbf{C}}(H_1(X, \mathbf{C}), \mathbf{C}) \cong \Gamma(X, d\mathcal{M}_X) \Big/ d\Gamma(X, \mathcal{M}_X). \tag{36}$$

In particular, when X is a compact Riemann surface (of genus two) defined by $F(x, y, t) = 0$, $\dfrac{x^{i-1}dx}{y} \bmod d\Gamma(X, \mathcal{M}_X)$ $(i = 1, \ldots, 4)$ are $\mathbf{C}$-basis of the right hand side space of (36). Hence $\det(\omega_{ij})_{i,j=1,\ldots,4} \neq 0$. ∎

6.3 Image of $\varphi \circ P$.

The aim of this subsection is to prove the following lemma.

Lemma 16 Image$(\varphi \circ P) = \mathbf{H}_2^*$.

Proof. First we recall two well-known facts.

Fact 17 (cf. [Wei]) *Suppose τ be an element of $\mathbf{H}_2$. Then there is a compact Riemann surface R of genus two with a symplectic basis $\{A_1, A_2, B_1, B_2\}$ of $H_1(R, \mathbf{Z})$ which gives τ as period matrix if and only if $\tau \in \mathbf{H}_2^*$.*

Fact 18 (cf. for example, [Gun]) *Suppose that R is an arbitrary compact Riemann surface of genus two. Then there exists $t \in S - D$ such that R is complex analytically isomorphic to $\overline{X_t}$.*

By the above two facts it is obvious that

$$\text{Image}(\varphi \circ P) \subset \{M \circ \tau | \tau \in \text{Image}(\varphi \circ P),\ M \in \Gamma_2(1)\} = \mathbf{H}_2^*\,.$$

Hence, for any $\tau' \in \mathbf{H}_2^*$ there exists $M \in \Gamma_2(1)$ such that $M \circ \tau' \in$ Image$(\varphi \circ P)$. But it is not trivial whether τ' itself is an element of Image$(\varphi \circ P)$ or not. To prove the lemma, we have only to show that, for any $\tau \in$ Image$(\varphi \circ P)$ the $\Gamma_2(1)$-orbit of τ is included in Image$(\varphi \circ P)$. This will be given by Claim 19 stated later on. To state the claims, now we give a little preparation.

Suppose that R is a compact Riemann surface of genus two which is given as a ramified covering over $\mathbf{P}^1 = \mathbf{P}^1(\mathbf{C})$ with six ordered ramified points $W_1, \ldots, W_6$. We take an oriented simple closed path on $\mathbf{P}^1$ which go through $W_1, \ldots, W_6$ in this order . For each $n \in \{1, \ldots, 6\}$ we denote by I_n, the segment of the path having W_n, W_{n+1} as its both ends, where $W_7 := W_1$. Then each $\kappa^{-1}(I_n)$ is a closed path on R. We give an orientation to each $\kappa^{-1}(I_n)$ and denote it by C_n such that $\langle C_n, C_{n+1}\rangle = 1$ for any $n \in \{1, \ldots, 5\}$, where $\langle\ ,\ \rangle$ is the intersection form on $H_1(R, \mathbf{Z})$. Note that here we identify, for each n, oriented closed path C_n with the element of $H_1(R, \mathbf{Z})$ having C_n as a representative. For the sake of convenience, we define W_n and C_n for any integer n such that $W_{n+6} = W_n$ and $C_{n+6} = C_n$ (for any $n \in \mathbf{Z}$). Now we define that

$$A_1^{(n)} := C_{n+1}, \quad A_2^{(n)} := C_{n+3}, \quad A_3^{(n)} = B_1^{(n)} := -C_n, \quad A_4^{(n)} = B_2^{(n)} := C_{n+4} \qquad (n \in \mathbf{Z})\,.$$

Then for each n, $A_1^{(n)}, A_2^{(n)}, B_1^{(n)}, B_2^{(n)}$ are symplectic basis of $H_1(R, \mathbf{Z})$. It can be easily seen that

$$(A_1^{(n-1)} \quad A_2^{(n-1)} \quad A_3^{(n-1)} \quad A_4^{(n-1)}) = (A_1^{(n)} \quad A_2^{(n)} \quad A_3^{(n)} \quad A_4^{(n)})S' \qquad (n \in \mathbf{Z})$$

$$\text{where} \qquad S' := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} . \tag{37}$$

On the other hand, the above R with a basis $A_1^{(n)}, \ldots, A_4^{(n)}$ gives a period matrix $\tau (\in \mathbf{H}_2^*)$. Obviously, this τ depends on the order of the branch points $\{W_i\}$ and the above simple closed path $\bigcup_n I_n$ (and, moreover, the ambiguity of the orientation of $\kappa^{-1}(I_1)$). But, if we consider the procedure of constructing the monodromy covering $(S - D)^\wedge$, it can be easily seen that Γ-orbit of the τ depends only on R and the sixth branch point W_6 but it doesn't depend on the order of the other five branch points, the oriented simple closed path and the ambiguity of the orientation of $\kappa^{-1}(I_1)$. Therefore we write the Γ-orbit as $\mathrm{orb}(R, W_6)$. Here we brought the preparation to an end.

Now we define that $S := K(S')^{-1}K^{-1}$. We note that $S^n(6) = n'$, where $n' \in \{1, \ldots, 6\}$ and $n' \equiv n \bmod 6$. Accordingly, we have $\Gamma S^n = \{M \in \Gamma_2(1) | M(n) = 6\}$ and hence $\Gamma_2(1) = \coprod_{n=0}^5 \Gamma S^n$. Therefore, to prove the lemma, we have only to show the following claim.

Claim 19 $S^n \circ \mathrm{orb}(\overline{X_t}, \infty) \subset \mathrm{Image}(\varphi \circ P)$ *for any* $n \in \mathbf{Z}$ *and* $t \in S - D$.

Proof. Suppose that t is any element of $S - D$. And suppose that $pr : \overline{X_t} \to \mathbf{P}^1$ is a map which is an extension of the projection $X_t \ni (x, y) \mapsto x \in \mathbf{C}$. pr is a ramified covering of $\mathbf{P}^1$ with six ramified points, say $W_1, \ldots, W_6 \in \mathbf{P}^1$ where W_6 is a point satisfying $x = \infty$. As in the preparation given above, we take an oriented simple closed path on $\mathbf{P}^1$ which go through $W_1, \ldots, W_6$ in this order, and using the path we take elements $C_n, A_j^{(n)} \in H_1(\overline{X_t}, \mathbf{Z})$ $(j, n \in \mathbf{Z},\ 1 \le j \le 4)$. Then we have

$$\mathrm{orb}(\overline{X_t}, W_6) = \{\varphi \circ P(h) | h \in \sigma^{-1}(t)\} \subset \mathrm{Image}(\varphi \circ P) .$$

Now we show that the inclusion

$$S^n \circ \mathrm{orb}(\overline{X_t}, W_0) \subset \mathrm{Image}(\varphi \circ P) \tag{38}$$

holds for any integer n. First note that, for any $n \in \mathbf{Z}$ we have

$$S^n \circ \mathrm{orb}(\overline{X_t}, W_0) = \mathrm{orb}(\overline{X_t}, W_{-n}) . \tag{39}$$

By (39), if $(\overline{X_t}, W_0) \not\sim (\overline{X_t}, W_{-n})$, then any point $t' \in S - D$ satisfying $(\overline{X_{t'}}, \infty) \sim (\overline{X_t}, W_{-n})$ is not included in the $\mathbf{C}^*$-orbit of t. Therefore,

$$\mathrm{orb}(\overline{X_t}, W_{-n}) = \{\varphi \circ P(h) | h \in \sigma^{-1}(t')\} \subset \mathrm{Image}(\varphi \circ P)$$

holds for the n. This inclusion and (39) imply (38). On the other hand, if $(\overline{X_t}, W_0) \sim (\overline{X_t}, W_{-n})$, then

$$S^n \circ \mathrm{orb}(\overline{X_t}, W_0) = \mathrm{orb}(\overline{X_t}, W_{-n}) = \mathrm{orb}(\overline{X_t}, W_0) \subset \mathrm{Image}(\varphi \circ P) .$$

Therefore (38) holds for any integer n. ∎

Here the proof of Lemma 16 is completed. ∎

6.4 $\mathbf{C}^*$-action on $(S–D)^\wedge$.

Lemma 20 *If $\lambda \in \mathbf{C}^*$, $\lambda \neq 1$, then λ-action on $(S–D)^\wedge$ has no fixed points.*

Proof. Suppose that there exists $\lambda \in \mathbf{C}^*$ and $h \in (S–D)^\wedge$ satisfying $\lambda \cdot h = h$. Then

$$\omega_{2j}(h) = \omega_{2j}(\lambda \cdot h) = \lambda^{-1}\omega_{2j}(h) \qquad (j = 1, 2, 3, 4)\ .$$

By an elementary result of the theory of compact Riemann surface that $(\omega_{21}(h), \ldots, \omega_{24}(h)) \neq 0$ for any $h \in (S–D)^\wedge$. Therefore $\lambda = 1$. ∎

Remark. To prove the above lemma, there is another way which doesn't use the period mapping. The proof is easy, but a little more complicated than the above proof. So we don't mention it here. □

6.5 Fiber of $\varphi \circ P$ at each point of $\mathbf{H}_2^*$.

First we take an element $h \in (S–D)^\wedge$. For the h, a symplectic basis $A_j(h) \in H_1(X_{\sigma(h)}, \mathbf{Z})$ ($j \in \{1, \ldots, 4\}$) is obtained. Using $\{A_j(h)\}$, an isomorphism (10) with $t = \sigma(h)$ is obtained. We denote the group of automorphisms of $\overline{X_{\sigma(h)}}$ by $\mathrm{Aut}(\overline{X_{\sigma(h)}})$. Any $f \in \mathrm{Aut}(\overline{X_{\sigma(h)}})$ determines an element M_f of $\mathrm{Aut}(H_1(X_{\sigma(h)}, \mathbf{Z}), \langle\ ,\ \rangle)$. M_f is regarded as an element of $Sp(4, \mathbf{Z})$ via (10). The following fact is an easy corollary of Fact12.

Fact 21 *The above homomorphism $\mathrm{Aut}(\overline{X_{\sigma(h)}}) \ni f \mapsto M_f \in Sp(4, \mathbf{Z})$ is injective. Its image coincides with $\mathrm{stab}_{Sp(4,\mathbf{Z})}(\tau)$.*

Proof. Omitted. ∎

As a preparation of Lemma 23, we prove the following lemma.

Lemma 22 *Using the above notations, it satisfies that*

$$f(\infty) = \infty \Longleftrightarrow M_f \in \Gamma\ .$$

Proof. [$\Rightarrow$] Suppose that $f(\infty) = \infty$. Then there exists $\lambda \in \mathbf{C}^*$ satisfying $\lambda \cdot \sigma(h) = \sigma(h)$ such that, f coincides with an automorphism of $\overline{X_{\sigma(h)}}$ defined by $(x, y) \mapsto (\lambda^2 x, \lambda^5 y)$. Note that $|\lambda| = 1$. Hence $\lambda = e^{\sqrt{-1}u}$ for some real u. Then, obviously, M_f is obtained by monodromy transformation given by the path $[0, 1] \ni \theta \mapsto e^{\sqrt{-1}u\theta} \cdot \sigma(h) \in S - D$. Hence $M_f \in \Gamma$.

[$\Leftarrow$] Suppose that $M_f \in \Gamma$. Then similar argument as Lemma 4 implies that each $[e_i, e_j]$ is mapped by M_f to $[e_f(i), e_f(j)]$ $(\mathrm{mod} 2H_1(X_{\sigma(h)}, \mathbf{Z}))$. Therefore, as the proof of Lemma 4, we have

$$M_f(i) = f(i) \qquad \text{for any } i \in \{1, \ldots, 6\}\ .$$

Since $M_f \in \Gamma$, we obtain $M_f(6) = 6$ by Lemma 3 and the following argument. Hence $f(6) = 6$. This completes the proof. ∎

Since φ absorb the $\mathbf{C}^*$-action on Image(P), composite map $\varphi \circ P$ induces a map $\mathbf{C}^* \backslash (S–D)^\wedge \to \mathbf{H}_2$. The aim of this subsection is to show the injectivity of the map.

