p-ADIC MULTIPLE ZETA VALUES I — p-ADIC MULTIPLE POLYLOGARITHMS AND THE p-ADIC KZ EQUATION

HIDEKAZU FURUSHO

Abstract. Our main aim in this paper is to give a foundation of the theory of p-adic multiple zeta values. We introduce (one variable) p-adic multiple polylogarithms by Coleman's p-adic iterated integration theory. We define p-adic multiple zeta values to be special values of p-adic multiple polylogarithms. We consider the p-adic KZ equation and introduce the p-adic Drinfel'd associator by using certain two fundamental solutions of the p-adic KZ equation. We show that our p-adic multiple polylogarithms appear on coefficients of a certain fundamental solution of the p-adic KZ equation and our p-adic multiple zeta values appear on coefficients of the p-adic Drinfel'd associator. We show various properties of p-adic multiple zeta values, which are sometimes analogous to the complex case and are sometimes peculiar to the p-adic case, via the p-adic KZ equation.

Contents

0. Introduction 2
1. Review of the complex case 5
1.1. Multiple polylogarithms and multiple zeta values 5
1.2. The KZ equation and the Drinfel'd associator 7
2. p-adic multiple polylogarithms and p-adic multiple zeta values 8
2.1. Review of Coleman's p-adic iterated integration theory 8
2.2. Analytic continuation of p-adic multiple polylogarithms 11
2.3. p-adic multiple zeta values and main results 13
3. The p-adic KZ equation 18
3.1. The p-adic Drinfel'd associator 19
3.2. Explicit formulae of the fundamental solution of the p-adic KZ equation 22
3.3. Explicit formulae of the p-adic Drinfel'd associator 25

2000 *Mathematics Subject Classification.* Primary: 11S80, Secondary: 33B30.

3.4. Proofs of main results and functional equations among p-adic multiple polylogarithms 29
References 33

0. Introduction

The aim of the present paper and the upcoming papers [F2] and [F3] is to enlighten p-adic crystalline aspects of the fundamental group of the projective line minus three points and add p-adic crystalline part to [F1]. In this paper, we will introduce the notions of (one-variable) p-adic multiple polylogarithms, p-adic multiple zeta values, p-adic KZ equation and p-adic Drinfel'd associator, which will be our basic foundations of [F2] and [F3], and show their various properties and their relationships.

Let $k_1, \cdots, k_m \in \mathbf{N}$. The (usual) multiple zeta value is the real number defined by the following series

$$\zeta(k_1, \cdots, k_m) = \sum_{\substack{0<n_1<\cdots<n_m \\ n_i \in \mathbf{N}}} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}} \quad . \tag{0.1}$$

Especially in the case when $m = 1$, the multiple zeta value coincides with the Riemann zeta value $\zeta(k)$. We can check easily that this series converges in the topology of $\mathbf{R}$ if and only if $k_m > 1$, however, this series never converges in the topology of $\mathbf{Q}_p$! Thus it is not so easy and not so straightforward to give a definition of p-adic version of multiple zeta value. To give a nice definition, we need another interpretation of multiple zeta values.

Suppose $z \in \mathbf{C}$. The (one variable) multiple polylogarithm is a function defined by the following series

$$Li_{k_1,\cdots,k_m}(z) = \sum_{\substack{0<n_1<\cdots<n_m \\ n_i \in \mathbf{N}}} \frac{z^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \quad .$$

Especially in the case when $m = 1$, the multiple polylogarithm coincides with the classical polylogarithm $Li_k(z)$. Easily we see that this series converges for $|z| < 1$. In §2.2, we will define the p-adic multiple polylogarithm to be the function defined by the above series just replacing $z \in \mathbf{C}$ by $z \in \mathbf{C}_p$. We remark that especially in the case when $m = 1$, the p-adic multiple polylogarithm is equal to the p-adic polylogarithm $\ell_k(z)$ which was studied by Coleman [C]. What is interesting is that this p-adic multiple polylogarithm converges for $|z|_p < 1$

similarly to the above complex case. Here $|\cdot|_p$ means the standard multiplicative valuation of $\mathbf{C}_p$.

An important relationship between the multiple polylogarithm and the multiple zeta value is the following formula:

$$\zeta(k_1, \cdots, k_m) = \lim_{\substack{z \to 1 \\ |z|<1}} Li_{k_1,\cdots,k_m}(z) \quad . \tag{0.2}$$

In this paper, we will define the p-adic multiple zeta value à la formula (0.2) instead of à la (0.1). But we note that here happens a serious problem because the open unit disk centered at 0 on $\mathbf{C}_p$ and the one centered at 1 on $\mathbf{C}_p$ are disjoint! Thus we cannot consider $\lim\limits_{z \to 1}$ of p-adic multiple polylogarithms which are functions defined on $|z|_p < 1$, i.e. on the open unit disk centered at 0. To give a meaning of this limit, we will make an analytic continuation of p-adic multiple polylogarithms by Coleman's p-adic iterated integration theory [C] and then define p-adic multiple zeta values to be a limit value at 1 of analytically continued p-adic multiple polylogarithms.

The organization of this paper is as follows. §1 is devoted to a short review of well-known results on (usual) multiple polylogarithms and multiple zeta values and definitions of the KZ equation and the Drinfel'd associator, which will play a role of prototype in the p-adic case in the following two sections.

In §2, we will introduce p-adic multiple zeta values and show their many nice properties. At first, we will review Coleman's p-adic iterated integration theory [C] in §2.1 and then in §2.2 we will give an analytic continuation of p-adic multiple polylogarithms (which is just a multiple analogue of that of his p-adic polylogarithms $\ell_k(z)$ in [C]) to the whole plane minus 1, i.e. $\mathbf{C}_p - \{1\}$, by his integration theory. But we will see that there happens a terrible problem that the analytically continued p-adic multiple polylogarithm admits too many (uncountably infinite) branches $Li^a_{k_1,\cdots,k_m}(z)$ $(k_1, \cdots, k_m \in \mathbf{N}, z \in \mathbf{C}_p - \{1\})$ which correspond to each branch parameter $a \in \mathbf{C}_p$, coming from branch $log^a(z)$ of p-adic logarithms (see §2.2). However the following theorem in §2.3 will remove our anxiety.

Theorem 2.13. *If* $\lim'_{\substack{z \to 1 \\ z \in \mathbf{C}_p - \{1\}}} Li^a_{k_1,\cdots,k_m}(z)$ *converges on* $\mathbf{C}_p$*, its limit does not depend on any choice of branch parameter* $a \in \mathbf{C}_p$*. For* $\lim'$*, see Notation 2.12.*

By this theorem, we can give a definition of the p-adic multiple zeta value $\zeta_p(k_1, \cdots, k_m)$ to be the above limit on $\mathbf{C}_p$ as follows.

Definition 2.17. $\zeta_p(k_1,\cdots,k_m) := \lim_{\substack{z\to 1 \\ z\in \mathbf{C}_p-\{1\}}}{}' \; Li^a_{k_1,\cdots,k_m}(z) \in \mathbf{C}_p$ *if it converges.*

This definition of p-adic multiple zeta value is actually independent of any choice of branch parameter $a \in \mathbf{C}_p$ by Theorem 2.13. Especially, in the case when $m = 1$, we shall see in Example 2.19 (due to Coleman [C]) that the p-adic multiple zeta value is equal to the p-adic L-value up to a certain constant multiple. The following three theorems in §2.3 are p-adic analogues of basic properties, Lemma 1.6, Lemma 1.9 and Proposition 1.11, in the complex case.

Theorem 2.18. $\lim_{\substack{z\to 1 \\ z\in \mathbf{C}_p-\{1\}}}{}' \; Li^a_{k_1,\cdots,k_m}(z)$ *converges on* $\mathbf{C}_p$ *if* $k_m > 1$.

Theorem 2.25. $\zeta_p(k_1,\cdots,k_m) \in \mathbf{Q}_p$.

Theorem 2.28. *The product of two p-adic multiple zeta values can be written as a* $\mathbf{Q}$*-linear combination of p-adic multiple zeta values.*

In §3, we will consider the p-adic KZ equation and introduce the p-adic Drinfel'd associator which will play a role of main tools to prove Theorem 2.18, Theorem 2.22 and Theorem 2.28. In §3.1, we will introduce the p-adic KZ equation in Definition 3.2 and prove the following.

Theorem 3.4. *Let* $a \in \mathbf{C}_p$. *Then there exists a unique solution* $G^a_0(A,B)(z)$ $(z \in \mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\})$ *of the p-adic KZ equation which is a formal power series whose coefficients are Coleman functions and are locally analytic on* $\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$ *and satisfies a certain asymptotic behavior* $G^a_0(z) \approx z^A$ $(z \to 0)$.

Then we will introduce a definition of p-adic Drinfel'd associator $\Phi^p_{KZ}(A,B)$ from two fundamental solutions, $G^a_0(u)$ and $G^a_1(u)$, of the p-adic KZ equation in Definition 3.12 and show its branch independency in Theorem 3.10. In §3.2, we will state precisely and prove the following.

Theorem 3.15. *Let* $a \in \mathbf{C}_p$. *The fundamental solution* $G^a_0(z)$ $(z \in \mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\})$ *of the p-adic KZ equation can be expressed in terms of (analytically continued) p-adic multiple polylogarithms* $Li^a_{k_1,\cdots,k_m}(z)$ *and the p-adic logarithm* $log^a(z)$ *explicitly.*

In §3.3, we will state precisely and prove the following.

Theorem 3.28. *The p-adic Drinfel'd associator* $\Phi^p_{KZ}(A,B)$ *can be expressed explicitly in terms of p-adic multiple zeta values.*

In §3.4, we will show functional equations of p-adic multiple polylogarithms at first.

Theorem 3.36.

$$Li^a_{k_1,\cdots,k_m}(1-z) = (-1)^m \sum_{\substack{W',W'':\text{words} \\ W=W'W''}} I_p(W'') \cdot J^a_p\big(\tau(W')\big)(z)$$

where $W = A^{k_m-1}BA^{k_{m-1}-1}B\cdots A^{k_1-1}B$. *Here each* $I_p(W'')$ *is a certain* $\mathbf{Q}$*-linear combination of* p*-adic multiple zeta values (see Theorem 3.28) and each* $J_p^a\big(\tau(W')\big)(z)$ *is a certain combination of* p*-adic multiple polylogarithms (see Theorem 3.15).*
Next we will see that especially Coleman-Sinnott's functional equation of the p-adic dilogarithm will be re-proved in Example 3.37 by Theorem 3.36 and then will prove Theorem 2.22 and Theorem 2.28.

In the upcoming paper [F2], we will relate the p-adic Drinfel'd associator with the p-adic crystalline Frobenius action on the p-adic rigid (unipotent) fundamental group of the projective line minus three points and compare the p-adic Drinfel'd associator with other corresponding objects in various realizations of motivic fundamental groups of the projective line minus three points. In [F3], many algebraic relations among p-adic multiple zeta values are shown. We will show that the p-adic Drinfel'd associator determines a point of the Grothendieck-Teichmüller group and discuss on this point. Finally we will add p-adic crystalline part to [F1].

Acknowledgments . The author wish to thank his advisor Professor Akio Tamagawa for his encouragement. The author is supported in part by JSPS Research Fellowships for Young Scientists.

1. Review of the complex case

We shall review definitions of multiple polylogarithms and multiple zeta values in §1.1 and shall recall notions of the KZ equation and the Drinfel'd associator briefly in §1.2, which may help to understand its p-adic version developed in the following two sections.

1.1. **Multiple polylogarithms and multiple zeta values.**

We review briefly definitions and properties of multiple polylogarithms and and multiple zeta values. For more details, consult [F0] and [Gon] for example.

Let $k_1, \cdots, k_m \in \mathbf{N}$ and $z \in \mathbf{C}$.

Definition 1.1. The (one variable) multiple polylogarithm (MPL for short) is defined to be the following series:

$$Li_{k_1,\cdots,k_m}(z) = \sum_{\substack{0<n_1<\cdots<n_m \\ n_i\in\mathbf{N}}} \frac{z^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}} \qquad .$$

Remark 1.2. This MPL is the special case of the multiple polylogarithm

$$Li_{k_1,\cdots,k_m}(z_1,\cdots,z_m)=\sum_{\substack{0<n_1<\cdots<n_m\\ n_i\in\mathbf{N}}}\frac{z_1^{n_1}\cdots z_m^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}$$

introduced in [Gon] where $z_1=\cdots=z_{m-1}=1$ and $z_m=z$

Easily we can check the following.

Lemma 1.3. *The MPL $Li_{k_1,\cdots,k_m}(z)$ converges for $|z|<1$.*

Lemma 1.4. *Suppose that $|z|<1$. Then*

$$\frac{d}{dz}Li_{k_1,\cdots,k_m}(z)=\begin{cases}\frac{1}{z}Li_{k_1,\cdots,k_m-1}(z) & k_m\neq 1,\\ \frac{1}{1-z}Li_{k_1,\cdots,k_{m-1}}(z) & k_m=1,\end{cases}$$

$$\frac{d}{dz}Li_1(z)=\frac{1}{1-z}\qquad .$$

By Lemma1.4, for $|z|<1$, we get $Li_1(z)=-log(1-z)$ and the following,

$$Li_{k_1,\cdots,k_m}(z)=\begin{cases}\int_0^z\frac{1}{t}Li_{k_1,\cdots,k_m-1}(t)dt & k_m\neq 1,\\ \int_0^z\frac{1}{1-t}Li_{k_1,\cdots,k_{m-1}}(t)dt & k_m=1,\end{cases}$$

from which we get an expression of the MPL by iterated integral of $\frac{dt}{t}$ and $\frac{dt}{1-t}$. Since $\frac{dt}{t}$ and $\frac{dt}{1-t}$ admit poles at $t=0,1$ and ∞, we cannot give an analytic continuation of the MPL to the whole complex plane due to monodromies around 0, 1 and ∞. However we can say that

Lemma 1.5. *The MPL $Li_{k_1,\cdots,k_m}(z)$ can be analytically continued to the universal unramified covering $\mathbf{P}^1(\mathbf{C})\widetilde{\backslash\{0,1,\infty\}}$ of $\mathbf{P}^1(\mathbf{C})\backslash\{0,1,\infty\}$.*

Since the simply-connected Riemann surface $\mathbf{P}^1(\mathbf{C})\widetilde{\backslash\{0,1,\infty\}}$ is an infinite covering of $\mathbf{P}^1(\mathbf{C})\backslash\{0,1,\infty\}$, each MPL admits (*countably*) infinite branches. The following are well-known (for example, see [Gon]).

