UNIFORMIZATION OF THE ORBIFOLD OF A FINITE REFLECTION GROUP

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Abstract. We try to understand the relationship between the $K(\pi, 1)$-property of the complexified regular orbit space of a finite reflection group and the flat structure on the orbit space via the uniformization equation attached to the flat structure.

1. Introduction

Let W be a finite reflection group of a real vector space V. If W is crystallographic, then the quotient space $V^*//W$ appears in several contexts in geometry: i) in Lie theory as the quotient space of a simple Lie algebra by the adjoint Lie group action [Ch1,2] and ii) in complex geometry as the base space of the universal unfolding of a simple singularity [Br1]. Having these backgrounds, $V^*//W$ carries some distinguished geometric properties and structures, which, fortunately and also amusingly, can be described only in terms of the reflection group regardless whether W is crystallographic or not. We recall two of them:

1. The complexified regular orbit space $(V^*//W)^{reg}_{\mathbf{C}}$ is a $K(\pi, 1)$-space (Brieskorn [Br3], Deligne [De]). In other words, $\pi_1((V^*//W)^{reg}_{\mathbf{C}})$ is an Artin group (i.e. a generalized braid group [B-S][De]) and the universal covering space of $(V^*//W)^{reg}_{\mathbf{C}}$ is contractible (c.f. also [Sa]).

2. The quotient space $V^*//W$ carries a *flat structure* (Saito [S3][S6])[1]. This means roughly that the tangent bundle of $V^*//W$ carries a flat metric J together with some additional structures. Nowadays, a flat structure without a primitive form is also called a *Frobenius manifold structure with gravitational descendent* (Dubrovin [Du], Manin [Ma1,2]).

Apparently, these two geometries on $V^*//W$ are of a quite different nature, one topological and the other differential geometric. Nevertheless, there is already a remarkable relationship between them on a combinatorial level: the *polyhedron dual to the system of real reflection*

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[1]The original construction of the flat structure on $V^*//W$ was given in [S3]. The description of the gravitational descendent was modified in [S6] to obtain the system of uniformization equations. The present article follows the latter style.

hyperplanes of W (which is the key in [Br2][De] to determine the topology of the complex regular orbit space) is reconstructed by a use of the *formal group action* $\exp(tD)$:=*the integral of the primitive vector field* D on $(V^*//W)_{\mathbf{R}}$ (which is a basic ingredient of the flat structure) [S8].

Inspired by the observation, the present article aims to construct a more direct relationship between the two geometries. The working hypothesis is that *a bridge between them is given by the topological behavior of the map (which, for brevity, we call the period map) obtained from solutions of the uniformization equation* $\mathcal{M}_{W,s}$ [S6] *on* $V^*//W$ *constructed from the flat structure* for a special s (see 6.1 Remark). [2]

Here we begin a program to examine this hypothesis. In the first half §2-4, we describe the uniformization equation. After fixing notation for finite reflection groups in §2, we give a detailed exposition of the flat structure in §3 and the uniformization equation $\mathcal{M}_{W,s}$ in §4. Although they are already known [S6], we renew and clarify several arguments and make them accessible for our purpose. (c.f. also [He][Sab][Ta]).

In the latter half (§5, 6), we begin to analyze the period map. In §5, solutions of the uniformization equation for the parameter $s = 1/2$ are partly given by *primitive Abelian integrals on a certain family of plane curves parameterized by* $V^*//W$. Although this fact is easy [S6], the attached period map is not studied from the view point of the primitive form, although some information is available in classical works [Th][Mu][Ko]. We examine examples of type A_1, A_2, A_3 and B_2.

In §6, first, we describe the monodromy group $\Gamma(W)$ in term of Coxeter diagram. Then we give a possible formulation of the *period domain* and the *inverse map to the period map*, and pose some conjectures. §6 is quite incomplete. It requires more work to verify or to modify the conjectures, which is beyond the scope of the present article.

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[2] *A historical note*: Before the theory of the primitive form and the flat structure reached its present form, the author suggested in [S2] to study the uniformization of the regular orbits $(V^*//W)^{reg}_{\mathbf{C}}$ by the horizontal sections of *logarithmic flat torsion-free connections on the logarithmic tangent bundle* on $V^*//W$. The torsion-free condition implies the existence of a *primitive function,* whose derivatives give a system of fundamental solutions. The primitive function for type A_1 is the logarithm. For type A_2, it is given by the elliptic integral of the first kind which gives the universal covering (up to center) of the regular orbit space. For type A_3, the space of all logarithmic flat torsion-free connections decomposes into two one-parameter families [S2,§3]. The first family gives the uniformization equation $\mathcal{M}_{W,s}$ of the present article. The meaning of the second family is unknown: for example, what is the Fourier-Laplace transform of the second family? (c.f. [A])

We note also that there is related work on certain integrable systems defined on the quotient space $V^*//W$ ([Gi] [Tak]). However the relationship with the flat structure still needs to be worked out.

2. Finite reflection group

This section gives a short summary of basic results on finite reflection groups used in the present article (see also [B]). Experienced readers are recommended to look only at the notation in 2.9 and skip to §3.

2.1. **Reflection.** Let V and V^* be a real vector space and its dual. An element $\alpha \in \mathrm{GL}(V) \simeq \mathrm{GL}(V^*)$ is a *reflection* if there exist a hyperplane H_α in V^* and a non-zero vector $f_\alpha \in V^*$ such that $\alpha|_{H_\alpha} = id_{H_\alpha}$ and $\alpha(f_\alpha) = -f_\alpha$. The H_α is called the *reflection hyperplane* of α. One has $\alpha(x) = x - f_\alpha(x)e_\alpha$ for $x \in V$ and $\alpha(x^*) = x^* - e_\alpha(x^*)f_\alpha$ for $x^* \in V^*$, where $e_\alpha \in V$ is a defining form of H_α with $\langle e_\alpha, f_\alpha\rangle = 2$.

2.2. **Finite reflection group W.** We shall mean by a *finite reflection group* W a finite group generated by reflections acting irreducibly on a real vector space V. Put $R(W) := \{\alpha \in W \mid \text{a reflection}\}$. There exist, unique up to a constant factor, W-invariant symmetric bilinear forms I and I^* on V and V^*, respectively.[3] One has $f_\alpha = 2I(e_\alpha, \cdot)/I(e_\alpha, e_\alpha)$ and $e_\alpha = 2I^*(f_\alpha, \cdot)/I^*(f_\alpha, f_\alpha)$. A connected component C of $V^* \setminus \cup_{\alpha\in R(W)} H_\alpha$ is called a *chamber*. A hyperplane H_α ($\alpha \in R(W)$) is called a *wall* of a chamber C, if $H_\alpha \cap \bar{C}$ contains an open subset of H_α.

2.3. **Coxeter group representation of W.**

We may present a finite reflection group as a Coxeter group [Co1].

A *Coxeter matrix* $M := (m(\alpha, \beta))_{\alpha,\beta\in\Pi}$ is a symmetric matrix with index set Π s.t. $m(\alpha, \alpha) = 1$ ($\alpha \in \Pi$) and $m(\alpha, \beta) \in \mathbf{Z}_{\geq 2} \cup \{\infty\}$ ($\alpha \neq \beta \in \Pi$). The group $W(M)$ generated by letters a_α ($\alpha \in \Pi$) and defined by fundamental relations: $(a_\alpha a_\beta)^{m(\alpha,\beta)} = 1$ ($\alpha, \beta \in \Pi$) is called a *Coxeter group*. The pair $(W(M), \{a_\alpha \mid \alpha \in \Pi\})$ is called a *Coxeter system*.

Theorem. *Let W be a finite reflection group acting on V and let C be a chamber of the W-action. Then the following 1.- 5. hold.*

1. *The pair $(W, \Pi(C))$ is a Coxeter system, where we put*

$$\Pi(C) := \{\alpha \in R(W) \mid H_\alpha \textit{ is a wall of the chamber } C\} \tag{2.3.1}$$

and the Coxeter matrix is given by $m(\alpha, \beta) :=$ the order of $\alpha\beta$ in W.

2. *W acts on the set of chambers simply and transitively. Hence, the Coxeter matrix does not depend on the choice of a chamber.*

3. *The closure $\bar{C}$ of a chamber is a fundamental domain for the action of W on V. That is: there is a homeomorphism: $\bar{C} \simeq V^*/W$.*

4. *Fix the sign of the vector e_α for $\alpha \in \Pi(C)$ in the manner:*

$$C = \{x \in V^* \mid \langle e_\alpha, x\rangle > 0 \textit{ for } \alpha \in \Pi(C)\}. \tag{2.3.2}$$

Then the off-diagonals of the matrix $(I(e_\alpha, e_\beta))_{\alpha,\beta\in\Pi(C)}$ are non-positive.

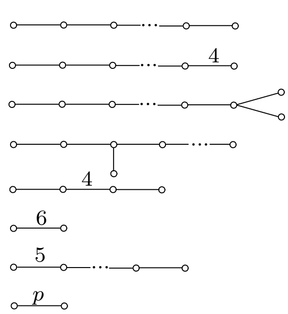
5. *$\Pi_W := \{e_\alpha | \alpha \in \Pi(C)\}$ forms a basis of V. The coefficients of $e_\beta = \sum_{\alpha\in\Pi} c_\alpha e_\alpha$ for $\beta \in R(W)$ are either all non-negative or non-positive.*

2.4. **Classification.**

We recall the classification of finite Coxeter groups ([B, ch.VI,§4]).

To a Coxeter matrix M, one attaches a Coxeter graph Γ, whose vertices are indexed by the set Π and two vertices α and β are connected by an edge iff $m(\alpha,\beta)\geq 3$. The edge is labeled by $m(\alpha,\beta)$ (omitted if $m(\alpha,\beta)=3$). The graph is called *simply-laced* if all labels are 3.

The following is the list of graphs associated to finite Coxeter groups.



A_l	$(l\geq 1)$
B_l	$(l\geq 2)$
D_l	$(l\geq 4)$
E_l	$(l=6,7$ or $8)$
F_4	
G_2	
H_l	$(l=2,3$ or $4)$
$I_2(p)$	$(p\geq 7)$.

Note. 1. Different Coxeter diagrams define non-isomorphic groups, i.e. the same group is not attached to different Coxeter matrices.

2. The group W is called crystallographic if it preserves a full-lattice in V. This condition rules out the groups of type H_l and $I_2(p)$.

3. The irreducibility of W implies the indecomposability of the Coxeter matrix M and, hence, the connectedness of the graph Γ.

2.5. **Polynomial invariants.**

Let $S(V)=\mathbf{R}\oplus V\oplus S(V)_2\oplus S(V)_3\oplus\cdots$ be the symmetric tensor algebra of V. The action of $g\in W$ on V induces the action on $S(V)$. Define the set of invariants:

$$S(V)^W := \{P\in S(V)\mid g(P)=P \text{ for } \ \forall g\in W\}. \tag{2.5.1}$$

Obviously, $S(V)^W$ is a graded subalgebra of $S(V)$.

Theorem. *(Chevalley [Ch 2]). Let W be a finite reflection group acting irreducibly on a real vector space V of rank l. Then $S(V)^W$, as an $\mathbf{R}$-algebra, is generated by l algebraically independent homogeneous elements, say $P_1,\cdots,P_l$. The set of degrees $d_1=\deg(P_1),\cdots,d_l=\deg(P_l)$ (with multiplicity) is independent of a choice of the generators.*

Note. The ring $S(V)$, viewed as a $S(V)^W$-module, is free of rank $\#W$, and $\dim(S(V)/S(V)S(V)^W_+)=\#W$, where $S(V)^W_+$ is the maximal ideal of $S(V)^W$ of all positively graded elements (c.f. (2.6.1) i)).

2.6. **Poincare series.**

The $S(V)^W$ is a graded subring of $S(V)$, i.e. $S(V)^W=\oplus_{d\in\mathbf{Z}_{\geq 0}}S(V)^W_d$ for $S(V)^W_d=S(V)_d\cap S(V)^W$. The Poincare series: $P_{S(V)^W}(t):=$

$\sum_{d=0}^{\infty}\dim_{\mathbf{R}}(S(V)_d^W)\ t^d$ is calculated in two different ways : i) Using $S(V)^W \simeq \mathbf{R}[P_1]\otimes\cdots\otimes\mathbf{R}[P_l]$, one has $P_{S(V)^W}(t) = \prod_{i=1}^{l} P_{\mathbf{R}[P_i]}(t) = \prod_{i=1}^{l}\frac{1}{(1-t^{d_i})}$ (this expression reproves the uniqueness of the $d_1,\cdots,d_l$), and ii) since $\dim_{\mathbf{R}}(S(V)_d^W) = \mathrm{tr}(\frac{1}{\#W}\sum_{w\in W} w|S(V)_d)$ for $d\in\mathbf{Z}_{\geq 0}$ and $\sum_{d=0}^{\infty}\mathrm{tr}(w|S(V)_d)t^d = \frac{1}{\det(1-tw)}$ (use the extension of V to $V_{\mathbf{C}}$), one has $P_{S(V)^W}(t) = \frac{1}{\#W}\sum_{w\in W}\frac{1}{\det(1-tw)}$. Comparing the values and derivatives at t=1 of the two expressions of $P_{S(V)^W}(t)$, one obtains:

(2.6.1) $\quad i)\ \#W = d_1\cdots d_l \quad$ and $\quad ii)\ \#R(W) = \sum_{i=1}^{l}(d_i-1).$

2.7. **Anti-invariants.**

An element $P\in S(V)$ is called an *anti-invariant* if $g\cdot P = \det(g)^{-1}P$ for all $g\in W$. The set of all anti-invariant shall be denoted by $S(V)^{-W}$.

Put $\delta_W := \prod_{\alpha\in R(W)} e_\alpha$. Let $\frac{\partial(P_1,\cdots,P_l)}{\partial(X_1,\cdots,X_l)}$ be the Jacobian for generator system $P_1,\cdots,P_l$ and $X_1,\cdots,X_l$ of the algebras $S(V)^W$ and $S(V)$, respectively. It is easy to see that δ_W and the Jacobian are anti-invariants. Using (2.6.1) ii), we further show an important lemma.

Lemma. 1. *Any anti-invariant is divisible by* δ_W*:* $S(V)^{-W} = S(V)^W\delta_W$.
2. *One has* $\frac{\partial(P_1,\cdots,P_l)}{\partial(X_1,\cdots,X_l)} = c\ \delta_W$ *for a nonzero constant* $c\in\mathbf{R}$.

2.8. **Coxeter elements and exponents.**

A *Coxeter element* is a product $c := \Pi_{\alpha\in\Pi(C)}\alpha$ for a linear ordering of elements of $\Pi(C)$. Its conjugacy class depends neither on C nor on the ordering (for Γ is a tree [B,Ch.V,n^{o}6.2.]). The order h of c is called the *Coxeter number*. Put $\det(\lambda 1 - c) = \prod_{i=1}^{l}(\lambda - \exp(2\pi\sqrt{-1}m_i/h))$ for some integers $m_1,\cdots,m_l$, called the *exponents* of W, such that

(2.8.1) $$0 < m_1 \leq m_2 \leq \cdots \leq m_l < h.$$

Here, 0 is not an exponent (i.e. 1 cannot be an eigenvalue of c), since I is nondegenerate. So $\exp(2\pi\sqrt{-1}m_i/h)$ and $\exp(2\pi\sqrt{-1}(m_{l-i+1})/h))$ should be complex conjugate to each other. Thus, we have

(2.8.2) $\quad m_i + m_{l-i+1} = h \quad$ and $\quad \sum_{i=1}^{l} m_i = \frac{1}{2}lh.$

In the rest of this §, we assume $l\geq 2$ (i.e. W is not of type A_1) although the resulting formula (2.8.3) is valid even that case.

Since the Coxeter graph Γ is a tree, one can find a unique decomposition $\Pi(C) = \Pi_1\cup\Pi_2$ such that any two elements in Π_i mutually commute for $i=1,2$ (ie. $\langle e_\alpha, f_\beta\rangle = 0$ for $\alpha\neq\beta\in\Pi_i$). Put $c_i := \prod_{\alpha\in\Pi_i}\alpha$ and $c = c_1c_2$. The mutual commutativity of elements in Π_i implies that $c_i(x) = x - \sum_{\alpha\in\Pi_i} e_\alpha(x)f_\alpha$ and that c_i is an involution, i.e. $c_i^2 = 1$.

We state a key lemma on the eigenvectors of the Coxeter elements (Kostant [K1] and Coleman [C]), which we shall use in §3-6 crucially.

Lemma. *There exists a real 2 dimensional subspace U of V^* satisfying:*

i) *U is invariant under the actions of c_1 and c_2, and $\{c_1|U, c_2|U\}$ forms a Coxeter system for the dihedral group $W(I_2(h))$ acting on U.*

ii) *$U \cap C = \mathbf{R}_{>0} \cdot \zeta_1 + \mathbf{R}_{>0} \cdot \zeta_2$ is a chamber of the group $\langle c_1|U, c_2|U\rangle = W(I_2(h))$, where ζ_j is a c_j-fixed vector in $\bigcap_{\alpha\in\Pi_j} H_\alpha \cap \bigcap_{\alpha\in\Pi\setminus\Pi_j}\{e_\alpha > 0\}$.*

The lemma implies in particular that no reflection hyperplane of W intersects the open cone $\mathbf{R}_{>0} \cdot \zeta_1 + \mathbf{R}_{>0} \cdot \zeta_2$.

Corollary. 1. *Any reflection hyperplane of W intersects U only along one of the h lines which are $W(I_2(h))$ orbits of $\mathbf{R}\zeta_1$ or $\mathbf{R}\zeta_2$. If a reflection hyperplane H_α contains the line $\mathbf{R}\zeta_i$, then $\alpha \in \Pi_i$.*

2. *Let W be a finite reflection group of rank l and Coxeternumber h.*

$$\text{(2.8.3)} \qquad i)\ m_1 = 1,\ m_l = h-1 \quad and \quad ii)\ \#R(W) \ = \ \tfrac{1}{2}lh$$

3. *The eigenvectors of the action $c|_U$ belonging to the eigenvalues $\exp(2\pi\sqrt{-1}/h)$ and $\exp(-2\pi\sqrt{-1}/h)$, respectively, do not belong to any complexified reflection hyperplane $H_{\alpha,\mathbf{C}} := H_\alpha \otimes \mathbf{C}$ for $\alpha \in R(W)$.*

Recall that $d_1, \cdots, d_l$ are the degrees of a generator system of $S(V)^W$. A study of the Jacobian J shows $d_j - 1 \equiv m_j \bmod h$ $(1 \le j \le l)$ for renewed index. This together with (2.6.1) ii), (2.8.2) and (2.8.3) implies

$$\text{(2.8.4)} \qquad d_i = m_i + 1 \qquad \text{for} \quad i = 1, \cdots, l.$$

Recall the W-invariant bilinear forms I and I^* on V and V^* such that $I(x, y) = I^*(I(x), I(y))$. The associated quadratic form

$$\text{(2.8.5)} \qquad P_1 := I^*(x, x)/2h = \textstyle\sum_{ij=1}^{l} X_i X_j I^*(x_i, x_j)/2h$$

(here $x = \sum_{i=1}^{l} X_i x_i$, and x_i and X_i are dual basis of V^* and V) gives an invariant in $S(V)^W$ of lowest degree $d = 2$ (unique up to constant since W-action is irreducible). This fact together with (2.8.4) implies

Corollary. 4. *The multiplicity of the smallest exponent* $(= 1)$ *is equal to 1. Hence, that of the largest exponent* $(= h - 1)$ *is also equal to 1.*

Remark. l and h cannot be simultaneously odd due to the second formula (2.8.2). More precisely (see 6.2 Assertion 1. for a proof): l is odd $\Rightarrow$ $\#\Pi_1 \neq \#\Pi_2$ $\Leftrightarrow$ h is even and $\frac{1}{2}h$ is an exponent $\Rightarrow$ h is even. Here, the two arrows are trivial. The converse of the first arrow does not hold for type D_l (even l). The converse of the second arrow does not hold for types B_l and C_l (even l), E_6, E_8, F_4, H_4 and $I_2(p)$ (even p).

2.9. **The quotient variety $V^*//W$ and the discriminant D_W.**

The categorical quotient variety of V^* by the action of W is given by

$$\text{(2.9.1)} \qquad S_W := V^*//W := \mathrm{Spec}(S(V)^W).$$

It has origin 0 defined by the maximal ideal $S(V)^W_+$ (recall 2.5).

Let K be either $\mathbf{R}$ or $\mathbf{C}$.The set of K-rational points of S_W is given by

$$S_{W,K} := \mathrm{Hom}^{alg}_{\mathbf{R}}(S(V)^W, K) \tag{2.9.2}$$

where $\mathrm{Hom}^{alg}_{\mathbf{R}}(*, *)$ means the set of all $\mathbf{R}$-algebra homomorphisms.

Put $V^*_{\mathbf{C}} := V^* \otimes_{\mathbf{R}} \mathbf{C}$. The action of W on $V^* = V^*_R$ extends complex linearly to $V^*_{\mathbf{C}}$. For any point $x \in V^*_K$, the evaluation homomorphism: $P \in S(V)^W \mapsto P(x) \in K$ induces the W-invariant morphism:

$$\pi_K : V^*_K \to S_{W,K}. \tag{2.9.3}$$

Put V^*_K/W :={W-orbits on V^*_K}, where a W-orbit on V^*_K means a subset of V^*_K of the form Wx for some $x \in V^*_K$. An element $P \in S(V)^W$ is naturally considered as a function on V^*_K/W since it is constant on each orbit. Since *for $x, y \in V_K$ one has $Wx = Wy$ if and only if $P(x) = P(y)$ for all $P \in S(V)^W$*, the morphism π_K (2.9.3) induces an injection $V^*_K/W \to S_{W,K}$ for $K = \mathbf{R}$ or $\mathbf{C}$. In fact, the $\pi_{\mathbf{C}}$ induces a homeomorphism: $V^*_{\mathbf{C}}/W \simeq S_{W,\mathbf{C}}$, but $\pi_{\mathbf{R}}$ induces an embedding $V^*_{\mathbf{R}}/W \subset S_{W,\mathbf{R}}$ onto a closed semi-algebraic set. Choosing a generator system $P_1, \cdots, P_l$ of $S(V)^W$ (with $\deg(P_1) \leq \cdots \leq \deg(P_l)$), one has a bijection $S_{W,K} \simeq K^l$ and the π_K is given by $(P_1, \cdots, P_l) : V^*_K \to K^l$.

The square δ^2_W of the anti-invariant δ_W (2.7) is an invariant. We call it the *discriminant* of W and denote by Δ_W. The *discriminant divisor* is defined by $\Delta_W = 0$. The *discriminant locus* in $S_{W,\mathbf{C}}$ is given by

$$D_{W,\mathbf{C}} := \{t \in S_{W,\mathbf{C}} \mid \Delta_W(t) = 0\}. \tag{2.9.4}$$

2.7 Lemma 2. implies i) the critical values of the morphism π lie in the discriminant D_W and ii) $(\pi_{\mathbf{C}})^{-1}D_{W,\mathbf{C}} = \bigcup_{\alpha \in R(W)} H_{\alpha,\mathbf{C}}$. Therefore,

Fact. 1. *Any W-fixed point in $V_{\mathbf{C}}$ lies in a reflection hyperplane.*

2. *The complement of the discriminant locus $S_{W,\mathbf{C}} \setminus D_{W,\mathbf{C}}$ is the space of regular (i.e. isotropy free) orbits of the W-action on $V_{\mathbf{C}}$.*

Let us express the discriminant Δ_W as a polynomial in $(P_1, \cdots, P_l)$. Since $\deg(\Delta_W) = hl$ (definition of δ_W and (2.8.3) ii)) and $\deg(P_l) = h$ ((2.8.3) i) and (2.8.4)), Δ_W is a polynomial in P_l of degree at most l:

$$\Delta_W = A_0 P_l^l + A_1 P_l^{l-1} + \cdots + A_l \tag{2.9.5}$$

where A_i is a polynomial in $P_1, \cdots, P_{l-1}$ of degree hi. Since $\Delta_W(\xi) \neq 0$ and $P_1(\xi) = \cdots = P_{l-1}(\xi) = 0$ for an eigenvector ξ of a Coxeter element belonging to $\exp(2\pi\sqrt{-1}/h)$ (use 2.8 Cor.3, 4 and (2.8.4)), one obtains the next goal of this section and the starting point of the present article:

Lemma. 1. *A_0 is non-zero. Hence, Δ_W is normalized to a monic polynomial of degree l in P_l and D_W has multiplicity l at the origin.*

2. *The eigenspace of a Coxeter element belonging to the eigenvalue $\exp(2\pi\sqrt{-1}/h)$ is mapped by $\pi_{\mathbf{C}}$ to a line $P_1 = \cdots = P_{l-1} = 0$ in $S_{W,\mathbf{C}}$.*

3. Flat structure

We describe the flat structure, the Frobenius manifold structure and the associated flat coordinates on the variety S_W in detail. The setting and the notation are the same as in §2, that is: W is a finite reflection group of a real vector space V and S_W is the quotient variety $V^*//W$ (recall 2.9). The flat structure is obtained by Fourier transform of the Levi-Civita connection for the W-invariant form I ([S3], [S6]).

3.1. **Logarithmic forms and logarithmic vector fields.**

We recall ([S4]) the definition and the basic properties of the modules of logarithmic forms and vector fields for the variety S_W with the divisor $D_W = \{\Delta_W = 0\}$ (see 2.9). In the sequel, we shall use coordinates $P_1, \cdots, P_l$ of S_W satisfying the degree conditions (2.8.1)-(2.8.5) by choosing a generator system of the invariants ring $S(V)^W$.

Let Der_{S_W} and $\Omega^1_{S_W}$ be the modules of $\mathbf{R}$-derivations of $S(V)^W$ and of 1-forms on S_W over $\mathbf{R}$, respectively. They are $S(V)^W$-free modules of rank l generated by the derivations $\partial/\partial P_i$ and by the differentials dP_i $(i = 1, \cdots, l)$, respectively. The logarithmic modules are defined by

$$\begin{array}{lcl} Der_{S_W}(-\log\Delta) & := & \{X \in Der_{S_W} \mid X\Delta_W \in \Delta_W S(V)^W\} \\ \Omega^1_{S_W}(\log\Delta) & := & \{\omega \in \frac{1}{\Delta_W}\Omega^1_{S_W} \mid d\omega \in \frac{1}{\Delta_W}\Omega^2_{S_W}\} \end{array} \tag{3.1.1}$$

where d is the exterior differentiation and $\Omega^2_{S_W} = \Omega^1_{S_W} \wedge \Omega^1_{S_W}$. It is easy to see that $Der_{S_W}(-\log\Delta)$ is closed under the bracket product and that $d\Omega^1_{S_W}(\log\Delta) \subset \Omega^1_{S_W}(\log\Delta) \wedge \Omega^1_{S_W}(\log\Delta)$.

The natural pairing $\langle\ \cdot\ ,\ \cdot\ \rangle$ between Der_{S_W} and $\Omega^1_{S_W}$ induces the $S(V)^W$-perfect-pairing: $Der_{S_W}(-\log\Delta) \times \Omega^1_{S_W}(\log\Delta) \to S(V)^W$ (i.e. they are $S(V)^W$-dual to each other) ([S4,(1.6) Lemma ii)]).

By identifying the (co-)tangent spaces T_xV^* or $T^*_xV^*$ at each point $x \in V^*$ with V^* or with the dual space V, respectively, the W-invariant forms I^* and I on V^* and V (recall 2.2) induce the $S(V)^W$-bilinear forms: $I^* : Der_{S_W} \times Der_{S_W} \to \frac{1}{\Delta}S(V)^W$ and $I : \Omega^1_{S_W} \times \Omega^1_{S_W} \to S(V)^W$, $I^*(\frac{\partial}{\partial P_i}, \frac{\partial}{\partial P_j}) = \Sigma_{pq}\frac{\partial X_p}{\partial P_i}\frac{\partial X_q}{\partial P_j}I^*(\frac{\partial}{\partial X_p}, \frac{\partial}{\partial X_q})$ and $I(dP_i, dP_j) = \Sigma_{pq}\frac{\partial P_i}{\partial X_p}\frac{\partial P_j}{\partial X_q}$ $I(dX_p, dX_q)$, where $X_1, \cdots, X_l$ is a linear coordinate system of V.

We now have the following important lemma.

Lemma. *The pairings I^* and I induce $S(V)^W$-perfect pairings:*

$$\begin{array}{l} I^* : Der_{S_W} \times Der_{S_W}(-\log\Delta) \to S(V)^W \\ I\ \ :\ \ \Omega^1_{S_W} \times\ \Omega^1_{S_W}(\log\Delta)\ \ \to\ \ S(V)^W \end{array} \tag{3.1.2}$$

This is equivalent to say that one has $S(V)^W$-isomorphisms

$$\begin{array}{lcl} I^* : & Der_{S_W} \simeq \Omega^1_{S_W}(\log\Delta) \\ I\ : & \Omega^1_{S_W} \simeq Der_{S_W}(-\log\Delta), \end{array} \tag{3.1.3}$$

which make the following diagram commutative:

$$\begin{array}{lccc} & Der_{S_W}(-\log\Delta) & \subset & Der_{S^W} \\ (3.1.4) & \uparrow I & & \downarrow I^* \\ & \Omega^1_{S_W} & \subset & \Omega^1_{S_W}(\log\Delta) \end{array}$$

Proof. We prove only the isomorphism $I : \Omega^1_{S_W} \simeq Der_{S_W}(-\log\Delta)$ since the other isomorphism I^* is obtained by taking its $S(V)^W$-dual. Recall δ_W such that $\delta_W^2 = \Delta_W$ 2.7. For any $\omega \in \Omega^1_{S_W}$, $I(\omega, d\delta) \in S(V)$ is W-anti-invariant, it is divisible by δ (2.7 Lemma 1). Thus, $I(\omega, d\Delta_W) = 2\delta_W I(\omega, d\delta_W)$ is divisible by δ_W^2, implying $I(\omega)$ belongs to $Der_{S_W}(-\log\Delta)$. To prove that the images $I(dP_i)$ $i = 1, \cdots, l$ form an $S(V)^W$-free basis of $Der_{S_W}(-\log\Delta)$, it is sufficient to show that the determinant of their coefficients matrix w.r.t. the basis $\frac{\partial}{\partial P_i}$ $i = 1, \cdots, l$ is a unit multiple of Δ_W (due to a theorem [S4,(1.7)Theorem ii)]). This is true due to 2.7 Lemma2:

$$(3.1.5) \qquad \begin{aligned} &\det((I(dP_i)(P_j))_{ij}) = \det((I(dP_i, dP_j))_{ij}) \\ = \ &\det\Big((\tfrac{\partial P_i}{\partial X_p})_{ip} \cdot (I(X_p, X_q))_{pq} \cdot (\tfrac{\partial P_j}{\partial X_q})_{jq}\Big) = c\delta^2 = c\Delta_W. \quad \Box \end{aligned}$$

Recalling (2.8.5), we have the following definition of *Euler operator*:

$$(3.1.6) \qquad E := I(dP_1) = \textstyle\sum_{i=1}^{l} \frac{m_i+1}{h} P_i \frac{\partial}{\partial P_i},$$

3.2. **The primitive vector field D and the invariants $S(V)^{W,\tau}$.**

We fix a particular vector field: the primitive vector field D ([S3,(2.2)]). The D is transversal to the discriminant locus (see Note below). This fact gives the quite important and key role to D in the sequel.

The Der_{S_W} is naturally a graded module since $S(V)^W$ is a graded algebra such that $\deg(\delta P) = \deg(\delta) + \deg(P)$ for any homogeneous $\delta \in Der_{S_W}$ and $P \in S(V)^W$. Due to the maximality $\deg(P_l) > \deg(P_i)$ for $i = 1, \cdots, l-1$ (c.f. 2.8 Corollary **4.**), the lowest graded piece of Der_{S_W} is a vector space of dimension 1 spanned by $\frac{\partial}{\partial P_l}$. We fix a base

$$(3.2.1) \qquad D := \frac{\partial}{\partial P_l} \quad \text{with the normalization} \quad DP_l = 1$$

and call it the *primitive vector field* or the *primitive derivation* (see 3.10 for the name). The D is unique, up to a scaling factor, independent of coordinates. We introduce the subring of $S(V)^W$ of D-invariants:

$$(3.2.2) \qquad S(V)^{W,\tau} := \{P \in S(V)^W \mid DP = 0\}.$$

One has $S(V)^{W,\tau} = \mathbf{R}[P_1, \cdots, P_{l-1}]$ and $S(V)^W = S(V)^{W,\tau}[P_l]$.

Note. The 1-parameter group action $\exp(tD)$ on S_W is denoted by τ_t [S8, (3.1)]. This justifies the notation (3.2.2) since $S(V)^{W,\tau} = \{P \in S(V)^W \mid P \circ \tau_t = P \ \forall t\}$. The τ-action is transversal to the discriminant locus D_W ([S8, (3.4) Lemma 6], recall also 2.9 Lemma 2.).

