Higher monodromy

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Abstract

For a given category C and a topological space X, the constant stack on X with stalk C is the stack of locally constant sheaves with values in C. Its global objects are classified by their monodromy, a functor from the Poincaré groupoid $\Pi_1(X)$ to C. In this paper we recall these notions from the point of view of higher category theory and define the 2-monodromy of locally constant stacks with values in a 2-category $\mathbf{C}$ as a 2-functor from the homotopy 2-groupoid $\Pi_2(X)$ to $\mathbf{C}$. We show that 2-monodromy classifies locally constant stacks on a reasonably well-behaved space X. As an application, we show how to get classical formulae of algebraic topology from this classification, and we extend them to the non abelian case.

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Contents

Introduction

A classical result in algebraic topology is the classification of the coverings of a (reasonably well-behaved) path-connected topological space X by means of the representations of its fundamental group $\pi_1(X)$. In the language of sheaves this generalizes to an equivalence between the category of locally constant sheaves of sets on X and that of representations of $\pi_1(X)$ on the stalks. The equivalence is given by the functor which assignes to each locally constant sheaf $\mathcal{F}$ with stalk S its monodromy $\mu(\mathcal{F})\colon \pi_1(X) \to \mathrm{Aut}(S)$.

Now, let C be a category. It makes sens to consider the representations of $\pi_1(X)$ on C, in other words functors from the Poincaré groupoid $\Pi_1(X)$ to C. One would then say that they classify locally constant sheaves on X "with stalk in the category C" even if there do not exist any sheaves with values in C. To state this assertion more precisely, one needs the language of stacks of Grothendieck and Giraud. A stack is, roughly speaking, a sheaf of categories and one may consider the constant stack C_X on X with stalk the category C (if C is the category of sets, one recovers the stack of locally constant sheaves of sets). Then one defines a locally constant sheaf on X with values in C to be a global section of the constant stack C_X. The monodromy functor establishes (on a locally relatively 1-connected space X) an equivalence of categories between global sections of C_X and functors $\Pi_1(X) \to \mathsf{C}$.

A question naturally arises: what classifies stacks on X which are locally constant? Or, on the other side, which geometrical objects are classified by representations (*i.e.* 2-functors) of the homotopy 2-groupoid $\Pi_2(X)$? We define a locally constant stack with values in a 2-category $\mathbf{C}$ as a global section of the constant 2-stack $\mathbf{C}_X$. In this paper, we give an explicit construction of the 2-monodromy of such a stack as a 2-functor $\mu^2(\mathfrak{S})\colon \Pi_2(X) \to \mathbf{C}$. We will show that, for locally relatively 2-connected topological spaces, a locally constant stack is uniquely determined (up to equivalence) by its 2-monodromy. We then use this result to extend classical formulae of algebraic topology to the non abelian case, relating Giraud's second non abelian cohomology set with constant coeffiecients of X to its first and second homotopy group.

During the preparation of this work, a paper [12] of B. Toen appeared, where a similar result about locally constant ∞-stacks and their ∞-monodromy is established. His approach is different from ours, since we do not use any model category theory and any simplicial techniques, but only classical 2-category (and enriched higher category) theory. Moreover, since we are only interested in the degree 2 monodromy, we need weaker hypothesis on the space X than *loc.cit.*, where the author works on the category of CW-complexes.

This paper is organised as follow. In Chapter 1 we recall the basic notions on stacks and give a functorial construction of the classical monodromy. Our approach appears at first view to be rather heavy, as we use more language and machinery in our definition as is usually done when one considers just monodromy for sheaves of sets or abelian groups. The reason for our category theoretical approach is to motivate the construction of the 2-monodromy (and to give a good idea how one could define n-monodromy of a locally constant n-stack with values in an n-category, for all n). As a byproduct we get the classification of locally constant sheaves with values in finite categories (*e.g.* in the category defined by a group) which yields an amusing way to recover some classical formulae relating the first non abelian cohomology set with constant coefficients to the representations of the fundamental group.

In Chapter 2 we introduce the 2-monodromy 2-functor of a locally constant stack with values in a 2-category. This construction is analogous to our approach to 1-monodromy, but the diagrams which should be checked for commutativity become rather large. One reason for our lengthy tale on 1-monodromy is to give good reasons to believe in our formulae, since we do not have the space to write down complete proofs. We also describe the 2-monodromy as a descent datum on the loop space at a fixed point. Finally, we give some explicit calculations about the classification of gerbes with locally constant bands. This is related to Giraud's second non abelian cohomology set with constant coeffiecients.

In Appendix A we review the construction of the stack of sheaves with values in a complete category and in Appendix B the construction of the 2-stack of stacks with values in a 2-complete 2-category.

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Notations and conventions

We assume that the reader is familiar with the basic notions of classical category theory, as those of category, functor between categories, transformation between functors (also called morphism of functors), equivalence of categories, monoidal category and monoidal functor. We will also use some notions from higher category theory, as 2-categories, 2-functors, 2-transformations, modifications, 2-limits and 2-colimits. Moreover, we will look at 3-categories and 3-functors, but only in the context of a "category enriched in 2-categories" which is much more elementary than the general theory of n-categories for $n \geqslant 3$. References are made to [10, 9, 1, 11][1].

We use the symbols C, D, etc., to denote categories. If C is a category, we denote by $\mathrm{Ob}\,\mathsf{C}$ (resp. $\pi_0(\mathsf{C})$) the collection of its objects (resp. of isomorphism classes of its objects), and by $\mathrm{Hom}_{\mathsf{C}}(P, Q)$ the set of morphisms between the objects P and Q (if $\mathsf{C} = \mathsf{Set}$, the category of all small sets, we will write $\mathrm{Hom}\,(P, Q)$ instead of $\mathrm{Hom}_{\mathsf{Set}}(P, Q)$). For a category C, its opposite category is denoted by C°.

We use the symbols $\mathbf{C}, \mathbf{D}$, etc., to denote 2-categories. If $\mathbf{C}$ is a 2-category, we denote by $\mathrm{Ob}\,\mathbf{C}$ (resp. $\boldsymbol{\pi}_0(\mathbf{C})$) the collection of its objects (resp. of equivalence classes of its objects), and by $\mathsf{Hom}_{\mathbf{C}}(\mathrm{P}, \mathrm{Q})$ the small category of arrows between the objects P and Q (if $\mathbf{C} = \mathbf{Cat}$, the strict 2-category of all small categories, we will use the shorter notation $\mathsf{Hom}\,(\mathsf{C}, \mathsf{D})$ to denote the category of functors between C and D). Given two 2-categories $\mathbf{C}$ and $\mathbf{D}$, the 2-category of 2-functors from $\mathbf{C}$ to $\mathbf{D}$ will be denoted by $\mathbf{Hom}\,(\mathbf{C}, \mathbf{D})$. If G is a commutative group, we will use the notation $\mathbf{Cat}_G$ for the 2-category of G-linear categories and $\mathsf{Hom}_G(\mathsf{C}, \mathsf{D})$ for the G-linear category of G-linear functors. If $\mathbf{C}$ is a 2-category, $\mathbf{C}^{\circ}$, denotes its opposite 2-category and for any object $\mathrm{Q} \in \mathrm{Ob}\,\mathbf{C}$, we set $\mathrm{Pic}_{\mathbf{C}}(\mathrm{Q}) = \pi_0(\mathsf{Aut}_{\mathbf{C}}(\mathrm{Q}))$[2], where $\mathsf{Aut}_{\mathbf{C}}(\mathrm{Q})$ denotes the monoidal category of auto-

[1]Note that 2-categories are called bicategories by some authors, for which a 2-category is what we call a strict 2-category. Similarly, a 2-functor is sometimes called a pseudo-functor

[2]This is consistent with the classical notion of strict gr- group. Indeed, if R is a ring, by the Morita theorem the group $\mathrm{Pic}_{\mathbf{Cat}_{\mathbb{Z}}}(\mathsf{Mod}(R))$ is isomorphic to the Picard group of R.

equivalences of Q in $\mathbf{C}$. Note that the group $\mathrm{Pic}_{\mathbf{C}}(\mathrm{Q})$ acts on the commutative group $\mathrm{Z}_{\mathbf{C}}(\mathrm{Q}) = \mathrm{Aut}_{\mathsf{Aut}_{\mathbf{C}}}(\mathrm{Q})(\mathrm{id}_{\mathrm{Q}})$ by conjugation. If there is no risk of confusion, for a category (resp. G-linear category) C, we will use the shorter notations $\mathrm{Pic}(\mathsf{C})$ (resp. $\mathrm{Pic}_G(\mathsf{C})$) and$\mathrm{Z}(\mathsf{C})$.

1 Locally constant sheaves with values in a category

1.1 General definitions

We start by recalling the definition of a stack on a topological space. The classical reference is Giraud's book [7]. Our presentation follows that of [8], to which we refer for more details.

Let X be a topological space. Denote by $\mathsf{Op}(X)$ the category of its open subsets with inclusion morphisms and by $\mathbf{Op}(X)$ the 2-category obtained by trivially enriching $\mathsf{Op}(X)$ with identity 2-arrows. Recall that a prestack of categories[3] on X is a 2-functor

$$\mathfrak{S} : \mathbf{Op}(X)^{\circ} \longrightarrow \mathbf{Cat}.$$

A functor between prestacks is a 2-transformation of 2-functors and transformations of functors of prestacks are modifications of 2-transformations of 2-functors. Two prestacks are called equivalent if there exist functors $F\colon \mathfrak{S} \to \mathfrak{S}'$ and $G\colon \mathfrak{S}' \to \mathfrak{S}$ such that $\mathrm{id}_{\mathfrak{S}} \simeq G \circ F$ and $\mathrm{id}_{\mathfrak{S}'} \simeq F \circ G$. We denote by $\mathbf{PSt}(X)$ the strict 2-category of prestacks on X.

Recall that a descent datum for $\mathfrak{S}$ on an open subset $U \subset X$ is a triplet

$$F = (\{U_i\}_{i\in I}, \{\mathcal{F}_i\}_{i\in I}, \{\theta_{ij}\}_{i,j\in I}), \tag{1.1.1}$$

where $\{U_i\}_{i\in I}$ is an open covering of U, $\mathcal{F}_i \in \mathfrak{S}(U_i)$, and $\theta_{ij}\colon \mathcal{F}_j|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_i|_{U_{ij}}$ are isomorphisms such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}_j|_{U_{ijk}} & \mathcal{F}_j|_{U_{ij}}|_{U_{ijk}} \arrow[l, "\mathfrak{S}", "\sim"'] \arrow[rr, "\theta_{ij}", "\sim"'] & & \mathcal{F}_i|_{U_{ij}}|_{U_{ijk}} \arrow[r, "\mathfrak{S}", "\sim"'] & \mathcal{F}_i|_{U_{ijk}} \\
\mathcal{F}_j|_{U_{jk}}|_{U_{ijk}} \arrow[u, "\mathfrak{S}", "\sim"'] & & & & \mathcal{F}_i|_{U_{ik}}|_{U_{ijk}} \arrow[u, "\mathfrak{S}", "\sim"'] \\
\mathcal{F}_k|_{U_{jk}}|_{U_{ijk}} \arrow[u, "\theta_{jk}", "\sim"'] \arrow[rr, "\mathfrak{S}", "\sim"'] & & \mathcal{F}_k|_{U_{ijk}} & & \mathcal{F}_k|_{U_{ik}}|_{U_{ijk}}. \arrow[ll, "\mathfrak{S}"', "\sim"] \arrow[u, "\theta_{ik}", "\sim"']
\end{tikzcd}$$

The descent datum F is called effective if there exist $\mathcal{F} \in \mathfrak{S}(U)$ and isomorphisms

[3] More generally, one may replace $\mathbf{Cat}$ by a 2-category $\mathbf{C}$ and get a prestack with values in $\mathbf{C}$. See Appendix B for more details.

$\theta_i \colon \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ for each i, such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}|_{U_j}|_{U_{ij}} \arrow[r, "\mathfrak{S}", "\sim"'] \arrow[d, "\theta_j"', "\sim"] & \mathcal{F}|_{U_{ij}} & \mathcal{F}|_{U_i}|_{U_{ij}} \arrow[l, "\mathfrak{S}"', "\sim"] \arrow[d, "\theta_i"', "\sim"] \\
\mathcal{F}_j|_{U_{ij}} \arrow[rr, "\theta_{ij}", "\sim"'] & & \mathcal{F}_i|_{U_{ij}}.
\end{tikzcd}$$

Definition 1.1.1. (i) A prestack $\mathfrak{S}$ is called separated if for each open subset U and $\mathcal{F}, \mathcal{G} \in \mathfrak{S}(U)$, the presheaf of sets on U

$$\mathcal{H}om_{\mathfrak{S}|_U}(\mathcal{F}, \mathcal{G}) \colon V \mapsto \mathrm{Hom}_{\mathfrak{S}(V)}(\mathcal{F}|_V, \mathcal{G}|_V)$$

is a sheaf.

