Generalized permutative representations of the Cuntz algebras. III
—Generalization of chain type—

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We introduce a generalization of permutative representations of the Cuntz algebras with chain. We show their existence, uniqueness, irreducibility, equivalence, decomposition and states associated with them.

1. Introduction

In our previous papers([**10**, **11**]), we introduce a class of representations of the Cuntz algebra $\mathcal{O}_N$ which is a generalization of permutative representations with *cycle* by [**5**, **7**, **8**]. As application, we have results about quantum field theory([**1**, **2**, **3**, **4**]), fractal sets([**14**]) and dynamical systems([**13**, **15**, **16**, **17**]). We continue to treat *chain* case in this paper. The remarkable results are that this class of representations is completely reducible. Furthermore the decomposition formula is possible to describe explicitly.

In § 2, we prepare several notions and symbols. In § 3, we define a generalized permutative representation of $\mathcal{O}_N$ with chain, review results in [**10**, **11**] and show its existence. In § 4, the construction of the canonical basis of representation is shown. In § 5, we show the condition of uniqueness, irreducibility and equivalence of representations and describe decomposition formula. In § 6, we show states and spectrums of $\mathcal{O}_N$ associated with generalized permutative representations. In § 7, we introduce several examples of them and applications in [**17**]. For example, the following application about states of $\mathcal{O}_2$ is obtained:

Theorem 1.1. *For $\xi \in U(1) \equiv \{c \in \mathbf{C} : |c| = 1\}$, put a state ω_ξ of $\mathcal{O}_2$ by the following conditions:*

$$\omega_\xi(s_{i_1}\cdots s_{i_k}) = \omega_\xi(s_{j_l}^*\cdots s_{j_1}^*) = 0,$$

$$\omega_\xi(s_{i_1}\cdots s_{i_k}s_{j_l}^*\cdots s_{j_1}^*) = \delta_{k,l}\,\xi^{j_1-i_1}\xi^{2(j_2-i_2)}\cdots\xi^{k(j_k-i_k)}/2^k$$

for each $i_1,\ldots,i_k,\ j_1,\ldots,j_l = 1,2,\ k,l \geq 1$. Then the followings hold:

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(i) *ω_ξ is pure if and only if ξ is not a root of unity.*

(ii) *For $\xi, \xi^{'} \in U(1)$, GNS representations of $\mathcal{O}_2$ by ω_ξ and $\omega_{\xi^{'}}$ are equivalent if and only if $\xi = \xi^{'}$.*

(iii) *If ξ is a root of unity, then there is a family $\{(\mathcal{K}_w, \pi_w)\}_{w \in U(1)}$ of mutually inequivalent irreducible representations of $\mathcal{O}_2$ such that the GNS representation of $\mathcal{O}_2$ by ω_ξ is equivalent to*

$$\int_{U(1)}^{\oplus} (\mathcal{K}_w, \pi_w)\, d\eta(w)$$

where η is the Haar measure of $U(1)$.

2. Preparation

In order to define a generalized permutative representation, we prepare the parameter space of representations. Fix $N \geq 2$.

We introduce several sets of multi indices which consist of numbers $1, \ldots, N$.

Put $\{1, \ldots, N\}^* \equiv \bigcup_{k \geq 0} \{1, \ldots, N\}^k$, $\{1, \ldots, N\}^0 \equiv \{0\}$, $\{1, \ldots, N\}^k \equiv \{(j_l)_{l=1}^k : j_l = 1, \ldots, N,\ l = 1, \ldots, k\}$ for $k \geq 1$. For $J \in \{1, \ldots, N\}^*$, the length $|J|$ of J is defined by $|J| \equiv k$ when $J \in \{1, \ldots, N\}^k$, $k \geq 0$. For $J_1, J_2 \in \{1, \ldots, N\}^*$, $J_1 \cup J_2 \equiv (j_1, \ldots, j_k, j_1^{'}, \ldots, j_l^{'})$ when $J_1 = (j_1, \ldots, j_k)$ and $J_2 = (j_1^{'}, \ldots, j_l^{'})$. Specially, we define $J \cup \{0\} = \{0\} \cup J = J$ for $J \in \{1, \ldots, N\}^*$ for convention. For $J_1, J_2 \in \{1, \ldots, N\}^*$, we denote $J_1 = * \cup J_2$ (*resp.* $J_1 = J_2 \cup *$) if there is $J_3 \in \{1, \ldots, N\}^*$ such that $J_1 = J_3 \cup J_2$ (*resp.* $J_1 = J_2 \cup J_3$). For $J \in \{1, \ldots, N\}^*$, $J^n = \underbrace{J \cup \cdots \cup J}_{n}$ for $n \geq 1$.

Next we define sets of infinite sequences of numbers $1, \ldots, N$.

Put $\{1, \ldots, N\}^\infty \equiv \{(i_n)_{n \in \mathbf{N}} : i_n \in \{1, \ldots, N\},\ n \in \mathbf{N}\}$ where $\mathbf{N} \equiv \{1, 2, 3, \ldots\}$. For $J \in \{1, \ldots, N\}^*$, $J^\infty \equiv J \cup J \cup \cdots \cup J \cup \cdots \in \{1, \ldots, N\}^\infty$. For $J_1, \ldots, J_k \in \{1, \ldots, N\}^*$,

$$\begin{aligned}(J_1^* \cdots J_k^*) &\equiv \bigcup_{n \geq 1} (J_1^n \cup \cdots \cup J_k^n) \\ &= J_1 \cup \cdots \cup J_k \cup J_1^2 \cup \cdots \cup J_k^2 \cup J_1^3 \cup \cdots \cup J_k^3 \cup \cdots .\end{aligned} \tag{2.1}$$

For example $(1)^\infty = (111\cdots), (1^*2^*) = (121122111222\cdots) \in \{1, \ldots, N\}^\infty$.

We introduce a continuous generalization of discrete parameters.

Denote $S(\mathbf{C}^N) \equiv \{z \in \mathbf{C}^N : \|z\| = 1\}$ is the unit complex sphere in $\mathbf{C}^N$. Put a set of sequences

$$S(\mathbf{C}^N)^\infty \equiv \{(z^{(n)})_{n \in \mathbf{N}} : z^{(n)} \in S(\mathbf{C}^N),\ n \in \mathbf{N}\}.$$

Furthermore, put

$$S(\mathbf{C}^N)^{\otimes k} \equiv \{z^{(1)}\otimes\cdots\otimes z^{(k)} \in (\mathbf{C}^N)^{\otimes k} : z^{(j)} \in S(\mathbf{C}^N),\, j=1,\ldots,k\} \quad (k \geq 1),$$

$$TS(\mathbf{C}^N) \equiv \coprod_{k\geq 1} S(\mathbf{C}^N)^{\otimes k}$$

where $(\mathbf{C}^N)^{\otimes k} \equiv \underbrace{\mathbf{C}^N \otimes \cdots \otimes \mathbf{C}^N}_{k}$ for $k \geq 1$. Note that $TS(\mathbf{C}^N)$ is a semigroup with respect to tensor product. We denote $z = (z^{(n)})_{n\in\mathbf{N}} \in S(\mathbf{C}^N)^{\infty}$, $z^{(n)} \in S(\mathbf{C}^N)$ for $n \in \mathbf{N}$. For $z = (z^{(n)}) \in S(\mathbf{C}^N)^{\infty}$, denote

$$z[k] \equiv z^{(1)} \otimes \cdots \otimes z^{(k)} \quad (k \geq 1). \tag{2.2}$$

Note $z[k] \in S(\mathbf{C}^N)^{\otimes k}$ for $k \geq 1$. Let $\{\varepsilon_1, \ldots, \varepsilon_N\}$ be the canonical basis of $\mathbf{C}^N$, that is, $z = z_1\varepsilon_1 + \cdots + z_N\varepsilon_N$ for $z = (z_1, \ldots, z_N) \in \mathbf{C}^N$. We denote $\varepsilon_J \equiv \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k}$ for $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, $k \geq 1$. Then $\varepsilon_{J_1} \otimes \varepsilon_{J_2} = \varepsilon_{J_1 \cup J_2}$ for each $J_1, J_2 \in \{1, \ldots, N\}^* \setminus \{0\}$. Clearly $\varepsilon_j \in S(\mathbf{C}^N)$ and $\varepsilon_J \in TS(\mathbf{C}^N)$ for $j = 1, \ldots, N$ and $J \in \{1, \ldots, N\}^* \setminus \{0\}$. For $J = (j_n)_{n\in\mathbf{N}} \in \{1, \ldots, N\}^{\infty}$, put $\varepsilon_J \equiv (\varepsilon_{j_n})_{n\in\mathbf{N}} \in S(\mathbf{C}^N)^{\infty}$.

Definition 2.1. *(Parameter of cycle)*

(i) *$z \in S(\mathbf{C}^N)^{\otimes k}$ is periodic if there is $\tau \in \mathbf{Z}_k \setminus \{id\}$ such that $\hat{\tau}(z) = z$ where $\hat{\cdot}$ is an action of the cyclic group $\mathbf{Z}_k$ on $(\mathbf{C}^N)^{\otimes k}$ by transposition of tensor factors. In this case, p is the period of z if p is the rank of of τ which is minimal among $\mathbf{Z}_k$.*

(ii) *$z \in S(\mathbf{C}^N)^{\otimes k}$ is non periodic if z is not periodic.*

(iii) *For $z, z^{'} \in TS(\mathbf{C}^N)$, $z \sim z^{'}$ if there are $k \geq 1$ and $\tau \in \mathbf{Z}_k$ such that $z, z^{'} \in S(\mathbf{C}^N)^{\otimes k}$ and $\hat{\tau}(z) = z^{'}$.*

Definition 2.2. *(Parameter of chain)*

(i) *For $z = (z^{(n)})$ and $y = (y^{(n)})$ in $S(\mathbf{C}^N)^{\infty}$, $z \sim y$ if there are non negative integers L and M, and a sequence $\{c_n\}_{n\geq 0}$ of complex numbers with absolute value 1 such that $y^{(n+L-1)} = c_n z^{(n)}$ for each $n \geq M$.*

(ii) *$z = (z^{(n)}) \in S(\mathbf{C}^N)^{\infty}$ is eventually periodic if there are positive integers p, M and a sequence $(c_n)_{n\geq M}$ in $U(1)$ such that $z^{(n+p)} = c_n z^{(n)}$ for any $n \geq M$. In this case, p is called the period of z if p is the minimal number which satisfies the above condition.*

(iii) *$z \in S(\mathbf{C}^N)^{\infty}$ is non eventually periodic if z is not eventually periodic.*

(iv) *For $y \in S(\mathbf{C}^N)^{\otimes n}$, $n \geq 1$, $y^{(1)}, \ldots, y^{(n)} \in S(\mathbf{C}^N)$ are the standard tensor components of y if $y^{(1)}, \ldots, y^{(n)}$ satisfy the following conditions: $y = y^{(1)}\otimes\cdots\otimes y^{(n)}$ and $y^{(j)}_{l_j} > 0$ for each $j = 1, \ldots, k$ when $l_j \equiv \min\{l \in \{1, \ldots, N\} : y^{(j)}_l \neq 0\}$ where $y^{(i)} = (y^{(i)}_1, \ldots, y^{(i)}_N)$ for each $i = 1, \ldots, n$.*

(v) *For* $y \in S(\mathbf{C}^N)^{\otimes p}$, $y^\infty \in S(\mathbf{C}^N)^\infty$ *is defined by* $z = (z^{(n)}) \in S(\mathbf{C}^N)^\infty$, $z^{(p(n-1)+i)} \equiv y^{(i)}$ *for each* $n \geq 1$ *and* $i = 1, \ldots, p$, *that is,*

$$y^\infty = (y^{(1)}, \ldots, y^{(p)}, y^{(1)}, \ldots, y^{(p)}, \ldots)$$

where $y^{(1)}, \ldots, y^{(p)}$ *are the standard tensor components of* y.

(vi) σ *is the shift on* $S(\mathbf{C}^N)^\infty$ *if* σ *is a transformation on* $S(\mathbf{C}^N)^\infty$ *which is defined by for* $z = (z^{(n)}) \in S(\mathbf{C}^N)^\infty$, $(y^{(n)}) = \sigma(z)$ *where* $y^{(n)} = z^{(n+1)}$ *for each* $n \geq 1$.

For example, for $J \equiv (1^*2^*)$ in (2.1), $\varepsilon_J \in S(\mathbf{C}^2)^\infty$ is non eventually periodic. Relations $\sim$ in Definition 2.2 (i) and Definition 2.1 (iii) are equivalence relations. The notion of eventually periodic is taken from theory of dynamical systems([**9**]). When $z \sim y$, we call that z and y are equivalent. These equivalences are corresponded to the notion of "tail equivalence" of permutative representation in [**5**].

The generalization of parameter space is corresponded to generalization of a class of representations of $\mathcal{O}_N$. Remark that a case $M = 0$ is possible in Definition 2.2 (i) but $p \geq 1$ in Definition 2.2 (ii).

3. Definition and existence of generalized permutative representations with chain

For $N \geq 2$, let $\mathcal{O}_N$ be the Cuntz algebra([**6**]), that is, it is a C*-algebra which is universally generated by generators $s_1, \ldots, s_N$ satisfying

$$s_i^* s_j = \delta_{ij} I \quad (i, j = 1, \ldots, N), \quad s_1 s_1^* + \cdots + s_N s_N^* = I. \tag{3.1}$$

In this paper, any representation means a unital $*$-representation. By simplicity and uniqueness of $\mathcal{O}_N$, it is sufficient to define operators $S_1, \ldots, S_N$ on an infinite dimensional Hilbert space which satisfy (3.1) in order to construct a representation of $\mathcal{O}_N$. Put α an action of a unitary group $U(N)$ on $\mathcal{O}_N$ defined by $\alpha_g(s_i) \equiv \sum_{j=1}^N g_{ji} s_j$ for $i = 1, \ldots, N$. Specially we denote $\gamma_w \equiv \alpha_{g(w)}$ when $g(w) = w \cdot I \subset U(N)$ for $w \in U(1) \equiv \{z \in \mathbf{C} : |z| = 1\}$.