We, further, introduce the "descent" modules of Der_{S_W} and $\Omega^1_{S_W}$:

$$
\begin{array}{rcl}
\mathcal{G} & := & \{\delta \in Der_{S_W} \mid [D,\delta]=0\} \\
\mathcal{F} & := & \{\omega \in \Omega^1_{S_W} \mid L_D\omega=0\}
\end{array}
\tag{3.2.3}
$$

where L_D is the Lie derivative given by $\langle L_D\omega,\delta\rangle = D\langle\omega,\delta\rangle - \langle\omega,[D,\delta]\rangle$. These $\mathcal{G}$ and $\mathcal{F}$ are $S(V)^{W,\tau}$-free modules of rank l with free dual basis $\frac{\partial}{\partial P_1},\cdots,\frac{\partial}{\partial P_l}$ and $dP_1,\cdots,dP_l$, respectively. One has the expressions:

$$
\begin{array}{rcl}
Der_{S_W} & = & \mathcal{G}\otimes_{S(V)^{W,\tau}} S(V)^W, \\
\Omega^1_{S_W} & = & \mathcal{F}\otimes_{S(V)^{W,\tau}} S(V)^W.
\end{array}
\tag{3.2.4}
$$

The $\mathcal{G}$ is closed under the bracket product and acts naturally on $S(V)^W$ as derivations. In fact, $\mathcal{G}$ is an Abelian extension of $Der_{S(V)^{W,\tau}}$:

$$
0 \to S(V)^{W,\tau}D \to \mathcal{G} \to Der_{S(V)^{W,\tau}} \to 0.
\tag{3.2.5}
$$

Combining (3.2.4) with (3.1.3), one gets the "descent expressions":

$$
\begin{array}{rcl}
Der_{S_W}(-\log\Delta) & = & I(\mathcal{F})\otimes_{S(V)^{W,\tau}} S(V)^W, \\
\Omega^1_{S_W}(\log\Delta) & = & I^*(\mathcal{G})\otimes_{S(V)^{W,\tau}} S(V)^W.
\end{array}
\tag{3.2.6}
$$

Note. The inclusion: $S(V)^{W,\tau}\subset S(V)^W$ induces the projection

$$
\pi_W : S_W \to \mathrm{Spec}(S(V)^{W,\tau})
\tag{3.2.7}
$$

forgetting the last coordinate P_l. By "descent", we mean that some geometric structure on S_W is a pull-back of that on $\mathrm{Spec}(S(V)^{W,\tau})$. We do not use explicitly the morphism π_W until §5 .

3.3. Metrics J and J^*.

We introduce non-degenerate symmetric bilinear forms J and J^* on the tangent and cotangent bundle of S_W, respectively. In fact, instead of introducing $S(V)^W$-bilinear forms on Der_{S_W} and $\Omega^1_{S_W}$, we introduce their descent $S(V)^{W,\tau}$-bilinear forms on the descent modules $\mathcal{G}$ and $\mathcal{F}$.

Definition. The Lie derivative L_DI defines a $S(V)^{W,\tau}$ -bilinear form:

$$
J^* : \mathcal{F}\times\mathcal{F} \to S(V)^{W,\tau}, \quad \omega_1\times\omega_2 \to DI(\omega_1,\omega_2).
\tag{3.3.1}
$$

Lemma. *The form J^* is nondegenerate everywhere on S_W. That is: $\det(J^*(dP_i,dP_j)_{ij=1,\cdots,l}$ is a non zero constant.*

Proof. One has an expression (recall 2.7 Lemma 2 and (2.9.5)):

$$
\det\left((I(dP_i,dP_j))_{ij}\right) = \Delta = A_0P_l^l + A_1P_l^{l-1} + \cdots + A_l
$$

where $A_i\in S(V)^{W,\tau}$. On the other hand, since $\deg(I^*(dP_i,dP_j)) = m_i+m_j < 2h\ (=2\deg(P_l))$ (recall (2.8.4), (2.8.1)), each entry $I^*(dP_i,dP_j)$ (as an element of $S(V)^W$) contains P_l at most linearly. Comparing these two facts, one obtains $\det\left((DI(dP_i,dP_j))_{ij}\right) = A_0$. But it was shown in 2.9 Lemma 1. that $A_0\neq 0$. □

The degree of the ij-entries of the matrix expression of J^* is given by

$$\deg(J^*(dP_i, dP_j)) = m_i + m_j - h. \tag{3.3.2}$$

So, if $m_i + m_j - h < 0$ then the entry vanishes. In view of the duality of the exponents (2.8.2), the matrix is a "skew lower triangular" matrix.

Since J^* is $S(V)^{W,\tau}$-nondegenerate, it induces $S(V)^{W,\tau}$-isomorphism

$$J^* : \mathcal{F} \simeq \mathcal{F}^* = \mathcal{G}, \ J^*(dP_i) := J^*(dP_i, d\cdot) = \textstyle\sum_{j=1}^{l} J^*(dP_i, dP_j)\frac{\partial}{\partial P_j}.$$

By this isomorphism, J^* induces a $S(V)^{W,\tau}$-symmetric-bilinear form on the dual module $\mathcal{G}$, which we shall denote by J. That is:

$$J : \mathcal{G} \times \mathcal{G} \to S(V)^{W,\tau}, \quad J(\delta_1, \delta_2) := J^*((J^*)^{-1}\delta_1, (J^*)^{-1}\delta_2). \tag{3.3.3}$$

Again, J is a nondegenerate form on $\mathcal{G}$. Due to (3.1.6), one has

$$J^*(dP_1) = D \quad \text{and} \quad J(D) = dP_1. \tag{3.3.4}$$

Note. The form J is identified with the residue pairing from a view point of the primitive form theory [S3] (c.f. (5.2.10)).

We shall show that *the metric J is flat, i.e. the curvature for J is zero.* This fact is a part of the *flat structure on S_W* given in 3.8 and 3.9. The following subsections 3.4 - 3.7 are devoted to the preparation.

3.4. **Relationship between I and J.**

The nondegeneracy of J^* implies the following quite important decomposition lemma, which leads to a reconstruction of I from J.

Lemma. *One has a direct sum decomposition as $S(V)^{W,\tau}$-module:*

$$Der_{S_W} \ = \ \mathcal{G} \oplus Der_{S_W}(-\log \Delta). \tag{3.4.1}$$

Proof. We prove a more precise formula: for $k \in \mathbf{Z}_{\geq 0}$, one has

$$S(V)^W_{\leq k}\ \mathcal{G} \ = \ \mathcal{G} \oplus S(V)^W_{\leq k-1} I(\mathcal{F}), \tag{3.4.2}$$

where $S(V)^W_{\leq k} := \{P \in S(V)^W \mid D^{k+1}P = 0\}$ $(k \in \mathbf{Z}_{\geq -1})$ is the module of polynomials in P_l of coefficients in $S(V)^{W,\tau}$ of degree $\leq k$.

Recall (3.1.3) that $I(dP_i) = \sum_{i=1}^{l} I(dP_i, dP_j)\frac{\partial}{\partial P_j}$ $(i = 1, \cdots, l)$ form $S(V)^W$-basis of $Der_{S_W}(-\log \Delta)$. Furthermore, the coefficient $I(dP_i, dP_j)$ is at most linear in P_l and $[D, I(dP_i)]$ $(i = 1, \cdots, l)$ are linearly independent over $S(V)^{W,\tau}$ (non-degeneracy of J^*).

We prove $\mathcal{G} \cap Der_{S_W}(-\log \Delta) = \{0\}$: if $\delta = \sum_{i=1}^{l} e_i I(dP_i) \in \mathcal{G}$ for $e_i \in S(V)^W_{\leq k}$ then we prove $e_i = 0$ by induction on k. The case $k = -1$ is true by definition $S(V)^W_{\leq -1} = 0$. Suppose $k \geq 0$. By assumption $\delta \in \mathcal{G}$, one has $0 = \mathrm{ad}(D)^{k+1}\delta = \sum_{i=1}^{l} \sum_{j=0}^{k+1} C(k+1, j)$ $(D^j e_i)(\mathrm{ad}(D)^{k+1-j} I(dP_i))$ where all terms except for $j = k$ vanishes. So, we obtain $(k+1)\sum_{i=1}^{l}(D^k e_i)[D, I(dP_i)] = 0$. Then the linear independence of $[D, I(dP_i)]$ implies the vanishing of the coefficients $D^k e_i$. So, $e_i \in S(V)^W_{\leq k-1}$ and the induction applies.

The LHS of (3.4.2) includes RHS since $I(dP_i)$ is at most linear in P_l. We prove the opposite inclusion relation by an induction on k. Case $k = 0$ is clear (note $S(V)^W_{\leq -1} = \{0\}$ and $S(V)^W_{\leq 0} = S(V)^{W,\tau}$). Let $\delta = \sum_{i=1}^l f_i \frac{\partial}{\partial P_i} \in S(V)^W_{\leq k}\mathcal{G}$ for $k > 0$, where the coefficient of P_l^k in f_i is denoted by $f_i^{(k)}$. One can find $g_j \in S(V)^{W,\tau}$ $(j = 1, \cdots, l)$ such that $f_i^{(k)} = \sum_{j=1}^l J^*(dP_i, dP_j)g_j$. Then $\delta - P_l^{k-1}\sum_{j=1}^l I(dP_j)g_j$ belongs to $S(V)^W_{\leq k-1}\mathcal{G}$, so that one applies the induction hypothesis. $\square$

The next corollaries shall be used in §4 to lift $\mathcal{G}$ to the normalization of the discriminant D_W. First, note that the ideal $(\partial\Delta) := Der_{S_W} \cdot \Delta$ in $S(V)^W$ contains the ideal (Δ) since $E\Delta = hl\Delta$.

Corollary. 1. *The expression $\mathcal{G} \simeq Der_{S_W}/Der_{S_W}(-\log\Delta)$ gives on $\mathcal{G}$ a $S(V)^W$-module structure of homological dimension ≤ 1.*

2. *The correspondence $\delta \in \mathcal{G} \mapsto \delta\Delta$ induces a $S(V)^W$-isomorphism:*

$$\mathcal{G} \simeq (\partial\Delta)/(\Delta). \tag{3.4.3}$$

Proof. 1. Trivial. 2. Surjectivity: Due to the decomposition (3.4.1), one has $Der_{S_W} \cdot \Delta = \mathcal{G} \cdot \Delta + (\Delta)$, which implies the surjectivity.

Injectivity: Suppose $\delta \in \mathcal{G}$ is mapped to 0. This means $\delta\Delta \in (\Delta)$ and $\delta \in Der_{S_W}(-\log\Delta)$. The direct sum (3.4.1) implies $\delta = 0$. $\square$

Denote by P_l* the multiplication of $P_l \in S(V)^W$ on $\mathcal{G}$. Define

$$w : \mathcal{G} \to Der_{S_W}(-\log\Delta),\ w(\delta) := P_l\delta - P_l * \delta \in I(\mathcal{F}). \tag{3.4.4}$$

So, the decomposition (3.4.2) of the element $P_l\delta$ for $\delta \in \mathcal{G}$ is given by

$$P_l\delta = P_l * \delta + w(\delta). \tag{3.4.5}$$

Assertion. *i) $w(\delta)$ is the unique element in $Der_{S_W}(-\log(\Delta))$ with*

$$[D, w(\delta)] = \delta. \tag{3.4.6}$$

ii) The w maps $S(V)^{W,\tau}$-free basis of $\mathcal{G}$ to $S(V)^W$-free basis of $Der_{S_W}(-\log(\Delta))$ (e.g. $w(D) = E$, c.f. (3.1.6) and (3.3.4)).

Proof. i) The $w(\delta)$ obviously satisfies (3.4.6). If $w_1, w_2 \in Der_{S_W}(-\log\Delta)$ satisfies $[D, w_1] = [D, w_2]$. Then $w_1 - w_2 \in \mathcal{G}$ and is 0 by (3.4.1).

ii) Due to i), one has $w(J^*(dP_i)) = I(dP_i)$. $\square$

Lemma. *For $\omega \in \mathcal{F}$ and for $\delta_1, \delta_2 \in \mathcal{G}$, one has the formulae*

$$I(\omega) = w(J^*(\omega)) \quad and \quad J(\delta_1, \delta_2) = I^*(w(\delta_1), \delta_2) \tag{3.4.7}$$

For a $S(V)^{W,\tau}$-basis $\delta_1, \cdots, \delta_l$ and its J-dual basis $\delta_1^*, \cdots, \delta_l^*$, one has

$$I = \textstyle\sum_{i=1}^l \delta_i \otimes w(\delta_i^*). \tag{3.4.8}$$

Proof. By definition of J^* (3.3.1), one has $J^*(\omega) = L_D I(\omega) = [D, I(\omega)]$. Then applying the characterization (3.4.6), one gets the first formula of (3.4.7). Using this, the second formula of (3.4.7) is calculated as: $I^*(w(\delta_1), \delta_2) = I^*(w(J^*J\delta_1), \delta_2) = I^*(IJ\delta_1, \delta_2) = J(\delta_1, \delta_2)$.

The coupling of the first tensor factor of RHS of (3.4.8) with $\omega \in \mathcal{F}$ is given by $\sum_{i=1}^{l} \delta_i(\omega) \cdot w(\delta_i^*) = w(\sum_{i=1}^{l} \delta_i(\omega) \cdot \delta_i^*) = w(J^*(\omega)) = I(\omega)$, which is equal to the coupling of $\omega \in \mathcal{F}$ with LHS of (3.4.8). $\square$

3.5. **The Levi-Civita connection ∇ on Der_{S_W}.**

The Levi-Civita connection attached to the metric I^* [3] induces a connection ∇ on Der_{S_W} which is singular along the discriminant. We describe the singularity of ∇ in terms of logarithmic vector fields.

First we list up the properties, which should be satisfied by ∇.

i) $\nabla : Der_{S_W} \times Der_{S_W} \to Der_{S_W}(*\Delta)$ *(= the localization of Der_{S_W} by Δ) is a covariant differentiation. That is: $\nabla_{\delta_1}\delta_2$ for $\delta_1,\ \delta_2 \in Der_{S_W}$ is $S(V)^W$-linear in δ_1 and additive in δ_2 satisfying the Leibniz rule:*

$$\nabla_{\delta_1}(P\delta_2) = (\delta_1 P)\delta_2 + P(\nabla_{\delta_1}\delta_2) \qquad (P \in S(V)^W). \tag{3.5.1}$$

ii) ∇ *preserves I^*: $\nabla I^* = 0$. That is: for $\delta_1, \delta_2, \delta_3 \in Der_{S_W}$ one has*

$$\delta_1 I^*(\delta_2, \delta_3) = I^*(\nabla_{\delta_1}\delta_2, \delta_3) + I^*(\delta_2, \nabla_{\delta_1}\delta_3). \tag{3.5.2}$$

iii) ∇ *is torsion free. That is: for $\delta_1, \delta_2 \in Der_{S_W}$ one has*

$$\nabla_{\delta_1}\delta_2 - \nabla_{\delta_2}\delta_1 = [\delta_1, \delta_2]. \tag{3.5.3}$$

The ∇ is determined uniquely by i),ii) and iii) by the formula:

$$\begin{aligned} 2I^*(\nabla_{\delta_1}\delta_2, \delta_3) = &+\delta_1 I^*(\delta_2, \delta_3) + \delta_2 I^*(\delta_3, \delta_1) - \delta_3 I^*(\delta_1, \delta_2) \\ &-I^*(\delta_1, [\delta_2, \delta_3]) + I^*(\delta_2, [\delta_3, \delta_1]) + I^*(\delta_3, [\delta_1, \delta_2]) \end{aligned} \tag{3.5.4}$$

for $\delta_1, \delta_2, \delta_3 \in Der_{S_W}$. Conversely, the (3.5.4) defines a connection ∇ satisfying i), ii) and iii): the *Levi-Civita connection* attached to I^*.

Using properties of logarithmic vector fields in 3.1, one checks that RHS of (3.5.4) belongs to $S(V)^W$ if two of δ_1, δ_2 and δ_3 are logarithmic. This means that the domain and the range of ∇ can be chosen as

$$\nabla\ :\ \begin{cases} Der_{S_W} \times Der_{S_W}(-\log(\Delta)) \to Der_{S_W}, \\ Der_{S_W}(-\log(\Delta)) \times Der_{S_W} \to Der_{S_W}, \\ Der_{S_W}(-\log(\Delta)) \times Der_{S_W}(-\log(\Delta)) \\ \qquad\qquad\qquad\qquad \to\ Der_{S_W}(-\log(\Delta)). \end{cases} \tag{3.5.5}$$

For short, we say that *the connection ∇ has logarithmic singularities.* In particular, the second line implies that the connection form for ∇ belongs to $End(Der_{S_W}) \otimes \Omega^1_{S_W}(\log(\Delta))$, i.e. the connection form for

[3] There is an unfortunate disagreement on the notation I and I^* between [S3] and [S6]. We employed notation of [S6] since it agree with that of root systems.

∇ as the connection on the tangent bundle of S_W has the logarithmic pole along the discriminant. We shall not use this fact explicitly.
iv) ∇ *is integrable:* $\nabla^2 = 0$. *That is: for* $\delta_1, \delta_2, \delta_3 \in Der_{S_W}$

$$\nabla_{\delta_1}\nabla_{\delta_2}\delta_3 - \nabla_{\delta_2}\nabla_{\delta_1}\delta_3 = \nabla_{[\delta_1,\delta_2]}\delta_3. \tag{3.5.6}$$

(*Proof.* This can be seen by extending the domain and range of ∇ to $Der_V(*\Delta) = \sum_{i=1}^l S(V)_\Delta \frac{\partial}{\partial X_i}$ (such extension exists as the metric connection on Der_V). Then RHS of (3.5.4) vanishes by the substitution of $\delta_1, \delta_2, \delta_3$ by $\frac{\partial}{\partial X_i}, \frac{\partial}{\partial X_j}$ and by $\frac{\partial}{\partial X_k}$. This implies $\nabla\frac{\partial}{\partial X_i} = 0$ for $i = 1, \cdots, l$. Expressing $Der_{S_W} \subset \sum_{i=1}^l S(V)_\Delta \frac{\partial}{\partial X_i}$, we obtain the result.)

3.6. **Key Lemma.**

We want to show that the action of ∇_D is invertible on Der_{S_W}. This is achieved by showing an isomorphism $\nabla_D : Der_{S_W}(-\log\Delta) \simeq Der_{S_W}$ and taking its inverse map in 3.8. For this end, we give a key lemma:

Lemma. 1. *The* ∇_δ *for* $\delta \in \mathcal{G}$ *maps* $I(\mathcal{F})$ *into* $\mathcal{G} \oplus I(\mathcal{F})$. *Furthermore,* ∇_D *induces an* $S(V)^{W,\tau}$*-isomorphism:*

$$\nabla_D : I(\mathcal{F}) \simeq \mathcal{G}. \tag{3.6.1}$$

2. *The image of the basis* $I(dP_i) \in I(\mathcal{F})$ $(i = 1, \cdots, l)$ *is given by*

$$\begin{aligned} &\nabla_D I(dP_i) \\ &= \sum_{j=1}^l \left(\tfrac{h+m_i-m_j}{2h} J^*(dP_i, dP_j) + \sum_{k=1}^{l-1} \tfrac{m_k+1}{2h}(DA_{ij}^k)P_k\right)\tfrac{\partial}{\partial P_j}. \end{aligned} \tag{3.6.2}$$

Here DA_{ij}^k *is an element of* $S(V)^{W,\tau}$ *given by*

$$DA_{ij}^k = \sum_{p,q=1}^l (J_{ip}J_{jq} - J_{jp}J_{iq})\tfrac{\partial}{\partial P_q} J^{pk} \tag{3.6.3}$$

with $J_{ij} := J^*(dP_i, dP_j)$ *and* $J^{ij} := J(\frac{\partial}{\partial P_i}, \frac{\partial}{\partial P_j})$. *One has*

$$\nabla_D E = \tfrac{1}{h} D. \tag{3.6.4}$$

Proof. 1. Let $\delta \in \mathcal{G}$. Because of (3.5.5), we know that $\nabla_\delta(I(\mathcal{F})) \subset Der_{S_W}$. In view of (3.2.6), it is sufficient to prove $I^*(\nabla_\delta(I(\mathcal{F})), I(\mathcal{F})) \subset S(V)^W_{\le 1}$ in order to prove $\nabla_\delta(I(\mathcal{F})) \subset \mathcal{G} \oplus I(\mathcal{F}) = \mathcal{G} \otimes_{S(V)^{W,\tau}} S(V)^W_{\le 1}$. So, for $U, V \in S(V)^W$, by using (3.5.4), we calculate

$$\begin{aligned} &2I(\nabla_\delta I(dU), I(dV)) \\ = \ &+\delta I(dU, dV) \ + I(dU)\delta V \ - I(dV)\delta U \\ &-I^*(\delta, [I(dU), I(dV)]) + [I(dV), \delta]U + [\delta, I(dU)]V \end{aligned}$$

Developing the last two brackets, one cancels the second and the third terms in the RHS. By eliminating I and I^* in the LHS, one obtains:

$$2\,\langle \nabla_\delta I(dU), dV\rangle \ = \ \delta I(dU, dV) - I^*(\delta, [I(dU), I(dV)]). \tag{3.6.5}$$

In particular, for $U = P_i$ and $V = P_j$, one obtains

$$((3.6.5)^*) \qquad 2\,\langle \nabla_\delta I(dP_i), dP_j\rangle \;=\; \delta I(dP_i, dP_j) - \textstyle\sum_{k=1}^{l} A_{ij}^k \cdot \delta P_k$$

where A_{ij}^k shall be defined in the formula (3.6.6) and the above formula is calculated since one has $I(\delta, [I(dP_i), I(dP_j)]) = \sum_{k=1}^{l} A_{ij}^k \cdot \delta P_k$.

Since the module of logarithmic vector fields are closed under bracket product 3.1, there exists $A_{ij}^k \in S(V)^W$ for $i, j, k = 1, \cdots, l$ such that

$$(3.6.6) \qquad [I(dP_i), I(dP_j)] = \textstyle\sum_{k=1}^{l} A_{ij}^k I(dP_k).$$

Since the degree in P_l of the coefficients in LHS is at most 2, the A_{ij}^k in the RHS has degrees in P_l at most 1 (recall (3.4.2)). Thus the RHS of $(3.6.5)^*$ has degree in P_l at most 1. This proves the first half of 1.

In order to prove $\nabla_D(I(\mathcal{F})) \subset \mathcal{G}$, we show $\langle \nabla_D(I(\mathcal{F})), \mathcal{F}\rangle \subset S(V)^{W,\tau}$. Substitute δ by D in $(3.6.5)^*$, and one has

$$(*) \qquad 2\,\langle \nabla_D I(dP_i), dP_j\rangle \;= J^*(dP_i, dP_j) \;-\; A_{ij}^l.$$

Therefore, we have only to show $A_{ij}^l \in S(V)^{W,\tau}$ for $i, j = 1, \cdots, l$. Apply the both hand side of (3.6.6) to the invariant P_1 of lowest degree (2.8.5). Recalling (3.1.6), the LHS gives $h[I(dP_i), I(dP_j)]P_1 = I(dP_i)((m_j+1)P_j) - I(dP_j)((m_i+1)P_i) = (m_j - m_i)I(dP_i, dP_j)$ and the RHS gives $h\sum_{k=1}^{l} A_{ij}^k (m_k+1)P_k$. So, one gets an equality:

$$(**) \qquad (m_j - m_i)I(dP_i, dP_j) \;=\; \textstyle\sum_{k=1}^{l} A_{ij}^k (m_k+1)P_k.$$

Apply D^2 to $(**)$. Since LHS vanishes and $D^2 A_{ij}^k = 0$, one obtains $0 = DA_{ij}^l(m_l+1)$. That is: $A_{ij}^l \in S(V)^{W,\tau}$ and $\nabla_D(I(dP_i)) \in \mathcal{G}$.

2. Apply D once to $(**)$ and noting $m_l + 1 = h$, one has the equality:

$$(3.6.7) \qquad A_{ij}^l \;=\; \tfrac{m_j - m_i}{h} J^*(dP_i, dP_j) - \textstyle\sum_{k=1}^{l-1} \tfrac{m_k+1}{h}(DA_{ij}^k)P_k.$$

Combining (3.6.7) with $(*)$, one obtains the formula (3.6.2).

Since ∇_D is a $S(V)^{W,\tau}$-homomorphism, in order to show the isomorphy (3.6.1), it is sufficient to show the non-degeneracy of the matrix $M = (M_{ij})_{ij=1}^{l}$, where M_{ij} is the coefficient of $\frac{\partial}{\partial P_j}$ of the RHS of (3.6.2). First, one notes that the weighted degree of M_{ij} is $m_i + m_j - h$ (3.3.2). So, $\det(M)$ should be of degree $\sum_{i=j=1}^{l}(m_i + m_j - h) = 0$, and hence, $\det(M)$ is a constant. Thus, the positive degree entries of M has no contributions to $\det(M)$ and, hence, one may calculate the determinant modulo the maximal ideal $S(V)_+^{W,\tau}$ in $S(V)^{W,\tau}$. So, the second term in (3.6.2) (containing the positive weighted factor P_k) can be ignored. Also, $J^*(dP_i, dP_j) \equiv 0$ modulo $S(V)_+^{W,\tau}$, if $m_i + m_j \neq h$. The remaining are only ij-entries of the matrix $(\frac{h+m_i-m_j}{2h} J^*(dP_i, dP_j))$ for

$m_i+m_j=h$. Then the factor $\frac{h+m_i-m_j}{2h}$ in the ij-entry is $\frac{m_i}{h}$. Therefore, $\det(M)=(\prod_{i=1}^l\frac{m_i}{h})\det((J^*(dP_i,dP_j))_{ij})\neq 0$ (c.f. 3.3 Lemma).

Finally, let us prove (3.6.3). Take the pairing of the vector field (3.6.6) with a form dP_p for a $1\leq p\leq l$. So, we obtain:

$$I(dP_i,dI(dP_j,dP_p))-I(dP_j,dI(dP_i,dP_p))=\textstyle\sum_{k=1}^l A_{ij}^k I(dP_k,dP_p).$$

This is of degree at most 2 in P_l. Apply D^2 and one obtains

$$\textstyle\sum_{q=1}^l J_{iq}\frac{\partial}{\partial P_q}J_{jp}-\sum_{q=1}^l J_{jq}\frac{\partial}{\partial P_q}J_{ip}=\sum_{k=1}^l(DA_{ij}^k)J_{kp} \tag{3.6.8}$$

Multiply J^{pm} and sum for $p=1,\cdots,l$. Since $\sum_{p=1}^l J_{jp}J^{pm}=\delta_j^m$ (Kronecker's delta), one replaces $\sum_{p=1}^l(\frac{\partial}{\partial P_q}J_{jp})J^{pm}$ by $-\sum_{p=1}^l J_{jp}(\frac{\partial}{\partial P_q}J^{pm})$. Finally, replacing the index m by k, one gets (3.6.3). The (3.6.4) is obtained by substituting $i=1$ in (3.6.2) and applying (3.1.6). □

Note. These A_{ij}^k and DA_{ij}^k depend on a choice of the coordinate system $P_1,\cdots,P_l$. In particular, $DA_{ij}^k=0$ if all J_{ij} are constant (c.f. (3.6.3)). This occurs if $P_1,\cdots,P_l$ is a *flat coordinate system* (c.f. 3.11).

3.7. Star product $*$, connection $\nabla\!\!\!/$ and exponent N.

As consequences of 3.6 Lemma, we introduce some structures on $\mathcal{G}$. They shall form building blocks of the flat structure in 3.9.

1. **Star product $*$ and Connection $\nabla\!\!\!/$.**

One has a bilinear map: $\mathcal{G}\times\mathcal{G}\to\mathcal{G}\oplus\nabla_D^{-1}\mathcal{G}$ by letting $\delta_1,\delta_2\mapsto\nabla_{\delta_1}\nabla_D^{-1}\delta_2$ (use 3.6 Lemma 1). Then, by decomposing the image into the direct summands as:

$$\nabla_{\delta_1}\nabla_D^{-1}\delta_2=\delta_1*\delta_2+\nabla_D^{-1}\nabla\!\!\!/_{\delta_1}\delta_2, \tag{3.7.1}$$

we introduce two binary operations:

$$*:\mathcal{G}\times\mathcal{G}\to\mathcal{G}, \tag{3.7.2}$$

$$\nabla\!\!\!/:\mathcal{G}\times\mathcal{G}\to\mathcal{G}. \tag{3.7.3}$$

We give two direct consequences of the definition. A complete list of properties of $*$ and $\nabla\!\!\!/$ (and of N defined below) shall be given in 3.9.

i) *The $*$ is $S(V)^{W,\tau}$-bilinear and, so, is regarded as a distributive product structure on $\mathcal{G}$. The primitive vector field D is the (left) unit.*

$$D*\delta=\delta. \tag{3.7.4}$$

Using (3.2.4), we extend the $*$ to a $S(V)^W$-algebra structure on Der_{S_W}.

ii) *The $\nabla\!\!\!/_{\delta_1}\delta_2$ is a covariant differentiation of δ_2 by δ_1. One has*

$$\nabla\!\!\!/_D\delta\equiv 0\quad \textit{for}\quad \delta\in\mathcal{G}. \tag{3.7.5}$$

By the extension (3.2.4), one may regard $\nabla\!\!\!/$ as a connection on Der_{S_W}.

2. **Exponent map N.**

Recall the principal part $w(\delta)$ of $P_l\delta$ for $\delta \in \mathcal{G}$ (see (3.4.4) and (3.4.5)). We introduce a $S(V)^{W,\tau}$-endomorphism N of $\mathcal{G}$ by the composition:

$$N : \mathcal{G} \to \mathcal{G}\ , \quad N(\delta) := \nabla_D(w(\delta)). \tag{3.7.6}$$

and call it the *exponent map* (in analogy with Fuchsian type equations). By a use of N, the decomposition formula (3.4.5) is rewritten as

$$P_l\ \delta \ = \ P_l * \delta \ + \ \nabla_D^{-1} N(\delta). \tag{3.7.7}$$

iii) *The exponent map N is independent of choices of P_l and D.*

Proof. One has to check that the changes $D \mapsto cD$ and $P_l \mapsto c^{-1}P_l + g$ for $c = const. \neq 0$ and $g \in S(V)^{W,\tau}$ may not cause a change of N. □

Lemma. *The eigenvalues of the exponent map N are $\frac{m_i}{h}$ $i = 1, \cdots, l$.*

Proof. Combining (3.7.6) with (3.4.7) and (3.6.2), one calculates:

$$\begin{aligned} N(J^*(dP_i)) = \nabla_D(w(J^*(dP_i))) = \nabla_D I(dP_i) = \ & \tfrac{m_i}{h} J^*(dP_i) \\ + \textstyle\sum_{j=1}^{l} \Big(\tfrac{h-m_j-m_i}{2h} J^*(dP_i, dP_j) + \sum_{k=1}^{l-1} \tfrac{m_k+1}{2h} (DA_{ij}^k) P_k \Big) \tfrac{\partial}{\partial P_j}. \end{aligned}$$

Coefficients of $\frac{\partial}{\partial P_j}$ in the RHS consist only of positively graded terms (since the degree of $J^*(dP_i, dP_j)$ equals to 0 if and only if $m_j + m_i = h$ (3.3.2)). This implies that $N - \mathrm{diag}[\frac{m_i}{h}]_{i=1}^l$ is nilpotent: the eigenvalues of N coincide with that of the diagonal matrix $\mathrm{diag}[\frac{m_i}{h}]_{i=1}^l$. □

3.8. **Fourier transformation (gravitational descendent).**

We transform the $S(V)^{W,\tau}[P_l]$-free-module Der_{S_W} to a $S(V)^{W,\tau}[D^{-1}]$-free-module. This is a formal Fourier-Laplace transformation (called the gravitational descendent in the topological field theory).

Theorem. 1. *The covariant derivative ∇_D with respect to the primitive vector field D induces a $S(V)^{W,\tau}$-isomorphism*

$$\nabla_D : Der_{S_W}(-\log \Delta) \ \xrightarrow{\sim} \ Der_{S_W}. \tag{3.8.1}$$

Therefore, ∇_D^{-1} is a well-defined $S(V)^{W,\tau}$-endomorphism of Der_{S_W}.

2. *The correspondence $\sum_k (\nabla_D)^{-k}\delta_k \mapsto \sum_k D^{-k} \otimes \delta_k$ ($\delta_k \in \mathcal{G}$ and is $= 0$ except for finite number of $k \in \mathbf{Z}_{\geq 0}$) gives isomorphisms:*

$$\begin{array}{ccc} Der_{S_W} & \simeq & \mathbf{Z}[D^{-1}] \otimes \mathcal{G} \quad \textit{and} \\ Der_{S_W}(-\log \Delta) & \simeq & D^{-1}\mathbf{Z}[D^{-1}] \otimes \mathcal{G} \end{array} \tag{3.8.2}$$

as $S(V)^{W,\tau}[D^{-1}]$-modules, where $\mathbf{Z}[D^{-1}]$ is the polynomial ring with the indeterminate D^{-1} which commutes with the elements of $S(V)^{W,\tau}$.

