Plane quartics and Fano threefolds of genus twelve*

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288 triangles are strictly biscribed by a plane quartic curve $C \subset \mathbf{P}^2 = \mathbf{P}^2(\mathbf{C})$. Two computations of this number will be presented. This number $288 = 36 \times 8$ is related with an even theta characteristic of C and with a Fano threefold V_{22} of genus twelve. In fact there is a natural birational correspondence between the moduli of V_{22}'s and that of plane quartics. This correspondence led the author to a description of those Fano threefolds as $VSP(6, \Gamma)$, the variety of sums of powers of another plane quartic $\Gamma : F_4(x, y, z) = 0$ ([6]).

1 Biscribed triangles

A triangle is *biscribed* by a curve C if it is both circumscribed and inscribed, that is, each vertex lies on C and each side is a tangent. It is interesting to count the number of such triangles for a given plane curve $C \subset \mathbf{P}^2$. The following is an easy exercise and we leave the proof to the readers.

Proposition The number of biscribed triangles of a smooth cubic is 24.

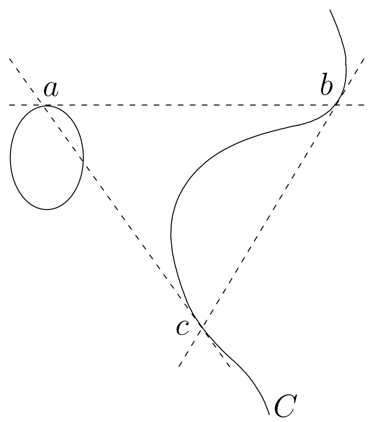

Figure 1 (Triangle biscribed by a cubic)

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A secant line $\overline{ab}$, $a \neq b \in C$, of a smooth plane curve C is a *strict tangent* of C if either
i) $\overline{ab}$ tangents to C at a point different from a, b, or
ii) $\overline{ab}$ is a triple tangent at a or b.

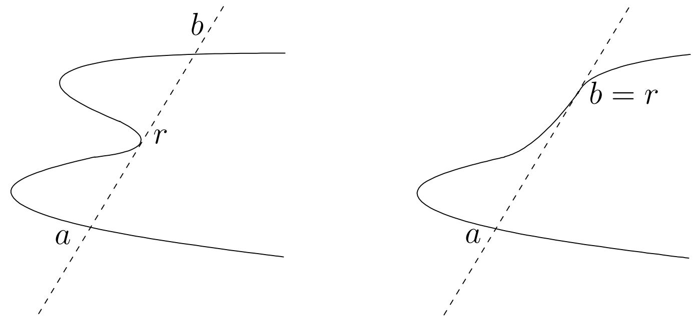

Figure 2

This is equivalent to the condition that the divisor class $h - a - b - 2r$ is effective for a point $r \in C$, where h is the linear equivalence class of the intersection of C with a line. A triangle is *strictly biscribed* if all sides are strict tangents. A plane cubic has no such triangles.

Problem Count the number of strictly biscribed triangles of a plane curve.

We consider the case where C is a smooth plane quartic. A triangle $\triangle = \triangle abc$ is strictly biscribed by C if and only if three distinct points $a, b, c \in C$ satisfy the linear equivalence

$$h - b - c \sim 2p, \ h - c - a \sim 2q \quad \text{and} \quad h - a - b \sim 2r \tag{1}$$

for some points $p, q, r \in C$.