Lemma 1.6. $\lim\limits_{\substack{z\to1\\|z|<1}}Li_{k_1,\cdots,k_m}(z)$ *converges if $k_m>1$.*

Lemma 1.7. $\lim\limits_{\substack{z\to1\\|z|<1}}Li_{k_1,\cdots,k_m}(z)$ *diverges if $k_m=1$.*

Definition 1.8. For $k_1,\cdots,k_m\in\mathbf{N}$, $k_m>1$, the multiple zeta value (MZV for short) is defined to be

$$\zeta(k_1,\cdots,k_m)=\lim_{\substack{z\to1\\|z|<1}}Li_{k_1,\cdots,k_m}(z)\left(=\sum_{0<n_1<\cdots<n_m}\frac{1}{n_1^{k_1}\cdots n_m^{k_m}}\right)\qquad .$$

Since MPL's are **C**-valued functions, MZV's lie on **C**. However we can say more.

Lemma 1.9. *For* $k_1, \cdots, k_m \in \mathbf{N}$, $k_m > 1$, $\zeta(k_1, \cdots, k_m) \in \mathbf{R}$.

Notation 1.10. For each natural number w, let Z_w be the **Q**-vector subspace of **R** generated by all MZV's $\zeta(k_1, \cdots, k_m)$ with $k_1 + \cdots + k_m = w$, i.e. $Z_w := \langle \zeta(k_1, \cdots, k_m) \mid k_1 + \cdots + k_m = w \rangle_{\mathbf{Q}} \subseteq \mathbf{R}$, and put $Z_0 = \mathbf{Q}$. Denote $Z_\bullet$ to be the *formal direct sum* of Z_w for all $w \geqslant 0$: $Z_\bullet := \bigoplus_{w \geqslant 0} Z_w$.

The following is one of the fundamental properties of MZV's.

Proposition 1.11. *The graded* **Q**-*vector space* $Z_\bullet$ *forms a graded* **Q**-*algebra, i.e.* $Z_a \cdot Z_b \subseteq Z_{a+b}$ *for* $a, b \geqslant 0$.

Proof . At least we have two proofs [Gon]. The first one is given by the harmonic product formulae, from which it follows, for example,

$$\zeta(m) \cdot \zeta(n) = \zeta(m, n) + \zeta(n, m) + \zeta(m + n).$$

The other one is given by the shuffle product formulae, from which it follows, for example,

$$\zeta(m) \cdot \zeta(n) = \sum_{i=0}^{m-1} \binom{n-1+i}{i} \zeta(m-i, n+i) + \sum_{j=0}^{n-1} \binom{m-1+j}{j} \zeta(n-j, m+j).$$

$\square$

1.2. **The KZ equation and the Drinfel'd associator.** In this subsection, we will briefly review the definition of the KZ equation and the Drinfel'd associator. For more detailed information on the KZ equation and the Drinfel'd associator, see [Dr], [F0] and [Kas].

Let $A^{\wedge}_{\mathbf{C}} = \mathbf{C}\langle\langle A, B \rangle\rangle$ be the non-commutative formal power series ring generated by two elements A and B with complex number coefficients.

Definition 1.12. The Knizhnik-Zamolodchikov equation [1] (KZ equation for short) is the differential equation

$$\text{(KZ)} \qquad \frac{\partial G}{\partial u}(u) = \left(\frac{A}{u} + \frac{B}{u-1}\right) \cdot G(u) \qquad ,$$

where $G(u)$ is an analytic function in complex variable u with values in $A^{\wedge}_{\mathbf{C}}$ where 'analytic' means each of whose coefficient is analytic.

[1] This is a special case of the KZ equation for $\mathbf{P}^1(\mathbf{C}) \backslash \{0, 1, \infty\}$ in [Kas].

The equation (KZ) has singularities only at $0, 1$ and ∞. Let $\mathbf{C}'$ be the complement of the union of the real half-lines $(-\infty, 0]$ and $[1, +\infty)$ in the complex plane. This is a simply-connected domain. The equation (KZ) has a unique analytic solution on $\mathbf{C}'$ having a specified value at any given points on $\mathbf{C}'$. Moreover, for the singular points 0 and 1, there exist unique solutions $G_0(u)$ and $G_1(u)$ of (KZ) such that

$$G_0(u) \approx u^A \ (u \to 0), \qquad G_1(u) \approx (1-u)^B \ (u \to 1),$$

where $\approx$ means that $G_0(u)\cdot u^{-A}$ (resp. $G_1(u)\cdot(1-u)^{-B}$) has an analytic continuation in a neighborhood of 0 (resp. 1) with value 1 at 0 (resp. 1). Here, $u^A := exp(Alogu) := 1 + \frac{Alogu}{1!} + \frac{(Alogu)^2}{2!} + \frac{(Alogu)^3}{3!} + \cdots$ and $logu := \int_1^u \frac{dt}{t}$ in $\mathbf{C}'$. In the same way, $(1-u)^B$ is well-defined on $\mathbf{C}'$. Since $G_0(u)$ and $G_1(u)$ are both non-zero unique solutions of (KZ) with the specified asymptotic behaviors, they must coincide with each other up to multiplication from the right by an invertible element of $A^{\wedge}_{\mathbf{C}}$.

Definition 1.13. The Drinfel'd associator [2] is the element $\Phi_{KZ}(A, B)$ of $A^{\wedge}_{\mathbf{C}}$ which is defined by

$$G_0(u) = G_1(u) \cdot \Phi_{KZ}(A, B) \quad .$$

By considering on $(A^{\wedge}_{\mathbf{C}})^{ab}$, the abelianization of $A^{\wedge}_{\mathbf{C}}$, we easily find that $\Phi_{KZ}(A, B) \equiv 1$ on $(A^{\wedge}_{\mathbf{C}})^{ab}$. We note that MZV's appear at each coefficient of the Drinfel'd associator $\Phi_{KZ}(A, B)$. For its explicit formulae, see [F0]Proposition 3.2.3.

2. p-adic multiple polylogarithms and p-adic multiple zeta values

In this section, we shall give the definition of p-adic multiple polylogarithms (§2.2) and p-adic multiple zeta values (§2.3) and state main results in §2.3, which will be proved in §3. The reader will find interesting analogies between §1.1 and §2.3.

2.1. Review of Coleman's p-adic iterated integration theory.

We will review the p-adic iterated integration theory by R. Coleman [C], following A. Besser's reformulation in [Bes1]. This theory will be employed in the analytic continuation of p-adic multiple polylogarithms in §2.3. For other nice expositions of his theory, see [Bes2]§5, [Br]§2.2.1 and [CdS]§2.

[2] To be precise, Drinfel'd defined $\varphi_{KZ}(A, B)$ instead of $\Phi_{KZ}(A, B)$ in [Dr], where $\varphi_{KZ}(A, B) = \Phi_{KZ}(\frac{1}{2\pi i}A, \frac{1}{2\pi i}B)$.

Assumption 2.1. Suppose that $X/\mathcal{O}_{\mathbf{C}_p}$ is a smooth projective and surjective scheme over the ring $\mathcal{O}_{\mathbf{C}_p}$ of integers of $\mathbf{C}_p$, of relative dimension 1 with its generic fiber $X_{\mathbf{C}_p}$ and its special fiber $X_{\overline{\mathbf{F}_p}}$. Let $Y = X - D$ where D is a closed subscheme of X which is relatively etale over $\mathcal{O}_{\mathbf{C}_p}$.

We denote $j : Y_{\overline{\mathbf{F}_p}} \hookrightarrow X_{\overline{\mathbf{F}_p}}$ to be an associated open embedding, where $Y_{\overline{\mathbf{F}_p}}$ is the special fiber of Y and denote the finite set $X(\overline{\mathbf{F}_p}) - Y(\overline{\mathbf{F}_p})$ by $\{e_1, \cdots, e_s\}$. For $0 \leqslant r < 1$, U_r stands for the rigid analytic space [3] obtained by *removing all closed discs of radius r around* e_i *from* $X(\mathbf{C}_p)$ $(1 \leqslant i \leqslant s)$ (see [Bes1]). For a subset $S \subset X(\overline{\mathbf{F}_p})$, we denote its tubular neighborhood (see [Ber]) in $X(\mathbf{C}_p)$ by $]S[$. For any rigid analytic space W, we mean by $A(W)$ the ring of global sections of the sheaf $\mathcal{O}_W^{\mathrm{rig}}$ of rigid analytic functions on W.

Fix $a \in \mathbf{C}_p$. It determines a branch of p-adic logarithm $log^a : \mathbf{C}_p^\times \to \mathbf{C}_p$ ([Bes1]Definition 2.6) which is characterized by $log^a(p) = a$. We call this $a \in \mathbf{C}_p$ by a branch parameter of p-adic logarithm. Define

$$A^a_{\mathrm{loc}} := \prod_{x \in X(\overline{\mathbf{F}_p})} A^a_{log}(U_x), \qquad \Omega^a_{\mathrm{loc}} := \prod_{x \in X(\overline{\mathbf{F}_p})} \Omega^a_{log}(U_x)$$

where

$$A^a_{log}(U_x) := \begin{cases} A(]x[) & x \in Y(\overline{\mathbf{F}_p}), \\ \underset{r \to 1}{\text{ind-lim}} A\Big(]x[\cap U_r\Big)\Big[log^a(z_x)\Big] & x \in \{e_1, \cdots, e_s\}, \end{cases}$$
$$\Omega^a_{log}(U_x) := A^a_{log}(U_x) dz_x \quad .$$

Here z_x means a local parameter

$$z_x :]x[\cap Y(\mathbf{C}_p) \xrightarrow{\sim} \{z \in \mathbf{C}_p \mid 0 < |z|_p < 1\}.$$

We note that $log^a(z_x)$ is a locally analytic function defined on $]x[\cap Y(\mathbf{C}_p)$ whose derivation is $\frac{1}{z_x}$ and it is transcendental over $\underset{r \to 1}{\text{ind-lim}} A\Big(]x[\cap U_r\Big)$ and $\underset{r \to 1}{\text{ind-lim}} A\Big(]x[\cap U_r\Big) \cong \Big\{ f(z) = \sum_{n=-\infty}^{n=\infty} a_n z^n \ (a_n \in \mathbf{C}_p)$ converging for $r < |z|_p < 1$ for some $0 \leqslant r < 1 \Big\}$ (see [Bes1]). We remark that these definitions of $A^a_{log}(U_x)$ and $\Omega^a_{log}(U_x)$ are independent of any choice of local parameters z_x. By taking a component-wise derivative, we obtain a $\mathbf{C}_p$-linear map $d : A^a_{\mathrm{loc}} \to \Omega^a_{\mathrm{loc}}$. Regard $A^\dagger := \Gamma(]X_{\overline{\mathbf{F}_p}}[, j^\dagger \mathcal{O}_{]X_{\overline{\mathbf{F}_p}}[})$ and

[3] As is explained in [Bes1] §2, while the definition of U_r depends on the choice of 'local lifts', the definitions of $A^a_{log}(U_x)$ and $\Omega^a_{log}(U_x)$ (see below) do not.

$\Omega^\dagger := \Gamma(]X_{\overline{\mathbf{F}_p}}[, j^\dagger\Omega^1_{]X_{\overline{\mathbf{F}_p}}[})$ to be a subspace of A^a_{loc} and Ω^a_{loc} respectively (for $j^\dagger$, see [Ber]).

In [C], Coleman constructed an $A^\dagger$-subalgebra A^a_{Col} of A^a_{loc}, which we call the ring of Coleman functions attached to a branch parameter $a \in \mathbf{C}_p$, and a $\mathbf{C}_p$-linear map $\int_{(a)} : A^a_{\mathrm{Col}} \underset{A^\dagger}{\otimes} \Omega^\dagger \to A^a_{\mathrm{Col}} \Big/ \mathbf{C}_p \cdot 1$ satisfying $d\,\big|_{A^a_{\mathrm{Col}}} \circ \int_{(a)} = id_{A^a_{\mathrm{Col}}\otimes\Omega^\dagger}$, which we call the p-adic (Coleman) integration attached to a branch parameter $a \in \mathbf{C}_p$. We often drop the subscript $_{(a)}$.

Actually Coleman's p-adic integration theory is essentially independent of any choice of branches, which may not be well-known, and we will try to explain this fact below:
Suppose that $a, b \in \mathbf{C}_p$. Consider the isomorphisms $\iota_{a,b} : A^a_{\mathrm{loc}} \xrightarrow{\sim} A^b_{\mathrm{loc}}$ and $\tau_{a,b} : \Omega^a_{\mathrm{loc}} \xrightarrow{\sim} \Omega^b_{\mathrm{loc}}$ obtained by replacing each $log^a(z_{e_i})$ by $log^b(z_{e_i})$ for $1 \leqslant i \leqslant s$.

Lemma 2.2. *These maps, $\iota_{a,b}$ and $\tau_{a,b}$, are independent of any choice of a local parameter z_{e_i}.*

Proof . Suppose that z'_{e_i} is another local parameter. Then we check easily that $\iota_{a,b}(z'_{e_i}) = z'_{e_i}$, $\iota_{a,b}(log^a(z'_{e_i})) = log^a(z'_{e_i})$ and $\tau_{a,b}(dz'_{e_i}) = dz'_{e_i}$, from which it follows the lemma. □

The following branch independency principle, which was not stated explicitly in [C], should be one of the important properties of Coleman's p-adic integration theory, but this principle just follows directly from his construction of A^a_{Col}.