3. *The left-multiplication of P_l and the covariant differentiation ∇_δ with respect to $\delta \in \mathcal{G}$ on the RHS of* (3.8.2) *are given as follows.*

$$(3.8.3) \qquad P_l(D^{-k}\otimes\delta) = D^{-k}\otimes(P_l*\delta) + D^{-k-1}\otimes(N(\delta)+k\delta)$$

$$(3.8.4) \qquad \nabla_{\delta_1}(D^{-k-1}\otimes\delta_2) = D^{-k}\otimes(\delta_1*\delta_2) + D^{-k-1}\otimes\nabla\!\!\!/_{\delta_1}\delta_2.$$

The $D^{-k}\mathbf{Z}[D^{-1}]\otimes\mathcal{G}$ *for* $k\in\mathbf{Z}_{\geq 0}$ *is a* $S(V)^W$*-free module of rank* l.

Proof. **1.** and **2.** Let us prove by induction on $k\in\mathbf{Z}_{>0}$ the following:

A(k): *There exists a sequence* $\mathcal{G}_0,\cdots,\mathcal{G}_k$ *of* $S(V)^{W,\tau}$*-free submodules of* Der_{S_W} *of rank* l *such that* $S(V)^W_{\leq i}\mathcal{G} = \mathcal{G}_0\oplus\cdots\oplus\mathcal{G}_i$, $S(V)^W_{\leq i-1}I(\mathcal{F}) = \mathcal{G}_1\oplus\cdots\oplus\mathcal{G}_i$ *and* $\nabla_D:\mathcal{G}_i\simeq\mathcal{G}_{i-1}$ *for* $i=1,\cdots,k$.

A(1) is proven by (3.6.1). Assume A(k) for $k\geq 1$. Put $\mathcal{G}_{k+1} := \{\delta\in S(V)^W_{\leq k}I(\mathcal{F}) \mid \nabla_D(\delta)\in\mathcal{G}_k\}$. In the following i)-iv), we prove that $\mathcal{G}_0,\cdots,\mathcal{G}_k,\mathcal{G}_{k+1}$ satisfies the conditions for A(k+1).

i) $(\mathcal{G}_1\oplus\cdots\oplus\mathcal{G}_k)\cap\mathcal{G}_{k+1}=\{0\}$. Suppose δ were non-trivial element of LHS. Then, by induction hypothesis, $\nabla_D\delta\in\mathcal{G}_0\oplus\cdots\oplus\mathcal{G}_{k-1}$ and $\neq 0$. This contradicts to the facts $\nabla_D\delta\in\mathcal{G}_k\cap\mathcal{G}_0\oplus\cdots\oplus\mathcal{G}_{k-1}=\{0\}$.

ii) $\mathcal{G}_1\oplus\cdots\oplus\mathcal{G}_k$ and $\mathcal{G}_{k+1}$ generates $S(V)^W_{\leq k}I(\mathcal{F})$. Let δ be any element of $S(V)^W_{\leq k}I(\mathcal{F})$. Using (3.4.2) and 3.6 Lemma, one observes $\nabla_D\delta\in S(V)^W_{\leq k}\mathcal{G}$. By the induction hypothesis, one can find $\delta'\in\mathcal{G}_1\oplus\cdots\oplus\mathcal{G}_k$ so that $\nabla_D(\delta-\delta')\in\mathcal{G}_k$. This means $\delta-\delta'\in\mathcal{G}_{k+1}$.

iii) The above i) and ii) implies: $S(V)^W_{\leq k}I(\mathcal{F}) = \mathcal{G}_1\oplus\cdots\oplus\mathcal{G}_k\oplus\mathcal{G}_{k+1}$ and then, by a use of (3.4.2), $S(V)^W_{\leq k+1}\mathcal{G} = \mathcal{G}_0\oplus\cdots\oplus\mathcal{G}_k\oplus\mathcal{G}_{k+1}$.

iv) $\nabla_D:\mathcal{G}_{k+1}\to\mathcal{G}_k$ is bijective. We prove this in two steps:

a) to construct a $S(V)^{W,\tau}$-isomorphism $w_k:\mathcal{G}_k\to\mathcal{G}_{k+1}$,

b) to show that $\nabla_D\cdot w_k$ is an $S(V)^{W,\tau}$-isomorphism of $\mathcal{G}_k$.

a) For $\delta_i\in\mathcal{G}_i$ $0\leq i\leq k$, put $w_i(\delta_i)$:= the projection of $P_l\ \delta_i\in S(V)^W_{\leq i+1}\mathcal{G}$ to the component $\mathcal{G}_{i+1}$ (w.r.t. the decomposition in iii)). In view of (3.4.2), w_i is a $S(V)^{W,\tau}$-isomorphism: $\mathcal{G}_i\simeq\mathcal{G}_{i+1}$.

b) By the induction hypothesis, one has the isomorphism: $\nabla_D^i:\mathcal{G}_i\simeq\mathcal{G}_0=\mathcal{G}$ for $0\leq i\leq k$. By a use of this isomorphism, let us show that $w_i(\nabla_D^{-i}(\delta)) = P_l\nabla_D^{-i}(\delta)-\nabla_D^{-i}(P_l*\delta)$ for $\delta\in\mathcal{G}$ by induction on $0\leq i\leq k$ (ie. one has to show that the RHS belongs to $\mathcal{G}_{i+1}$). For the purpose, it is sufficient to show a formula: $\nabla_D\left(P_l\nabla_D^{-i}(\delta)-\nabla_D^{-i}(P_l*\delta)\right) = \nabla_D^{-i}((N+i)(\delta))$ for $0\leq i\leq k$. But this is shown by induction on i, where the case $i=0$ is the formula (3.7.7). Then for $i\geq 1$, $\nabla_D\left(P_l\nabla_D^{-i}(\delta)-\nabla_D^{-i}(P_l*\delta)\right) = \nabla_D^{-i}(\delta)+P_l\nabla_D^{-i+1}(\delta)-\nabla_D^{-i+1}(P_l*\delta) = \nabla_D^{-i}((N+i)(\delta))$. On the other hand, N is an endomorphism of $\mathcal{G}$ whose eigenvalues are non-negative rational numbers (3.7 Lemma). So, $\nabla_D\cdot w_i = \mathrm{ad}(\nabla_D^i)(N+i)$ is an invertible endomorphism of G_i. In particular, the case for $i=k$ proves the isomorphism in iv). Thus A(k) for $k\in\mathbf{Z}_{>0}$ is proven. This, in particular, implies the isomorphism

(3.8.1). Then, as a consequence, the subspaces $\mathcal{G}_k$ are unique since $\mathcal{G}_0 = \mathcal{G}$, $\mathcal{G}_1 = I(\mathcal{F})$ and $\mathcal{G}_{k-1} = \nabla_D(\mathcal{G}_k)$ for $k \in \mathbf{Z}_{>0}$.

3. For the case $k = 0$, use (3.7.7) and (3.7.1). Then, a successive applications of ∇_D^{-1} with the rule: $[\nabla_D, P_l] = 1$ and $[\nabla_D, \nabla_\delta] = 0$ for $\delta \in \mathcal{G}$ (use the definitions (3.2.1) and (3.2.3) of D and $\mathcal{G}$, respectively, and the integrality (3.5.6) of ∇) implies the formulae (3.8.3) and (3.8.4). □

Note. 1. The formula (3.6.4) implies $E = \frac{1}{h}D^{-1} \otimes D$. Since $E = w(D)$ (3.4 Assertion ii)), this means $N(D) = \frac{1}{h}D$. That is:
Fact. *The D is an eigenvector of N of the eigenvalue $\frac{1}{h}$.*
2. (3.8.3) and (3.8.4) decompose into two left-multiplicative formulae:

$$(3.8.3)* \qquad \begin{array}{lll} P_l\, D^{-k} & = D^{-k}P_l + kD^{-k-1} & \text{for } k \in \mathbf{Z} \\ P_l\, \delta & = P_l * \delta + D^{-1} \otimes N(\delta) & \text{for } \delta \in \mathcal{G} \end{array}$$

$$(3.8.4)* \qquad \begin{array}{lll} \nabla_\delta D^{-k} & = D^{-k}\nabla_\delta & \text{for } k \in \mathbf{Z} \text{ and } \delta \in \mathcal{G} \\ \nabla_{\delta_1}\delta_2 & = D \otimes (\delta_1 * \delta_2) + \nabla\!\!\!/_{\delta_1}\delta_2 & \text{for } \delta_1, \delta_2 \in \mathcal{G}. \end{array}$$

3. Filtration $D^{-k}\mathbf{Z}[D^{-1}] \otimes \mathcal{G}$ for $k \in \mathbf{Z}$ is identified with the Hodge filtration attached to the universal unfolding of simple singularities [S3] (c.f. §5). They can be also interpreted as the sequence of modules of vector fields having higher order contact with the discriminant [Te][Yo].

3.9. **Flat structure.**

Recall that $S_W := \mathrm{Spec}(S(V)^W)$ is the quotient variety of a vector space V by a finite reflection group W (with a W-invariant symmetric bilinear form I^* on V). On S_W, we have introduced several structures and objects such as $D, E, J, P_l*, w, *, \nabla\!\!\!/$ and N (3.1-3.7). In this paragraph, we clarify the relationships among them and reduce them to some basic ones. Regarded as a differential geometric structure on S_W, we call them altogether the *flat structure* on S_W (see [S3] and [S6]).
First, we recall once again the above mentioned structures and objects.

1. The *primitive vector field* D on S_W (3.2.1). Using D, we reduce the coefficient $S(V)^W$ to the subring $S(V)^{W,\tau}$ (3.2.2) and the tangent sheaf Der_{S_W} of S_W to a Lie algebra $\mathcal{G}$ (3.2.3) $S(V)^{W,\tau}$-free of rank l.

2. The *Euler vector field* E (3.1.6) attached to the graded ring structure on $S(V)^W$. It is normalized as $L_E(D) = -D$ and $L_E(I) = -\frac{2}{h}I$.

3. The *nondegenerate symmetric bilinear form* J (3.3.3) on $\mathcal{G}$, defined as the dual form of $J^* := L_D(I)$ (3.3.1).

4. The $S(V)^W = S(V)^{W,\tau}[P_l]$-*module structure on* $\mathcal{G}$ (3.4.3) and the attached section $w : \mathcal{G} \to Der_{S_W}(-\log\Delta)$ (3.4.4).

5. The graded $S(V)^{W,\tau}$-*algebra structure* $*$ *on* $\mathcal{G}$ (3.7.2).

6. The *covariant differentiation* $\nabla\!\!\!/ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ (3.7.3).

7. The *exponent map* $N := \nabla_D w \in End_{S(V)^{W,\tau}}(\mathcal{G})$ (3.7.6).

We shall show in the next theorem the following reductions:

a) The 4. and 5. are unified and strengthened to a $S(V)^W$*-algebra structure* on $\mathcal{G}$ (Theorem I.1 and (3.9.7)).

b) The section w is given by $E*$ (Theorem I.6).

c) The $\nabla\!\!\!/$ is the Levi-Civita connection of J (Theorem I.3).

d) The map N is given by $-\nabla\!\!\!/ E + \frac{1+h}{h}\mathrm{id}_{\mathcal{G}}$ (Theorem I.9).

Thus, after these reductions, w, $\nabla\!\!\!/$ and N become superfluous, and *what remains as the basic ingredients of the flat structure are the vector fields* D *and* E*, a flat metric* J *and the* $S(V)^W$*-algebra structure* $*$.

Theorem. *(Flat structure.) One has the relations I.1-I.9 among* $D, E, J, P_l*, w, *, \nabla\!\!\!/$ *and* N *(where,* $\delta, \delta_1, \delta_2, \delta_3$ *are arbitrary elements in* $\mathcal{G}$*).*

I.1. *The product* $*$ *and the* $S(V)^W$*-module structure* P_l* *on* $\mathcal{G}$ *together define a commutative and associative algebra structure over* $S(V)^W$.

I.2. *The primitive vector field* D *is a) the unit element of the algebra, b) horizontal w.r.t.* $\nabla\!\!\!/$*, i.e.* $\nabla\!\!\!/ D = 0$ *and c) an eigenvector for* N.

I.3. *The* $\nabla\!\!\!/$ *is the Levi-Civita connection for the metric* J*, which is integrable. That is: the* J *is a flat metric and the* $\nabla\!\!\!/$ *has the properties*

i) the torsion freeness of $\nabla\!\!\!/$*:* $\nabla\!\!\!/_{\delta_1}\delta_2 - \nabla\!\!\!/_{\delta_2}\delta_1 = [\delta_1, \delta_2]$,

ii) the horizontality of J *with respect to connection* $\nabla\!\!\!/$*:* $\nabla\!\!\!/ J = 0$,

iii) the integrability of $\nabla\!\!\!/$*:* $\nabla\!\!\!/_{\delta_1}\nabla\!\!\!/_{\delta_2} - \nabla\!\!\!/_{\delta_2}\nabla\!\!\!/_{\delta_1} = \nabla\!\!\!/_{[\delta_1,\delta_2]}$.

I.4. *The* $\nabla\!\!\!/ *$ *is a symmetric (3,1)-tensor. That is:*

$$T(\delta_1, \delta_2, \delta_3) := \nabla\!\!\!/_{\delta_1}(\delta_2 * \delta_3) - (\nabla\!\!\!/_{\delta_1}\delta_2) * \delta_3 - \delta_2 * (\nabla\!\!\!/_{\delta_1}\delta_3) \tag{3.9.1}$$

is a symmetric $S(V)^{W,\tau}$*-tri-linear form in* $\delta_1, \delta_2, \delta_3$ *(values in* $\mathcal{G}$*).*

I.5. *The product* $*$ *is self-adjoint with respect to* J*. That is:*

$$J(\delta_1, \delta_2 * \delta_3) = J(\delta_1 * \delta_2, \delta_3). \tag{3.9.2}$$

I.6. *Let us extend the domain of the product* $*$ *from* $\mathcal{G}$ *to* Der_{S_W} *by a use of* (3.2.4) $S(V)^W$*-linearly. Then the section* w *is given by*

$$w(\delta) = E * \delta. \tag{3.9.3}$$

I.7. *Let us extend the domain of* $\nabla\!\!\!/$ *from* $\mathcal{G}$ *to* Der_{S_W} *by a use of* (3.2.4) *and Leibniz rule. Then the exponent map* N *is given by*

$$N(\delta) = -\nabla\!\!\!/_{\delta}E + \tfrac{1+h}{h}\delta. \tag{3.9.4}$$

I.8. *The* N *is horizontal with respect to the connection* $\nabla\!\!\!/$*. That is:*

$$\nabla\!\!\!/ N = 0. \tag{3.9.5}$$

I.9. *Let* N^* *be the adjoint action of* N *with respect to* J *(i.e.* $J(N(\delta_1), \delta_2) = J(\delta_1, N^*(\delta_2))$*). Then, one has the duality formula:*

$$N + N^* = 1. \tag{3.9.6}$$

Proof. A guiding principle of the proof: We develop in powers of D^{-1} the properties of the connection ∇ using (3.8.3) and (3.8.4). The termwise equality (due to the uniqueness (3.8.2)) yields the seeked results.

a) Consider the torsion-free condition of ∇: for $\delta_1,\ \delta_2 \in \mathcal{G}$

$$\nabla_{\delta_1}(D^{-1} \otimes \delta_2) - \nabla_{\delta_2}(D^{-1} \otimes \delta_1) = D^{-1} \otimes [\delta_1, \delta_2].$$

By a use of (3.8.4), the LHS is $\delta_1 * \delta_2 - \delta_2 * \delta_1 + \big(\nabla\!\!\!/_{\delta_1}\delta_2 - \nabla\!\!\!/_{\delta_2}\delta_1\big)D^{-1}$.

The constant term implies the commutativity of $*$ (I.1). This and the left-unitness (3.7.4) imply D is the unit of the $*$-algebra (I.2 a)).

The linear term implies the torsion-freeness of $\nabla\!\!\!/$ (I.3. i)). This and $\nabla\!\!\!/_D\mathcal{G} = 0$ (3.7.5) imply that the horizontality of D (I.2 b)).

That D is an eigenvector of N is shown in 3.8 Note 1. Fact. (I.2 c)).

b) Consider the integrability condition of ∇: for $\delta_1,\ \delta_2,\ \delta_3 \in \mathcal{G}$

$$(\nabla_{\delta_1}\nabla_{\delta_2} - \nabla_{\delta_2}\nabla_{\delta_1})\,\delta_3 D^{-2} = \nabla_{[\delta_1,\delta_2]}\delta_3 D^{-2}$$

By a use of (3.8.4), the LHS becomes

$$\begin{aligned}&\delta_1 * (\delta_2 * \delta_3) - \delta_2 * (\delta_1 * \delta_3) + \big(\nabla\!\!\!/_{\delta_1}(\delta_2 * \delta_3) - \nabla\!\!\!/_{\delta_2}(\delta_1 * \delta_3)\\ &+\delta_1 * \nabla\!\!\!/_{\delta_2}\delta_3 - \delta_2 * \nabla\!\!\!/_{\delta_1}\delta_3\big)D^{-1} + \big(\nabla\!\!\!/_{\delta_1}\nabla\!\!\!/_{\delta_2}\delta_3 - \nabla\!\!\!/_{\delta_2}\nabla\!\!\!/_{\delta_1}\delta_3\big)D^{-2}.\end{aligned}$$

The RHS is $[\delta_1, \delta_2] * \delta_3 D^{-1} + \nabla\!\!\!/_{[\delta_1,\delta_2]}\delta_3 D^{-2}$.

The constant terms imply the associativity of $*$ (I.1).

The quadratic terms imply integrability of $\nabla\!\!\!/$ (I.3. iii)).

The linear terms imply a relation: for $\delta_1,\ \delta_2,\ \delta_3 \in \mathcal{G}$

1) $\nabla\!\!\!/_{\delta_1}(\delta_2 * \delta_3) - \nabla\!\!\!/_{\delta_2}(\delta_1 * \delta_3) + \delta_1 * \nabla\!\!\!/_{\delta_2}\delta_3 - \delta_2 * \nabla\!\!\!/_{\delta_1}\delta_3 = [\delta_1, \delta_2] * \delta_3.$

This means $T(\delta_1, \delta_2, \delta_3) = T(\delta_2, \delta_1, \delta_3)$, where $T(\delta_1, \delta_2, \delta_3)$ is the tensor (3.9.1). Since $T(\delta_1, \delta_2, \delta_3) = T(\delta_1, \delta_3, \delta_2)$ by definition, I.4 is shown.

c) Recall the Leibniz rule for ∇: for $\delta_1,\ \delta_2 \in \mathcal{G}$

$$\nabla_{\delta_1}(P_l\nabla_D^{-1}\delta_2) = (\delta_1 P_l)\nabla_D^{-1}\delta_2 + P_l\nabla_{\delta_1}\nabla_D^{-1}\delta_2.$$

By a use of (3.8.3), the LHS and RHS are developed in

$$\delta_1 * (P_l * \delta_2) + \big(\nabla\!\!\!/_{\delta_1}(P_l * \delta_2) + \delta_1 * (N+1)(\delta_2)\big)D^{-1} + \nabla\!\!\!/_{\delta_1}(N+1)(\delta_2)D^{-2},$$

$$P_l*(\delta_1*\delta_2) + \big((\delta_1 P_l)\delta_2 + N(\delta_1*\delta_2) + P_l*(\nabla\!\!\!/_{\delta_1}\delta_2)\big)D^{-1} + (N+1)(\nabla\!\!\!/_{\delta_1}\delta_2)D^{-2},$$

respectively. The constant terms give the equality:

$$P_l * (\delta_1 * \delta_2) = \delta_1 * (P_l * \delta_2). \tag{3.9.7}$$

This means that *the $S(V)^W$-module structure on $\mathcal{G}$ was in fact a $S(V)^W$-algebra structure.* We note that the following are equivalent:

i) *The multiplication P_l* defines a $S(V)^W$-algebra structure on $\mathcal{G}$.*

ii) *There exists an element $[P_l] \in \mathcal{G}$ such that $P_l * \delta = [P_l] * \delta$.*

iii) *The section w is given by $w(\delta) = E * \delta$ for $\delta \in \mathcal{G}$.*

(*Proof.* i) $\Rightarrow$ ii): Substitute δ_2 by D and put $[P_l] := P_l * D$. ii) $\Rightarrow$ i): Due to the associativity of $*$. ii) $\Rightarrow$ iii): $E * \delta = w(D) * \delta = (P_l D - P_l * D) * \delta = P_l \delta - P_l * \delta = w(\delta)$. iii) $\Rightarrow$ ii): $P_l * \delta = P_l \delta - w(\delta) = P_l \delta - E * \delta = P_l \delta - (P_l D - P_l * D) * \delta) = (P_l * D) * \delta$. $\square$)

Therefore, the I.6 (3.9.3) is also shown.

The quadratic terms imply the horizontality of N (I.7. (3.9.5)).

The linear terms give an equality:

$$\text{2)} \quad \begin{aligned} & N(\delta_1 * \delta_2) - \delta_1 * (N+1)(\delta_2) \\ = \;& \nabla\!\!\!/_{\delta_1}(P_l * \delta_2) - (\delta_1 P_l)\delta_2 - P_l * (\nabla\!\!\!/_{\delta_1}\delta_2). \end{aligned}$$

Specializing this for $\delta_2 = D$ and noting the fact $\nabla\!\!\!/ D = 0$, one obtains

$$N(\delta) = -(\delta P_l)D + \nabla\!\!\!/_\delta([P_l]) + \delta * (1+N)(D) = -\nabla\!\!\!/_\delta E + (1 + \tfrac{1}{h})\delta.$$

This proves (3.9.4). Using (3.9.4), the equality 2) is rewritten as:

$$\text{3)} \quad \begin{aligned} T(\delta_1, \delta_2, [P_l]) \;&=\; \tfrac{1}{h}\delta_1 * \delta_2 + N(\delta_1 * \delta_2) - \delta_1 * N(\delta_2) - \delta_2 * N(\delta_1) \\ &= -\delta_1 * \delta_2 - \nabla\!\!\!/_{\delta_1 * \delta_2} E + \delta_1 * \nabla\!\!\!/_{\delta_2} E + \delta_2 * \nabla\!\!\!/_{\delta_1} E. \end{aligned}$$

Using the symmetry $T(\delta_1, \delta_2, [P_l]) = T([P_l], \delta_1, \delta_2)$ (I.4), this reduces to the homogeneity of $*$-produce: $[E, \delta_1 * \delta_2] - \delta_2 * [E, \delta_1] - \delta_1 * [E, \delta_2] = \delta_1 * \delta_2$.

d) For $\delta_1, \delta_2 \in \mathcal{G}$, using w (3.4.4) and applying (3.4.7), one has

$$\text{4)} \quad I^*(w(\delta_1), w(\delta_2)) = I^*(P_l\delta_1 - P_l * \delta_1, w(\delta_2)) = P_l J(\delta_1, \delta_2) - J(P_l * \delta_1, \delta_2).$$

Apply D to BHS of 4). Using (3.7.6) and (3.4.7), the LHS is equal to

$$\begin{aligned} & I^*(\nabla_D w(\delta_1), w(\delta_2)) + I^*(w(\delta_1), \nabla_D w(\delta_2)) \\ = \;& I^*(N(\delta_1), w(\delta_2)) + I^*(w(\delta_1), N(\delta_2)) = J(N(\delta_1), \delta_2) + J(\delta_1, N(\delta_2)). \end{aligned}$$

The RHS is equal to $J(\delta_1, \delta_2)$. This implies the duality (3.9.6).

e) We show the horizontality of J (I.3.ii)). Let us apply $\delta \in \mathcal{G}$ to the both hand sides of 4) in d). Using (3.7.7), (3.7.1), (3.9.5) and apply 4) again for $\nabla\!\!\!/_\delta \delta_1$ and δ_2,...etc., LHS is calculated as

$$\begin{aligned} & \delta I^*(w(\delta_1), w(\delta_2)) = I^*(\nabla_\delta \nabla_D^{-1} N(\delta_1), w(\delta_2)) + I^*(w(\delta_1), \nabla_\delta \nabla_D^{-1} N(\delta_2)) \\ & = I^*(\delta * N(\delta_1) + \nabla_D^{-1}(\nabla\!\!\!/_\delta N(\delta_1)), w(\delta_2)) \\ & \quad + I^*(w(\delta_1), \delta * N(\delta_2) + \nabla_D^{-1}(\nabla\!\!\!/_\delta N(\delta_2))) \\ & = J(\delta * N(\delta_1), \delta_2) + I^*(\nabla_D^{-1}(N(\nabla\!\!\!/_\delta \delta_1), w(\delta_2)) \\ & \quad + J(\delta_1, \delta * N(\delta_2)) + I^*(w(\delta_1), \nabla_D^{-1}(N(\nabla\!\!\!/_\delta \delta_2)) \\ & = J(\delta * N(\delta_1), \delta_2) + I^*(w(\nabla\!\!\!/_\delta \delta_1), w(\delta_2)) \\ & \quad + J(\delta_1, \delta * N(\delta_2)) + I^*(w(\delta_1), w(\nabla\!\!\!/_\delta \delta_2)) \\ & = J(\delta * N(\delta_1), \delta_2) + P_l J(\nabla\!\!\!/_\delta \delta_1, \delta_2) - J(\nabla\!\!\!/_\delta \delta_1, P_l * \delta_2) \\ & \quad + J(\delta_1, \delta * N(\delta_2)) + P_l J(\delta_1, \nabla\!\!\!/_\delta \delta_2) - J(P_l * \delta_1, \nabla\!\!\!/_\delta \delta_2). \end{aligned}$$

The RHS is given by $P_l\delta J(\delta_1,\delta_2)+(\delta P_l)J(\delta_1,\delta_2)-\delta J(P_l*\delta_1,\delta_2)$. The leading terms in P_l in BHS give the horizontality of J (I.3. ii)). The remaining terms: $(\delta P_l)J(\delta_1,\delta_2)-\delta J(P_l*\delta_1,\delta_2)=J(\delta*N(\delta_1),\delta_2)-J(\nabla\!\!\!/_\delta\delta_1,P_l*\delta_2)+J(\delta_1,\delta*N(\delta_2))-J(P_l*\delta_1,\nabla\!\!\!/_\delta\delta_2)$ is reduced to 2).

f) We show the self-adjointness of the $*$-product (I.5.(3.9.2)). We have only to prove the special case when $\delta_3=D$ (and $\delta_1,\delta_2\in\mathcal{G}$):

$$J(\delta_1,\delta_2)=J(\delta_1*\delta_2,D), \tag{3.9.8}$$

since, by a use of (3.9.8), the LHS and RHS of (3.9.2) are equal to $J(\delta_1*(\delta_2*\delta_3),D)$ and $J((\delta_1*\delta_2)*\delta_3,D)$, respectively, and then, the associativity of the $*$-product implies the equality and hence (3.9.2).

First, we show a formula (see 3.10 1. for a meaning): for $\delta\in\mathcal{G}$:

$$\delta=\nabla_\delta\nabla_D^{-1}D \tag{3.9.9}$$

Proof. This is the torsion-freeness of ∇. According to (3.7.1), the RHS is equal to $\delta*D+\nabla_D^{-1}\nabla\!\!\!/_\delta D$. The first term is equal to δ, since D is the unit. The second term vanishes due to the horizontality of D. $\square$

We return to a proof of (3.9.8). Using (3.9.9) and (3.4.7), one has

$$\begin{aligned} J(\delta_1,\delta_2) &= I^*(\nabla_{\delta_1}\nabla_D^{-1}D,w(\delta_2))\\ &= \delta_1 I^*(\nabla_D^{-1}D,w(\delta_2))-I^*(\nabla_D^{-1}D,\nabla_{\delta_1}w(\delta_2))\end{aligned}$$

Since $\nabla_D^{-1}D=hE=hw(D)$ where $\frac{1}{h}$ is the eigenvalue of N on D (recall 3.8 Note 1.), one calculates further (again using (3.4.7))

$$\begin{aligned} \tfrac{1}{h}J(\delta_1,\delta_2) &= \delta_1 J(D,w(\delta_2))-J(D,\nabla_{\delta_1}w(\delta_2))\\ &= J(\nabla\!\!\!/_{\delta_1}D,w(\delta_2))+J(D,\nabla\!\!\!/_{\delta_1}w(\delta_2))-J(D,\nabla_{\delta_1}w(\delta_2))\\ &= J(D,\nabla\!\!\!/_{\delta_1}w(\delta_2)-\nabla_{\delta_1}w(\delta_2))\end{aligned}$$

where the first term in the second line vanishes due to the horizontality of D. Here the form J in the RHS is extended $S(V)^W$-linearly (3.2.4). Using the expression $w(\delta)=P_l\delta-P_l*\delta$ (3.4.4) and (3.7.1), one has

$$\begin{aligned} \nabla\!\!\!/_{\delta_1}w(\delta_2) &= \delta_1P_l\cdot\delta_2+P_l\nabla\!\!\!/_{\delta_1}\delta_2-\nabla\!\!\!/_{\delta_1}(P_l*\delta_2)\\ \text{and}\qquad \nabla_{\delta_1}w(\delta_2) &= \delta_1P_l\cdot\delta_2+P_l(\nabla_D(\delta_1*\delta_2)+\nabla\!\!\!/_{\delta_1}\delta_2)\\ &\quad -(\nabla_D(\delta_1*(P_l*\delta_2))+\nabla\!\!\!/_{\delta_1}(P_l*\delta_2))\end{aligned}$$

(In fact, these calculations should be justified. See §4). So, we obtain

$$\begin{aligned} \nabla\!\!\!/_{\delta_1}w(\delta_2)-\nabla_{\delta_1}w(\delta_2) &= -P_l\nabla_D(\delta_1*\delta_2)+\nabla_D(\delta_1*(P_l*\delta_2))\\ &= -\nabla_D(P_l(\delta_1*\delta_2)-P_l*(\delta_1*\delta_2))+\delta_1*\delta_2\\ &= -\nabla_D(w(\delta_1*\delta_2))+\delta_1*\delta_2\\ &= -N(\delta_1*\delta_2)+\delta_1*\delta_2 \;=\; (1-N)(\delta_1*\delta_2).\end{aligned}$$

Substituting this in the previous formula, one obtains

$$\tfrac{1}{h}J(\delta_1,\delta_2)=J(D,(1-N)(\delta_1*\delta_2))=J((1-N^*)(D),\delta_1*\delta_2)$$

Recalling the duality (3.9.6) and the fact that $\frac{1}{h}$ is the eigenvalue of N for the vector D (3.8 Note 1.), one has $(1-N^*)(D) = N(D) = \frac{1}{h}D$. Multiplying h on BHS, we obtain (3.9.8). $\square$

Remark. 1. The $S(V)^W$-algebra on $\mathcal{G}$ defines the normalization: $\widetilde{D}_W = \mathrm{Spec}(\mathcal{G}) \to D_W = \mathrm{Spec}(S(V)^W/(\Delta)$ of the discriminant (c.f. §4).

2. The (3.9.8) together with the facts $w(D) = E = I(dP_1)$ implies

$$J(\delta_1, \delta_2) = \delta_1 * \delta_2\ (P_1). \tag{3.9.10}$$

3. The $*, P_l*, \nabla\!\!\!/, N$ and the metric J on the module $\mathcal{G}$ are sufficient to recover the $S(V)^W$-module Der_{S_W} with the connection ∇ and the metric I through (3.8.2), (3.8.3), (3.8.4) and (3.4.8).

3.10. **Relations to primitive forms and Frobenius manifold.**

We discuss some relations of the flat structure in 3.9 to primitive forms and to Frobenius manifold structure.

1. Primitive form. Since the ∇ is the Levi-Civita connection and torsion free, we have confused the module $\mathcal{H}^{(0)} := Der_{S_W}$ of operands (sections of a bundle) with the algebra Der_{S_W} of operators (covariant differentiation). In order to separate two roles, we introduce a sequence:

$$\mathcal{H}^{(-k)} := \nabla_D^{-k} Der_{S_W} \simeq D^{-k}\mathbf{Z}[D^{-1}] \otimes \mathcal{G}$$

of the modules of the operands (recall (3.8.2)) for $k \in \mathbf{Z}_{\geq 0}$, on which one has the action of the covariant differentiations:

$$\begin{aligned} \nabla : \quad\quad\quad & Der_{S_W} \times \mathcal{H}^{(-k-1)} \to \mathcal{H}^{(-k)}, \\ \nabla : Der_{S_W}&(-\log(\Delta^2)) \times \mathcal{H}^{(-k)} \to \mathcal{H}^{(-k)}. \end{aligned}$$

Then the formula (3.9.9) (based on the torsion-freeness of ∇, used at several crucial steps of proof of Theorem) implies the following key fact:

Assertion. *The covariant differentiation of the element* $\zeta^{(-1)} := \nabla_D^{-1} D$ *(i.e. the correspondence:* $\delta \mapsto \nabla_\delta \zeta^{(-1)}$*) induces a bijection between the Lie algebra* Der_{S_W} *of operators and the module* $\mathcal{H}^{(0)}$ *of operands. More generally, the covariant differentiation of the element* $\zeta^{(-k)} := \nabla_D^{-k} D$ $(k \in \mathbf{Z}_{\geq 0})$ *induces an isomorphism of* $S(V)^W$*-modules:*

$$\nabla\zeta^{(-k-1)} \ : \quad Der_{S_W} \ \simeq \ \mathcal{H}^{(-k)}. \tag{$*$}$$

Such $\zeta^{(-k)}$ (satisfying some additional conditions corresponding to (3.8.3), (3.8.4) and orthogonality, see [S6]) is called a *primitive form.*

The isomorphism $*$) has meanings in two directions:

1. Through the correspondence, cohomological structures on $\mathcal{H}^{(0)}$ (such as the polarization I, Hodge filtration) and the algebra structure on Der_{S_W} (such as the Lie algebra str., integrable system) are combined together in one object. This amalgamation creates some rich structures

such as the flat structure in 3.9, the Frobenius structure and the flat coordinates (discussed in the following).