3.1. Definition. We give the definition of generalized permutative representations with chain here by using parameters in § 2. In order to show decomposition theorem of them in § 5.2, we review the definition and properties of generalized permutative representations with cycle([**10**, **11**]), too.

For $z = (z_1, \ldots, z_N) \in S(\mathbf{C}^N)$, denote

$$s(z) \equiv z_1 s_1 + \cdots + z_N s_N.$$

For $z = z^{(1)} \otimes \cdots \otimes z^{(k)} \in S(\mathbf{C}^N)^{\otimes k}$,

$$s(z) \equiv s(z^{(1)}) \cdots s(z^{(k)}), \quad s(z)^* \equiv s(z^{(k)})^* \cdots s(z^{(1)})^*.$$

Definition 3.1. *Let* $(\mathcal{H}, \pi)$ *be a representation of* $\mathcal{O}_N$.

(i) *For $z \in TS(\mathbf{C}^N)$, a unit vector $\Omega \in \mathcal{H}$ satisfies the cycle condition with respect to z if $\pi(s(z))\Omega = \Omega$.*
(ii) *For $z \in S(\mathbf{C}^N)^\infty$, a unit vector $\Omega \in \mathcal{H}$ satisfies the chain condition with respect to z if $\mathcal{R}(z) \equiv \{\pi(s(z[n]))^*\Omega : n \geq 1\}$ is an orthonormal family in $\mathcal{H}$. $\mathcal{R}(z)$ is called the chain of Ω by z.*

Definition 3.2. (i) *For $z \in TS(\mathbf{C}^N)$, $(\mathcal{H}, \pi, \Omega)$ is a generalized permutative(=GP) representation of $\mathcal{O}_N$ with cycle by z if $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_N$ with a unit cyclic vector $\Omega \in \mathcal{H}$ which satisfies the cycle condition with respect to z. p is the period of $(\mathcal{H}, \pi, \Omega)$ if z has the period p.*
(ii) *For $z \in S(\mathbf{C}^N)^\infty$, $(\mathcal{H}, \pi, \Omega)$ is a GP representation of $\mathcal{O}_N$ with chain by z if $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_N$ with a unit cyclic vector $\Omega \in \mathcal{H}$ which satisfies the chain condition with respect to z. p is the period of $(\mathcal{H}, \pi, \Omega)$ if z has the period p.*

We call Ω in Definition 3.2 both (i) and (ii) the *GP vector* of a GP representation. We denote $GP(z) = (\mathcal{H}, \pi, \Omega)$ for (i), (ii) simply. We explain meanings of cycle and chain in Example 3.4. We do not assume the completeness of the set $\{\pi(s(z[n]))^*\Omega : n \geq 1\}$ in Definition 3.1 (ii). For two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of $\mathcal{O}_N$, $(\mathcal{H}_1, \pi_1) \sim (\mathcal{H}_2, \pi_2)$ means the unitary equivalence between $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$. Specially, $GP(z) \sim GP(z^{'})$ means that two cyclic representations of $\mathcal{O}_N$ are unitarily equivalent.

We review results about GP representation with cycle.

Theorem 3.3. *([**10**])*
(i) *(Existence) For any $z \in TS(\mathbf{C}^N)$, there exists $GP(z)$, that is, there exists a cyclic representation $(\mathcal{H}, \pi, \Omega)$ of $\mathcal{O}_N$ which satisfies the cycle condition with respect to z.*
(ii) *(Uniqueness and irreducibility) If $z \in TS(\mathbf{C}^N)$ is non periodic, then $GP(z)$ is unique up to unitary equivalences, and irreducible.*
(iii) *(Equivalence) For non periodic elements $z, z^{'} \in TS(\mathbf{C}^N)$, $GP(z) \sim GP(z^{'})$ if and only if $z \sim z^{'}$.*

Proof. (i) Proposition 3.4 in [**10**]. (ii) The uniqueness is in Proposition 5.4 in [**10**]. The irreducibility is in Proposition 5.5 in [**10**]. (iii) Proposition 5.11 in [**10**]. □

By Theorem 3.3 (ii), we can regard a symbol $GP(z)$ as the representative element of an equivalence class of irreducible representations of $\mathcal{O}_N$ which satisfies the cycle condition with respect to non periodic $z \in TS(\mathbf{C}^N)$. From this, we see that the statement (iii) has no ambiguity. Note that our results in Theorem 3.3 (ii), (iii) are assumed the non-periodicity with respect to a parameter $z \in TS(\mathbf{C}^N)$. About decomposition of periodic cycle, see [**11**].

We show examples of them here.

Example 3.4. (i) The *standard representation* $(l_2(\mathbf{N}), \pi_S)$ of $\mathcal{O}_N$ is defined by $\pi_S(s_i)e_n \equiv e_{N(n-1)+i}$ for $n \in \mathbf{N}$, $i = 1, \ldots, N$ where $\{e_n\}_{n\in\mathbf{N}}$ is the canonical basis of $l_2(\mathbf{N})$([**1**, **14**]). Then $(l_2(\mathbf{N}), \pi_S, e_1)$ satisfies the condition of $GP((1, 0, \ldots, 0))$. Because $(1, 0, \ldots, 0) \in S(\mathbf{C}^N)$ is non periodic, $(l_2(\mathbf{N}), \pi_S)$ is irreducible.

(ii) The *barycentric representation* $(L_2[0, 1], \pi_B)$ of $\mathcal{O}_N$ is defined by

$$(\pi_B(s_i)\phi)(x) \equiv \chi_{[(i-1)/N, i/N]}(x)\phi(Nx - i + 1)$$

for $\phi \in L_2[0, 1]$, $x \in [0, 1]$ and $i = 1, \ldots, N$ where χ_Y is the characteristic function of a subset Y of $[0, 1]$([**15**]). Then $(L_2[0, 1], \pi_B, \Omega)$ is $GP((N^{-1/2}, \ldots, N^{-1/2}))$ where Ω is the constant function on $[0, 1]$ with value 1. $(L_2[0, 1], \pi_B)$ is irreducible, too.

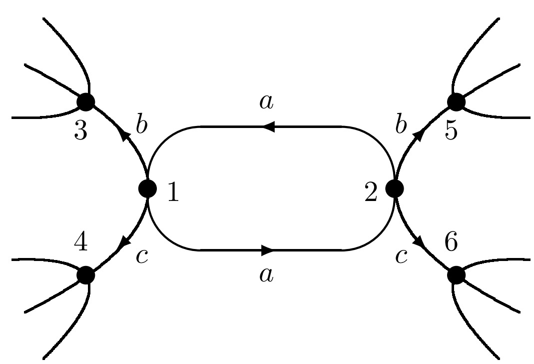
(iii) In (ii), $(L_2[0, 1], \pi_B \circ \gamma_{\bar{w}}, \Omega)$ is $GP((w/N^{1/2}, \ldots, w/N^{1/2}))$ for $w \in U(1)$.

(iv) Define a representation $(l_2(\mathbf{N}), \pi)$ of $\mathcal{O}_3$ by

$$\pi(s_1)e_1 \equiv e_2, \quad \pi(s_1)e_2 \equiv e_1, \quad \pi(s_2)e_1 \equiv e_3, \quad \pi(s_3)e_1 \equiv e_4,$$

$$\pi(s_2)e_2 \equiv e_5, \quad \pi(s_3)e_2 \equiv e_6, \quad \pi(s_i)e_n \equiv e_{3(n-1)+i} \quad (i = 1, 2, 3,\ n \geq 3).$$

Then $(l_2(\mathbf{N}), \pi)$ is cyclic with cyclic vector e_1 and $\pi(s_1s_1)e_1 = e_1$. Therefore $(l_2(\mathbf{N}), \pi, e_1)$ is $GP((1, 0, 0) \otimes (1, 0, 0))$. The tree of the representation $(l_2(\mathbf{N}), \pi)$ is following:



where vertices and edges mean the canonical basis $\{e_x\}_{x\in\mathbf{N}}$ of $l_2(\mathbf{N})$ and the action of operators $\pi(s_1), \pi(s_2), \pi(s_3)$ on $\{e_x\}_{x\in\mathbf{N}}$, respectively. For example, if $\pi(s_1)e_x = e_y$ for $x, y \in \mathbf{N}$, then it is represented as

$$x \bullet \xrightarrow{\ a\ } \bullet\ y$$

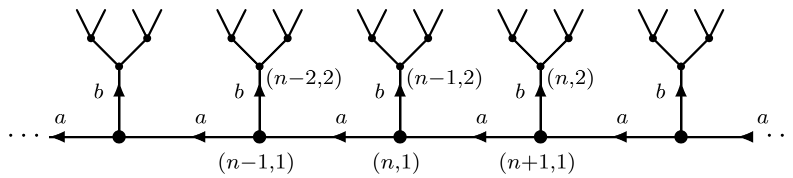
where labels a, b,c of edges correspond to $\pi(s_1), \pi(s_2), \pi(s_3)$, respectively. Since $(1, 0, 0) \otimes (1, 0, 0) \in S(\mathbf{C}^3)^{\otimes 2}$ is periodic, $(l_2(\mathbf{N}), \pi)$ is not irreducible.

(v) Put $R_i \equiv \mathbf{Z} \times \mathbf{N}_i, \quad \mathbf{N}_i \equiv \{N(n - 1) + i : n \in \mathbf{N}\}$ for $i = 1, \ldots, N$. Then we have a decomposition $\mathbf{Z} \times \mathbf{N} = R_1 \sqcup \cdots \sqcup R_N$. Consider a branching function system $f \equiv \{f_i\}_{i=1}^N$ on $\mathbf{Z} \times \mathbf{N}$ defined by

$$f_i : \mathbf{Z} \times \mathbf{N} \to R_i; \quad f_i(n,m) \equiv (n-1,\, N(m-1)+i) \tag{3.2}$$

for $i = 1,\ldots,N$. Then $f_1(n,1) = (n-1,1)$ for each $n \in \mathbf{Z}$. From this, we have $f_1^k(n,1) = (n-k,1)$ for $k \geq 1$ and $n \in \mathbf{Z}$. Put a representation $(l_2(\mathbf{Z} \times \mathbf{N}), \pi_f)$ of $\mathcal{O}_N$ by $\pi_f(s_i)e_x \equiv e_{f_i(x)}$ for $x \in \mathbf{Z} \times \mathbf{N}$, $i = 1,\ldots,N$. From this, we have $\pi_f(s_1^*)e_{n,1} = e_{n+1,1}$ for $n \in \mathbf{Z}$. Hence $\{\pi_f((s_1^*)^n)e_{0,1} : n \in \mathbf{N}\} = \{e_{n,1} : n \in \mathbf{N}\}$ is an orthonormal family.

When $N = 2$, the *tree* of the representation $(l_2(\mathbf{Z} \times \mathbf{N}), \pi_f)$ is following:



$(l_2(\mathbf{Z} \times \mathbf{N}), \pi_f, e_{0,1})$ is $GP((1,0,\ldots,0)^\infty)$ of $\mathcal{O}_N$.

It is easy to show that cyclicities and eigen equations in Example 3.4 follow from their definitions, respectively. (i), (iv) and (v) are (cyclic)permutative representations in [**5**, **7**, **8**]. Other examples are introduced in § 7.

3.2. Existence. Fix $z = (z^{(n)}) \in S(\mathbf{C}^N)^\infty$. We show the existence of $GP(z)$ by constructing $(\mathcal{H}, \pi, \Omega)$ concretely. Denote $z^{(n)} = (z_1^{(n)}, \ldots, z_N^{(n)}) \in S(\mathbf{C}^N)$ for each $n \in \mathbf{N}$.

For convenience, we extend $z = (z^{(n)})_{n\in\mathbf{Z}}$ by $z^{(-n)} \equiv \varepsilon_1$ for each $n \geq 0$. Choose a set $\{g[n]\}_{n\in\mathbf{Z}}$ of unitary matrices in $U(N)$ such that they satisfy the following conditions:

$$(g[n])_{1j} = \overline{z_j^{(n-1)}} \quad (j = 1,\ldots,N,\, n \in \mathbf{Z}). \tag{3.3}$$

Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$ in Example 3.4 (v) where $\mathcal{H} \equiv l_2(\mathbf{Z} \times \mathbf{N})$. For a set $\{g[n]\}_{n\in\mathbf{Z}} \subset U(N)$ in (3.3), define a family $\{\pi'(s_i)\}_{i=1}^N$ of operators on $\mathcal{H}$ by

$$\pi'(s_i)e_{n,m} \equiv \pi\left(\alpha_{g[n]}(s_i)\right) e_{n,m} \quad ((n,m) \in \mathbf{Z} \times \mathbf{N}). \tag{3.4}$$

We see that $\pi'(s_i)^* e_{n-1,N(m-1)+q} = \overline{g[n]_{qi}} e_{n,m}$ for $(n,m) \in \mathbf{Z} \times \mathbf{N}$, $i,q = 1,\ldots,N$ and $(\mathcal{H}, \pi')$ is a representation of $\mathcal{O}_N$.

Proposition 3.5. *For any $z \in S(\mathbf{C}^N)^\infty$, there always exists $GP(z)$.*

Proof. Let $(\mathcal{H}, \pi')$ be a representation of $\mathcal{O}_N$ in (3.4). By (3.3), $(g[n]^*)_{j1} = z_j^{(n-1)}$ for $n \in \mathbf{Z}$ and $j = 1,\ldots,N$. By Lemma A.7 (iv), we have $s(z^{(n-1)}) = \alpha_{g[n]^*}(s_1)$ for each $n \in \mathbf{Z}$. Then $\pi'(s(z^{(k-1)}))e_{k,1} = (\pi \circ$

$\alpha_{g[k]})(\alpha_{g[k]^*}(s_1))e_{k,1} = \pi(s_1)e_{k,1} = e_{k-1,1}$. From this, $\pi^{'}(s(z[k-1]))e_{k,1} = e_{0,1}$. Hence $\pi^{'}(s(z[k-1]))^*e_{0,1} = e_{k,1}$ for $k \in \mathbf{N}$. Therefore $\{\pi^{'}(s(z[k]))^*e_{0,1} : k \in \mathbf{N}\}$ is an orthonormal family in $\mathcal{H}$. Put $V \equiv \pi^{'}(\mathcal{O}_N)e_{0,1}$. Then $(V, \pi^{'}, e_{0,1})$ is $GP(z)$. □

We show this construction of representations in the point of view from group theory in [**12**].