Proposition 2.3 (Branch Independency Principle)**.** *Suppose that $a, b \in \mathbf{C}_p$. Then $\iota_{a,b}(A^a_{\mathrm{Col}}) = A^b_{\mathrm{Col}}$, $\tau_{a,b}(A^a_{\mathrm{Col}} \otimes \Omega^\dagger) = A^b_{\mathrm{Col}} \otimes \Omega^\dagger$ and $\iota_{a,b} \circ \int_{(a)} = \int_{(b)} \circ \tau_{a,b} \mod \mathbf{C}_p \cdot 1$. Namely the following diagram is commutative.*

$$\begin{array}{ccc}
A^a_{\mathrm{Col}} \underset{A^\dagger}{\otimes} \Omega^\dagger & \xrightarrow{\tau_{a,b}} & A^b_{\mathrm{Col}} \underset{A^\dagger}{\otimes} \Omega^\dagger \\
{\scriptstyle\int_{(a)}}\big\downarrow & & \big\downarrow{\scriptstyle\int_{(b)}} \\
A^a_{\mathrm{Col}} \Big/ \mathbf{C}_p \cdot 1 & \xrightarrow{\iota_{a,b}} & A^b_{\mathrm{Col}} \Big/ \mathbf{C}_p \cdot 1
\end{array}$$

Proof . It follows directly because both $(A^b_{\mathrm{Col}}, \int_{(b)})$ and $(\iota_{a,b}(A^b_{\mathrm{Col}}), \iota_{a,b} \circ \int_{(a)} \circ \tau^{-1}_{a,b})$ satisfies the same axioms (A)–(F) of logarithmic F-crystals on [C] (see also [CdS]), which is a characterization of $(A^b_{\mathrm{Col}}, \int_{(b)})$. □

Other important properties of Coleman's functions are the uniqueness principle and the functorial property below.

Proposition 2.4 (Uniqueness Principle ; [C]Ch IV). *Let $a \in \mathbf{C}_p$. Let $f \in A^a_{\mathrm{Col}}$ be a Coleman function which is defined on an admissible open subset U of $X(\mathbf{C}_p)$. Suppose that $f|_U \equiv 0$. Then $f \equiv 0$ on $X(\mathbf{C}_p)$.*

Especially this proposition yields the fact that a locally constant Coleman function is globally constant.

Proposition 2.5 (Functorial Property ; [C]Theorem 5.11 and [Bes2]Definition 4.7). *Let $a \in \mathbf{C}_p$. Let (X', Y') be another pair satisfying Assumption 2.1. Suppose that $f : X' \to X$ is a morphism defined over $\mathcal{O}_{\mathbf{C}_p}$ such that $f(Y') \subset Y$. Then the pull-back morphism $f^\sharp : A^a_{\mathrm{loc}} \to A'^a_{\mathrm{loc}}$ induces the morphism $f^* : A^a_{\mathrm{Col}} \to A'^a_{\mathrm{Col}}$ of rings of Coleman functions, where A'^a_{loc} (resp. A'^a_{Col}) stands for A^a_{loc} (resp. A^a_{Col}) for (X', Y').*

Precisely speaking, Coleman showed this theorem in more general situation in [C] Theorem 5.1.

Notation 2.6. Let $a \in \mathbf{C}_p$ and $\omega \in A^a_{\mathrm{Col}} \underset{A^\dagger}{\otimes} \Omega^\dagger$. Then by Coleman's integration theory, there exists (uniquely modulo constant) a Coleman function F_ω such that $\int \omega \equiv F_\omega$ (modulo constant). For $x, y \in]Y(\overline{\mathbf{F}_p})[$, we define $\int_x^y \omega$ to be $F_\omega(y) - F_\omega(x)$. It is clear that $\int_x^y \omega$ does not depend on any choice of F_ω (although it may depend on $a \in \mathbf{C}_p$). If $F_\omega(x)$ and $F_\omega(y)$ make sense for some $x, y \in X(\mathbf{C}_p)$, we also denote $F_\omega(y) - F_\omega(x)$ by $\int_x^y \omega$. When we let y vary, we regard $\int_x^y \omega$ as the Coleman function which is characterized by $dF_\omega = \omega$ and $F_\omega(x) = 0$.

2.2. **Analytic continuation of p-adic multiple polylogarithms.** We will define p-adic multiple polylogarithms to be Coleman functions which admits an expansion around 0 similar to the complex case.

Let $k_1, \cdots, k_m \in \mathbf{N}$ and $z \in \mathbf{C}_p$. Consider the following series

$$Li_{k_1,\cdots,k_m}(z) = \sum_{\substack{0<n_1<\cdots<n_m \\ n_i \in \mathbf{N}}} \frac{z^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \quad . \tag{2.1}$$

Lemma 2.7. *This series $Li_{k_1,\cdots,k_m}(z)$ converges on the open unit disk $D(0:1) = \{z \in \mathbf{C}_p \mid |z|_p < 1\}$ around 0 with radius 1.*

Proof . Easy. □

Lemma 2.8. *Let $z \in \mathbf{C}_p$ such that $|z|_p < 1$. Then*

$$\frac{d}{dz} Li_{k_1,\cdots,k_m}(z) = \begin{cases} \frac{1}{z} Li_{k_1,\cdots,k_m - 1}(z) & k_m \neq 1, \\ \frac{1}{1-z} Li_{k_1,\cdots,k_{m-1}}(z) & k_m = 1, \end{cases}$$

$$\frac{d}{dz} Li_1(z) = \frac{1}{1-z} \quad .$$

Proof . It follows from a direct calculation. □

Lemma 2.7 and Lemma 2.8 are p-adic analogue of Lemma 1.3 and Lemma 1.4 respectively.

From now on, we fix a branch parameter $a \in \mathbf{C}_p$ and employ Coleman's p-adic integration theory attached to this branch parameter $a \in \mathbf{C}_p$ for $X = \mathbf{P}^1_{\mathcal{O}_{\mathbf{C}_p}}$ and $Y = Spec\ \mathcal{O}_{\mathbf{C}_p}[t, \frac{1}{t}, \frac{1}{1-t}]$.

Definition 2.9. We define recursively the (one variable) p-adic multiple polylogarithm (p-adic MPL for short) $Li^a_{k_1,\cdots,k_m}(z) \in A^a_{\mathrm{Col}}$ attached to $a \in \mathbf{C}_p$ which is the Coleman function characterized below:

$$Li^a_{k_1,\cdots,k_m}(z) := \begin{cases} \int_0^z \frac{1}{t} Li^a_{k_1,\cdots,k_m-1}(t)dt & k_m \neq 1, \\ \int_0^z \frac{1}{1-t} Li^a_{k_1,\cdots,k_{m-1}}(t)dt & k_m = 1, \end{cases}$$

$$Li^a_1(z) = -log^a(1-z) := \int_0^z \frac{dt}{1-t} \quad .$$

Remark 2.10. (1) It is easy to see that $Li^a_{k_1,\cdots,k_m}(z) = \sum_{0<n_1<\cdots<n_m} \frac{z^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}$ if $|z|_p < 1$.

(2) Because we get $\frac{dt}{t}, \frac{dt}{1-t} \in A^a_{\mathrm{Col}}$ and we know that $Li^a_{k_1,\cdots,k_m}(z)$ is analytic on $|z|_p < 1$ and takes value 0 at $z = 0$, it is easy to see that each p-adic MPL is well defined on A^a_{Col}.

(3) Our construction of p-adic MPL is just a multiple analogue of Coleman's construction [C] of p-adic polylogarithm $\ell_k(z)$. His p-adic polylogarithm $\ell_k(z)$ can be written as $Li^a_k(z)$ in our notation.

The following is a p-adic version of Lemma 1.5.

Proposition 2.11. *The p-adic MPL $Li^a_{k_1,\cdots,k_m}(t)$ is locally analytic on $\mathbf{P}^1(\mathbf{C}_p)\backslash\{1,\infty\}$. More precisely, $Li^a_{k_1,\cdots,k_m}(t)\ \big|_{]0[} \in A(]0[)$, $Li^a_{k_1,\cdots,k_m}(t)\ \big|_{]1[} \in A(]1[)\big[log^a(t-1)\big]$ and $Li^a_{k_1,\cdots,k_m}(t)\ \big|_{]\infty[} \in A(]\infty[)\big[log^a(\frac{1}{t})\big]$.*

Proof . By construction, we can prove the claim inductively. □

This proposition means that the series (2.1) can be analytically continued to $\mathbf{P}^1(\mathbf{C}_p)\backslash\{1,\infty\}$, although the complex MPL cannot be analytically continued to $\mathbf{P}^1(\mathbf{C})\backslash\{0,1,\infty\}$ but only to its universal unramified covering $\widetilde{\mathbf{P}^1(\mathbf{C})}\backslash\{0,1,\infty\}$ instead by Lemma 1.5. We note that the p-adic MPL admits *uncountably* infinite branches which correspond to branches of p-adic logarithms $log^a(z)$ although complex MPL admits *countably* infinite branches. We call $Li^a_{k_1,\cdots,k_m}(z)$ the branch of p-adic MPL corresponding to $a \in \mathbf{C}_p$.

2.3. **p-adic multiple zeta values and main results.** We will state main results of this paper, whose proof will be given in §3.3 and §3.4 and will introduce p-adic multiple zeta values, whose definitions themselves are highly non-trivial.

Notation 2.12. Let $\alpha \in \mathbf{C}_p$ and let $f(z)$ be a function defined on $\mathbf{C}_p$. We denote $\lim'_{z\to\alpha} f(z)$ to be $\lim_{n\to\infty} f(z_n)$ if this limit converges to the same value for any sequence $\{z_n\}_{n=1}^{\infty}$ which satisfies $z_n \to \alpha$ in $\mathbf{C}_p$ and $e(\mathbf{Q}_p(z_1, z_2, \cdots)/\mathbf{Q}_p) < \infty$ (which means that the field generated by $z_1, z_2, \cdots$ over $\mathbf{Q}_p$ is a finitely ramified (possibly infinite) extension field over $\mathbf{Q}_p$). If the latter limit converges (resp. does not converge) to the same value, we call $\lim'_{z\to\alpha} f(z)$ converges (resp. diverges).

Theorem 2.13. *Fix $k_1, \cdots, k_m \in \mathbf{N}$ and a prime p. Then the statement whether $\lim'_{\substack{z\to 1\\ z\in\mathbf{C}_p-\{1\}}} Li^a_{k_1,\cdots,k_m}(z)$ converges or diverges [4] on $\mathbf{C}_p$ is independent of any choice of branch parameter $a \in \mathbf{C}_p$. Moreover if it converges on $\mathbf{C}_p$, this limit value is independent of any choice of branch parameter $a \in \mathbf{C}_p$.*

Proof . Since $Li^a_{k_1,\cdots,k_m}(z)$ is locally analytic on $\mathbf{P}^1(\mathbf{C}_p)\backslash\{1,\infty\}$ by Proposition 2.11, its $A^a_{log}(]1[)$-component can be written as follows.

$$\begin{aligned} Li^a_{k_1,\cdots,k_m}(z) =& f_0(z-1) + f_1(z-1)log^a(z-1) + f_2(z-1)(log^a(z-1))^2 \\ &+ \cdots\cdots + f_m(z-1)(log^a(z-1))^m, \end{aligned}$$

where $f_i(z) \in A(D(0:1))$ for $i = 0, \cdots, m$. By Proposition 2.3, we see that these $f_i(z)$'s are independent of any choice of branch parameter $a \in \mathbf{C}_p$. Saying $\lim'_{z\to 1} Li^a_{k_1,\cdots,k_m}(z)$ converges is equivalent to saying $f_i(0) = 0$ for all $i = 1, \cdots, m$ by Lemma 2.14 and Lemma 2.15, which is a statement independent of any choice of branch parameter $a \in \mathbf{C}_p$. Thus we get the first half of this theorem. If $\lim'_{z\to 1} Li^a_{k_1,\cdots,k_m}(z)$ converges, then $\lim'_{z\to 1} Li^a_{k_1,\cdots,k_m}(z) = f_0(0)$. Since $f_0(z)$ was independent of any choice of branches, the second half of this theorem follows. $\square$

Lemma 2.14. *For $n \geqslant 0$, $\lim'_{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}} \epsilon(log^a\epsilon)^n = 0$.*

Proof . If $n = 0$, it is clear.
If $n = 1$, suppose that $\epsilon_n \in L$ $(n \geqslant 1)$ and $\epsilon_n \to 0$ as $n \to \infty$, where L is a finitely ramified (possibly infinite) extension of $\mathbf{Q}_p$ with ramification

[4] This makes sense because p-adic MPL's are locally analytic on $\mathbf{P}^1(\mathbf{C}_p)\backslash\{1,\infty\}$ by Proposition 2.11.

index $e_L(<\infty)$ and a uniformizer π_L. Take $c_L\in\mathbf{N}$ such that $p^{c_L}>e_L$. Decompose $\epsilon_n=u_n\cdot\pi_L^{r_n}$ where $u_n\in\mathcal{O}_L^\times$ and $r_n\in\mathbf{Z}$. Take $s_n\in\mathbf{N}$ such that $(s_n,p)=1$ and $u_n^{s_n}\equiv 1 \mod \pi_L\mathcal{O}_L$. Put $\alpha_n:=u_n^{s_n}-1\in\pi_L\mathcal{O}_L$. Then $(u_n^{s_n})^{p^{c_L}}=(1+\alpha_n)^{p^{c_L}}\equiv 1+\alpha_n^{p^{c_L}}\equiv 1 \mod p\mathcal{O}_{\mathbf{C}_p}$. Therefore $log^a u_n=\frac{1}{s_n\cdot p^{c_L}}log^a(u_n^{s_n\cdot p^{c_L}})\in\frac{1}{p^{c_L}}\mathcal{O}_{\mathbf{C}_p}$. So we get

$$\lim_{n\to\infty}\epsilon_n log^a\epsilon_n=\lim_{n\to\infty}\{\epsilon_n log^a u_n+\epsilon_n r_n log^a\pi_L\}=\lim_{n\to\infty}\epsilon_n log^a u_n=0.$$

In a similar way, we can prove the case for $n>1$. □

Lemma 2.15. *Let $a\in\mathbf{C}_p$, $l\geqslant 0$ and $g(z)=\sum_{k=0}^{l}a_k\big(log^a(z)\big)^k$ ($a_k\in\mathbf{C}_p$). Then $\lim'_{z\to1}g(z)$ converges if and only if $a_k=0$ for $1\leqslant k\leqslant l$.*

Proof . Take $z_n=\alpha^n$ such that $|\alpha|_p<1$ and $log^a(\alpha)\neq 0$. Then we get the claim by an easy calculation. □

Remark 2.16. (1) As for another proof of the last statement of Theorem 2.13 for $k_m>1$, see Remark 3.27.