Lemma 23 *The map $\mathbf{C}^* \backslash (S–D)^\wedge \to \mathbf{H}_2$ is injective.*

Proof. Suppose that h, h' are elements of $(S–D)^\wedge$ satisfying $\varphi \circ P(h) = \varphi \circ P(h')$. Then there exist $\lambda, \lambda' \in \mathbf{C}^*$ such that

$$\lambda \cdot \sigma(h) = F \circ \varphi \circ P(h) \qquad \text{and} \qquad \lambda' \cdot \sigma(h') = F \circ \varphi \circ P(h')$$

are valid. Then, if we denote $\varphi \circ P(h)$ by τ, we have

$$\sigma(\lambda \cdot h) = \sigma(\lambda' \cdot h') \qquad \text{and} \qquad \varphi \circ P(\lambda \cdot h) = \varphi \circ P(\lambda' \cdot h') = \tau \ . \tag{40}$$

The first equality of (40) implies the existence of $M \in \Gamma$ satisfying $P(\lambda' \cdot h') = P(\lambda \cdot h)M$. Therefore,

$$(KM^{-1}K^{-1}) \circ \tau = \varphi(P(\lambda \cdot h)M) = \varphi(P(\lambda' \cdot h')) = \tau \ .$$

Hence, by Fact 21 there exist $f \in \mathrm{Aut}(\overline{X_{\sigma(\lambda\cdot h)}})$ such that $M_f = M$ where M_f is an element of $\mathrm{Aut}(H_1(X_{\sigma(\lambda\cdot h)}, \mathbf{Z}), \langle\ ,\ \rangle)$ induced by f. And using $A_j(\lambda \cdot h)$ $(j \in \{1,\ldots,4\})$ as a basis of $H_1(X_{\sigma(\lambda\cdot h)}, \mathbf{Z})$, via (10) with $t = \sigma(\lambda \cdot h)$, M_f is regarded as an element of $Sp(4,\mathbf{Z})$. Since $M_f = M \in \Gamma$, Lemma 22 implies that $f(\infty) = \infty$. Hence there exists $\lambda'' \in \mathbf{C}^*$ satisfying $\lambda'' \cdot \sigma(\lambda \cdot h) = \sigma(\lambda \cdot h)$ such that f coincides with an element of $\mathrm{Aut}(\overline{X_{\sigma(\lambda\cdot h)}})$ defined by $(x,y) \mapsto ((\lambda'')^2 x, (\lambda'')^5 y)$. Therefore, $\lambda' \cdot h'$ is on the $\mathbf{C}^*$-orbit of $\lambda \cdot h$. That is, h' is on the $\mathbf{C}^*$-orbit of h. ∎

7 Triviality of the bundle $(S\text{–}D)^\wedge \to \mathbf{H}_2^*$.

The aim of this section is to prove the triviality of the bundle $\varphi : \mathrm{Image}(P) \to \mathbf{H}_2^*$, that is, to prove the Theorem 1 mentioned later. Before proving the theorem, as a preparation, we show some lemmas as follows.

Lemma 24 *Suppose that τ is any element of $\mathbf{H}_2^*$ and U is a sufficiently small neighborhood of τ in $\mathbf{H}_2^*$. Then there exist exactly two maps $\widehat{F}_\tau^{(i)} : U \to (S\text{–}D)^\wedge$ $(i = 1,2)$ satisfying*

$$\sigma \circ \widehat{F}_\tau^{(i)} = F|_U \qquad \text{and} \qquad \varphi \circ P \circ \widehat{F}_\tau^{(i)} = id_U \ .$$

Proof. Here we use the following well-known facts (cf. [Fre]).

(i) Since $Sp(4,\mathbf{Z})$ acts on $\mathbf{H}_2$ discontinuously, for any subgroup $G \subset Sp(4,\mathbf{Z})$, and for any element $\tau \in \mathbf{H}_2$, there exists a neighborhood U of τ in $\mathbf{H}_2$ such that

$$\left[\begin{array}{l} \text{if } M \in \mathrm{stab}_G(\tau), \text{ then } M \circ U = U, \\ \text{if } M \in G - \mathrm{stab}_G(\tau), \text{ then } (M \circ U) \cap U = \emptyset. \end{array}\right] \tag{41}$$

Moreover, each element $\tau' \in U$ satisfies $\mathrm{stab}_G(\tau') \subset \mathrm{stab}_G(\tau)$.

(ii) For each τ, $\mathrm{stab}_{Sp(4,\mathbf{Z})}(\tau)$ is finite set including $\pm I$. Moreover, $\{\tau \in \mathbf{H}_2 | \, \mathrm{stab}_{Sp(4,\mathbf{Z})}(\tau) \neq \{\pm I\}\}$ is proper analytic subset of $\mathbf{H}_2$.

Now we take $\tau \in \mathbf{H}_2^*$. Since σ is covering map, there exists a neighborhood $U_{F(\tau)}$ of $F(\tau)$ in $S - D$ such that, each connected component of $\sigma^{-1}(U_{F(\tau)})$ is isomorphic to $U_{F(\tau)}$ by σ:

$$\sigma|_{(\text{the component})} : (\text{the component}) \xrightarrow{\sim} U_{F(\tau)}$$

Then the conditions of the lemma implies that $\hat{F}_\tau$ must be the composition $(\sigma|_{U_h})^{-1} \circ (F|_{U'})$, where

$$\begin{array}{l} h \in \sigma^{-1}(F(\tau)) \cap (\varphi \circ P)^{-1}(\tau) \ , \\ U_h \text{ is a connected component of } \sigma^{-1}(U_{F(\tau)}) \text{ containing } h \ , \\ U' \text{ is a neighborhood of } \tau \text{ in } \mathbf{H}_2^* \text{ satisfying } F(U') \subset U_{F(\tau)} \ . \end{array} \tag{42}$$

In this case, moreover, it must satisfy that

$$\hat{F}_\tau(\tau') \in \sigma^{-1}(F(\tau')) \cap (\varphi \circ P)^{-1}(\tau') \tag{43}$$

for any $\tau' \in U'$. By the way, it can be easily seen from (ii) that for any $\tau' \in \mathbf{H}_2^*$, $\#\mathrm{stab}_\Gamma(\tau') = \#\sigma^{-1}(F(\tau')) \cap (\varphi \circ P)^{-1}(\tau')$ holds, and that $\{\tau' \in \mathbf{H}_2^* | \, \mathrm{stab}_\Gamma(\tau') = \{\pm I\}\}$ is open dense subset of $\mathbf{H}_2^*$. Therefore, if we note (43), for each $\tau \in \mathbf{H}_2^*$, there exist at most two local sections $\hat{F}_\tau$ satisfying the conditions of the lemma.From now on, we show that there exists just two local sections $\hat{F}_\tau$ satisfying the conditions. Suppose U_τ be a neighborhood of τ in $\mathbf{H}_2^*$ satisfying $F(U_\tau) \subset U_{F(\tau)}$ and the condition (41) with $U = U_\tau$, $G = \Gamma$. Then we take h, U_h, U' satisfying (42). Moreover, suppose that $U' \subset U_\tau$ and that

$$\varphi \circ P \circ (\sigma|_{U_h})^{-1} \circ (F|_{U_\tau})(U') \subset U_\tau \tag{44}$$

holds. Then by Lemma 7, for each $\tau' \in U'$ there exists $M \in \Gamma$ such that

$$\varphi \circ P \circ (\sigma|_{U_h})^{-1} \circ (F|_{U_\tau})(\tau') = M \circ \tau' . \tag{45}$$

Then, the conditions

$$\begin{array}{l} M \circ \tau' \in M \circ U' \subset M \circ U_\tau, \qquad \text{and} \\ M \circ \tau' = \varphi \circ P \circ (\sigma|_{U_h})^{-1} \circ (F|_{U_\tau})(\tau') \in \varphi \circ P \circ (\sigma|_{U_h})^{-1} \circ (F|_{U_\tau})(U') \subset U_\tau \end{array}$$

imply that $(M \circ U_\tau) \cap U_\tau \neq \emptyset$, hence, by (41) with $U = U_\tau$ and $G = \Gamma$, we conclude that $M \in \mathrm{stab}_\Gamma(\tau)$.

If $\mathrm{stab}_\Gamma(\tau) = \{\pm I\}$, then $M = \pm I$, which implies that $\varphi \circ P \circ (\sigma|_{U_h})^{-1} \circ (F|_{U_\tau})(\tau') = \tau'$. Therefore, $\varphi \circ P \circ (\sigma|_{U_h})^{-1} \circ (F|_{U'}) = id_{U'}$.

On the other hand, suppose that $\mathrm{stab}_\Gamma(\tau) \neq \{\pm I\}$. In this case we assume moreover that U' is connected and that, not only U_τ, but also U' satisfies the condition (41) with $U = U'$ and $G = \Gamma$. Furthermore, we assume that $\mathrm{stab}_\Gamma(\tau') = \{\pm I\}$. Since $M \in \mathrm{stab}_\Gamma(\tau)$, then (41) with $U = U'$ and $G = \Gamma$ implies that

$$\varphi \circ P \circ (\sigma|_{U_h})^{-1} \circ (F|_{U_\tau})(\tau') = M \circ \tau' \in M \circ U' = U' , \tag{46}$$

that is,

$$\varphi \circ P \circ (\sigma|_{U_h})^{-1} \circ (F|_{U_\tau})(U') \subset U' \tag{47}$$

holds. Here we denote by M' an element of Γ satisfying $M' \circ (M \circ \tau') = \tau'$. Note that, since $\mathrm{stab}_\Gamma(\tau') = \{\pm I\}$, we have $M' = \pm M^{-1} \in \mathrm{stab}_\Gamma(\tau)$. From now on, we write $\psi := \varphi \circ P \circ (\sigma|_{M'(U_h)})^{-1} \circ (F|_{U'})$ for short. Then the conditions $M' \in \mathrm{stab}_\Gamma(\tau)$, (41) with $U = U'$ and $G = \Gamma$, and (47) imply $\psi(U') \subset M' \circ U' = U'$, and the conditions (46) and $M' \circ (M \circ \tau') = \tau'$ imply $\psi(\tau') = M' \circ (M \circ \tau') = \tau'$. The following arguments are devoted to proving that $\psi = id_{U'}$. We write $U'' := \{\tau'' \in U' | \, \mathrm{stab}_\Gamma(\tau'') = \{\pm I\}\}$ for short. Since we assume that U' is connected, U'' is also connected, open dense subset by (ii). Therefore, since U' is Hausdorff space, $\psi : U' \to U'$ is continuous, and U'' is dense in U', we have only to show that $\psi|_{U''} = id_{U''}$. Moreover, since $\tau' \in U''$, $\psi(\tau') = \tau'$ and U'' is connected, we have only to show that the fixed point set of $\psi|_{U''}$: $\{\tau'' \in U'' | \psi(\tau'') = \tau''\}$ is open and closed subset of U''. By the way, since U'' is Hausdorff space, and $\psi|_{U''} : U'' \to U''$ is continuous, the set $\{\tau'' \in U'' | \psi(\tau'') = \tau''\}$ is closed subset of U''. Therefore, we have only to show that the set $\{\tau'' \in U'' | \psi(\tau'') = \tau''\}$ is open subset of U''. Suppose $\tau'' \in U''$ satisfies $\psi(\tau'') = \tau''$. Then by the argument in the case $\mathrm{stab}_\Gamma(\tau) = \{\pm I\}$, there exists a neighborhood U''' of τ'' in U'' such that $\psi(\tau''') = \tau'''$ for any $\tau''' \in U'''$. Therefore, $\{\tau'' \in U'' | \psi(\tau'') = \tau''\}$ is open subset of U''. Hence We conclude that $\psi = id_{U'}$. ∎

Lemma 25 *Suppose that $\tau \in \mathbf{H}_2$ is diagonal matrix: $\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$, and M is any element of $\Gamma_2(1)$. Then there exists a permutation i, j, k, l, m of $1, 2, 3, 4, 5$ such that the following 1,2 hold.*

1. *For $\tau_{12} \in \mathbf{C}$ satisfying $|\tau_{12}| \ll 1$, we write $\tau := \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix}$. Then we have*

$$\alpha_\nu(M \circ \tau) = C + C'\tau_{12} + C_\nu \tau_{12}^2 + O(\tau_{12}^3) \qquad (\nu = k, l, m)$$

where C, C', C_k, C_l, C_m are independent to τ_{12} and C_k, C_l, C_m are different from each other.