4. Construction of the canonical basis

In order to show the uniqueness of GP representation in § 5, we construct the canonical basis of GP representations with chain. The construction is given by making a "tree" of representation in Example 3.4.

4.1. Construction of the tree of a representation. At fist, we prepare *roots* of the tree.

Fix $z = (z^{(n)}) \in S(\mathbf{C}^N)^\infty$. We extend $z = (z^{(n)})_{n\in\mathbf{Z}}$ as $z^{(-n)} \equiv \varepsilon_1$ for $n \geq 0$. Choose $g = (g[n])_{n\in\mathbf{Z}}$ which satisfies (3.3) and $g[-n] = I$ for $n \geq 0$. Let $(\mathcal{H}, \pi, \Omega)$ be $GP(z)$. We denote $\pi(s_i)$ by s_i simply in this section. Put

$$\Omega_{-n} \equiv s_1^n\Omega, \quad \Omega_0 \equiv \Omega, \quad \Omega_n \equiv s(z[n])^*\Omega \quad (n \geq 1). \tag{4.1}$$

By Definition 3.2 (ii), $\mathcal{R}(z) \equiv \{\Omega_n\}_{n\in\mathbf{N}}$ is an orthonormal family in $\mathcal{H}$.

Lemma 4.1. (i) $s(z^{(n)})\Omega_n = \Omega_{n-1}$ *for each* $n \in \mathbf{Z}$.
(ii) $\{\Omega_n\}_{n\in\mathbf{Z}}$ *is an orthonormal family in* $\mathcal{H}$.

Proof. (i) We see $< s(z^{(n)})\Omega_n|\Omega_{n-1} >= 1$ for each $n \in \mathbf{Z}$. Hence $\|s(z^{(n)})\Omega_n - \Omega_{n-1}\|^2 = \|s(z^{(n)})\Omega_n\|^2 - 2\mathrm{Re} < s(z^{(n)})\Omega_n|\Omega_{n-1} > + \|\Omega_{n-1}\|^2 = 0$ for each $n \in \mathbf{Z}$.
(ii) For each $n \in \mathbf{Z}$, we see $\|\Omega_n\| = 1$. Let $n, m \geq 1$. Then $< \Omega_{-n}|\Omega_{-(n+m)} >=< s_1^n\Omega|s_1^{n+m}\Omega >$. By (i), $\Omega = s(z[m])\Omega_m$. Hence $< s_1^n\Omega|s_1^{n+m}\Omega >=< \varepsilon_1^{\otimes n} \otimes z[m]|\varepsilon_1^{\otimes(n+m)} >< \Omega_m|\Omega >$. Since $< \Omega_m|\Omega >= 0$, $< \Omega_{-n}|\Omega_{-(n+m)} >= 0$.

Let $n \geq 1$ and $m \geq 0$. By (i), $\Omega_m = s(z[m+1]) \cdots s(z[m+n])\Omega_{m+n}$. Hence $< \Omega_{-n}|\Omega_m >=< \varepsilon_1^{\otimes n}\,|\,z[m+1] \otimes \cdots \otimes z[m+n] >< \Omega|\Omega_{m+n} >= 0$. Therefore the statement holds. □

$$\begin{tikzcd} \cdots & \Omega_{n-1} \arrow[l] & \Omega_n \arrow[l, "s(z^{(n)})"'] & \Omega_{n+1} \arrow[l, "s(z^{(n+1)})"'] & \cdots \arrow[l] \end{tikzcd}$$

On the orthonormal family $\{\Omega_n\}_{n\in\mathbf{Z}}$, we construct $N-1$ *trunks* at each $n \in \mathbf{Z}$.

Lemma 4.2. *Put*

$$e_{n,j} \equiv s(y_j^{(n+1)})\Omega_n$$

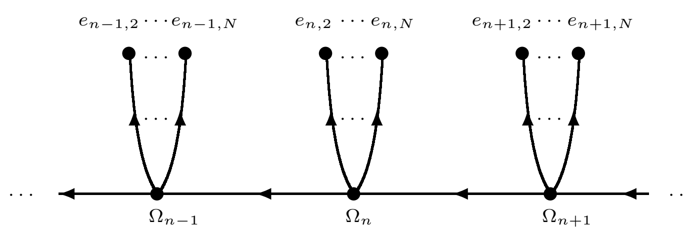
where $y_j^{(n)} \equiv (\,(g[n])^*_{1j}, \ldots, (g[n])^*_{Nj}\,) \in S(\mathbf{C}^N)$ *for* $n \in \mathbf{Z}$ *and* $j = 1, \ldots, N$. *Then* $\{e_{n,j} : n \in \mathbf{Z},\, j = 1, \ldots, N\}$ *is an orthonormal family in* $\mathcal{H}$. *Specially,* $e_{n,1} = \Omega_{n-1}$ *for each* $n \in \mathbf{Z}$.

Proof. By definition of $y_j^{(n)}$, $\{y_j^{(n)}\}_{j=1}^N$ is an orthonormal basis of $\mathbf{C}^N$ for each $n \in \mathbf{Z}$. From this, we see $< e_{n,j}|e_{m,i} >= \delta_{j,i}\delta_{n,m}$ for each $n, m \in \mathbf{Z}$ and $i, j = 1, \ldots, N$. Hence the first statement holds. Because $y_1^{(n)} = (\,(g[n])^*_{11}, \ldots, (g[n])^*_{N1}\,) = (z_1^{(n-1)}, \ldots, z_N^{(n-1)})$ by (3.3) and Lemma 4.1 (i), $e_{n,1} = s(y_1^{(n+1)})\Omega_n = s(z^{(n)})\Omega_n = \Omega_{n-1}$ for each $n \in \mathbf{Z}$. □

Corollary 4.3. (i) $< e_{n,j}|\Omega_m >= 0$ *for each* $n, m \in \mathbf{Z}$ *and* $j = 2, \ldots, N$.

(ii) *For* $k \geq 1$, $m \in \mathbf{Z}$ *and* $i = 1, \ldots, N$, *there are* $m' \in \mathbf{Z}$ *and* $x \in S(\mathbf{C}^N)^{\otimes k}$ *such that* $e_{m,i} = s(x)\Omega_{m'}$.

Proof. (i) By Lemma 4.2, $\Omega_m = e_{m+1,1}$. Hence the statement holds.
(ii) If $k = 1$, then put $x \equiv y_i^{(m+1)}$ and $m' \equiv m$. If $k \geq 2$, then put $m' \equiv m + k - 1$ and $x \equiv y_i^{(m+1)} \otimes z^{(m+1)} \otimes \cdots \otimes z^{(m+k-1)}$. Then $s(x)\Omega_{m'} = s(y_i^{(m+1)} \otimes z^{(m+1)} \otimes \cdots \otimes z^{(m+k-1)})\Omega_{m+k-1} = s(y_i^{(m+1)})\Omega_m = e_{m,i}$. □



By putting N *branches* on a trunk successively, we complete trees.

Lemma 4.4. *Let*

$$e_{J,n,j} \equiv s_J e_{n,j} \quad (J \in \{1, \ldots, N\}^*,\, n \in \mathbf{Z},\, j = 2, \ldots, N).$$

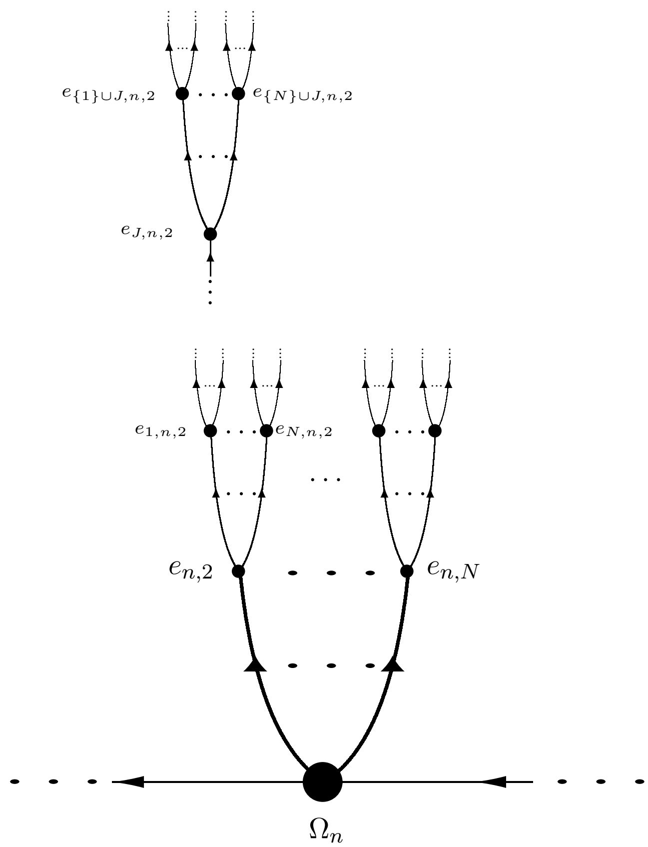
Then the followings hold:

(i) $< e_{J,n,i}|e_{m,j} >= 0$ *for each* $J \in \{1, \ldots, N\}^k$, $k \geq 1$, $n, m \in \mathbf{Z}$, $j = 1, \ldots, N$ *and* $i = 2, \ldots, N$.

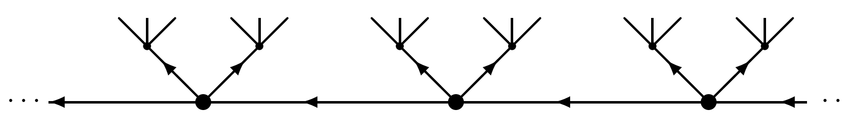
(ii) $\{\Omega_n : n \in \mathbf{Z}\} \cup \{e_{J,n,j} : J \in \{1, \ldots, N\}^*,\, n \in \mathbf{Z},\, j = 2, \ldots, N\}$ *is an orthonormal family.*

Proof. (i) By Corollary 4.3 (ii), there are $x \in S(\mathbf{C}^N)^{\otimes k}$ and $m' \in \mathbf{Z}$ such that $e_{m,j} = s(x)\Omega_{m'}$. From this, $< e_{J,n,i}|e_{m,j} >=< \varepsilon_J|x >< e_{n,i}|\Omega_{m'} >= 0$ by Corollary 4.3 (i).
(ii) By (i), it is sufficient to show $< e_{J,n,i}|e_{J',m,j} >= 0$ when $(J, n, i) \neq (J', m, j)$ for $J, J' \in \{1, \ldots, N\}^* \setminus \{0\}$, $n, m \in \mathbf{Z}$ and $i, j = 2, \ldots, N$. If

$|J^{'}| = |J|$, then $< e_{J,n,i}|e_{J^{'},m,j} >= \delta_{J,J^{'}} < e_{n,i}|e_{m,j} >= \delta_{J,J^{'}}\delta_{ij}\delta_{nm}$ by Lemma 4.2. If $l \equiv |J^{'}| - |J| > 0$, then there are $m^{'} \geq 1$ and $x \in S(\mathbf{C}^N)^{\otimes |J|}$ and $e_{J^{'},m,j} = s(x)\Omega_{m^{'}}$. Hence $< e_{J,n,i}|e_{J^{'},m,j} >=< \varepsilon_J|x >< e_{n,i}|\Omega_{m^{'}} >= 0$ by (i). $\square$



When $N = 3$, we have the following tree:



See the figure in Example 3.4 (v).

4.2. Completeness. We show the completeness of the family $\{\Omega_n : n \in \mathbf{Z}\} \cup \{e_{J,n,j} : J \in \{1, \ldots, N\}^*, n \in \mathbf{Z}, j = 2, \ldots, N\}$ of vectors in $\mathcal{H}$ which are constructed until the previous subsection. For this purpose, we prepare an index set of basis.

For $z \in S(\mathbf{C}^N)^\infty$, put

$$\Lambda(z) \equiv \coprod_{n \in \mathbf{Z}} \Lambda^{(n)}(z), \quad \Lambda^{(n)}(z) \equiv \coprod_{k \geq 1} \Lambda_k^{(n)}(z),$$

$$\Lambda_1^{(n)}(z) \equiv \{(n,\emptyset)\}, \quad \Lambda_2^{(n)}(z) \equiv \{(n, y_j^{(n+1)}) : j = 2,\ldots,N\},$$

$$\Lambda_k^{(n)}(z) \equiv \{(n, \varepsilon_J \otimes y_j^{(n+1)}) : j = 2,\ldots,N,\ J \in \{1,\ldots,N\}^{k-2}\}$$

for $k \geq 3$.

Lemma 4.5. (i) *Put*

$$e(n,x) \equiv s(x)\Omega_n \quad ((n,x) \in \Lambda^{(n)}(z),\ n \in \mathbf{Z})$$

where we define $s(\emptyset) \equiv I$. *Then* $\{e(x) : x \in \Lambda(z)\} = \{\Omega_n \in \mathbf{Z}\} \cup \{e_{J,n,j} : J \in \{1,\ldots,N\}^*,\ n \in \mathbf{Z},\ j = 2,\ldots,N\}$.

(ii) *For* $J \in \{1,\ldots,N\}^*$ *and* $n \in \mathbf{Z}$, $s_J\Omega_n \in \mathrm{Lin} < \{e(x) : x \in \Lambda(z)\} >$.

(iii) $\{e(x) : x \in \Lambda(z)\}$ *is complete in* $\mathcal{H}$.