(ii) A separated prestack $\mathfrak{S}$ is called a stack if each descent datum is effective.

(iii) A functor (resp. transformation of functors, resp. equivalence) of stacks is a functor (resp. transformation of functors, resp. equivalence) of the underlying prestacks. One denotes by $\mathbf{St}(X)$ the strict 2-category of stacks on X.

(iv) A functor of stacks is called faithful (resp. fully faithful) if it is faithful (resp. fully faithful) on each open subset.

Note that (i) implies that if $\{U_i\}_{i \in I}$ is an open covering of an open subset U and if $\mathcal{F}, \mathcal{G} \in \mathfrak{S}(U)$ are such that $\mathcal{F}|_{U_i} \simeq \mathcal{G}|_{U_i}$ for all $i \in I$, then $\mathcal{F} \simeq \mathcal{G}$.

For $\mathfrak{S}$ and $\mathfrak{S}'$ stacks on X, we will denote by $\mathfrak{Hom}(\mathfrak{S}, \mathfrak{S}')$ the prestack, which is actually a stack, associating to an open subset $U \subset X$ the category $\mathsf{Hom}(\mathfrak{S}|_U, \mathfrak{S}'|_U)$ of functors from $\mathfrak{S}|_U$ to $\mathfrak{S}'|_U$.

As for presheaves, to any prestack $\mathfrak{S}$ one naturally associates a stack $\mathfrak{S}^{\ddagger}$. Precisely, one has the following

Proposition 1.1.2. *The forgetful 2-functor*

$$\mathsf{For}: \ \mathbf{St}(X) \longrightarrow \mathbf{PSt}(X)$$

has a 2-left adjoint 2-functor

$$\ddagger: \ \mathbf{PSt}(X) \longrightarrow \mathbf{St}(X).$$

Let us fix an adjunction 2-transformation

$$\eta_X : \ \mathsf{Id}_{\mathbf{PSt}(X)} \longrightarrow \mathsf{For} \circ \ \ddagger.$$

Note that there is an obvious fully faithful[4] 2-functor of 2-categories $\mathbf{Cat} \longrightarrow \mathbf{PSt}(X)$ which associates to a category C the constant prestack on X with stalk C.

[4]Recall that a 2-functor $\mathsf{F} \colon \mathbf{C} \to \mathbf{D}$ is faithful (resp. full, resp. fully faithful) if for any objects $X, Y \in \mathrm{Ob}\,\mathbf{C}$, the induced functor $\mathsf{F} \colon \mathsf{Hom}_{\mathbf{C}}(X,Y) \to \mathsf{Hom}_{\mathbf{D}}(\mathsf{F}(X), \mathsf{F}(Y))$ is faithful (resp. full and essentially surjective, resp. an equivalence of categories).

Definition 1.1.3. Let C be a category. The constant stack on X with stalk C is the image of C by the 2-functor

$$(\,\cdot\,)_X \colon \mathbf{Cat} \longrightarrow \mathbf{PSt}(X) \xrightarrow{\ddagger} \mathbf{St}(X).$$

Note that the 2-functor $(\,\cdot\,)_X$ conserves faithful and fully faithful functors (hence sends (full) subcategories to (full) substacks). Moreover, the 2-transformation η_X induces on global sections a natural faithful functor

$$\eta_{X,\mathsf{C}} \colon \mathsf{C} \longrightarrow \mathsf{C}_X(X).$$

Definition 1.1.4. An object $\mathcal{F} \in \operatorname{Ob} \mathsf{C}_X(X)$ is called a locally constant sheaf on X with values in C. A locally constant sheaf is constant with stalk M if it is isomorphic to $\eta_{X,\mathsf{C}}(M)$ for some object $M \in \operatorname{Ob} \mathsf{C}$.

Let $\mathsf{C} = \mathsf{Set}$, the category of all small sets (resp. $\mathsf{C} = \mathsf{Mod}(A)$, the category of left A-modules for some ring A). Then it is easy to see that C_X is naturally equivalent to the stack of locally constant sheaves of sets (resp. A_X-module). Moreover the functor $\eta_{X,\mathsf{C}} \colon \mathsf{C} \to \mathsf{C}_X(X)$ is canonically equivalent to the functor which associates to a set S (resp. an A-module M) the constant sheaf on X with stalk S (resp. M). More generally we have

Proposition 1.1.5. *Let C be a complete category*[5] *and X a locally connected topological space. Denote by $\mathfrak{LcSh}_X(\mathsf{C})$ the full substack of the stack $\mathfrak{Sh}_X(\mathsf{C})$ of sheaves with values in C, whose objects are locally constant. Then there is a natural equivalence of stacks*

$$\mathsf{C}_X \xrightarrow{\sim} \mathfrak{LcSh}_X(\mathsf{C}).$$

For a detailed construction of the stacks $\mathfrak{Sh}_X(\mathsf{C})$ and $\mathfrak{LcSh}_X(\mathsf{C})$, see Appendix A.

Remark 1.1.6. Suppose now that C is a category that is not necessarily complete. The category $\widehat{\mathsf{C}} = \mathsf{Hom}\,(\mathsf{C}^\circ, \mathsf{Set})$ of contravariant functors with values in Set is complete (and cocomplete), and the Yoneda embedding

$$Y \colon \mathsf{C} \longrightarrow \widehat{\mathsf{C}}$$

commutes to small limits. Then one usually defines sheaves with values in C as presheaves that are sheaves in the category $\mathsf{PSh}_X(\widehat{\mathsf{C}})$. Note that in general there does not exist a sheaf with values in C (take for example C the category of finite sets), but if $\mathsf{C} \neq \varnothing$ then the category $\mathsf{C}_X(X)$ of locally constant sheaves with values in C is always non-empty. More precisely we get a fully faithful functor of stacks

$$Y_X \colon \mathsf{C}_X \longrightarrow (\widehat{\mathsf{C}})_X \simeq \mathfrak{LcSh}_X(\widehat{\mathsf{C}}).$$

Let $\mathcal{F} \in \mathfrak{LcSh}_X(\widehat{\mathsf{C}})$. Then $\mathcal{F}$ is in the essential image of C_X if and only if all stalks are representable.

[5] Recall that a complete category is a category admitting all small limits.

1.2 Operations on constant stacks

Let $f\colon X \to Y$ be a continuous map of topological spaces, $\mathfrak{S}$ a prestack on X and $\mathfrak{D}$ a prestack on Y.

Notation 1.2.1. (i) Denote by $f_*\mathfrak{S}$ the prestack on Y such that, for any open set $V \subset Y$, $f_*\mathfrak{S}(V) = \mathfrak{S}(f^{-1}(V))$. If $\mathfrak{S}$ is a stack on X, then $f_*\mathfrak{S}$ is a stack on Y.

(ii) Denote by $f_p^{-1}\mathfrak{D}$ the prestack on X such that, for any open set $U \subset X$, $f_p^{-1}\mathfrak{D}(U) = \underset{f(U)\subset V}{2\varinjlim}\,\mathfrak{D}(V)$. If $\mathfrak{D}$ is a stack on Y, we set $f^{-1}\mathfrak{D} = (f_p^{-1}\mathfrak{D})^{\ddagger}$.

Recall that the category $\underset{f(U)\subset V}{2\varinjlim}\,\mathfrak{D}(V)$ is described as follows:

$$\mathrm{Ob}(\underset{f(U)\subset V}{2\varinjlim}\,\mathfrak{D}(V)) = \bigsqcup_{f(U)\subset V} \mathrm{Ob}(\mathfrak{D}(V)),$$

$$\mathrm{Hom}_{\underset{f(U)\subset V}{2\varinjlim}\,\mathfrak{D}(V)}(G_V, G_{V'}) = \varinjlim_{f(U)\subset V''\subset V\cap V'} \mathrm{Hom}_{\mathfrak{D}(V'')}(G_V|_{V''}, G_{V'}|_{V''}).$$

Proposition 1.2.2. *The 2-functors*

$$f_*\colon \mathbf{St}(X) \longrightarrow \mathbf{St}(Y) \qquad f^{-1}\colon \mathbf{St}(Y) \longrightarrow \mathbf{St}(X)$$

are 2-adjoint, f_* being the right 2-adjoint of f^{-1}.

Moreover, if $g\colon Y \to Z$ is another continuous map, one has natural equivalences[6] of 2-functors

$$g_* \circ f_* \simeq (g \circ f)_*, \qquad f^{-1} \circ g^{-1} \simeq (g \circ f)^{-1}.$$

For each continuous map $f\colon X \to Y$, the following diagram commutes up to equivalence

$$\begin{tikzcd} & \mathbf{PSt}(Y) \arrow[r, "\ddagger"] \arrow[d, "f_p^{-1}"] & \mathbf{St}(Y) \arrow[d, "f^{-1}"] \\ \mathbf{Cat} \arrow[ur] \arrow[dr] & & \\ & \mathbf{PSt}(X) \arrow[r, "\ddagger"] & \mathbf{St}(X), \end{tikzcd} \tag{1.2.1}$$

hence f^{-1} preserves constant stacks (up to natural equivalence).

Definition 1.2.3. Denote by $\Gamma(X, \cdot\,)$ the 2-functor of global sections

$$\mathbf{St}(X) \longrightarrow \mathbf{Cat} \quad ; \quad \mathfrak{S} \mapsto \Gamma(X, \mathfrak{S}) = \mathfrak{S}(X)$$

and set $\Gamma_X = \Gamma(X, \cdot) \circ (\cdot)_X$.

[6]For sake of simplicity, here and in the sequel we use the word "equivalence" for a coherently invertible 2-transformation. In the case of the inverse image, to be natural means that if we consider $h = f_3 \circ f_2 \circ f_1$, $g_1 = f_2 \circ f_1$ and $g_2 = f_3 \circ f_2$, then the two equivalences $h^{-1} \simeq f_1^{-1} g_2^{-1} \simeq f_1^{-1} f_2^{-1} f_3^{-1}$ and $h^{-1} \simeq g_1^{-1} f_3^{-1} \simeq f_1^{-1} f_2^{-1} f_3^{-1}$ are naturally isomorphic by a modification, in the sense that, if we look at $k = f_4 \circ f_3 \circ f_2 \circ f_1$, we get the obvious commutative diagram of modifications.

Note that for any stack $\mathfrak{S}$, $\mathfrak{S}(\varnothing)$ is the terminal category (which consists of precisely one morphism). Hence the 2-functor

$$\Gamma(\{pt\}, \cdot):\ \mathbf{St}(\{pt\}) \longrightarrow \mathbf{Cat}$$

is an equivalence of 2-categories.

Proposition 1.2.4. *The 2-functor* $\Gamma(X, \cdot)$ *is right 2-adjoint to* $(\cdot)_X$.

Proof. Consider the map $a_X :\ X \to \{pt\}$. Then $\Gamma(X, \cdot)$ is canonically equivalent to

$$\mathbf{St}(X) \xrightarrow{a_{X*}} \mathbf{St}(\{pt\}) \xrightarrow{\sim} \mathbf{Cat}.$$

From the diagram (1.2.1) one gets that the functor $(\cdot)_X$ is canonically equivalent to

$$\mathbf{Cat} \xrightarrow{\sim} \mathbf{St}(\{pt\}) \xrightarrow{a_X^{-1}} \mathbf{St}(X).$$

Then the result follows from Proposition 1.2.2. $\square$

It is not hard to see that we can choose the functors $\eta_{X,\mathsf{C}} \colon \mathsf{C} \longrightarrow \Gamma(X, \mathsf{C}_X)$ to define the adjunction 2-transformation

$$\eta_X \colon \mathsf{Id}_{\mathbf{Cat}} \longrightarrow \Gamma_X.$$

Consider the commutative diagram of topological spaces

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "a_X"'] & & Y \arrow[dl, "a_Y"] \\ & \{pt\} & \end{tikzcd}$$

and the induced 2-transformation of 2-functors

$$a_{Y*} \circ (\cdot)_Y \longrightarrow a_{Y*} \circ f_* \circ f^{-1} \circ (\cdot)_Y \simeq a_{X*} \circ f^{-1} \circ (\cdot)_Y \simeq a_{X*} \circ (\cdot)_X.$$

Hence we get a 2-transformation of 2-functors f^{-1} compatible with η_X and η_Y, *i.e.* the following diagram commutes up to a natural invertible modification:

$$\begin{tikzcd} \Gamma_Y \arrow[rr, "f^{-1}"] & & \Gamma_X. \\ & \mathsf{Id}_{\mathbf{Cat}} \arrow[ul, "\eta_Y"] \arrow[ur, "\eta_X"'] & \end{tikzcd}$$

Note that this implies that for any point $x \in X$ and any $\mathcal{F}, \mathcal{G} \in \mathsf{C}_X(X)$, the natural morphism

$$i_x^{-1}\mathcal{H}om_{\mathsf{C}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(i_x^{-1}\mathcal{F}, i_x^{-1}\mathcal{G})$$

is an isomorphism (here $i_x \colon \{pt\} \to X$ denotes the natural map sending $\{pt\}$ to x and we identify C with global sections of C_{pt}). Therefore, for any map $f \colon X \to Y$, we get a natural isomorphism

$$f^{-1}\mathcal{H}om_{\mathsf{C}_Y}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_{\mathsf{C}_X}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G}).$$

Lemma 1.2.5. *Let $I = [0,1]$ and $t \in I$. Consider the maps*

$$X \underset{p}{\overset{\iota_t}{\rightleftarrows}} X \times I,$$

where $\iota_t(x) = (x,t)$ and p is the projection. Then the 2-transformations

$$\mathsf{\Gamma}_X \underset{\iota_t^{-1}}{\overset{p^{-1}}{\rightleftarrows}} \mathsf{\Gamma}_{X\times I}$$

are equivalences of 2-functors, quasi-inverse one to each other.