(2) It is striking that $\lim'_{\epsilon\to0}Li^a_{k_1,\cdots,k_m}(1-\epsilon)$ does not depend on any choice of branch parameter $a\in\mathbf{C}_p$ although $Li^a_{k_1,\cdots,k_m}(1-\epsilon)$ takes whole vales on $\mathbf{C}_p$ if we fix ϵ ($0<|\epsilon|_p<1$) and let a vary on $\mathbf{C}_p$.

Definition 2.17. For any index $(k_1,\cdots,k_m)$ whose $\lim'_{z\to1}Li^a_{k_1,\cdots,k_m}(z)$ converges, we define the corresponding p-adic multiple zeta vale (p-adic MZV for short) $\zeta_p(k_1,\cdots,k_m)$ to be its limit in $\mathbf{C}_p$, i.e.

$$\zeta_p(k_1,\cdots,k_m):=\lim_{\substack{z\to1\\ z\in\mathbf{C}_p-\{1\}}}{}' \; Li^a_{k_1,\cdots,k_m}(z)\in\mathbf{C}_p \qquad \text{if it converges.}$$

If this limit diverges for $(k_1,\cdots,k_m)$, we do not give a definition of the corresponding p-adic MZV $\zeta_p(k_1,\cdots,k_m)$.

We note that this definition of p-adic MZV is independent of any choice of branches by Theorem 2.13.

Theorem 2.18. *If $k_m>1$, $\lim'_{\substack{z\to1\\ z\in\mathbf{C}_p-\{1\}}} Li^a_{k_1,\cdots,k_m}(z)$ always converges on $\mathbf{C}_p$.*

Thus we get a definition of p-adic MZV $\zeta_p(k_1,\cdots,k_m)$ for $k_m>1$. This theorem is a p-adic analogue of Lemma 1.6 and it will be proved in §3.3.

Examples 2.19. Coleman made the following calculation in [C] (stated in Ch I (4) and proved in Ch VII):

$$\lim_{z\to1}{}' Li^a_n(z)=\frac{p^n}{p^n-1}L_p(n,\omega^{1-n}) \quad \text{for } n>1. \tag{2.2}$$

Here L_p is the Kubota-Leopoldt p-adic L-function and ω is the Teichmüller character. In particular this formula (2.2) shows that this limit value is actually independent of any choice of branch parameter $a \in \mathbf{C}_p$ although this fact is a special case of Theorem 2.13. We remark that, in the case of p-adic polylogarithms, this branch independency also follows from the so-called distribution relation ([C] Proposition 6.1). By (2.2), we get

$$\zeta_p(n) = \frac{p^n}{p^n - 1} L_p(n, \omega^{1-n}) \quad \text{for } n > 1. \tag{2.3}$$

(a) When n is even (i.e. $n = 2k$ for some $k \geqslant 1$), by (2.3) we get the equality

$$\zeta_p(2k) = 0.$$

We will see that this equality proved arithmetically here will be also deduced from geometric identities, 2-cycle relation and 3-cycle relation, among p-adic MZV's, proved in [F3].

(b) On the other hand, when n is odd (i.e. $n = 2k+1$ for some $k \geqslant 0$), it does not look so easy to show $\zeta_p(2k+1) \neq 0$:
Suppose that p is an odd prime. Then by [KNQ] Theorem 3.1, we see that saying $L_p(2k+1, \omega^{-2k}) \neq 0$ is equivalent to saying

$$H^2_{\mathrm{et}}(\mathbf{Z}, \mathbf{Q}_p/\mathbf{Z}_p(-k)) = 0, \tag{L$_{2k+1}$}$$

which is one of standard conjectures [5] in Iwasawa theory and a higher version of Leopoldt conjecture (cf. [KNQ] Remark 3.2.(ii)). We know that $\zeta_p(2k+1) \neq 0$ in the case where p is regular or $(p-1)|2k$ by [Sou]§3.3.

We have another statement which is equivalent to saying $\zeta_p(2k+1) \neq 0$.

Remark 2.20. Suppose that p is an odd prime. Let G denote the standard Iwasawa module for $\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}$ (μ_{p^∞}: the group of roots of unity whose order is a power of p), i.e. $G = \pi_1\left(\mathbf{Z}[\mu_{p^\infty}, \frac{1}{p}]\right)^{p,\mathrm{ab}}$. Let I denote the inertia subgroup of one prime $\mathfrak{p}$ in $\mathbf{Q}(\mu_{p^\infty})$. Soulé [Sou] has constructed a specific non-zero character $\chi^{\text{Soulé}}_{p,m} \in Hom_{\mathbf{Z}_p}(G, \mathbf{Z}_p(m))$ for $m \geqslant 1$: odd. Let $\mathcal{G}$ denote the standard Iwasawa module for $\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p$, i.e. $\mathcal{G} = \pi_1(\mathbf{Q}_p(\mu_{p^\infty}))^{p,\mathrm{ab}}$. Let $\mathcal{I}$ denote the inertia subgroup of $\mathcal{G}$. For $m \geqslant 1$, the Coates-Wiles homomorphism $\chi^{\mathrm{CW}}_{p,m}$ is a specific non-zero element in $Hom_{\mathbf{Z}_p}(\mathcal{I}, \mathbf{Z}_p(m))$. According to [KNQ],

[5] In [KNQ], (L$_{2k+1}$) was denoted by (C$_k$).

Coleman—using his reciprocity law—has shown the following formula for all odd $m > 1$:

$$\chi_{p,m}^{\text{Soulé}} \big|_{\mathcal{I}} = (p^{m-1}-1)\cdot L_p(m,\omega^{1-m})\cdot \chi_{p,m}^{\text{CW}}. \tag{2.4}$$

Here we identify $\mathcal{I}$ with the image of I. By (2.4), we get the following statement in algebraic number theory side which is equivalent to saying $\zeta_p(2k+1)\neq 0$ (equivalently (L_{2k+1})):

(P_{2k+1})
the prime ideal $\mathfrak{p}$ ramify on the kernel field $K_{p,m}^{\text{Soulé}}$ of $\chi_{p,m}^{\text{Soulé}}$.

The author guess that problems on p-adic MZV's related to p-adic transcendental number theory, such as the p-adic Apéry's result $\zeta_p(3)\notin \mathbf{Q}$, could be translated into problems on the algebraic number theory.

As for a p-adic analogue of Lemma 1.7, at present, we have nothing to say except the following.

Note 2.21. The limit $\lim\limits_{\substack{z\to 1\\ z\in\mathbf{C}_p-\{1\}}}{}' \; Li^a_{k_1,\cdots,k_m}(z)$ sometimes converges and sometimes diverges on $\mathbf{C}_p$ for $k_m=1$.

For example, see Example 2.23.(a) and (b) below.

Theorem 2.22. *Suppose that* $\lim\limits_{\substack{z\to 1\\ z\in\mathbf{C}_p-\{1\}}}{}' \; Li^a_{k_1,\cdots,k_m}(z)$ *converges on* $\mathbf{C}_p$ *for* $k_m=1$*. Then it converges to a p-adic version of the regularized MZV, i.e.*

$$\zeta_p(k_1,\cdots,k_{m-1},1)=(-1)^m I_p(W) \text{ where } W=BA^{k_{m-1}-1}B\cdots A^{k_1-1}B.$$

See §3.2 for $I_p(W)$ and Remark 3.29(2) for the regularized p-adic MZV. This theorem will be proved in §3.4. Therefore p-adic MZV $\zeta_p(k_1,\cdots,k_m)$ for $k_m=1$ can be written as a $\mathbf{Q}$-linear combination of p-adic MZV's corresponding to the same weight indexes with $k_m>1$.

Examples 2.23. (a) $\lim\limits_{z\to 1}{}' Li^a_{2,1}(z)$ converges to $-2\zeta_p(1,2)$, i.e. $\zeta_p(2,1)=-2\zeta_p(1,2)$. This follows from the functional equation in Example 3.37.(a), $\zeta_p(2)=0$ by Example 2.19.(a) and Lemma 2.14.

(b) $\lim\limits_{z\to 1}{}' Li^a_{3,1}(z)$ diverges if and only if $\zeta_p(3)\neq 0$ (equivalently if and only if p satisfies the 3rd Leopoldt conjecture (L_3) above).

Suppose that 3rd Leopoldt conjecture (L_3) fails at a prime p. Then we get $\zeta_p(3,1)=-2\zeta_p(1,3)-\zeta_p(2,2)$ for this prime p. This follows from the functional equation in Example 3.37.(b) combined with Lemma 2.14.

(c) We will show many identities between p-adic MZV's in [F3], from which we will deduce, for example, $\zeta_p(3)=\zeta_p(1,2)$ and $\zeta_p(1,3)=\zeta_p(2,2)=\zeta_p(1,1,2)=0$.

Remark 2.24. The author guesses that to know whether $\lim'_{\substack{z\to 1\\ z\in\mathbf{C}_p-\{1\}}} Li^a_{k_1,\cdots,k_m}(z)$ converges or diverges might be to tell something deep in number theory, such as Example 2.23.(b).

Those p-adic MZV's *were* defined to be elements of $\mathbf{C}_p$, but actually we can say more.

Theorem 2.25. *All p-adic MZV's are p-adic numbers, i.e. $\zeta_p(k_1,\cdots,k_m)\in\mathbf{Q}_p$.*

Proof . Suppose that $\lim'_{\substack{z\to 1\\ z\in\mathbf{C}_p-\{1\}}} Li^a_{k_1,\cdots,k_m}(z)$ converges. Recall that p-adic MPL $Li^a_{k_1,\cdots,k_m}(z)$ $(a\in\mathbf{C}_p)$ is an iterated integral of $\frac{dt}{t}$ and $\frac{dt}{1-t}$ which is a rational 1-form defined over $\mathbf{Q}_p$ and notice that $Li^a_{k_1,\cdots,k_m}(z)\in\mathbf{Q}_p$ for all $z\in p\mathbf{Z}_p$. Then from the Galois equivariancy stated in [BdJ] Remark 2.3, it follows that $Li^a_{k_1,\cdots,k_m}(z)$ is $Gal(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$-invariant for $z\in\mathbf{P}^1(\mathbf{Q}_p)\backslash\{1,\infty\}$ if we take $a\in\mathbf{Q}_p$. Therefore in this case, we get $Li^a_{k_1,\cdots,k_m}(z)\in\mathbf{Q}_p$ for $z\in\mathbf{P}^1(\mathbf{Q}_p)\backslash\{1,\infty\}$. Thus we get $\lim_{\substack{z\to 1\\ z\in\mathbf{Q}_p-\{1\}}} Li^a_{k_1,\cdots,k_m}(z)\in\mathbf{Q}_p$, which yields the theorem (Recall that this limit is independent of any choice of branch parameter $a\in\mathbf{C}_p$ by Theorem 2.13). □

It may be better to say that this theorem is a p-adic version of Lemma 1.9. The author poses the following question, which he wants to study in the future.

Question 2.26. Are all p-adic MZV's p-adic integers? Namely $\zeta_p(k_1,\cdots,k_m)\in\mathbf{Z}_p$ for all primes p?

Definition 2.27. For each natural number w, let $Z^{(p)}_w$ be the finite dimensional $\mathbf{Q}$-linear subspace of $\mathbf{Q}_p$ generated by all p-adic MZV's of indices with weight w, and put $Z^{(p)}_0=\mathbf{Q}$. Define $Z^{(p)}_{\bullet}$ to be the *formal direct sum* of $Z^{(p)}_w$ for all $w\geqslant 0$: $Z^{(p)}_{\bullet}:=\bigoplus_{w\geqslant 0} Z^{(p)}_w$.

By Theorem 2.22, we see that $Z^{(p)}_w=\langle\zeta_p(k_1,\cdots,k_m)\mid k_1+\cdots+k_m=w, k_m>1, m\in\mathbf{N}\rangle_{\mathbf{Q}}\subset\mathbf{Q}_p$

Theorem 2.28. *The graded $\mathbf{Q}$-vector space has a structure of $\mathbf{Q}$-algebra, i.e. $Z^{(p)}_a\cdot Z^{(p)}_b\subseteq Z^{(p)}_{a+b}$ for $a,b\geqslant 0$.*

This is a p-adic analogue of Lemma 1.11, whose proof will be given in §3.4. Unfortunately we do not have so simple proof as Lemma 1.11. Our proof is based on showing the shuffle product formulae (Corollary 3.42) coming from the shuffle-like multiplication of iterated integrals, from which it follows, for example,

$$\zeta_p(m)\cdot\zeta_p(n)=\sum_{i=0}^{m-1}\binom{n-1+i}{i}\zeta_p(m-i,n+i)+\sum_{j=0}^{n-1}\binom{m-1+j}{j}\zeta_p(n-j,m+j).$$

In [BF], we shall discuss on the harmonic product formulae [H] coming from the shuffle-like multiplication of series in Definition 1.1, from which it should follow, for example,

$$\zeta_p(m)\cdot\zeta_p(n)=\zeta_p(m,n)+\zeta_p(n,m)+\zeta_p(m+n).$$

We note that the validity of the harmonic product formulae for p-adic MZV's is non-trivial because we do not have a series expansion of p-adic MZV such as §0 (0.1).

As for Question 2.26, the author guesses that we need a nice definition of p-adic multiple zeta (or L-)functions.