Proof. (i) We see $e(n,\emptyset) = \Omega_n$, $e(n, y_j^{(n+1)}) = s(y_j^{(n+1)})\Omega_n = e_{n,j}$, $e(n, \varepsilon_J \otimes y_j^{(n+1)}) = e_{J,n,j}$ for $j = 2,\ldots,N$, $J \in \{1,\ldots,N\}^*$ and $n \in \mathbf{Z}$.
(ii) Put $\mathcal{H}_1 \equiv \mathrm{Lin} < \{e(x) : x \in \Lambda(z)\} >$. By (i), we identify $\mathcal{H}_1$ and $\mathrm{Lin} < \{\Omega_n, e_{J,n,j} : J \in \{1,\ldots,N\}^*,\ n \in \mathbf{Z},\ j = 2,\ldots,N\} >$. If $J = 0$, then $s_J\Omega_n = \Omega_n \in \mathcal{H}_1$. By Lemma 4.2, $\dim \mathrm{Lin} < \{e_{n,j} : j = 1,\ldots,N\} >= N$ for each $n \in \mathbf{Z}$. Hence $\mathrm{Lin} < \{e_{n,j} : j = 1,\ldots,N\} >= \mathrm{Lin} < \{s_j\Omega_n : j = 1,\ldots,N\} >$. Therefore if $j = 1,\ldots,N$, then $s_j\Omega \in \mathrm{Lin} < \{e_{n,j'} : j' = 1,\ldots,N\} >\subset \mathcal{H}_1$. From this, for $J \in \{1,\ldots,N\}^k$, $k \geq 1$, $j = 1,\ldots,N$, $s_Js_j\Omega_n \in \mathrm{Lin} < \{s_Je_{n,j'} : j' = 1,\ldots,N\} >\subset \mathcal{H}_1$ for each $n \in \mathbf{Z}$.
(iii) By Lemma B.3 (ii) and definition of $\{e_{J,n,i}\}$, $\{e_{J,n,i}\}$ is complete in $\mathcal{H}$. By (i), the assertion holds. □

Theorem 4.6. *For* $z \in S(\mathbf{C}^N)^\infty$, $\{e(x) : x \in \Lambda(z)\}$ *is a complete orthonormal basis of* $GP(z)$.

Proof. By Lemma 4.4 (ii) and Lemma 4.5, the statement holds. □

We call $\{e(x) : x \in \Lambda(z)\}$ the *GP basis* of $GP(z)$. Note that the GP basis depends on the choice of $\{g[n]\}_{n\in\mathbf{Z}}$ in (3.3).

5. Properties of GP representations

5.1. Uniqueness, irreducibility and equivalence.

Theorem 5.1. *(Uniqueness) For* $z \in S(\mathbf{C}^N)^\infty$, $GP(z)$ *is unique up to unitary equivalences.*

Proof. For $z \in S(\mathbf{C}^N)^\infty$, fix $\{g[n]\}_{n\in\mathbf{Z}}$ in (3.3). $\Lambda(z)$ in § 4.2 is uniquely determined by them. For representations $(\mathcal{H},\pi,\Omega)$ and $(\mathcal{H}',\pi',\Omega')$ of $\mathcal{O}_N$ which are $GP(z)$, take canonical basis $\{e(x) : x \in \Lambda(z)\}$ and $\{e'(x) :$

$x \in \Lambda(z)\}$ in Theorem 4.6, respectively. Then we can define a unitary operator U from $\mathcal{H}$ to $\mathcal{H}'$ by $Ue(x) \equiv e'(x)$ for $x \in \Lambda(z)$. We see $\mathrm{Ad}U \circ \pi = \pi'$. $\square$

By Theorem 5.1, we can use the symbol $GP(z)$ as both a representation and an equivalence class of representations of $\mathcal{O}_N$ for each $z \in S(\mathbf{C}^N)^\infty$.

Recall Definition 2.2.

Theorem 5.2. *(Irreducibility I) If $z \in S(\mathbf{C}^N)^\infty$ is non eventually periodic, then $GP(z)$ is irreducible.*

Proof. Assume that z is non eventually periodic and $(\mathcal{H}, \pi, \Omega) = GP(z)$. Fix $v_0 \in \mathcal{H}$, $v_0 \neq 0$. By cyclicity, there are $n \geq 1$ and $J \in \{1, \ldots, N\}^*$ such that $< \pi(s_J^*)v_0|\Omega_n > \neq 0$. Since $< \pi(s_J^*)v_0|\Omega_n >=< \pi(s(z[n])s_J^*)v_0|\Omega >$, we can assume $c \equiv < v_0|\Omega > \neq 0$. By Lemma B.5, $\pi(s(z[n])\,\{s(z[n])\}^*)v_0$ goes to $c\Omega$ when $n \to \infty$. Hence $\Omega \in \mathcal{O}_N v_0$. Therefore $\mathcal{H} = \mathcal{O}_N\Omega \subset \mathcal{O}_N v_0 \subset \mathcal{H}$. We see that v_0 is a cyclic vector of $\mathcal{H}$. Because any non zero vector in $\mathcal{H}$ is cyclic, $\mathcal{H}$ is irreducible. $\square$

The inverse of Theorem 5.2 is shown in § 5.2.

Theorem 5.3. *(Equivalence) For $z, y \in S(\mathbf{C}^N)^\infty$, $GP(z) \sim GP(y)$ if and only if $z \sim y$.*

Proof. If $z \sim y$, then $GP(z) \sim GP(y)$ by Lemma C.3.

Assume $z \not\sim y$. If $GP(z) \sim GP(y)$, then we can assume that there is a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ with unit cyclic vectors Ω and Ω' which satisfy chain conditions with respect to z and y, respectively. By Lemma C.1 (ii) and Lemma C.2, $\pi(\mathcal{O}_N)\Omega$ and $\pi(\mathcal{O}_N)\Omega'$ are orthogonal. Because Ω and Ω' are cyclic, this is contradiction. Therefore $GP(z) \not\sim GP(y)$. $\square$

We consider a relation between chain and cycle here.

Lemma 5.4. *Assume that $z \in S(\mathbf{C}^N)^\infty$ is non eventually periodic and $y \in TS(\mathbf{C}^N)$. If $(\mathcal{H}, \pi)$ is a representation of $\mathcal{O}_N$ with $\Omega, \Omega' \in \mathcal{H}$ which satisfy the chain condition with respect to z, and the cycle condition with respect to y, respectively, then $< \Omega|\Omega' >= 0$.*

Proof. Assume $y \in S(\mathbf{C}^N)^{\otimes p}$ for $p \geq 1$.

$$< \Omega|\Omega' >=< \pi(s(z[np]))\Omega_{np}|\pi(s(y^{\otimes n}))\Omega' >=< z[np]|y^{\otimes n} >< \Omega_{np}|\Omega' > .$$

By Schwarz inequality, $| < \Omega|\Omega' > | \leq | < z[np]|y^{\otimes n} > |$. By Lemma A.1 (iv), $z \not\sim y^\infty$. Hence $| < \Omega|\Omega' > | \leq \lim_{n\to\infty} | < z[np]|y^{\otimes n} > | = 0$ by Lemma A.6 (i). $\square$

Proposition 5.5. *Let $z \in S(\mathbf{C}^N)^\infty$ and $y \in TS(\mathbf{C}^N)$. If z is non eventually periodic, then $GP(z) \not\sim GP(y)$.*

Proof. Assume that $GP(z) \sim GP(y)$. We derive contradiction. By assumption, we can assume that $(\mathcal{H}, \pi, \Omega)$ is $GP(z)$ and $(\mathcal{H}, \pi, \Omega^{'})$ is $GP(y)$. By Lemma 5.4, $< \Omega|\Omega^{'} >= 0$. In the same way, $< \Omega_n|\Omega^{'} >= 0$ for each root vectors $\{\Omega_n\}$ of $GP(z)$. Assume that $y \in S(\mathbf{C}^N)^{\otimes p}$ for $p \geq 1$. For $J \in \{1, \ldots, N\}^m$, $m \geq 0$, $n \in \mathbf{Z}$, there are $k, l \geq 0$ such that $m + k = lp$ and $< \pi(s_J)\Omega_n|\Omega^{'} >=< \varepsilon_J \otimes z^{(n+1)} \otimes \cdots \otimes z^{(n+k)} \,|\, y^{\otimes l} >< \Omega_{n+k}|\Omega^{'} >= 0$. By Lemma B.3 (ii), $\Omega^{'} = 0$. This is contradiction. Hence $GP(z) \not\sim GP(y)$. □

Corollary 5.6. *Any irreducible GP representation with chain and that with cycle are inequivalent.*

5.2. Decomposition. In order to decompose GP representation with chain by eventually periodic $z \in S(\mathbf{C}^N)^\infty$, we prepare a structure theorem of eventually periodic chains.

For a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ and a unitary operator U on a Hilbert space $\mathcal{K}$, we have a new representation $(\mathcal{K} \otimes \mathcal{H},\, U \boxtimes \pi)$ of $\mathcal{O}_N$ which is defined by

$$(U \boxtimes \pi)(s_i) \equiv U \otimes \pi(s_i) \quad (i = 1, \ldots, N). \tag{5.1}$$

In this subsection, an equality between representations means their unitary equivalence.

Lemma 5.7. *Let $\varphi \in L_\infty(U(1))$ such that $|\varphi(w)| = 1$ almost everywhere $w \in U(1)$, M_φ the multiplication operator on $L_2(U(1))$ by φ, and $(\mathcal{H}, \pi)$ a representation of $\mathcal{O}_N$. Then we have the followings:*

(i)
$$M_\varphi \boxtimes \pi = \int_{U(1)}^{\oplus} \pi \circ \gamma_{\varphi(w)}\, d\eta(w)$$

where γ is the $U(1)$-gauge action on $\mathcal{O}_N$ in § 3 and η is the Haar measure of $U(1)$.

(ii) $M_\varphi \boxtimes \pi = M_{\overline{\varphi}} \boxtimes \pi$ *where $\bar{\varphi}(w) \equiv \overline{\varphi(w)}$ for $w \in U(1)$.*

Proof. (i) Define a unitary W from $L_2(U(1)) \otimes \mathcal{H}$ to $L_2(U(1), \mathcal{H})$ by $W(\phi \otimes v) \equiv \phi \cdot v$ for $\phi \in L_2(U(1))$ and $v \in \mathcal{H}$. Then $W(U \boxtimes \pi)(s_i)W^*(\phi \cdot v) = (M_\varphi \phi) \cdot (\pi(s_i)v)$. From this, $(W(M_\varphi \boxtimes \pi)(s_i)W^*\psi)(w) = \varphi(w)\pi(s_i)\psi(w) = \pi(\gamma_{\varphi(w)}(s_i))\psi(w)$ for $\psi \in L_2(U(1), \mathcal{H})$, $w \in U(1)$, and $i = 1, \ldots, N$. Therefore $\{(\mathrm{Ad}W \circ (M_\varphi \boxtimes \pi))(x)\psi\}(w) = (\pi \circ \gamma_{\varphi(w)})(x)\psi(w)$ for each $x \in \mathcal{O}_N$, $\psi \in L_2(U(1), \mathcal{H})$ and $w \in U(1)$. By definition of direct integral, we have the statement.
(ii) Define an operator T on $L_2(U(1))$ by $(T\phi)(w) \equiv \phi(\bar{w})$. Then $TM_\varphi T^* = M_{\bar{\varphi}}$. From this, $(T \otimes I)(M_\varphi \boxtimes \pi)(s_i)(T^* \otimes I) = (M_{\bar{\varphi}} \boxtimes \pi)(s_i)$ for each $i = 1, \ldots, N$. Hence the assertion holds. □

Corollary 5.8. *Let $p \geq 1$ and $\varphi(w) \equiv w^{1/p}$ for $w \in U(1)$ where $w^{1/p} = e^{2\pi\sqrt{-1}\theta/p}$ when $w = e^{2\pi\sqrt{-1}\theta}$, $0 \leq \theta < 1$. For a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$, we have*

$$M_\varphi \boxtimes \pi = \int_{U(1)}^{\oplus} \pi \circ \gamma_{w^{1/p}}\, d\eta(w).$$

We denote M_φ by $M_{w^{1/p}}$ simply.

Lemma 5.9. *Let $(\mathcal{H}, \pi, \Omega) = GP(z)$ for $z = z^{(1)} \otimes \cdots \otimes z^{(p)} \in S(\mathbf{C}^N)^{\otimes p}$, $p \geq 1$. Assume that a set $\{\Omega_j\}_{j=0}^{p-1}$ of vectors which are defined by $\Omega_j \equiv \pi(s(z^{(j+1)} \otimes \cdots \otimes z^{(p)}))\Omega$ for $j = 0, \ldots, p-1$, is an orthonormal family.*

(i) *$\pi(s(z[j]))^*\Omega = \Omega_j$ for $0 = 1, \ldots, p-1$.*

(ii) *Let $\zeta_c(w) \equiv w^c$ for $c \in \mathbf{R}$, $y \equiv z^\infty = (z^{(1)}, z^{(2)}, \ldots)$ and*

$$v_{np+j} \equiv \zeta_{n+j/p} \otimes \Omega_j$$

for $j = 0, \ldots, p-1$, $n \in \mathbf{Z}$. If $\pi' \equiv M_{\bar{w}^{1/p}} \boxtimes \pi$, then $\pi'(s(y[n])^*)v_0 = v_n$ for each $n \in \mathbf{N}$.