Proof. Let C be a category. It is sufficent to show that

$$\mathsf{\Gamma}(X, \mathsf{C}_X) \underset{\iota_t^{-1}}{\overset{p^{-1}}{\rightleftarrows}} \mathsf{\Gamma}(X \times I, \mathsf{C}_{X\times I})$$

are natural equivalences of categories. Since $\iota_t^{-1} \circ p^{-1} \simeq (p \circ \iota_t)^{-1} = \mathrm{id}_{\mathsf{\Gamma}(X,\mathsf{C}_X)}$, it remains to check that for each $\mathcal{F} \in \mathsf{\Gamma}(X \times I, \mathsf{C}_{X\times I})$ there is a natural isomorphism $p^{-1}\iota_t^{-1}\mathcal{F} \simeq \mathcal{F}$.

First let us prove that if $\mathcal{F}$ is a locally constant sheaf of sets on $X \times I$, then the natural morphism

$$\Gamma(X \times I, \mathcal{F}) \longrightarrow \Gamma(X, \iota_t^{-1}\mathcal{F}) \tag{1.2.2}$$

is an isomorphism. Indeed, let s and s' be two sections in $\Gamma(X \times I, \mathcal{F})$ such that $s_{x,t} = s'_{x,t}$. Since $\mathcal{F}$ is locally constant, the set $\{t' \in I \mid s_{x,t'} = s_{x,t}\}$ is open and closed, hence equal to I. Therefore the map is injective. Now let $s \in \Gamma(X, i_t^{-1}\mathcal{F})$. Then s is given by sections s_j of $\mathcal{F}$ on on a family $(U_j \times I_j)_{j\in J}$ where I_j is an open interval containing t, the U_j cover X and the sheaf $\mathcal{F}$ is constant on $U_j \times I_j$. It is not hard to see that by refining the covering the sections s_j can be extended to $U_j \times I$ and using the injectivity of the map one sees that we can patch the extensions of the s_j to get a section of $\mathcal{F}$ on $X \times I$ that is mapped to s. Hence the morphism (1.2.2) is an isomorphism.

Now let $\mathcal{F} \in \mathsf{\Gamma}(X \times I, \mathsf{C}_{X\times I})$. Since $\iota_t^{-1}\mathcal{H}om_{\mathsf{C}_{X\times I}}(p^{-1}\iota_t^{-1}\mathcal{F}, \mathcal{F}) \simeq \mathcal{H}om_{\mathsf{C}_X}(\iota_t^{-1}\mathcal{F}, \iota_t^{-1}\mathcal{F})$, by (1.2.2) we have an isomorphism

$$\Gamma(X \times I, \mathcal{H}om_{\mathsf{C}_{X\times I}}(p^{-1}\iota_t^{-1}\mathcal{F}, \mathcal{F})) \xrightarrow{\sim} \Gamma(X, \mathcal{H}om_{\mathsf{C}_X}(\iota_t^{-1}\mathcal{F}, \iota_t^{-1}\mathcal{F})).$$

It is an easy exercise to verify that this morphism and its inverse are compatible with the composition of morphisms of locally constant sheaves and preserve isomorphisms. Then the isomorphism $p^{-1}\iota_t^{-1}\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ is defined by the identity section in the set $\Gamma(X, \mathcal{H}om_{\mathsf{C}_X}(\iota_t^{-1}\mathcal{F}, \iota_t^{-1}\mathcal{F}))$. $\square$

Corollary 1.2.6. *The adjunction 2-transformation*

$$\eta_I \colon \mathsf{Id}_{\mathbf{Cat}} \longrightarrow \mathsf{\Gamma}_I$$

is an equivalence, i.e. for each category C, the functor $\eta_{I,\mathsf{C}} \colon \mathsf{C} \longrightarrow \mathsf{\Gamma}(I, \mathsf{C}_I)$ is a natural equivalence.

Proof. Take $X = \{pt\}$ in Lemma 1.2.5. Then one gets a diagram of equivalences

$$\begin{tikzcd} \Gamma_{pt} \arrow[r, "p^{-1}"] & \Gamma_{\{pt\}\times I} \\ \mathrm{Id}_{\mathbf{Cat}} \arrow[u, "\eta_{pt}"] \arrow[ur, "\eta_{\{pt\}\times I}"'] \arrow[r, "\eta_I"'] & \Gamma_I \arrow[u] \end{tikzcd}$$

which commutes up to invertible modification. □

Remark 1.2.7. For each X and each $t \in I$, we have an invertible modification $\mathrm{id}_{\Gamma_X} \simeq \iota_t^{-1}p^{-1}$. Then for any $s, t \in I$ there exists a unique invertible modification $\iota_t^{-1} \simeq \iota_s^{-1}$ compatible with these modifications. Recall that there exists a unique invertible modification $\mathrm{id}_{\Gamma_X} \simeq p^{-1}i_t^{-1}$ such that our conventions are satisfied. Then the invertible modification $\iota_t^{-1} \simeq \iota_s^{-1}$ is also caracterised by the fact that its image by p^{-1} is precisely $p^{-1}\iota_t^{-1} \simeq \mathrm{id}_{\Gamma_X} \simeq p^{-1}\iota_s^{-1}$. This can be used to prove the following two lemmas by diagram chases.

Lemma 1.2.8. *The diagram of continuous maps on the left induces for each $t' \in I$ the commutative diagram of modifications on the right*

$$\begin{tikzcd} X \arrow[r, "\iota_t"] \arrow[d, "\iota_s"'] & X \times I \arrow[d, "\iota_s \times \mathrm{id}_I"] \\ X \times I \arrow[r, "j_t"'] & X \times I^2 \end{tikzcd} \qquad \begin{tikzcd} \iota_t^{-1}(\iota_s \times \mathrm{id}_I)^{-1} \arrow[r, "\sim"] \arrow[d, "\sim"'] & \iota_{t'}^{-1}(\iota_s \times \mathrm{id}_I)^{-1} \arrow[d, "\sim"] \\ \iota_s^{-1}j_t^{-1} \arrow[r, "\sim"] & \iota_s^{-1}j_{t'}^{-1}. \end{tikzcd}$$

Lemma 1.2.9. *Let $H\colon X \times I \to Y$ be a continuous map that factors through the projection $p\colon X \times I \to X$. Then the composition of isomorphisms*

$$(H \circ \iota_t)^{-1} \simeq \iota_t^{-1}H^{-1} \simeq \iota_{t'}^{-1}H^{-1} \simeq (H \circ \iota_{t'})^{-1}$$

is the identity.

Let **Top** denote the strict 2-category of topological spaces and continuous maps, where 2-arrows are homotopy classes of homotopies between functions (see for example [1] cap. 7 for explicit details). Hence, homotopic spaces are equivalent in **Top**. Then homotopy invariance of locally constant sheaves may be expressed as the following

Proposition 1.2.10. *The assignment $(\mathsf{C}, X) \mapsto \Gamma(X, \mathsf{C}_X)$ defines a 2-functor*

$$\Gamma : \ \mathbf{Cat} \times \mathbf{Top}^{\circ} \longrightarrow \mathbf{Cat}.$$

Moreover, the natural functors $\eta_{X,\mathsf{C}}\colon \mathsf{C} \to \Gamma(X, \mathsf{C})$ define a 2-transformation

$$\eta : \mathsf{Q}_1 \longrightarrow \Gamma$$

where $\mathsf{Q}_1\colon \mathbf{Cat} \times \mathbf{Top}^{\circ} \to \mathbf{Cat}$ is the projection.

Proof. Let $H : f_0 \to f_1$ be a homotopy, *i.e.* a continous map $X \times I \to I$ such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$. Then $\alpha_H : f_0^{-1} \to f_1^{-1}$ is defined by the chain of natural invertible modifications

$$f_0^{-1} = (H \circ \iota_0)^{-1} \simeq \iota_0^{-1} H^{-1} \simeq \iota_1^{-1} H^{-1} \simeq (H \circ \iota_1)^{-1} = f_1^{-1}.$$

Consider the constant homotopy at $f \colon X \to Y$

$$H_f \colon X \times I \longrightarrow Y \quad ; \quad (x,t) \mapsto f(x).$$

Since we may factor H_f as $X \times I \xrightarrow{p} X \xrightarrow{f} Y$, by Lemma 1.2.9 we get $\alpha_{H_f} = \mathrm{id}_{\mathrm{f}^{-1}}$.

Now let $H_0, H_1 \colon f_0 \xrightarrow{\sim} f_1$ be two homotopies and $K \colon H_0 \xrightarrow{\sim} H_1$ a homotopy. Then consider the commutative diagram

$$\begin{tikzcd}
f_0^{-1} \arrow[r, "\sim"] \arrow[d, "\alpha_{H_0}"] & \iota_0^{-1} H_0^{-1} \arrow[r, "\sim"] \arrow[d, "\sim"] & \iota_0^{-1} j_0^{-1} K^{-1} \arrow[r, "\sim"] \arrow[d, "\sim"] & \iota_0^{-1} j_1^{-1} K^{-1} \arrow[d, "\sim"] & \iota_0^{-1} H_1^{-1} \arrow[l, "\sim"'] \arrow[d, "\sim"] & f_0^{-1} \arrow[l, "\sim"'] \arrow[d, "\alpha_{H_1}"] \\
f_1^{-1} \arrow[r, "\sim"'] & \iota_1^{-1} H_0^{-1} \arrow[r, "\sim"'] & \iota_1^{-1} j_0^{-1} K^{-1} \arrow[r, "\sim"'] & \iota_1^{-1} j_1^{-1} K^{-1} & \iota_1^{-1} H_1^{-1} \arrow[l, "\sim"] & f_1^{-1}. \arrow[l, "\sim"]
\end{tikzcd}$$

Then we have to check that the horizontal lines are identity modifications. This is a consequence of Lemma 1.2.8, which allows us by a diagram chase to identify the two lines with the modifications induced by constant homotopies.

The fact that α is compatible with the composition of homotopies is finally a very easy diagram chase. $\square$

1.3 The monodromy functor

Definition 1.3.1. The homotopy groupoid (or Poincaré groupoid) of X is the small groupoid $\Pi_1(X) = \mathsf{Hom}_{\mathbf{Top}}(\{pt\}, X)$, where $\{pt\}$ is the terminal object in **Top**.

Roughly speaking, objects of $\Pi_1(X)$ are the points of X and if $x, y \in X$, $\mathrm{Hom}_{\Pi_1(X)}(x,y)$ is the set of homotopy classes of paths starting from x and ending at y. The composition law is the opposite of the composition of paths. Note that, in particular, $\pi_0(\Pi_1(X)) = \pi_0(X)$, the set of arcwise connected components of X, and, for each $x \in X$, $\mathrm{Aut}_{\Pi_1(X)}(x) = \pi_1(X,x)$. Moreover, since by definition $\Pi_1(X)$ is functorial in X, it defines a 2-functor

$$\Pi_1 \colon \mathbf{Top} \longrightarrow \mathbf{Gr},$$

where **Gr** denotes the strict 2-category of small groupoids.