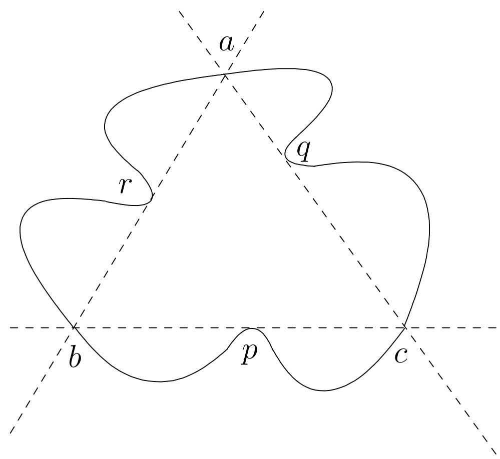

Figure 3 (Triangle strictly biscribed by a quartic)

The image of the morphism $\Phi_{|2h-a-b-c|} : C \longrightarrow \mathbf{P}^2$, the restriction of the quadratic Cremona transformation with center $\triangle$, is a quintic curve with three cusps. Namba[10] makes use of this for the classification of singular plane quintics.

2 First computation

The following is the author's computation in 1982 (cf. Remark 2.3.6 at p. 159 in [10]). We put

$$D = \{(a,b) \,|\, h - a - b \sim 2r \quad \exists r \in C\} \subset C \times C,$$

which is a divisor. Since $(D.\ C \times pt.) = (C.\ pt. \times C) \ = 10$ and $(D.\Delta) = 2 \times 28$, D is numerically equivalent to $16(C \times pt. + pt. \times C) - 6\Delta$. Here 10 is the number of the tangent lines of C passing through a general point $a \in C$, excluding the tangent line itself at a, 28 is the number of bitangents of C and $\Delta \subset C \times C$ is the diagonal. Let D_{ij}, $1 \le i < j \le 3$, be the pull-backs of D by three projections $C \times C \times C \longrightarrow C \times C$. The intersection $D_{12} \cap D_{13} \cap D_{23}$ consists of three parts:

$$\{a, b, c \text{ are distinct}\} \cup \{\text{two of } a, b, c \text{ are the same}\} \cup \{a = b = c\}.$$

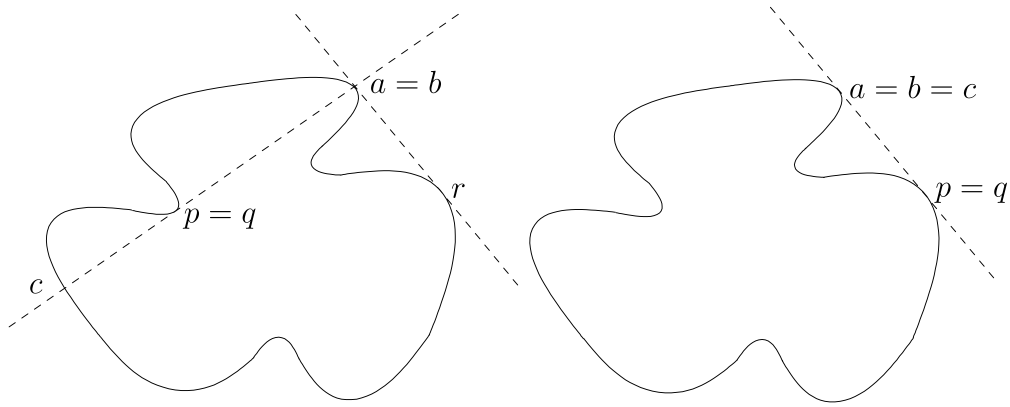

Figure 4 (2nd and 3rd parts)

(a, b, c) belongs to the first part if and only if the secant lines $\overline{ab}$, $\overline{bc}$ and $\overline{ac}$ are strict tangents of C. Hence the number of the strictly biscribed triangles of C is equal to

$$\begin{aligned} &\{(D_{12}.D_{13}.D_{23}) - \# \text{ of 2nd part} - \# \text{ of 3rd part}\}/3! \\ &= (3296 - 28 \times 2 \times 9 \times 3 - 28 \times 2)/6 = 288 \end{aligned} \tag{2}$$

counted with multiplicities.