2. $\alpha_i(M\circ\tau)$*,* $\alpha_j(M\circ\tau)$*,* C *are different from each other.*

Proof. First we note that

$$\vartheta_\varepsilon\begin{pmatrix}\tau_1 & \tau_{12}\\ \tau_{12} & \tau_2\end{pmatrix} = \sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}\frac{d^n\vartheta_{\varepsilon_1'\varepsilon_1''}}{d\tau_1^n}(\tau_1)\frac{d^n\vartheta_{\varepsilon_2'\varepsilon_2''}}{d\tau_2^n}(\tau_2)\tau_{12}^{2n} = \vartheta_{\varepsilon_1'\varepsilon_1''}(\tau_1)\vartheta_{\varepsilon_2'\varepsilon_2''}(\tau_2) + O(\tau_{12}^2) \tag{48}$$

for $\varepsilon\neq(1111)$ and

$$\vartheta_{1111}\begin{pmatrix}\tau_1 & \tau_{12}\\ \tau_{12} & \tau_2\end{pmatrix} = \frac{\pi}{2\sqrt{-1}}\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n+1)!}\frac{d^n\Theta_1}{d\tau_1^n}(\tau_1)\frac{d^n\Theta_1}{d\tau_2^n}(\tau_2)\tau_{12}^{2n+1} = \frac{\pi}{2\sqrt{-1}}\Theta_1(\tau_1)\Theta_1(\tau_2)\tau_{12} + O(\tau_{12}^3)\ , \tag{49}$$

where $\Theta_1(\tau) := \vartheta_{00}(\tau)\vartheta_{01}(\tau)\vartheta_{10}(\tau)$ for $\tau\in\mathbf{H}$. On the other hand, (18), (61), and (23) imply that for any $\tau\in\mathbf{H}_2$, for any $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp(4,\mathbf{Z})$ and for any distinct $i,j\in\{1,\ldots,5\}$ it satisfies that

$$\begin{aligned}\alpha_i(M\circ\tau)-\alpha_j(M\circ\tau) &= \operatorname{sign}(i-j)\vartheta_{ijk}^2(M\circ\tau)\vartheta_{ijl}^2(M\circ\tau)\vartheta_{ijm}^2(M\circ\tau)\\ &= \Phi(M;i,j)\det(C\tau+D)^3\vartheta^2_{M^{-1}(ijk)}\vartheta^2_{M^{-1}(ijl)}\vartheta^2_{M^{-1}(ijm)}\ ,\end{aligned}$$

where $\{k,l,m\}$ is complement of $\{i,j\}$ in $\{1,\ldots,5\}$, and

$$\begin{aligned}\Phi(M;i,j) &:= \operatorname{sign}(i-j)\kappa(M)^6\\ &\quad\times\exp[2\pi\sqrt{-1}(\phi(M,M^{-1}(ijk))+\phi(M,M^{-1}(ijl))+\phi(M,M^{-1}(ijm)))]\ .\end{aligned}$$

Here we note that $\Phi(M;i,j)$ is non-zero constant, which depends M,i,j but is independent to τ. Furthermore, for any $M\in Sp(4,\mathbf{Z})$, the function

$$\mathbf{H}_2\ni\tau\longmapsto\det(C\tau+D)\in\mathbf{C}$$

is holomorphic, and has no zeros on $\mathbf{H}_2$. Therefore, to prove the lemma, we have only to show that, for any $M\in Sp(4,\mathbf{Z})$, there exists a permutation i,j,k,l,m of $1,2,3,4,5$ satisfying the following (50), …, (55).

$$\vartheta^2_{M^{-1}(ijk)}\vartheta^2_{M^{-1}(ijl)}\vartheta^2_{M^{-1}(ijm)}(\tau(0))\neq 0\ , \tag{50}$$
$$\vartheta^2_{M^{-1}(ilj)}\vartheta^2_{M^{-1}(ilk)}\vartheta^2_{M^{-1}(ilm)}(\tau(0))\neq 0\ , \tag{51}$$
$$\vartheta^2_{M^{-1}(jli)}\vartheta^2_{M^{-1}(jlk)}\vartheta^2_{M^{-1}(jlm)}(\tau(0))\neq 0\ , \tag{52}$$
$$\vartheta^2_{M^{-1}(kli)}\vartheta^2_{M^{-1}(klj)}\vartheta^2_{M^{-1}(klm)} = C_m'\tau_{12}^2+O(\tau_{12}^4)\ , \tag{53}$$
$$\vartheta^2_{M^{-1}(kmi)}\vartheta^2_{M^{-1}(kmj)}\vartheta^2_{M^{-1}(kml)} = C_l'\tau_{12}^2+O(\tau_{12}^4)\ , \tag{54}$$
$$\vartheta^2_{M^{-1}(lmi)}\vartheta^2_{M^{-1}(lmj)}\vartheta^2_{M^{-1}(lmk)} = C_k'\tau_{12}^2+O(\tau_{12}^4)\ , \tag{55}$$

where C_m', C_l', C_k' are non-zero constants, which depend on τ_1,τ_2 but are independent to τ_{12}, and $\tau(0) := \begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}$. Now let us check that those conditions hold for any $M\in Sp(4,\mathbf{Z})$. In the following, we write $\Psi(\tau_1,\tau_2) := \frac{\pi}{2\sqrt{-1}}\Theta_1(\tau_1)\Theta_1(\tau_2)$ for short. Checks are divided into six cases as follows.

Case 1. Suppose that $M(1)=6$. In this case, $\{M^{-1}(1),\ldots,M^{-1}(5)\}=\{2,3,4,5,6\}$. By (48), (49) it can be easily seen that

$$\begin{aligned}
&\vartheta^2_{342}\vartheta^2_{345}\vartheta^2_{346} = (\vartheta_{01}\vartheta_{10})^2(\tau_1)\vartheta^4_{10}(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2+O(\tau_{12}^4)\ ,\\
&\vartheta^2_{352}\vartheta^2_{354}\vartheta^2_{356} = (\vartheta_{01}\vartheta_{10})^2(\tau_1)\vartheta^4_{00}(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2+O(\tau_{12}^4)\ ,\\
&\vartheta^2_{452}\vartheta^2_{453}\vartheta^2_{456} = (\vartheta_{01}\vartheta_{10})^2(\tau_1)\vartheta^4_{01}(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2+O(\tau_{12}^4)\ ,\\
&\vartheta^2_{263}\vartheta^2_{264}\vartheta^2_{265}(\tau(0)) = \vartheta^6_{00}(\tau_1)\Theta_1^2(\tau_2)\ ,\\
&\vartheta^2_{234}\vartheta^2_{235}\vartheta^2_{236}(\tau(0)) = \vartheta^2_{243}\vartheta^2_{245}\vartheta^2_{246}(\tau(0)) = \vartheta^2_{253}\vartheta^2_{254}\vartheta^2_{256}(\tau(0)) = (\vartheta^4_{01}\vartheta^2_{00})(\tau_1)\Theta_1^2(\tau_2)\ ,\\
&\vartheta^2_{362}\vartheta^2_{364}\vartheta^2_{365}(\tau(0)) = \vartheta^2_{462}\vartheta^2_{463}\vartheta^2_{465}(\tau(0)) = \vartheta^2_{562}\vartheta^2_{563}\vartheta^2_{564}(\tau(0)) = (\vartheta^4_{10}\vartheta^2_{00})(\tau_1)\Theta_1^2(\tau_2)\ .
\end{aligned}$$

Here we note that $\vartheta_{00}(\tau)\vartheta_{01}(\tau)\vartheta_{10}(\tau) \neq 0$ for any $\tau \in \mathbf{H}$. Therefore, taking

$$\{k,l,m\} = \{M(3), M(4), M(5)\} \quad \text{and} \quad \{i,j\} = \{M(2), M(6)\} ,$$

conditions (50)-(55) are satisfied.

Case 2. Suppose that $M(2) = 6$. In this case, $\{M^{-1}(1), \dots, M^{-1}(5)\} = \{1,3,4,5,6\}$. By (48), (49) it can be easily seen that

$$\begin{aligned}
&\vartheta_{341}^2\vartheta_{345}^2\vartheta_{346}^2 = (\vartheta_{00}\vartheta_{10})^2(\tau_1)\vartheta_{10}^4(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{351}^2\vartheta_{354}^2\vartheta_{356}^2 = (\vartheta_{00}\vartheta_{10})^2(\tau_1)\vartheta_{00}^4(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{451}^2\vartheta_{453}^2\vartheta_{456}^2 = (\vartheta_{00}\vartheta_{10})^2(\tau_1)\vartheta_{01}^4(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{163}^2\vartheta_{164}^2\vartheta_{165}^2(\tau(0)) = \vartheta_{01}^6(\tau_1)\Theta_1^2(\tau_2) , \\
&\vartheta_{134}^2\vartheta_{135}^2\vartheta_{136}^2(\tau(0)) = \vartheta_{143}^2\vartheta_{145}^2\vartheta_{146}^2(\tau(0)) = \vartheta_{153}^2\vartheta_{154}^2\vartheta_{156}^2(\tau(0)) = (\vartheta_{00}^4\vartheta_{01}^2)(\tau_1)\Theta_1^2(\tau_2) , \\
&\vartheta_{361}^2\vartheta_{364}^2\vartheta_{365}^2(\tau(0)) = \vartheta_{461}^2\vartheta_{463}^2\vartheta_{465}^2(\tau(0)) = \vartheta_{561}^2\vartheta_{563}^2\vartheta_{564}^2(\tau(0)) = (\vartheta_{10}^4\vartheta_{01}^2)(\tau_1)\Theta_1^2(\tau_2) .
\end{aligned}$$

Therefore, taking $\{k,l,m\} = \{M(3), M(4), M(5)\}$ and $\{i,j\} = \{M(1), M(6)\}$, conditions (50)-(55) are satisfied.

Case 3. Suppose that $M(3) = 6$. In this case, $\{M^{-1}(1), \dots, M^{-1}(5)\} = \{1,2,4,5,6\}$. By (48), (49) it can be easily seen that

$$\begin{aligned}
&\vartheta_{124}^2\vartheta_{125}^2\vartheta_{126}^2 = \vartheta_{10}^4(\tau_1)(\vartheta_{00}\vartheta_{10})^2(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{162}^2\vartheta_{164}^2\vartheta_{165}^2 = \vartheta_{01}^4(\tau_1)(\vartheta_{00}\vartheta_{10})^2(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{261}^2\vartheta_{264}^2\vartheta_{265}^2 = \vartheta_{00}^4(\tau_1)(\vartheta_{00}\vartheta_{10})^2(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{451}^2\vartheta_{452}^2\vartheta_{456}^2(\tau(0)) = \Theta_1^2(\tau_1)\vartheta_{01}^6(\tau_2) , \\
&\vartheta_{142}^2\vartheta_{145}^2\vartheta_{146}^2(\tau(0)) = \vartheta_{241}^2\vartheta_{245}^2\vartheta_{246}^2(\tau(0)) = \vartheta_{461}^2\vartheta_{462}^2\vartheta_{465}^2(\tau(0)) = \Theta_1^2(\tau_1)(\vartheta_{00}^4\vartheta_{01}^2)(\tau_2) , \\
&\vartheta_{152}^2\vartheta_{154}^2\vartheta_{156}^2(\tau(0)) = \vartheta_{251}^2\vartheta_{254}^2\vartheta_{256}^2(\tau(0)) = \vartheta_{561}^2\vartheta_{562}^2\vartheta_{564}^2(\tau(0)) = \Theta_1^2(\tau_1)(\vartheta_{10}^4\vartheta_{01}^2)(\tau_2) .
\end{aligned}$$

Therefore, taking $\{k,l,m\} = \{M(1), M(2), M(6)\}$ and $\{i,j\} = \{M(4), M(5)\}$, conditions (50)-(55) are satisfied.