Denote by Y_{Π_1} the Yoneda 2-functor

$$\mathbf{Cat} \times \mathbf{Top}^{\circ} \longrightarrow \mathbf{Cat}, \quad (\mathsf{C}, X) \mapsto \mathsf{Hom}\,(\Pi_1(X), \mathsf{C}).$$

Definition 1.3.2. The monodromy 2-transformation

$$\mu\colon \mathsf{\Gamma} \longrightarrow \mathsf{Y}_{\Pi_1}$$

is defined as follows. For any topological space X and any category C, the functor

$$\mathsf{\Gamma}_{X,\mathsf{C}}\colon \mathsf{Hom}_{\mathbf{Top}}(\{pt\}, X) \longrightarrow \mathsf{Hom}\,(\mathsf{\Gamma}(X, \mathsf{C}_X), \mathsf{C})$$

induces by evaluation a natural functor

$$\mathsf{\Gamma}(X, \mathsf{C}_X) \times \Pi_1(X) \longrightarrow \mathsf{C},$$

hence by adjunction a functor

$$\mu_{X,\mathsf{C}}: \ \mathsf{\Gamma}(X, \mathsf{C}_X) \longrightarrow \mathsf{Hom}\,(\Pi_1(X), \mathsf{C}).$$

We will sometimes use the shorter notation μ instead of the more cumbersome $\mu_{X,\mathsf{C}}$. We will also extend μ to pointed spaces without changing the notations.

Let us briefly illustrate that we have constructed the well-known classical monodromy functor. Recall that, for any $x \in X$ one has a natural stalk 2-functor

$$\mathsf{F}_x\colon \mathbf{St}(X) \longrightarrow \mathbf{Cat} \quad ; \quad \mathfrak{S} \mapsto \mathfrak{S}_x = 2\varinjlim_{x \in U} \mathfrak{S}(U).$$

Let $i_x : \{x\} \to X$ denote the natural embedding. Since F_x is canonically equivalent to $\mathsf{\Gamma}(\{x\}, \cdot) \circ i_x^{-1}$, one gets a 2-transformation $\mathsf{\Gamma}(X, \cdot) \to \mathsf{F}_x$ and then a 2-transformation

$$\rho_x\colon \mathsf{\Gamma}_X \to \mathsf{F}_{X,x} = \mathsf{F}_x \circ (\cdot)_X.$$

Let $\mathbf{Top}_*$ be the 2-category of pointed topological spaces, pointed continuous maps and homotopy classes of pointed homotopies. One can prove by diagram chases similar to those in the proof of Proposition 1.2.10, that

Proposition 1.3.3. *The assignement* $(\mathsf{C}, (X, x)) \mapsto (\mathsf{C}_X)_x$ *defines a 2-functor*

$$\mathsf{F}: \ \mathbf{Cat} \times \mathbf{Top}_*^\circ \longrightarrow \mathbf{Cat}.$$

Moreover, the natural functors $\rho_{x,\mathsf{C}}\colon \mathsf{\Gamma}(X, \mathsf{C}_X) \longrightarrow (\mathsf{C}_X)_x$ *define a 2-transformation*

$$\rho: \ \mathsf{\Gamma} \longrightarrow \mathsf{F}.$$

Since the stalks of the stack associated to a prestack do not change, we see that, for each category C and each pointed space (X, x), the functor

$$\mathsf{C} \xrightarrow{\eta_{X,\mathsf{C}}} \mathsf{C}_X(X) \xrightarrow{\rho_{x,\mathsf{C}}} (\mathsf{C}_X)_x$$

is an equivalence of categories. Hence the composition

$$\rho \circ \eta : \ \mathsf{Q}_1 \longrightarrow \mathsf{F}$$

is an equivalence of 2-functors. Let $\varepsilon\colon \mathsf{F} \to \mathsf{Q}_1$ denote a fixed quasi-inverse to $\rho \circ \eta$, *i.e.* for each category C and each pointed space (X, x), we fix a natural equivalence

$$\varepsilon_{x,\mathsf{C}} : \ (\mathsf{C}_X)_x \xrightarrow{\sim} \mathsf{C}$$

such that if we have a pointed continuous map $f\colon (X, x) \to (Y, y)$ we get a diagram

$$\begin{tikzcd}
\mathsf{C}_Y(Y) \arrow[rr, "f^{-1}"] \arrow[dd, "\rho_{y,\mathsf{C}}"'] & & \mathsf{C}_X(X) \arrow[dd, "\rho_{x,\mathsf{C}}"] \\
& \mathsf{C} \arrow[ul, "\eta_{Y,\mathsf{C}}"'] \arrow[ur, "\eta_{X,\mathsf{C}}"] & \\
(\mathsf{C}_Y)_y \arrow[rr, "\sim", "f^{-1}"'] \arrow[ur, "\sim", "\varepsilon_{y,\mathsf{C}}"'] & & (\mathsf{C}_X)_x \arrow[ul, "\sim"', "\varepsilon_{x,\mathsf{C}}"]
\end{tikzcd}$$

that commutes up to natural isomorphism. Denote by ω the composition $\varepsilon \circ \rho\colon \Gamma \to \mathsf{Q}_1$. Fix a topological space X and a category C, and let $\mathcal{F} \in \Gamma(X, \mathsf{C}_X)$. Then a direct comparison shows that up to natural isomorphism we have

$$\mu_{X,\mathsf{C}}(\mathcal{F})(x) = \omega_{x,\mathsf{C}}(\mathcal{F})$$

(if $\mathsf{C} = \mathsf{Set}$, then ω_x is just the usual stalk-functor) and if $\gamma\colon x \to y$ is a path, then $\mu_{X,\mathsf{C}}(\mathcal{F})(\gamma)$ is defined by the chain of isomorphisms

$$\omega_{x,\mathsf{C}}(\mathcal{F}) \simeq \omega_{0,\mathsf{C}}(\gamma^{-1}\mathcal{F}) \simeq \eta^{-1}_{I,\mathsf{C}}(\gamma^{-1}\mathcal{F}) \simeq \omega_{1,\mathsf{C}}(\gamma^{-1}\mathcal{F}) \simeq \omega_{y,\mathsf{C}}(\mathcal{F})$$

(and if $\mathsf{C} = \mathsf{Set}$, we usually choose $\eta_I^{-1} = \Gamma(I, \,\cdot\,)$).
In particular, this means that the following diagram commutes up to natural invertible modification:

$$\begin{tikzcd}
\Gamma \arrow[rr, "\mu"] \arrow[dr, "\omega"'] & & \mathsf{Y}_{\Pi_1} \arrow[dl, "ev"] \\
& \mathsf{Q}_1 &
\end{tikzcd} \tag{1.3.1}$$

where ev is the evaluation 2-transformation, *i.e.* for each pointed space (X, x) and each functor $\alpha\colon \Pi_1(X) \to \mathsf{C}$, $ev_x(\alpha) = \alpha(x)$.

Let $\Delta\colon \mathsf{Q}_1 \longrightarrow \mathsf{Y}_{\Pi_1}$ denote the diagonal 2-transformation: for each topological space X, each category C and each $M \in \operatorname{Ob}\mathsf{C}$, $\Delta_{X,\mathsf{C}}(M)$ is the constant functor $x \mapsto M$ (*i.e.* the trivial representation with stalk M). Clearly $ev \circ \Delta = \mathrm{id}_{\mathsf{Q}_1}$. Moreover, we get

Proposition 1.3.4. *The diagram of 2-transformations*

$$\begin{tikzcd} \mathsf{\Gamma} \arrow[rr, "\mu"] & & \mathsf{Y}_{\Pi_1} \\ & \mathsf{Q}_1 \arrow[ul, "\eta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

commutes up to invertible modification.

Proof. We have to show that for each topological space X, each category C and every $M \in \mathrm{Ob}\,\mathsf{C}$, the representation $\mu_{X,\mathsf{C}}(\eta_{X,\mathsf{C}}(M))$ is naturally isomorphic to the trivial representation with stalk M. But this is easily verified by a diagram chase. □

Proposition 1.3.5. *For each topological space X and each category C, the functor*

$$\mu_{X,\mathsf{C}} : \ \mathsf{\Gamma}(X, \mathsf{C}_X) \longrightarrow \mathsf{Hom}\,(\Pi_1(X), \mathsf{C}).$$

is faithful and conservative.

Proof. Since the diagram 1.3.1 commutes, we get that if $f, g\colon \mathcal{F} \to \mathcal{G}$ are two morphisms of locally constant sheaves such that $\mu_{X,\mathsf{C}}(f) = \mu_{X,\mathsf{C}}(g)$ then $f_x = g_x$ in $(\mathsf{C}_X)_x$ for all $x \in X$ and since C_X is a stack this implies that $f = g$, hence $\mu_{X,\mathsf{C}}$ is faithful.
The same diagram implies that if $\mu_{X,\mathsf{C}}(f)$ is an isomorphism, then f_x is an isomorphism in $(\mathsf{C}_X)_x$ for all $x \in X$ and therefore f is an isomorphism. □

Proposition 1.3.6. *Let X be locally arcwise connected. Then for each category C, the functor*

$$\mu_{X,\mathsf{C}} : \ \mathsf{\Gamma}(X, \mathsf{C}_X) \longrightarrow \mathsf{Hom}\,(\Pi_1(X), \mathsf{C}).$$

is full.

Proof. Let $\mathcal{F}, \mathcal{G} \in \mathsf{C}_X(X)$. A morphism $\phi\colon \mu(\mathcal{F}) \to \mu(\mathcal{G})$ is given by a family of morphisms

$$\phi_x \colon ev_x(\mu(\mathcal{F})) \longrightarrow ev_x(\mu(\mathcal{G}))$$

such that for every path $\gamma\colon x \to y$ the diagram

$$\begin{tikzcd} ev_x(\mu(\mathcal{F})) \arrow[r, "\phi_x"] \arrow[d, "\mu(\mathcal{F})(\gamma)"'] & ev_x(\mu(\mathcal{G})) \arrow[d, "\mu(\mathcal{G})(\gamma)"] \\ ev_y(\mu(\mathcal{F})) \arrow[r, "\phi_y"'] & ev_y(\mu(\mathcal{G})) \end{tikzcd}$$

is commutative. By the diagram and the definition (of the stalks of a stack) we get an arcwise connected open neighborhood U_x of x and a morphism

$$\varphi^x \colon \mathcal{F}|_{U_x} \longrightarrow \mathcal{G}|_{U_x}$$

such that $ev_x(\mu(\varphi^x)) = \phi_x$.
In order to patch the φ^x we have to show that for any $z \in U_x \cap U_y$ we have $(\varphi^x)_z = (\varphi^y)_z$. Since ε_z is an equivalence, it is sufficent to check that $ev_z(\mu(\varphi^x)) = ev_z(\mu(\varphi^y))$.

Choose a path $\gamma\colon x \to z$ and $\gamma'\colon y \to z$, then

$$\mu(\mathcal{G})(\gamma) \circ ev_x(\varphi^x) \circ \mu(\mathcal{F})(\gamma)^{-1} = ev_z\mu(\varphi^x).$$

But by definition the lefthand side is just ϕ_z. Hence

$$(\varphi^x)_z = \phi_z = (\varphi^y)_z$$

and by definition of the stalks this means that φ^x and φ^y coincide in a neighborhood of z. Since z was chosen arbitrarily, they coincide on $U_x \cap U_y$. Since C_X is a stack, we can lift the morphisms φ_x to a unique $\varphi\colon \mathcal{F} \to \mathcal{G}$ such that $\mu(\varphi) = \phi$. $\square$

Corollary 1.3.7. *Let X be 1-connected* [7]. *Then*

$$\eta_X \colon \mathsf{Id}_{\mathbf{Cat}} \longrightarrow \Gamma_X$$

is an equivalence of 2-functors, i.e. for each category C, the functor $\eta_{X,\mathsf{C}}\colon \mathsf{C} \longrightarrow \Gamma(X, \mathsf{C}_X)$ is an equivalence of categories, natural in C.

Proof. Fix $x_0 \in X$. Since the groupoid $\Pi_1(X)$ is trivial (*i.e.* $\Pi_1(X) \simeq 1$), the 2-transformation ev_{x_0} is an equivalence, quasi-inverse to $\mu_X \circ \eta_X$. Since μ_X is fully faithful, this implies by abstract nonsense that μ_X and η_X are equivalences. $\square$

Denote by $\mathbf{CSt}(X)$ the full 2-subcategory of $\mathbf{St}(X)$ of constant stacks on X. We get

Corollary 1.3.8. *If X is 1-connected, the functors*

$$\mathbf{Cat} \underset{\Gamma(X,\cdot)}{\overset{(\cdot)_X}{\rightleftarrows}} \mathbf{CSt}(X)$$

are equivalences of 2-categories, inverse one to each other.