By adjunction the divisor class h is the canonical class K_C of a plane quartic C. Hence if a line tangents to C at two points a and r, then the divisor $a + r$ is a theta characteristic. This is an odd theta characteristic since C is not hyperelliptic. Generally a divisor class η with $2\eta \sim K_C$ is called a *theta characteristic* of a curve C. Their cardinality is equal to 4^g, where g is the genus of C. A theta characteristic η is called *even* or *odd* according as the parity of $h^0(\eta)$, the dimension of the vector space $\{f \in \mathbf{C}(C) \mid (f) + \eta \ge 0\}$. The following is well-known:

Proposition *The number of even (resp. odd) theta characteristics is* $2^{g-1}(2^g+1)$ *(resp.* $2^{g-1}(2^g-1)$*).*

Since a plane quartic curve C is of genus 3, these numbers are equal to 36 and 28, respectively.

3 Second computation

Let $\triangle = \triangle abc$ be a triangle strictly biscribed by a quartic C and p, q, r as in Figure 3. We give another computation using the tangent points p, q, r of $\triangle$ instead of vertices and using even theta characteristics. For $\triangle$, we denote the divisor class

$$a + b + c + p + q + r - h \in \operatorname{Pic}^2 C$$

of degree 2 by $\eta(\triangle)$. Then $\eta(\triangle)$ is a theta characteristic and we have

$$\eta(\triangle) \sim a - p + q + r \sim b + p - q + r \sim c + p + q - r \tag{3}$$

by (1). It is easy to see that $\eta(\triangle)$ is *ineffective*, that is, $h^0(\eta(\triangle)) = 0$. In particular, $\eta(\triangle)$ is an even theta characteristic.

Proposition *For an even theta characteristic* η *of a plane quartic* C*, the number of strictly biscribed triangles* $\triangle$ *with* $\eta(\triangle) \simeq \eta$ *is equal to* 8 *(counted with multiplicities).*

Proof. For a pair of points $p, q \in C$, we denote $p \overset{\eta}{\cap} q$ if $h^0(\eta - p + q) \neq 0$. By the Riemann-Roch theorem, $p \overset{\eta}{\cap} q$ implies $q \overset{\eta}{\cap} p$. Hence $\overset{\eta}{\cap}$ defines a symmetric divisor in $C \times C$, which we denote by

$$E(\eta) = \{(p, q) \,|\, p \overset{\eta}{\cap} q\} \tag{4}$$

and call the *incidence relation* induced by η. This divisor is linearly equivalent to $p_1^*\eta + p_2^*\eta + \Delta$. Three tangent lines at p, q, r form a strictly biscribed triangle if and only if $(p, q, r) \in C \times C \times C$ belongs to the intersection $E_{12} \cap E_{13} \cap E_{23}$, where $E_{ij}, 1 \leq i < j \leq 3$, are the pull-backs of $E(\eta)$. Hence the number is equal to

$$(E_{12}.E_{13}.E_{23})/3! = (p_1^*\eta + p_2^*\eta + \Delta_{12}.p_1^*\eta + p_3^*\eta + \Delta_{13}.p_2^*\eta + p_3^*\eta + \Delta_{23})/6$$

$$= (-4 + 3 \times 4 + 3 \times 8 + 16)/6 = 8. \qquad \square$$

Since the number of even theta characteristics is equal to 36, the number of strictly biscribed triangle is equal to $36 \times 8 = 288$, which agrees with (2).

4 Fano threefolds (of genus twelve)

A compact complex manifold X is called *Fano* if the anticanonical class $-K_X \in \operatorname{Pic} X$ is ample, or equivalently, the first Chern class $c_1(X) \in H^2(X, \mathbf{Z})$ is positive. The most typical example is the projective space $\mathbf{P}^n$. Its anticanonical class is $n+1$ times the hyperplane class. The projective line $\mathbf{P}^1$ is the unique Fano manifold in dimension one. In dimension two, a Fano manifold is called also a del Pezzo surface. There are 10 deformation types of them and the projective plane $\mathbf{P}^2$ is characterized among them by the property that $B_2 = 1$, where B_2 is the second Betti number. For Fano manifolds the Picard group is torsion free and the Chern class map $\operatorname{Pic} X \longrightarrow H^2(X, \mathbf{Z})$ is an isomorphism. In particular, B_2 is equal to the Picard number.