Remark 2.29. Since the p-adic L-function is related to the Bernoulli numbers, the author expects that the multiple Bernoulli number (MBN for short) $B^n_{k_1,\cdots,k_m}\in\mathbf{Q}$ ($k_1,\cdots,k_m\in\mathbf{Z}$, $n\in\mathbf{N}$) given by the following generating series

$$\frac{Li_{k_1\cdots,k_m}(1-e^{-x})}{1-e^{-x}}=:\sum_{n=0}^{\infty}B^n_{k_1,\cdots,k_m}\frac{x^n}{n!}\qquad,$$

where $Li_{k_1\cdots,k_m}(z)$ and e^z means the following formal power series

$$\sum_{\substack{0<n_1<\cdots<n_m\\ n_i\in\mathbf{N}}}\frac{z^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}\in\mathbf{Q}[[z]]\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{z^n}{n!}\in\mathbf{Q}[[z]]$$

respectively, would help to describe a p-adic behavior of p-adic MZV's and constructions of p-adic multiple zeta (or L-)functions above. We remark that this definition of MBN is just a multiple version of that of the poly-Bernoulli number in [AK] and [Kan] (especially B^1_n is the usual Bernoulli number). We stress that the definition of MBN is independent of any prime p.

3. The p-adic KZ equation

In this section, we will introduce and consider the p-adic KZ equation. We will give the definition of the p-adic Drinfel'd associator $\Phi^p_{KZ}(A,B)$ in §3.1. In §3.2 (resp. §3.3), we will give an explicit formula of a certain fundamental solution $G_0^p(z)$ of the p-adic KZ equation (resp. an explicit formula of $\Phi^p_{KZ}(A,B)$). In §3.4, we will show the functional equation among p-adic MPL's and will give proofs of Theorem 2.22 and Theorem 2.28.

3.1. The p-adic Drinfel'd associator.

Notation 3.1. Let $A^\wedge_{\mathbf{C}_p}=\mathbf{C}_p\langle\langle A,B\rangle\rangle$ be the non-commutative formal power series ring with $\mathbf{C}_p$ coefficients generated by two elements A and B.

Definition 3.2. The p-adic Knizhnik-Zamolodchikov equation (p-adic KZ equation for short) is the differential equation

$$\frac{\partial G}{\partial u}(u)=(\frac{A}{u}+\frac{B}{u-1})\cdot G(u) \qquad , \tag{KZ^p}$$

where $G(u)$ is an analytic function in variable $u\in\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$ with values in $A^\wedge_{\mathbf{C}_p}$ where 'analytic' means each of whose coefficient is locally p-adic analytic.

Unfortunately, because $\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$ is topologically totally disconnected, the equation (KZ^p) does not have a unique solution on $\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$ even locally as in the complex analytic function case. But fortunately we get the following nice property on Coleman functions.

Proposition 3.3. *Fix $a\in\mathbf{C}_p$, $z_0\in\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$ and $g_0\in A^\wedge_{\mathbf{C}_p}$. Then there exists a unique solution $H^a(u)\in A^a_{\mathrm{Col}}\widehat{\otimes}A^\wedge_{\mathbf{C}_p}$ of (KZ^p) which satisfies $H^a(z_0)=g_0$. Here $A^a_{\mathrm{Col}}\widehat{\otimes}A^\wedge_{\mathbf{C}_p}$ means the non-commutative two variable formal power series ring with A^a_{Col}-coefficients, i.e. $A^a_{\mathrm{Col}}\widehat{\otimes}A^\wedge_{\mathbf{C}_p}=A^a_{\mathrm{Col}}\langle\langle A,B\rangle\rangle$.*

Proof . This immediately follows from Theorem 3.4 by taking $H_a(u)=G_0^a(u)\cdot G_0^a(z_0)^{-1}\cdot g_0$. $\square$

Theorem 3.4. *Fix $a\in\mathbf{C}_p$. Then there exists a unique solution $G_0^a(u)\in A^a_{\mathrm{Col}}\widehat{\otimes}A^\wedge_{\mathbf{C}_p}$ of (KZ^p) which is locally analytic on $\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$ and satisfies $G_0^a(u)\approx u^A$ $(u\to 0)$.*

Here $u^A:=1+\frac{log^a(u)}{1!}A+\frac{(log^a(u))^2}{2!}A^2+\cdots$ and $G_0^a(u)\approx u^A$ $(u\to 0)$ means that the $A^a_{log}(]0[)\widehat{\otimes}A^\wedge_{\mathbf{C}_p}$-component of $P^a(u):=G_0^a(u)\cdot u^{-A}=G_0^a(u)\cdot\left\{1-\frac{log^a(u)}{1!}A+\frac{(log^a(u))^2}{2!}A^2-\frac{(log^a(u))^3}{3!}A^3+\cdots\right\}$ lies on $1+A(]0[)\widehat{\otimes}A^\wedge_{\mathbf{C}_p}\cdot A+A(]0[)\widehat{\otimes}A^\wedge_{\mathbf{C}_p}\cdot B$ and takes value $1\in A^\wedge_{\mathbf{C}_p}$ at $u=0$.

Lemma 3.5. *Fix $a \in \mathbf{C}_p$. Let $G(u)$ and $H(u)$ be solutions of (KZ^p) in $A^a_{\mathrm{Col}}\widehat{\otimes}A^\wedge_{\mathbf{C}_p}$. Suppose that $H(u)$ is invertible. Then $H(u)^{-1}G(u)$ is a constant function, i.e. an element of $A^\wedge_{\mathbf{C}_p}$.*

Proof .

$$\begin{aligned}\frac{d}{du}H(u)^{-1}G(u) &= -H(u)^{-1}\cdot\frac{d}{du}H(u)\cdot H(u)^{-1}G(u) + H(u)^{-1}\frac{d}{du}G(u)\\ &= -H(u)^{-1}(\frac{A}{u}+\frac{B}{u-1})H(u)\cdot H(u)^{-1}G(u) + H(u)^{-1}(\frac{A}{u}+\frac{B}{u-1})G(u)\\ &= -H(u)^{-1}(\frac{A}{u}+\frac{B}{u-1})G(u) + H(u)^{-1}(\frac{A}{u}+\frac{B}{u-1})G(u) = 0\ .\end{aligned}$$

Therefore $H(u)^{-1}G(u) \in A^a_{\mathrm{Col}}\widehat{\otimes}A^\wedge_{\mathbf{C}_p}$ is constant by Proposition 2.4. □

Proof of Theorem 3.4.

Uniqueness: Suppose that $H^a_0(u)$ is another solution satisfying above properties. Then $H^a_0(u)$ is invertible (i.e. $H^a_0(u) \in (A^a_{\mathrm{Col}}\widehat{\otimes}A^\wedge_{\mathbf{C}_p})^\times$) because its constant term is $1 \in A^a_{\mathrm{Col}}$. By Lemma 3.5, there exists a unique constant series $c \in A^\wedge_{\mathbf{C}_p}$ such that $G^a_0(u) = H^a_0(u)\cdot c$. By the assumption, we get $u^A\cdot c\cdot u^{-A} \to 1$ as $u \to 0$, from which we can deduce $c = 1$.

Existence: By substituting $G(u) = P(u)\cdot u^A$ into (KZ^p), we get

$$\frac{dP}{du}(u) = \Big[\frac{A}{u}, P(u)\Big] + \frac{B}{u-1}P(u)\ . \tag{3.1}$$

At first, we construct a local solution $P_0(u)$ of (3.1) which lies on $1 + A(]0[)\widehat{\otimes}A^\wedge_{\mathbf{C}_p}\cdot A + A(]0[)\widehat{\otimes}A^\wedge_{\mathbf{C}_p}\cdot B$ and takes value $1 \in A^\wedge_{\mathbf{C}_p}$ at $u = 0$ by induction with respect to the degree of monomial (i.e. the weight of word). By Kiehl's theorem (see [Sh] Theorem 1.3.1, for example), we get $H^1(]0[, \mathcal{O}_{]0[}) = 0$, which means that all rigid analytic one forms on $]0[$ are integrable (because $dim\ ]0[= 1$). Therefore, by induction, we get that, to show the existence of such solution $P_0(u)$ of (3.1) above, it is enough to show the existence of local solution of (3.1) which is analytic at $u = 0$, takes value on $1 + A^\wedge_{\mathbf{C}_p}\cdot A + A^\wedge_{\mathbf{C}_p}\cdot B$ and takes value $1 \in A^\wedge_{\mathbf{C}_p}$ at $u = 0$ because $u = 0$ is only one simple pole of (3.1) on $]0[$. But this is a local property, we can say more strongly that it is enough to show the existence of formal solution of (3.1) in $\mathbf{Q}[[u]]\langle\langle A, B\rangle\rangle$ which lies on $1+\mathbf{Q}[[u]]\langle\langle A, B\rangle\rangle\cdot A+\mathbf{Q}[[u]]\langle\langle A, B\rangle\rangle\cdot B$ and takes value $1 \in \mathbf{Q}\langle\langle A, B\rangle\rangle$ at $u = 0$.

In the complex case, it was shown in [Dr] (see also [Kas] Ch XIX) that there exists a unique local solution $Q(u) \in 1 + \Big(\mathbf{C}[[u]]\langle\langle A, B\rangle\rangle\Big)\cdot A + \Big(\mathbf{C}[[u]]\langle\langle A, B\rangle\rangle\Big)\cdot B$ of (3.1) which takes value $1 \in A^\wedge_{\mathbf{C}}$ at $u = 0$

and each of whose coefficients is analytic in a neighborhood of $u=0$. This solution $Q(u)$ was constructed inductively. From its construction, it follows immediately that $Q(u)$ lies on $\mathbf{Q}[[u]]\langle\langle A,B\rangle\rangle$. Therefore this determines a required local solution $P_0(u)$ in $1+A(]0[)\widehat{\otimes}A^{\wedge}_{\mathbf{C}_p}\cdot A+A(]0[)\widehat{\otimes}A^{\wedge}_{\mathbf{C}_p}\cdot B$.

Secondly we construct a required solution G_0^a of (KZ^p) in Theorem 3.4 from the above local solution $P_0(u)$ of (3.1). Since $\frac{du}{u}$ and $\frac{du}{1-u}\in A^a_{\mathrm{Col}}$, we can show that actually $P_0(u)$ lies on $\left(A^a_{\mathrm{Col}}\mid_{]0[}\cap A(]0[)\right)\langle\langle A,B\rangle\rangle$ because each coefficients of $P_0(u)$ is an iterated integration of Coleman functions. Then by the uniqueness property of Coleman functions in Proposition 2.4, there exists a unique Coleman function $P^a(u)\in A^a_{\mathrm{Col}}$ such that $P^a(u)\mid_{]0[}=P_0(u)$. By putting $G_0^a(u)=P^a(u)\cdot u^A$, we get a required solution in Theorem 3.4.

Finally we check its local analyticity on $]0[$, $]1[$ and $]\infty[$. Since the constant term of $G_0^a\in A^a_{\mathrm{Col}}\langle\langle A,B\rangle\rangle$ is $1\in A^a_{\mathrm{Col}}$, we can show inductively that $G_0^a\mid_{]0[}\in A^a_{log}(U_0)\langle\langle A,B\rangle\rangle$ lies on $\left(A(]0[)[log^a(u)]\right)\langle\langle A,B\rangle\rangle$, $G_0^a\mid_{]1[}\in A^a_{log}(U_1)\langle\langle A,B\rangle\rangle$ lies on $\left(A(]1[)[log^a(u-1)]\right)\langle\langle A,B\rangle\rangle$ and $G_0^a\mid_{]\infty[}\in A^a_{log}(U_\infty)\langle\langle A,B\rangle\rangle$ lies on $\left(A(]\infty[)[log^a(\frac{1}{u})]\right)\langle\langle A,B\rangle\rangle$. □

Proposition 3.6. *Let $a,b\in\mathbf{C}_p$. Then $\iota_{a,b}(G_0^a)=G_0^b$.*

Proof . This follows from the unique characterization of G_0^b in Theorem 3.4. □

Proposition 3.7. *Fix $a\in\mathbf{C}_p$. Then there exists a unique solution $G_1^a(u)\in A^a_{\mathrm{Col}}\widehat{\otimes}A^{\wedge}_{\mathbf{C}_p}$ of (KZ^p) which is locally analytic on $\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$ and satisfies $G_1^a(u)\approx(1-u)^B$ ($u\to 1$).*

Here the meanings of notations $(1-u)^B$ and $G_1^a(u)\approx(1-u)^B$ ($u\to1$) is similar to those of u^A and $G_0^a(u)\approx u^A$ ($u\to 0$) in Theorem 3.4.

Proof . By a similar argument to Theorem 3.4, we get the claim. □

Proposition 3.8. $G_1^a(u)=G_0^a(B,A)(1-u)$.

Here for any $g\in A^{\wedge}_{\mathbf{C}_p}$, $g(B,A)$ stands for the image of g by the automorphism $A^{\wedge}_{\mathbf{C}_p}$ induced from $A\mapsto B$, $B\mapsto A$ and, for $f(u)\in A^a_{\mathrm{loc}}$, $f(1-u)$ means its image by the algebra homomorphism $\tau^\sharp: A^a_{\mathrm{loc}}\to A^a_{\mathrm{loc}}$ induced from the automorphism $\tau: t\mapsto 1-t$ of $\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$.