Theorem 1.3.9. *Let X be locally relatively 1-connected* [8]. *Then*

$$\mu_X \colon \Gamma_X \longrightarrow \mathsf{Y}_{\Pi_1(X)}$$

is an equivalence of 2-functors.

[7] Here and for the sequel, a topological space X is n-connected if $\pi_i(X) \simeq 1$ for all $0 \leqslant i \leqslant n$, and locally n-connected if each neighborhood of each point conteins an n-connected neighborhood.

[8] Recall that a topological space X is locally relatively 1-connected if each point $x \in X$ has a fundamental system of arcwise connected neighborhoods U such that each loop γ in U is homotopic in X to a constant path.

Proof. We may assume without loss of generality that X is connected. Fix a category C. We have to show that $\mu_{X,\mathsf{C}}$ is essentially surjective. Let us first suppose that C is complete, hence we can work in the category of sheaves with values in C.
Let $\alpha \in \mathsf{Hom}\,(\Pi_1(X), \mathsf{C})$. By choosing a base point $x_0 \in X$, α is given by a pair (M, α) where M is an object of C and $\alpha\colon \pi_1(X, x_0) \to \mathrm{Aut}_{\mathsf{C}}(M)$ is a morphism of groups. Note that if we choose for each $x \in X$ a path $x \to x_0$ then each path $\gamma\colon x \to y$ defines a unique automorphism $\alpha(\gamma) : M \to M$ compatible with the composition of paths.

Define

$$\mathcal{V} = \Big\{(V, x) \mid x \in V,\ V \text{ relatively 1-connected open subset of } X\Big\}$$

and set $(V, x) \leqslant (W, y)$ if and only if $x \in W \subset V$ which turns $\mathcal{V}$ into a filtered category.

Let $U \subset X$ be an open subset. We set

$$M_\alpha(U) = \varprojlim_{\substack{(V,x)\in\mathcal{V} \\ V\subset U}} M$$

where for any $(V, x) \leqslant (W, y)$ we chose a path $\gamma_{xy}\colon x \to y$ in W and use the isomorphism $\alpha(\gamma_{xy})\colon M \to M$ in the projective system. Note that since W is relatively 1-connected, this automorphism does not depend on the choice of γ_{xy}.

Now let $U = \bigcup_{i\in I} U_i$ be a covering stable by finite intersection. In order to prove that M_α is a sheaf we have to show that the natural morphism

$$\varprojlim_{\substack{(V,x)\in\mathcal{V} \\ V\subset U}} M \longrightarrow \varprojlim_{\substack{(V,x)\in\mathcal{V} \\ \exists i\ V\subset U_i}} M \simeq \varprojlim_{i\in I} M_\alpha(U_i)$$

is an isomorphism, but this follows from a simple cofinality argument.

By definition it is clear that, if U is relatively 1-connected, then for any choice of $x \in U$ we get a natural isomorphism $M_\alpha(U) \simeq M$ (this isomorphism being compatible with restrictions). Hence, since relatively 1-connected open subsets form a base of the topology of X, we get that M_α is a locally constant sheaf that is constant on every relatively 1-connected open subset of X.

To calculate the monodromy, consider first a path $\gamma\colon x \to y$ such that there exists a relatively 1-connected open neighborhood of γ. Obviously we get that $\mu(M_\alpha)(\gamma)$ is naturally isomorphic to $\alpha(\gamma)$. For a general γ, we decompose γ in a finite number of paths that can be covered by relatively 1-connected open subsets to get the result.

Now the general case. Embed C into $\widehat{\mathsf{C}}$ by the Yoneda-functor

$$Y\colon \mathsf{C} \longrightarrow \widehat{\mathsf{C}}$$

Then given a representation α, we can construct M_α as a locally constant sheaf with values in $\widehat{\mathsf{C}}$. Then M_α has representable stalks (isomorphic to M) and is therefore in the image of the fully faithful functor

$$\Gamma(X, \mathsf{C}_X) \longrightarrow \Gamma(X, \widehat{\mathsf{C}}_X).$$

Since by construction the monodromy of a locally constant sheaf with values in C can be calculated by considering it as a locally constant sheaf with values in $\widehat{\mathsf{C}}$, we can conclude. $\square$

Corollary 1.3.10. *Suppose that X is connected and locally relatively 1-connected. Then there is an isomorphism of groups*

$$\mathrm{Pic}(\Gamma_X) \simeq \mathrm{Out}_{\mathsf{Gr}}(\pi_1(X)),$$

where $\mathrm{Out}_{\mathsf{Gr}}(\pi_1(X))$ denotes the group of outer automorphisms of $\pi_1(X)$.

Proof. By Theorem 1.3.9 and the 2-Yoneda lemma (see for example [1] cap. 7), one gets the following chain of equivalences of monoidal categories

$$\mathsf{Aut}\,(\Gamma_X) \simeq \mathsf{Aut}\,(\mathsf{Y}_{\Pi_1(X)}) \simeq \mathsf{Aut}\,(\Pi_1(X)).$$

Applying π_0, we get an isomorphism of groups $\mathrm{Pic}(\Gamma_X) \simeq \mathrm{Pic}(\Pi_1(X))$. Since X is connected, this last group is isomorphic to $\mathrm{Out}_{\mathsf{Gr}}(\pi_1(X))$ (see below for details). $\square$

Note that, for each category C the previous isomorphism induces a morphism of groups $\mathrm{Out}_{\mathsf{Gr}}(\pi_1(X)) \to \mathrm{Pic}(\mathsf{C}_X(X))$.

1.4 Degree 1 non abelian cohomology with constant coefficients

Let M be a (not necessarily commutative) monoid. Denote by $M[1]$ the small category with 1 as single object and $\mathrm{End}_{M[1]}(1) = M$. Note that if G is a group then $G[1]$ is a groupoid. Then it is easy to check that we get fully faithful functors

$$[1]:\ \mathsf{Mon} \longrightarrow \mathbf{Cat} \qquad\qquad [1]:\ \mathsf{Gr} \longrightarrow \mathbf{Gr}.$$

Also note that if G is a connected groupoid, *i.e.* $\pi_0(\mathsf{G}) \simeq 1$, then for each $P \in \mathrm{Ob}(\mathsf{G})$, the inclusion functor $\mathrm{Aut}_{\mathsf{G}}(P)[1] \longrightarrow \mathsf{G}$ is an equivalence.

Consider the category $\mathsf{Set}(G)$ of right G-sets and G-linear maps. Then $G[1]$ is equivalent to the full sub-category of $\mathsf{Set}(G)$ with G as single object. Hence the stack $G[1]_X$ is equivalent to the stack $\mathfrak{Tors}(G_X)$ of torsors over the sheaf G_X, *i.e.* the stack which associates to each open subset $U \subset X$ the category $\mathsf{Tors}(G_U)$ of right G_U-sheaves locally free of rank one[9]. Assume that X is locally relatively 1-connected. By Theorem 1.3.9 there is an equivalence of categories

$$\mathsf{Tors}(G_X) \simeq \mathsf{Hom}\,(\Pi_1(X), G[1]).$$

A standard cocycle argument shows that there is an isomorphism

$$\pi_0(\mathsf{Tors}(G_X)) \simeq H^1(X; G_X)^{10}.$$

[9]Recall that $\mathsf{Tors}(G_X)$ is equivalent to the category of G-coverings over X.

[10]In the non commutative case, this is sometimes taken as the definition of $H^1(X; G_X)$.

Assume that the space X is connected. Let us calculate the set $\pi_0(\mathsf{Hom}(\Pi_1(X), G[1]))$. Since $\Pi_1(X)$ is connected, $\Pi_1(X)$ is equivalent to $\pi_1(X)[1]$ for a choice of a base point in X. Hence there is a natural surjective map

$$\mathrm{Hom}_{\mathsf{Gr}}(\pi_1(X), G) \longrightarrow \pi_0(\mathsf{Hom}(\Pi_1(X), G[1])),$$

and one checks immediately that two morphisms of groups $\varphi, \psi\colon \pi_1(X) \to G$ give isomorphic functors if and only if there exists $g \in G$ such that $\varphi = \mathrm{ad}(g) \circ \psi$, where $\mathrm{ad}(g)$ is the group automorphism of G given by $h \mapsto ghg^{-1}$ for each $h \in G$. Hence

$$\mathrm{Hom}_{\mathsf{Gr}}(\pi_1(X), G)/\operatorname{Int}(G) \simeq \pi_0(\mathsf{Hom}(\Pi_1(X), G[1]))$$

where Int(G) is the groups of inner automorphisms of G (*i.e.* automorphism of the form $\mathrm{ad}(g)$ for some $g \in G$), which acts on the left on $\mathrm{Hom}_{\mathsf{Gr}}(\pi_1(X), G)$ by composition[11]. We get the classical

Proposition 1.4.1. *Let X be connected and locally relatively 1-connected. Then for any group G there is an isomorphism of pointed sets*

$$H^1(X; G_X) \simeq \mathrm{Hom}_{\mathsf{Gr}}(\pi_1(X), G)/\operatorname{Int}(G).$$

In particular, if G is commutative one recovers the isomorphism of groups

$$H^1(X; G_X) \simeq \mathrm{Hom}_{\mathsf{Gr}}(\pi_1(X), G).$$

More generally, to each complex of groups $G^{-1} \xrightarrow{d} G^0$ one associates a small groupoid, which we denote by the same symbol, as follows: objects are the elements $g \in G^0$ and morphisms $g \to g'$ are given by $h \in G^{-1}$ such that $d(h)g = g'$. If moreover $G^{-1} \xrightarrow{d} G^0$ has the structure of crossed module[12] the associated category is a strict gr-category, *i.e.* a group object in the category of groupoids. In fact, all strict gr-categories arise in this way (see for example [5], and [SGA4] for the commutative case). In particular, if G is a group, the groupoid $G[1]$ is identified with $G \to 1$ and it has the structure of a strict gr-category if and only if G commutes. Moreover, the category $\mathsf{Aut}(G[1])$ of auto-equivalence of $G[1]$ is equivalent to the strict gr-category $G \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G)$.

Now consider the constant stack $(G \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G))_X$. It is equivalent to the strict gr-stack $\mathfrak{Bitors}(G_X)$ of G_X-bitorsors, *i.e.* G_X-torsors with an additional compatible structure of left G_X-torsors (see [2] for more details). Suppose that X is locally relatively 1-connected. Then, by Theorem 1.3.9, there is an equivalence of strict gr- categories

$$\mathsf{Bitors}(G_X) \simeq \mathsf{Hom}(\Pi_1(X), G \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G)).$$

[11] Note that Int(G) $\simeq G/\operatorname{Z}(G)$, where $\operatorname{Z}(G)$ denotes the center of G.

[12] Recall that a complex of groups $G^{-1} \xrightarrow{d} G^0$ is a crossed module if it is endowed with a (left) action of G^0 on G^{-1} such that (i) $d({}^g h) = \mathrm{ad}(g)(d(h))$ and (ii) ${}^{d(\tilde h)}h = \mathrm{ad}(\tilde h)(h)$ for any $h, \tilde h \in G^{-1}$ and $g \in G^0$.

One can show (see *loc. cit.*) that

$$\pi_0(\mathsf{Bitors}(G_X)) \simeq H^1(X; G_X \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G_X))$$

where the right hand side is the cohomology group of X with values in the crossed module $G_X \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G_X)$. Suppose that X is connected. Then $\Pi_1(X) \simeq \pi_1(X)[1]$ and a similar calculation as above leads to the isomorphism of groups

$$\pi_0(\mathsf{Hom}\,(\Pi_1(X), G \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G))) \simeq \mathrm{Hom}_{\mathsf{Gr}}(\pi_1(X), \mathrm{Z}(G)) \rtimes \mathrm{Out}_{\mathsf{Gr}}(G),$$

where $\mathrm{Out}_{\mathsf{Gr}}(G) = \mathrm{Aut}_{\mathsf{Gr}}(G)/\operatorname{Int}(G)$ is the group of outer automorphisms of G, which acts on the left on $\mathrm{Hom}_{\mathsf{Gr}}(\pi_1(X), \mathrm{Z}(G))$ by composition. We get

Proposition 1.4.2. *Let X be connected and locally relatively 1-connected. Then for any group G there is an isomorphism of groups*

$$H^1(X; G_X \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G_X)) \simeq \mathrm{Hom}_{\mathsf{Gr}}(\pi_1(X), \mathrm{Z}(G)) \rtimes \mathrm{Out}_{\mathsf{Gr}}(G).$$

A similar result holds replacing $G \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G)$ by a general crossed module $G^{-1} \xrightarrow{d} G^0$. More precisely, noting that $\ker d$ is central in G^{-1}, one gets an isomorphism of groups

$$H^1(X; G_X^{-1} \xrightarrow{d} G_X^0) \simeq \mathrm{Hom}_{\mathsf{Gr}}(\pi_1(X), \ker d) \rtimes \operatorname{coker} d.$$

2 Classification of locally constant stacks

Following our presentation of 1-monodromy, we will approach the theory of 2-monodromy in the setting of 2-stacks. We refer to [2] for the basic definitions. Let X be a topological space and let $2\mathbf{Cat}$, $2\mathbf{PSt}(X)$ and $2\mathbf{St}(X)$ denote the strict 3-category[13] of 2-categories, 2-prestacks and that of 2-stacks on X, respectively. As for sheaves and stacks, there exists a 2-stack associated to a 2-prestack:

Proposition 2.0.3. *The forgetful 3-functor*

$$\mathbf{For}\colon 2\mathbf{St}(X) \longrightarrow 2\mathbf{PSt}(X)$$

has a left adjoint 3-functor

$$\ddagger\colon 2\mathbf{PSt}(X) \longrightarrow 2\mathbf{St}(X).$$

Hence, similarly to the case of sheaves we can associate to any 2-category a constant 2-prestack on X and set

[13]Here and in the following, we will use the terminology of 3-categories and 3-functors only in the framework of strict 3-categories, *i.e.* categories enriched in $2\mathbf{Cat}$.