In dimension three, there are 105 deformation types of Fano threefolds ([3], [5]). The property $B_2 = 1$ does not characterize the projective space any more. In fact there are 17 deformation types of Fano threefolds with $B_2 = 1$ and some are even irrational. Some of the readers may ask how the additional topological property $B_3 = 0$ is. The third Betti number B_3 is very important invariant and equal to zero for the projective space $\mathbf{P}^3$. But even this additional property does not characterize $\mathbf{P}^3$. There are four types of Fano threefolds with $B_2 = 1$ and $B_3 = 0$:

$$\mathbf{P}^3, \quad Q^3 \subset \mathbf{P}^4, \quad V_5 \subset \mathbf{P}^6 \quad \text{and} \quad V_{22} \subset \mathbf{P}^{13}. \tag{5}$$

All are rational and the first three are easy to describe: $Q^3 \subset \mathbf{P}^4$ is a hyperquadric and $V_5 \subset \mathbf{P}^6$ is a quintic del Pezzo threefold. A quick description of V_5 is the intersection of the 6-dimensional Grassmannian $G(2,5) \subset \mathbf{P}^9$ with a transversal linear subspace of codimension three ([2]). The final one is not very easy to describe. It was very mysterious at least for me in early 80's. I carried out the computation of §2 in order to understand this V_{22}.

Let h be an ample generator of $\operatorname{Pic} X \simeq \mathbf{Z}$ for a Fano threefold X with $B_2 = 1$. The positive integer r defined by $-K_X = rh$ is called the (Fano) *index* of X. This measures a certain complexity of a Fano manifold: when r (or more precisely the nonpositive integer $r - n - 1$) becomes smaller and smaller a Fano manifold becomes more and more complicated. The indices of four Fano threefolds in (5) are equal to 4, 3, 2 and 1, respectively. What is new and makes a classification of Fano manifolds hard in dimension three and higher is the appearance of those with $B_2 = r = 1$. Such Fano manifolds are called *prime*. Their Picard groups are generated by $-K_X$. In dimension three, Iskovskih[3] classified prime Fano threefolds into 10 deformation types, which are distinguished by the *degree*:

$$(-K_X)^3 = 2, 4, 6, 8, 10, 12, 14, 16, 18 \text{ or } 22. \tag{6}$$

The V_{22} is the prime Fano threefolds with the largest degree. It is embedded into $\mathbf{P}^{13}$ by the anticanonical linear system $|h| = |-K_X|$. The degree $(-K_X)^3$ is always even and the integer $g := \frac{1}{2}(-K_X)^3 + 1 \geq 2$, called the *genus*, is more convenient for a Fano threefold. For example, the above (6) is equivalent to $g = 2, 3, 4, 5, 6, 7, 8, 9, 10$ or 12.

A Fano threefold was initiated by G. Fano in connection with the Lüroth problem: is a unirational variety rational? But it is interesting in many other ways including moduli. V_{22} is not interesting from the Lüroth view point since it is rational. But but it is the most interesting Fano threefold from moduli view point since it has continuous moduli in spite of its trivial period mapping, or trivial intermediate Jacobian. [1] For a Fano threefold X the (virtual) number of moduli is given by the formula

$$\mu := h^1(T_X) - h^0(T_X) = 19 - B_2 - g + \frac{1}{2}B_3$$

by virtue of the vanishing of Akizuki and Nakano: $H^2(\Omega_X^2(-K_X)) = H^3(\Omega_X^2(-K_X)) = 0$. This number is equal to $-15, -10, -3$ and 6 for the Fano threefolds in (5). The first three are (locally) *rigid*, that is, $H^1(T_X) = 0$ and $-\mu$ is the dimension of their automorphism groups, $PGL(4)$, $PSO(5)$ and $PGL(2)$. But surprisingly the final one V_{22} has a 6-dimensional family of deformations. Around 1982, the following was known on this variety:

1. (Shokurov[12]) $V_{22} \subset \mathbf{P}^{13}$ contains a line l.