Proof . By the functorial property of Coleman functions (Proposition 2.5), $\tau^\sharp(A^a_{\mathrm{Col}})\subseteq A^a_{\mathrm{Col}}$. Therefore the proposition follows immediately from the uniqueness of $G_1^a(u)$ because $G_0^a(B,A)(1-u)$ lies on

$A^a_{\mathrm{Col}}\widehat{\otimes}A^{\wedge}_{\mathbf{C}_p}$, satisfies ($\mathrm{KZ}^p$) and admits the same asymptotic behavior to that of $G_1(u)$ at $u=1$. □

Remark 3.9. As in the same way to the above proof of the uniqueness on Theorem 3.4, we see that $G^a_0(u)$ and $G^a_1(u)$ are both invertible and they must coincide with each other up to a multiplication from the right by an invertible element $\Phi^{(a),p}_{KZ}(A,B)\in A^{\wedge}_{\mathbf{C}_p}=\mathbf{C}_p\langle\langle A,B\rangle\rangle$ (which is independent of $u\in\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$). Namely

$$G^a_0(u)=G^a_1(u)\cdot\Phi^{(a),p}_{KZ}(A,B).$$

Theorem 3.10. *Actually $\Phi^{(a),p}_{KZ}(A,B)$ is independent of any choice of branch parameter $a\in\mathbf{C}_p$.*

Proof . Put $z_0\in\left]\mathbf{P}^1_{\overline{\mathbf{F}}_p}\backslash\{0,1,\infty\}\right[$. Since $\left]\mathbf{P}^1_{\overline{\mathbf{F}}_p}\backslash\{0,1,\infty\}\right[$ is a branch independent region, the special value of $G^a_0(u)$ at $u=z_0$ actually does not depend on any choice of branch parameter $a\in\mathbf{C}_p$. Similarly we see that the value of $G^a_1(u)$ at $u=z_0$ does not depend on any choice of branch parameter. Therefore $\Phi^{(a),p}_{KZ}(A,B)=G^a_1(z_0)^{-1}\cdot G^a_0(z_0)$ is actually independent of any choice of branch parameter $a\in\mathbf{C}_p$. □

Remark 3.11. We have another proof of Theorem 3.10. Put $a,b\in\mathbf{C}_p$. Then we get $\iota_{a,b}(G^a_0(u))=G^b_0(u)$ and $\iota_{a,b}(G^a_1(u))=G^b_1(u)$ by Proposition 3.6. Therefore we get $\iota_{a,b}(\Phi^{(a),p}_{KZ})=\Phi^{(b),p}_{KZ}$, which implies $\Phi^{(a),p}_{KZ}=\Phi^{(b),p}_{KZ}$ because $\Phi^{(a),p}_{KZ}$ and $\Phi^{(b),p}_{KZ}\in A^{\wedge}_{\mathbf{C}_p}$.

Definition 3.12. The p-adic Drinfel'd associator $\Phi^p_{KZ}(A,B)$ is the element of $\mathbf{C}_p\langle\langle A,B\rangle\rangle^{\times}$, which is defined by $G^a_0(u)=G^a_1(u)\cdot\Phi^p_{KZ}(A,B)$.

Remark 3.13. (1) This definition of the p-adic Drinfel'd associator $\Phi^p_{KZ}(A,B)$ is independent of $u\in\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$ by Remark 3.9 and any choice of branch parameter $a\in\mathbf{C}_p$ by Theorem 3.10.

(2) In §3.3, we will see that each coefficient of $\Phi^p_{KZ}(A,B)$ is expressed in terms of p-adic MZV's, from which we know that actually $\Phi^p_{KZ}(A,B)$ belongs to $\mathbf{Q}_p\langle\langle A,B\rangle\rangle$ by Theorem 2.25.

(3) We shall prove many identities, such as 2-, 3- and 5-cycle relation of the p-adic Drinfel'd associator $\Phi^p_{KZ}(A,B)$ in [F3].

3.2. **Explicit formulae of the fundamental solution of the p-adic KZ equation.** In this subsection, we will give a calculation to express each coefficient of the fundamental solution $G^a_0(z)$ of the p-adic KZ equation (KZ^p).

Notation 3.14. (1) Let $\mathbf{A}_{\bullet} = \underset{w\geqslant 0}{\oplus}\mathbf{A}_w = \mathbf{Q}\langle A, B\rangle (\subset A^{\wedge}_{\mathbf{C}_p})$ be the non-commutative graded polynomial ring over $\mathbf{Q}$ with two variables A and B with $degA = degB = 1$. Here $\mathbf{A}_w$ is the homogeneous degree w part of $\mathbf{A}_{\bullet}$. We call an element of $\mathbf{A}_{\bullet}$ which is monic and monomial by a word. But exceptionally we shall not call 1 a word. For each word W, the weight and depth of W are as follows.

$wt(W) :=$ 'the sum of exponents of A and B in W'
$dp(W) :=$ 'the sum of exponent of B in W'

(2) Put $M' = \mathbf{A}_{\bullet}\cdot B = \{F\cdot B | F \in \mathbf{A}_{\bullet}\}$ which is the $\mathbf{Q}$-linear subspace of $\mathbf{A}_{\bullet}$. Note that there is a natural surjection from $\mathbf{A}_{\bullet}$ to $\mathbf{A}_{\bullet} \Big/ \mathbf{A}_{\bullet} A$. By identifying the latter space with $\mathbf{Q}\cdot 1 + M' (= \mathbf{Q}\cdot 1 + \mathbf{A}_{\bullet}\cdot B)$ we obtain the $\mathbf{Q}$-linear map $f' : \mathbf{A}_{\bullet} \twoheadrightarrow \mathbf{A}_{\bullet} \Big/ \mathbf{A}_{\bullet} A \overset{\sim}{\to} \mathbf{Q}\cdot 1 + M' \hookrightarrow \mathbf{A}_{\bullet}$. Abbreviately we denote $f' : A^{\wedge}_{\mathbf{C}_p} \to A^{\wedge}_{\mathbf{C}_p}$ to be the $\mathbf{C}_p$-linear map induced from $f' : \mathbf{A}_{\bullet} \to \mathbf{A}_{\bullet}$.

(3) For each word $W = B^{q_0}A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_k}B^{q_k}$ ($k \geqslant 0$, $q_0 \geqslant 0$, $p_i, q_i \geqslant 1$ for $i \geqslant 1$) in M', we define

$$Li^a_W(z) := Li^a_{\underbrace{1,\dots 1}_{q_k-1},p_k+1,\underbrace{1,\dots 1}_{q_{k-1}-1},p_{k-1}+1,\dots\dots,1,p_1+1\underbrace{1,\dots 1}_{q_0}}(z) \in A^a_{\mathrm{Col}} \ .$$

By extending linearly, we get the $\mathbf{Q}$-linear map $Li^a(z) : M' \to A^a_{\mathrm{Col}}$ which sends each word W in M' to $Li^a_W(z)$.

Theorem 3.15 (Explicit Formulae)**.** *Put $a \in \mathbf{C}_p$. Let $G^a_0(z)$ be the fundamental solution of the p-adic KZ equation (KZ^p) in Theorem 3.4. Expand $G^a_0(z) = 1 + \sum_{W:words} J^a_p(W)(z)\ W$. Then each coefficient $J^a_p(W)(z)$ can be expressed as follows.*

(a) *When W is in M',* $J^a_p(W)(z) = (-1)^{dp(W)}Li^a_W(z)$.

(b) *When W is written as $VA^r (r \geqslant 0, V \in M')$,*

$$J^a_p(W)(z) = \sum_{\substack{s+t=r\\ 0\leqslant s, 0\leqslant t}} (-1)^{dp(W)+s} Li^a_{f'(V\circ A^s)}(z)\frac{\{log^a(z)\}^t}{t!} \quad .$$

(c) *When W is written as $A^r (r \geqslant 0)$,* $J^a_p(W)(z) = \frac{\{log^a(z)\}^r}{r!}$.

For the definition of the shuffle product '$\circ$', see [F1] Definition 3.2.2. By applying f' to $G^a_0(z)$ term by term, we get

Lemma 3.16. $f'(G^a_0(z)) = 1 + \sum_{W\in M':words} J^a_p(W)(z)\ W.$

Proof . Apply f' term by term. $\square$

Lemma 3.17. $f'(G_0^a(z)) = 1 + \sum_{W\in M':words} (-1)^{dp(W)} Li_W^a(z)\ W.$

Proof . By the definition of p-adic MPL and the p-adic KZ equation and $\lim_{z\to 0} G_0^a(z)\cdot z^{-A} = 1$, we can show this formula inductively. □

By combining Lemma 3.16 with Lemma 3.17, we get Theorem 3.15.(a).

Notation 3.18. Let $A^\wedge_{\mathbf{C}_p}[[\alpha]] := A^\wedge_{\mathbf{C}_p}\widehat{\otimes}\mathbf{C}_p[[\alpha]]$ be the one variable formal power series ring with the non-commutative algebra $A^\wedge_{\mathbf{C}_p}$ coefficient. Let $g'_1 : A^\wedge_{\mathbf{C}_p} \to A^\wedge_{\mathbf{C}_p}[[\alpha]]$ be the algebra homorphism which sends A, B to $A-\alpha$, B respectively and let $g'_2 : A^\wedge_{\mathbf{C}_p}[[\alpha]] \to A^\wedge_{\mathbf{C}_p}$ be the well-defined $\mathbf{C}_p$-linear map which sends $W\otimes\alpha^q$ to WA^q for each word W and $q\geqslant 0$.

Consider the $\mathbf{C}_p$-linear map $g'_2\circ g'_1 : A^\wedge_{\mathbf{C}_p} \to A^\wedge_{\mathbf{C}_p}$.

Lemma 3.19. $g'_2\circ g'_1(VA) = 0$ *for* $V\in A^\wedge_{\mathbf{C}_p}$.

Proof . It is clear by definition. □

Lemma 3.20. *Put* $z\in\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$. *Then* $G_0^a(z) = g'_2\circ g'_1\Big(f'\big(G_0^a(z)\big)\Big)\cdot z^A$.

Proof . By Lemma 3.19, we get

$$g'_2\circ g'_1\Big(f'\big(G_0^a(z)\big)\Big) = g'_2\circ g'_1\big(G_0^a(z)\big) \quad . \tag{3.2}$$

Both $G_0^a(A-\alpha,B)(u)$ and $u^{-\alpha}G_0^a(A,B)(u)$ are solutions of the p-adic differential equation $\frac{dG}{du}(u) = (\frac{A-\alpha}{u}+\frac{B}{u-1})G(u)$ in $A^a_{\mathrm{Col}}\widehat{\otimes}A^\wedge_{\mathbf{C}_p}[[\alpha]]$ and satisfies the same asymptotic behavior $G(u)\approx u^{A-\alpha}$ as $u\to 0$. Therefore the uniqueness of solution of the above p-adic differential equation (which can be shown in a similar way to Proposition 3.3), we get $G_0^a(A-\alpha,B)(u) = G_0^a(A,B)(u)\cdot u^{-\alpha}$, from which it follows that

$$g'_2\circ g'_1\big(G_0^a(z)\big) = G_0^a(z)\cdot z^A \quad . \tag{3.3}$$

By (3.2) and (3.3), we get $G_0^a(z) = g'_2\circ g'_1\Big(f'\big(G_0^a(z)\big)\Big)\cdot z^A$. □

Therefore we see that $P^a(z) = g'_2\circ g'_1\Big(f'\big(G_0^a(z)\big)\Big)$ (for $P^a(z)$, see the proof of Theorem 3.4).

Proof of Theorem 3.15. By combining Lemma 3.17 with Lemma 3.20, we get the explicit formula in Theorem 3.15. □

Notation 3.21. We denote the involution of $A^\wedge_{\mathbf{C}_p}$ which exchanges A and B by $\tau : A^\wedge_{\mathbf{C}_p} \to A^\wedge_{\mathbf{C}_p}$.

Corollary 3.22.

$$G_1^a(z) = 1 + \sum_{W:words} J_p^a\big(\tau(W)\big)(1-z)\cdot W \quad .$$

Proof . It follows from Theorem 3.8 combined with Theorem 3.15. □

Examples 3.23. The following is a low degree part of $G_0^a(A,B)(z)$.

$$\begin{aligned}
G_0^a(A,B)(z) = &1 + (log^a z)A + log^a(1-z)B + \frac{(log^a z)^2}{2}A^2 - Li_2^a(z)AB \\
&+ \{Li_2^a(z) + (log^a z)log^a(1-z)\}\, BA + \frac{\{log^a(1-z)\}^2}{2}B^2 + \frac{(log^a z)^3}{6}A^3 \\
&- Li_3^a(z)A^2B + \{2Li_3^a(z) + (log^a z)Li_2^a(z)\}\, ABA + Li_{1,2}^a(z)AB^2 \\
&- \left[Li_3^a(z) - (log^a z)Li_2^a(z) - \frac{(log^a z)^2 log^a(1-z)}{2}\right] BA^2 + Li_{2,1}^a(z)BAB \\
&- \left[Li_{1,2}^a(z) + Li_{2,1}^a(z) - \frac{log^a z\{log^a(1-z)\}^2}{2}\right] B^2A \\
&+ \frac{\{log^a(1-z)\}^3}{6}B^3 + \cdots \quad .
\end{aligned}$$

3.3. **Explicit formulae of the p-adic Drinfel'd associator.** In this subsection, first we give a proof of Theorem 2.18 and then give an explicit formula to express each coefficient of the p-adic Drinfel'd associator $\Phi_{KZ}^p(A,B)$. The technique employed here is essentially a p-adic analogue of these appeared in [LM] Appendix A.

Lemma 3.24.

$$\lim_{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{}' \epsilon^{-A}G_0^a(\epsilon) = 1 \ .$$

Proof . Since $P^a(u) = G_0^a(u)\cdot u^{-A}$ lies on $A(]0[)\widehat{\otimes}A^\wedge_{\mathbf{C}_p}$ and takes value 1 at $u=0$ by Theorem 3.4, we get an expression $P^a(u) = 1 + uk(u)$ where $k(u)\in A(]0[)\widehat{\otimes}A^\wedge_{\mathbf{C}_p}$. Thus

$$\begin{aligned}
\lim_{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{}' \epsilon^{-A}G_0^a(\epsilon) &= \lim_{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{}' \epsilon^{-A}P^a(\epsilon)\epsilon^A \\
&= \lim_{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{}' 1 + \epsilon\cdot exp\{-log^a(\epsilon)A\}\cdot k(\epsilon)\cdot exp\{log^a(\epsilon)A\}.
\end{aligned}$$

By taking its word expansion and applying Lemma 3.23 in each term, we get the lemma. □

Although $\underset{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{\lim{}'}\epsilon^{-A}G_0^a(\epsilon)=1$, we cannot say $\lim_{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}\epsilon^{-A}G_0^a(\epsilon)=1$.