2. The module $\mathcal{H}^{(-k)}$ (the Hodge filter) is obtained by covariant differentiation of a single element $\zeta^{(-k-1)}$. That is: the data of the module $\mathcal{H}^{(-k)}$ are condensed in the single element $\zeta^{(-k-1)}$, the primitive form, and by differentiating the primitive form, they are resolved again. This view point is an origin of the word "primitive form". In fact, before the primitive form was introduced, one had studied the "primitive (i.e. potential) function" as a solution of second order uniformization equation such that its derivatives give the system of solutions of a torsion free connection [S2].

In the present paper, reversing the history, we start from the flat structure, and then we shall reconstruct the uniformization equations in §4, whose solutions (i.e. the primitive functions) are partially obtained by the period integral of the primitive form §5.

2. Frobenius manifold. A tuple $(M, \circ, e, E, g)$ of a manifold M with a metric g, two global vector fields e and E on M and a ring structure $\circ$ on the tangent bundle of M, subject to some axioms, is called a Frobenius manifold ([Do],[Ma, 0.4.1, I.1.3],[He, def 9.1],[Sab],[Ta]). As an immediate consequence of 3.9 Theorem, we show the next.

Theorem. *The tuple $(S_W, *, D, E, J)$ form a Frobenius manifold.*

Proof. In view that $\nabla\!\!\!/$ is the Levi-Civita connection of J I.3, we check the the axioms 1) - 5) for a Frobenius manifold [He def 9.1] as follows.

1) *The $*$ is self-adjoint with respect J.* This is shown in (3.9.2).

2) *The tensor $\nabla\!\!\!/ *$ is symmetric.* This is shown in I.4.

3) *The metric J is flat, i.e. $\nabla\!\!\!/^2 = 0$.* This is shown in I.3. iii).

4) *The D is horizontal, i.e. $\nabla\!\!\!/ D = 0$.* This is shown in I.2.b).

5) *The $*$ is homogeneous of degree 1 and the J is homogeneous w.r.t. the Euler vector field E, respectively.* The first half follows from the defining formula (3.7.1) in view of the normalizations (2.8.5), (3.1.6) and (3.2.1) (see also the formula at the end of c) of the proof of 3.9 Theorem). Alternatively, one shows as follows: since $D* = 1$ one has $0 = \deg(D*) = \deg(D) + \deg(*) = -1 + \deg(*)$. The J is homogeneous since J^* is homogeneous by definition (3.3.1). One calculates: $\deg(J) = -\deg(J^*) = -\deg(L_D I) = -(\deg(D) + \deg(I)) = 1 + \frac{2}{h}$. □

3.11. **Flat coordinates.**

We introduce coordinates of S_W whose differentials are horizontal sections of the dual connection $\nabla\!\!\!/^*$. We call them *flat coordinates*.

The dual connection $\nabla\!\!\!/^* : \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ (3.2.3) is defined by $\delta\langle\delta', \omega\rangle = \langle\nabla\!\!\!/_\delta \delta', \omega\rangle + \langle\delta, \nabla\!\!\!/^*_\delta \omega\rangle$ for $\delta, \delta' \in \mathcal{G}$ and $\omega \in \mathcal{F}$. Let us denote by $\wedge\nabla\!\!\!/^*$

the composition of $\nabla\!\!\!/^*$ with the wedge product $\mathcal{F}\otimes\mathcal{F}\to\mathcal{F}\wedge\mathcal{F}$. The torsion freeness of $\nabla\!\!\!/$ I.3. i) implies $\wedge\nabla\!\!\!/^*(\omega)=d\omega$ for $\omega\in\mathcal{F}$, since

$$\begin{array}{rcl}\wedge\nabla\!\!\!/^*(\omega) & = & \sum_{i=1}^{l} dP_i\wedge\nabla\!\!\!/^*_{\frac{\partial}{\partial P_i}}(\omega)=\sum_{ij=1}^{l} dP_i\wedge dP_j\langle\frac{\partial}{\partial P_j},\nabla\!\!\!/^*_{\frac{\partial}{\partial P_i}}(\omega)\rangle\\ & = & \sum_{ij=1}^{l} dP_i\wedge dP_j(\frac{\partial}{\partial P_i}\langle\frac{\partial}{\partial P_j},\omega\rangle-\langle\nabla\!\!\!/_{\frac{\partial}{\partial P_i}}\frac{\partial}{\partial P_j},\omega\rangle)\\ & = & d\omega-\sum_{ij=1}^{l} dP_i\wedge dP_j\langle\nabla\!\!\!/_{\frac{\partial}{\partial P_i}}\frac{\partial}{\partial P_j},\omega\rangle=d\omega.\end{array}$$

The same formula holds for $\omega\in\mathcal{O}_{S_W}\otimes_{S(V)^W}\mathcal{F}$ where $\mathcal{O}_{S_W}$ is the sheaf of local (w.r.t. the classical topology on $S_{W,K}$) analytic functions on $S_{W,K}$ where K is either $\mathbf{R}$ or $\mathbf{C}$ (recall (2.9.2)). Thus, *a local analytic horizontal section* $\omega\in\ker(\nabla\!\!\!/^*)$ *is a closed form:* $d\omega=0$ *and there is a local analytic function* u *such that* $\omega=du$, where u satisfies the second order differential equation:

$$\nabla\!\!\!/^* du=0. \tag{3.11.1}$$

The space of global analytic solutions (including constants) on $S_{W,K}$ of the equation has rank $l+1$, since $\nabla\!\!\!/^*$ is integrable and non-singular on $S_{W,K}$ and $S_{W,K}$ is simply connected. A solution, say u, is, in fact, a polynomial: develop u in a Taylor series $u=\sum_{d\in\mathbf{Z}_{\geq 0}}u_d$ for $u_d\in S(V)^W_d$ at the origin. Since $\nabla\!\!\!/$ is homogeneous and continuous w.r.t. the adic topology on the space of power serieses on S_W at the origin, each graded piece u_d is again a solution of (3.11.1). Since the solution space is finite dimensional, the expansion of u should be a finite sum (i.e. u is a polynomial). These imply the part i) of the next lemma.

Lemma. *Consider the space of the polynomial solutions of* (3.11.1)*:*

$$\Omega:=\{u\in S(V)^W\mid\nabla\!\!\!/^* du=0\}. \tag{3.11.2}$$

Then i) the Ω *is a graded vector space of rank* $l+1$ *with the splitting:* $\Omega=\mathbf{R}\cdot 1\oplus\Omega_+$ *for* $\Omega_+:=\Omega\cap S(V)^W_+$*, ii)* Ω_+ *generates freely the algebra* $S(V)^W$ *over* $\mathbf{R}$*, and iii) one has the bijections:*

$$\begin{array}{rcl}S_{W,K} & \simeq & \mathrm{Hom}^{alin}_{\mathbf{R}}(\Omega,K)\\ & \simeq & \mathrm{Hom}_{\mathbf{R}}(\Omega^+,K).\end{array} \tag{3.11.3}$$

Here, $\mathrm{Hom}^{alin}_{\mathbf{R}}$ *is the set of all* $\mathbf{R}$*-linear maps bringing* $1\in\Omega$ *to* $1\in K$ *("alin" stands for "affine linear"), and* $\mathrm{Hom}_{\mathbf{R}}$ *is the set of all* $\mathbf{R}$*-linear maps. By this bijection, we regard* S_W *the linear flat space dual to* Ω_+.

Proof. ii) Let $1,Q_1,\cdots,Q_l\in S(V)^W$ be homogeneous $\mathbf{R}$-basis of Ω with $\deg(Q_1)\leq\cdots\leq\deg(Q_l)$. The differentials $dQ_1,\cdots,dQ_l$ span $\ker(\nabla\!\!\!/^*)$ over $\mathbf{R}$. As a general fact, one has $S(V)^{W,\tau}\otimes_{\mathbf{R}}\ker(\nabla\!\!\!/^*)=\mathcal{F}$. Since dP_i (for coordinates $P_1,...,P_l$ of S_W) are $S(V)^{W,\tau}$-basis of $\mathcal{F}$, the determinant$(\frac{\partial(Q_1,\cdots,Q_l)}{\partial(P_1,\cdots,P_l)})$ is a unit on S_W, and is a non-zero constant.

Then, one sees that $\deg(Q_i) = \deg(P_i)$ $(i = 1, ..., l)$ and that each homogeneous block of the Jacobi matrix is constant invertible. Then the ring homomorphism: $\mathbf{R}[Q_1, ..., Q_l] \to S(V)^W$ is an isomorphism.

iii) The map $S_{W,K} \to \mathrm{Hom}^{alin}_{\mathbf{R}}(\Omega, K)$ is defined by the correspondence $\varphi \mapsto \varphi|\Omega$ for $\varphi \in \mathrm{Hom}^{alg}_{\mathbf{R}}(S(V)^W, K)$. Since Ω generates the ring of invariants $S(V)^W$, the map is injective. Conversely, since $Q_1, \cdots, Q_l$ are algebraically independent (2.5 Theorem), the map is surjective. $\square$

The (3.11.1) is explicitly given by the system of second order equations

$$(3.11.4) \qquad (\delta_1\delta_2 - \nabla\!\!\!/_{\delta_1}\delta_2)u = 0 \qquad \text{for } \ \delta_1, \delta_2 \in \mathcal{G}.$$

Elements of Ω are called *flat invariants*. Homogeneous basis $Q_1, \cdots, Q_l$ of Ω_+ are called a system of *flat coordinates* of S_W. They are unique up to a weighted linear transformation. The flat coordinates for all finite reflection groups except for types E_7 and E_8 are calculated in [S-Y-S]. The flat coordinates for type E_7 is calculated in [Ya]. The flat coordinates for type E_8 is calculated in [Ka][No][No].

A translation $\xi \in \Omega_+^* := \mathrm{Hom}^{lin}_K(\Omega/K\cdot 1, K)$ of the affine space S_W extends naturally to a derivation of $S(\Omega_+) \simeq S(V)^W$. So, one has the embedding $\Omega^*_+ \subset Der_{S_W}$ with the characterization: $\Omega^*_+ = \{\delta \in \mathcal{G} \mid \nabla\!\!\!/\delta = 0\}$ (*Proof.* For an element $\delta \in \mathcal{G}$, one has: $\nabla\!\!\!/\delta = 0$ $\Leftrightarrow \langle \nabla\!\!\!/_\xi \delta, \omega\rangle = 0$ for $\forall \xi \in \mathcal{G}$ and $\forall \omega \in \Omega \Leftrightarrow$ $\xi\langle\delta,\omega\rangle - \langle \delta, \nabla\!\!\!/^*_\xi \omega\rangle = 0$ for $\forall \xi \in \mathcal{G}$ and $\forall\omega \in \Omega \Leftrightarrow$ $\langle\delta,\omega\rangle$ is a constant for $\forall\omega\in\Omega \Leftrightarrow$ $\delta \in \Omega^*_+$. $\square$) Therefore, we shall sometimes call an element of Ω^*_+ (regarded as an element of Der_{S_W}) *flat tangent vector* of S_W (e.g. D is a flat tangent).

Here are some immediate uses of the flat coordinates.

1. *The quadratic form* P_1 (2.8.5) *is a flat invariant.*

This follows from (3.3.4) and 3.9 Theorem I.2.b) as below.

$$\begin{aligned}(\delta_1\delta_2 - \nabla\!\!\!/_{\delta_1}\delta_2)P_1 &= \delta_1 J^*(J(\delta_2), dP_1) - J^*(J(\nabla\!\!\!/_{\delta_1}\delta_2)), dP_1) \\ &= \delta_1 J(\delta_2, D) - J(\nabla\!\!\!/_{\delta_1}\delta_2, D) = J(\delta_2, \nabla\!\!\!/_{\delta_1} D) = 0.\end{aligned}$$

2. *Let* $\Omega^*_+ = \oplus \Omega^*_d$ *be the graded decomposition of* Ω^*_+. *Then,*

a) The piece Ω^*_d *is a vector space of rank* $\#\{1 \le i \le l \mid d = -\frac{m_i+1}{h}\}$.

b) If $d_1 + d_2 + 1 + \frac{2}{h} = 0$ *then* $J|_{\Omega^*_{d_1}\times\Omega^*_{d_2}}$ *is the duality between the two pieces. If* $d_1 + d_2 + 1 + \frac{2}{h} \neq 0$ *then* $J|_{\Omega^*_{d_1}\times\Omega^*_{d_2}} = 0$.

c) The N *is constant on each piece:* $N\,|_{\Omega^*_d} = (d + \frac{1+h}{h})\mathrm{id}_{\Omega^*_d}$.

3. *The star product* $*$ *is calculated by an explicit formula:*

(3.11.5)

$$\frac{\partial}{\partial Q_i} * \left(\frac{h-m_j}{h} J^*(dQ_j)\right) = \sum_{k=1}^l \left(\frac{\partial}{\partial Q_i} I(dQ_j, dQ_k) - A^i_{jk}\right)\frac{\partial}{\partial Q_k}$$

Here, the coefficients A^i_{jk} is given by (3.6.6) by replacing $P_1, \cdots, P_l$ by the flat $Q_1, \cdots, Q_l$. Note $A^i_{jk} \in S(V)^{W,\tau}$ (i.e. $DA^i_{jk} = 0$ for (3.6.3)).

4. Flat potential. *Put* $J_3(\delta_1,\delta_2,\delta_3) := J(\delta_1 * \delta_2, \delta_3)$ *for* δ_1, δ_2 *and* $\delta_3 \in \mathcal{G}$. *Then a)* J_3 *is a symmetric tensor in the three variables, and b) there exists a polynomial* $F \in \S(V)^W$ *such that*

$$\delta_1\delta_2\delta_3 F = J(\delta_1,\delta_2,\delta_3) \tag{3.11.6}$$

for δ_1, δ_2 *and* $\delta_3 \in \Omega_+^*$.

Proof. Due to the commutativity and associativity of $*$ and the self-adjointness of $*$ (3.9.2), J_3 is a symmetric tensor on $\mathcal{G}$ over $S(V)^{W,\tau}$. Therefore, we have only to show the following

Fact. *The 4-forms* $\delta_0 J_3(\delta_1,\delta_2,\delta_3)$ *in* $\delta_0, \cdots, \delta_3 \in \Omega_+^*$ *is symmetric.*

Put $J_4(\delta_0,\delta_1,\delta_2,\delta_3) := \delta_0 J_3(\delta_1,\delta_2,\delta_3)$. Let us specialize (3.9.1) to flat tangent vectors $\delta_1,\delta_2,\delta_3 \in \Omega_+^*$. Then one obtains $\nabla\!\!\!/_{\delta_1}(\delta_2 * \delta_3)$. This implies that $\nabla\!\!\!/_{\delta_1}(\delta_2 * \delta_3)$ is a symmetric cubic form on Ω_+^*. On the other hand, by using the metric property I.3. ii) of $\nabla\!\!\!/$ the definition of J_4 is rewritten as: $J_4(\delta_0,\delta_1,\delta_2,\delta_3) = J(\nabla\!\!\!/_{\delta_0}(\delta_1 * \delta_2), \delta_3)$. This expression is now symmetric w.r.t. the letters $\{0,1,2\}$. Since it was already symmetric w.r.t. the letters $\{1,2,3\}$, altogether, J_4 is symmetric w.r.t. all letters $\{0,1,2,3\}$. $\square$

3.12. **Dimensions.**

Before ending this section, we clarify that some data, such as E, $\nabla\!\!\!/$, N and exponents, are absolutely defined independent of scaling factors.

Recall that in the study of the flat structure on the variety S_W (except for type A_1), there were two ambiguities of scaling factors for

1. a choice of an invariant form I in 2.2, and
2. a choice of a primitive derivation D in (3.2.1).

Let us call an object X has dimension (α,β) if the scale changes $I^* \mapsto \lambda I^*$ and $D \mapsto \mu D$ induce the scale change $X \mapsto \lambda^\alpha \mu^\beta X$ (in case of type A_1, one has $\lambda\mu = 1$ and, hence, only the difference $\alpha - \beta$ has a meaning).

The following table is easy to calculate.

object	α	β	deg	*object*	α	β	deg
I^*	1	0	$\frac{2}{h}$	E	0	0	0
D	0	1	-1	$\nabla\!\!\!/$	0	0	0
I	-1	0	$-\frac{2}{h}$	$*$	0	-1	1
J	1	-1	$1+\frac{2}{h}$	N	0	0	0
J^*	-1	1	$-1-\frac{2}{h}$	Δ^2	0	$-l$	l
P_l	0	-1	1	P_1	1	0	$\frac{2}{h}$
$P_l *$	0	-1	1	J_3	1	-2	$2+\frac{2}{h}$
$[P_l]$	0	0	0	F	1	-1	$2+\frac{2}{h}$

The table shows that E, $\nabla\!\!\!/$, N and exponents are absolutely defined.

4. Uniformization equations

Following [S6,§5], we introduce the system of uniformizing equation $\mathcal{M}_{W,s}$ on S_W attached to the flat structure in 3.9. It is a system of second order holonomic system such that the first order derivatives of its solution gives the system of the solutions of the Levi-Civita connection ∇ in §3 on the lattice $\mathcal{H}^{(s)}$. The $\mathcal{M}_{W,s}$ has new features:

i) the equation contains a spectral parameter s shifting exponents,

ii) the equation is a $\mathcal{D}_{S_W}$-module instead of a $\mathcal{E}_{S_W}$-module,

iii) the homogeneity condition is omitted from the equation to allow non-constant solutions for the invariant cycles.

Compared with [S6,§5], we skipped a construction of the integral lattice on the local system of the solutions and the monodromy group action on it for the case of crystallographic reflection group (for details of them, see [ibd]). Explicit equations $\mathcal{M}_{W,s}$ and some solutions for types A_1, A_2 and A_3 were discussed in [S2] (see footnote 2.).

4.1. $S(V)^W$-modules $\mathcal{H}_W^{(s)}$ for $s \in \mathbf{C}$.

Let $\{D^{\mathbf{C}}\} := \{D^s \mid s \in \mathbf{C}\}$ be a group isomorphic to the complex number field $\mathbf{C}$ with the multiplicative rule: $D^sD^t = D^{s+t}$. Let $\mathbf{Z}\cdot\{D^{\mathbf{C}}\}$ be its group ring, which contains the Laurent algebra $\mathbf{Z}[D, D^{-1}]$. Put

$$\mathcal{H}_W := \mathbf{Z} \cdot \{D^{\mathbf{C}}\} \otimes_{\mathbf{Z}} \mathcal{G}, \tag{4.1.1}$$

where one recalls (3.2.3) for the $S(V)^{W,\tau}$-module $\mathcal{G}$. The $\mathcal{H}_W$ has natural $S(V)^{W,\tau}$-module structure by a commutation rule

$$f \cdot D^s - D^s \cdot f = 0 \qquad \text{for } \ f \in S(V)^{W,\tau} \text{ and } \ s \in \mathbf{C}. \tag{4.1.2}$$

Further, the $\mathcal{H}_W$ is equipped with a $S(V)^W = S(V)^{W,\tau}[P_l]$-module structure and a covariant differentiation action ∇_δ for $\delta \in \mathcal{G}$ as follows:
i) the multiplication of P_l is defined by the formula (3.8.3) and (3.8.3)*,
ii) the action ∇_δ for $\delta \in \mathcal{G}$ is defined by the formula (3.8.4) and (3.8.4)*,
where we replace $k \in \mathbf{Z}_{\geq 0}$ by a complex number $s \in \mathbf{C}$. It is a routine to check that the $S(V*)^W$-module structure does not depend on a choice of P_l, and the ∇-action satisfies the Leibniz rule and the integrability condition (i.e. $\nabla^2 = 0$).

For any $s \in \mathbf{C}$, let us introduce the submodule of $\mathcal{H}_W$:

$$\mathcal{H}_W^{(s)} := D^s\mathbf{Z}[D^{-1}] \otimes_{\mathbf{Z}} \mathcal{G}. \tag{4.1.3}$$

The $\mathcal{H}_W^{(s)}$ is a $S(V)^W$-submodule (i.e. closed under the left multiplication of P_l) easily seen from (3.8.3) and (3.8.3)*. Thus, by definition, one has an increasing sequence of the $S(V)^W$-modules:

$$\cdots \subset \mathcal{H}_W^{(s-2)} \subset \mathcal{H}_W^{(s-1)} \subset \mathcal{H}_W^{(s)} \subset \mathcal{H}_W^{(s+1)} \subset \mathcal{H}_W^{(s+2)} \subset \cdots, \tag{4.1.4}$$

and a short exact sequence of $S(V)^W$-modules:

$$0 \to \mathcal{H}_W^{(s-1)} \to \mathcal{H}_W^{(s)} \to \mathcal{G} \to 0, \tag{4.1.5}$$

where $\mathcal{G}$ is equipped with $S(V)^W$-module structure by the action of P_l* (3.9 Theorem I.1). One observes $\Delta^2\mathcal{H}_W^{(s)} \subset \mathcal{H}_W^{(s-1)}$ since $\Delta^2\mathcal{G} = 0$ due to the 3.4 Corollary. An explicit formula can be given as follows.

$$\begin{aligned} \Delta^2(D^s \otimes \partial_j) &= \textstyle\sum_{i=1}^l b_{ji}(P_l D^s \otimes \partial_i - D^s \otimes P_l * \partial_i) \\ &= \textstyle\sum_{i=1}^l b_{ji} D^{s-1} \otimes (N-s)\partial_i). \end{aligned} \tag{4.1.6}$$

Here $\partial_i = \frac{\partial}{\partial P_i}$ and (b_{ji}) is the adjoint matrix of $(P_l\delta_{ij} - a_{ij})$ with $w(\partial_i) = P_l\partial_i - P_l * \partial_i = \sum_{j=1}^l (P_l\delta_{ij} - a_{ij})\partial_j$ so that $\sum_{i=1}^l b_{ji} w(\partial_i) = \Delta^2\partial_j$.

Note. The modules $\mathcal{H}_W^{(-i)}$ for $i \in \mathbf{Z}_{\geq 0}$ in 3.10 are identified with the above defined one through (3.8.2). But the identification changes according to a choice of a primitive form (c.f. (4.3.1)).

The covariant differentiation by $\delta \in \mathcal{G}$ defines a map $\nabla_\delta : \mathcal{H}_W^{(s)} \to \mathcal{H}_W^{(s+1)}$ (use (3.8.4) and (3.8.4)*). Since $Der_{S_W} = S(V)^W \otimes \mathcal{G}$, the action extends to a covariant differentiation $\nabla : Der_{S_W} \times \mathcal{H}_W^{(s)} \to \mathcal{H}_W^{(s+1)}$. Particularly, ∇_D induces a $S(V)^{W,\tau}$-bijection:

$$\nabla_D : \mathcal{H}_W^{(s)} \simeq \mathcal{H}_W^{(s+1)}, \quad D^s \otimes \delta \mapsto D^{s+1} \otimes \delta. \tag{4.1.7}$$

Note. The power action $\nabla_D^t : \mathcal{H}_W^{(s)} \to \mathcal{H}_W^{(s+t)}$ for $t \in \mathbf{C}$ can be defined by letting $\nabla_D^t(D^s \otimes \delta) := D^{s+t} \otimes \delta$ satisfying the relation $[\nabla_D^t, P_l] = t\nabla_D^{t-1}$.

4.2. **Polarization element I_s.**

We introduce the *polarization element* $I_s \in \mathcal{H}_W \otimes_{S(V)^W} \mathcal{H}_W$ (the tensor of left $S(V)^W$-modules) for $s \in \mathbf{C}$ by the next three formulae:

$$\begin{aligned} I_s &:= \textstyle\sum_{i=1}^l (D^s \otimes \delta_i) \otimes (P_l D^{-s} \otimes \delta_i^* - D^{-s} \otimes P_l * \delta_i^*) \\ &= \textstyle\sum_{i=1}^l (D^s \otimes \delta_i) \otimes (D^{-s} \otimes w(\delta_i^*) + sD^{-s-1} \otimes \delta_i^*) \\ &= \textstyle\sum_{i=1}^l (D^s \otimes \delta_i) \otimes (D^{-s-1} \otimes (N+s)(\delta_i^*)), \end{aligned} \tag{4.2.1}$$

where i) $\delta_1, \cdots, \delta_l$ are $S(V)^{W,\tau}$-free basis of $\mathcal{G}$ and $\delta_1^*, \cdots, \delta_l^*$ are their dual basis w.r.t. J so that the definition does not depend on the basis, and ii) the equivalence of the three definitions follows from (3.8.3)*.

Note. Through the embedding $Der_{S_W} \subset \mathcal{H}_W$ (3.8.2), the polarization I (3.4.8) is identified with the above defined I_0.

Lemma. 1. *For any $s \in \mathbf{C}$, one has the formula:*

$$I_s = {}^tI_{-s} \quad \text{and} \quad I_s = -{}^tI_{-s-1}, \tag{4.2.2}$$

where the transposition ${}^tI_s$ of the polarization I_s is defined by transposing the tensor factors in the definition (4.2.1).

2. *Let the covariant differentiation ∇_δ for $\delta \in \mathcal{G}$ act on the tensor $\mathcal{H}_W \otimes_{S(V)^W} \mathcal{H}_W$ naturally (by the Leibniz rule). Then one has*

$$\nabla I_s = 0. \tag{4.2.3}$$

Proof. First, we note the following general formula:

$*)\qquad \sum_{i=1}^l f(A\delta_i) \otimes_{S(V)^{W,\tau}} g(\delta_i^*) = \sum_{i=1}^l f(\delta_i) \otimes_{S(V)^{W,\tau}} g(A^*\delta_i^*)$

where f, g are $S(V)^{W,\tau}$-homomorphisms defined on $\mathcal{G}$, A and A^* are $S(V)^{W,\tau}$-endomorphism on $\mathcal{G}$ and its adjoint w.r.t. J, and $\delta_1, \cdots, \delta_l$ and $\delta_1^*, \cdots, \delta_l^*$ are $S(V)^{W,\tau}$-free basis of $\mathcal{G}$ and the dual basis w.r.t. J.

1. Take the first line of (4.2.1). The multiplication of P_l in the first term can be transposed to that in the left side of the tensor. The left action $P_l *$ in the second term can be transposed to the left due to $*)$ and the self adjointness of P_l* ((3.9.2) and II.6. i)). So, one obtains

$$\begin{aligned} I_s &= \textstyle\sum_{i=1}^l (D^s \otimes \delta_i) \otimes (P_l D^{-s} \otimes \delta_i^* - D^{-s} \otimes P_l * \delta_i^*) \\ &= \textstyle\sum_{i=1}^l \big(P_l D^s \otimes \delta_i - D^s \otimes (P_l * \delta_i)\big) \otimes (D^{-s} \otimes \delta_i^*) = {}^t I_{-s} \end{aligned}$$

This proves the first formula in (4.2.2).

Let us employ the third line of (4.2.1). Use the fact that $(N+s)^* = N^* + s = -N + s + 1$ ((3.9.6)), and apply $*)$ so that one obtains:

$$\begin{aligned} I_s &= \textstyle\sum_{i=1}^l (D^s \otimes \delta_i) \otimes (D^{-s-1} \otimes (N+s)(\delta_i^*)) \\ &= \textstyle\sum_{i=1}^l (D^s \otimes (-N+s+1)\delta_i) \otimes (D^{-s-1} \otimes \delta_i^*) \\ &= -\textstyle\sum_{i=1}^l (D^{-u-1} \otimes (N+u)\delta_i) \otimes (D^u \otimes \delta_i^*) \quad h \text{ for } -u := s+1 \\ &= -{}^t I_u = -{}^t I_{-s-1}. \end{aligned}$$

This proves the second formula in (4.2.2).

2. Let δ_i ($i = 1, \cdots, l$) be flat basis (i.e. $\nabla\!\!\!/ \delta_i = 0$) of $\mathcal{G}$. Then the dual basis δ_i^* and their N-images $N(\delta_i)$, $N(\delta_i^*)$ are also flat. The covariant differentiation of I_s vanishes using the duality (3.9.6) again.

$$\begin{aligned} &\nabla_\delta I_s \\ &= \textstyle\sum_{i=1}^l \nabla_\delta (D^s \otimes \delta_i) \otimes (P_l D^{-s} \otimes \delta_i^* - D^{-s} \otimes P_l * \delta_i^*) \\ &\quad + \textstyle\sum_{i=1}^l \big(D^s \otimes \delta_i\big) \otimes \nabla_\delta \big(D^{-s-1} \otimes (N+s)(\delta_i^*)\big) \\ &= \textstyle\sum_{i=1}^l (D^{s+1} \otimes (\delta * \delta_i) + D^s \otimes \nabla\!\!\!/_\delta \delta_i) \otimes (P_l D^{-s} \otimes \delta_i^* - D^{-s} \otimes P_l * \delta_i^*) \\ &+ \textstyle\sum_{i=1}^l \big(D^s \otimes \delta_i\big) \otimes \big(D^{-s} \otimes (\delta * (N+s)\delta_i^*) + D^{-s-1} \otimes \nabla\!\!\!/_\delta (N+s)\delta_i^*)\big) \\ &= \textstyle\sum_{i=1}^l \big(P_l D^{s+1} \otimes (\delta * \delta_i) - D^{s+1} \otimes (P_l * \delta * \delta_i)\big) \otimes \big(D^{-s} \otimes \delta_i^*\big) \\ &\quad + \textstyle\sum_{i=1}^l \big(D^s \otimes (\delta * (N+s))^* \delta_i\big) \otimes \big(D^{-s} \otimes \delta_i^*\big) \\ &= \textstyle\sum_{i=1}^l \big(D^s \otimes (N-s-1)(\delta * \delta_i)\big) \otimes \big(D^{-s} \otimes \delta_i^*\big) \\ &\quad + \textstyle\sum_{i=1}^l \big(D^s \otimes (N^*+s)(\delta * \delta_i)\big) \otimes \big(D^{-s} \otimes \delta_i^*\big) \\ &= \textstyle\sum_{i=1}^l \big(D^s \otimes (N+N^*-1)(\delta * \delta_i)\big) \otimes \big(D^{-s} \otimes \delta_i^*\big) = 0. \end{aligned}$$

These complete a proof of Lemma. $\square$

Combining the two formulae (4.2.2), one obtains the semi-periodicity $I_{s+1} = -I_s$. On the other hand, the formulae (4.2.2) imply, $I_0 = {}^tI_0$ and $I_{-\frac{1}{2}} = -{}^tI_{-\frac{1}{2}}$. Therefore, we obtain the next corollary.

Corollary. 1. *One has $I_{s+n} = (-1)^n I_s$ for $s \in \mathbf{C}$ and $n \in \mathbf{Z}$.*

2. *The polarization I_s is a symmetric form for an integer $s \in \mathbf{Z}$ and is a skew symmetric form for a half integer $s \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$.*

4.3. **Connection $\nabla^{(s)}$ and uniformization equation.**

Put $\zeta^{(s-1)} := D^{s-1} \otimes D \in \mathcal{H}_W$ for $s \in \mathbf{C}$ (c.f. 3.10 Assertion.). The covariant differentiation of $\zeta^{(s-1)}$ induces an $S(V)^W$-homomorphism:

$$\varphi^{(s)} \ : \ Der_{S_W} \to \mathcal{H}_W^{(s)}, \quad \delta \mapsto \nabla_\delta \zeta^{(s-1)}. \tag{4.3.1}$$

Lemma. *If $s \notin \{exponents\} + \mathbf{Z}_{\geq 0}$, then $\varphi^{(s)}$ is an isomorphism.*

Proof. By induction on $n \in \mathbf{Z}_{\geq 0}$, we prove that $\varphi^{(s)}$ induces isomorphy:

$$S(V)^W_{\leq n} \otimes_{S(V)^{W,\tau}} \mathcal{G} \ \simeq \ D^s \otimes \mathcal{G} \oplus D^{s-1} \otimes \mathcal{G} \oplus \cdots \oplus D^{s-n} \otimes \mathcal{G}.$$

For $n = 0$, one has $\nabla_\delta(D^{s-1} \otimes D) = D^s \otimes \delta$ for $\delta \in \mathcal{G}$ and hence $\varphi^{(s)} : \mathcal{G} \simeq D^s \otimes \mathcal{G}$. Applying (3.8.3) n-times for $n > 0$, one obtains

$$\begin{array}{rcl} P_l^n(D^s \otimes \delta) & = & D^s \otimes (P_l *)^n \delta \\ & + & D^{s-1} \otimes ((P_l *)^{n-1}(N-s) + \cdots)\delta \\ & + & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad + \\ & + & D^{s-n} \otimes (N-s+n-1)\cdots(N-s+1)(N-s)\delta \end{array}$$

where $(N-s+n-1)\cdots(N-s+1)(N-s)$ is an isomorphism of $\mathcal{G}$ to itself since each factor is invertible by the assumption on $s \in \mathbf{C}$. $\square$

Note. The isomorphism for $s = 0$ is the identification (3.8.2). In general, *for any $s \in \mathbf{C}$, $\mathcal{H}_W^{(s)}$ is a finitely generated $S(V)^W$-module of homological dimension ≤ 1.* (*Proof.* For a large $n \in \mathbf{Z}_{\geq 0}$, $\varphi^{(s-n)}$ is an isomorphism due to Lemma above. Then one obtains the statement on $\mathcal{H}_W^{(s-n+i)}$ by induction on $i \geq 0$, applying (4.1.5) and 3.4 Corollary 1.)