Definition 2.0.4. Let $\mathbf{C}$ be a 2-category. The constant 2-stack on X with stalk $\mathbf{C}$ is the image of $\mathbf{C}$ by the 3-functor

$$(\,\cdot\,)_X \colon 2\mathbf{Cat} \longrightarrow 2\mathbf{PSt}(X) \xrightarrow{\ddagger} 2\mathbf{St}(X).$$

An object $\mathfrak{S} \in \operatorname{Ob} \mathbf{C}_X(X)$ is called a locally constant stack on X with values in $\mathbf{C}$. A locally constant stack is constant with stalk P, if it is equivalent to $\boldsymbol{\eta}_{X,\mathbf{C}}(\mathrm{P})$ for some object $\mathrm{P} \in \operatorname{Ob} \mathbf{C}$, where the 3-functor

$$\boldsymbol{\eta}_{X,\mathbf{C}} \colon \mathbf{C} \longrightarrow \mathbf{C}_X(X)$$

is induced by the 3-adjunction of Proposition 2.0.3.

Let us look at the case when $\mathbf{C}$ is $\mathbf{Cat}$, the 2-category of all small categories. It is easy to see that a stack $\mathfrak{S}$ on X is locally constant if and only if there exists an open covering $X = \bigcup U_i$ such that $\mathfrak{S}|_{U_i}$ is equivalent to a constant stack (as defined in the first part). We denote by $\mathbf{LcSt}(X)$ the full 2-subcategory of $\mathbf{St}(X)$ whose objects are the locally constant stacks.

Similarly, suppose that $\mathbf{C}$ is a 2-category that admits all small 2-limits. Then one can define the notion of a stack with values in $\mathbf{C}$ similarly to the case of sheaves (see Appendix B). It is not difficult to see that $\mathbf{C}_X(X)$ is naturally equivalent to the category of locally constant stacks with values in $\mathbf{C}$. For a more general $\mathbf{C}$, we can alway embed $\mathbf{C}$ by the Yoneda 2-Lemma into the 2-category of contravariant 2-functors from $\mathbf{C}$ to $\mathbf{Cat}$ which admits all small 2-limits. Then $\mathbf{C}_X(X)$ can be identified with the (essentially) full sub-2-category of $\mathsf{Hom}\,(\mathbf{C}^\circ, \mathbf{Cat})_X(X)$ defined by objects whose stalk is 2-representable.

Now let $f \colon X \to Y$ be a continous map. We leave to the reader to define operations f_* and f^{-1} such that

Proposition 2.0.5. *The 3-functors*

$$f_* \colon 2\mathbf{St}(X) \longrightarrow 2\mathbf{St}(Y) \qquad f^{-1} \colon 2\mathbf{St}(Y) \longrightarrow 2\mathbf{St}(X)$$

are 3-adjoint, f_* being the right 3-adjoint of f^{-1}.

Definition 2.0.6. Denote by $\mathbf{\Gamma}(X,\,\cdot\,)$ the 3-functor of global sections

$$2\mathbf{St}(X) \longrightarrow 2\mathbf{Cat} \quad ; \quad \boldsymbol{\mathfrak{S}} \mapsto \mathbf{\Gamma}(X, \boldsymbol{\mathfrak{S}}) = \boldsymbol{\mathfrak{S}}(X)$$

and set $\mathbf{\Gamma}_X = \mathbf{\Gamma}(X, \cdot) \circ (\cdot)_X$.

Note that the 3-functor

$$\mathbf{\Gamma}(\{pt\}, \cdot) \colon 2\mathbf{St}(\{pt\}) \longrightarrow 2\mathbf{Cat}$$

is an equivalence of 3-categories. Hence, using Proposition 2.0.5, we have

Proposition 2.0.7. *The 3-functor* $\mathbf{\Gamma}(X, \,\cdot\,)$ *is right 3-adjoint to* $(\,\cdot\,)_X$.

It is not hard to see that we can choose the 3-functors $\boldsymbol{\eta}_{X,\mathbf{C}}\colon \mathbf{C} \longrightarrow \mathbf{\Gamma}(X, \mathbf{C}_X)$ to define the adjunction 3-transformation

$$\boldsymbol{\eta}_X\colon \mathbf{Id}_{\mathbf{2Cat}} \longrightarrow \mathbf{\Gamma}_X.$$

Consider the commutative diagram of topological spaces

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "a_X"'] & & Y \arrow[dl, "a_Y"] \\ & \{pt\} & \end{tikzcd}$$

Hence we get a 3-transformation of 3-functors f^{-1} compatible with $\boldsymbol{\eta}_X$ and $\boldsymbol{\eta}_Y$, *i.e.* the following diagram commutes up to a natural invertible 2-modification:

$$\begin{tikzcd} \mathbf{\Gamma}_Y \arrow[rr, "f^{-1}"] & & \mathbf{\Gamma}_X. \\ & \mathbf{Id}_{\mathbf{2Cat}} \arrow[ul, "\boldsymbol{\eta}_Y"] \arrow[ur, "\boldsymbol{\eta}_X"'] & \end{tikzcd}$$

Recall that, if $f\colon X \to Y$ is a continous map and $\mathfrak{T}$ a locally constant stack on Y, by diagram 1.2.1 the stack $f^{-1}\mathfrak{T}$ on X is locally constant. It is then easy to check that we get a 2-functor

$$f^{-1}\colon \mathbf{LcSt}(Y) \longrightarrow \mathbf{LcSt}(X).$$

Note that this implies that for any point $x \in X$ and any $\mathfrak{S}, \mathfrak{T} \in \mathbf{C}_X(X)$, the natural functor

$$i_x^{-1}\mathfrak{Hom}_{\mathbf{C}_X}(\mathfrak{S}, \mathfrak{T}) \longrightarrow \mathsf{Hom}_{\mathbf{C}}(i_x^{-1}\mathfrak{S}, i_x^{-1}\mathfrak{T})$$

is an equivalence (here $i_x\colon \{pt\} \to X$ denotes the natural map to x and we identify $\mathbf{C}$ with global sections of $\mathbf{C}_{pt}$). Therefore, for any map $f\colon X \to Y$, we get a natural equivalence of stacks of categories

$$f^{-1}\mathfrak{Hom}_{\mathbf{C}_Y}(\mathfrak{S}, \mathfrak{T}) \xrightarrow{\sim} \mathfrak{Hom}_{\mathbf{C}_X}(f^{-1}\mathfrak{S}, f^{-1}\mathfrak{T}).$$

2.1 The 2-monodromy 2-functor

Lemma 2.1.1. *Consider the maps* $X \underset{p}{\overset{\iota_t}{\rightleftarrows}} X \times I$ *as in Lemma 1.2.5. Then the 3-transformations*

$$\mathbf{\Gamma}_X \underset{\iota_t^{-1}}{\overset{p^{-1}}{\rightleftarrows}} \mathbf{\Gamma}_{X\times I}$$

are equivalences of 2-functors, quasi-inverse one to each other.

Proof. Let $\mathbf{C}$ be a 2-category. Since $\iota_t^{-1} \circ p^{-1} \simeq \mathsf{Id}_{\mathbf{\Gamma}_X}$, it remains to check that for each $\mathfrak{S} \in \mathbf{\Gamma}(X \times I, \mathbf{C}_{X\times I})$ there is a natural equivalence of stacks $p^{-1}\iota_t^{-1}\mathfrak{S} \simeq \mathfrak{S}$.

First, let us suppose that $\mathfrak{S}$ is a locally constant stack of categories and let us prove that the natural functor

$$\Gamma(X \times I, \mathfrak{S}) \longrightarrow \Gamma(X, \iota_t^{-1}\mathfrak{S})$$

is an equivalence. Since the sheaves $\mathcal{H}om_{\mathfrak{S}}$ are locally constant, by Proposition 1.2.5, it is clear that this functor is fully faithful. Let us show that it is essentially surjective.

First note that, since $\mathfrak{S}$ is locally constant, it is easy to see that for every open neighborhood $U \times I_j \ni (x,t)$ such that I_j is an interval and $\mathfrak{S}|_{U\times I_j}$ is constant, there exists an open subset $\tilde{U} \ni x$ so that for every locally constant sheaf $\mathcal{F} \in \mathfrak{S}(\tilde{U} \times I_j)$ there exists $\tilde{\mathcal{F}} \in \mathfrak{S}(\tilde{U} \times I)$ such that $\tilde{\mathcal{F}}|_{\tilde{U}\times I_j} \simeq \mathcal{F}$.

Now take $\mathcal{F} \in \Gamma(X, \iota_t^{-1}\mathfrak{S})$. Then we can find a covering $X \times \{t\} \subset \bigcup_{j\in J} U_j \cap I_j$ where I_j are open intervals containing t such that $\mathfrak{S}|_{U_j \cap I_j}$ is constant and we can find objects $\mathcal{F}_j \in \mathfrak{S}(U_j \times I)$ such that $\iota_t^{-1}\mathcal{F}_j \simeq \mathcal{F}|_{U_j}$. Then the isomorphism

$$\mathcal{H}om\,(\mathcal{F}_i|_{U_{ij}\times I}, \mathcal{F}_j|_{U_{ij}\times I}) \xrightarrow{\sim} \mathcal{H}om\,(\mathcal{F}|_{U_i}|_{U_{ij}}, \mathcal{F}|_{U_j}|_{U_{ij}})$$

implies that we may use the descent data of $\mathcal{F}$ to patch the $\mathcal{F}_i$ to a global object on $X \times I$ that is mapped to $\mathcal{F}$ by ι_t^{-1}.

The rest of the proof is similar to that of 1.2.5. Consider the stack of functors $\mathfrak{Hom}_{\mathbf{C}_{X\times I}}(p^{-1}\iota_t^{-1}\mathfrak{S}, \mathfrak{S})$. Since $\mathfrak{S}$ is locally constant, the $\mathfrak{Hom}_{\mathbf{C}_X}$ stack is locally constant and the natural functor

$$\iota_t^{-1}\mathfrak{Hom}_{\mathbf{C}_{X\times I}}(p^{-1}\iota_t^{-1}\mathfrak{S}, \mathfrak{S}) \longrightarrow \mathfrak{Hom}_{\mathbf{C}_X}(\iota_t^{-1}\mathfrak{S}, \iota_t^{-1}\mathfrak{S})$$

is an equivalence. We have thus shown that the natural functor

$$\Gamma(X \times I, \mathfrak{Hom}_{\mathbf{C}_{X\times I}}(p^{-1}\iota_t^{-1}\mathfrak{S}, \mathfrak{S})) \longrightarrow \Gamma(X, \mathfrak{Hom}_{\mathbf{C}_X}(\iota_t^{-1}\mathfrak{S}, \iota_t^{-1}\mathfrak{S}))$$

is an equivalence. We can therefore lift the identity of $\iota_t^{-1}\mathfrak{S}$ to get an equivalence

$$p^{-1}\iota_t^{-1}\mathfrak{S} \xrightarrow{\sim} \mathfrak{S}.$$

□

Corollary 2.1.2. *For each 2-category* $\mathbf{C}$*, the 2-functor*

$$(\,\cdot\,)_I \colon \mathbf{C} \longrightarrow \mathbf{C}_I(I)$$

is an equivalence.