(iii) *If $\pi' \equiv M_{\bar{w}^{1/p}} \boxtimes \pi$, then $\zeta_n \otimes \pi(s_J)\Omega_j = \pi'(s_J)v_{np+j}$ for $J \in \{1, \ldots, N\}^*$, $n \in \mathbf{Z}$ and $j = 0, \ldots, p-1$.*

(iv) *$M_{\bar{w}^{1/p}} \boxtimes \pi$ is cyclic.*

Proof. (i) $\pi(s(z[j]))\Omega_j = \pi(s(z^{(1)}) \cdots s(z^{(j)}))\pi(s(z^{(j+1)}) \cdots s(z^{(p)})\Omega = \Omega$ for $0 = 1, \ldots, p-1$. Then the assertion holds.
(ii) Since $y[np] = z^{\otimes n}$, $(\pi'(s(z))(\phi \otimes \Omega))(w) = \bar{w}\phi(w)\Omega$ for $\phi \in L_2(U(1))$ and $w \in U(1)$. Hence $\pi(s(y[np+j])^*)\Omega_0 = \pi(s(z[j])^*(s(z)^p)^*)\Omega = \pi(s(z[j])^*)\Omega = \Omega_j$ for $n \in \mathbf{N}$ and $j = 0, \ldots, p-1$ by (i). Therefore

$$\pi'(s(y[np+j])^*)v_0 = \zeta_{n+j/p} \otimes \Omega_j = v_{np+j}.$$

From this, the statement holds.
(iii) For $w \in U(1)$, $J \in \{1, \ldots, N\}^k$, $k \geq 1$, $c \in \mathbf{R}$, and $j = 0, \ldots, p-1$,

$$(\pi'(s_J)(\zeta_c \otimes \Omega_j))(w) = \bar{w}^{k/p}\zeta_c(w) \otimes \pi(s_J)\Omega_j = \zeta_{c-k/p}(w) \otimes \pi(s_J)\Omega_j.$$

From this, $\zeta_c \otimes \pi(s_J)\Omega_j = \pi'(s_J)(\zeta_{c+k/p} \otimes \Omega_j) = \pi'(s_J)$. Hence we have the assertion.
(iv) Put $\pi' \equiv M_{\bar{w}^{1/p}} \boxtimes \pi$. We extend $y = (y^{(n)})_{n\in\mathbf{Z}}$ by $y^{(-np+j)} \equiv z^{(j)}$ for $n \geq 1$ and $j = 0, \ldots, p-1$. Note $\pi'(s(y[n]))v_0 = v_{-n}$ for $n \geq 1$. Hence $\{v_n\}_{n\in\mathbf{Z}} \subset \mathcal{V}$. Since Lin $< \{\pi(s_J)\Omega_j : J \in \{1, \ldots, N\}^*, j = 0, \ldots, p-1\} >$ is dense in $\mathcal{H}$, Lin $< \{\zeta_n \otimes \pi(s_J)\Omega_j : n \in \mathbf{Z}, J \in \{1, \ldots, N\}^*, j = 0, \ldots, p-1\} >$ is dense in $L_2(U(1)) \otimes \mathcal{H}$. By (iii), $\mathcal{V} = L_2(U(1)) \otimes \mathcal{H}$. Therefore π' is cyclic. $\square$

When $(\mathcal{H}, \pi, \Omega) = GP(z)$, we denote $U \boxtimes GP(z)$ and $GP(z) \circ \gamma_w$ instead of $U \boxtimes \pi$ and $\pi \circ \gamma_w$ for convenience.

Proposition 5.10. *(Structure of eventually periodic chain) If $z \in S(\mathbf{C}^N)^{\otimes p}$, $p \geq 1$, is non periodic, then $M_{w^{1/p}} \boxtimes GP(z) = GP(z^\infty)$.*

Proof. Because z is non periodic, the condition in Lemma 5.9 is satisfied by Lemma B.1. By Lemma 5.9 (iii), $v_n \in \mathcal{V} \equiv \pi(\mathcal{O}_N)v_0$ for each $n \in \mathbf{N}$. Since $\{v_n\}_{n\geq 1}$ is an orthonormal family, π' contains $GP(z^\infty)$ as subrepresentation. By Lemma 5.9 (iv), $M_{\bar{w}^{1/p}} \boxtimes GP(z) = GP(z^\infty)$. By Lemma 5.7 (ii), $M_{\bar{w}^{1/p}} \boxtimes GP(z) = M_{w^{1/p}} \boxtimes GP(z)$. Hence the assertion holds. □

Theorem 5.11. *If $z \in S(\mathbf{C}^N)^{\otimes p}$, $p \geq 1$, is non periodic, then*

$$GP(z^\infty) = \int_{U(1)}^{\oplus} GP(z) \circ \gamma_{w^{1/p}}\, d\eta(w).$$

Proof. By Proposition 5.10 and Corollary 5.8, the statement holds. □

Corollary 5.12. *(Decomposition of eventually periodic chain)*

(i) *If $z \in S(\mathbf{C}^N)^\infty$ is eventually periodic, then there are $p \geq 1$ and $y \in S(\mathbf{C}^N)^{\otimes p}$ such that y is non periodic and*

$$GP(z) = \int_{U(1)}^{\oplus} GP(y) \circ \gamma_{w^{1/p}}\, d\eta(w). \tag{5.2}$$

(ii) *If there are $q \geq 1$ and $y_0 \in S(\mathbf{C}^N)^{\otimes q}$ which satisfies the statement (i) with respect to z, then $p = q$ and there is $c \in U(1)$ such that $y_0 \sim cy$.*

Proof. (i) By Lemma A.1 (ii) and Theorem 5.3, $GP(z) \sim GP(y^\infty)$ for non periodic $y \in S(\mathbf{C}^N)^{\otimes p}$. By Theorem 5.11, the statement holds.
(ii) By Theorem 5.11,

$$GP(y_0^\infty) = \int_{U(1)}^{\oplus} GP(y_0) \circ \gamma_{w^{1/q}}\, d\eta(w) = GP(z) = GP(y^\infty).$$

By Theorem 5.3, $y_0^\infty \sim y^\infty$. By Lemma A.1 (iii), there is $c \in U(1)$ such that $p = q$ and $y_0 \sim cy$. □

For $z \in TS(\mathbf{C}^N)$ and $w \in U(1)$, we consider wz by the scalar product of a vector z by a scalar w. We see $wz \in TS(\mathbf{C}^N)$ again.

Proposition 5.13. *If $z \in TS(\mathbf{C}^N)$ is non periodic, then*

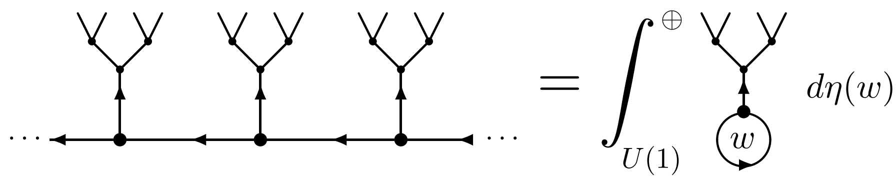
$$GP(z^\infty) = \int_{U(1)}^{\oplus} GP(wz)\, d\eta(w).$$

Proof. By Lemma B.2 (i), $GP(wz) = GP(z) \circ \gamma_{\bar{w}^{1/p}}$. By Lemma 5.7 (ii) and Corollary 5.12, the statement holds. □

By Proposition 5.13, if the period of chain is 1(that is, $z \in S(\mathbf{C}^N)$), then

$$GP(z^\infty) = \int_{U(1)}^{\oplus} GP(z) \circ \gamma_w \, d\eta(w).$$

We illustrate this by the tree of representations in Example 3.4 when $N = 2$:



Theorem 5.14. *(Irreducibility II) For $z \in S(\mathbf{C}^N)^\infty$, $GP(z)$ is irreducible if and only if z is non eventually periodic.*

Proof. If z is non eventually periodic, then $GP(z)$ is irreducible by Theorem 5.2. If z is eventually periodic, then $GP(z)$ is not irreducible by Corollary 5.12. Hence the statement holds. □

6. States and spectrums of $\mathcal{O}_N$ associated with GP representations

6.1. States. We show a relation between GP representations with chain and states of $\mathcal{O}_N$.

Theorem 6.1. *For $z \in S(\mathbf{C}^N)^\infty$, the GNS representation $(\mathcal{H}_z, \pi_z, \Omega_z)$ of $\mathcal{O}_N$ by the following state ω_z of $\mathcal{O}_N$:*

$$\omega_z(s_I s_J)^* \equiv \delta_{k,l} < z[k]|\varepsilon_I >< \varepsilon_J|z[l] > \tag{6.1}$$

for $I \in \{1,\ldots,N\}^k$, $J \in \{1,\ldots,N\}^l$, $k,l \geq 0$, is equivalent to $GP(z)$. Furtheremore the followings hold:

(i) *ω_z is pure if and only if z is non eventually periodic.*
(ii) *For $z, z^{'} \in S(\mathbf{C}^N)^\infty$, $(\mathcal{H}_z, \pi_z)$ and $(\mathcal{H}_{z'}, \pi_{z'})$ are equivalent if and only if $z \sim z^{'}$.*
(iii) *If $z \in S(\mathbf{C}^N)^\infty$ is eventually periodic, then there is $y \in TS(\mathbf{C}^N)$ such that*

$$(\mathcal{H}_z, \pi_z) = \int_{U(1)}^{\oplus} GP(wy) \, d\eta(w).$$

Proof. Put $\hat{\omega}_z(x) \equiv < \Omega|\pi(x)\Omega >$ for $x \in \mathcal{O}_N$. By Lemma B.3 (i), we have $\hat{\omega}_z(s_I s_J^*) = \omega_z(s_I s_J^*)$. Hence $\hat{\omega}_z = \omega_z$. By uniqueness of GNS

representation and cyclicity of GP representation, $(\mathcal{H}_z, \pi_z)$ is equivalent to $GP(z)$.
(i) By Theorem 5.2, $(\mathcal{H}_z, \pi_z)$ is irreducible if and only if z is non eventually periodic.
(ii) By Theorem 5.3, the assertion holds.
(iii) By Corollary 5.12, the statement holds. □

We call ω_z the *GP state* of $\mathcal{O}_N$ by $z \in S(\mathbf{C}^N)^\infty$.

Corollary 6.2. *The following state ω_z of $\mathcal{O}_N$ is pure if and only if $z \in S(\mathbf{C}^N)^\infty$ is non eventually periodic:*

$$\omega_z(s_I s_J^*) \equiv \delta_{|I|,|J|} \bar{z}_I z_J \quad (I, J \in \{1, \ldots, N\}^*)$$

where $z_J \equiv z_{j_1}^{(1)} \cdots z_{j_k}^{(k)}$ for $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$ for $k \geq 1$ where $z = (z^{(n)})$ and $z^{(n)} = (z_1^{(n)}, \ldots, z_N^{(n)})$.

In this way, we see that GP states are defined on the(gauge fixing)UHF subalgebra of $\mathcal{O}_N$.

States associated with permutative representations with chain([**5**, **7**, **8**]) are given as follows: For $J \equiv (j_n)_{n\geq 1} \in \{1, \ldots, N\}^\infty$, define a state ω of $\mathcal{O}_N$ by

$$\omega(s_{J'} s_{J''}^*) \equiv 0 \quad (J' \neq J''), \quad \omega(s_{J'} s_{J'}^*) \equiv \delta_{J', J[n]} \quad (J' \in \{1, \ldots, N\}^n)$$

where $J[n] \equiv (j_1, \ldots, j_n)$ for $n \geq 1$.

Recall that the GP state is defined for GP representation with cycle([**10**]).

Proposition 6.3. *If ω and ω_0 are GP states of $\mathcal{O}_N$ by eventually periodic $z \in S(\mathbf{C}^N)^\infty$ and non periodic $y \in S(\mathbf{C}^N)^{\otimes p}$, $p \geq 1$, respectively, such that $GP(z) = M_{w^{1/p}} \boxtimes GP(y)$, then*

$$\omega(x) = \int_{U(1)} \omega_0(\gamma_{w^{1/p}}(x))\, d\eta(w) \quad (x \in \mathcal{O}_N)$$

where η is the normalized Haar measure of $U(1)$ defined by $d\eta(e^{2\pi\sqrt{-1}\theta}) \equiv d\theta$ for $0 \leq \theta < 1$.

Proof. Let $(\mathcal{H}, \pi, \Omega) = GP(z)$ and $(\mathcal{H}_0, \pi_0, \Omega_0) = GP(y)$. By Corollary 5.12, we can realize $(\mathcal{H}, \pi, \Omega)$ as $\mathcal{H} = L_2(U(1), \mathcal{H}_0)$ and $\mathcal{H}_w = \{\phi(w) : \phi \in L_2(U(1), \mathcal{H}_0)\}$ and $\Omega(w) = \Omega_0$ for $w \in U(1)$. By Corollary 5.12 (i),

$$\pi = \int_{U(1)}^{\oplus} \pi_0 \circ \gamma_{w^{1/p}}\, d\eta(w).$$

Because $\omega = < \Omega|\pi(\cdot)\Omega >$ and $\omega_0 = < \Omega_0|\pi_0(\cdot)\Omega_0 >$, we have

$$\omega(x) = \int_{U(1)} < \Omega_0|(\pi_0 \circ \gamma_{w^{1/p}})(x)\Omega_0 > \, d\eta(w) = \int_{U(1)} \omega_0(\gamma_{w^{1/p}}(x))\, d\eta(w)$$

for each $x \in \mathcal{O}_N$. $\square$

In Proposition 6.3, we can check the following equation: For $I, J \in \{1, \ldots, N\}^*$,

$$\omega(s_I s_J^*) = \begin{cases} \omega_0(s_I s_J^*) & (|I| = |J|), \\ \\ \dfrac{p(e^{2\pi\sqrt{-1}(|I|-|J|)/p} - 1)}{2\pi\sqrt{-1}(|I| - |J|)}\omega_0(s_I s_J^*) & (|I| \neq |J|). \end{cases}$$

This is verified by Theorem 6.1 and § 6 in [**10**].

By Proposition 6.3, we have the following:

Corollary 6.4. *Let $z \in TS(\mathbf{C}^N)$. Assume that ω_{z^∞} and ω_{wz} are GP states by z^∞ and wz for $w \in U(1)$, respectively.*

(i) *If z is non periodic, then*

$$\omega_{z^\infty} = \int_{U(1)} \omega_{wz}\, d\eta(w).$$

(ii) *If $z \in S(\mathbf{C}^N)$, then*

$$\omega_{z^\infty} = \int_{U(1)} \omega_z \circ \gamma_w\, d\eta(w).$$

6.2. Spectrum. We consider the spectrum of $\mathcal{O}_N$. The spectrum Spec$\mathcal{O}_N$ of $\mathcal{O}_N$ is the set of all equivalence classes of irreducible representations of $\mathcal{O}_N$. One of our aim is a classification of elements of Spec$\mathcal{O}_N$ and a constructive understanding of them.