Case 4. Suppose that $M(4) = 6$. In this case, $\{M^{-1}(1), \dots, M^{-1}(5)\} = \{1,2,3,5,6\}$. By (48), (49) it can be easily seen that

$$\begin{aligned}
&\vartheta_{123}^2\vartheta_{125}^2\vartheta_{126}^2 = \vartheta_{10}^4(\tau_1)(\vartheta_{01}\vartheta_{10})^2(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{162}^2\vartheta_{163}^2\vartheta_{165}^2 = \vartheta_{01}^4(\tau_1)(\vartheta_{01}\vartheta_{10})^2(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{261}^2\vartheta_{263}^2\vartheta_{265}^2 = \vartheta_{00}^4(\tau_1)(\vartheta_{01}\vartheta_{10})^2(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{351}^2\vartheta_{352}^2\vartheta_{356}^2(\tau(0)) = \Theta_1^2(\tau_1)\vartheta_{00}^6(\tau_2) , \\
&\vartheta_{132}^2\vartheta_{135}^2\vartheta_{136}^2(\tau(0)) = \vartheta_{231}^2\vartheta_{235}^2\vartheta_{236}^2(\tau(0)) = \vartheta_{361}^2\vartheta_{362}^2\vartheta_{365}^2(\tau(0)) = \Theta_1^2(\tau_1)(\vartheta_{00}^2\vartheta_{01}^4)(\tau_2) , \\
&\vartheta_{152}^2\vartheta_{153}^2\vartheta_{156}^2(\tau(0)) = \vartheta_{251}^2\vartheta_{253}^2\vartheta_{256}^2(\tau(0)) = \vartheta_{561}^2\vartheta_{562}^2\vartheta_{563}^2(\tau(0)) = \Theta_1^2(\tau_1)(\vartheta_{10}^4\vartheta_{00}^2)(\tau_2) .
\end{aligned}$$

Therefore, taking $\{k,l,m\} = \{M(1), M(2), M(6)\}$ and $\{i,j\} = \{M(3), M(5)\}$, conditions (50)-(55) are satisfied.

Case 5. Suppose that $M(5) = 6$. In this case, $\{M^{-1}(1), \dots, M^{-1}(5)\} = \{1,2,3,4,6\}$. By (48), (49) it can be easily seen that

$$\begin{aligned}
&\vartheta_{123}^2\vartheta_{124}^2\vartheta_{126}^2 = \vartheta_{10}^4(\tau_1)(\vartheta_{00}\vartheta_{01})^2(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{162}^2\vartheta_{163}^2\vartheta_{164}^2 = \vartheta_{01}^4(\tau_1)(\vartheta_{00}\vartheta_{01})^2(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{261}^2\vartheta_{263}^2\vartheta_{264}^2 = \vartheta_{00}^4(\tau_1)(\vartheta_{00}\vartheta_{01})^2(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4) , \\
&\vartheta_{341}^2\vartheta_{342}^2\vartheta_{346}^2(\tau(0)) = \Theta_1^2(\tau_1)\vartheta_{10}^6(\tau_2) , \\
&\vartheta_{132}^2\vartheta_{134}^2\vartheta_{136}^2(\tau(0)) = \vartheta_{231}^2\vartheta_{234}^2\vartheta_{236}^2(\tau(0)) = \vartheta_{361}^2\vartheta_{362}^2\vartheta_{364}^2(\tau(0)) = \Theta_1^2(\tau_1)(\vartheta_{10}^2\vartheta_{01}^4)(\tau_2) , \\
&\vartheta_{142}^2\vartheta_{143}^2\vartheta_{146}^2(\tau(0)) = \vartheta_{241}^2\vartheta_{243}^2\vartheta_{246}^2(\tau(0)) = \vartheta_{461}^2\vartheta_{462}^2\vartheta_{463}^2(\tau(0)) = \Theta_1^2(\tau_1)(\vartheta_{10}^2\vartheta_{00}^4)(\tau_2) .
\end{aligned}$$

Therefore, taking $\{k,l,m\} = \{M(1), M(2), M(6)\}$ and $\{i,j\} = \{M(3), M(4)\}$, conditions (50)-(55) are satisfied.

Case 6. Suppose that $M(6) = 6$. In this case, $\{M^{-1}(1), \dots, M^{-1}(5)\} = \{1,2,3,4,5\}$. By (48), (49) it can be easily seen that

$$\begin{aligned}
&\vartheta_{341}^2\vartheta_{342}^2\vartheta_{345}^2 = (\vartheta_{00}\vartheta_{01})^2(\tau_1)\vartheta_{10}^4(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4)\,,\\
&\vartheta_{351}^2\vartheta_{352}^2\vartheta_{354}^2 = (\vartheta_{00}\vartheta_{01})^2(\tau_1)\vartheta_{00}^4(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4)\,,\\
&\vartheta_{451}^2\vartheta_{452}^2\vartheta_{453}^2 = (\vartheta_{00}\vartheta_{01})^2(\tau_1)\vartheta_{01}^4(\tau_2)\Psi^2(\tau_1,\tau_2)\tau_{12}^2 + O(\tau_{12}^4)\,,\\
&\vartheta_{123}^2\vartheta_{124}^2\vartheta_{125}^2(\tau(0)) = \vartheta_{10}^6(\tau_1)\Theta_1^2(\tau_2)\,,\\
&\vartheta_{132}^2\vartheta_{134}^2\vartheta_{135}^2(\tau(0)) = \vartheta_{142}^2\vartheta_{143}^2\vartheta_{145}^2(\tau(0)) = \vartheta_{152}^2\vartheta_{153}^2\vartheta_{154}^2(\tau(0)) = (\vartheta_{00}^4\vartheta_{10}^2)(\tau_1)\Theta_1^2(\tau_2)\,,\\
&\vartheta_{231}^2\vartheta_{234}^2\vartheta_{235}^2(\tau(0)) = \vartheta_{241}^2\vartheta_{243}^2\vartheta_{245}^2(\tau(0)) = \vartheta_{251}^2\vartheta_{253}^2\vartheta_{254}^2(\tau(0)) = (\vartheta_{01}^4\vartheta_{10}^2)(\tau_1)\Theta_1^2(\tau_2)\,.
\end{aligned}$$

Therefore, taking $\{k,l,m\} = \{M(3), M(4), M(5)\}$ and $\{i,j\} = \{M(1), M(2)\}$, conditions (50)-(55) are satisfied. The proof is completed. ∎

Theorem 1 *There exists a holomorphic map $\widehat{F} : \mathbf{H}_2^* \to (S\text{–}D)^\wedge$ such that*

$$\sigma \circ \widehat{F} = F|_{\mathbf{H}_2^*} \qquad \textit{and} \qquad \varphi \circ P \circ \widehat{F} = \mathrm{id}_{\mathbf{H}_2^*}\,.$$

Proof. First we take a local section given in Lemma 24. Then using analytic continuation we have a section over $\mathbf{H}_2^*$, which may be multi-valued. Again by Lemma 24, this section is at most two-valued. In the following we show that the section is in fact single valued. Since $\mathbf{H}_2$ is simply connected, it suffices to show that for any diagonal $\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \in \mathbf{H}_2$ and $M \in Sp(4, \mathbf{Z})$, there exists a neighborhood U of $M \circ \tau$ in $\mathbf{H}_2$ such that the local section given in Lemma 24 on neighborhood of a point of $U - A$ can be analytically continuated to single-valued section on $U - A$. Here we note that, for sufficiently small $\varepsilon > 0$,

$$U_\tau(\varepsilon) := \left\{ \begin{pmatrix} \tau_1' & \tau_{12}' \\ \tau_{12}' & \tau_2' \end{pmatrix} \middle| \tau_1', \tau_2', \tau_{12}' \in \mathbf{C},\ |\tau_1' - \tau_1| < \varepsilon,\ |\tau_2' - \tau_2| < \varepsilon \text{ and } |\tau_{12}'| < \varepsilon \right\}$$

is a subset of $\mathbf{H}_2$, and it satisfies that

$$U_\tau(\varepsilon) \cap A := \left\{ \begin{pmatrix} \tau_1' & 0 \\ 0 & \tau_2' \end{pmatrix} \middle| \tau_1', \tau_2' \in \mathbf{C},\ |\tau_1' - \tau_1| < \varepsilon \text{ and } |\tau_2' - \tau_2| < \varepsilon \right\}\,.$$

We fix this ε. Then $M \circ U_\tau(\varepsilon)$ is a neighborhood of $M \circ \tau$ in $\mathbf{H}_2$, and the equality

$$(M \circ U_\tau(\varepsilon)) \cap A := \left\{ M \circ \begin{pmatrix} \tau_1' & 0 \\ 0 & \tau_2' \end{pmatrix} \middle| \tau_1', \tau_2' \in \mathbf{C},\ |\tau_1' - \tau_1| < \varepsilon \text{ and } |\tau_2' - \tau_2| < \varepsilon \right\}\,.$$

holds. Here we take $\tau_{12}' \in \mathbf{C}$ satisfying $0 < |\tau_{12}'| < \varepsilon$, and denote $\tau' := \begin{pmatrix} \tau_1 & \tau_{12}' \\ \tau_{12}' & \tau_2 \end{pmatrix}$. τ' is an element of $U_\tau(\varepsilon) - A$. Note that $\pi_1((M \circ U_\tau(\varepsilon)) - A,\ M \circ \tau') \cong \mathbf{Z}$ and it is generated by an element having

$$[0,1] \ni \theta \longmapsto M \circ \tau'(\theta) \in (M \circ U_\tau(\varepsilon)) - A \tag{56}$$

as its representative, where $\tau'(\theta) := \begin{pmatrix} \tau_1 & \tau_{12}' e^{2\pi\sqrt{-1}\theta} \\ \tau_{12}' e^{2\pi\sqrt{-1}\theta} & \tau_2 \end{pmatrix}$. By Lemma 25, it can be easily seen that the monodromy transformation given by the path

$$[0,1] \ni \theta \longmapsto (t_2(M \circ \tau'(\theta)), \dots, t_5(M \circ \tau'(\theta))) \in S - D\,,$$

where $t_2, \dots, t_5$ are regarded as functions on $\mathbf{H}_2$ by (20), is identity in $\mathrm{Aut}(H_1(X_t, \mathbf{Z}), \langle\ ,\ \rangle)$ where $t = (t_2(M \circ \tau'), \dots, t_5(M \circ \tau'))$. This means that, from analytic continuation of the local section $\hat{F}_{M\circ\tau'}^{(i)}$ along the path (56), multi-valuedness doesn't occure. Hence, analytic continuation of $\hat{F}_{M\circ\tau'}^{(i)}$ gives a single-valued section of the bundle $(S\text{–}D)^\wedge \xrightarrow{\varphi\circ P} \mathbf{H}_2^*$ on $(M \circ U_\tau(\varepsilon)) - A$. Therefore, from $\hat{F}_\tau$, the aimed section $\hat{F}$ is obtained. ∎

8 The factor of automorphy j.

In section 6 we have checked that the monodromy covering $(S\text{–}D)^\wedge$ is total space of a $\mathbf{C}^*$-bundle in the strict sense. Moreover, Theorem 1 implies that the following isomorphism as $\mathbf{C}^*$-bundles:

$$\mathbf{C}^* \times \mathbf{H}_2^* \xrightarrow{\sim} (S\text{–}D)^\wedge , \qquad (\lambda, \tau) \longmapsto \lambda \cdot \hat{F}(\tau) .$$

Under this isomorphism, the monodromy group action on $(S\text{–}D)^\wedge$ induces Γ-action on $\mathbf{C}^* \times \mathbf{H}_2^*$. The aim of this section is to describe the Γ-action. Since the monodromy group action on $(S\text{–}D)^\wedge$ commutes to the $\mathbf{C}^*$-action on the space, Γ-action on $\mathbf{C}^* \times \mathbf{H}_2^*$ also commutes to the $\mathbf{C}^*$-action on $\mathbf{C}^* \times \mathbf{H}_2^*$. Therefore we can apply **Diagram-1** to the bundle $\mathbf{C}^* \times \mathbf{H}_2^*$, where $X := \mathbf{H}_2^*$, $G := \Gamma$. And the factor of automorphy j appeared in **Diagram-1** is now given to satisfy the following equality:

$$\hat{F}(M \circ \tau) = j(M, \tau) \cdot \gamma(\hat{F}(\tau)) \qquad (\tau \in \mathbf{H}_2^*,\ M = \rho(\gamma) \in \Gamma) .$$

Taking the images of both sides of the equality by σ, we have

$$F(M \circ \tau) = j(M, \tau) \cdot F(\tau) \qquad (\tau \in \mathbf{H}_2^*, M \in \Gamma) .$$

Hence

$$j(M, \tau)^2 = \chi(M) \det(C\tau + D)^3 = (j_{1101}(M, \tau) \det(C\tau + D))^2 \qquad (\tau \in \mathbf{H}_2^*, M \in \Gamma) .$$

More exactly, the following theorem holds.