Remark 2.1.3. For any X and any $t \in I$, we have the equivalence $\mathsf{Id}_{\mathbf{\Gamma}_X} = (p \circ \iota_t)^{-1} \simeq \iota_t^{-1}p^{-1}$. Then, for any $s,t \in I$, there exists an equivalence $\iota_t^{-1} \simeq \iota_s^{-1}$, unique up to a unique invertible modification, compatible with these equivalences.

Lemma 2.1.4. *(i) The topological diagram on the left induces for each $t' \in I$ the diagram of equivalences on the right, which is commutative up to natural invertible modification:*

$$\begin{tikzcd} X \arrow[r, "\iota_t"] \arrow[d, "\iota_s"'] & X \times I \arrow[d, "\iota_s \times \mathrm{id}_I"] \\ X \times I \arrow[r, "j_t"'] & X \times I^2 \end{tikzcd} \qquad \begin{tikzcd} \iota_t^{-1}(\iota_s \times \mathrm{id}_I)^{-1} \arrow[r, "\sim"] \arrow[d, "\sim"'] & \iota_{t'}^{-1}(\iota_s \times \mathrm{id}_I)^{-1} \arrow[d, "\sim"] \\ \iota_s^{-1} j_t^{-1} \arrow[r, "\sim"] & \iota_s^{-1} j_{t'}^{-1} \end{tikzcd}$$

(ii) The topological diagram

$$\begin{tikzcd} & X \arrow[rr, "\iota_r"] \arrow[dl, "\iota_s"'] \arrow[dd, "\iota_t"] & & X \times I \arrow[dl, "\iota_s \times \mathrm{id}_I"] \arrow[dd, "\iota_t \times \mathrm{id}_I"] \\ X \times I \arrow[rr, "j_r"'] \arrow[dd, "j_t"'] & & X \times I^2 \arrow[dd, "j_t \times \mathrm{id}_I"] & \\ & X \times I \arrow[rr, "j_r"'] \arrow[dl, "\iota_s \times \mathrm{id}_I"'] & & X \times I^2 \arrow[dl, "\iota_s \times \mathrm{id}_{I^2}"] \\ X \times I^2 \arrow[rr, "k_r"'] & & X \times I^3 & \end{tikzcd}$$

induces a (very big) commutative diagram of the corresponding modifications.

Lemma 2.1.5. *Let $f\colon X \to Y$ be a continous map and $H_f\colon X \times I \to Y$ the constant homotopy of f. Then, for any $t, t' \in I$, the diagram*

$$\begin{tikzcd} f^{-1} \arrow[r, "\mathrm{id}_{f^{-1}}"] \arrow[d, "\sim"'] & f^{-1} \arrow[d, "\sim"] \\ \iota_t^{-1} H_f^{-1} \arrow[r, "\sim"'] & \iota_{t'}^{-1} H_f^{-1} \end{tikzcd}$$

commutes up to natural invertible modification.

Proof. Compose with the functor p^{-1}, where $p\colon X \times I \to Y$ is the projection. $\square$

Proposition 2.1.6. *The assignement $(\mathbf{C}, X) \mapsto \mathbf{\Gamma}(X, \mathbf{C}_X)$ defines a 3-functor*

$$\mathbf{\Gamma}\colon 2\mathbf{Cat} \times 2\mathbf{Top}^\circ \longrightarrow 2\mathbf{Cat}.$$

Proof. Let $f_0, f_1\colon X \to Y$ be continous maps and $H\colon f_0 \xrightarrow{\sim} f_1$ a homotopy. Then the equivalence of 2-functors $\alpha_H\colon f_0^{-1} \xrightarrow{\sim} f_1^{-1}$ is defined by the chain of equivalences

$$f_0^{-1} = (H \circ \iota_0)^{-1} \simeq \iota_0^{-1} H^{-1} \simeq \iota_1^{-1} H^{-1} \simeq (H \circ \iota_1)^{-1} = f_1^{-1}.$$

Now let $H_0, H_1\colon f_0 \xrightarrow{\sim} f_1$ be homotopies and $K\colon H_0 \xrightarrow{\sim} H_1$ a homotopy. We get a diagram of equivalences

$$\begin{tikzcd}
f_0^{-1} \arrow[r, "\sim"] \arrow[d, "\alpha_{H_0}"] & \iota_0^{-1}H_0^{-1} \arrow[r, "\sim"] \arrow[d, "\sim"] & \iota_0^{-1}j_0^{-1}K^{-1} \arrow[r, "\sim"] \arrow[d, "\sim"] & \iota_0^{-1}j_1^{-1}K^{-1} \arrow[d, "\sim"] & \iota_0^{-1}H_1^{-1} \arrow[l, "\sim"'] \arrow[d, "\sim"] & f_0^{-1} \arrow[l, "\sim"'] \arrow[d, "\alpha_{H_1}"] \\
f_1^{-1} \arrow[r, "\sim"'] & \iota_1^{-1}H_0^{-1} \arrow[r, "\sim"'] & \iota_1^{-1}j_0^{-1}K^{-1} \arrow[r, "\sim"'] & \iota_1^{-1}j_1^{-1}K^{-1} & \iota_1^{-1}H_0^{-1} \arrow[l, "\sim"] & f_1^{-1}. \arrow[l, "\sim"]
\end{tikzcd}$$

By the two Lemma 2.1.4 (i) and 2.1.5, the horizontal compositions are naturally equivalent to the identity. Hence K defines a natural invertible modification $\alpha_K\colon \alpha_{H_0} \xrightarrow{\sim} \alpha_{H_1}$.

Let $H\colon f_0 \xrightarrow{\sim} f_1$ be a homotopy and $K_H\colon X \times I^2 \to Y$ the constant homotopy of H. Composing with the functor p^{-1}, where $p\colon X \times I \to Y$ is the projection, one gets that the modification $\alpha_{K_H}\colon \alpha_H \to \alpha_H$ is the identity.

With a little patience (and a very large piece of paper) the reader can check that all compatibility conditions of a 3-functor are verified (using Lemma 2.1.4). □

Definition 2.1.7. The homotopy 2-groupoid of X is the 2-groupoid[14]

$$\Pi_2(X) = \mathbf{Hom}_{2\mathbf{Top}}(\{pt\}, X).$$

Roughly speaking, its objects are the points of X and, if $x, y \in X$, $\mathsf{Hom}_{\Pi_2(X)}(x,y)$ is the category $\Pi_1(P_{x,y}X)$, where $P_{x,y}X$ is the space of paths starting from x and ending in y, endowed with the compact-open topology. Compositions laws are defined in the obvious way. Note that, in particular, $\boldsymbol{\pi}_0(\Pi_2(X)) = \pi_0(X)$ and for each $x \in X$, $\mathrm{Pic}_{\Pi_2(X)}(x) = \pi_0(\Pi_1(\Omega_x X)) = \pi_1(X,x)$, where we denote by $\Omega_x X$ the loop space $P_{x,x}X$ with base point x, and $Z_{\Pi_2(X)}(x) = \pi_2(X,x)$[15]. Since by definition $\Pi_2(X)$ is functorial in X, it defines a 3-functor

$$\Pi_2\colon 2\mathbf{Top} \longrightarrow 2\mathbf{Gr},$$

where $2\mathbf{Gr}$ denotes the strict 3-category of 2-groupoids.

Denote by $\mathbf{Y}_{\Pi_2}$ the Yoneda 3-functor

$$2\mathbf{Cat} \times 2\mathbf{Top}^\circ \longrightarrow 2\mathbf{Cat}, \quad (\mathbf{C}, X) \mapsto \mathbf{Hom}(\Pi_2(X), \mathbf{C}).$$

Definition 2.1.8. The 2-monodromy 3-transformation

$$\mu^2\colon \mathbf{\Gamma} \longrightarrow \mathbf{Y}_{\Pi_2}$$

is defined in the following manner. For each topological space X, the 2-functor

$$\mathbf{\Gamma}_{X,\mathbf{C}}\colon \mathbf{Hom}_{2\mathbf{Top}}(\{pt\}, X) \longrightarrow \mathbf{Hom}(\mathbf{\Gamma}(X, \mathbf{C}_X), \mathbf{C})$$

[14] Recall that a 2-groupoid is a 2-category whose 1-arrows are invertible up to an invertible 2-arrow and whose 2-arrows are invertible.

[15] Note that the categorical action of $\mathrm{Pic}_{\Pi_2(X)}(x) = \pi_1(X,x)$ on $Z_{\Pi_2(X)}(x) = \pi_2(X,x)$ is exactly the classical one of algebraic topology.

induces by evaluation a natural 2-functor

$$\mathbf{\Gamma}(X, \mathbf{C}_X) \times \Pi_2(X) \longrightarrow \mathbf{C},$$

hence by adjunction a 2-functor

$$\mu^2_{X,\mathbf{C}} \colon \mathbf{\Gamma}(X, \mathbf{C}_X) \longrightarrow \mathbf{Hom}\,(\Pi_2(X), \mathbf{C}).$$

As in the case of 1-monodromy, let us visualize this construction using stalks. To every 2-stack $\boldsymbol{\mathfrak{S}}$ we can associate its stalk at $x \in X$, which is the 2-category

$$\boldsymbol{\mathfrak{S}}_x = 3\varinjlim_{x \in U} \boldsymbol{\mathfrak{S}}(U).$$

If $\mathbf{C}$ is a 2-category, then $(\mathbf{C}_X)_x \simeq \mathbf{C}$. In the case that $\boldsymbol{\mathfrak{S}}$ is the 2-stack $\boldsymbol{\mathfrak{S}}\mathbf{t}_X$ of all stacks on X, we get the natural 3-functor

$$2\mathbf{St}_X \longrightarrow 2\mathbf{Cat} \quad ; \quad \boldsymbol{\mathfrak{S}} \mapsto \boldsymbol{\mathfrak{S}}_x$$

that induces an equivalence

$$\mathbf{LcSt}_x \xrightarrow{\sim} 2\mathbf{Cat}.$$

Then similarly as in the case of sheaves, one proves

Proposition 2.1.9. *The assignment* $(\mathbf{C}, (X, x)) \mapsto (\mathbf{C}_X)_x$ *defines a 3-functor*

$$\mathbf{F} \colon 2\mathbf{Cat} \times 2\mathbf{Top}^\circ_* \longrightarrow 2\mathbf{Cat}$$

and the 2-functors $\boldsymbol{\rho}_{X,\mathbf{C}}$ *define a 3-transformation*

$$\boldsymbol{\rho} \colon \mathbf{\Gamma} \longrightarrow \mathbf{F}.$$

We find that

$$\boldsymbol{\rho} \circ \boldsymbol{\eta} \colon \mathbf{Q_1} \longrightarrow \mathbf{F}$$

is an equivalence of 3-functors. Let $\boldsymbol{\varepsilon} \colon \mathbf{F} \to \mathbf{Q}_1$ be a fixed quasi-inverse to $\boldsymbol{\rho} \circ \boldsymbol{\eta}$ and set $\boldsymbol{\omega} = \boldsymbol{\varepsilon} \circ \boldsymbol{\rho}$.

Fix a topological space X and a locally constant stack $\mathfrak{S}$. Then (up to a natural equivalence)

$$\mu^2_{X,\mathbf{C}}(\mathfrak{S})(x) = \boldsymbol{\omega}_{x,\mathbf{C}}(\mathfrak{S})$$

(if $\mathfrak{S}$ is a locally constant stack with values in $\mathbf{Cat}$, then $\boldsymbol{\omega}_{x,\mathbf{Cat}}(\mathfrak{S})$ can be canonically identified with $\mathfrak{S}_x$). If $\gamma \colon x \to y$ is a path, then the equivalence $\mu^2_{X,\mathbf{C}}(\mathfrak{S})(\gamma)$ is defined by the chain of equivalences

$$\boldsymbol{\omega}_{x,\mathbf{C}}(\mathfrak{S}) \simeq \boldsymbol{\omega}_{0,\mathbf{C}}(\gamma^{-1}\mathfrak{S}) \simeq \boldsymbol{\eta}_{I,\mathbf{C}}(\gamma^{-1}\mathfrak{S}) \simeq \boldsymbol{\omega}_{1,\mathbf{C}}(\gamma^{-1}\mathfrak{S}) \simeq \boldsymbol{\omega}_{y,\mathbf{C}}(\mathfrak{S}),$$

where $\boldsymbol{\eta}_{I,\mathbf{C}}$ is just the global section functor in the case of ordinary stacks, *i.e.* for $\mathbf{C} = \mathbf{Cat}$. If $H\colon \gamma_0 \xrightarrow{\sim} \gamma_1$ is an homotopy, then the invertible transformation $\mu^2_{X,\mathbf{C}}(\mathfrak{S})(H)$ is defined by the diagram of equivalences

$$\begin{tikzcd}
\boldsymbol{\omega}_{(0,0)}(H^{-1}\mathfrak{S}) \arrow[rr, "\mu^2(\mathfrak{S})(H(\cdot,0))"] \arrow[dd, "\mu^2(\mathfrak{S})(\gamma_0)"'] & & \boldsymbol{\omega}_{(1,0)}(H^{-1}\mathfrak{S}) \arrow[dd, "\mu^2(\mathfrak{S})(\gamma_1)"] \\
& \boldsymbol{\eta}_{I^2}(H^{-1}\mathfrak{S}) \arrow[ul, "\rho_{(0,0)}"'] \arrow[ur, "\rho_{(1,0)}"] \arrow[dl, "\rho_{(0,1)}"] \arrow[dr, "\rho_{(1,1)}"'] & \\
\boldsymbol{\omega}_{(0,1)}(H^{-1}\mathfrak{S}) \arrow[rr, "\mu^2(\mathfrak{S})(H(\cdot,1))"'] & & \boldsymbol{\omega}_{(1,1)}(H^{-1}\mathfrak{S}).
\end{tikzcd}$$

In particular, the following diagram of 3-transformations commutes (up to a natural 2-modification)

$$\begin{tikzcd}
\boldsymbol{\Gamma} \arrow[rr, "\mu^2"] \arrow[dr, "\omega"'] & & \mathbf{Y}_{\Pi_2}, \arrow[dl, "\mathsf{ev}"] \\
& \mathbf{Q}_1 &
\end{tikzcd}$$

where ev denotes the evaluation 3-transformation.