One has an embedding $\varphi^{(s)} : \mathcal{H}_W^{(s+n)}/\Delta^{2n}$-torsions $\subset \frac{1}{\Delta^{2n}} Der_{S_W}$ for $s \notin \{exp.\} + \mathbf{Z}_{\geq 0}$ and $n \in \mathbf{Z}_{\geq 0}$. In the sequel, we study this "generic part" as a logarithmic connection along the discriminant D_W.

Let $\nabla^{(s)}$ be the one parameter family of connection on Der_{S_W} satisfying $\varphi^{(s)} \nabla_\delta^{(s)} \delta' = \nabla_\delta(\varphi^{(s)} \delta')$. It is *a logarithmic flat torsion free connection on the logarithmic tangent bundle of S_W*, i.e. one has (3.5.5), (3.5.6) and (3.5.3) replacing ∇ by $\nabla^{(s)}$. In fact $\nabla^{(s)}$ depends on s only linearly shifting the exponent. Let us consider the dual connection: $\nabla^{(s)*} : Der_{S_W} \times \Omega^1_{S_W} \to \Omega^1_{S_W}(\log(\Delta))$ defined on $\Omega^1_{S_W}$ by the rule $\delta\langle \delta', \omega \rangle = \langle \nabla_\delta^{(s)} \delta', \omega \rangle + \langle \delta', \nabla_\delta^{(s)*} \omega \rangle$ for the natural $S(V)^W$-pairing $\langle * \rangle$.

Lemma. *A local analytic 1-form ω on S_W is horizontal w.r.t. $\nabla^{(s)*}$ (i.e. $\nabla^{(s)*}\omega = 0$) iff there exists a local analytic function u satisfying i) $\omega = du$ (the primitivity), and ii) the system of equations*

$$P(\delta_1,\delta_2)u = 0 \quad \text{and} \quad Q_s(\delta_1,\delta_2)u = 0 \quad \text{for} \quad \delta_1,\delta_2 \in \mathcal{G}, \tag{4.3.2}$$

where

$$\begin{aligned} P(\delta_1,\delta_2) &:= \delta_1\delta_2 - (\delta_1 * \delta_2)D - \nabla\!\!\!/_{\delta_1}\delta_2 && (4.3.3)\\ Q_s(\delta_1,\delta_2) &:= \delta_1 w(\delta_2) - w(\nabla\!\!\!/_{\delta_1}\delta_2) - \delta_1 * (N-s)\delta_2. && (4.3.4)\end{aligned}$$

Proof. We still denote by $\langle\cdot,\cdot\rangle$ the extension of the natural pairing between the module Der_{S_W} and the module $\Omega^{1an}_{S_W}$ of local analytic 1-forms (which takes values in the ring of local analytic functions). We denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the pairing between $\mathcal{H}_W$ and $\Omega^{1an}_{S_W}$ via $\varphi^{(s)}$ (4.3.1) so that $\langle\delta,\omega\rangle = \langle\langle D^s\otimes\delta,\omega\rangle\rangle$ for $\delta\in\mathcal{G}$.

The horizontal section $\omega\in\Omega^{1an}_{S_W}$ of $\nabla^{(s)*}$ is a closed form, since

$$\begin{aligned} d\omega &= \textstyle\sum_{ij=1}^{l} dP_i\wedge dP_j\ \partial_i\big\langle\partial_j,\omega\big\rangle \\ &= \textstyle\sum_{ij=1}^{l} dP_i\wedge dP_j\ \big\langle\big\langle\nabla_{\partial_i}\nabla_{\partial_j}D^{s-1}\otimes D,\omega\big\rangle\big\rangle \\ &\quad + \textstyle\sum_{ij=1}^{l} dP_i\wedge dP_j\ \big\langle\partial_j,\nabla^{(s)*}_{\partial_i}\omega\big\rangle \quad = \quad 0\end{aligned}$$

where the second line vanishes since $\nabla_{\partial_i}\nabla_{\partial_j}D^{s-1}\otimes D$ is symmetric w.r.t. the indices i and j, and the third line vanishes due to the horizontality: $\nabla^{(s)*}_{\partial_i}\omega = 0$ of ω.

That ω is horizontal implies $\nabla^{(s)*}\omega$ is perpendicular to Der_{S_W}. So,

$$\begin{aligned} 0 &= \langle\delta',\nabla^{(s)*}_{\delta}\omega\rangle - \langle\delta*\delta',\nabla^{(s)*}_{D}\omega\rangle \\ &= \delta\langle\delta',\omega\rangle - \langle\langle\nabla_\delta(D^s\otimes\delta'),\omega\rangle\rangle - \langle\delta*\delta',\nabla^{(s)*}_{D}\omega\rangle \\ &= \delta\langle\delta',\omega\rangle - \langle\langle D^{s+1}\otimes(\delta*\delta') + D^s\otimes\nabla\!\!\!/_{\delta'}\delta,\omega\rangle\rangle \\ &\qquad -\langle\langle D^s\otimes(\delta*\delta'),\nabla^{(s)*}_{D}\omega\rangle\rangle \\ &= \delta\langle\delta',\omega\rangle - D\langle\langle D^s\otimes(\delta*\delta'),\omega\rangle\rangle - \langle\nabla\!\!\!/_{\delta'}\delta,\omega\rangle \\ &= \delta\langle\delta',\omega\rangle - D\langle\delta*\delta',\omega\rangle - \langle\nabla\!\!\!/_{\delta'}\delta,\omega\rangle\end{aligned}$$

This yields (4.3.3). One obtains (4.3.4) as follows.

$$\begin{aligned} 0 &= \langle w(\delta),\nabla^{(s)*}_{\delta'}\omega\rangle \\ &= \delta'\langle w(\delta),\omega\rangle - \langle\langle\nabla_{\delta'}(P_lD^s\otimes\delta - D^s\otimes(P_l*\delta)),\omega\rangle\rangle \\ &= \delta'\langle w(\delta),\omega\rangle - \langle\langle\nabla_{\delta'}(D^{s-1}\otimes(N(\delta)-s\delta)),\omega\rangle\rangle \\ &= \delta'\langle w(\delta),\omega\rangle - \langle\langle D^s\otimes(\delta'*(N-s)\delta)),\omega\rangle\rangle \\ &\qquad -\langle\langle D^{s-1}\otimes(\nabla\!\!\!/_{\delta'}(N-s)\delta),\omega\rangle\rangle.\end{aligned}$$

The last term is modified as: $D^{s-1}\otimes(\nabla\!\!\!/_{\delta'}(N-s)\delta) = D^{s-1}\otimes((N-s)\nabla\!\!\!/_{\delta'}\delta) = P_l(D^s\otimes\nabla\!\!\!/_{\delta'}\delta) - D^s\otimes(P_l*\nabla\!\!\!/_{\delta'}\delta) = \varphi^{(s)}(w(\nabla\!\!\!/_{\delta'}\delta))$. $\square$

Remark. 1. One can prove that $\zeta^{(s-1)}$ satisfies the following equations by similar calculations as in the proof of the lemma (c.f. [S6,§5]).

$$
(4.3.5) \qquad \begin{array}{llcll}
i) & P(\delta_1,\delta_2)\,\zeta^{(s-1)} & = & 0 & \text{for } \delta_1,\delta_2\in\mathcal{G}\\
ii) & Q_s(\delta_1,\delta_2)\,\zeta^{(s-1)} & = & 0 & \text{for } \delta_1,\delta_2\in\mathcal{G}\\
iii) & (E-(1/h-s))\,\zeta^{(s-1)} & = & 0. &
\end{array}
$$

2. We remark that the equations (4.3.4) includes in particular

$$
(4.3.6) \qquad Q_s(\delta,D)=\delta(E-(1/h-s)) \quad \text{for } \delta\in\mathcal{G}.
$$

3. For each fixed $s\in\mathbf{C}$, one may regard (4.3.2) as a family of ordinary differential equations in the indeterminate P_l parameterized by $\mathrm{Spec}(S(V)^{W,\tau})$ where the monodromy is preserved by the deformation on the complement of the bifurcation locus in $\mathrm{Spec}(S(V)^{W,\tau})$.

4.4. **The characteristic variety of $\mathcal{M}_{W,s}$.**

We call (4.3.2) *the one-parameter family of system of uniformization equations of type* W. The system is holonomic since the characteristic variety is the conormal bundle of the discriminant. The homogeneity condition iii) is deleted from (4.3.5), and the space of local solutions of the system has rank $l+1$. This gives a quite important freedom in a study of the system when 0 is an exponent and there is a nilpotent monodromy (see 4.5 Example). We refer to [S-K-K] for terminologies.

Let $\mathcal{D}_{S_W}$ be the algebra of polynomial coefficients differential operators on S_W (= the enveloping algebra of Der_{S_W} over $S(V)^W$= the algebra generated by P_i and ∂_j ($1\le i,j\le l$) with the commutation relations $[\partial_i,P_j]=\delta^i_j$ (Kronecker's delta)). Let $\mathcal{D}_{S_W,\le d}$ be the $S(V)^W$ submodule of differential operators of degree (i.e. the total degree w.r.t. $\partial_1,\cdots,\partial_l$) less or equal than $d\in\mathbf{Z}$. Then the $gr\mathcal{D}_{S_W}:=\oplus_{d=0}^{\infty}\mathcal{D}_{S_W,\le d}/\mathcal{D}_{S_W,\le d-1}$ is a commutative graded algebra isomorphic to $S(V)^W[\xi_1,\cdots,\xi_l]$ (where ξ_i stands for the image of ∂_i), and is the coordinate ring for the cotangent space $T^*_{S_W}$ of S_W.

Let $\mathcal{I}_{W,s}:=\Sigma_{\delta_1,\delta_2\in\mathcal{G}}P(\delta_1,\delta_2)+\Sigma_{\delta_1,\delta_2\in\mathcal{G}}Q_s(\delta_1,\delta_2)$ be the left ideal of $\mathcal{D}_{S_W}$ generated by $P(\delta_1,\delta_2)$ and $Q_s(\delta_1,\delta_2)$ for $\delta_1,\delta_2\in\mathcal{G}$. Put

$$
(4.4.1) \qquad \mathcal{M}_{W,s}:=\mathcal{D}_{S_W}/\mathcal{I}_{W,s}
$$

and call it also the one parameter family of the system of uniformization equations of type W, confusing with the system (4.3.2).

The *characteristic variety* of $\mathcal{M}_{W,s}$ is the subvariety of the cotangent space $T^*_{S_W}$ defined by the ideal $\sigma(\mathcal{I}_{W,s})$ in $S(V)^W[\xi_1,\cdots,\xi_l]$ generated by the principal symbols of all elements of $\mathcal{I}_{W,s}$, where the principal symbol $\sigma(X)$ for $X\in\mathcal{D}_{S_W}$ with $\deg(X)=d$ is, as usual, defined as the degree d-part of X, associating to ∂_i the commutative variable ξ_i. Recall that a left $\mathcal{D}_{S_W}$-module is called *holonomic* if its characteristic variety is a Lagrangian subvariety, and is *simple holonomic* if all, except for the zero section of the cotangent space, of irreducible components of its characteristic variety are simple.

Theorem. *For each fixed $s \in \mathbf{C}$, the $\mathcal{M}_{W,s}$ is a simple regular holonomic system, whose characteristic variety is the union of the conormal bundle $N^*(D_W)$ of the discriminant D_W and the zero section $T^*_{S_W}(0)$ of the cotangent bundle $T^*_{S_W}$ of the multiplicity $l+1$.*

Proof. The conormal bundle $N^*(D_W)$ of the D_W is defined as the Zariski closure in $T^*_{S_W}$ of the conormal bundle $N^*(D_W \setminus Sing(D_W))$ of the smooth part $D_W \setminus Sing(D_W)$, where the conormal bundle of a smooth subvariety $X \subset S_W$ is the subvariety of $T^*_{S_W}|_X$ consisting of the covectors perpendicular to the subbundle T_X of $T_{S_W}|_X$ at each point of X. Conormal bundle is always a middle dimensional subvariety of the cotangent bundle $T^*_{S_W}$, and is a Lagrangian subvariety.

For the proof of the theorem, we prepare two lemmas, whose proofs use the flat structure 3.9 on S_W. The first lemma states about the involutivity of the generator systems of the ideal $\mathcal{I}_{W,s}$.

Lemma. *Let P be an element of $\mathcal{I}_{W,s}$ of degree $m \in \mathbf{Z}_{\geq 0}$. Then $m \geq 2$ and there exists elements R_{ij} $(ij = 1, \cdots, l-1)$ and S_i $(i = 1, \cdots, l)$ of $\mathcal{D}_{S_W}$ of degrees less or equal than $m-2$ such that*

$$(4.4.2) \qquad P = \textstyle\sum_{1 \leq i \leq j \leq l-1} R_{ij} P(\partial_i, \partial_j) + \sum_{1 \leq i \leq l} S_i Q_s(\partial_i).$$

Proof. The $P(\delta_1, \delta_2)$ is a $S(V)^{W,\tau}$-symmetric bilinear form in $\delta_1, \delta_2 \in \mathcal{G}$ and $P(D, \delta) = 0$ due to 3.9 Theorem I.1, I.2 and I.3. Put

$$(4.3.4)* \qquad Q_s(\delta) := Q_s(D, \delta) = Dw(\delta) - (N-s)\delta.$$

Then, by a use of 2) in the proof of 3.9 Theorem, the bilinear operator $Q_s(\delta_1, \delta_2)$ is reduced to a single variable operator $Q_s(\delta * \delta_2)$:

$$(4.3.4)** \qquad Q_s(\delta_1, \delta_2) = Q_s(\delta_1 * \delta_2) + P_l P(\delta_1, \delta_2) - P(\delta_1, P_l * \delta_2).$$

Due to the symmetric $S(V)^{W,\tau}$-bilinearities of $P(\delta_1, \delta_2)$ and the reduction (4.3.4)**, the ideal $\mathcal{I}_{W,s}$ is generated by $P(\partial_i, \partial_j)$ $(1 \leq i \leq j \leq l-1)$ and $Q_s(\partial_i)$ $(1 \leq i \leq l)$. We show that the generator system is involutive: the symbol ideal $\sigma(\mathcal{I}_{W,s})$ is generated by the symbols of them.

For a $P \in \mathcal{I}_{W,s}$, let $P = \sum_{1 \leq i \leq j \leq l-1} R_{ij} P(\partial_i, \partial_j) + \sum_{1 \leq i \leq l} S_i Q_s(\partial_i)$ be an expression. Consider the maximal degree of the coefficients of the expression: $d := \max\{\deg(R_{ij})(1 \leq i \leq j \leq l-1), \deg(S_i)(1 \leq i \leq l)\}$. Of course $\deg(P) \leq d+2$. We want to show that if $d+2 > \deg(P)$, then there exists a new expression for P such that the maximal degree

d' of the new coefficients is strictly less than d. For the purpose, we consider two relations among the generators of the ideal $\mathcal{I}_{W,s}$:

$$\begin{array}{ll} *) & \delta P(\delta',\delta'') - \delta' P(\delta,\delta'') + D\left(P(\delta,\delta' * \delta'') - P(\delta',\delta * \delta'')\right) \\ & \quad + P(\delta, \nabla\!\!\!/_{\delta'}\delta'') - P(\delta', \nabla\!\!\!/_{\delta}\delta'') = 0, \\ **) & \delta Q_s(\delta') - DQ_s(\delta * \delta') - Q_s(\nabla\!\!\!/_{\delta}\delta') \\ & \quad - EP(\delta,\delta') - \delta P(P_l * D, \delta') + DP(P_l * D, \delta * \delta') \\ & \quad - P([P_l * D, \delta], \delta') - P(\delta, \nabla\!\!\!/_{P_l * D}\delta') \\ & \quad + P(\delta, (N+s-1)\delta') + P(P_l * D, \nabla\!\!\!/_{\delta}\delta') = 0 \end{array}$$

for $\delta, \delta', \delta'' \in \mathcal{G}$. The proof of them is easily reduced to 3.9 Theorem.

We return to an expression (4.4.2) of an element P of the ideal $\mathcal{I}_{W,s}$. If $d := \max\{\deg(R_{ij}), \deg(S_i)\} > \deg(P) - 2$, then one has

$$\textstyle\sum_{ij=1}^{l-1} \sigma_d(R_{ij})\sigma(P(\partial_i,\partial_j)) + \sum_{i=1}^{l} \sigma_d(S_i)\sigma(Q_s(\partial_i)) = 0,$$

where $\sigma_d(X)$ means the homogeneous of degree d part of $X \in \mathcal{D}_{S_W,\le d}$. Subtracting a suitable linear combination of the above elements $*$) and $**$), we obtain new coefficients R'_{ij} and S'_i such that $\sigma_d(R'_{ij}) = r_{ij}\xi_D^d$ and $\sigma_d(S'_i) = s_i\xi_D^d$ for $r_{ij}, s_i \in S(V)^W$. The new symbols satisfy the relation $\sum_{ij=1}^{l-1} r_{ij}\sigma(P(\partial_i,\partial_j)) + \sum_{i=1}^{l} s_i\sigma(Q_s(\partial_i)) = 0$. Because of the algebraic independence of $\xi_1, \cdots, \xi_l$, this implies $r_{ij} = s_i = 0$. So, $\sigma_d(R'_{ij}) = \sigma_d(S'_i) = 0$, i.e. the degrees of R'_{ij} and S'_i are less than d. □

Corollary. *The symbol ideal $\sigma(\mathcal{I}_{W,s})$ is generated by principal symbols:*

$$(4.4.3)\qquad \begin{array}{lcll} \sigma(P(\partial_i,\partial_j)) & = & \xi_i\xi_j - \xi_D(\xi_i * \xi_j) & (1 \le i,j \le l-1), \\ \sigma(Q_s(\partial_i)) & = & \xi_D(P_l\xi_i - P_l * \xi_i) & (1 \le i \le l), \end{array}$$

where the $*$-product on ξ_i is defined by the identification $\xi_i \leftrightarrow \partial_i \in \mathcal{G}$, and we put $\xi_D := \sigma(D)$. This induces a $S(V)^W$-algebra isomorphism:

$$(4.4.4)\qquad \mathcal{G} \simeq \mathcal{O}_{Ch(\mathcal{M}_{W,s}) \cap \{\xi_D = 1\}}$$

The next lemma determines the symbol ideal geometrically.

Lemma. *Let $(\xi_1, \cdots, \xi_l)$ and $\mathcal{I}(N^*(D_W))$ be ideals in $S(V)^W[\xi_1, \cdots, \xi_l]$ defining the zero section of the cotangent bundle $T^*_{S_W}$ and the conormal bundle of the discriminant, respectively. Then one has:*

$$(4.4.5)\qquad \mathcal{I}(N^*(D_W)) \cap (\xi_1, \cdots, \xi_l)^2 \;=\; \sigma(\mathcal{I}_{W,s})\;.$$

Proof. We first show the inclusion $\supset$. Clearly, $\sigma(P(\delta_1,\delta_2))$ and $\sigma(Q_s(\delta))$ belong to $(\xi_1, \cdots, \xi_l)^2$ (see (4.4.3)). To show that the symbols vanish on the conormal bundle $N^*(D_W)$, it is sufficient to show

$$\delta_1(\Delta)\ \delta_2(\Delta) - (\delta_1 * \delta_2)(\Delta)\ D(\Delta),\ P_l\delta(\Delta) - P_l * \delta(\Delta) \ \in\ (\Delta)$$

for $\delta_1, \delta_2, \delta \in \mathcal{G}$, since the conormal vectors at the smooth points of the discriminant are given by constant multiple of the differential $d\Delta =$

$\sum_i \partial_i \Delta dP_i$. The second formula is obvious since the difference $P_l\delta - P_l * \delta = w(\delta)$ is a logarithmic vector field (c.f. (3.4.4) and (3.1.1)).

The first formula: it is enough to show the formula for $\delta_1 = \delta \in \mathcal{G}$ and $\delta_2 = \delta_U := \nabla_D I(dU)$ for $U \in S(V)^W_{\le 1}$. Recalling (3.6.5), one has

$$2\delta_U(\Delta) = 2\langle \nabla_D I(dU), d\Delta\rangle = DI(dU, d\Delta) - I(D, [I(dU), I(d\Delta)]).$$

Recall (3.7.1) whose first term is the $*$-product and second term belongs to $Der_{S_W}(-\log(\Delta)$. Therefore, using the formula (3.6.5) again, one has

$$\begin{aligned} 2(\delta * \delta_U)(\Delta)\ mod\ (\Delta) &= 2\langle \nabla_\delta \nabla_D^{-1}\delta_U, d\Delta\rangle \\ &= 2\langle \nabla_\delta I(dU), d\Delta\rangle \\ &= \delta I(dU, d\Delta) - I(\delta, [I(dU), I(d\Delta)]). \end{aligned}$$

So,

$$\begin{aligned} & 2\delta(\Delta)\ \delta_U(\Delta) - 2(\delta * \delta_U)(\Delta)\ D(\Delta) \quad mod \quad (\Delta) \\ = \ & \delta(\Delta)\,(DI(dU, d\Delta) - I(D, [I(dU), I(d\Delta)])) \\ + \ & D(\Delta)\,(\delta I(dU, d\Delta) - I(\delta, [I(dU), I(d\Delta)])) \\ = \ & XI(dU, d\Delta) - I(X, [I(dU), I(d\Delta)])) \end{aligned}$$

where $X := \delta(\Delta)D - D(\Delta)\delta$. By definition, $X\Delta = 0$ and hence X is a logarithmic vector field. Then, each of the last two terms is zero modulo the ideal (Δ) as follows. The first term: using I (3.1.3), one has $I(dU, d\Delta) \in (\Delta)$ and then $X(\Delta) = (\Delta)$. The second term: put $X = I(\omega)$ for some $\omega = \sum_{i=1}^l F_i dP_i \in \Omega^1_{S_W}$ (3.1.3). Then

$$\begin{aligned} & I^*(X, [I(dU), I(d\Delta)]) = \langle [I(dU), I(d\Delta)], \omega\rangle \\ = \ & \textstyle\sum_{i=1}^l F_i \langle [I(dU), I(d\Delta)], dP_i\rangle \\ = \ & \textstyle\sum_{i=1}^l F_i\,(I(dU)(I(d\Delta, dP_i)) - I(d\Delta, I(dU, dP_i)))\,. \end{aligned}$$

Using (3.1.3), one checks that each term belongs to the ideal (Δ).

Let us prove now the opposite inclusion: any homogeneous element X of degree $d \ge 2$ w.r.t.(ξ) in the LHS of (4.4.3) belongs to the RHS. First, by a successive application of the first relations of (4.4.3), we may reduce X to such element that any monomial in $\xi_1, \cdots, \xi_l$ contains (at least) $d-1$th power of ξ_D. So, put $X = \xi_D^{d-1}Y$, where Y is a linear form in (ξ). If the coefficients of the linear form Y contains the variable P_l, then by successive applications of the second relations of (4.4.3), we may reduce that the coefficients of Y belongs to $S(V)^{W,\tau}$. So, $X = \xi_D^{d-1}Y$ for $Y = \sigma(\delta)$ with $\delta \in \mathcal{G}$. The fact $X \in \mathcal{I}(N^*_{D_W})$ means $(D(\Delta))^{d-1}\delta(\Delta) \in (\Delta)$. Since $D(\Delta)$ is coprime to the discriminant Δ, this implies $\delta(\Delta) \in (\Delta)$. This is possible only when $\delta = 0$ due to the splitting (3.4.1) (c.f. (3.4.3)). These end the proof of Lemma. $\square$

As a consequence, one has an exact sequence: $0 \to \mathcal{O}(T^*_{S_W}) \oplus \mathcal{O}_{S_W} \to gr(\mathcal{M}_{W,s}) \to \mathcal{O}_{N^*(D_W)} \to 0$. These complete a proof of Theorem. $\square$

Remark. The conormal bundle $N^*(D_W)\backslash\{$zero section of $T^*_{S_W}\}$ is smooth.

4.5. **Duality between the solutions $Sol(\mathcal{M}_{W,s})$ and $Sol(\mathcal{M}_{W,-s})$.**

We consider the solution sheaf $Sol(\mathcal{M}_{W,s}) := \mathcal{E}xt^0_{\mathcal{D}_{S_W}}(\mathcal{M}_{W,s}, \mathcal{O}_{S_W})$. Due to Theorem in the previous subsection, its restriction to $S_W \setminus D_W$ is a local system of rank $l+1$ containing the constant function sheaf. The exterior differentiation induces an exact sequence at each $x \in S_W \setminus D_W$

$$(4.5.1) \qquad 0 \ \rightarrow \ \mathbf{C}_{S_W,x} \ \rightarrow \ Sol(\mathcal{M}_{W,s})_x \ \xrightarrow{d} \ T^*_{S_W,x} \ \rightarrow \ 0.$$

Assertion 1. *One has the following splitting of the local system:*

$$(4.5.2) \qquad Sol(\mathcal{M}_{W,s}) = \begin{cases} \mathbf{C}_{S_W} \oplus Sol(\widetilde{\mathcal{M}}_{W,s}) & if \quad 1/h - s \neq 0 \\ \mathbf{C}_{S_W}\lambda \oplus Sol(\widetilde{\mathcal{M}}_{W,S}) & if \quad 1/h - s = 0, \end{cases}$$

where $\widetilde{\mathcal{M}}_{W,s} := \mathcal{M}_{W,s}/\mathcal{D}_{S_W}(E - (1/h - s))$ is the equation for $\zeta^{(s-1)}$ (recall (4.3.5)) and λ is a solution of the equation $\mathcal{M}_{W,s}$ with $E\lambda = 1$.

Proof. Let u be a (local) solution of $\mathcal{M}_{W,s}$. In view of (4.3.6) one has $(E - (1/h - s))u = c$ for $c \in \mathbf{C}$. If $1/h - s \neq 0$, then put $u' := u + c/(1/h-s)$ and one has $(E-(1/h-s))u' = 0$. That is: $u' \in Sol(\widetilde{\mathcal{M}}_{W,s})$. If $1/h - s = 0$, then one always has a solution with $c \neq 0$. □

Remark. The same proof of (4.4.5) shows the formula:

$$\mathcal{I}(N^*(D_W)) = (\sigma(P(\partial_i, \partial_j)), \sigma(w(\partial_i)) \ (1 \leq i, j \leq l)) = \sigma(\mathcal{I}(\tilde{M}_{W,s})).$$

Example. We illustrate the difference of $\mathcal{M}_{W,s}$ and $\widetilde{\mathcal{M}}_{W,s}$ by the example of type A_1. Since $l = 1$, $S(V)^W$ is generated by a single element $P_1 := \frac{1}{2}X^2$ with $I(dP_1, dP_1) = 2P_1$. Put $z := P_1$ and $D = \frac{d}{dz}$. Then,

$$\mathcal{M}_{A_1,s} : \tfrac{d}{dz}(z\tfrac{d}{dz} - (\tfrac{1}{2} - s))u = 0 \ \text{ and } \ \widetilde{\mathcal{M}}_{A_1,s} : (z\tfrac{d}{dz} - (\tfrac{1}{2} - s))u = 0.$$

Therefore, the solutions are given by

$$Sol(\mathcal{M}_{A_1,s}) : \quad u = \begin{cases} c \ z^{\frac{1}{2}-s} + d & (c, d \in \mathbf{C}) & \text{for } s \neq \frac{1}{2}, \\ c \ \log(z) + d & (c, d \in \mathbf{C}) & \text{for } s = \frac{1}{2}, \end{cases}$$

$$Sol(\widetilde{\mathcal{M}}_{A_1,s}) : \quad u = \begin{cases} c \ z^{\frac{1}{2}-s} & (c \in \mathbf{C}) & \text{for } s \neq \frac{1}{2}, \\ c \ 2\pi\sqrt{-1} & (c \in \mathbf{C}) & \text{for } s = \frac{1}{2}. \end{cases}$$

Remark. We shall see (5.3 Example) that the non-constant solution λ of $\mathcal{M}_{W,s}$ in (4.5.2) for type A_1 is given by the indefinite integral of the primitive form $\frac{dz}{z}$ of type A_1 and the constant solution $2\pi\sqrt{-1}$ of $\widetilde{\mathcal{M}}_{W,s}$ is its period (a similar description by a use of $\mathfrak{p}$-function holds for type A_3, and presumably for type D_4). We introduced the system $\mathcal{M}_{W,s}$ in order to include such log-type function λ (=an indefinite integral of a primitive form) in the construction of the period map. The relationship between λ and its period goes much deeper. E.g. the transcendentality of $2\pi\sqrt{-1}$ is shown by a use exponential function = the inverse of log function for type A_1. How about the $\mathfrak{p}$-functions for types A_3 and D_4?

Assertion 2. *One has a left $\mathcal{D}_{S_W}$-homomorphism:*

$$\mathcal{M}_{W,s+1} \to \mathcal{M}_{W,s} \tag{4.5.3}$$

by the right multiplication of D. It induces a homomorphism:

$$D : Sol(\mathcal{M}_{W,s}) \to Sol(\mathcal{M}_{W,s+1}). \tag{4.5.4}$$

The kernel of this homomorphism is a vector subspace spanned by the flat coordinates of degree s and by the constant function 1_{S_W}.

Proof. In order to show the well definedness of (4.5.3) and (4.5.4), it is sufficient to prove the relations:

$$\begin{aligned} P(\delta_1,\delta_2)D &= DP(\delta_1,\delta_2),\\ Q_{s+1}(\delta_1,\delta_2)D &= DQ_s(\delta_1,\delta_2). \end{aligned}$$

The verifications are left to the reader. The kernel of (4.5.4) is given by solutions of the system of the equations (4.3.3), (4.3.4)* and D. It is easy to see that the system is reduced to the system of equations:

$$(\delta_1\delta_2 - \nabla\!\!\!/_{\delta_1}\delta_2)u = 0,\ ((N-s)\delta)u = 0 \text{ and } Du = 0.$$

This is the equation for a flat coordinate of degree s (if $s \neq 0$). $\square$

Assertion 3. *The polarization $I \in Der_{S_W} \otimes Der_{S_W}$ (3.4.8) induces a pairing between the two solution systems for s and $-s$:*

$$\begin{array}{rcl} I : Sol(\mathcal{M}_{W,s})/\mathbf{C}_{S_W} \times Sol(\mathcal{M}_{W,-s})/\mathbf{C}_{S_W} & \to & \mathbf{C}_{S_W} \\ du \times dv & \mapsto & I(du,dv) = \sum_{i=1}^{l} \delta_i u \cdot w(\delta_i^*)v \end{array} \tag{4.5.5}$$

which is nondegenerate at every point of $S_W \setminus D_W$. One has:

$$I(dDu, dv) = -I(du, dDv) \tag{4.5.6}$$

for $u \in Sol(\mathcal{M}_{W,s-1})$ and $v \in Sol(\mathcal{M}_{W,-s})$.

Proof. Combining the homomorphism $\varphi^{(s)}$ (4.3.1) , one obtains

$$I_s = \varphi^{(s)} \otimes \varphi^{(-s)}(I) = \varphi^{(-s)} \otimes \varphi^{(s)}(I).$$

Then the horizontality of I_s (4.2.3) implies further

$$\nabla^{(s)} \otimes \nabla^{(-s)}(I) = \nabla^{(-s)} \otimes \nabla^{(s)}(I) = 0.$$

This implies the value $I(du, dv)$ is a constant and (4.5.5) is defined. Recall $\det(I(dP_i, dP_j)) = \Delta$ (3.1.5) so that I is nondegenerate on $S_W \setminus D_W$. So, the exact sequence (4.5.1) implies the nondegeneracy of the pairing (4.5.5). We show (4.5.6) by recalling (4.3.4)* and (3.9.6):

$$\begin{aligned} & I(du, dDv) = \textstyle\sum_{i=1}^{l} \delta_i u \cdot w(\delta_i^*)Dv = \sum_{i=1}^{l} \delta_i u \cdot ((N+s-1)\delta_i^*)v \\ = \ & \textstyle\sum_{i=1}^{l} ((N+s-1)^*\delta_i)u \cdot \delta_i^* v = -\sum_{i=1}^{l}((N-s)\delta_i)u \cdot \delta_i^* v \\ = \ & -\textstyle\sum_{i=1}^{l} w(\delta_i)Du \cdot \delta_i^* v = -I(dDu, dv). \end{aligned} \qquad \square$$

Remark. The nondeneracy does not hold for the pairing on $Sol(\widetilde{\mathcal{M}}_{W,s}) \times Sol(\widetilde{\mathcal{M}}_{W,-s})$ when $s \in \{exponents\} - \mathbf{Z}_{\geq 0}$ due to the splitting (4.5.2).