Theorem 2 *On trivialization of* $(S\text{–}D)^\wedge \xrightarrow{\varphi \circ P} \mathbf{H}_2^*$ *using the global section* $\hat{F}$ *given in the Theorem 1, the following equality holds:*

$$j(M, \tau) = j_{1101}(M, \tau) \det(C\tau + D) \qquad (\tau \in \mathbf{H}_2^*, M \in \Gamma) . \tag{57}$$

Since $\rho(\gamma_i)$ $(i = 1, 2, 3, 4)$ generate Γ, to prove the theorem needs only to check (57) for the generators. If $M = \rho(\gamma_i)$, the left hand side of (57) is obtained by investigating the behavior of values of $\alpha_1, \ldots, \alpha_5$ when τ go through a path in $\mathbf{H}_2^*$ with τ as initial point and $\rho(\gamma_i) \circ \tau$ as end point. On the other hand, the right hand side of (57) is obtained by simple computations using (61). Before proving the theorem, as a preparation, we give some lemmas.

Lemma 26 *Suppose that* $(e_1, \ldots, e_5) = (-\sqrt{3}, -1/\sqrt{3}, 0, 1/\sqrt{3}, \sqrt{3})$. *Then period matrix given by the curve* $X(e)$ *with the basis* $A_1, A_2, B_1, B_2 \in H_1(X(e), \mathbf{Z})$ *as in the* **Figure-1**, *is* $\frac{\sqrt{-1}}{\sqrt{3}} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Proof. This curve has automorphism $\varphi : (x, y) \mapsto (\frac{1+\sqrt{3}x}{\sqrt{3}-x}, \frac{-8\sqrt{-1}}{(\sqrt{3}-x)^3} y)$, which induces an automorphism φ_* of $H_1(X(e), \mathbf{Z})$. Using the basis A_1, A_2, B_1, B_2 mentioned above, φ_* is expressed as follows:

$$(\varphi_*(A_1), \varphi_*(A_2), \varphi_*(B_1), \varphi_*(B_2)) = (A_1, A_2, B_1, B_2) \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} .$$

Since period matrix given from $X(e)$ with A_1, A_2, B_1, B_2 coincides with the one given from $X(e)$ with $\varphi_*(A_1), \varphi_*(A_2), \varphi_*(B_1), \varphi_*(B_2)$. Therefore τ satisfies the following equalities.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix}^{-1} \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix} ,$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix} ,$$

where the second equalityis given by using $\varphi \circ \varphi \circ \varphi$ instead of φ. As a solution of the equalities, we obtain that $\tau = \frac{\sqrt{-1}}{\sqrt{3}} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. ∎

Lemma 27 *Suppose that $e_1, \ldots, e_5 \in \mathbf{R}$, $e_1 < \cdots < e_5$. And suppose that A_1, A_2, B_1, B_2 are basis of $H_1(X(e), \mathbf{Z})$, which are given as in the* **Figure-1**. *Then for the period matrix τ given from $X(e)$ with A_1, A_2, B_1, B_2, all elements $\tau_1, \tau_2, \tau_{12}$ are in $\sqrt{-1}\mathbf{R}_+$.*

Proof. First note that the period matrix τ of a compact Riemann surface X with positive genus g depends on the choice of symplectic basis $A_1, \ldots, A_g, B_1 \ldots, B_g$ of $H_1(X, \mathbf{Z})$, but is independent to the choice of basis $\omega_1, \ldots, \omega_g$ of g-dimentional $\mathbf{C}$-vector space $\Gamma(X, \Omega^1_X)$. Here we use $y^{-1}(x-e_3)^{i-1}dx$ $(i = 1, 2)$ as basis of $\Gamma(X(e), \Omega^1_{X(e)})$, and we denote that $\eta_{ij} := \int_{A_j} y^{-1}(x-e_3)^{i-1}dx \quad (i \in \{1,2\},\ j \in \{1,2,3,4\},\ A_3 := B_1,\ A_4 := B_2)$, then

$$\tau = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}^{-1} \begin{pmatrix} \eta_{13} & \eta_{14} \\ \eta_{23} & \eta_{24} \end{pmatrix} = \frac{1}{\eta_{11}\eta_{22} - \eta_{12}\eta_{21}} \begin{pmatrix} \eta_{22}\eta_{13} - \eta_{12}\eta_{23} & \eta_{22}\eta_{14} - \eta_{12}\eta_{24} \\ \eta_{11}\eta_{23} - \eta_{21}\eta_{13} & \eta_{11}\eta_{24} - \eta_{14}\eta_{21} \end{pmatrix} .$$

Since A_1, A_2, B_1, B_2 are taken as in the **Figure-1**, the period η_{11} is an element of $\mathbf{R}_+$. In fact,

$$\eta_{11} = 2\int_{e_1}^{e_2} \frac{dx}{\sqrt{x-e_1}\sqrt{x-e_2}\sqrt{x-e_3}\sqrt{x-e_4}\sqrt{x-e_5}} ,$$

where in the above integrand, if $e_1 < x < e_2$, then $\sqrt{x-e_1} \in \mathbf{R}_+$, $\sqrt{x-e_j} \in \sqrt{-1}\mathbf{R}_+$ (for $\forall j \in \{2,3,4,5\}$). Hence $\eta_{11} \in \mathbf{R}_+$. Similar argument implies that $\eta_{12}, \eta_{21}, \eta_{22} \in \mathbf{R}_-$, $\eta_{13} \in \sqrt{-1}\mathbf{R}_+$ and $\eta_{14}, \eta_{23}, \eta_{24} \in -\sqrt{-1}\mathbf{R}_+ = \sqrt{-1}\mathbf{R}_-$. Therefore $\eta_{11}\eta_{22} - \eta_{12}\eta_{21} \in \mathbf{R}_-$, $\eta_{22}\eta_{13} - \eta_{12}\eta_{23} \in \sqrt{-1}\mathbf{R}_-$, $\eta_{11}\eta_{24} - \eta_{14}\eta_{21} \in \sqrt{-1}\mathbf{R}_-$ and $\eta_{22}\eta_{14} - \eta_{12}\eta_{24} \in \sqrt{-1}\mathbf{R}$. Therefore $\tau_1, \tau_2 \in \sqrt{-1}\mathbf{R}_+$ and $\tau_{12} \in \sqrt{-1}\mathbf{R}$. Moreover, by theorem of Weil, the element τ_{12} is not equal to zero. Hence $(\tau_1, \tau_2, \tau_{12}) \in ((\sqrt{-1}\mathbf{R}_+) \times (\sqrt{-1}\mathbf{R}_+) \times (\sqrt{-1}\mathbf{R}_+)) \coprod ((\sqrt{-1}\mathbf{R}_+) \times (\sqrt{-1}\mathbf{R}_+) \times (\sqrt{-1}\mathbf{R}_-))$. On the other hand, $\mathcal{F} := \{(e_1, \ldots, e_5) \in \mathbf{R}^5 \mid e_1 < \cdots < e_5\}$ is connected subset of $\mathbf{R}^5$, and the map

$$\mathcal{F} \ni e \longmapsto (\tau_1, \tau_2, \tau_{12}) \in \mathbf{C}^3 \tag{58}$$

is continuous. Therefore, the image of the map is connected subset of $\mathbf{C}^3$. By the way, lemma 26 implies that the intersection of $(\sqrt{-1}\mathbf{R}_+) \times (\sqrt{-1}\mathbf{R}_+) \times (\sqrt{-1}\mathbf{R}_+)$ and the image of the map (58) is not empty. Hence, the image of the map (58) is contained in $(\sqrt{-1}\mathbf{R}_+) \times (\sqrt{-1}\mathbf{R}_+) \times (\sqrt{-1}\mathbf{R}_+)$. The proof is completed. ∎

Lemma 28 *Suppose that $\zeta_8 := \exp(2\pi\sqrt{-1}/8)$. Then*

$$j_{1101}(\rho(\gamma_1), \tau) = \zeta_8 , \qquad j_{1101}(\rho(\gamma_2), \tau) = \zeta_8\sqrt{1 - \tau_1 - \tau_2 + 2\tau_{12}} ,$$
$$j_{1101}(\rho(\gamma_3), \tau) = \zeta_8 , \qquad j_{1101}(\rho(\gamma_4), \tau) = \zeta_8\sqrt{1 - \tau_2} ,$$

where $\sqrt{1 - \tau_1 - \tau_2 + 2\tau_{12}}$ and $\sqrt{1 - \tau_2}$ are both lie in the fourth quadrant.

Proof. It can be easily obtained from theorem TFTC that the values of $j_{1101}(\rho(\gamma_k), \tau)$ for $k = 1, 3, 4$ are as above. So now we have only to show the equality for $k = 2$. Since $\rho(\gamma_2) = {}^-C_2{}^+B_2{}^-C_2{}^+A_{12}{}^+C_2{}^-B_2{}^-C_1$, using (64), we can easily see that the following equality holds.

$$\begin{aligned} \kappa(\rho(\gamma_2), \tau) &= \kappa({}^-C_2{}^+B_2{}^-C_2{}^+A_{12}{}^+C_2{}^-B_2{}^-C_1, \tau) \\ &= \kappa({}^-C_2, ({}^+B_2{}^-C_2{}^+A_{12}{}^+C_2{}^-B_2{}^-C_1) \circ \tau)\kappa({}^+B_2, ({}^-C_2{}^+A_{12}{}^+C_2{}^-B_2{}^-C_1) \circ \tau) \\ &\quad \times \kappa({}^-C_2, ({}^+A_{12}{}^+C_2{}^-B_2{}^-C_1) \circ \tau)\kappa({}^+A_{12}, ({}^+C_2{}^-B_2{}^-C_1) \circ \tau) \\ &\quad \times \kappa({}^+C_2, ({}^-B_2{}^-C_1) \circ \tau)\kappa({}^-B_2, {}^-C_1 \circ \tau)\kappa({}^-C_1, \tau) . \end{aligned}$$

By theorem TFTC, first we have

$$\kappa({}^+B_2, ({}^-C_2{}^+A_{12}{}^+C_2{}^-B_2{}^-C_1) \circ \tau) = \kappa({}^+A_{12}, ({}^+C_2{}^-B_2{}^-C_1) \circ \tau) = \kappa({}^-B_2, {}^-C_1 \circ \tau) = 1 ,$$
$$\kappa({}^-C_1, \tau) = \sqrt{1 - \tau_1} \qquad \text{where } \sqrt{1 - \tau_1} \text{ lies in the fourth quadrant.}$$

Moreover, since

$$ {}^{-}B_2{}^{-}C_1 = \frac{1}{1-\tau_1}\begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_{12}^2-(1-\tau_1)(1-\tau_2) \end{pmatrix} , $$

the inequality $\Im(\frac{\tau_{12}^2-(1-\tau_1)(1-\tau_2)}{1-\tau_1}) > 0$ holds, and

$$ \kappa({}^{+}C_2, ({}^{-}B_2{}^{-}C_1)\circ\tau) = \sqrt{1+\frac{\tau_{12}^2-(1-\tau_1)(1-\tau_2)}{1-\tau_1}} , \quad \text{which lies in the first quadrant.} $$

Since

$$ ({}^{+}A_{12}{}^{+}C_2{}^{-}B_2{}^{-}C_1)\circ\tau = \frac{1}{(1-\tau_1)\tau_2+\tau_{12}^2}\begin{pmatrix} \tau_1\tau_2-\tau_{12}^2 & \tau_{12}-\tau_1\tau_2+\tau_{12}^2 \\ \tau_{12}-\tau_1\tau_2+\tau_{12}^2 & \tau_1+\tau_2-2\tau_{12}-1 \end{pmatrix} , $$

the inequality $\Im(\frac{\tau_1+\tau_2-2\tau_{12}-1}{(1-\tau_1)\tau_2+\tau_{12}^2}) > 0$ holds, and

$$ \kappa({}^{-}C_2, ({}^{+}A_{12}{}^{+}C_2{}^{-}B_2{}^{-}C_1)\circ\tau) = \sqrt{1-\frac{\tau_1+\tau_2-2\tau_{12}-1}{(1-\tau_1)\tau_2+\tau_{12}^2}} , \quad \text{which lies in the fourth quadrant.} $$