We review results about cycle case in [**10**]. Put

$$TS_{NP}(\mathbf{C}^N) \equiv \{z \in TS(\mathbf{C}^N) : z \text{ is non periodic}\}.$$

Then $TS_{NP}(\mathbf{C}^N)/\sim$ is identified with a subset of Spec$\mathcal{O}_N$ by Theorem 3.3.

On the other hand, let

$$S_{NP}(\mathbf{C}^N)^\infty \equiv \{z \in S(\mathbf{C}^N)^\infty : z \text{ is non eventually periodic}\}.$$

Then $S_{NP}(\mathbf{C}^N)^\infty/\sim$ is identified with a subset of Spec$\mathcal{O}_N$ by Theorem 5.14 and Theorem 5.3. By Corollary 5.6, $S_{NP}(\mathbf{C}^N)^\infty/\sim$ and $TS_{NP}(\mathbf{C}^N)/\sim$ have no intersection as subsets of Spec$\mathcal{O}_N$.

In consequence, GPSpec$\mathcal{O}_N \equiv \left(TS_{NP}(\mathbf{C}^N)/\sim\right) \sqcup \left(S_{NP}(\mathbf{C}^N)^\infty/\sim\right)$ is identified with a subset of Spec$\mathcal{O}_N$. In [**12**], we show that GPSpec$\mathcal{O}_N$ is closed under $U(N)$-action arising from the canonical action of $U(N)$ on $\mathcal{O}_N$. Next problem is a study of Spec$\mathcal{O}_N \setminus$ GPSpec$\mathcal{O}_N$, that is, (i) whether Spec$\mathcal{O}_N \setminus$ GPSpec$\mathcal{O}_N$ is empty or not. If it is not empty, then (ii) construct all of them concretely.

7. Examples

7.1. Correspondence with ordinary permutative representations. Let $E \equiv \{\varepsilon_1, \ldots, \varepsilon_N\}$ be the canonical basis of $\mathbf{C}^N$ and a subset

$$E_N^\infty \equiv \{(\varepsilon_{i_n})_{n\geq 1} : i_n \in \{1,\ldots,N\}\} = \{\varepsilon_J : J \in \{1,\ldots,N\}^\infty\}$$

of $S(\mathbf{C}^N)^\infty$. Then the GP representation of $\mathcal{O}_N$ by $z \in E_N^\infty \cap S(\mathbf{C}^N)^\infty$ is a (cyclic)permutative representation with chain by [**5**, **7**, **8**]. For instance, Example 3.4 (v) is associated with $\varepsilon_J \in E_N^\infty$, $J \equiv (1)^\infty \in \{1,\ldots,N\}^\infty$.

By Theorem 5.3, we see that a class of GP representation is properly wider than ordinary permutative representation by [**5**, **7**, **8**].

7.2. Representations of $\mathcal{O}_2$ parameterized by $U(1)$. Fix $\xi \in U(1)$ and let $z_\xi^{(n)} \equiv \frac{1}{\sqrt{2}}(1, \xi^n) \in S(\mathbf{C}^2)$ for $n \in \mathbf{N}$. Then $z_\xi \equiv (z_\xi^{(n)})_{n\in\mathbf{N}} \in S(\mathbf{C}^2)^\infty$. z_ξ is eventually periodic if and only if ξ is a root of unity, that is, there is $p \geq 1$ such that $\xi^p = 1$.

Proposition 7.1. *We have the following statement about representations of $\mathcal{O}_2$:*

(i) *$GP(z_\xi)$ is not a permutative representation by* [**5**, **7**, **8**] *for any $\xi \in U(1)$.*
(ii) *$GP(z_\xi)$ is irreducible if and only if ξ is not a root of unity.*
(iii) *If there is a positive integer p such that $\xi^p = 1$ and $\xi^q \neq 1$ for each $1 \leq q < p$, then,*

$$GP(z_\xi) = \int_{U(1)}^{\oplus} GP(wz_\xi[p])\, d\eta(w)$$

where $z_\xi[p] \equiv z_\xi^{(1)}\otimes\cdots\otimes z_\xi^{(p)} \in S(\mathbf{C}^N)^{\otimes p}$. Furthermore $\{GP(wz_\xi[p])\}_{w\in U(1)}$ is a family of mutually inequivalent irreducible representations of $\mathcal{O}_2$.
(iv) *For $\xi, \xi' \in U(1)$, $GP(z_\xi) = GP(z_{\xi'})$ if and only if $\xi = \xi'$.*
(v) *Put a state ω_ξ of $\mathcal{O}_2$ by*

$$\omega_\xi(s_{i_1}\cdots s_{i_k}) = \omega_\xi(s_{j_l}^*\cdots s_{j_1}^*) = 0,$$

$$\omega_\xi(s_{i_1}\cdots s_{i_k}s_{j_l}^*\cdots s_{j_1}^*) = 2^{-k}\delta_{k,l}\xi^{j_1-i_1}\xi^{2(j_2-i_2)}\cdots\xi^{k(j_k-i_k)}$$

for each $i_1,\ldots,i_k, j_1,\ldots,j_l = 1,2,\ k,l \geq 1$. Then the GNS representation of $\mathcal{O}_2$ by ω_ξ is equivalent to $GP(z_\xi)$.
(vi) *The set of all equivalence classes of irreducible representations of $\mathcal{O}_2$ by $\xi \in U(1)$ is one to one corresponded to $\{e^{2\pi\sqrt{-1}\theta} \in U(1) : \theta \notin \mathbf{Q}\}$.*

Proof. (i) Because z_ξ is not equivalent to any element in E_2^∞ in § 7.1 for each $\xi \in U(1)$, the statement holds.
(ii) By Theorem 5.2, $GP(z_\xi)$ is irreducible if and only if z_ξ is non eventually periodic. By definition, z_ξ is non eventually periodic if and only if $z_\xi^{(1)} \neq z_\xi^{(n)}$

for each $n \geq 2$ if and only if $\xi \neq \xi^n$ for each $n \geq 2$. From this, $GP(z_\xi)$ is irreducible if and only if ξ is not a root of unity.
(iii) By assumption, $z_\xi[p] = 2^{-p/2}(1,\xi) \otimes \cdots \otimes (1,\xi^{p-1}) \otimes (1,1)$ is non periodic. $GP(z_\xi[p])$ is irreducible. By Lemma B.2, the statement about $\{GP(wz_\xi[p])\}_{w\in U(1)}$ holds. Note $z_\xi = (z_\xi[p])^\infty$. By Corollary 5.12, we have the first assertion.
(iv) $GP(z_\xi) = GP(z_{\xi'})$ if and only if $z_\xi \sim z_{\xi'}$. Because phase factors are determined, $z_\xi \sim z_{\xi'}$ if and only if there are M and L such that $\xi^{n+L} = (\xi')^n$ for each $n \geq M$. This holds only if $(\overline{\xi'}\xi)^n = \xi^{-L}$ for each $n \geq M$. If $L \neq 0$, then $\xi = \xi'$. If $L = 0$, then $\xi = \xi'$, too. Hence $GP(z_\xi) = GP(z_{\xi'})$ if and only if $\xi = \xi'$.
(v) Note $(z_\xi)_J = \xi^{j_1-1}\xi^{2(j_2-1)}\cdots\xi^{k(j_k-1)}$ for each $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$, in Corollary 6.2. By Corollary 6.2,

$$\begin{aligned}\omega(s_I s_J^*) &= 2^{-k}\delta_{k,l}\xi^{j_1-1}\xi^{2(j_2-1)}\cdots\xi^{k(j_k-1)}\overline{\xi^{i_1-1}\xi^{2(i_2-1)}\cdots\xi^{l(i_l-1)}} \\ &= 2^{-k}\delta_{k,l}\xi^{j_1-i_1}\xi^{2(j_2-i_2)}\cdots\xi^{k(j_k-i_k)}\end{aligned}$$

for $I = (i_1,\ldots,i_l)$ and $I = (j_1,\ldots,j_k)$. Hence we have the assertion.
(vi) The statement holds by (ii) and (iv). □

¿From this, Theorem 1.1 is proved.

For example, if $\xi = e^{2\pi\sqrt{-1}/3}$, then $(\xi^n)_{n\in\mathbf{N}} = (\xi,\xi^2,1,\xi,\xi^2,1,\ldots)$, $((\xi^2)^n)_{n\in\mathbf{N}} = (\xi^2,\xi,1,\xi^2,\xi,1,\ldots)$. Therefore $z_\xi \not\sim z_{\xi^2}$. Hence $GP(z_\xi) \not\sim GP(z_{\xi^2})$.

7.3. Representations arising from real numbers. We define a permutative representation of $\mathcal{O}_N$ with chain arising from a real number.

For a real number $a \in [0,1)$, consider the N-adic expansion $a = \sum_{k\geq 1} a_k/N^k$.

Definition 7.2. (i) *For $a \in [0,1)$, $b(a) \equiv (b^{(k)}(a))_{k\in\mathbf{N}} \in \{1,\ldots,N\}^\infty$ is defined by $b^{(k)}(a) \equiv a_k + 1 \in \{1,\ldots,N\}$.*
(ii) *For $a, a' \in [0,1)$, $a \sim a'$ if there are $k,l \geq 0$ such that $N^k a = N^l a'$* mod 1.

For $\varepsilon_{b(a)} = (\varepsilon_{b^{(k)}(a)})_{k\in\mathbf{N}} \in E_N^\infty \subset S(\mathbf{C}^N)^\infty$, we have a GP representation $GP(\varepsilon_{b(a)})$ of $\mathcal{O}_N$. This class of representations of $\mathcal{O}_N$ is well known by [**5**, **7**, **8**].

Proposition 7.3. (i) *For $a \in [0,1)$, $GP(\varepsilon_{b(a)})$ is irreducible if and only if $a \notin \mathbf{Q}$.*
(ii) *For $a, a' \in [0,1)$, $GP(\varepsilon_{b(a)}) = GP(\varepsilon_{b(a')})$ if and only if $a \sim a'$.*

Proof. (i) $b(a)$ is non eventually periodic if and only if $a \notin \mathbf{Q}$. Hence $GP(\varepsilon_{b_a})$ is irreducible if and only if $a \notin \mathbf{Q}$ by Theorem 5.2.

(ii) By Theorem 5.3, $GP(\varepsilon_{b(a)}) = GP(\varepsilon_{b(a')})$ if and only if $\varepsilon_{b(a)} \sim \varepsilon_{b(a')}$. We can check that $\varepsilon_{b(a)} \sim \varepsilon_{b(a')}$ if and only if $a \sim a^{'}$. □

Any representation of $\mathcal{O}_2$ in § 7.2 and that in § 7.3 are disjoint.

7.4. Representations arising from dynamical systems on projective spaces. Let F be a transformation on a complex projective space $\mathbf{C}P^{N-1} \equiv (\mathbf{C}^N \setminus \{0\})/\mathbf{C}^{\times}$. Consider the orbit of the dynamical system $(\mathbf{C}P^{N-1}, F)$ at $a \in \mathbf{C}P^{N-1}$. Put $a_n \equiv F^n(a)$ for $n \in \mathbf{N}$. Then we have a sequence $a_* \equiv (a_n)_{n\in\mathbf{N}}$ in $\mathbf{C}P^{N-1}$. Choose $b^{(n)}(a) \in S(\mathbf{C}^N)$ such that $[b^{(n)}(a)] \equiv \{cb^{(n)}(a) : c \in \mathbf{C}^{\times}\} = a_n$. Then $b(a) \equiv (b^{(n)}(a))_{n\in\mathbf{N}} \in S(\mathbf{C}^N)^{\infty}$. A sequence $(a_n)_{n\in\mathbf{N}}$ in $\mathbf{C}P^{N-1}$ is eventually periodic if there are M, p such that $a_{n+p} = a_n$ for each $n \geq M$. The equivalence class of $b(a)$ in $S(\mathbf{C}^N)^{\infty}$ is independent in the choice of $b^{(n)}(a)$ with respect to a. In this sense, the equivalence class of $GP(b(a))$ essentially depends on the orbit of F starting from a.

Proposition 7.4. (i) *$GP(b(a))$ is irreducible if and only if a_* is non eventually periodic.*

(ii) *For $a, a^{'} \in \mathbf{C}P^{N-1}$, $GP(b(a)) = GP(b(a^{'}))$ if and only if there are L, M such that $a_{L+n} = a^{'}_n$ for each $n \geq M$.*

Proof. (i) $b(a)$ is non eventually periodic if and only if the orbit a_* is non eventually periodic in $\mathbf{C}P^{N-1}$. By Theorem 5.2, it follows.
(ii) By Theorem 5.3, the assertion holds. □

On the other hand, if $z = (z^{(n)}) \in S(\mathbf{C}^N)^{\infty}$, we have a sequence $\{[z^{(n)}]\}$ in $\mathbf{C}P^{n-1}$. Hence the parameter space $S(\mathbf{C}^N)^{\infty}$ can be regarded as a set of all sequences of points in $\mathbf{C}P^{n-1}$.

In the same way, we can obtain a representation of $\mathcal{O}_N$ from a dynamical system on a sphere S^{N-1}.

7.5. Others.

Example 7.5. The representation of $\mathcal{O}_2$ arising from a dynamical system $(\mathbf{C}, Q)$, $Q(z) \equiv z^2$ gives a direct sum of chains in [**17**]. Put a representation $(L_2(\mathbf{C}), \pi_0)$ of $\mathcal{O}_2$ arising from Q by

$$(\pi_0(s_i)\phi)(z) \equiv m_i(z)\phi(Q(z)) \tag{7.1}$$

for $\phi \in L_2(\mathbf{C})$ and $z \in \mathbf{C}$ where $m_i(z) \equiv 2|z| \cdot \chi_{E_i}(z)$, $i = 1, 2$, $E_1 \equiv \{z \in \mathbf{C} : \mathrm{Im}\, z \geq 0\}$, $E_2 \equiv \{z \in \mathbf{C} : \mathrm{Im}\, z < 0\}$, χ_Y is the characteristic function on $Y \subset \mathbf{C}$, $L_2(\mathbf{C})$ is taken by a measure $d\mu(z) = dxdy$ on $\mathbf{C}$ for $z = x + \sqrt{-1}y$, and s_1, s_2 are generators of $\mathcal{O}_2$. Then $(L_2(\mathbf{C}), \pi_0)$ is equivalent to

$$(GP((2^{-1/2}, 2^{-1/2})^{\infty}))^{\oplus\infty} = \left\{ \int_{U(1)}^{\oplus} GP((2^{-1/2}, 2^{-1/2})) \circ \gamma_w \, d\eta(w) \right\}^{\oplus\infty}.$$

Example 7.6. Let $z^{(n)} \equiv (1/2^{n/2}, (2^n-1)^{1/2}/2^{n/2})$ for $n \geq 1$. Then $z \equiv (z^{(n)}) \in S(\mathbf{C}^2)^\infty$ and z is non eventually periodic.