We are interested in the case when the application $D^n : Sol(\mathcal{M}_{W,s}) \to Sol(\mathcal{M}_{W,s+n})$ for $n \in \mathbf{Z}_{\geq 0}$ and $s \in \mathbf{C}$ brings a solution space to its dual space, that is the case $-s = s+n$, since then we obtain a bilinear form:

$$\begin{array}{rccc} I : Sol(\mathcal{M}_{W,s})/\mathbf{C}_{S_W} \times Sol(\mathcal{M}_{W,s})/\mathbf{C}_{S_W} & \to & \mathbf{C}_{S_W} \\ du \times dv & \mapsto & I(du, dD^n v). \end{array} \tag{4.5.7}$$

So, $s = -\frac{n}{2} \in \frac{1}{2}\mathbf{Z}_{\leq 0}$. According to the parity of n, there are two cases:

Case i) $s = 0, -1, -2, \cdots$. In this case (4.5.7) is a nondegenerate symmetric bilinear form as follows: applying the operator D, one has a sequence of isomorphism (for $\{$exponents$\} \cap \mathbf{Z} = \emptyset$) of the local systems:

$$\begin{array}{l} \cdots \xrightarrow{\sim} Sol(\mathcal{M}_{W,-2})/\mathbf{C}_{S_W} \xrightarrow{\sim} Sol(\mathcal{M}_{W,-1})/\mathbf{C}_{S_W} \xrightarrow{\sim} Sol(\mathcal{M}_{W,0})/\mathbf{C}_{S_W} \\ \quad \xrightarrow{\sim} Sol(\mathcal{M}_{W,1})/\mathbf{C}_{S_W} \xrightarrow{\sim} Sol(\mathcal{M}_{W,2})/\mathbf{C}_{S_W} \xrightarrow{\sim} \cdots \end{array}$$

which is equivariant with the inner product (4.5.7) (up to sign). One has the isomorphism: $Sol(\mathcal{M}_{W,0})/\mathbf{C}_{S_W} \simeq Sol(\widetilde{\mathcal{M}}_{W,0}) \simeq V_{\mathbf{C}}^*$ where (4.5.7) is identified with the Killing form I on V. For any covector $X \in V_{\mathbf{C}}$, let $X^{(n)} := D^n X \in Sol(\widetilde{\mathcal{M}}_{W,n})/\mathbf{C}_{S_W}$ for $n \in \mathbf{Z}$ so that $I(X^{(n)}, Y^{(n)}) = I_n(X^{(n)}, Y^{(-n)}) = (-1)^n I(X, Y)$ for $X, Y \in V_{\mathbf{C}}$. In fact, these spaces are identified with the middle homology group

Case ii) $s = -1/2, -3/2, -5/2, \cdots$. In this case (4.5.7) is a skew-symmetric bilinear form which may degenerate as follows: applying the operator D, one has a sequence of homomorphism of the local systems:

$$\begin{array}{l} \cdots \xrightarrow{\sim} Sol(\mathcal{M}_{W,-\frac{5}{2}})/\mathbf{C}_{S_W} \xrightarrow{\sim} Sol(\mathcal{M}_{W,-\frac{3}{2}})/\mathbf{C}_{S_W} \xrightarrow{\sim} Sol(\mathcal{M}_{W,-\frac{1}{2}})/\mathbf{C}_{S_W} \\ \quad \to Sol(\mathcal{M}_{W,\frac{1}{2}})/\mathbf{C}_{S_W} \xrightarrow{\sim} Sol(\mathcal{M}_{W,\frac{3}{2}})/\mathbf{C}_{S_W} \xrightarrow{\sim} \cdots \end{array}$$

which is equivariant with the inner product (4.5.7) (up to sign). The homomorphisms are isomorphic except at the middle step:

$$Sol(\mathcal{M}_{W,-\frac{1}{2}})/\mathbf{C}_{S_W} \to Sol(\mathcal{M}_{W,\frac{1}{2}})/\mathbf{C}_{S_W}.$$

The rank of the kernel of this step is equal to the multiplicity of exponent $\frac{h}{2}$ due to Assertion 2. So, it is non-degenerate iff $\frac{h}{2} \notin \{m_1, \cdots, m_l\}$. Recall 2.8 *Remark* i) a criterion for this condition in terms of the Coxeter graph. We shall investigate these cases more in details in §6.

Fix a base point, say $*$, of the universal covering space $(S_{W,\mathbf{C}} \backslash D_{W,\mathbf{C}})^\sim$ and branches at $*$ of the functions in $Sol(\mathcal{M}_{W,s})$. Then the analytic continuation of the evaluation homomorphism induces a map

$$(S_{W,\mathbf{C}} \setminus D_{W,\mathbf{C}})^\sim \;\to\; \mathrm{Hom}_{\mathbf{C}}(Sol(\mathcal{M}_{W,s})_*, \mathbf{C}) \tag{4.5.8}$$

which we shall call the period map of type W and weight s. By choosing a basis $u_1, \cdots, u_l$ and 1 of $Sol(\mathcal{M}_{W,s})$, the period map is given by the l-tuple $(u_1, \cdots, u_l)$ of functions and the jacobian is calculated by

$$\frac{\partial(u_1, \cdots, u_l)}{\partial(P_1, \cdots, P_l)} = c\Delta_W^{-s-\frac{1}{2}} \quad \text{for some } c \in \mathbf{C} \setminus \{0\}. \tag{4.5.9}$$

5. Period integrals for primitive forms

The connection $\nabla^{(s)}$ for $s = n/2$ $(n \in \mathbf{Z}_{\geq 0})$ is realized as the Gauss-Manin connection for a universal unfolding of a simple singularity of dimension n if W is a Weyl group of a Lie algebra. Then the solutions of the uniformization equation $\mathcal{M}_W^{(s)}$ are realized by *the period integrals of the primitive form* of the unfolding. Thus the period map is defined.

5.1. **Universal unfoldings of simple singularities.**

We recall briefly the description of universal unfoldings of simple singularities due to Brieskorn [Br1] (see [Sl] for non-simply-laced cases).

A simply-laced simple singularity of dimension n $(n \in \mathbf{Z}_{\geq 1})$ is the singular point of a hypersurface of dimension n defined by the equation (5.1.1) in coordinates $(x, y, z_3, \cdots, z_{n+1})$. The type, in LHS of Table, is given by the dual graph Γ of the exceptional divisors of the minimal resolution of the singularity for $n = 2$ (c.f. 2.4). We note that the equation is a weighted homogeneous polynomial of weight $(w_x, w_y, \frac{1}{2}, \cdots, \frac{1}{2} : 1)$ for suitable weights $w_x, w_y \in \mathbf{Q}_{>0}$ for x and y such that $w_x + w_y = \frac{1}{2} + \frac{1}{h}$.

$$
(5.1.1) \qquad
\begin{array}{lccccc}
A_l : & x^{l+1} & - \ y^2 & + & z_3^2 + \cdots + z_{n+1}^2 \\
D_l : & x^{l-1} & - \ xy^2 & + & z_3^2 + \cdots + z_{n+1}^2 \\
E_6 : & x^4 & + \ y^3 & + & z_3^2 + \cdots + z_{n+1}^2 \\
E_7 : & x^3 y & + \ y^3 & + & z_3^2 + \cdots + z_{n+1}^2 \\
E_8 : & x^5 & + \ z^3 & + & z_3^2 + \cdots + z_{n+1}^2
\end{array}
$$

Let G be a group acting linearly on the two variables x, y such that the G-action leaves the equation for a simple singularity (5.1.1) invariant and the induced G-action on the dual graph Γ is faithful. The simple singularity with the G-action is called a simple singularity of non simply-laced type Γ/G, where Γ/G is given in the next table ([Ar2][Sl]).

$$
(5.1.2) \qquad
\begin{array}{llll}
\Gamma/G & \Gamma & G & \text{The action of a generator of } G \\
B_l & A_{2l-1} & \mathbf{Z}/2\mathbf{Z} & : \ (x, y) \mapsto (-x, y) \\
C_l & D_{l+1} & \mathbf{Z}/2Z & : \ (x, y) \mapsto (x, -y) \\
F_4 & E_6 & \mathbf{Z}/2Z & : \ (x, y) \mapsto (-x, y) \\
G_2 & D_4 & \mathbf{Z}/3Z & : \ (x, y) \mapsto \frac{1}{2}(x + y, -3x + y)
\end{array}
$$

Let $\mathfrak{g}$, $\mathfrak{h}$ and W be a simple Lie algebra over $\mathbf{C}$, a Cartan subalgebra of $\mathfrak{g}$ and its Weyl group, respectively. The ring $S(\mathfrak{g}^*)^{\mathrm{ad}(\mathfrak{g})}$ of invariant polynomials on $\mathfrak{g}$ by the adjoint group action is isomorphic to $S(\mathfrak{h}^*)^W$ (Chevalley), and one obtains the flat adjoint quotient morphism: $\mathfrak{g} \to \mathfrak{h}//W \simeq S_W$, whose fiber over 0 is the nilpotent variety $N(\mathfrak{g})$ of $\mathfrak{g}$. For a subregular element $x \in N(\mathfrak{g})$, consider an affine subspace $\mathfrak{X}$ of $\mathfrak{g}$ which is transversal at x to the adjoint group orbit of x in $\mathfrak{g}$ (e.g. put $\mathfrak{X} := x + \mathfrak{z}_{\mathfrak{g}}(y)$ where $y \in N$ such that $\langle x, y, [x, y] \rangle$ form a sl_2 triplet).

Theorem. *(Brieskorn* [Br1][Sl]*) Let $\varphi_2 : \mathfrak{X} \to S_W$ be the restriction of the adjoint quotient map to the slice $\mathfrak{X}$ as defined above. Then the fiber $X_0 := \varphi_2^{-1}(0)$ is the corresponding simple singularity of dimension $n = 2$ and φ_2 is its semi-universal unfolding. If $\mathfrak{g}$ is of a non-simply-laced type, then there is a G-action on $\mathfrak{X}$ such that φ_2 is G-invariant and the G-action on X_0 defines the singularity of non-simply-laced type.*

We collect some facts on the universal unfolding $\varphi_2 : \mathfrak{X} \to S_W$, which we shall use in 5.2 as a building (or supporting) data for a primitive form. Their proofs are either referred to or easy and omitted.

i) *The φ_2 is a weighted homogeneous polynomial map of the weights $(w_x, w_y, 1/2, d_1/h, \cdots, d_{l-1}/h : d_1/h, \cdots, d_l/h)$ (see 2.5 for d_i,* [Br1]*).*

ii) *The composition $\pi_W \circ \varphi_2$ with π_W* (3.2.7) *is a submersion. Hence the fiber product $Z := \mathfrak{X} \times_{\mathrm{Spec}(S(\mathfrak{h}^*)^{W,\tau})} S_W \simeq \mathbf{C}^3 \times S_W$ is a smooth affine variety with projections $p : Z \to S_W$ and $\hat{\pi}_W : Z \to \mathfrak{X}$. We realize $\mathfrak{X}$ as a hypersurface in Z: define $F_2(x, y, z_3, P_1, \cdots, P_l) := P_l \circ p - P_l \circ \varphi_2 \circ \hat{\pi}_W$. Then, a) $\hat{\pi}_W$ identifies the hypersurface $\{F_2 = 0\}$ in Z with $\mathfrak{X}$.*

b) φ_2 is given by the restriction of the projection $p : Z \to S_W$ to $\mathfrak{X}$.

iii) *Let $\mathcal{O}_C := \mathcal{O}_Z/(F_2, \partial_x F_2, \partial_y F_2, \partial_{z_3} F_2)$ be the ring of polynomial functions on the critical set C of φ_2. Then, $\delta \mapsto \delta F_2 |_C$ induces a $S(V)^{W,\tau}$-isomorphism: $\mathcal{G} \simeq \mathcal{O}_C$ ($\simeq \mathcal{O}_C^G$ for non-simply-laced case).*

iv) *Put $r := w_x + w_y + (n-1)\frac{1}{2} = \frac{1}{h} + \frac{n}{2}$ and $s := \max\{d_i/h - d_j/h \mid i, j = 1, \cdots, l\} = \deg(P_l) - \deg(P_1)$. Then one has* [S1]*: $2r + s = n + 1$ (duality) and $s < 1$ (a characterization of a simple singularity).*

5.2. **The primitive form ζ_{F_n}.**

We introduce the primitive form attached to the universal unfolding of a simple singularity. By a use of the covariant differentiation of the primitive form, the module of relative abelian differentials of the family is identified with the module of vector fields on S_W. This identifies the flat structure studied in §3 with that defined by the primitive form[S5-7].

Let $n \in \mathbf{Z}_{\geq 1}$. Consider a weighted homogeneous polynomial

$$F_n(x, y, z_3, \cdots, z_{n+1}, P_1, \cdots, P_l) \tag{5.2.1}$$

of total degree 1 defined on the space $Z_n := \mathbf{C}^{n+1} \times S_W$ with the weight $(w_x, w_y, 1/2, \cdots, 1/2, d_1/h, \cdots, d_l/h)$. We call F_n a universal unfolding of a simple singularity of dimension n, if it satisfies the i), ii) and iii):

i) The restriction $F_n |_{P_1 = \cdots = P_l = 0}$ is a polynomial given in (5.1.1).

ii) Put $\mathcal{O}_{C_n} := \mathcal{O}_{Z_n}/(F_n, \partial_x F_n, \partial_y F_n, \partial_{z_1} F_n, \cdots, \partial_{z_{n+1}} F_n)$. Then, one has a $S(V)^{W,\tau}$-isomorphism: $\mathcal{G} \simeq \mathcal{O}_{C_n}$ by the correspondence $\delta \mapsto \delta F_n |_{C_n}$.

iii) The constant factor of F_n is normalized to: $\frac{\partial}{\partial P_l} F_n = 1$.

Under the data i), ii) and iii), the universal unfolding morphism

$$\varphi_n : \mathfrak{X}_n \to S_W \tag{5.2.2}$$

is defined by the restriction $\varphi_n := p_n|\mathfrak{X}_n$ of the projection $p_n : Z_n \to S_W$ to the hypersurface $\mathfrak{X}_n := \{F_n = 0\} \subset Z_n$. The φ_n defines a flat family of n-dim. affine variety parameterized by S_W, where the 0-fiber $X_0 := \varphi_n^{-1}(0)$ is the simple singularity of dimension n. It is well-known (Milnor) that over the complement of the discriminant $S_W \setminus D_W$, φ_n defines a locally trivial fibration whose fiber, called a Milnor fiber, is homotopic to a bouquet of l-copies of n-dimensional spheres. In case of non-simply-laced type, the G-action (5.1.2) extends to Z_n leaving F_n invariant such that the map in ii) induces $\mathcal{G} \simeq \mathcal{O}_{C_n}^G$. So, φ_n is G-invariant.

Introduce the *module of Abelian differentials of* deg n *relative to* φ_n:

(5.2.3) $$\mathcal{H}_{F_n}^{(0)} := \Omega_{Z_n}^{n+l+1}/dF_n \wedge dP_1 \wedge \cdots \wedge dP_l \wedge d\Omega_{Z_n}^{n-1}$$

where $\Omega_{Z_n}^p$ is the module of polynomial coefficient p-forms on Z_n. In fact, an element $\omega \in \mathcal{H}_{F_n}^{(0)}$ defines a n-form (modulo closed forms)

(5.2.4) $$\mathrm{Res}_t[\omega] := \mathrm{Res}\big[\tfrac{\omega}{F_n, P_1 - t_1, \cdots, P_l - t_l}\big]$$

on each fiber $X_t := \varphi_n^{-1}(t)$ for $t = (t_1, \cdots, t_l) \in S_{W,\mathbf{C}}$, and hence the de-Rham cohomology class $\mathrm{Res}_t[\omega] \in H^n(X_t, \Omega_{X_t}^*)$ (see [Ha,§4] for a definition of the residue symbol Res). So, we denote ω also by $\mathrm{Res}[\omega]$.

Remark. A justification to call $\mathcal{H}_{F_n}^{(0)}$ the module of abelian differentials is the following fact (which is an easy result of a study on the family φ_1, but, since we shall not use it, we do not give a proof of it) for the case $n = 1$ when the Milnor fiber X_t is a punctured curve.

Fact. *Let $\zeta_1, \cdots, \zeta_l$ be homogeneous $S(V)^W$-free basis of $\mathcal{H}_{F_1}^{(0)}$ such that* $\deg(\zeta_i) = -\frac{1}{2} + \frac{m_i}{h}$ *$(i = 1, \cdots, l)$. Then* $\mathrm{Res}_t[\zeta_i]$ *for* $0 < m_i < \frac{h}{2}$ *form basis of abelian differential of the first kind on $\bar{X}_t$,* $\mathrm{Res}_t[\zeta_i]$ *for* $\frac{h}{2} < m_i < h$ *form basis of abelian differential of the second kind on $\bar{X}_t$, and* $\mathrm{Res}_t[\zeta_i]$ *for* $m_i = \frac{h}{2}$ *form basis of abelian differential of the third kind having poles on the punctures on $\bar{X}_t$.*

We recall some structures a)-f) equipped on the module $\mathcal{H}_{F_n}^{(0)}$ as a lattice in $\mathbf{R}^n q_*(\Omega^{\cdot}_{\mathfrak{X}_n/T}[D, D^{-1}])$ (see [S7,§2(2.6.2)] for details and proofs).

a) *The $S(V)^W$-module structure on* $\mathcal{H}_{F_n}^{(0)}$ free of rank l.

b) *A decreasing filtration*: $\mathcal{H}_{F_n}^{(0)} \supset \mathcal{H}_{F_n}^{(-1)} \supset \mathcal{H}_{F_n}^{(-2)} \supset \cdots$ by free $S(V)^W$-modules of rank l (which we call the Hodge filtration).

c) *the Gauss-Manin connection*: $\nabla : Der_{S_W} \times \mathcal{H}_{F_n}^{(-k-1)} \to \mathcal{H}_{F_n}^{(-k)}$ such that $\nabla_D : \mathcal{H}_{F_n}^{(-k-1)} \to \mathcal{H}_{F_n}^{(-k)}$ is an $S(V)^{W,\tau}$-isomorphism.

d) *The $S(V)^{W,\tau}[D^{-1}]$-module structure* on $\mathcal{H}_{F_n}^{(0)}$ by putting $D^{-1} := \nabla_D^{-1}$, so that one has $\mathcal{H}_{F_n}^{(-k)} = D^{-k}\mathcal{H}_{F_n}^{(0)}$ $(k \in \mathbf{Z}_{\geq 0})$.

e) *The identification of graded pieces of the filtration with the module* *of Kähler differentials of degree $n+1$ on $\mathfrak{X}_n$ relative to φ_n:*

$$\mathcal{H}^{(-k)}_{F_n}/\mathcal{H}^{(-k-1)}_{F_n} \overset{D^k}{\simeq} \mathcal{H}^{(0)}_{F_n}/\mathcal{H}^{(-1)}_{F_n} \simeq \Omega^{n+1}_{\mathfrak{X}_n/S_W} := \Omega^{n+1}_{\mathfrak{X}_n}/\Omega^1_{S_W}\wedge\Omega^n_{\mathfrak{X}_n}.$$

f) *The higher residue pairing*:

$$K_{F_n} = \sum_{i=0}^{\infty} K^{(i)}_{F_n} D^{-i} : \mathcal{H}^{(0)}_{F_n}\times\mathcal{H}^{(0)}_{F_n} \to S(\mathfrak{h}^*)^{W,\tau}[D^{-1}] \ ,$$

which is a $S(V)^{W,\tau}[D^{-1}]$-sesqui-linear form compatible (in a suitable sense) with a)-e),whose leading term is given by the residue pairing:

$$K^{(0)}_{F_n}(\omega_1,\omega_2) = \mathrm{Res}\big[\tfrac{\omega_1\cdot(\omega_2/dP_1\wedge\cdots\wedge dP_l\wedge dx\wedge dy\wedge dz_3\wedge\cdots\wedge dz_{n+1})}{\partial_x F_n,\ \partial_y F_n,\ \partial_{z_3}F_n,\cdots,\ \partial_{z_{n+1}}F_n}\big]$$

inducing a perfect selfdual-pairing on the piece $\Omega^{n+1}_{\mathfrak{X}_n/S_W} \simeq \mathcal{H}^{(0)}_{F_n}/\mathcal{H}^{(-1)}_{F_n}$.

We now consider the most basic element among all Abelian differentials:

(5.2.5) $\quad \zeta_{F_n} := \mathrm{Res}[dP_1\wedge\cdots\wedge dP_l\wedge dx\wedge dy\wedge dz_3\wedge\cdots\wedge dz_{n+1}],$

which is the element in $\mathcal{H}^{(0)}_{F_n}$ of lowest degree. Using notation 5.1 iv),

$*)\quad \deg(\zeta_{F_n}) = \sum_{i=1}^{l}\frac{d_i}{h}+w_x+w_y+\frac{n-1}{2}-1-\sum_{i=1}^{l}\frac{d_i}{h} = r-1 = \frac{1}{h}+\frac{n-2}{2}.$

Denote by $\zeta^{(-k)}_{F_n} := \nabla_D^{-k}\zeta_{F_n}\in\mathcal{H}^{(-k)}_{F_n}$ the element shifted k-times by D^{-1}. Under these setting and notation, we have theorems ([S5-7]).

Theorem. *The element ζ_{F_n} is a primitive form for the family F_n.*

Proof. This means that ζ_{F_n} satisfies the following properties 0)-iv).

0) One has the bijection: $\mathcal{G}\simeq\Omega^{n+1}_{\mathfrak{X}_n/S_W},\ \delta\mapsto\nabla_\delta\zeta^{(-1)} \bmod \mathcal{H}^{(-1)}_{F_n}$.

i) ζ_{F_n} is homogeneous: $\nabla_E\,\zeta^{(0)}_{F_n} = (r-1)\,\zeta^{(0)}_{F_n}$.

ii) Orthogonality: $K^{(k)}_{F_n}(\nabla_\delta\zeta^{(-1)}_{F_n},\nabla_{\bar\delta}\zeta^{(-1)}_{F_n}) = 0$ for $\delta,\bar\delta\in\mathcal{G}$ and $k\geq 1$.

iii) $K^{(k)}_{F_n}(\nabla_\delta\nabla_{\delta'}\zeta^{(-2)}_{F_n},\nabla_{\delta''}\zeta^{(-1)}_{F_n}) = 0$ for $\delta,\delta',\delta''\in\mathcal{G}$ and $k\geq 2$.

iv) $K^{(k)}_{F_n}(P_l\nabla_\delta\zeta^{(-1)}_{F_n},\nabla_{\delta'}\zeta^{(-1)}_{F_n}) = 0$ for $\delta,\delta'\in\mathcal{G}$ and $k\geq 2$.

Let us briefly veryfy the conditions. The 0) follows directly from the next formula of covariant differentiation of ζ_{F_n}, and i) follows from $*$).

$**)\quad \nabla_\delta\zeta^{(-1)} = \mathrm{Res}[\ \delta F_n\ dP_1\wedge\cdots\wedge dP_l\wedge dx\wedge dy\wedge dz_3\wedge\cdots dz_{n+1}].$

The remaining ii), iii) and iv) are verified by degree check as follows.

One has $\deg(K^{(k)}_{F_n}) = -n-1-k$ and $\deg(\zeta^{(0)}_{F_n}) = r$. One may assume $\delta,\delta',\delta''\in\mathcal{G}$ is $\frac{\partial}{\partial P_i}$ $(i=1,\cdots,l)$. Then $\deg(\nabla_\delta\nabla_D^{-1}) = \deg P_l - \deg P_i \leq s$. Recalling the duality $2r+s=n+1$ and the characterization $s<1$ (see 5.1 iv)), and taking the range of k in account, one calculate

$$\begin{aligned}
&\deg(K^{(k)}_{F_n}(\nabla_\delta\zeta^{(-1)}_{F_n},\nabla_{\delta'}\zeta^{(-1)}_{F_n}))\\
&\qquad\leq\ (-n-1-k)+(s+r)+(s+r) = s-k<0,\\
&\deg(K^{(k)}_{F_n}(\nabla_\delta\nabla_{\delta'}\zeta^{(-2)}_{F_n},\nabla_{\delta''}\zeta^{(-1)}_{F_n}))\\
&\qquad\leq\ (-n-1-k)+(2s+r)+(s+r) = 2s-k<0,\\
&\deg(K^{(k)}_{F_n}(P_l\nabla_\delta\zeta^{(-1)}_{F_n},\nabla_{\delta'}\zeta^{(-1)}_{F_n}))\\
&\qquad\leq\ (-n-1-k)+(1+s+r)+(s+r) = s+1-k<0.
\end{aligned}$$

The negativity of degrees implies that these elements are zero. □

Let us state some immediate consequences of Theorem.

The 0) and ii) of the condition for a primitive form implies that one has a $S(V)^{W,\tau}[D^{-1}]$-module isomorphism:

$$\mathcal{G}\otimes \mathbf{Z}[D^{-1}] \simeq \mathcal{H}^{(0)}_{F_n}. \tag{5.2.6}$$

defined by the correspondence $\sum_{i\geq 0}\delta_i\otimes D^{-i}\mapsto \sum_{i\geq 0}\nabla_{\delta_i}D^{-i}\zeta^{(-1)}_{F_n}$.

Note. The formula (5.2.6) implies that the choice of a primitive form defines a splitting of the Hodge filtration (in the sense in b)) into an orthogonal sum of graded pieces $\mathfrak{g}\otimes D^{-k}\simeq \mathcal{H}^{(-k)}_{F_n}/\mathcal{H}^{(-k-1)}_{F_n}$. In analogy with the classical Hodge theory, where a choice of a Kähler metric defines an orthogonal splitting of the Hodge filtration, we may call (5.2.6) a Hodge decomposition. Note that we have not yet discussed on the $S(V)^W$- and Der_{S_W}- module structures on the modules in (5.2.6).

The iii) and iv) implies that the multiplication of P_l and the covariant differentiation by an element $\delta\in\mathcal{G}$ in the RHS of (5.2.6) is given by two terms in the LHS. Let us write down explicitly the actions:

$$P_l(D^{-k}\otimes\delta) = D^{-k}\otimes(P_l *_n \delta) + D^{-k-1}\otimes(N_n(\delta)+k\delta), \tag{5.2.7}$$

$$\nabla_{\delta_1}(D^{-k-1}\otimes\delta_2) = D^{-k}\otimes(\delta_1 *_n \delta_2) + D^{-k-1}\otimes \nabla\!\!\!/_{n\ \delta_1}\delta_2. \tag{5.2.8}$$

where $*_n : \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ is a $S(V)^{W,\tau}$-bilinear map, $N_n:\mathcal{G}\to\mathcal{G}$ is a $S(V)^{W,\tau}$-endomorphism and $\nabla\!\!\!/_n:\mathcal{G}\times\mathcal{G}\to\mathcal{G}$ is a covariant differentiation over $S(V)^{W,\tau}$.

1. *The $*_n$-product is induced from the ring structure in $\mathcal{O}_{C_n}$:*

$$(\delta_1 *_n \delta_2)F_n \equiv \delta_1 F_n\cdot\delta_2 F_n \ \ in\ \ \mathcal{O}_{C_n} \tag{5.2.9}$$

using the identification $\mathcal{G}\simeq\mathcal{O}_{C_n}$.

2. *Using the identification: $\mathcal{G}\simeq\Omega^{n+1}_{\mathfrak{X}_n/S_W}$, the flat metric J is given by the first residue pairing $K^{(0)}$.* That is: for $\delta_1,\delta_2\in\mathcal{G}$, one has

$$J(\delta_1,\delta_2) = c\cdot\mathrm{Res}\Big[\frac{\delta_1F_n\ \delta_2F_n\ \zeta_{F_n}}{\partial_xF_n,\partial_yF_n,\partial_{z_3}F_n,\cdots,\partial_{z_{n+1}}F_n}\Big] \tag{5.2.10}$$

3. *The intersection form on the homology group $H_n(X_t,\mathbf{Z})$ of a Milnor fiber X_t is now calculated by the flat metric (i.e. the residue pairing) and by the period of primitive forms:*

$$\langle\gamma,\gamma'\rangle = \frac{(-1)^{\frac{n}{2}-k}}{(2\pi)^n}\sum_{i=1}^n\delta_i\int_\gamma\zeta^{(k-1)}_{F_n}\cdot w(\delta^{i*})\int_{\gamma'}\zeta^{(\frac{n}{2}-k-1)}_{F_n} \tag{5.2.11}$$

where $\delta_1,\cdots,\delta_l$ and $\delta^{1*},\cdots,\delta^{l*}$ are $S(V)^{W,\tau}$-dual basis of $\mathcal{G}$ w.r.t. the inner product J_{F_n} [S6 §3].

4. *Let W be the Weyl group of the Lie algebra $\mathfrak{g}$ corresponding to the simple singularity F_n. The identification of the primitive form ζ_{F_n} with the $\zeta^{(-n/2)}_W = D^{-n/2}\otimes D$ in $\mathcal{H}_W$ 4.3 induces an isomorphism*

$$\mathcal{H}^{(k)}_{F_n}\simeq\mathcal{H}^{(k-n/2)}_W \tag{5.2.12}$$

compatible with the structures a)-f) up to the shift by $\frac{n}{2}$. *In particular, one has* $*_n = *$, $N_n = N + \frac{n}{2}$ *and* $\nabla\!\!\!/_{F_n} = \nabla\!\!\!/$.

We first show the result for $n = 2$, where Brieskorn [Br1] has identified the intersection form on $H_2(X_t, \mathbf{Z})$ with the Killing form (up to a constant) of the corresponding Lie algebra $\mathfrak{g}$. For general n, the result is reduced to an unpublished result [S5,§5-6]: *Let* $F_n + z_{n+2}^2 + z_{n+3}^2 = F_{n+2}$. *The correspondence* $\omega \in \mathcal{H}_{F_n}^{(k)} \mapsto \nabla_D(\omega \wedge dz_{n+2} \wedge dz_{n+3}) \in \mathcal{H}_{F_{n+2}}^{(k+1)}$ *for* $k \leq -1$ *and* $n \geq 1$ *defines a bijection:* $\rho : \mathcal{H}_{F_n}^{(k)} \simeq \mathcal{H}_{F_{n+2}}^{(k+1)}$, *which is compatible with the structures i)-iv) and* $\rho(\zeta_{F_n}^{(k)}) = \zeta_{F_{n+2}}^{(k+1)}$.

Remark. Another direct proof of the theorem for $n = 1$ without using the result of [S5] may conjecturely be given by the following approach.

Let $\mathfrak{g}$ be a semi-simple algebra over $\mathbf{C}$. Let θ be an involution of $\mathfrak{g}$ inducing the eigenspace decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The restriction of the adjoint quotient morphism to the subspace $\mathfrak{p}$ induces a flat morphism whose fiber over 0 is the nilpotent variety $N(\mathfrak{p})$ (c.f. [K-R]). Then one wants to study again the restriction of the morphism on the subspace $\mathfrak{X}$ of $\mathfrak{p}$ which is transversal to singularity of $N(\mathfrak{p})$. This was studied in details by Sekiguchi [Se], whose result is not so simple as in Brieskorn's theorem. One reason is that $N(\mathfrak{p})$ may no longer be irreducible, and the other reason is that the singularity $N(\mathfrak{p})_{sing}$ may not be irreducible and each component may have different codimension in $N(\mathfrak{p})$. Nevertheless, we expect Brieskorn type theorem for the real split case as follows.