2. (Iskovskih[3]) The double projection

$$\Phi_{|h-2l|} : V_{22} \cdots \longrightarrow \mathbf{P}^6$$

   from a line l is birational onto a smooth quintic del Pezzo threefold V_5.

3. (M.-Umemura[9]) There is a special V_{22}, denoted by U_{22}, which has an almost homogeneous action of $PGL(2)$. U_{22} is the closure of a certain $PGL(2)$-orbit in $\mathbf{P}^{12}$.

These are very analogous to the following facts for V_5:

1. $V_5 \subset \mathbf{P}^6$ contains a line l.

2. The (single) projection $\Phi_{|h-l|} : V_5 \cdots \longrightarrow \mathbf{P}^4$ from a line l is birational onto a smooth quadric Q^3 and induces an isomorphism between the blow-up of V_5 along l and that of $Q^3 \subset \mathbf{P}^4$ along a twisted cubic.

3. V_5 is the closure of a certain $PGL(2)$-orbit in $\mathbf{P}^6$.

[1] The study of Brill-Noether loci of vector bundles on a curve in [8] has its origin in the analysis of the fiber of the period map of a prime Fano threefold of genus $7 \leq g \leq 10$.

But there are two differences:

i) the birational map between V_{22} and V_5 are more complicated than that between V_5 and Q. The double projection induces a *strong* rational map between the blow-up of V_{22} along l and the blow-up of V_5 along a quintic rational curve, where strong means an isomorphism in codimension one. But the map is not an isomorphism.

ii) V_5 is rigid but U_{22} has 6-dimensional deformations.

Using the results $1 \sim 3$ for V_{22}, the author was able to prove the following:

Theorem (1982, unpublished) *The moduli space of V_{22}'s is birationally equivalent to that of the pairs (C, η) of a plane quartic C and an even theta characteristic η of C.*

The correspondence between V_{22} and (C, η) and the outline of the proof are as follows:

1. The Hilbert scheme $\mathcal{L}(V_{22})$ of lines on $V_{22} \subset \mathbf{P}^{13}$ is a plane quartic. $\mathcal{L}(V_{22})$ is smooth for a general V_{22}.

2. The incidence relation

$$\{(l, l') \mid l \cap l' \neq \emptyset\} \subset \mathcal{L}(V_{22}) \times \mathcal{L}(V_{22})$$

   of lines is equal to the incidence relation $E(\eta)$ induced from an even theta characteristic η of $\mathcal{L}(V_{22})$ if V_{22} is general.

3. A general V_{22} is reconstructed from the pair $(\mathcal{L}(V_{22}), \eta)$ using the description of the inverse of the double projection $\Phi_{|h-2l|}$.

A strictly biscribed triangle of $(\mathcal{L}(V_{22}), \eta)$ corresponds to a *trilinear point* of V_{22}, that is, a point where three lines pass through. By the computation of the previous section, we have

Proposition *The number of trilinear point of a general V_{22} is equal to* 8.

Remark The normal bundle of a line $l \subset V_{22}$ is isomorphic to either $\mathcal{O} \oplus \mathcal{O}(-1)$ or $\mathcal{O}(1) \oplus \mathcal{O}(-2)$. $\mathcal{L}(V_{22})$ is smooth at the point $[l]$ if and only if the former holds. In general $\mathcal{L}(V_{22})$ is not smooth. For example, $\mathcal{L}(U_{22})$ is a double conic in $\mathbf{P}^2$

5 Covariant quartics

Not only a Fano threefold V_{22} but also a plane quartic itself produces a pair (C, η) of a quartic C and an even theta η. The reference of this section is [4].