Example 7.7. For $\theta \in [0,1)$, define $z_\theta = (z_\theta^{(n)}) \in S(\mathbf{C}^2)^\infty$ by

$$z_\theta^{(n)} \equiv (\cos 2\pi n\theta, \sin 2\pi n\theta) \in S^1 \equiv \{(x,y) \in \mathbf{R}^2 : x^2+y^2=1\} \quad (n \in \mathbf{N}).$$

Then z_θ is non eventually periodic if and only if $\theta \notin \mathbf{Q}$. Hence $GP(z_\theta)$ is irreducible if and only if $\theta \notin \mathbf{Q}$. If there is $p \in \mathbf{N}$ such that $p = \min\{q \in \mathbf{N} : q\theta \in \mathbf{N} \cup \{0\}\}$, then

$$GP(z_\theta) = \int_{U(1)}^{\oplus} GP(wz_\theta[p])\, d\eta(w).$$

If $\theta \neq 0$, then $GP(z_\theta)$ is neither equivalent to any representation in § 7.2 nor that in § 7.3.

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Appendix A. Technical lemmata

A.1. Parameters of representations. Recall notations in § 2.

Lemma A.1. (i) *$\sim$ is an equivalence relation in $S(\mathbf{C}^N)^\infty$.*

(ii) *If $z \in S(\mathbf{C}^N)^\infty$ is eventually periodic, then there is $y \in TS(\mathbf{C}^N)$ such that y is non periodic and $z \sim y^\infty$.*

(iii) *Let $z \in S(\mathbf{C}^N)^\infty$. If there are non periodic elements $x, y \in TS(\mathbf{C}^N)$ such that $z \sim y^\infty$ and $z \sim x^\infty$, then there is $c \in U(1)$ such that $x \sim cy$.*

(iv) *Let $z, y \in S(\mathbf{C}^N)^\infty$. If z is non eventually periodic and y is eventually periodic, then $z \not\sim y$.*

Proof. (i) Reflection law and Symmetric law. are trivial. We show transitive law. If $x \sim y$ and $y \sim z$, then there are L, L', M, M' and $\{c_n\}, \{c_n'\} \subset U(1)$ such that $x^{(n+L)} = c_n y^{(n)}$, $n \geq M$ and $y^{(n+L')} = c_n' z^{(n)}$, $n \geq M'$. From these, $x^{(n+L+L')} = c_{n+L'} y^{(n+L')} c_n' z^{(n)}$ for each $n \geq M'' \equiv \max\{M, M'\}$. Hence $x \sim z$. $\sim$ is transitive.
(ii) Assume that $z = (z^{(n)}) \in S(\mathbf{C}^N)^\infty$ is eventually periodic. Then there are positive integers M, p and a sequence $\{c_n\} \subset \mathbf{C}$, $|c_n| = 1$ for each $n \geq M$ such that $z^{(n+p)} = c_n z^{(n)}$ for each $n \geq M$. Define $y \equiv (y^{(i)})_{i=1}^p \in S(\mathbf{C}^N)^{\otimes p}$ by $y^{(i)} \equiv z^{(M+i-1)}$ for $i = 1, \ldots, p$. Then

$$z^{(M+kp+i-1)} = c_{M+(k-1)p+i-1} z^{(M+(k-1)p+i-1)} = \cdots = C_{k,i} z^{(M+i-1)} = C_{k,i} y^{(i)}$$

for $i=1,\ldots,p$ and $k\geq 0$ where $C_{k,i}\equiv\prod_{l=0}^{k-1}c_{M+lp+i-1}$. Hence $z\sim y^{\infty}$. If y is periodic, then there are $q\geq 1$ and non periodic $y_1\in TS(\mathbf{C}^N)$ such that $y=y_1^{\otimes q}$. Hence $z\sim y_1^{\infty}$.
(iii) Assume that $x\in S(\mathbf{C}^N)^{\otimes k}$,$y\in S(\mathbf{C}^N)^{\otimes l}$ for $k,l\geq 1$ and $(x^{(1)},\ldots,x^{(k)})$, $(y^{(1)},\ldots,y^{(l)})$ are standard tensor components of x and y in Definition 2.2 (iv). Furthermore we assume $k\geq l$. If $k=al+b$ for $a\geq 0$ and $0\leq b\leq l-1$. By (i), $x^{\infty}\sim y^{\infty}$. By definition of equivalence, there are L and $\{c_n\}_{n\geq 1}\subset U(1)$ such that $x^{(1+L)}=c_1y^{(1)},\ldots,x^{(l+L)}=c_ly^{(l)},x^{(l+1+L)}=c_{l+1}y^{(1)},\ldots,x^{(k-b+L)}=c_{k-b}y^{(l)},x^{(k-b+1+L)}=c_{k-b+1}y^{(1)}\ldots,x^{(k+L)}=c_ky^{(b)}$. From this, $x^{(1+l+L)}=c_{l+1}\overline{c_1}x^{(1+L)},\ldots,x^{(1+2l+L)}=c_{2l}\overline{c_l}x^{(1+l+L)}$. x is periodic when $k>l$. Therefore $k=l$. Then $x^{(1+L)}=c_1y^{(1)},\ldots,x^{(k+L)}=c_ky^{(k)}$. From this, $\sigma^L(x)=x^{(1+L)}\otimes\cdots\otimes x^{(k+L)}=(c_1\cdots c_k)y^{(1)}\otimes\cdots\otimes y^{(k)}=cy$ where $c\equiv c_1\cdots c_k\in U(1)$. Hence $x\sim cy$.
(iv) Denote $z=(z^{(n)})$ and $y=(y^{(n)})$. Assume $z\sim y$. Then there are L and $\{c_n\}\subset U(1)$ such that $z^{(n+L)}=c_ny^{(n)}$ for $n\geq M$. Because y is eventually periodic, there are p, M' and $\{c_n'\}\subset U(1)$ such that $y^{(n+p)}=c_n'y^{(n)}$ for each $n\geq M'$. Hence $z^{(n+L+p)}=c_{n+p}y^{(n+p)}=c_{n+p}c_n'y^{(n)}=\overline{c_n}c_{n+p}c_n'z^{(n+L)}$ for $n\geq M''\equiv\max\{M,M'\}$. Hence z is eventually periodic. This contradicts against the choice of z. Therefore $z\not\sim y$. $\square$

We consider the value of the inner product among $S(\mathbf{C}^N)^{\otimes n}$ as vectors in $(\mathbf{C}^N)^{\otimes n}$ for $n\geq 1$.

Lemma A.2. *If $x\sim y$, then there are $L\geq 0$, $M\geq 1$ and $\{C_k\}\subset U(1)$ such that $x[k]=cy[k+L]$ for each $k\geq M$.*

Proof. By assumption, there are $L\geq 0$, $M\geq 1$ and $\{c_k\}\subset U(1)$ such that $x^{(k)}=c_ky^{(k+L)}$ for each $k\geq M$. Hence $x[k]=x^{(1)}\otimes\cdots\otimes x^{(k)}=(c_1y^{(1+L)})\otimes\cdots\otimes(c_ky^{(k+L)})=C_ky[k]$ where $C_k=c_1\cdots c_k$ for $k\geq 1$. $\square$

Lemma A.3. (i) *For $z,y\in S(\mathbf{C}^N)^{\infty}$, $z\sim y$ if and only if there are non negative integers L and M such that $|<z^{(n+L)}|y^{(n)}>|=1$ for each $n\geq M$ where $z=(z^{(n)})$ and $y=(y^{(n)})$.*

(ii) *For $z\in S(\mathbf{C}^N)^{\infty}$, z is eventually periodic if and only if there are positive integers p and M such that $|<z^{(n+p)}|z^{(n)}>|=1$ for each $n\geq M$ where $z=(z^{(n)})$.*

(iii) *If $z,y\in S(\mathbf{C}^N)^{\infty}$ and $z\not\sim y$, then there is a positive integer M such that $|<z^{(M)}|y^{(M)}>|<1$ where $z=(z^{(n)})$ and $y=(y^{(n)})$.*

Proof. Two unit vectors in a vector space with inner product are linearly dependent if and only if the absolute value of the inner product of them are 1. By Definition 2.2, (i) and (ii) follow immediately. (iii) is a corollary of (i). $\square$

Lemma A.4. *If $z \in S(\mathbf{C}^N)^\infty$ is non eventually periodic, then there is a positive integer M such that $| < z[k]|z[k'] > | < 1$ when $k, k' > M$ and $k \neq k'$.*

Proof. For $L \geq 1$, put $y \equiv \sigma^L(z)$. Because z is non eventually periodic, $y \not\sim z$. By Lemma A.3 (ii), we have the statement. $\square$

Lemma A.5. (i) *If $z, y \in S(\mathbf{C}^N)^\infty$ are not equivalent, then z and $\sigma(y)$ are not equivalent, too.*
(ii) *For each $z = (z^{(n)})_{n\geq 1} \in S(\mathbf{C}^N)^\infty$ and $\{c_n \in \mathbf{C} : |c_n| = 1,\, n \geq 1\} \subset U(1)$, $z' \equiv (c_n z^{(n)})_{n\geq 1}$ is equivalent to z.*

Proof. By Definition 2.2, they hold immediately. $\square$

Lemma A.6. (i) *If $z, y \in S(\mathbf{C}^N)^\infty$ are not equivalent, then*

$$\lim_{n\to\infty} < z[n]|y[n+p] >= 0$$

for each $p \geq 0$ where $< z[n]|y[n] >$ is the inner product of vectors $z[n]$, $y[n]$ in $(\mathbf{C}^N)^{\otimes k}$ and $z[n]$ is the symbol in (2.2).
(ii) *If $z \in S(\mathbf{C}^N)^\infty$ is non eventually periodic, then*

$$\lim_{n\to\infty} < z[n]|z[n+p] >= 0$$

for each $p \geq 1$.

Proof. (i) Denote $z = (z^{(n)})$ and $y = (y^{(n)})$.

We show $p = 0$ case at first. By Lemma A.3 (iii), there is $M \geq 1$ such that $| < z^{(M)}|y^{(M)} > | < 1$. We denote $M_1 \equiv M$. If $| < z^{(n)}|y^{(n)} > | = 1$ for each $n > M_1$, then this contradicts $z \not\sim y$ by Lemma A.3 (i). Hence there is $M_2 > M_1$ such that $| < z^{(M_2)}|y^{(M_2)} > | < 1$. In this way, we can takes a monotone increasing sequence $(M_n)_{n\in\mathbf{N}}$ of positive integers such that $| < z^{(M_n)}|y^{(M_n)} > | < 1$ for each $n \geq 1$. From this, $0 \leq | < z[M_n]|y[M_n] > | = | < z^{(1)}|y^{(1)} > | \cdots | < z^{(M_n)}|y^{(M_n)} > | \leq \prod_{k=1}^n | < z^{(M_k)}|y^{(M_k)} > | < 1$. Clearly, $| < z[M_{n+1}]|y[M_{n+1}] > | < | < z[M_n]|y[M_n] > |$ for each $n \geq 1$. Therefore $\lim_{n\to\infty} | < z[n]|y[n] > | \leq \lim_{n\to\infty} | < z[M_n]|y[M_n] > | = 0$. Hence we have $p = 0$ case.

By Lemma A.5 (i), $z \not\sim y' \equiv \sigma^p(y)$ for each $p \geq 0$. Hence

$$\lim_{n\to\infty} < z[n]|y[n+p] >= \lim_{n\to\infty} < z[n]|y'[n] >= 0.$$

(ii) By Lemma A.3, we can obtain a monotone increasing sequence $(M_n)_{n\in\mathbf{N}}$ of positive integers such that $| < z[M_{n+1}]|z[M_{n+1+p}] > | < | < z[M_n]|z[M_n+$

$p] > | < 1$ for each $n \geq 1$. Hence the assertion is verified. $\square$

A.2. Properties of $s(z)$. Recall notations in § 3.

Lemma A.7. (i) *$s(\varepsilon_J) = s_J \equiv s_{j_1} \cdots s_{j_k}$ for $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$ for $k \geq 1$.*

(ii) *A map s from $TS(\mathbf{C}^N)$ to $\mathrm{Iso}\mathcal{O}_N$ is a homomorphism as semigroup, that is, $s(z \otimes y) = s(z)s(y)$ for $z, y \in TS(\mathbf{C}^N)$ where $\mathrm{Iso}\mathcal{O}_N$ is the semigroup of all isometries in $\mathcal{O}_N$. Specially, $s(z[k]) = s(z^{(1)}) \cdots s(z^{(k)})$ for $z = (z^{(n)}) \in S(\mathbf{C}^N)^\infty$, $k \geq 1$.*

(iii) *For $z \in S(\mathbf{C}^N)^{\otimes n}$ and $y \in S(\mathbf{C}^N)^{\otimes m}$, $n, m \geq 1$,*

$$s(z)^* s(y) = \begin{cases} < z|y > I & (n = m) \\ \\ < z|y_1 > s(y_2) & (n < m,\ y = y_1 \otimes y_2,\ y_1 \in S(\mathbf{C}^N)^{\otimes n}), \\ \\ < z_1|y > s(z_2)^* & (n > m,\ z = z_1 \otimes z_2,\ z_1 \in S(\mathbf{C}^N)^{\otimes m}), \end{cases}$$

(iv) *Let $z = (z_1, \ldots, z_N) \in S(\mathbf{C}^N)$. If $g = (g_{ij}) \in U(N)$ satisfies $g_{j1} = z_j$ for $j = 1, \ldots, N$, then we have $s(z) = \alpha_g(s_1)$.*

Proof. (i),(ii),(iii) follow by simple computation.
(iv) $s(z) = z_1 s_1 + \cdots + z_N s_N = g_{11} s_1 + \cdots + g_{N1} s_N = \alpha_g(s_1)$. $\square$

Lemma A.8. (i) *If $z, y \in S(\mathbf{C}^N)^\infty$ are not equivalent, then*

$$\lim_{k \to \infty} \|s(z[k])^* s(y[k])\| = 0.$$

(ii) *If $z \in S(\mathbf{C}^N)^\infty$ is non eventually periodic, then*

$$\lim_{k \to \infty} \|s(z[k])^* s(z[k+p])\| = 0$$

for each $p \geq 1$.