Conjecture: if θ *is the involution associated to a real split form of* $\mathfrak{g}$, *then the restriction of the adjoint quotient morphism to a transversal slice at a point of a subregular nilpotent orbit in* $N(\mathfrak{p})$ *gives a real form of the universal unfolding of a simple singularity of dimension 1.*

5.3. **Period integral and period map.**

As a consequence of 5.1 and 5.2, we can express the solutions of the equation $\widetilde{\mathcal{M}}_{W,-n/2}$ for $n \in \mathbf{Z}_{\geq 0}$ by the integrals of the primitive form for the family φ_n. Namely, let $\gamma(t)$ be a horizontal section $\mathbf{R}^n\varphi_{n*}(\mathbf{Z}_{\mathfrak{X}})$ where t runs in the universal covering $(S_{W,\mathbf{C}} \backslash D_W)^\sim$. Then the integral $\int_{\gamma(t)} \mathrm{Res}_t[\zeta_{F_n}^{(k)}]$ as a function on $(S_{W,\mathbf{C}} \backslash D_W)^\sim$ is a solution of $\widetilde{\mathcal{M}}_{W,k-n/2}$ for $\delta \int_{\gamma(t)} \mathrm{Res}_t[\zeta_{F_n}^{(k)}] = \int_{\gamma(t)} \nabla_\delta \mathrm{Res}_t[\zeta_{F_n}^{(k)}]$ $(\delta \in Der_{S_W})$. We regard $\int_{\gamma(t)} \mathrm{Res}_t[\zeta_{F_n}^{(k)}]$ defines further, at each point t, a linear functional on the flat tangent vector space Ω_+^*: $\delta \in \Omega_+^* \mapsto \delta \int_{\gamma(t)} \mathrm{Res}_t[\zeta_{F_n}^{(k)}]$. This defines the *period map* $t \in (S_{W,\mathbf{C}} \backslash D_W)^\sim \mapsto \int \mathrm{Res}_t[\zeta_{F_n}^{(k)}] \in \mathrm{Hom}(\mathbf{R}^n\varphi_{n*}(\mathbf{Z}_{\mathfrak{X}}), \Omega_{+\mathbf{C}})$. Applying D on Ω, we obtain *the projection to the first factor* $\mathrm{Hom}(\mathbf{R}^n\varphi_{n*}(\mathbf{Z}_{\mathfrak{X}}), \mathbf{C})$.

A description of the period map for $n = 2$ is achieved by [Br1][Lo] and [Y]. Namely, the simultaneous resolution $\tilde{\mathfrak{X}} \to \mathfrak{h}$ of the family $\mathfrak{X}_2 \to S_W$ is constructed in terms of Lie theory in [Br1], where the Kostant-Kirillov form on the (co-)adjoint orbit is identified with the primitive form [Y]. This implies that $\mathfrak{h}$ is the period domain and the inverse to the period map is nothing but the quotient morphism $\mathfrak{h} \to \mathfrak{h}/W \simeq S_W$. (For details, see the references. For an analytic treatment, see [Lo]).

We describe now the period map and its inverse for $n = 1$ of type A_1,A_2,A_3 or B_2 (where the projection to the first factor suffices). The types A_1 and A_2 are classical. The type A_3 is taken from [S2], and B_2 is a consequence of A_3 (which recover the classical well-known results).

Example. 1. $A_1 :\ F := xy - t$. The primitive form is given by the differential $\zeta = dx/x$. The base $\gamma \in H_1(X_t, \mathbf{Z})$ is invariant (under the monodromy action) and the period integral $\int_\gamma \zeta = 2\pi\sqrt{-1}$ is a constant independent of t. The indefinite integral $u = \int_1^t \zeta$ and 1 form basis of $Sol(\mathcal{M}_{A_1,-\frac{1}{2}})$. The period domain of u is the plane $\mathbf{C}$ with the translation action by $2\pi\sqrt{-1}\mathbf{Z}$. The inverse map is given by the exponential function $t = \exp(u)$.

2. $A_2 :\ F := y^2 - (4x^3 - g_2x - g_3)$ with $\Delta_{A_2} = g_3^2 - \frac{1}{27}g_2^3$. The primitive form is given by the elliptic integral of the first kind $\zeta = dx/\sqrt{4x^2 - g_2x - g_3}$. The Milnor fiber X_g is a punctured (at ∞) elliptic curve. The integrals $u_i = \int_{\gamma_i} \zeta$ $(i = 1, 2)$ over (oriented) basis $\gamma_1, \gamma_2 \in H_1(X_g, \mathbf{Z})$ and 1 give basis of $Sol(\mathcal{M}_{A_2,-\frac{1}{2}})$. The period domain of (u_1, u_2) is $\tilde{\mathcal{H}} := \{(u_1, u_2) \in C^2 \mid \Im(u_1/u_2) > 0\}$ with the modular group $\Gamma(A_2) = \mathrm{SL}(2, \mathbf{Z})$ action. The inverse map $\tilde{\mathcal{H}} \to S_{a_2} \backslash D_{A_2}$ to the period map is given by $g_2 := 60E_4(u_1, u_2)$ and $g_3 := 140E_6(u_1, u_2)$ where $E_{2i} := \sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}}(nu_1 + mu_2)^{-2i}$ is the elliptic Eisenstein series. The discriminant Δ_{A_2}, given by $\eta(\tau)^{24}u_2^{-12}$, generates the ideal of cusp forms, where $\eta(\tau) = \exp(2\pi\sqrt{-1}\tau/24)\Pi_{n=1}^{\infty}(1 - q^n)$ is the Dedekind η-function, $\tau = u_1/u_2$ and $q = \exp(2\pi\sqrt{-1}\tau))$.

3. $A_3 : F := s^2 - (t^4 + x_2t^2 + x_3t + x_4)$. The primitive form is given by the hyperelliptic integral of the lowest degree $\zeta = dt/\sqrt{t^4 + x_2t^2 + x_3t + x_4}$. The Milnor fiber $X_{\underline{x}}$ is a two distinct punctured elliptic curve. Let γ_1, γ_2 and γ be the basis of $H_1(X_x, \mathbf{Z})$ such that γ_1, γ_2 form (oriented) basis of the first homology group of the compactified elliptic curve and γ is the (monodromy) invariant cycle presented by a closed path in X_t turning once around one of the punctures. Then the integrals $\int_{\gamma_i} \zeta$ $(i = 1, 2)$ are the periods on $\bar{X}_{\underline{x}}$ but $\int_\gamma \zeta$ is realized by an integral over an interval connecting the two punctures on $X_{\underline{x}}$. In order to get a description of the third integral and the inverse map, we proceed the following reduction to A_2 [S2,Theorem]: regarding the polynomial $t^4 + x_2t^2 + x_3t + x_4$ as a binary quartic form, the invariants of weight 4 and 6 are given by $L(\underline{x}) := 16x_4 + \frac{4}{3}x_2^2$ and $-\frac{1}{9}M(\underline{x}) := -\frac{8}{27}x_2^3 - 4x_3^2 + \frac{32}{3}x_2x_4$. The two punctures on the elliptic curve $X_{\underline{x}}$ correspond to the two points $p(\underline{x}) := (z = -\frac{2}{3}x_2, w = 2x_3)$ and ∞ on the elliptic curve $w^2 = 4z^3 - L(\underline{x})z + \frac{1}{9}M(\underline{x})$. The correspondence $\underline{x} \in S_{A_3} \mapsto (p(\underline{x}), g_2 =$

$L(\underline{x}), g_3 = -\frac{1}{9}M(\underline{x}))$ gives a biregular morphism from S_{A_3} to the total space of Weierstrass family of (affine) elliptic curves: $w^2 = 4z^3 - g_2 z - g_3$ such that $\Delta_{A_3} = -\frac{27}{16}\Delta_{A_2}$. Consider the associated elliptic integrals:

$$u_i = \int_{\gamma_i} dz/\sqrt{4z^3 - Lz + \tfrac{1}{9}M}\ (i=1,2)\ \&\ v = \int_{\infty}^{p(\underline{x})} dz/\sqrt{4z^3 - Lz + \tfrac{1}{9}M}.$$

Theorem. ([S2, Theorem])

1. The v, u_1, u_2 and 1 form basis of $Sol(\mathcal{M}_{A_3,-\frac{1}{2}})$.

2. The period domain of (v, u_1, u_2) and the modular group are given by

$$\begin{aligned} \mathcal{B}(A_3) &= \{(v,u_1,u_2)\in \mathbf{C}^3 \mid \Im(u_1/u_2) > 0,\ v \neq mu_1 + nu_2 \forall m,n\in\mathbf{Z}\},\\ \Gamma(A_3) &= \mathrm{SL}(2,\mathbf{Z}) \mid\times\ \mathbf{Z}^2. \end{aligned}$$

3. The inverse map $\mathcal{B}(A_3) \to S_{A_3} \setminus D_{A_3}$ is given by the following system:

$$\begin{aligned} -\tfrac{2}{3}x_2 &= \mathfrak{p}(v,u_1,u_2) = v^{-2} + {\textstyle\sum}'\{(v - mu_1 - nm_2)^{-2} - (nu_1 + mu_2)^{-2}\},\\ -x_3 &= -\tfrac{1}{2}\mathfrak{p}'(v,u_1,u_2) = v^{-3} + {\textstyle\sum}'(v - mu_1 - nm_2)^{-3},\\ L &= 16x_4 + \tfrac{4}{3}x_2^2 = 60E_4(u_1,u_2). \end{aligned}$$

4. $B_2 : F := s^2 - (t^4 + x_2 t^2 + x_4)$. This is a subfamily of A_3 fixed by the $\mathbf{Z}/2$-action $(s,t)\mapsto(s,-t)$ with $\Delta_{B_2} = x_4(x_4 - x_2^2/4)$. The cycles vanishing along $x_4 = 0$ defines the short root and the cycles vanishing along $x_4 - x_2^2/4 = 0$ defines the long root of type B_2, respectively. The primitive form is given by the classical Legendre-Jacobi form of elliptic integral $dt/\sqrt{t^4 + x_2 t^2 + x_4}$. Let us use the same reduction and notation as in the previous example A_3. Because of the $\mathbf{Z}_2$-symmetry, the difference, in the elliptic curve, of the two punctures on $X_{\underline{x}}$ is a two-torsion element, and, hence, the point $p(\underline{x})$ in the elliptic curve $w^2 = 4z^3 - g_2 z - g_3$ is a branching point of the double cover to the z-plane (i.e. $-\frac{2}{3}x_2$ is a solution of the cubic equation $4z^3 - g_2 z - g_3 = 0$). Then we can choose a base γ_1 of the homology group such that $2v = u_1$. As immediate consequences of this description, we obtain:

Theorem. *1. The u_1, u_2 and 1 form basis of $Sol(\mathcal{M}_{B_2,-\frac{1}{2}})$.*

2. The period domain and the modular group acting on it are given by

$$\begin{aligned} \mathcal{B}(B_2) &= \{(v,u_1,u_2)\in\mathcal{B}(A_3) \mid 2v = u_1\} \simeq \{(u_1,u_2)\in\mathbf{C}^2 \mid \Im(u_1/u_2)\geq 0\},\\ \Gamma(B_2) &= \Gamma_0(2) := \{\begin{bmatrix} a & b\\ c & d\end{bmatrix} \in \mathrm{SL}(2,\mathbf{Z}) \mid b\equiv 0 \bmod 2\}. \end{aligned}$$

3. The inverse map $\mathcal{B}(B_2) \to S_{B_2}\setminus D_{B_2}$ is given by the following system:

$$-\tfrac{2}{3}x_2 = \mathfrak{p}(\tfrac{1}{2}u_1, u_1, u_2) \quad \textit{and} \quad 16x_4 + \tfrac{4}{3}x_2^2 = 60E_4(u_1,u_2).$$

4. There are two orbits of cusps on the boundary of $\mathcal{B}(B_2)$. Each factor $x_4 = \frac{3}{16}(20E_4(u_1,u_2) - \mathfrak{p}^2(\frac{1}{2}u_1,u_1,u_2))$ and $4x_4 - x_2^2 = 15E_4(u_1,u_2) - 3\mathfrak{p}^2(\frac{1}{2}u_1,u_1,u_2)$ of the discriminant vanishes on each orbit of cusps, respectively, but vanishes nowhere on the period domain. Therefore, the factor is (up to a constant factor) given by $\eta(\tau)^8 u_2^{-4}$ and $\eta(-\frac{1}{2\tau})^8 u_1^{-4} = 2^4\eta(2\tau)^8 u_2^{-4}$, respectively, and the discriminant $\Delta_{B_2} = x_4(x_4 - x_2^2/4)$ generates the ideal of cusp forms.

5. The flat coordinates are given by $x_2 = -\frac{2}{3}\mathfrak{p}$ and $x_4 - \frac{1}{8}x_2^2 = \frac{15}{32}(4E_4 - \mathfrak{p}^2)$.

6. Inverse maps from the Period domain

We try to understand the inverse map to the period map for n=1. There are some partial results for the types A_1, A_2, A_3 and B_2 (recall 5.3 Example). However, we are still unsure how to give a general conjectural description of the ring of inverse functions and of the flat structures in 3.9 in terms of the inverse functions on the period domain.

We first study in 6.1 an abstract group $\Gamma(M)$ attached to a Coxeter matrix M, which shall play the role of the modular group in 6.3 by letting it act on the symplectic vector space introduced in 6.2. The period domain for the period map and its inverse map are conjecturely described in 6.4 and 6.5. The goal of this section is 6.6 Conjecture 6, which states that certain power root (prescribed in 6.1) of the discriminant is automorphic with the character ϑ_W on $\Gamma(W)$ given in 6.1.

6.1. **Group $\Gamma(M)$.**

Let $M = (m(\alpha,\beta))_{\alpha,\beta\in\Pi}$ be a Coxeter matrix (2.3). We introduce a group $\Gamma(M)$ attached to M by the relations (6.1.1), (6.1.2) and (6.1.3).

Generators: γ_α for $\alpha \in \Pi$

Relations: 1. For the pair $\alpha,\beta \in \Pi$ such that $m(\alpha,\beta) = 2$

(6.1.1) $$\gamma_\alpha\gamma_\beta = \gamma_\beta\gamma_\alpha.$$

2. For the pair $\alpha,\beta \in \Pi$ such that $m(\alpha,\beta) \geq 3$

(6.1.2) $$\underbrace{\gamma_\alpha\gamma_\beta\gamma_\alpha\cdots}_{m(\alpha,\beta)} = \underbrace{\gamma_\beta\gamma_\alpha\gamma_\beta\cdots}_{m(\alpha,\beta)}$$

3. Let Γ' be any irreducible finite type subdiagram of the Coxeter graph Γ of M, and let h' be the Coxeter number of Γ'. Suppose there does not exist an exponent of Γ' which is equal to $h'/2$, then

(6.1.3) $$(\Pi_{\alpha\in\Gamma'}\ \gamma_\alpha)^{\langle h'\rangle} = 1$$

for any order of the product (see Remark 1. below), where

(6.1.4) $$\langle h'\rangle := \begin{cases} 2h' & \text{if } h' \text{ is odd,} \\ h' & \text{if } h' \text{ is even and } h'/2 \text{ is even,} \\ \frac{h'}{2} & \text{if } h' \text{ is even and } h'/2 \text{ is odd.} \end{cases}$$

Remark. 1. Different orders in the product in (6.1.3) define conjugate elements due to (6.1.1) [B,Ch.v,§6,1.Lemme 1.].

2. In the above Relations 3., the condition that there does not exist an exponent of Γ' equal to the half of its Coxeter number h' is equivalent to that the symplectic form I'_{odd} attached to Γ' (see 6.2 Assertion 1.) is nondegerate (c.f. 2.8 Remark and 6.3 Assertion 4.).

By definition, we have some obvious homomorphisms and characters.

i) The correspondence $\gamma_\alpha \mapsto -1$ ($\alpha \in \Pi$) induces a character

$$\theta_M : \Gamma(M) \to \{\pm 1\}. \tag{6.1.5}$$

Proof. We need to verify that $\langle h' \rangle \#(\Gamma')$ is an even number for any irreducible finite type Coxeter graph Γ' without an exponent equal to the half-Coxeter number. In view of (6.1.4), we have to check only the third case in the formula. According to the classification, one check that this occurs only for the types E_8, H_4 and $I_2(p)$ for $p = 2 \cdot odd$. In all cases, $\#(\Gamma')$ is an even number (c.f. also 6.2 Assertion 1.). $\square$

ii) Put $k(M) := \gcd\{\langle h' \rangle \#(\Gamma')/2 \mid \Gamma'$: irreducible finite type subdiagrams of M s.t. $h'/2$ is not an exponent of Γ' (h':the Coxeter # of Γ')$\}$. The correspondence $\gamma_\alpha \mapsto \exp(\frac{\pi\sqrt{-1}}{k(M)})$ ($\alpha \in \Pi$) induces a character

$$\vartheta_M : \Gamma(M) \to \mathbf{C}^\times \tag{6.1.6}$$

such that $\vartheta_M^{k(M)} = \theta_M$ and $\vartheta_M^{2k(M)} = 1$. If the graph Γ of M does not contain $I_2(p)$ for $p = 2 \cdot odd$, then $k(M)$ is an even number. Except for $k(A_1) = \infty, k(A_2) = k(A_3) = 6, k(B_2) = k(C_2) = 4, k(D_4) = 6, k(G_2) = 3$ and $k(I_2(p)) = \langle p \rangle$, one has $k(W) = 2$ for all finite Coxeter group W.

iii) The Artin group $A(M)$ is defined by the relations (6.1.1) and (6.1.2) on the generator system g_α ($\alpha \in \Pi$) [B-S]. Then, the correspondence $g_\alpha \mapsto \gamma_\alpha$ for $\alpha \in \Pi$ induces a homomorphism from $A(M)$ onto $\Gamma(M)$:

$$\gamma : A(M) \to \Gamma(M). \tag{6.1.7}$$

iv) Suppose the Coxeter graph of M contains neither E_8, H_4 nor $I_2(p)$ for $p = 2 \cdot odd$ (crystallographic groups except E_8 and G_2 satisfy the assumption). Then the correspondence $\gamma_\alpha \mapsto \sigma_\alpha$ ($\alpha \in \Pi$) induces a homomorphism σ from $\Gamma(M)$ onto the Coxeter group with the diagram:

$$\begin{array}{ccc} \Gamma(M) & \xrightarrow{\sigma} & W(M) \\ \downarrow \theta_M & & \downarrow \det \\ \{\pm 1\} & = & \{\pm 1\}. \end{array} \tag{6.1.8}$$

(*Proof.* Except for the third case of (6.1.4), all defining relations of $\Gamma(M)$ are satisfied in $W(M)$ (recall 2.8).)

Let us call the kernel of σ *the principal congruence subgroup of* $\Gamma(M)$.

Remark. Our original intension, explained at the introduction, was to use the flat structure to understand the Artin group $A(M)$. By the use of the flat structure, we arrived at a group $\Gamma(M)$ which lies between $A(M)$ and $W(M)$. Except for the first few types A_2, B_2, G_2 or $I_2(p)$, the group $\Gamma(M)$ is close to the group $W(M)$ and we are only at the first stage (after $W(M)$) to understand the Artin group. It is interesting to construct Eilenberg-MacLane space for the group $\Gamma(M)$ (c.f. [D-S]).

6.2. **Symplectic space** $(\tilde{F}, \tilde{I}_{odd})$**.**

We construct a symplectic vector space on which $\Gamma(M)$ acts symplectic linearly. First, we recall the orthogonal representation of the Coxeter group $W(M)$ ([B, Ch.V,§4, theor.1]). Consider vector spaces

$$F := \oplus_{\alpha\in\Pi}\mathbf{R}e_\alpha$$

equipped with the symmetric bilinear form I defined by

$$I(e_\alpha, e_\beta) \quad = -2\cos(\pi/m(\alpha,\beta))/d_\alpha d_\beta,$$

where $d_\alpha \in \mathbf{R}\setminus\{0\}$ $(\alpha\in\Pi)$ are arbitrary scaling constants (we regard that the pair (F, I) is determined by M independent of the scaling constants and the basis e_α moves in F according to the scaling). A cobasis is introduced by $e_\alpha^\vee := 2e_\alpha/I(e_\alpha, e_\alpha)$ for $\alpha \in \Pi$. Then the Coxeter group $W(M)$ acts freely on F by letting a generator $a_\alpha \in W(M)$ act on it by reflections $\sigma_\alpha : u \mapsto u - I(u, e_\alpha^\vee)e_\alpha$ so that $I(\sigma(u), \sigma(v)) = I(u, v)$.

¿From now on, we assume that the Coxeter graph (2.4) associated to M is a tree. Up to order, Π decomposes uniquely into a disjoint union:

$$\Pi = \Pi_1 \cup \Pi_2, \tag{6.2.1}$$

such that each Π_i is discrete, i.e. e_α for $\alpha \in \Pi_i$ are mutually orthogonal to each other. We introduce a skew symmetric form I_{odd} on V.

$$I_{odd}(e_\alpha, e_\beta) := \begin{cases} I(e_\alpha, e_\beta) & \text{if } \alpha \in \Pi_1 \text{ and } \beta \in \Pi_2, \\ -I(e_\alpha, e_\beta) & \text{if } \alpha \in \Pi_2 \text{ and } \beta \in \Pi_1, \\ 0 & \text{if } \alpha, \beta \in \Pi_1 \text{ or } \alpha, \beta \in \Pi_2. \end{cases} \tag{6.2.2}$$

Note that the I_{odd} does not depend on the scaling factors but only on (F, I) and (6.2.1). Interchanges of Π_i $(i = 1, 2)$ induces a sign change $I_{odd} \mapsto -I_{odd}$. In the sequel, we fix a decomposition (6.2.1) once for all.

Example. Let W be a finite reflection group acting on (V, I) as in §2. By the choice of a chamber C, one obtains a Coxeter system $(W, \Pi(C))$ with the basis Π_W (2.3 Theo.5.), whose associated graph is a tree (2.4). Thus the above (6.2.2) defines the form $I_{odd,C}$ depending on C, where one has $I_{odd,C}(x, y) = I_{odd,wC}(wx, wy)$ for $w \in W$. Accordingly, many of our later constructions (such as the symplectic space $\tilde{F}$, the realization ρ of $\Gamma(M)$ in the symplectic group, the lattice $\tilde{Q}$, the period domain $\tilde{B}$, etc) depend on the choice of a chamber. However, they are conjugate to each other in suitable sense by the W-action. So, once for all, we choose and fix one chamber and denote the associated form by I_{odd}. We shall not mention explicitly the dependence on the chamber.

In general, I_{odd} may not be nonsingular. A more precise formula is:

Assertion 1. *Let the notation be as above. One has*

$$\operatorname{rank}(I_{odd}) = 2\min\{\#\Pi_1, \#\Pi_2\}, \tag{6.2.3}$$

$$\operatorname{corank}(I_{odd}) = |\#\Pi_1 - \#\Pi_2| = \operatorname{rank}(\ker(c+1)). \tag{6.2.4}$$

Here $c := \Pi_{\alpha\in\Pi}\sigma_\alpha$ *is a Coxeter element in a generalized sense. In case* $W(M)$ *is a finite reflection group, the number* (6.2.4) *is equal to*

$$\#\{1 \leq i \leq l \mid m_i = h/2\}. \tag{6.2.5}$$

Proof. Put $\ker(I_{odd}) := \{\xi \in F \mid I_{odd}(\xi, x) = 0\ \forall x \in F\}$ and $\mathbf{R}e_{\Pi_i} := \sum_{\alpha\in\Pi_i}\mathbf{R}e_\alpha$ $(i = 1, 2)$. We may assume $\#\Pi_1 \geq \#\Pi_2$ without loss of generality. For a proof of (6.2.3) and (6.2.4), it is sufficient to prove

$$\ker(I_{odd}) = \mathbf{R}e_{\Pi_1} \cap (\mathbf{R}e_{\Pi_2})^\perp = \ker(c+1).$$

The inclusion $\ker(I_{odd}) \supset \mathbf{R}e_{\Pi_1} \cap (\mathbf{R}e_{\Pi_2})^\perp$ is obvious. We show the converse (i.e. $\xi = \sum_{\beta\in\Pi_2} c_\beta e_\beta$ and $I(\xi, e_\alpha) = 0$ for all $\alpha \in \Pi_1$ implies $\xi = 0$) by induction on $\#\Pi_2$. From the inequality $\#\Pi_2 \leq \#\Pi_1$, one sees that there is a vertex $\alpha \in \Pi_1$ which is connected with only one vertex, say β, of Π_2. Then $I(\xi, e_\alpha) = 0$ implies that the coefficient c_β vanishes. The diagram $\Pi \setminus \{\alpha, \beta\}$ may decompose into components but for each component Γ, one still has $\#(\Pi_2 \cap \Gamma) \leq \#(\Pi_1 \cap \Gamma)$ so that one can proceed further by induction to show $\xi = 0$.

Let us show $\ker(c+1) \subset \ker(I_{odd})$. Choose the Coxeter element to be $c = c_1c_2$ for $c_i = \prod_{\alpha\in\Pi_i}\sigma_\alpha$. Let $\xi \in \ker(c+1)$. Then $c\xi = -\xi$ implies $c_1\xi + c_2\xi = 0$ (*). Let $\xi = \xi_1 + \xi_2$ with $\xi_i \in \mathbf{R}e_{\Pi_i}$ $(i = 1, 2)$ so that $c_i\xi_i = -\xi_i$ $(i = 1, 2)$ and $c_1\xi_2 - \xi_2 \in \mathbf{R}e_{\Pi_1}$, $c_2\xi_1 - \xi_1 \in \mathbf{R}e_{\Pi_2}$. Then the condition (*) implies $(c_1\xi_2 - \xi_2) + (c_2\xi_1 - \xi_1) = 0$ and hence $c_1\xi_2 - \xi_2 = -(c_2\xi_1 - \xi_1) = 0$. These imply that $I(\xi_2, e_\alpha) = 0$ for all $\alpha \in \Pi_1$ and $I(\xi_1, e_\alpha) = 0$ for all $\alpha \in \Pi_2$. Thus $I_{odd}(\xi_1, e_\alpha) = I_{odd}(\xi_2, e_\alpha) = 0$ for all $\alpha \in \Pi$, i.e. $\xi \in \ker(I_{odd})$. Conversely, suppose $\xi \in \mathbf{R}e_{\Pi_1} \cap (\mathbf{R}e_{\Pi_2})^\perp$. Then $c_1\xi = -\xi$ and $c_2\xi = \xi$ and hence $c\xi = -\xi$.
The (6.2.5) follows from the definition of the exponents 2.8. $\square$

Assertion 1. implies that the form I_{odd} is non-degenerate $\Leftrightarrow$ $\#\Pi_1 = \#\Pi_2$ $\Leftrightarrow$ there is no eigenvalue -1 of the Coxeter element. In fact, $\operatorname{corank}(I_{odd})$ is positive for types A_l, B_l, C_l with l =even, D_l with $l \geq 4$, E_7 and H_3 (recall the remark at the end of 2.8).

The *nonsingular-hull* $(\tilde{F}, \tilde{I}_{odd})$ of (F, I_{odd}) is the smallest nonsingular symplectic vector space containing (F, I_{odd}) (unique up to isomorphism). The explicit model of the nonsingular-hull is constructed as follows.

Assertion 2. *Consider the vector space*

$$\tilde{F} := F \oplus \ker(I_{odd}) \tag{6.2.6}$$

equipped with the skew symmetric form:

(6.2.7) $\quad \tilde{I}_{odd}(x_1 \oplus y_1, x_2 \oplus y_2) := I_{odd}(x_1, x_2) + I(x_1, y_2) - I(x_2, y_1).$

Then, $\tilde{I}_{odd}$ is nondegenerate, and $(\tilde{F}, \tilde{I}_{odd})$ is the nonsingular-hull.

Proof. Without loss of generality, we assume $\#\Pi_1 \geq \#\Pi_2$. We saw in the proof of Assertion 1 that $\ker I_{odd} = \mathbf{R}e_{\Pi_1} \cap (\mathbf{R}e_{\Pi_2})^\perp$. Let us see that

$$\tilde{F} \quad = \quad \mathbf{R}e_{\Pi_1} \ \oplus \ (\mathbf{R}e_{\Pi_2} \oplus \ker I_{odd})$$

is a holonomic decomposition. That is: i) $\mathbf{R}e_{\Pi_1}$ and $\mathbf{R}e_{\Pi_2} \oplus \ker I_{odd}$ are totally isotropic w.r.t. $\tilde{I}_{odd}$ (*proof*: each of the spaces $\mathbf{R}e_{\Pi_1}$, $\mathbf{R}e_{\Pi_2}$ and $\ker I_{odd} \subset \mathbf{R}e_{\Pi_1}$ are totally isotropic. Then $I \mid \mathbf{R}e_{\Pi_2} \times \ker I_{odd} = 0$ implies the statement). ii) The coupling $I \mid \mathbf{R}e_{\Pi_1} \times (\mathbf{R}e_{\Pi_2} \oplus \ker I_{odd})$ is non-degenerate (*proof*: it suffices to show $I | \ker I_{odd}$ is non-degenerate, which follows since it is a restriction of the positive definite form $I | \mathbf{R}e_{\Pi_1}$). $\square$

6.3. Symplectic linear representations of $\Gamma(M)$.

For each $\alpha \in \Pi$, we consider two transvections

(6.3.1) $$s_\alpha^\pm : \quad u \in \tilde{F} \ \mapsto \ u \mp \tilde{I}_{odd}(e_\alpha^\vee, u) e_\alpha \in \tilde{F}.$$

One can directly show: i) $\tilde{I}_{odd}(s_\alpha^\pm(u), s_\alpha^\pm(v)) = \tilde{I}_{odd}(u, v)$, ii) $s_\alpha^+ s_\alpha^- = 1$, and therefore the transvections belong to the symplectic group:

$$s_\alpha^\pm \in \mathrm{Sp}(\tilde{F}, \tilde{I}_{odd}) := \{ g \in \mathrm{GL}(\tilde{F}) \mid \tilde{I}_{odd} \circ g = \tilde{I}_{odd} \}.$$

Lemma. *The correspondences $\gamma_\alpha \mapsto s_\alpha^\pm$ ($\alpha \in \Pi$) induce representations:*

(6.3.2) $$\rho^\pm : \Gamma(M) \to \mathrm{Sp}(\tilde{F}, \tilde{I}_{odd}).$$

Note. ρ^+ and ρ^- are different and $\rho^- \neq (\rho^+)^{-1}$ (except for type A_1).

Proof. We prove that the system $s_\alpha := s_\alpha^+$ ($\alpha \in \Pi$) satisfy the relations (6.1.1), (6.1.2) and (6.1.3). The other case is proven similarly.

We normalize the basis e_α for $\alpha \in \Pi$ as $I(e_\alpha, e_\alpha) = 2$ and $e_\alpha^\vee = e_\alpha$. Then one has $\quad I(e_\alpha, e_\beta) = -2\cos(\pi/m(\alpha, \beta)) \quad$ for $\alpha, \beta \in \Pi$

and $\quad s_\alpha(u) = u - I_{odd}(e_\alpha, u) e_\alpha \quad$ for $u \in \tilde{F}$.

If $m(\alpha, \beta) = 2$, then $\tilde{I}_{odd}(e_\alpha, e_\beta) = 0$ and one has $s_\alpha s_\beta(u) = s_\beta s_\alpha(u) = u - \tilde{I}_{odd}(e_\alpha^\vee, u) e_\alpha - \tilde{I}_{odd}(e_\beta^\vee, u) e_\beta$. This proves (6.1.1).

For a proof of (6.1.2) and (6.1.3), we prepare two Assertions. The first one studied the case of two vertices in a slightly generalized form.

Let h and m be a pair of integers with $h \geq 3$ and $\gcd(h, m) = 1$. Consider a two dimensional vector space $\mathbf{R}e_1 + \mathbf{R}e_2$ together with a symplectic form I_{odd} with $I_{odd}(e_1, e_2) = -\lambda$ for $\lambda := 2\cos(\pi m/h)$. Let

$$A := \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix}$$

be the matrix expression of the transvections s_1 and s_2 on $\mathbf{R}e_1 + \mathbf{R}e_2$.