Since

$$ ({}^{+}B_2{}^{-}C_2{}^{+}A_{12}{}^{+}C_2{}^{-}B_2{}^{-}C_1)\circ\tau = \frac{1}{-\tau_1-\tau_1\tau_2+(\tau_{12}+1)^2}\begin{pmatrix} \tau_1 & \tau_{12}+\tau_{12}^2-\tau_1\tau_2 \\ \tau_{12}+\tau_{12}^2-\tau_1\tau_2 & \tau_2+\tau_{12}^2-\tau_1\tau_2 \end{pmatrix} , $$

the inequality $\Im(\frac{\tau_2+\tau_{12}^2-\tau_1\tau_2}{-\tau_1-\tau_1\tau_2+(\tau_{12}+1)^2}) > 0$ holds, and

$$ \kappa({}^{-}C_2, ({}^{+}B_2{}^{-}C_2{}^{+}A_{12}{}^{+}C_2{}^{-}B_2{}^{-}C_1)\circ\tau) = \sqrt{1-\frac{\tau_2+\tau_{12}^2-\tau_1\tau_2}{-\tau_1-\tau_1\tau_2+(\tau_{12}+1)^2}} , $$

which lies in the fourth quadrant. Especially, if $\tau_2=\sqrt{-1}$, $\tau_{12}=0$, and $|\tau_1| \ll 1$, then

$$ \begin{aligned} &\kappa({}^{-}C_1,\tau) \asymp 1 , \qquad \kappa({}^{+}C_2, ({}^{-}B_2{}^{-}C_1)\circ\tau) = \zeta_8 , \\ &\kappa({}^{-}C_2, ({}^{+}A_{12}{}^{+}C_2{}^{-}B_2{}^{-}C_1)\circ\tau) = \sqrt{\frac{1}{\sqrt{-1}}+\frac{-\tau_1}{1-\tau_1}} \asymp \sqrt{-\sqrt{-1}} = \zeta_8^{-1} , \\ &\kappa({}^{-}C_2, ({}^{+}B_2{}^{-}C_2{}^{+}A_{12}{}^{+}C_2{}^{-}B_2{}^{-}C_1)\circ\tau) = \sqrt{\frac{1-\sqrt{-1}+\tau_1}{1-\tau_1(1+\sqrt{-1})}} \asymp \sqrt{1-\sqrt{-1}} \end{aligned} $$

where $\sqrt{1-\sqrt{-1}}$ lies in the fourth quadrant. Hence, for the τ, we have that $\kappa(\rho(\gamma_2),\tau) \asymp \sqrt{1-\sqrt{-1}}$, which lies in the fourth quadrant. On the other hand, theorem TFTC implies that $\kappa(\rho(\gamma_2),\tau) = \kappa(\rho(\gamma_2))\sqrt{1+2\tau_{12}-\tau_1-\tau_2}$, where $\kappa(\rho(\gamma_2)) = \zeta_8^n$ for some integer n independent to τ, and the value $\sqrt{1+2\tau_{12}-\tau_1-\tau_2}$ lies in the forth quadrant. Hence $\kappa(\rho(\gamma_2))=1$, and the proof is completed. ∎

Proof of theorem 2

We denote that $\rho(\gamma_\mu) = \begin{pmatrix} A_\mu & B_\mu \\ C_\mu & D_\mu \end{pmatrix}$ for any $\mu \in \{1,2,3,4\}$. Moreover, we use the following notations.

$$ \begin{aligned} &M_{\mu,t} := (1-t)I + t\rho(\gamma_\mu) \qquad \mu\in\{1,2,3,4\} , \quad 0\le t\le 1 , \\ &\alpha(\mu,i,j,t) := \exp(-\pi\sqrt{-1}t/2)\det(tC_\mu\tau^{(\mu)}+I)^{-3}(\alpha_j(M_{\mu,t}\circ\tau^{(\mu)})-\alpha_i(M_{\mu,t}\circ\tau^{(\mu)})) , \\ &\hat{F}(\mu,t) := \exp(-\pi\sqrt{-1}t/4)\left(\sqrt{\det(tC_\mu\tau^{(\mu)}+I)}\right)^{-3}\cdot\hat{F}(M_{\mu,t}\circ\tau^{(\mu)}) \end{aligned} $$

where in the definition of $\hat{F}(\mu,t)$, the value $\sqrt{\det(tC_\mu\tau^{(\mu)}+I)}$ is 1 if $t=0$. For $\mu\in\{1,2,3,4\}$, $\tau^{(\mu)}$ is an element of $\mathbf{H}_2^*$, which is chosen later to satisfy the following condition.

Condition 29 *For any pairs (i,j) satisfying $1 \le i < j \le 5$ and $(i,j) \ne (\mu, \mu+1)$, if t runs trough $[0,1]$, the following three hold.*

1. *The value $\alpha(\mu,i,j,t)$ remains on a neighborhood of the non-zero value $\alpha(\mu,i,j,0)$.*

2. *The ratio $\alpha(\mu,\mu,\mu+1,t)/\alpha(\mu,i,j,t)$ remains on a neighborhood of zero.*

3. *$\alpha(\mu,\mu,\mu+1,t)$ starts $\alpha(\mu,\mu,\mu+1,0)$ and rotates $1/2$-times around zero.*

Note that, the C-block and D-block of $M_{\mu,t}$ are tC_μ and I, respectively. For any μ, $\hat{F}(\mu,1)$ equals to $\gamma_\mu(\hat{F}(\tau^{(\mu)}))$ or $(-1)\cdot\gamma_\mu(\hat{F}(\tau^{(\mu)}))$. To prove the theorem, we have only to show that $\hat{F}(\mu,1) = \gamma_\mu(\hat{F}(\tau^{(\mu)}))$ for any μ. So now we investigate behaviors of $\alpha(\mu,i,j,t)$ and $\hat{F}(\mu,t)$ when t run through $[0,1]$. For each $\mu \in \{1,2,3,4\}$, we take $\tau^{(\mu)} \in \mathbf{H}_2^*$, by which it is easy to investigate the behavior of $\alpha(\mu,i,j,t)$ when the real parameter t runs from 0 to 1. Suppose that, for each $\mu \in \{1,2,3,4\}$, all elements of $\tau^{(\mu)}$ are taken from $\sqrt{-1}\mathbf{R}_+$. We write $\tau_1 = \sqrt{-1}t_1$, $\tau_2 = \sqrt{-1}t_2$, $\tau_{12} = \sqrt{-1}t_{12}$ where t_1, t_2, t_{12} are elements of $\mathbf{R}_+$.

When $\mu = 1$, we take t_1 sufficiently large.

When $\mu = 3$, we take t_2 sufficiently large.

To investigate the behavior of $\alpha(\mu,i,j,t)$, we use the Fourier expansion of the ten theta constants ϑ_ε. The expansion is written as follows:

$$\vartheta_\varepsilon(\tau) = \exp[\pi\sqrt{-1}(\frac{1}{4}\varepsilon'\tau{}^t\varepsilon' + \frac{1}{2}\varepsilon'{}^t\varepsilon'')] \times \sum_{n\in\mathbf{Z}^2} (-1)^{\varepsilon''{}^tn} q_1^{n_1^2+\varepsilon_1'n_1} q_2^{n_2^2+\varepsilon_2'n_2} r^{2n_1n_2+\varepsilon_2'n_1+\varepsilon_1'n_2} ,$$

where $q_1^x := \exp[\pi\sqrt{-1}\tau_1 x]$, $q_2^x := \exp[\pi\sqrt{-1}\tau_2 x]$, $r^x := \exp[\pi\sqrt{-1}\tau_{12}x]$ for any $x \in \mathbf{C}$. That is,

$$\begin{aligned}
\vartheta_{0000}(\tau) &= 1 + 2\sum_{0<n\in\mathbf{Z}}(q_1^{n^2} + q_2^{n^2}) + 2\sum_{n_1,n_2=1}^{\infty} q_1^{n_1^2}q_2^{n_2^2}(r^{2n_1n_2} + r^{-2n_1n_2}) , \\
\vartheta_{0001}(\tau) &= 1 + 2\sum_{0<n\in\mathbf{Z}}(q_1^{n^2} + (-1)^n q_2^{n^2}) + 2\sum_{n_1,n_2=1}^{\infty} (-1)^{n_2} q_1^{n_1^2}q_2^{n_2^2}(r^{2n_1n_2} + r^{-2n_1n_2}) , \\
\vartheta_{0010}(\tau) &= 1 + 2\sum_{0<n\in\mathbf{Z}}((-1)^n q_1^{n^2} + q_2^{n^2}) + 2\sum_{n_1,n_2=1}^{\infty} (-1)^{n_1} q_1^{n_1^2}q_2^{n_2^2}(r^{2n_1n_2} + r^{-2n_1n_2}) , \\
\vartheta_{0011}(\tau) &= 1 + 2\sum_{0<n\in\mathbf{Z}}(-1)^n(q_1^{n^2} + q_2^{n^2}) + 2\sum_{n_1,n_2=1}^{\infty} (-1)^{n_1+n_2} q_1^{n_1^2}q_2^{n_2^2}(r^{2n_1n_2} + r^{-2n_1n_2}) , \\
\vartheta_{0100}(\tau) &= q_2^{\frac14}\left[\sum_{n_2\in\mathbf{Z}} q_2^{n_2(n_2+1)} + 2\sum_{0<n_1\in\mathbf{Z},0\le n_2\in\mathbf{Z}} q_1^{n_1^2}q_2^{n_2(n_2+1)}(r^{n_1(2n_2+1)} + r^{-n_1(2n_2+1)})\right] , \\
\vartheta_{0110}(\tau) &= q_2^{\frac14}\left[\sum_{n_2\in\mathbf{Z}} q_2^{n_2(n_2+1)} + 2\sum_{0<n_1\in\mathbf{Z},0\le n_2\in\mathbf{Z}} (-1)^{n_1} q_1^{n_1^2}q_2^{n_2(n_2+1)}(r^{n_1(2n_2+1)} + r^{-n_1(2n_2+1)})\right] , \\
\vartheta_{1000}(\tau) &= q_1^{\frac14}\left[\sum_{n_1\in\mathbf{Z}} q_1^{n_1(n_1+1)} + 2\sum_{0\le n_1\in\mathbf{Z},0<n_2\in\mathbf{Z}} q_1^{n_1(n_1+1)}q_2^{n_2^2}(r^{(2n_1+1)n_2} + r^{-(2n_1+1)n_2})\right] , \\
\vartheta_{1001}(\tau) &= q_1^{\frac14}\left[\sum_{n_1\in\mathbf{Z}} q_1^{n_1(n_1+1)} + 2\sum_{0\le n_1\in\mathbf{Z},0<n_2\in\mathbf{Z}} (-1)^{n_2} q_1^{n_1(n_1+1)}q_2^{n_2^2}(r^{(2n_1+1)n_2} + r^{-(2n_1+1)n_2})\right] , \\
\vartheta_{1100}(\tau) &= 2q_1^{\frac14}q_2^{\frac14}r^{\frac12}\sum_{n_1,n_2=0}^{\infty} q_1^{n_1(n_1+1)}q_2^{n_2(n_2+1)}(r^{2n_1n_2+n_1+n_2} + r^{-2n_1n_2-n_1-n_2-1}) , \\
\vartheta_{1111}(\tau) &= -2q_1^{\frac14}q_2^{\frac14}r^{\frac12}\sum_{n_1,n_2=0}^{\infty} (-1)^{n_1+n_2} q_1^{n_1(n_1+1)}q_2^{n_2(n_2+1)}(r^{2n_1n_2+n_1+n_2} - r^{-2n_1n_2-n_1-n_2-1}) .
\end{aligned}$$

Then $\alpha_j(\tau) - \alpha_i(\tau)$ are written as follows.