Proof. (i) By Lemma A.7 (iii), $s(z[k])^* s(y[k]) =< z[k]|y[k] > I$. Hence $\|s(z[k])^* s(y[k])\| = | < z[k]|y[k] > |$ for $k \geq 1$. By Lemma A.6 (i), the assertion holds.
(ii) By Lemma A.6 (ii), the statement holds. $\square$

Appendix B. Lemmata on GP representations

B.1. Cycles. The following lemma is shown in [**10**]. We show this here for convenience again. Recall Definition 3.2 (i).

Lemma B.1. *Let $(\mathcal{H}, \pi, \Omega) = GP(z)$ for $z = z^{(1)} \otimes \cdots \otimes z^{(p)} \in S(\mathbf{C}^N)^{\otimes p}$ for $p \geq 1$. Put $\Omega_j \equiv \pi(s(z^{(j+1)}) \cdots s(z^{(p)}))\Omega$ for $j = 0, \ldots, p-1$. If z is non periodic, then $\{\Omega_j\}_{j=0}^{p-1}$ is an orthonormal family.*

Proof. We identify $\pi(s_j)$ and s_j here. We see $\|\Omega_j\| = 1$ for $j = 0,\ldots,p-1$. Put $\sigma(y) = y^{(2)} \otimes \cdots \otimes y^{(p)} \otimes y^{(1)}$ for $y = y^{(1)} \otimes \cdots \otimes y^{(p)} \in S(\mathbf{C}^N)^{\otimes p}$ and $z_j \equiv \sigma^j(z)$ for $j = 0,\ldots,p-1$. Then $s(z_j)\Omega_j = \Omega_j$ for $j = 0,\ldots,p-1$. From this, $< \Omega_i|\Omega_j >=< z_i|z_j >< \Omega_i|\Omega_j >$. Furthermore $< \Omega_i|\Omega_j >=< z_i|z_j >^n< \Omega_i|\Omega_j >$ for each $n \in \mathbf{N}$. By Schwarz inequality, $| < \Omega_i|\Omega_j > | \leq | < z_i|z_j > |^n$. If z is non periodic and $i \neq j$, $| < z_i|z_j > |^n$ goes to 0 when $n \to \infty$. Hence $< \Omega_i|\Omega_j >= 0$ when $i \neq j$. $\square$

Lemma B.2. *Let $z \in S(\mathbf{C}^N)^{\otimes p}$ for $p \geq 1$. Then the followings hold:*

(i) *For $c \in U(1)$, $GP(cz) = GP(z) \circ \gamma_{\bar{c}^{1/p}}$.*

(ii) *Any two elements in $\{GP(cz)\}_{c \in U(1)}$ are mutually inequivalent.*

Proof. Assume $(\mathcal{H}, \pi, \Omega) = GP(z)$.
(i) Put $\pi' \equiv \pi \circ \gamma_{\bar{c}^{1/p}}$. Then $\pi'(s(cz))\Omega = c\,\{\pi(\gamma_{\bar{c}^{1/p}}(s(z)))\Omega\} = c\pi(\bar{c}(s(z)))\Omega = \Omega$. Because π' is cyclic, too, $(\mathcal{H}, \pi', \Omega) = GP(cz)$.
(ii) For $c, c' \in U(1)$, $cz \sim c'z$ if and only if $c = c'$. Hence the statement holds by Theorem 5.3. $\square$

B.2. Chains.

Lemma B.3. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$ and $z \in S(\mathbf{C}^N)^{\infty}$. Assume that there is a unit vectors $\Omega \in \mathcal{H}$ such that Ω satisfies the chain condition with respect to z in Definition* 3.1 *(ii). Then the followings hold:*

(i) *$\pi(s_J^*)\Omega =< \varepsilon_J|z[k] > \Omega_k$ for $J \in \{1,\ldots,N\}^k$ and $k \geq 1$ where $\Omega_k \equiv \pi(s(z[k])^*)\Omega$ for $k \geq 1$.*

(ii) *A subspace $\mathcal{H}_1 \equiv \mathrm{Lin} < \{\pi(s_J)\Omega_n : J \in \{1,\ldots,N\}^*, n \in \mathbf{Z}\} >$ of $\pi(\mathcal{O}_N)\Omega$ is dense in $\pi(\mathcal{O}_N)\Omega$.*

Proof. (i) By definition, $\Omega_k = \pi(s(z[k]))\Omega$ for $k \geq 1$. By Lemma A.7 (iii), $\pi(s_J^*)\Omega = \pi(s_J^*)\pi(s(z[k]))\Omega_k = \pi(s_J^* s(z[k]))\Omega_k =< \varepsilon_J|z[k] > \Omega_k$.
(ii) $\mathcal{H}_0 \equiv \mathrm{Lin} < \{\pi(s_I s_J^*)\Omega : I, J \in \{1,\ldots,N\}^*\} >$ is dense in $\pi(\mathcal{O}_N)\Omega$. By (i), $\mathcal{H}_0 \subset \mathcal{H}_1$. Hence $\mathcal{H}_1$ is dense in $\pi(\mathcal{O}_N)\Omega$. $\square$

Lemma B.4. *Let $(\mathcal{H}, \pi, \Omega) = GP(z)$ for non eventually periodic $z \in S(\mathbf{C}^N)^{\infty}$. If $< v|\Omega >= 0$, then*

$$\lim_{n\to\infty} \pi(s(z[n])^* v = 0.$$

Proof. By Lemma B.3 (ii), it is sufficient to show the case $v = s_J\Omega_n$ for $J \in \{1,\ldots,N\}^*$ and $n \in \mathbf{Z}$. Assume $J \in \{1,\ldots,N\}^k$ for $l \geq 0$. For $k \geq l$, $y^{(k)} \equiv \varepsilon_J \otimes z^{(n+1)} \otimes \cdots \otimes z^{(n+k-l)} \in S(\mathbf{C}^N)^{\otimes k}$. Hence $y \equiv (y^{(l)}) \in S(\mathbf{C}^N)^{\infty}$. From this, $\pi(s(z[n])^* v =< z[k]|y[k] > \Omega_{n+k-l}$. Hence $\|\pi(s(z[n])^* v\| = | < z[k]|y[k] > |$. By Lemma A.6 (ii), we have the assertion. $\square$

Lemma B.5. *Let* $(\mathcal{H}, \pi, \Omega) = GP(z)$ *for non eventually periodic* $z \in S(\mathbf{C}^N)^\infty$. *If* $v_0 \in \mathcal{H}$ *satisfies* $< v_0|\Omega > \neq 0$, *then there is* $c \in \mathbf{C}$, $c \neq 0$ *such that*

$$\lim_{n\to\infty} \pi(s(z[n])\,\{s(z[n])\}^*)v_0 = c\Omega.$$

Proof. We simply denote $\pi(s_i)$ and s_i here. Assume $c \equiv < \Omega|v_0 > \neq 0$ and put $v \equiv v_0 - c\Omega$. Then we have

$$\|s(z[n])\,\{s(z[n])\}^*\, v_0 - s(z[n])\,\{s(z[n])\}^*\,(c\Omega)\| \leq \|s(z[n])\,\{s(z[n])\}^*\, v\|.$$

By $< \Omega|v >= 0$ and Lemma B.4,

$$\lim_{n\to\infty} s(z[n])\,\{s(z[n])\}^*\, v_0 = \lim_{n\to\infty} s(z[n])\,\{s(z[n])\}^*\,(c\Omega).$$

Since $s(z[n])\,\{s(z[n])\}^*\,(c\Omega) = cs(z[n])\Omega_n = c\Omega$, we have the assertion. $\square$

Appendix C. Lemmata for Theorem 5.3

Lemma C.1. *Let* $(\mathcal{H}, \pi)$ *be a representation of* $\mathcal{O}_N$ *and* $z, y \in S(\mathbf{C}^N)^\infty$. *Assume that there are unit vectors* Ω *and* $\Omega^{'}$ *in* $\mathcal{H}$ *such that* Ω *and* $\Omega^{'}$ *satisfy the chain condition with chains* $\{\Omega_n\}_{n\in\mathbf{N}}$ *and* $\{\Omega^{'}_n\}_{n\in\mathbf{N}}$ *with respect to* z *and* y *in Definition* 3.1 *(ii), respectively. Then we have the followings:*

- (i) *If* $z \not\sim y$, *then* $< \Omega|\Omega^{'} >= 0$.
- (ii) *If* $z \not\sim y$, *then* $< \Omega_k|\Omega^{'}_l >= 0$ *for each* $k, l \geq 1$.
- (iii) *If* $\{\Omega_n\}_{n\in\mathbf{N}}$ *and* $\{\Omega^{'}_n\}_{n\in\mathbf{N}}$ *are orthogonal, then* $< \pi(s_I)\Omega_k|\pi(s_J)\Omega^{'}_l >= 0$ *for each* $I, J \in \{1, \ldots, N\}^*$ *and* $k, l \geq 1$.

Proof. (i) By Lemma 4.1, $< \Omega|\Omega^{'} >=< z[k]|y[k] >< \Omega_k|\Omega^{'}_k >$ for each $k \geq 1$. By Schwarz inequality and Lemma A.6 (i), $| < \Omega|\Omega^{'} > | \leq | < z[k]|y[k] > | \to 0$ when $k \to \infty$. Hence $< \Omega|\Omega^{'} >= 0$.
(ii) For each $k, l \geq 1$, Ω_k and $\Omega^{'}_l$ satisfy chain conditions with respect to $\sigma^k(z)$ and $\sigma^l(y)$. Because $z \not\sim y$, we see $\sigma^k(z) \not\sim \sigma^l(y)$. By (i), we have the assertion.
(iii) If $|I| = |J|$, then $< \pi(s_I)\Omega_k|\pi(s_J)\Omega^{'}_l >= \delta_{IJ} < \Omega_k|\Omega^{'}_l >= 0$ by (ii). If $|I| = k^{'} + l^{'} > k^{'} \equiv |J|$, $l^{'} \geq 1$, then $< \pi(s_I)\Omega_k|\pi(s_J)\Omega^{'}_l >= \delta_{I_1 J} < \pi(s_{I_2})\Omega_k|\Omega^{'}_l >$. Hence $< \pi(s_{I_2})\Omega_k|\Omega^{'}_l >=< \varepsilon_{I_2}|\, y[l+1, 1+k^{'}] >< \Omega_k|\Omega^{'}_{k^{'}+l} >= 0$ by (ii). $\square$

Lemma C.2. *Let* $(\mathcal{H}, \pi)$ *be a representation of* $\mathcal{O}_N$ *and* $z, y \in S(\mathbf{C}^N)^\infty$. *Assume that there are unit vectors* Ω *and* $\Omega^{'}$ *in* $\mathcal{H}$ *which satisfy the chain condition with respect to* z *and* y, *and* $\{\Omega_n\}$ *and* $\{\Omega^{'}_n\}$ *are their chains, respectively. If* $\{\Omega_n\}$ *and* $\{\Omega^{'}_n\}$ *are mutually orthogonal, then* $\pi(\mathcal{O}_N)\Omega$ *and* $\pi^{'}(\mathcal{O}_N)\Omega^{'}$ *are orthogonal each other.*

Proof. By Lemma B.3 (ii), Lin $< \{\pi(s_J)\Omega : J \in \{1,\ldots,N\}^*\} >$ and Lin $< \{\pi^{'}(s_J)\Omega^{'} : J \in \{1,\ldots,N\}^*\} >$ are dense in $\pi(\mathcal{O}_N)\Omega$ and $\pi^{'}(\mathcal{O}_N)\Omega^{'}$, respectively. By Lemma C.1 (iii), the assertion holds. □

Lemma C.3. (i) *Let $z \in S(\mathbf{C}^N)^\infty$. Then $GP(\sigma^L(z)) = GP(z)$ for each $L \geq 0$.*

(ii) *For $z \in S(\mathbf{C}^N)^\infty$ and $\{c_n\} \subset U(1)$, put $y \equiv (y^{(n)})_{n\in\mathbf{N}} \in S(\mathbf{C}^N)^\infty$ by $y^{(n)} \equiv c_n z^{(n)}$. Then $GP(y) \sim GP(z)$.*

Proof. (i) Let $(\mathcal{H},\pi,\Omega) = GP(z)$. Then $\{\pi(s(z[k])^*)\Omega : k \geq 1\}$ is an orthonormal family. Hence $\{\pi(s(z[k+L])^*)\Omega : k \geq 1\}$ is, too. Hence $(\mathcal{H},\pi,\Omega)$ satisfies the chain condition of $GP(\sigma(z))$. By Theorem 5.1, $GP(\sigma^L(z)) \sim GP(z)$.
(ii) Let $(\mathcal{H},\pi,\Omega) = GP(z)$. By Lemma A.2, there is $\{C_k\} \subset U(1)$ such that $z[k] = C_k y[k]$ for each $k \geq 1$. Hence $\pi(s(y[k])^*)\Omega = \pi(s(\bar{C}_k z[k])^*)\Omega = C_k\pi(s(z[k])^*)\Omega$. Therefore $\{\pi(s(y[k])^*)\Omega : k \geq 1\}$ is an orthonormal family in $\mathcal{H}$. Hence $(\mathcal{H},\pi,\Omega) = GP(y)$. By Theorem 5.1, $GP(y) \sim GP(z)$. □

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