Let

$$D : F_3(x, y, z) = \sum_{i+j+k=3} a_{ijk} x^i y^j z^k = 0$$

be a plane cubic. The ring of invariants of ternary cubics, that is, that of the natural action of $SL(3)$ on the polynomial ring $\mathbf{C}[a_{300}, \dots, a_{003}]$ of 10 coordinates of F_3 is generated by two homogeneous polynomials S and T, which are of degree 4 and 6, respectively. A cubic D has a cusp or a worse singularity if and only if $S = T = 0$. D is of *Fermat* type, *i.e.*, projectively equivalent to $x^3 + y^3 + z^3 = 0$, if and only if $S = 0$ and $T \neq 0$.

Now let $\Gamma : F_4(x, y, z) = 0 \subset \mathbf{P}^2$ be a plane quartic. For a point $p \in \mathbf{P}^2$ with homogeneous coordinate $(a : b : c)$, we consider the cubic

$$\Gamma_p : a\frac{\partial F_4}{\partial x} + b\frac{\partial F_4}{\partial y} + c\frac{\partial F_4}{\partial z} = 0$$

defined by a linear combination of partials. Then Γ_p does not depend on the choice of a system of homogeneous coordinates and is called the (first) *polar* of Γ at p. So a quartic Γ produces a (linear) family $\{\Gamma_p\}$ of plane cubics parameterized by $\mathbf{P}^2$.

Since the invariant S is quartic, so is the curve

$$C = \{p \in \mathbf{P}^2 \mid S(\Gamma_p) = 0\},$$

which is the locus of points p at which the polar Γ_p is Fermat in rough but usual expression. This curve C is called the *covariant quartic* of Γ. The following is easy to prove but a crucial observation:

Lemma *Assume that Γ_p, $p \in C$, is Fermat, that is, $\Gamma_p : l_1^3 + l_2^3 + l_3^3 = 0$ for linearly independent three liner forms l_1, l_2 and l_3. Then the three vertices of the triangle $l_1 l_2 l_3 = 0$ also lie on the covariant quartic C.*

This gives a self correspondence of C, that is, a curve E in $C \times C$. If Γ is general, then C is smooth and E is the incidence relation $E(\eta)$ induced by an even theta characteristic η of C. By Scorza[11], a general Γ is reconstructed from the pair (C, η). (See also Dolgachev-Kanev[1].) Therefore, by the theorem in the previous section, there is a birational map between the moduli space of V_{22}'s and that of plane quartics. In 80's the author sought a direct construction of V_{22} from a plane quartic and reached to the following:

Theorem([6]§3) *The variety of sums of powers $VSP(6, F_4)$, which is the closure of*

$$\{([l_1], \dots, [l_6]) \mid F_4 \in \langle l_1^4, \dots, l_6^4 \rangle_{\mathbf{C}}\} \subset [(\mathbf{P}^{2,\vee})^6 - \text{diagonals}]/S_6$$

in the Hilbert scheme of six points in the dual projective plane $\mathbf{P}^{2,\vee}$, is a prime Fano threefold of genus twelve for a general ternary quartic form $F_4 = F_4(x, y, z)$. Conversely, every smooth Fano threefold V_{22} is isomorphic to $VSP(6, F_4)$ for a ternary quartic form F_4.

The proof, which involves a development of vector bundle technique (cf. [7]), will be given elsewhere. The eight strictly biscribed triangles of the covariant quartic C correspond to the expression of $F_4(x, y, z)$ in the special form

$$ax^4 + by^4 + cz^4 + d(y-z)^4 + e(z-x)^4 + f(x-\alpha x)^4$$

for constants $a, b, \ldots, f$ and $\alpha \in \mathbf{C}$ in eight ways (up to projective equivalence).

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