Assertion 3. *Put* $e := \exp(\pi\sqrt{-1}m/h)$. *Then, one has*

$$\left[\begin{array}{cc} 1 & 1 \\ e^{-1} & e \end{array}\right]^{-1} AB \left[\begin{array}{cc} 1 & 1 \\ e^{-1} & e \end{array}\right] = \left[\begin{array}{cc} -e^2 & 0 \\ 0 & -e^{-2} \end{array}\right] \tag{6.3.3}$$

$$\left[\begin{array}{cc} 1 & 1 \\ -e & -e^{-1} \end{array}\right]^{-1} BA \left[\begin{array}{cc} 1 & 1 \\ -e & -e^{-1} \end{array}\right] = \left[\begin{array}{cc} -e^2 & 0 \\ 0 & -e^{-2} \end{array}\right] \tag{6.3.4}$$

$$\operatorname{order}(AB) = \operatorname{order}(BA) = \langle h \rangle. \tag{6.3.5}$$

$$\underbrace{ABA\cdots}_{h} = \underbrace{BAB\cdots}_{h} = \begin{cases} 1 & \text{if } h \text{ is even and } h/2 \text{ is odd,} \\ -1 & \text{if } h \text{ is even and } h/2 \text{ is even,} \\ (-1)^{(h-1)/2}\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right] & \text{if } h \text{ is odd.} \end{cases} \tag{6.3.6}$$

Proof. For (6.3.3) and (6.3.4), we calculate eigenvectors of AB and BA:

$$AB\left[\begin{array}{c} 1 \\ e^{\mp 1} \end{array}\right] = -e^{\pm 2}\left[\begin{array}{c} 1 \\ e^{\mp 1} \end{array}\right] \text{ and } BA\left[\begin{array}{c} 1 \\ -e^{\pm 1} \end{array}\right] = -e^{\pm 2}\left[\begin{array}{c} 1 \\ -e^{\pm 1} \end{array}\right].$$

The (6.3.5) is an immediate consequence of (6.3.3) and (6.3.4). If h is even, use (6.3.3) and (6.3.4) to calculate $(AB)^{h/2} = (BA)^{h/2} = \operatorname{diag}[-(-1)^{h/2}, -(-1)^{h/2}]$ where we applied $e^h = -1$. This gives the first two formulae of (6.3.6). If h is odd, put $h = 2n+1$. Then,

$$\begin{aligned}
\underbrace{ABA\cdots}_{2n+1} &= \left[\begin{array}{cc} 1 & 1 \\ e^{-1} & e \end{array}\right]\left[\begin{array}{cc} -e^2 & 0 \\ 0 & -e^{-2} \end{array}\right]^n \left[\begin{array}{cc} 1 & 1 \\ e^{-1} & e \end{array}\right]^{-1} A \\
&= \tfrac{(-1)^n}{e-e^{-1}}\left[\begin{array}{cc} 0 & e^{-1}-e \\ e-e^{-1} & e^{-2}-e^2 \end{array}\right]\left[\begin{array}{cc} 1 & e+e^{-1} \\ 0 & 1 \end{array}\right] \\
&= (-1)^n\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right]. \\
\underbrace{BAB\cdots}_{2n+1} &= B\left[\begin{array}{cc} 1 & 1 \\ e^{-1} & e \end{array}\right]\left[\begin{array}{cc} -e^2 & 0 \\ 0 & -e^{-2} \end{array}\right]^n \left[\begin{array}{cc} 1 & 1 \\ e^{-1} & e \end{array}\right]^{-1} \\
&= \tfrac{(-1)^n}{e-e^{-1}}\left[\begin{array}{cc} 1 & 0 \\ -e-e^{-1} & 1 \end{array}\right]\left[\begin{array}{cc} 0 & e^{-1}-e \\ e-e^{-1} & e^{-2}-e^2 \end{array}\right] \\
&= (-1)^n\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right].
\end{aligned}$$

Comparing two expressions, we obtains the last formula of (6.3.6). □

The (6.3.6) proves (6.1.2). For a proof of (6.1.3), we prepare second Assertion which reduces the general problem to the case of two vertices.

Assertion 4. *Let* Γ' *be an irreducible finite type subdiagram of the Coxeter graph of* M *with the Coxeter number* h'. *Put* $V' := \Sigma_{\alpha\in\Gamma'}\mathbf{R}e_\alpha$. *Then i)* $\operatorname{order}((\Pi_{\alpha\in\Gamma'}s_\alpha)|V') = \langle h' \rangle$ *except for* $2\langle h' \rangle$ *if* Γ' *is of type* A_{4k+1}, *and ii)* $\operatorname{order}(\Pi_{\alpha\in\Gamma'}s_\alpha) = \langle h' \rangle$ *if* $h'/2$ *is not an exponent of* Γ' *and* $= \infty$ *if* $h'/2$ *is an exponent of* Γ' *(recall* (6.1.4) *for* $\langle h \rangle$*).*

Proof. Since Γ' is a tree, we decompose its vertices into a disjoint union $\Gamma'_1 \amalg \Gamma'_2$ such that each Γ'_i is totally disconnected. Put $c = c_1c_2$ for $c_i := \Pi_{\alpha\in\Gamma'_i} s_\alpha$ and $V' = V'_1 \oplus V'_2$ for $V'_i := \Sigma_{\alpha\in\Gamma'_i}\mathbf{R}e_\alpha$ for $i = 1, 2$. Since $I_{odd}(e_\alpha, e_\beta) = 0$ for $\alpha, \beta \in \Gamma'_i$, one has $c_i(u) = u - \sum_{\alpha\in\Gamma'_i} \tilde{I}_{odd}(e^\vee_\alpha, u)e_\alpha$.

Since $(c_1-c_2)(e_\beta) = -\sum_{\alpha\in\Gamma'_1} I_{odd}(e^\vee_\alpha, e_\beta)e_\alpha + \sum_{\alpha\in\Gamma'_2} I_{odd}(e^\vee_\alpha, e_\beta)e_\alpha = 2e_\beta - \sum_{\alpha\in\Gamma'} I(e^\vee_\alpha, e_\beta)e_\alpha$, the matrix expression of $2 - c_1 + c_2$ w.r.t. the basis e_α ($\alpha \in \Gamma'$) is the Cartan matrix of type Γ', which is up to a multiplication of a diagonal matrix $(I(e_\alpha, e_\alpha))_{\alpha\in\Gamma'}$ symmetric. Due the classical result on Cartan matrices (e.g. [B,chV,§6.2]) (or, it is easy to see directly), the eigenvalues are $2 - 2\cos(\pi m'_i/h')$ where m'_i are the exponents for Γ'. Let $e = e_1 + e_2$ with $e_i \in V'_i$ be an eigenvector ($\neq 0$) of c_1-c_2 belonging to an eigenvalue $\lambda = 2\cos(\pi m'_i/h')$. Combining the equalities: $c_1(e_1) = e_1$, $c_2(e_2) = e_2$ and $(c_1 - c_2)(e_1 + e_2) = \lambda(e_1 + e_2)$, one has $c_1e_2 = e_2 + \lambda e_1$ and $c_2e_1 = e_1 - \lambda e_2$ and, hence, $e_1 - e_2$ is an eigenvector of $c_1 - c_2$ belonging to the eigenvalue $-\lambda$. If we assume $\lambda \neq 0$ (i.e. $2m'_i \neq h'$), then $e_1 \neq 0 \neq e_2$ and $V_\lambda := \mathbf{R}e_1 + \mathbf{R}e_2$ has rank 2. Then, c_1 and c_2 act on the space V_λ as transvections of the vectors e_1 and e_2 with respect to the skew symmetric form $\left[\begin{smallmatrix} 0 & -\lambda \\ \lambda & 0 \end{smallmatrix}\right]$. We saw in Assertion 3. that the order of $c|V_\lambda$ is equal to $\langle h'/\gcd(h', m'_i)\rangle$.

Put $\lambda_i := 2\cos(\pi m'_i/h')$ for $0 < m'_i < h'/2$. One has the decomposition:

$$V' = \ker(I'_{odd}) \oplus V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_m},$$

where $I'_{odd} := I_{odd}|V'$ so that i) $\ker(I'_{odd})$ is the fixed point subspace of V' by the action of c, and ii) the action of c preserves the splitting.

i) Since the smallest exponent is $m'_1 = 1$ (recall (2.8.3)), one has $\mathrm{order}(c|V_{\lambda_1}) = \langle h'\rangle$. The order $\langle h'/\gcd(h', m'_i)\rangle$ of another factor $c|V_{\lambda_i}$ is not a divisor of $\langle h'\rangle$, only when h' is $2\times$odd and $\gcd(h', m'_i) = 2$. This occurs only when Γ' is of type A_{4k+1} ($k \geq 2$).

ii) If $\ker(I'_{odd}) = \{0\}$, then $\tilde{F} = V' \oplus V'^\perp$ so that $\mathrm{order}(c|V') = \mathrm{order}(c)$. Assume $\ker(I'_{odd}) = V'_1 \cap V'^\perp_2 \neq \{0\}$. Then for $x \in \cap_i V^\perp_{\lambda_i} \setminus (\ker(I'_{odd}))^\perp$ one has $c^n x - x = n(\sum_{\alpha\in\Gamma'_1} I_{odd}(e^\vee_\alpha, x)e_\alpha) \neq 0$ for $n \neq 0$. □

Assertion 4. proves (6.1.3) and, hence, Lemma is proven. □

We state another consequence (on odd roots) of Assertion 3.

Corollary. *Let $m(\alpha, \beta) \in \mathbf{Z}_{\geq 3}$ be odd. Then $s_{\alpha\beta} := s_\alpha s_\beta s_\alpha \cdots$ (m-factors) is of order 4. It acts transitively on the set $\{\pm e_\alpha, \pm e_\beta\}$.*

Proof. Use the third line of the formula (6.3.6). □

Conjecture 1. *The homomorphisms* (6.3.2) *are injective.*

Assertion 5. *Conjecture 1. is true for Coxeter matrices with $l \leq 2$.*

Proof. This is trivial for types A_1 and $A_1 \times A_1$. Thus, we prove for type $I_2(m)$ ($m \geq 3$) (including A_2, B_2, C_2, H_2 and G_2). It is achieved by a use of the fundamental domain on the complex upper-half-plane.

Assertion 6. *Let $\bar\rho : \Gamma(I_2(m)) \to \mathrm{PSp}(F_{I_2(m)}) = \mathrm{Sp}(F_{I_2(m)})/\{\pm 1\}$ be the projectivization of $\rho = \rho^+$ for $m \geq 3$. Then the image of $\bar\rho$ is isomorphic to the group* $\langle \bar s_\alpha, \bar s_\beta \mid i)\ (\bar s_\alpha \bar s_\beta)^{m/2} = 1$ if m is even or $(\bar s_\alpha \bar s_\beta)^m = 1$ if m is odd, and $ii)$ $\bar s_\alpha \bar s_\beta \bar s_\alpha \cdots = \bar s_\beta \bar s_\alpha \bar s_\beta \cdots$ (m-factors in both sides)$\rangle$.

Proof. It is obvious that the images $\bar s_\alpha$ and $\bar s_\beta$ in $\mathrm{PSp}(F_{I_2(m)})$ of s_α and s_β for $\Pi = \{\alpha, \beta\}$ satisfy the relations i) and ii) due to Assertion 3.

Let $\omega_1 = \langle e_\alpha, z\rangle$ and $\omega_2 = \langle e_\beta, z\rangle$ be homogeneous coordinates of $\mathbf{P}(\mathrm{Hom}_{\mathbf{R}}(F, \mathbf{C}))$. Consider the action of $\Gamma(I_2(m))$ on a connected component $\mathcal{H}$ of $\mathbf{P}(\mathrm{Hom}_{\mathbf{R}}(F, \mathbf{C})) \setminus \mathbf{P}(F^*)$, which is isomorphic to the complex upper half plane $\mathcal{H}$ with respect to the inhomogeneous coordinate $\tau := \omega_1/\omega_2$. Define a subset of $\mathcal{H}$:

$$\begin{aligned} \mathcal{F} \;:=\; & \{\tau \in \mathcal{H} \mid -\cos(\pi/m) \leq \Re(\tau) < \cos(\pi/m), |\tau| > 1\} \\ & \cup \{e^{\sqrt{-1}\theta} \mid \pi/2 \leq \theta \leq \pi(1 - 1/m)\}. \end{aligned}$$

Then, the fundamental domain of the action is either $\mathcal{F}$ if m is odd or $\mathcal{F} \cup \mathcal{F}'$ if m is even where $\mathcal{F}'$ is the image of $\mathcal{F}$ by the transformation $\tau \mapsto -1/\tau$. (In fact, the two vertices e and $-e^{-1}$ of $\mathcal{F}$ are the fixed point of the action of $\bar s_\alpha \bar s_\beta$ and $\bar s_\beta \bar s_\alpha$, respectively, and the cusps $\sqrt{-1}\infty$ and 0 are fixed point by the actions of $\bar s_\alpha$ and $\bar s_\beta$, respectively.) □

We return to a proof of Assertion 5. Let us "lift" the fundamental relations in $\mathrm{PSp}(F_{I_2(m)})$ to the elements in $\mathrm{Sp}(F_{I_2(m)})$. Then, due to Assertion 3, one has $s_\alpha s_\beta s_\alpha \cdots = s_\beta s_\alpha s_\beta \cdots$ (m-times) for all m, $(s_\alpha s_\beta)^{m/2} = 1$ if m is even and $m/2$ is odd, $(s_\alpha s_\beta)^{m/2} = -1$ if m is even and $m/2$ is even and $(s_\alpha s_\beta)^m = -1$ if m is odd. Then, by killing the sign factor, one obtains (6.1.2) and (6.1.3) as the fundamental relations for the image of $\Gamma(I_2(m))$ in $\mathrm{Sp}(\tilde F_{I_2(m)})$. So, Assertion 5. is proven. □

Example. There are three cases when $I_2(p)$ is crystallographic.

$$\begin{aligned} &\Gamma(I_2(3)) = \Gamma(A_2) = \Gamma_0(1) := \mathrm{SL}_2(\mathbf{Z}), \\ &\Gamma(I_2(4)) = \Gamma(B_2) = \Gamma_0(2) := \{[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in \mathrm{SL}_2(\mathbf{Z}) \mid b \equiv 0 \bmod 2\}, \\ &\Gamma(I_2(6)) = \Gamma(G_2) = \Gamma_0(3) := \{[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in \mathrm{SL}_2(\mathbf{Z}) \mid b \equiv 0 \bmod 3\}. \end{aligned}$$

Remark. Let Z be a central element of the Artin group $A(W)$. Then $\rho(Z)|V$ is either 1 or -1. *Proof.* Put $z := \rho(Z)$. The relation $g_\alpha Z = Z g_\alpha$ for $\alpha \in \Pi$ implies $s_\alpha(z(u)) = z(u - \tilde I_{odd}(e_\alpha^\vee, u) e_\alpha)$ for $u \in \tilde V$. This implies $\tilde I_{odd}(e_\alpha^\vee, z(u)) e_\alpha = \tilde I_{odd}(e_\alpha^\vee, u) z(e_\alpha)$. Choosing u such that $\tilde I_{odd}(e_\alpha^\vee, u) \neq 0$, one observes that $z(e_\alpha) = c_\alpha e_\alpha$ for some constant c_α. Substituting u by e_β for $\beta \in \Pi$ and assuming $z(e_\beta) = c_\beta e_\beta$ for some constant c_β, one has $c_\beta \tilde I_{odd}(e_\alpha^\vee, e_\beta) = c_\alpha \tilde I_{odd}(e_\alpha^\vee, e_\beta)$. Since the graph for Π is connected, $c_\alpha = \varepsilon$ is independent of $\alpha \in \Pi$ and $\varepsilon \in \{\pm 1\}$.

6.4. **Period domain $\mathcal{B}(W)$.**

Suppose W is crystallographic. That is: there is a finite root system w.r.t. (F, I) whose Weyl group coincides with W (see [B,chap.VI,§2,n^{o}5 and §4] for details). Then, for a suitable choice of the scaling constants d_α ($\alpha \in \Pi$) in 6.2, the matrix $(I(e_\alpha^\vee, e_\beta))_{\alpha,\beta\in\Pi}$, so called the Cartan matrix, is integral. In fact, up to an overall constant factor on the scaling constants, there are one or two choices according as M is simply laced or not. Choose one such scaling and consider the set $R := \cup_{\alpha\in\Pi} W\, e_\alpha$ and a lattice $Q := \Sigma_{\alpha\in\Pi}\mathbf{Z}e_\alpha$ in V. Then R forms a root system with simple basis $\{e_\alpha \mid \alpha \in \Pi\}$, $W = W(R)$ is the Weyl group and $Q = Q(R)$ is the root lattice of the root system. By a choice of an over all constant factor on the scaling constants, $I \mid Q \times Q$ is integral valued. We normalize $\min\{I(e_\alpha, e_\alpha) \mid \alpha \in \Pi\} = 2$. We shall sometimes denote $\Gamma(R)$ instead of $\Gamma(M)$ or $\Gamma(W)$ (here recall that the action of $\Gamma(R)$ on $Q(R)$ depends on a choice of a chamber (5.2 Example)). The group

$$\tilde{Q} := Q \oplus (Q \cap \ker(I_{odd})) \tag{6.4.1}$$

is a full sublattice in $\tilde{F}$ and $\tilde{I}_{odd} \mid \tilde{Q} \times \tilde{Q}$ is an integral symplectic form.

Problem. Let $\tilde{t}$ be a point in the chamber C and t be its image in S_W. Show that the first homology lattice $H_1(X_t, \mathbf{Z})$ of the Milnor fiber of φ_1 is canonically isomorphic to the lattice $(Q, I_{odd,C})$ (use [S8]).

Recall the flat vector space Ω_+ (3.11.2) attached to the reflection group W. Due to 1., the period map attached to the primitive form ζ_{F_1} (recall 5.3) is a holomorphic local submersion given by

$$P_W \;\; : \;\; t \in (S_W \backslash D_W)^\sim \;\; \mapsto \;\; \textstyle\int \mathrm{Res}_t[\zeta_{F_1}^{(0)}] \in \mathrm{Hom}(Q, \Omega_{+\mathbf{C}}).$$

The projection to the first factor is given by local covering map

$$\langle D, P_W\rangle : \;\; t \in (S_W \backslash D_W)^\sim \;\; \mapsto \;\; \textstyle\int \mathrm{Res}_t[\zeta_{F_1}^{(0)}] \in \mathrm{Hom}(Q, \mathbf{C}).$$

Conjecture 2. *The period map P_W is a global immersion of $(S_W \setminus D_W)^\sim/\ker(\rho\circ\gamma)$ into a locally closed submanifold $\mathcal{B}(W)$ of an open cone $\tilde{\mathcal{B}}(W)$ in* $\mathrm{Hom}(Q, \Omega_{+\mathbf{C}})$. *Determine $\mathcal{B}(W)$ and $\tilde{\mathcal{B}}(W)$ explicitly without using period integral but only in terms of the reflection group.*

These problems are somehow asking for analogues of the Riemann's relations and Schottkey's relations for the classical periods of Abelian integrals on compact Riemann surfaces. Here the period domain is "homogenized" (i.e. admits $\mathbf{C}^\times$-action) due to the flat structure on Ω.

6.5. **Inverse functions on $\mathcal{B}(W)$.**

We observed in 5.3 Example that for types A_2, A_3 and B_2 the inverse map from the period domain $\mathcal{B}(W)$ (to be exact, from its projection to the first factor) to $S_W\backslash D_W$ is described by a system of elliptic Eisenstein series (including some specializations of the $\mathfrak{p}$-function).

There are four more types D_4, B_3, C_3 and G_2, where the similar considerations work to construct the inverse map. Namely, the Milnor fibers for types D_4, C_3 and G_2 are elliptic curves with 3 punctures either without symmetry, with $\mathbf{Z}/2$-symmetry or with $\mathbf{Z}/3$-symmetry, respectively. The Milnor fiber for type B_3 is a double cover of a once punctured elliptic curve.

Problem Work out explicitly the inversion problem for the period maps of types D_4, B_3, C_3 and G_2.

Beyond these examples, the genus of Milnor fiber increases (without a "reduction" to an elliptic curve). Then the indefinite integral $\int_{\infty_1}^{\infty_2} \zeta$ becomes multi-valued by the periods $\int_\gamma \zeta$ for $\gamma \in H_1(\bar{X}_t, \mathbf{Z})$, where the periods are dense in the 1-dimensional complex plane. Then, it is already a problem to find a reasonable formulation for the period map and the inverse map. We need to introduce a concept of a regular function on the period domain $\mathcal{B}(W)$ (with a suitable polynomial growth condition on the boundary) which is automorphic for the group $\Gamma(W)$, and also a concept of pull-back to a function on $(V^*//W)_{\mathbf{C}}$. For instance, one may consider either a single indefinite integral $\int_\infty^p$ and its (multi-valued-)inverse function as in the classical $\mathfrak{p}$-function theory, or a system of derivatives $\delta_i \int_\infty^p \zeta$ of the periods to obtain the Jacobian variety as in the classical Abel-Jacobi theory? (See also a related work [Ko][Mu][To]).

Conjecture 3. *There exists a ring* $\mathbf{C}[E]$ *of regular functions on* $\mathcal{B}(W)$ *which are automorphic for the group* $\Gamma(W)$*. Any element of* $\mathbf{C}[E]$ *is a pull-back of a polynomial function in* $S(V)^W$ *to the period domain. This correspondence induces a natural isomorphism:*

$$\mathbf{C}[E] \simeq S(V)^W \otimes \mathbf{C}. \tag{6.5.1}$$

The isomorphism implies that the $\mathrm{Spec}(\mathbf{C}[E])$ carries the flat structure studied in §3. Then we may naturally ask how to describe the flat structure in terms of $\mathbf{C}[E]$. In particular, $\mathbf{C}[E]$ is generated by algebraically independent homogeneous elements, say $E_4, \cdots, E_{2h}$ of degree $2d_1 = 4, \cdots, 2d_l = 2h$ such that $J(dE_{2d_i}, dE_{2d_j}) = const$, which we may call *flat automorphic forms* or *primitive automorphic forms.*

Conjecture 4. *Let* W *be crystallographic, which is not of type* G_2*. Then there exists an extension* $\mathbf{C}[E] \subset \mathbf{C}[\tilde{E}]$ *by a ring of regular functions on* $\mathcal{B}(W)$ *which are automorphic for the principal congruence subgroup of* $\Gamma(W)$ *(c.f. 6.1iv)) such that the commutative diagram holds:*

$$
\begin{array}{ccc}
\mathbf{C}[E] & \simeq & S(V)^W \otimes \mathbf{C} \\
\cap & & \cap \\
\mathbf{C}[\tilde{E}] & \simeq & S(V) \otimes \mathbf{C}.
\end{array} \tag{6.5.5}
$$

We remark that, by definition, $\mathrm{Spec}(\mathbf{C}[\tilde{E}])$ carries a linear space structure with a flat metric, which is isomorphic to the pair (V^*, I^*).

6.6. **Power root of the Discriminant Δ_W.**

We turn our attention to the discriminant Δ_W and its power root. After a choice of the primitive vector field D in (3.2.1), the discriminant was normalized to be a monic polynomial of degree l in P_l (c.f. 2.9 Lemma 1.). However, in the sequel, we disregard tentatively the constant factors and proceeds the caluculations up to constant factor.

Recall the anti-invariants $\delta_W \in S(V)$ in 2.7 and the disriminant $\Delta_W = \delta_W^2 \in S(V)^W$ in 2.9. Let us denote by the same notation the corresponding elements in $\mathbf{C}[\tilde{E}]$ (resp. $\mathbf{C}[E]$) in RHS of (6.5.5). Owing to Conjectures 3 and 4, we have Jacobian expressions of Δ_W and δ_W as below. Namely, let $\tilde{E}_1, \cdots, \tilde{E}_l$ be a system of generator of $\mathbf{C}[\tilde{E}]$ corresponding to a linear coordinate system of V, and let $e_1, \cdots, e_l \in \Pi$ be the linear coordinate system on $\tilde{\mathcal{B}}(W)$. Then by a use of (4.5.9), up to a constant factor c, one has

$$dE_4 \wedge \cdots \wedge dE_{2h} = c\ \Delta_W\ de_1 \wedge \cdots \wedge de_l, \tag{6.6.1}$$

$$d\tilde{E}_1 \wedge \cdots \wedge d\tilde{E}_l = c\ \delta_W\ de_1 \wedge \cdots \wedge de_l. \tag{6.6.2}$$

One should define the meaning of "boundary components" of $\mathcal{B}(W)$ and "cusp forms" in $\mathbf{C}[E]$ or in $\mathbf{C}[\tilde{E}]$ such that the discriminant Δ_W (resp. δ_W) generates the ideal of cusp forms in $\mathbf{C}[E]$ (resp. $\mathbf{C}[\tilde{E}]$). The discriminant, as a function on $\mathcal{B}(W)$, vanishes nowhere. The δ_W is a square root of Δ_W, and is anti-invariant with respect to θ_W:

$$\gamma^* \delta_W = \theta_W(\gamma)\ \delta_W. \tag{6.6.3}$$

Here we recall (6.1.5) for the definition of the character θ, and recall the commutative diagram (6.1.8).

Conjecture 5. *Does δ_W have a suitable infinite product expression?*

Remark. The conjectures and problems in 6.5 and 6.6 seem to have close relations with odd the root system: $R_{odd} := \cup_{\alpha \in \Pi_W} \Gamma(W) e_\alpha$. For instance, as a generalization of Eisenstein series, one may consider, for $d \in \mathbf{Z}_{\geq 2}$, the sum of partial fractions such as $\sum_{e \in R_{odd}} e^{-2d}$, $\sum_{e \in R_{odd,s}} e^{-2d}$, $\sum_{e \in R_{odd,l}} e^{-2d}$ or a sum of suitable combinations $e_s^{-2d} - e_l^{-2d}$ for $e_s \in R_{odd,s}$ and $e_l \in R_{odd,l}$ and also $e \in \Gamma(W) \cdot (\ker(I_{odd}) \cap Q)$ in case whenI_{odd} is degenerate on the period domain. There are supporting examples that the boundary component and the infinite product might reasonably be described in terms of the odd root system. In spite of the examples, it is still unclear to the author what are natural formulation of the partial fractional sums fitting to our setting and we leave them as an open problem.

Example. **1.** Type A_2. We have the expression $\mathcal{B}(A_2) = \{(u_1, u_2) \in \mathbf{C}^2 \mid Im(u_1\bar{u}_2 - \bar{u}_1 u_2) > 0\}$ where (u_1, u_2) are the homogeneous linear coordinates of the period domain. The ring $\mathbf{C}[E]$ is the ring of automorphic forms for the full modular group $\Gamma_0(1)$, and is generated by the classical Eisenstein series E_4 and E_6 (e.g. [Ko,p111]). The ring $\mathbf{C}[\tilde{E}]$ is the ring of automorphic forms for the principal congruence subgroup $\Gamma(2)$ of level 2. All conjectures are positively solved for type A_2.

The discriminant Δ_W (in homogeneous form) is given by

$$\Delta_{A_2}(u_1, u_2) = \eta(\tau)^{24} u_2^{-12},$$

where $\tau := u_1/u_2$ and $\eta(\tau)$ is the Dedekind eta-function (e.g. [Ko,p121]). It generates the ideal of cusp forms in the ring $\mathbf{C}[E]$. The generator δ_{A_2} of the anti-invariants with respect to the character θ_{A_2} is given by

$$\delta_{A_2}(u_1, u_2) = \eta(\tau)^{12} u_2^{-6},$$

(for this proof and for a finer statement see *Assertion 7.* below).

2. Type B_2. We have seen that the discriminant for the family φ_{B_2} decomposes into irreducible components corresponding to short and long roots (5.3 Example 4), and, hence, it is given by

$$\Delta_{B_2} = \eta(\tau)^8 \eta(2\tau)^8 u_2^{-8}.$$

Then one can show by a similar calculation as in type A_2 that

$$\delta_{B_2} = \eta(\tau)^4 \eta(2\tau)^4 u_2^{-4}.$$

is an anti-invariant with respect to the character θ_{B_2} on $\Gamma(B_2) = \Gamma_0(2)$.

3. Type G_2. Since the discriminant for the family φ_{G_2} decomposes into components corresponding to short and long roots, one may apriori describe the discriminant (up to a constant factor) to be of the form

$$\Delta_{G_2} = \eta(\tau)^6 \eta(3\tau)^6 u_2^{-6}.$$

Then one can show by a similar calculation as in type A_2 that

$$\delta_{G_2} = \eta(\tau)^3 \eta(3\tau)^3 u_2^{-3}$$

is an anti-invariant with respect to the character θ_{G_2} on $\Gamma(G_2) = \Gamma_0(3)$.

Throughout the examples, we observe further the following. Recall the integer $k(W)$ and the character ϑ_W of the group $\Gamma(W)$ (see 6.1 ii)).

Assertion 7. *For type A_2, B_2 and G_2, consider the $k(W)$th power root, say λ_W, of δ_W. Since $k(A_2) = 6$, $k(B_2) = 4$ and $k(G_2) = 3$, up to a constant factor, they are explicitly given by*

$$\lambda_{A_2} := \eta^2(\tau)u_2^{-1},\ \lambda_{B_2} := \eta(\tau)\eta(2\tau)u_2^{-1} \text{ and } \lambda_{G_2} := \eta(\tau)\eta(3\tau)u_2^{-1}.$$

Then, λ_W is an automorphic form for $\Gamma(W)$ with the character ϑ_W. That is:

$$g^*(\lambda_W) = \vartheta_W(g)\lambda_W \quad \text{for } g \in \Gamma(W).$$

Proof. We have only to verify that λ_W is equivariant with the character ϑ_W. The verification of this fact is achieved by an elementary but slightly subtle use of the transformation formula for the eta-function, where we use the sign conventions in [Kob, p121].

Put $W_1 := W_{A_2}$, $W_2 := W_{B_2}$ and $W_3 := W_{G_2}$. Then, the modular group $\Gamma(W_p)$ is given by $\Gamma_0(p)$ for $p = 1, 2$ and 3, and is generated by $s_\alpha : (u_1, u_2) \mapsto (u_1, u_2)\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $s_\beta^{-1} : (u_1, u_2)\begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}$ (recall 6.3 Example). On the other hand, the explicit formula (in the above examples) says that $\lambda_{W_p} = \eta(\tau)\eta(p\tau)u_2^{-1}$ $(p = 1, 2, 3)$.

Further more, one observes directly $k(W_p) = \frac{12}{1+p}$ $(p = 1, 2, 3)$. Therefore, by putting $\zeta := \exp(\pi\sqrt{-1}/12)$, the character ϑ_{W_p} is given by $\vartheta_{W_p}(s_\alpha) = \vartheta_{W_p}(s_\beta) = \zeta^{p+1}$ (the notation here is incoherent with that in 6.3). So,

$$\begin{aligned} s_\alpha^*(\lambda_{W_p}) &:= \eta(\tau+1)\ \eta(p\tau+p)\ u_2^{-1} \\ &= \zeta\eta(\tau)\ \zeta^p\eta(p\tau)\ u_2^{-1} = \zeta^{p+1}\lambda_{W_p}, \end{aligned}$$

$$\begin{aligned} s_\beta^{*-1}(\lambda_{W_p}) &:= \eta(\tau/(p\tau+1))\ \eta(p\tau/(p\tau+1))\ (pu_1+u_2)^{-1} \\ &= \sqrt{\tfrac{-(p\tau+1)/\tau}{\sqrt{-1}}}\ \eta(-p-1/\tau)\ \zeta\eta(-1/(p\tau+1))\ (pu_1+u_2)^{-1} \\ &= \sqrt{\tfrac{-(p\tau+1)/\tau}{\sqrt{-1}}\tfrac{\tau}{\sqrt{-1}}\tfrac{p\tau+1}{\sqrt{-1}}}\ \zeta^{-p}\eta(\tau)\ \zeta\eta(p\tau+1)\ (p\tau+1)^{-1}u_2^{-1} \\ &= \zeta^{-p+2}\sqrt{\tfrac{1}{\sqrt{-1}}}\ \eta(\tau)\ \eta(p\tau)\ u_2^{-1} \\ &= \zeta^{-p-1}\eta(\tau)\ \eta(p\tau)\ u_2^{-1} = \zeta^{-p-1}\lambda_{W_p}. \end{aligned}$$

□

We observe also the following nearly trivial remark, which is still interesting since the form I_{odd} is degenerate for the type A_3.

Fact. *A $k(A_3)$th power root $\lambda_{A_3} := \eta(\tau)^2u_2^{-1}$ of δ_{A_3} is automorphic for $\Gamma(A_3)$ with the character ϑ_{A_3} (here, recall $k(A_3)=k(A_2)=6$).*

Proof. This is shown by a reduction to A_2. Recall 5.3 Ex.3. for the setting. The morphism $S_{A_3} \setminus D_{A_3} \to S_{A_2} \setminus D_{A_2}$ induces an equality $\Delta_{A_3} = c\Delta_{A_2}$ and a homomorphism $A(A_3) \to A(A_2)$ bringing the generators a_1, a_2, a_3 of $A(A_3)$ to b_1, b_2, b_1 of $A(A_2) = \langle b_1, b_2 \mid b_1b_2b_1 = b_2b_1b_2\rangle$

([S2,Theorem III]) so that the characters ϑ_{A_3} and ϑ_{A_2} commutes with the homomorphism $\Gamma(A_3) \to \Gamma(A_2)$. This implies the result. $\square$

This fact encourages us to ask the next question:

Conjecture 6. *Let W be a crystallographic finite reflection group. Is the $k(W)$th power root of δ_W, say λ_W (up to a constant factor), an automorphic form for the group $\Gamma(W)$ with the character ϑ_W? Can one find an infinite product expression for λ_W compatible with Conjecture 4?*

In view of the fact that $k(D_4)=6$ is the last largest power (among all finite crystallographic group) and that the period map for D_4 can still be described by elliptic integrals, it is interesting to have an exact and explicit expression of λ_{D_4} as a distinguished Jacobi form on $\tilde{\mathcal{B}}(D_4)$.

A concluding Remark.

Except for the discussed types $A_1, A_2, A_3, B_2, B_3, C_2, C_3, D_4$ and G_2, in all further cases, the character ϑ_W takes values in $\mathbf{Z}_4=\{\pm1,\pm\sqrt{-1}\}$. However, we have no information on the primitive automorphic forms for them at this stage, since they are beyond elliptic integrals. Actually, Conjecture 6 seems not be understandable only from the geometry of the family $\varphi_W : \mathfrak{X}_W \to S_W$. Instead, the conjecture seems reasonably understandable if one finds a suitable "mirror object" to the family φ_W such that it gives arise a suitable construction of the inversion maps and the flat structure on S_W. Perhaps, finding such mirror object (based on odd root systems?) may be the main question and goal of the present section and hence the main problem of the the present article.

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