Lemma 3.25.

$$\underset{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{\lim{}'}\epsilon^{-B}G_0^a(1-\epsilon)=\Phi^p_{KZ}(A,B)\ .$$

Proof . By Proposition 3.8 and Lemma 3.24, we get

$$\underset{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{\lim{}'}\epsilon^{-B}G_1^a(1-\epsilon)=\underset{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{\lim{}'}\epsilon^{-B}G_0^a(B,A)(\epsilon)=1\ .$$

Thus

$$\underset{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{\lim{}'}\epsilon^{-B}G_0^a(1-\epsilon)=\underset{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{\lim{}'}\epsilon^{-B}G_1^a(1-\epsilon)\Phi^p_{KZ}(A,B)=\Phi^p_{KZ}(A,B)\ .$$

□

Compare Lemma 3.25 with Lemma 3.24.

Proof of Theorem 2.18. From the explicit formula of $G_0^a(z)$ (Theorem 3.15.(a)) and Lemma 3.25, it follows that $\underset{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{\lim{}'}Li^a_{k_1,\cdots,k_m}(1-\epsilon)$ converges for $k_m>1$. □

Notation 3.26. (1) Put $M=A\cdot\mathbf{A}_{\bullet}\cdot B=\{A\cdot F\cdot B|F\in\mathbf{A}_{\bullet}\}$, which is a $\mathbf{Q}$-linear subspace of $\mathbf{A}_{\bullet}$. Note that there is a natural surjection from $\mathbf{A}_{\bullet}$ to $\mathbf{A}_{\bullet}\Big/(B\cdot\mathbf{A}_{\bullet}+\mathbf{A}_{\bullet}\cdot A)$. By identifying the latter space with $\mathbf{Q}\cdot 1+M(=\mathbf{Q}\cdot 1+A\cdot\mathbf{A}_{\bullet}\cdot B)$, we obtain the $\mathbf{Q}$-linear map $f:\mathbf{A}_{\bullet}\twoheadrightarrow\mathbf{A}_{\bullet}\Big/(B\cdot\mathbf{A}_{\bullet}+\mathbf{A}_{\bullet}\cdot A)\xrightarrow{\sim}\mathbf{Q}\cdot 1+M\hookrightarrow\mathbf{A}_{\bullet}$. Abbreviately we denote $f:A^{\wedge}_{\mathbf{C}_p}\to A^{\wedge}_{\mathbf{C}_p}$ to be the $\mathbf{C}_p$-linear map induced from $f:\mathbf{A}_{\bullet}\to\mathbf{A}_{\bullet}$.

(2) For each word $A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_k}B^{q_k}$ ($k\geqslant 0$, $p_i,q_i\geqslant 1$ for $i\geqslant 1$) in M, we define

$$\begin{aligned}Z_p(W):&=\underset{\substack{\epsilon\to 0\\ \epsilon\in\mathbf{C}_p}}{\lim{}'}Li^a_W(1-\epsilon)\\ &=\zeta_p(\underbrace{1,\dots 1}_{q_k-1},p_k+1,\underbrace{1,\dots 1}_{q_{k-1}-1},p_{k-1}+1,\dots\dots,1,p_1+1)\ .\end{aligned}$$

By extending linearly, we get the $\mathbf{Q}$-linear map $Z_p:M\to\mathbf{C}_p$ which sends each word W in M to $Z_p(W)\in\mathbf{C}_p$.

We knew that $Z_p(W)$ is independent of any choice of branch parameter $a\in\mathbf{C}_p$ by Theorem 2.13 and lies on $\mathbf{Q}_p$ by Theorem 2.25.

Remark 3.27. By combining Lemma 3.25 with Theorem 3.10, we get another proof of branch independency (Theorem 2.13) of the value $\lim'_{\substack{z\to 1\\ z\in \mathbf{C}_p-\{1\}}} Li^a_{k_1,\cdots,k_m}(z)$ for $k_m>1$.

Theorem 3.28 (Explicit Formulae). *Expand the p-adic Drinfel'd associator $\Phi^p_{KZ}(A,B)=1+\sum_{W:words} I_p(W)W$. Then each coefficient $I_p(W)$ can be expressed as follows.*

(a) *When W is in M,* $\quad I_p(W)=(-1)^{dp(W)}Z_p(W)$.

(b) *When W is written as $B^rVA^s(r,s\geqslant 0, V\in M)$,*

$$I_p(W)=(-1)^{dp(W)}\sum_{0\leqslant a\leqslant r, 0\leqslant b\leqslant s}(-1)^{a+b}Z_p\Big(f(B^a\circ B^{r-a}VA^{s-b}\circ A^b)\Big).$$

(c) *When W is written as $B^rA^s(r,s\geqslant 0)$,*

$$I_p(W)=(-1)^{dp(W)}\sum_{0\leqslant a\leqslant r, 0\leqslant b\leqslant s}(-1)^{a+b}Z_p\Big(f(B^a\circ B^{r-a}A^{s-b}\circ A^b)\Big).$$

Remark 3.29. (1) These explicit formulae are p-adic version of those given in [F1] Proposition 3.2.3.

(2) Suppose that $k_i\geqslant 1$ and $k_m=1$. In the complex case, $(-1)^m I(A^{k_m-1}B\cdots A^{k_1-1}B)$ (for $I(\cdot)$, see [F1] Proposition 3.2.3) is called the regularized MZV corresponding to $A^{k_m-1}B\cdots A^{k_1-1}B$ (just something like a modification of the divergent series $\zeta(k_1,\cdots,k_{m-1},1)$), see for example [IKZ]. Therefore we may call $(-1)^m I_p(A^{k_m-1}B\cdots A^{k_1-1}B)$ by the regularized p-adic MZV corresponding to $A^{k_m-1}B\cdots A^{k_1-1}B$.

By applying f to $\Phi^p_{KZ}(A,B)$ term by term, we get

Lemma 3.30.

$$f(\Phi^p_{KZ}(A,B))=1+\sum_{W\in M:words} I_p(W)W.$$

Proof . Apply f term by term. □

Lemma 3.31.

$$f(\Phi^p_{KZ}(A,B))=1+\sum_{W\in M:words}(-1)^{dp(W)}Z_p(W)\cdot W$$

Proof . It follows from Theorem 2.13, Theorem 3.15.(a) and Lemma 3.25. □

By combining Lemma 3.30 and Lemma 3.31, we get Theorem 3.28.(a).

Notation 3.32. Let $A^{\wedge}_{\mathbf{C}_p}[[\alpha,\beta]] := A^{\wedge}_{\mathbf{C}_p}\widehat{\otimes}\mathbf{C}_p[[\alpha,\beta]]$ be the two variable formal power series ring with the non-commutative algebra $A^{\wedge}_{\mathbf{C}_p}$ coefficient. Let $g_1 : A^{\wedge}_{\mathbf{C}_p} \to A^{\wedge}_{\mathbf{C}_p}[[\alpha,\beta]]$ be the algebra homomorphism which sends A, B to $A-\alpha$, $B-\beta$ respectively and let $g_2 : A^{\wedge}_{\mathbf{C}_p}[[\alpha,\beta]] \to A^{\wedge}_{\mathbf{C}_p}$ be the well-defined $\mathbf{C}_p$-linear map which sends $W\otimes\alpha^p\beta^q$ to B^qWA^p for each word W and $p,q\geqslant 0$.

Consider the $\mathbf{C}_p$-linear map $g_2\circ g_1 : A^{\wedge}_{\mathbf{C}_p} \to A^{\wedge}_{\mathbf{C}_p}$.

Lemma 3.33. *$g_2\circ g_1(VA)=0$ and $g_2\circ g_1(BV)=0$ for $V\in A^{\wedge}_{\mathbf{C}_p}$.*

Proof . It follows immediately from the definition. □

Lemma 3.34. $\Phi^p_{KZ}(A,B) = g_2\circ g_1\Big(f\left(\Phi^p_{KZ}(A,B)\right)\Big)$.

Proof . By Lemma 3.33, we get

$$g_2\circ g_1\Big(f\left(\Phi^p_{KZ}(A,B)\right)\Big)= g_2\circ g_1\left(\Phi^p_{KZ}(A,B)\right) . \tag{3.4}$$

Both $G^a_0(A-\alpha,B-\beta)(u)$ and $u^{-\alpha}(1-u)^{-\beta}G_0(A,B)(u)$ are solutions of the p-adic differential equation $\frac{dG}{du}(u) = (\frac{A-\alpha}{u}+\frac{B-\beta}{u-1})G(u)$ in $A^a_{\mathrm{Col}}\widehat{\otimes}A^{\wedge}_{\mathbf{C}_p}[[\alpha,\beta]]$ and satisfy the same asymptotic behavior $G(u)\approx u^{A-\alpha}$ as $u\to 0$. By the uniqueness of solutions of the above p-adic differential equation (which can be shown in a similar way to Proposition 3.3), we get

$$G^a_0(A-\alpha,B-\beta)(u) = u^{-\alpha}(1-u)^{-\beta}G_0(A,B)(u).$$

Similarly we get

$$G^a_1(A-\alpha,B-\beta)(u) = u^{-\alpha}(1-u)^{-\beta}G_1(A,B)(u).$$

Therefore

$$G^a_1(A-\alpha,B-\beta)(u)^{-1}G^a_0(A-\alpha,B-\beta)(u) = G^a_1(A,B)(u)^{-1}G^a_0(A,B)(u),$$

from which it follows that

$$\Phi^p_{KZ}(A-\alpha,B-\beta) = \Phi^p_{KZ}(A,B).$$

Thus we get,

$$g_2\circ g_1\left(\Phi^p_{KZ}(A,B)\right) = \Phi^p_{KZ}(A,B). \tag{3.5}$$

From (3.4) and (3.5), it follows that $\Phi^p_{KZ}(A,B) = g_2\circ g_1\Big(f\left(\Phi^p_{KZ}(A,B)\right)\Big)$. □

Proof of Theorem 3.28. By combining Lemma 3.31 with Lemma 3.34, we get the explicit formulae in Theorem 3.28. □

Examples 3.35. The following is a low degree part of the p-adic Drinfel'd associator $\Phi^p_{KZ}(A,B)$.

$$\begin{aligned}\Phi^p_{KZ}(A,B) = {} & 1 - \zeta_p(2)AB + \zeta_p(2)BA - \zeta_p(3)A^2B + 2\zeta_p(3)ABA \\ & + \zeta_p(1,2)AB^2 - \zeta_p(3)BA^2 - 2\zeta_p(1,2)BAB + \zeta_p(1,2)B^2A \\ & - \zeta_p(4)A^3B + 3\zeta_p(4)A^2BA + \zeta_p(1,3)A^2B^2 - 3\zeta_p(4)ABA^2 \\ & + \zeta_p(2,2)ABAB - (2\zeta_p(1,3) + \zeta_p(2,2))AB^2A - \zeta_p(1,1,2)AB^3 \\ & + \zeta_p(4)BA^3 - (2\zeta_p(1,3) + \zeta_p(2,2))BA^2B + (4\zeta_p(1,3) + \zeta_p(2,2))BABA \\ & + 3\zeta_p(1,1,2)BAB^2 - \zeta_p(1,3)B^2A^2 - 3\zeta_p(1,1,2)B^2AB \\ & + \zeta_p(1,1,2)B^3A + \cdots .\end{aligned}$$

3.4. **Proofs of main results and functional equations among p-adic multiple polylogarithms.** Here we show functional equations among p-adic MPL's in Theorem 3.36 and give proofs of Theorem 2.22 and Theorem 2.28.

Theorem 3.36 (Functional Equation among p-adic MPL's)**.** *Let W be a word and $z \in \mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$, then*

$$J^a_p(W)(1-z) = \sum_{\substack{W',W'':words \\ W=W'W''}} J^a_p\big(\tau(W')\big)(z)\cdot I_p(W'') \ .$$

For J^a_p, I_p, τ, see Theorem 3.15, Theorem 3.28 and Notation 3.21 respectively. This formulae may be regarded as a functional equation among p-adic MPL's because $J^a_p(\tau(W'))(z)$ (resp. $I_p(W'')$) is expressed in terms of p-adic MPL's (resp. p-adic MZV's).

Proof . It follows from $G^a_0(A,B)(z) = G^a_1(A,B)(z)\cdot\Phi^p_{KZ}(A,B)$ and Corollary 3.22. □

Examples 3.37. Put $a \in \mathbf{C}_p$.

(a) Take $W = BAB$. Then we get

$$Li^a_{2,1}(1-z) = 2Li^a_3(z) - log^a(z)Li^a_2(z) - \zeta_p(2)log^a(z) - 2\zeta_p(3).$$

(b) Take $W = BA^2B$. Then we get

$$\begin{aligned}Li^a_{3,1}(1-z) = {} & -2Li^a_{1,3}(z) - Li^a_{2,2}(z) + log^a(z)Li^a_{1,2}(z) \\ & + \zeta_p(2)Li^a_2(z) - \zeta_p(3)log^a(z) - 2\zeta_p(1,3) - \zeta_p(2,2).\end{aligned}$$

(c) Take $W = AB$. Then we get

$$Li^a_2(1-z) = -Li^a_2(z) - log^a(z)log^a(1-z) + \zeta_p(2).$$

This formula is equal Coleman-Sinnott's functional equation of the p-adic dilogarithm ([C] Proposition 6.4.(iii)) because $\zeta_p(2) = 0$ by Example 2.19.(a).