$$
\begin{aligned}
\alpha_2(\tau)-\alpha_1(\tau) &\in 64q_1^{\frac{3}{2}}q_2^{\frac{1}{2}}r^{-1}(r+1+(q_1,q_2))^2\,,\\
\alpha_3(\tau)-\alpha_1(\tau) &\in 16q_1^{\frac{1}{2}}q_2^{\frac{1}{2}}(1+(q_1,q_2))\,,\\
\alpha_4(\tau)-\alpha_1(\tau) &\in 16q_1^{\frac{1}{2}}q_2^{\frac{1}{2}}(1+(q_1,q_2))\,,\\
\alpha_5(\tau)-\alpha_1(\tau) &\in 4q_1^{\frac{1}{2}}q_2^{\frac{1}{2}}r^{-1}(r+1+(q_1,q_2))^2\,,\\
\alpha_3(\tau)-\alpha_2(\tau) &\in 16q_1^{\frac{1}{2}}q_2^{\frac{1}{2}}(1+(q_1,q_2))\,,\\
\alpha_4(\tau)-\alpha_2(\tau) &\in 16q_1^{\frac{1}{2}}q_2^{\frac{1}{2}}(1+(q_1,q_2))\,,\\
\alpha_5(\tau)-\alpha_2(\tau) &\in 4q_1^{\frac{1}{2}}q_2^{\frac{1}{2}}r^{-1}(r+1+(q_1,q_2))^2\,,\\
\alpha_4(\tau)-\alpha_3(\tau) &\in 64q_1^{\frac{1}{2}}q_2^{\frac{3}{2}}r^{-1}(1-r+(q_1,q_2))^2\,,\\
\alpha_5(\tau)-\alpha_3(\tau) &\in 4q_1^{\frac{1}{2}}q_2^{\frac{1}{2}}r^{-1}(1-r+(q_1,q_2))^2\,,\\
\alpha_5(\tau)-\alpha_4(\tau) &\in 4q_1^{\frac{1}{2}}q_2^{\frac{1}{2}}r^{-1}(1-r+(q_1,q_2))^2\,.
\end{aligned}
$$

On the other hand, to consider the cases $\mu = 2$ and 4, we use the Jacobi transform of τ, that is, $\sigma = \begin{pmatrix} \sigma_1 & \sigma_{12} \\ \sigma_{12} & \sigma_2 \end{pmatrix} := J\circ\tau = -\tau^{-1}$. For any real, symmetric, 2×2 matrix S, the transformation formula of $\alpha_j - \alpha_i$ under the action of $\begin{pmatrix} I & 0 \\ S & I \end{pmatrix}$ is, by theorem TFTC, written as follows:

$$
\alpha_j(\begin{pmatrix} I & 0 \\ S & I \end{pmatrix}\circ\tau)-\alpha_i(\begin{pmatrix} I & 0 \\ S & I \end{pmatrix}\circ\tau) = -\det(\sigma-s)^3\vartheta^2_{J(ijk)}(\sigma-s)\vartheta^2_{J(ijl)}(\sigma-s)\vartheta^2_{J(ijm)}(\sigma-s)\,. \quad (59)
$$

Especially, using the above formula with $S=0$, the differences $\alpha_j-\alpha_i$ can be written as follows.

$$
\begin{aligned}
\alpha_2(\tau)-\alpha_1(\tau) &= -(\det\sigma)^3\vartheta^2_{234}(\sigma)\vartheta^2_{235}(\sigma)\vartheta^2_{245}(\sigma) \in -(\det\sigma)^3\cdot 4p_2^{\frac{1}{2}}(1+(p_1,p_2))\,,\\
\alpha_3(\tau)-\alpha_1(\tau) &= -(\det\sigma)^3\vartheta^2_{234}(\sigma)\vartheta^2_{145}(\sigma)\vartheta^2_{135}(\sigma) \in -(\det\sigma)^3\cdot 4p_2^{\frac{1}{2}}(1+(p_1,p_2))\,,\\
\alpha_4(\tau)-\alpha_1(\tau) &= -(\det\sigma)^3\vartheta^2_{235}(\sigma)\vartheta^2_{145}(\sigma)\vartheta^2_{134}(\sigma) \in -(\det\sigma)^3\cdot 4p_2^{\frac{1}{2}}(1+(p_1,p_2))\,,\\
\alpha_5(\tau)-\alpha_1(\tau) &= -(\det\sigma)^3\vartheta^2_{245}(\sigma)\vartheta^2_{135}(\sigma)\vartheta^2_{134}(\sigma) \in -(\det\sigma)^3\cdot 4p_2^{\frac{1}{2}}(1+(p_1,p_2))\,,\\
\alpha_3(\tau)-\alpha_2(\tau) &= -(\det\sigma)^3\vartheta^2_{234}(\sigma)\vartheta^2_{123}(\sigma)\vartheta^2_{124}(\sigma) \in -(\det\sigma)^3\cdot 64p_1p_2^{\frac{1}{2}}(1+(p_1,p_2))\,,\\
\alpha_4(\tau)-\alpha_2(\tau) &= -(\det\sigma)^3\vartheta^2_{235}(\sigma)\vartheta^2_{123}(\sigma)\vartheta^2_{125}(\sigma) \in -(\det\sigma)^3\cdot 16p_1p_2^{\frac{1}{2}}s(1+s^{-1}+(p_1,p_2))^2\,,\\
\alpha_5(\tau)-\alpha_2(\tau) &= -(\det\sigma)^3\vartheta^2_{245}(\sigma)\vartheta^2_{124}(\sigma)\vartheta^2_{125}(\sigma) \in -(\det\sigma)^3\cdot 16p_1p_2^{\frac{1}{2}}s(1+s^{-1}+(p_1,p_2))^2\,,\\
\alpha_4(\tau)-\alpha_3(\tau) &= -(\det\sigma)^3\vartheta^2_{145}(\sigma)\vartheta^2_{123}(\sigma)\vartheta^2_{345}(\sigma) \in -(\det\sigma)^3\cdot 16p_1p_2^{\frac{1}{2}}s(1-s^{-1}+(p_1,p_2))^2\,,\\
\alpha_5(\tau)-\alpha_3(\tau) &= -(\det\sigma)^3\vartheta^2_{135}(\sigma)\vartheta^2_{124}(\sigma)\vartheta^2_{345}(\sigma) \in -(\det\sigma)^3\cdot 16p_1p_2^{\frac{1}{2}}s(1-s^{-1}+(p_1,p_2))^2\,,\\
\alpha_5(\tau)-\alpha_4(\tau) &= -(\det\sigma)^3\vartheta^2_{134}(\sigma)\vartheta^2_{125}(\sigma)\vartheta^2_{345}(\sigma) \in -(\det\sigma)^3\cdot 64p_1p_2^{\frac{3}{2}}s^2(1-s^{-2}+(p_1,p_2))^2\,,
\end{aligned}
$$

where $p_1^x := \exp[\pi\sqrt{-1}\sigma_1 x]$, $p_2^x := \exp[\pi\sqrt{-1}\sigma_2 x]$, $s^x := \exp[\pi\sqrt{-1}\sigma_{12}x]$ for any $x\in\mathbf{C}$. If $\tau = \sqrt{-1}\begin{pmatrix} t_1 & t_{12} \\ t_{12} & t_2 \end{pmatrix}$ as above, then $\sigma = \sqrt{-1}\begin{pmatrix} u_1 & u_{12} \\ u_{12} & u_2 \end{pmatrix}$, where $u_1, u_2\in\mathbf{R}_+$, $u_{12}\in\mathbf{R}_-$.

When $\mu=2$, suppose that $u_1=u_2=1-u_{12}(=:u)$, and u is sufficiently large.

When $\mu=4$, suppose that $u_1=-u_{12}$ and that u_2 is sufficiently large.

9 Relation to Siegel modular forms.

It is obvious, but remarkable fact that by the theorem, and by the expressions of t_i as functions on $\mathbf{H}_2^*$, the $\mathbf{C}^*$-bundle $(S\text{–}D)^\wedge\to\mathbf{H}_2^*$ with Γ-action is naturally extended to a bundle on $\mathbf{H}_2$ with Γ-action.

That is, by the theorem, j is naturally regarded as defined not only on $\mathbf{H}_2^*$, but also on $\mathbf{H}_2$. Similarly, by the definition, functions t_i on $\mathbf{H}_2^*$ are naturally regarded as holomorphic functions on $\mathbf{H}_2$. Note that $j(M,\cdot)$ is, as function on $\mathbf{H}_2$, holomorphic and has no zeros on $\mathbf{H}_2$. This means that the $\mathbf{C}^*$-bundle $(S\text{–}D)^\wedge \to \mathbf{H}_2^*$ with Γ-action is naturally extended to a bundle on $\mathbf{H}_2$ with Γ-action.

Γ is generated by four elements $\rho(\gamma_i)$ ($i \in \{1,\dots,4\}$). By (25), (26) and the definition of χ in Lemma 8, it can be easily seen that $\chi(\rho(\gamma_i)) = \sqrt{-1}$ for any $i \in \{1,\dots,4\}$.

On the other hand, it is well known (cf. [Ig2]) that

$$\Theta(M\circ\tau) = \mathrm{sign}(M)\det(C\tau+D)^5\Theta(\tau) \qquad (\text{ for any } \tau\in\mathbf{H}_2,\ \text{any } M\in Sp(4,\mathbf{Z})) \tag{60}$$

where $\mathrm{sign}(M) = -1$ (resp. 1) when the image of $M \in Sp(4,\mathbf{Z})$ under the homomorphism $b : Sp(4,\mathbf{Z}) \to S_6$ in (68) is odd (resp. even) permutation.

Remark 30 (cf. for example, [Fre]) *It is well-known that* $\{\tau\in\mathbf{H}_2|\Theta(\tau)=0\} = A = \mathbf{H}_2 - \mathbf{H}_2^*$, *and* Θ *has simple zero at each generic point of* A.

Since each $\rho(\gamma_i)$ is mapped to $(i,i+1)$ by the homomorphism $Sp(4,\mathbf{Z})\to S_6$, it satisfy that $\chi(\rho(\gamma_i))^2 = \mathrm{sign}(\rho(\gamma_i))(=-1)$ for any $i\in\{1,\dots,4\}$, hence $\chi(M)^2 = \mathrm{sign}(M)$ for any $M\in\Gamma$. Therefore, we have

$$j(M,\tau)^4 = \mathrm{sign}(M)\det(C\tau+D)^5\cdot\det(C\tau+D) \qquad \text{for any } \tau\in\mathbf{H}_2,\ \text{any } M\in\Gamma\ .$$

A Appendix.

A.1 Notation.

Suppose $\mathbf{Z},\mathbf{Q},\mathbf{R},\mathbf{C}$ be the sets of integer, rational, real, complex numbers, respectivly. $\mathbf{R}_\pm := \{x|x\in\mathbf{R}, \pm x>0\}$, $\sqrt{-1}\mathbf{R}_\pm := \{\sqrt{-1}x|x\in\mathbf{R},\pm x>0\}$. If $R=\mathbf{Z},\mathbf{R}$ or $\mathbf{C}$, for any positive integers m,n, we write

$$M_{m,n}(R) := \{M|M \text{ is } m\times n \text{ matrix with coeffitients in } R\}\ , \qquad M_n(R) := M_{n,n}(R)\ .$$

We denote $n\times n$ identity matrix by I_n. We write transpose of matrix M by ${}^tM$. For any positive integer g, we define

$$Sp(2g,\mathbf{Z}) := \{M\in M_{2g}(\mathbf{Z})|MJ{}^tM = J\} \qquad \text{where} \qquad J := \begin{pmatrix} 0 & -I_g \\ I_g & 0\end{pmatrix}\ .$$

We usually write $M\in Sp(2g,\mathbf{Z})$ as $M = \begin{pmatrix} A & B \\ C & D\end{pmatrix}$ where $A,B,C,D\in M_g(\mathbf{Z})$. Moreover,

$$\mathbf{H}_g := \{\tau\in M_g(\mathbf{C})|\tau={}^t\tau,\quad \Im(\tau) \text{ is positive definite}\}\ , \qquad \mathbf{H} := \mathbf{H}_1\ .$$

For any $M = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in Sp(2g,\mathbf{Z})$, $\tau\in\mathbf{H}_g$, and $\varepsilon = (\varepsilon'\ \varepsilon'') = (\varepsilon'_1\dots\varepsilon'_g\varepsilon''_1\dots\varepsilon''_g)\in\mathbf{Z}^{2g}$, we define

$$M\circ\tau := (A\tau+B)(C\tau+D)^{-1}\ , \qquad M\circ\varepsilon := \varepsilon M^{-1} + ((C{}^tD)_0\ (A{}^tB)_0)\ ,$$

where $(C{}^tD)_0$ (resp. $(A{}^tB)_0$) is $1\times g$ matrix whose i-th element is (i,i) element of $C{}^tD$ (resp. $A{}^tB$) for each i. Moreover, $A := \{M\circ\tau|M\in Sp(4,\mathbf{Z}),\ \tau\in\mathbf{H}_2,\ \tau \text{ is diagonal matrix}\}$. $\mathbf{H}_2^* := \mathbf{H}_2 - A$. For any positive integers g,n, we define $\Gamma_g(n) := \{M\in Sp(2g,\mathbf{Z})|M-I_{2g}\in M_{2g}(n\mathbf{Z})\}$. Note that $\Gamma_g(1) = Sp(2g,\mathbf{Z})$.

Suppose that a group G acts on a set X. Then for any $x\in X$, $\mathrm{stab}_G(x) := \{g\in G|g(x)=x\}$.

For any positive integer n, S_n denotes the n-th symmetric group.

A.2 Theta constants and their transformation formula.

The aim of this section is to review transformation formula of theta constants according to [R-F]. (Notations are slightly modified.) In this article we use theta constants with characteristics, which is defined as follows. For any $\varepsilon = (\varepsilon'\varepsilon'') = (\varepsilon'_1...\varepsilon'_g\varepsilon''_1...\varepsilon''_g) \in \mathbf{Z}^{2g}$ and $\tau \in \mathbf{H}_g$, we define

$$\vartheta_\varepsilon(\tau) := \sum_{n\in\mathbf{Z}^g} \exp\left[\pi\sqrt{-1}(n+\frac{\varepsilon'}{2})\tau{}^t(n+\frac{\varepsilon'}{2}) + 2\pi\sqrt{-1}(n+\frac{\varepsilon'}{2})\frac{{}^t\varepsilon''}{2}\right] .$$

If there is no fear of confusion, we write $\vartheta_\varepsilon(\tau)$ as ϑ_ε for short.

For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbf{Z})$, $\tau \in \mathbf{H}_g$, $\varepsilon = (\varepsilon'\,\varepsilon'') \in \mathbf{Z}^{2g}$, it is well known that the following equality holds:

$$\vartheta_{M\circ\varepsilon}(M\circ\tau) = \kappa(M)\exp(\pi\sqrt{-1}\phi(M,\varepsilon))\sqrt{\det(C\tau+D)}\vartheta_\varepsilon(\tau) \tag{61}$$

where

$$\phi(M,\varepsilon) := \frac{1}{4}\{-\varepsilon'{}^tDB{}^t\varepsilon' + 2\varepsilon''{}^tCB{}^t\varepsilon' - \varepsilon''{}^tCA{}^t\varepsilon'' + 2(\varepsilon'{}^tD - \varepsilon''{}^tC){}^t(A{}^tB)_0\} . \tag{62}$$

$\kappa(M)^2$ is a constant, which depends on M, but is independent to ε and τ. It is known that $\kappa(M)^8 = 1$ for any $M \in Sp(2g,\mathbf{Z})$. Note that, by (61), $\kappa(M)\sqrt{\det(C\tau+D)}$ has no ambiguity. We write that $\kappa(M,\tau) := \kappa(M)\sqrt{\det(C\tau+D)}$.

Next we review some property of κ. Suppose I_g be $g\times g$ identity matrix. And suppose E_{ij} be $g\times g$ matrix whose (i,j) element is 1 and all other elements are 0. Then for any $i,j \in \{1,...,g\}$ with $i \neq j$, we define

$$ {}^{\pm}A_{ij} := \begin{pmatrix} I_g \mp E_{ji} & 0 \\ 0 & I_g \pm E_{ij} \end{pmatrix}, \quad {}^{\pm}B_i := \begin{pmatrix} I_g & \pm E_{ii} \\ 0 & I_g \end{pmatrix},$$
$$ {}^{\pm}C_i := \begin{pmatrix} I_g & 0 \\ \pm E_{ii} & I_g \end{pmatrix}, \quad D_i := \begin{pmatrix} I_g - 2E_{ii} & 0 \\ 0 & I_g - 2E_{ii} \end{pmatrix}.$$

Note that ${}^+C_i{}^-C_i = {}^+B_i{}^-B_i = {}^+A_{ij}{}^-A_{ij} = D_i^2 = I_{2g}$.

Fact 31 (cf. [R-F] p89) *$Sp(2g,\mathbf{Z})$ is generated by the following $g(2g+3)$ matrices:*

$$ {}^{\pm}A_{ij}, {}^{\pm}B_i, {}^{\pm}C_i, D_i \quad (i,j \in \{1,...,g\}, i\neq j).$$

Fact 32 (cf. [R-F] p90) *It satisfy that*

$$\kappa({}^{\pm}A_{ij})^2 = \kappa({}^{\pm}B_i)^2 = \kappa({}^{\pm}C_i)^2 = 1, \quad \kappa(D_i)^2 = -1 . \tag{63}$$

Fact 33 *For any $M_1, M_2 \in Sp(2g,\mathbf{Z})$, the following equality holds:*

$$\kappa(M_2M_1,\tau) = \kappa(M_2, M_1\circ\tau)\kappa(M_1,\tau)(-1)^x\exp[\pi\sqrt{-1}\phi(M_2, M_1\circ 0)] . \tag{64}$$

where

$$x = \frac{1}{2}(C_3{}^tD_3)_0{}^t[(A_3{}^tB_3)_0 - (-(C_1{}^tD_1)_0{}^tB_2 + (A_1{}^tB_1)_0{}^tA_2 + (A_2{}^tB_2)_0)] , \tag{65}$$

$M_3 := M_2M_1$, $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ *for $k \in \{1,2,3\}$. Note that $[\cdots]$ in (65) is the right half of $(M_2M_1)\circ 0 - M_2\circ(M_1\circ 0)$. Therefore, $[\cdots]$ is an element of $(2\mathbf{Z})^g$ and x is an integer.*

A.3 More on $Sp(4, \mathbf{Z})$.

In this section we explain the following well-known fact.

Fact 34 *There exists a homomorphism $b := Sp(4, \mathbf{Z}) \to S_6$ such that*

$$1 \longrightarrow \Gamma_2(2) \xrightarrow{\iota} Sp(4, \mathbf{Z}) \xrightarrow{b} S_6 \longrightarrow 1 \tag{66}$$

is exact, where ι is natural inclusion.

The isomorphism b is given by an action of $Sp(4, \mathbf{Z})$ over the six odd theta characteristics of genus two:

$$\mathbf{OTC} := \{(0101),\ (0111),\ (1011),\ (1010),\ (1110),\ (1101)\}\ . \tag{67}$$

Since explicit description of the isomorphism is needed in the article, here we show the proof of the fact according to [Ig2], [Koe].

Proof. We define $Sp(4, \mathbf{Z})$-action on $\{0,1\}^4$ as

$$\{0,1\}^4 \ni \varepsilon \longmapsto \tilde{\varepsilon} \in \{0,1\}^4 \qquad \text{where } \tilde{\varepsilon} \equiv M \circ \varepsilon \bmod (2\mathbf{Z})^4\ ,$$

for each $M \in Sp(4, \mathbf{Z})$. Then **OTC** is stable under the action. Therefore, if we call the elements of **OTC** simply as $1, \ldots, 6$ as in the order written in (67), each $M \in Sp(4, \mathbf{Z})$ is regarded as an element of S_6. Thus we have a homomorphism

$$b : Sp(4, \mathbf{Z}) \to S_6\ . \tag{68}$$

Since the images of (13), (37) under the map (68) obviously generate S_6, (68) is surjective. On the other hand, it can be easily seen that $\mathrm{Image}(\iota) \subset \mathrm{Kernel}(b)$. Therefore (66) with (68) give

$$Sp(4, \mathbf{Z})/\Gamma_2(2) \to S_6 \to 1 \qquad \text{(exact)}.$$

Then , by the fact that $[Sp(4, \mathbf{Z}) : \Gamma_2(2)] = 720 = \# S_6$ (cf. [Koe]), we obtain the exactness of (66). ∎

A.4 Coding.

In this section, suppose that $g = 2$. It is well known (cf. for example [R-F] p22 or [Krz] p336) that for each even theta characteristic a, there exist three odd theta characteristics p, q, r satisfying

$$p + q + r \equiv a \qquad \bmod (2\mathbf{Z})^4\ . \tag{69}$$

Note that, in the above equality,

- p, q, r are different from each other.
- The complement $\{s, t, u\}$ of $\{p, q, r\}$ in **OTC** also satisfy $s + t + u \equiv a \bmod (2\mathbf{Z})^4$.
- For the a, there is no solution other than $\{p, q, r\}$ and $\{s, t, u\}$.

Therefore, we denote a by symbols pqr or stu: $a = pqr = stu$. Note that, For any permutation p', q', r' of p, q, r, the equality $p'q'r' = pqr$ holds. For example, since

$$\begin{aligned} (0100) &\equiv (0101) + (1011) + (1010) = \text{“1”} + \text{“3”} + \text{“4”} \\ &\equiv (0111) + (1110) + (1101) = \text{“2”} + \text{“5”} + \text{“6”}\ , \end{aligned}$$

we write, as symbols,

$$(0100) = 134 = 143 = 314 = 341 = 413 = 431 = 256 = 265 = 526 = 562 = 625 = 652\ .$$

Those expression for all even theta characteristics are written in the following table.

$$\begin{array}{llll} (0000)=135=246 & (0001)=145=236 & (0100)=134=256 & (1111)=345=126 \\ (0010)=235=146 & (0011)=245=136 & (0110)=234=156 & \\ (1000)=124=356 & (1001)=123=456 & (1100)=125=346 & \end{array} \tag{70}$$

As for theta constants, we use these expressions. For example, $\vartheta_{134}(\tau) := \vartheta_{0100}(\tau)$, etc.

For any $M \in Sp(4, \mathbf{Z})$ and for any $x, y, z \in \mathbf{Z}^4$, the equality

$$M \circ (x + y + z) \equiv M \circ x + M \circ y + M \circ z \qquad \mod (2\mathbf{Z})^4 \tag{71}$$

holds. We apply this fact to the above notations. For example, $M(134) := M \circ (0100)$. We note that $M(ijk) = M(i)M(j)M(k)$ for three distinct $i, j, k \in \{1, \dots, 6\}$.

References

[ACa] Norbert A'Campo.: Tresses, monodromie et le groupe symplectique. Comment. Math. Helvetici, vol. **54** (1979), pp. 318-327.

[F-K] H.M.Farkas, I.Kra.: Riemann Surfaces, Springer

[Fre] E.Freitag.: Siegelsche Modulfunktionen. Springer-Verlag. (1983).

[Gun] Gunning, R.C.: Lectures on Riemann Surfaces. Princeton Mathematical Notes, Princeton University Press, 1966

[Ig1] Igusa, J. : On Siegel modular forms of genus two, American Journal of Mathematics, vol.84 (1962), pp.175-200.

[Ig2] Igusa, J.: On Siegel modular forms of genus two (II), Am. J. Math. **86** (1964), 392-412.

[Ig3] Igusa, J.: On the graded ring of theta-constants, Am. J. Math. **86** (1964), 219-246.

[Ig4] Igusa, J.: Theta Functions, Springer-Verlag. (1972).

[Koe] Koecher, M.: Zur Theorie de Modulfunktionen n-ten Grades, I, Mathematische Zeitschrift, vol. 59 (1954), PP. 399-416.

[Koi] Koizumi, S.: Remarks on K. Takase's Paper "A Generalization of Rosenhain's Nomal Form with an Application" Proc. Japan Acad., 73, (1997), 12-13.

[Krz] Krazer, A.: Lehrbuch der Thetafunktionen, Leipzig, (1903).

[Mar] Martens. H. H.: Torelli's Theorem and a Generalization for Hyper-Elliptic Surfaces. Communications, on Pure and Applied Mathematics, vol XVI, (1963) 97-110.

[Mum1] Mumford, D.: Tata Lectures on Theta I, *Birkhäuser* (1983).

[Mum2] Mumford, D.: Tata Lectures on Theta II, *Birkhäuser* (1984).

[R-F] H.E.Rauch, H.M.Farkas : Theta Functions with Applications to Riemann Surfaces, Baltimore 1974.

[Ros] Rosenhain, G. : Abhandlung über die Functionen zweier Variabler mit vier Perioden, 1851, Ostwald's Klassiker der Exacten Wissenschaften, no. 65 (1895).

[Sai] Saito, K.: Elliptic Integrals, Preliminary lecture notes intended for College on Singularity Theory (19 August - 6 September 1991 Trieste).

[Tak] Takase, K.: A Generalization of Rosenhain's Nomal Form for Hyperelliptic Curves with an Application, Proc. Japan Acad., 72, Ser. A (1996), 162-165.

[Wei] Weil, A.: Zum Beweis des Torellischen Satzes. Göttingen Nachrichten, Nr.2 (1957), pp.33-53.