Proof of Theorem 2.22. By the explicit formulae in Theorem 3.15 and the functional equation of p-adic MPL's in Theorem 3.36 combined with Lemma 2.14, it is immediate to see that

$$\lim_{\substack{z\to 1\\ z\in \mathbf{C}_p-\{1\}}}{}' \quad J_p^a(W)(1-z)=I_p(W) \qquad \text{if it converges.}$$

Theorem 2.22 is a special case for $W=BA^{k_{m-1}-1}B\cdots A^{k_1-1}B$. $\square$

Notation 3.38. (1) Denote the subset of $A^{\wedge}_{\mathbf{C}_p}$ consisting of formal Lie series in $A^{\wedge}_{\mathbf{C}_p}$ by $\mathbb{L}^{\wedge}_{\mathbf{C}_p}$ and its topological commutator by $[\mathbb{L}^{\wedge}_{\mathbf{C}_p},\mathbb{L}^{\wedge}_{\mathbf{C}_p}]$. We denote $exp\,\mathbb{L}^{\wedge}_{\mathbf{C}_p}$ (resp. $exp\,[\mathbb{L}^{\wedge}_{\mathbf{C}_p},\mathbb{L}^{\wedge}_{\mathbf{C}_p}]$) to be the subset of $A^{\wedge}_{\mathbf{C}_p}$ consisting of formal power series in $A^{\wedge}_{\mathbf{C}_p}$ which is an exponential of an element in $\mathbb{L}^{\wedge}_{\mathbf{C}_p}$ (resp. $[\mathbb{L}^{\wedge}_{\mathbf{C}_p},\mathbb{L}^{\wedge}_{\mathbf{C}_p}]$).

(2) We define the (non-commutative) A^a_{Col}-algebra homomorphism

$$\Delta : A^a_{\mathrm{Col}}\widehat{\underset{\mathbf{C}_p}{\otimes}} A^{\wedge}_{\mathbf{C}_p} \to A^a_{\mathrm{Col}}\widehat{\underset{\mathbf{C}_p}{\otimes}}(A^{\wedge}_{\mathbf{C}_p}\widehat{\underset{\mathbf{C}_p}{\otimes}}A^{\wedge}_{\mathbf{C}_p})$$

to be the homomorphism which is deduced from $\Delta(A)=A\otimes 1+1\otimes A$ and $\Delta(B)=B\otimes 1+1\otimes B$.

Proposition 3.39. $\Delta(\Phi^p_{KZ})=\Phi^p_{KZ}\widehat{\otimes}\Phi^p_{KZ}$.

Proof . Put $a\in\mathbf{C}_p$. By the following calculations,

$$\begin{aligned}
&\Delta\big(G_0^a(A,B)(u)\big)=G_0^a\big(\Delta(A),\Delta(B)\big)(u)\approx u^{\Delta(A)} \qquad \text{as} \;\; u\to 0,\\
&\frac{d\Delta\big(G_0^a(A,B)\big)}{du}(u)=\frac{dG_0^a\big(\Delta(A),\Delta(B)\big)}{du}(u)\\
&\qquad\qquad\qquad\quad =(\frac{\Delta(A)}{u}+\frac{\Delta(B)}{u-1})G_0^a\big(\Delta(A),\Delta(B)\big)(u)\\
&\qquad\qquad\qquad\quad =(\frac{\Delta(A)}{u}+\frac{\Delta(B)}{u-1})\Delta\big(G_0^a(A,B)(u)\big),
\end{aligned}$$

$$\begin{aligned}
G_0^a(A,B)(u)\widehat{\otimes}G_0^a(A,B)(u)&=\big(G_0^a(A,B)(u)\widehat{\otimes}1\big)\cdot\big(1\widehat{\otimes}G_0^a(A,B)(u)1\big)\\
&\approx u^A\widehat{\otimes}u^A \qquad \text{as}\;\; u\to 0,
\end{aligned}$$

$$\frac{d\big(G_0^a(A,B)(u)\widehat{\otimes}G_0^a(A,B)(u)\big)}{du}=\frac{d}{du}\Big\{\big(G_0^a(A,B)(u)\widehat{\otimes}1\big)\cdot\big(1\widehat{\otimes}G_0^a(A,B)(u)\big)\Big\}$$
$$=\Big\{\frac{d}{du}\big(G_0^a(A,B)(u)\big)\widehat{\otimes}1\Big\}\cdot\Big\{1\widehat{\otimes}G_0^a(A,B)(u)\Big\}$$
$$+\Big\{G_0^a(A,B)(u)\widehat{\otimes}1\Big\}\cdot\Big\{1\widehat{\otimes}\frac{d}{du}\big(G_0^a(A,B)(u)\big)\Big\}$$
$$=\Big\{\big(\frac{A}{u}+\frac{B}{u-1}\big)\cdot G_0^a(A,B)(u)\widehat{\otimes}1\Big\}\cdot\Big\{1\widehat{\otimes}G_0^a(A,B)(u)\Big\}$$
$$+\Big\{G_0^a(A,B)(u)\widehat{\otimes}1\Big\}\cdot\Big\{1\widehat{\otimes}\big(\frac{A}{u}+\frac{B}{u-1}\big)\cdot G_0^a(A,B)(u)\Big\}$$
$$=\big(\frac{A\widehat{\otimes}1+1\widehat{\otimes}A}{u}+\frac{B\widehat{\otimes}1+1\widehat{\otimes}1}{u-1}\big)\cdot\Big\{G_0^a(A,B)(u)\widehat{\otimes}G_0^a(A,B)(u)\Big\},$$

we see that both $\Delta\big(G_0^a(A,B)(u)\big)$ and $G_0^a(A,B)(u)\widehat{\otimes}G_0^a(A,B)(u)$ are solutions in $A^a_{\mathrm{Col}}\widehat{\otimes}_{\mathbf{C}_p}(A^\wedge_{\mathbf{C}_p}\widehat{\otimes}_{\mathbf{C}_p}A^\wedge_{\mathbf{C}_p})$ of the p-adic differential equation

$$\frac{dH}{dt}(t)=\big(\frac{\Delta(A)}{t}+\frac{\Delta(B)}{t-1}\big)\cdot H(t)$$

which satisfies $H(t)\approx t^A\widehat{\otimes}t^A$ as $t\to 0$. Because of the uniqueness of solution for above p-adic differential equation (which can be shown in a similar way to Proposition 3.3), we get

$$\Delta\big(G_0^a(A,B)(u)\big)=G_0^a(A,B)(u)\widehat{\otimes}G_0^a(A,B)(u).$$

Similarly we get

$$\Delta\big(G_1^a(A,B)(u)\big)=G_1^a(A,B)(u)\widehat{\otimes}G_1^a(A,B)(u).$$

Therefore

$$\Delta\big(\Phi^p_{KZ}\big)=\Delta\Big(G_1^a(A,B)(u)^{-1}\cdot G_0^a(A,B)(u)\Big)$$
$$=\Delta\Big(G_1^a(A,B)(u)\Big)^{-1}\cdot\Delta\Big(G_0^a(A,B)(u)\Big)$$
$$=\Big(G_1^a(A,B)(u)\widehat{\otimes}G_1^a(A,B)(u)\Big)^{-1}\cdot\Big(G_0^a(A,B)(u)\widehat{\otimes}G_0^a(A,B)(u)\Big)$$
$$=\Big(G_1^a(A,B)(u)^{-1}\cdot G_0^a(A,B)(u)\Big)\widehat{\otimes}\Big(G_1^a(A,B)(u)^{-1}\cdot G_0^a(A,B)(u)\Big)$$
$$=\Phi^p_{KZ}\widehat{\otimes}\Phi^p_{KZ}\qquad.$$

□

Lemma 3.40. *Put $a\in\mathbf{C}_p$. Denote $g_0^a(\alpha,\beta)(z)$ and $g_1^a(\alpha,\beta)(z)$ to be the images of $G_0^a(A,B)(z)$ and $G_1^a(A,B)(z)$ by the natural projection $A^a_{\mathrm{Col}}[[A,B]]\twoheadrightarrow A^a_{\mathrm{Col}}[[\alpha,\beta]]$ sending A (resp. B) to α (resp. β), where*

$A^a_{\mathrm{Col}}[[\alpha,\beta]]$ is the two variable commutative formal power series ring with A^a_{Col}-coefficients. Then

$$g^a_0(\alpha,\beta)(z)=g^a_1(\alpha,\beta)(z)=z^\alpha(1-z)^\beta \quad \textit{in} \quad A^a_{\mathrm{Col}}[[\alpha,\beta]]\ .$$

Proof . Both $g^a_0(\alpha,\beta)(z)$ and $z^\alpha(1-z)^\beta$ are solutions in $A^a_{\mathrm{Col}}[[\alpha,\beta]]$ of the p-adic differential equation

$$\frac{dH}{dt}(t)=(\frac{\alpha}{t}+\frac{\beta}{t-1})\cdot H(t)$$

which satisfy $H(t)\approx t^\alpha$ as $t\to 0$. By the uniqueness of solution of the above p-adic differential equation (which can be shown in a similar way to Proposition 3.3), we get

$$g^a_0(\alpha,\beta)(z)=z^\alpha(1-z)^\beta.$$

Similarly we get

$$g^a_1(\alpha,\beta)(z)=z^\alpha(1-z)^\beta.$$

Therefore

$$g^a_0(\alpha,\beta)(z)=g^a_1(\alpha,\beta)(z)=z^\alpha(1-z)^\beta. \qquad \square$$

Theorem 3.41. $\Phi^p_{KZ}(A,B)\in exp\ [\mathbb{L}^\wedge_{\mathbf{C}_p},\mathbb{L}^\wedge_{\mathbf{C}_p}]$.

Proof . By Proposition 3.39, we see that Φ^p_{KZ} is group-like, which means that $\Phi^p_{KZ}(A,B)\in exp\ \mathbb{L}^\wedge_{\mathbf{C}_p}$ (see [Se] Part I Ch IV §7). It follows from Lemma 3.40 that

$$\Phi^p_{KZ}(A,B)=G^a_1(A,B)(z)^{-1}G^a_0(A,B)(z)\in exp\ [\mathbb{L}^\wedge_{\mathbf{C}_p},\mathbb{L}^\wedge_{\mathbf{C}_p}]\ . \qquad \square$$

Corollary 3.42 (Shuffle Product Formulae). *For each word W and $W'\in M$,*

$$Z_p(W)\cdot Z_p(W')=Z_p(W\circ W')\ .$$

Proof . Consider the graded $\mathbf{Q}$-linear map $I_p:\mathbf{A}_{\bullet}\to Z^{(p)}_{\bullet}$ which sends each word W to $I_p(W)\in Z^{(p)}_{\bullet}$ (for $I_p(W)$, see Theorem 3.28). Then by Proposition 3.39, we obtain the shuffle product formulae $I_p(W)\cdot I_p(W')=I_p(W\circ W')$ on $\mathbf{Q}_p$ for each word W' and W''. By Theorem 3.28, we get the corollary. $\square$

Proof of Theorem 2.28. It follows immediately from Corollary 3.42. $\square$

References

[AK] Arakawa, T. and Kaneko, M.; On poly-Bernoulli numbers, Comment. Math. Univ. St. Paul. 48 (1999), no. 2, 159–167.

[Ber] Berthelot, P.; Cohomologie rigide et cohomologie rigide à support propre, Première partie, Prépublication IRMAR 96-03, 89 pages (1996).

[Bes1] Besser, A.; Syntomic regulators and p-adic integration II: K_2 of curves, Israel Journal of Math. 120 (2000), 335-360.

[Bes2] ______ Coleman integration using the Tannakian formalism, Mathematische Annalen 322 (2002) 1, 19-48.

[BdJ] Besser, A. and de Jeu, R.; The syntomic regulator for K-theory of fields, available at `arXiv:math.AG/0005069`.

[BF] Besser, A. and Furusho, H.; The double shuffle relation for p-adic multiple zeta values, in preparation.

[Br] Breuil, C.; Intégration sur les variétés p-adiques (d'aprés Coleman, Colmez), Séminaire Bourbaki, Vol. 1998/99, Astérisque No. 266, (2000), Exp. No. 860, 5, 319–350.

[C] Coleman, R.; Dilogarithms, regulators and p-adic L-functions, Invent. Math. 69 (1982), no. 2, 171–208.

[CdS] Coleman, R. and de Shalit, E.; p-adic regulators on curves and special values of p-adic L-functions, Invent. Math. 93 (1988),no. 2, 239–266.

[Dr] Drinfel'd, V. G.; On quasitriangular quasi-Hopf algebras and a group closely connected with $\mathrm{Gal}(\overline{Q}/Q)$, Leningrad Math. J. 2 (1991), no. 4, 829–860.

[F0] Furusho, H.; The multiple zeta value algebra and the stable derivation algebra, to be apeared in Publ. Res. Inst. Math. Sci, also available at `arXiv:math.NT/0011261`.

[F1] ______ Multiple zeta values and Grothendieck-Teichmüller groups, RIMS-1357, preprint (2002).

[F2] ______ p-adic multiple zeta values II — various realizations of motivic fundamental groups of the projective line minus three points, in preparation.

[F3] ______ p-adic multiple zeta values III — Grothendieck-Teichmüller groups, in preparation.

[Gon] Goncharov, A. B.; Multiple polylogarithms, cyclotomy and modular complexes, Math. Res. Lett. 5 (1998), no. 4, 497–516.

[H] Hoffman, M.; The algebra of multiple harmonic series, J. Algebra 194 (1997), no. 2, 477–495.

[IKZ] Ihara, K., Kaneko, M. and Zagier, D.; Derivation and double shuffle relations for multiple zeta values, preprint (2002).

[Kan] Kaneko, M.; Poly-Bernoulli numbers, J. Théor. Nombres Bordeaux 9 (1997), no. 1, 221–228.

[Kas] Kassel, C.; Quantum groups, Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995.

[KNQ] Kolster, M. and Nguyen Quang Do, T.; Syntomic regulators and special values of p-adic L-functions, Invent. Math. 133 (1998), no. 2, 417–447.

[LM] Le, T.T.Q. and Murakami, J.; Kontsevich's integral for the Kauffman polynomial, Nagoya Math. J. 142 (1996), 39–65.

[Se] Serre, J-P.; Lie algebras and Lie groups, lectures given at Harvard University (1964), Second edition, Lecture Notes in Mathematics, 1500, Springer-Verlag, Berlin, 1992.

[Sh] Shiho, A.; Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology, J. Math. Sci. Univ. Tokyo 9 (2002), no. 1, 1–163.

[Sou] Soulé, C.; On higher p-adic regulators, Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980), pp. 372–401, Lecture Notes in Math., 854, Springer, Berlin-New York, 1981.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 Japan

E-mail address: furusho@kurims.kyoto-u.ac.jp