2.2 Classifying locally constant stacks

Let $\Delta\colon \mathbf{Q}_1 \longrightarrow \mathbf{Y}_{\Pi_2}$ denote the diagonal 3-transformation. Exactly as in the sheaf case, one gets

Proposition 2.2.1. *The image of a constant stack is equivalent to a trivial representation. In other words, the diagram of 3-transformations*

$$\begin{tikzcd}
\boldsymbol{\Gamma} \arrow[rr, "\mu^2"] & & \mathbf{Y}_{\Pi_2} \\
& \mathbf{Q}_1 \arrow[ul, "\boldsymbol{\eta}"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

commutes up to 2-modifications.

Proposition 2.2.2. *For any topological space X and 2-category $\mathbf{C}$, the 2-functor*

$$\mu^2_{X,\mathbf{C}}\colon \boldsymbol{\Gamma}(X, \mathbf{C}_X) \longrightarrow \mathbf{Hom}\,(\Pi_2(X), \mathbf{C})$$

is faithful and conservative.

Proof. We have to show that, for any locally constant stack $\mathfrak{S}$ and $\mathfrak{S}'$, the induced functor

$$\mu^2_{X,\mathbf{C}}\colon \mathsf{Hom}_{\mathbf{C}_X}(\mathfrak{S}, \mathfrak{S}') \longrightarrow \mathsf{Hom}_{\mathbf{Hom}\,(\Pi_2(X),\mathbf{C})}(\mu^2_{X,\mathbf{C}}(\mathfrak{S}), \mu^2_{X,\mathbf{C}}(\mathfrak{S}'))$$

is faithful and conservative. Fix $F, G\colon \mathfrak{S} \to \mathfrak{S}'$ two functors of stacks. Since for each $x \in X$, there is a natural isomorphism $\mathrm{Hom}\,(\mu^2(F)(x), \mu^2(F)(x)) \simeq \mathrm{Hom}\,(F_x, G_x)$, we get the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}\,(F,G) \arrow[r, "\mu^2"] \arrow[d, "\mathsf{F}_x"'] & \mathrm{Hom}\,(\mu^2(F), \mu^2(G)) \arrow[d, "ev_x"] \\
\mathcal{H}om\,(F,G)_x \arrow[r, "\sim"] & \mathrm{Hom}\,(F_x, G_x).
\end{tikzcd}$$

Let $\varphi, \psi\colon F \to G$ be two morphisms of functors. If $\mu^2(\varphi) = \mu^2(\psi)$, then $\varphi_x = \psi_x$ for all $x \in X$ and, since $\mathcal{H}om\,(F,G)$ is a sheaf, we get $\varphi = \psi$. Similarly, if $\mu^2(\varphi)$ is an isomorphism, it follows that the morphism φ is an isomorphism. $\square$

Proposition 2.2.3. *Let X be locally 1-connected. Then for each 2-category $\mathbf{C}$, the 2-functor*

$$\mu^2_{X,\mathbf{C}}\colon \mathbf{\Gamma}(X, \mathbf{C}_X) \longrightarrow \mathbf{Hom}\,(\Pi_2(X), \mathbf{C})$$

is full.

Proof. We have to show that, for any locally consant stack $\mathfrak{S}$ and $\mathfrak{S}'$, the induced functor

$$\mu^2_{X,\mathbf{C}}\colon \mathsf{Hom}_{\mathbf{C}_X}(\mathfrak{S}, \mathfrak{S}') \longrightarrow \mathsf{Hom}_{\mathbf{Hom}\,(\Pi_2(X),\mathbf{C})}(\mu^2_{X,\mathbf{C}}(\mathfrak{S}), \mu^2_{X,\mathbf{C}}(\mathfrak{S}'))$$

is full and essentially surjective. Since the stack of functors $\mathfrak{Hom}_{\mathbf{C}_X}(\mathfrak{S}, \mathfrak{S}')$ is locally constant, for each pair of functors $F, G\colon \mathfrak{S} \to \mathfrak{S}'$, the sheaf $\mathcal{H}om\,(F,G)$ is locally constant. Let $\psi\colon \mu^2(F) \to \mu^2(G)$ be a morphism of functor. It defines a morphism of functors $\psi_x\colon F_x \to G_x$, hence a morphism $F \to G$. $\square$

Corollary 2.2.4. *Let X be 2-connected. Then the 2-functor*

$$(\,\cdot\,)_X\colon \mathbf{C} \longrightarrow \mathbf{C}_X$$

is an equivalence of 2-categories (for $\mathbf{C} = \mathbf{Cat}$, a quasi-2-inverse is given by $\Gamma(X, \cdot)$).

Proof. Fix $x_0 \in X$. Since $\Pi_2(X) \simeq 1$, the evaluation ev_{x_0} is a 2-equivalence, quasi-inverse to $\mu^2_{X,\mathbf{C}} \circ (\cdot)_X$. Since $\mu^2_{X,\mathbf{C}}$ is fully faithful this implies by abstract nonsense that $\mu^2_{X,\mathbf{C}}$ and $(\cdot)_X$ are 2-equivalences, with F_{x_0} a quasi-inverse of $(\cdot)_X$. Then, the 2-functor $\Gamma(X, \cdot)$ provides another quasi-inverse, thanks to the natural 2-transformation $\Gamma(X, \cdot) \to \mathsf{F}_{x_0}$. $\square$

Theorem 2.2.5. *Let X be locally relatively 2-connected*[16]*. Then*

$$\mu^2_X\colon \mathbf{\Gamma}_X \longrightarrow \mathbf{Y}_{\Pi_2(X)}$$

is an equivalence of 3-functors.

[16]Recall that a topological space X is locally relatively 2-connected if each point $x \in X$ has a fundamental system of 1-connected neighborhoods U such that every homotopy of a path in U is homotopic to the constant homotopy in X.

Proof. We have to show that for each 2-category $\mathbf{C}$, the 2-functor $\mu^2_{X,\mathbf{C}}\colon \mathbf{\Gamma}(X, \mathbf{C}_X) \longrightarrow \mathbf{Hom}\,(\Pi_2(X), \mathbf{C})$ is essentially surjective. We may assume without loss of generality that X is connected.

Suppose first that $\mathbf{C} = \mathbf{Cat}$. and let $\boldsymbol{\alpha} \in \mathbf{Hom}\,(\Pi_2(X), \mathbf{Cat})$. Fix a base point $x_0 \in X$, and denote by ΩX the loop space at x_0. Then the 2-functor $\boldsymbol{\alpha}$ is given by a pair (C, α), where C is a category and $\alpha\colon \Pi_1(\Omega X) \longrightarrow \mathsf{Aut}\,(\mathsf{C})$ a monoidal functor.

Set

$$\mathcal{V} = \Big\{(V, x) \mid x \in V,\ V \text{ relatively 2-connected open subset of } X\Big\}$$

and define $(V, x) \leqslant (W, y)$ if and only if $x \in W \subset V$. For any $(V, x) \leqslant (W, y)$, we chose a path $\gamma_{xy}\colon x \to y$ in W and for any $(V, x) \leqslant (W, y) \leqslant (Z, z)$ we chose an homotopy $H_{\gamma_{xy},\gamma_{yz},\gamma_{xz}}\colon \gamma_{xy}\gamma_{yz} \xrightarrow{\sim} \gamma_{xz}$ in Z.

Let $U \subset X$ be an open subset. We set

$$\mathsf{C}_\alpha(U) = \underset{\substack{(V,x)\in\mathcal{V} \\ V\subset U}}{2\varprojlim}\ \mathsf{C}$$

where in the projective system we use the equivalence $\alpha(\gamma_{xy})\colon \mathsf{C} \xrightarrow{\sim} \mathsf{C}$ for any points x, y and the invertible transformations of functors $\alpha(H_{\gamma_{xy},\gamma_{yz},\gamma_{xz}})$ for any paths γ_{xy}, γ_{yz} and γ_{xz}. Note that since W is relatively 2-connected, the equivalences $\alpha(\gamma_{xy})$ are unique up to invertible transformation and the invertible transformations $\alpha(H_{\gamma_{xy},\gamma_{yz},\gamma_{xz}})$ does not depend on the choice of the homotopy $H_{\gamma_{xy},\gamma_{yz},\gamma_{xz}}$.

One argues as in the proof of Theorem 1.3.9 to show that the pre-stack defined by $X \supset U \mapsto \mathsf{C}_\alpha(U)$ is actually a stack. By definition it is clear that, if U is relatively 2-connected, then for any choice of $x \in U$ we get an equivalence of categories $\mathsf{C}_\alpha(U) \simeq \mathsf{C}$ compatible with restriction functors in a natural sense. Hence, the stack C_α is constant on every relatively 2-connected open subset of X. Since relatively 2-connected open subsets form a base of the topology of X, we get that C_α is locally constant with stalk C.

The calcul of the 2-monodromy of C_α is similar to that of 1-monodromy in the proof of Theorem 1.3.9.

The same proof holds for any 2-complete 2-category and for a general 2-category $\mathbf{C}$ we can use the Yoneda 2-Lemma to reduce to this case. ☐

Suppose that X is connected and locally relatively 2-connected, and let ΩX be the loop space at a fixed point $x_0 \in X$. Consider the following diagram of topological space and continuous maps

$$(\Omega X)^3 \overset{\xrightarrow{q_{23}}}{\underset{\xrightarrow[q_{12}]{id\times m}}{\xrightarrow{m\times id}}} (\Omega X)^2 \overset{\xrightarrow{q_2}}{\underset{\xrightarrow[q_1]{}}{\xrightarrow{m}}} \Omega X \longrightarrow \{x_0\}$$

where the q_i's, the q_{ij}'s and the q_{ijk}'s are the natural projections, a the action of ΩX on PX and m the composition of paths in ΩX[17].

[17]Note that ΩX does not define a simplicial topological space, since the maps $m \circ (id \times m)$ and $m \circ (m \times id)$ are not equal but only homotopic. What one gets is a 2-simplicial object in the 2-category **Top**. This will not cause particular difficulties, since for locally constant objects there is a natural invertible transformation of functors $(m \times id)^{-1}m^{-1} \xrightarrow{\sim} (id \times m)^{-1}m^{-1}$.

Let $\mathfrak{S}$ be a locally constant stack on X with values in $\mathbf{C}$. Theorem 2.2.5 asserts that $\mathfrak{S}$ is completely and uniquely (up to equivalence) determined by its 2-monodromy $\mu^2_{X,\mathbf{C}}(\mathfrak{S})\colon \Pi_2(X) \to \mathbf{C}$. Since X is connected, the stalks of $\mathfrak{S}$ are all equivalent. Let us denote by P the stalk at x_0. Then the 2-monodromy reads as a monoidal functor $\mu^2_{X,\mathbf{C}}(\mathfrak{S})\colon \Pi_1(\Omega X) \to \mathsf{Aut}_{\mathbf{C}}(\mathrm{P})$. Since the topological space ΩX satisfies the hypothesis of Theorem 1.3.9, there is a chain of equivalences of categories

$$\mathsf{Hom}\,(\Pi_1(\Omega X), \mathsf{Aut}_{\mathbf{C}}(\mathrm{P})) \underset{\mu}{\overset{\sim}{\leftarrow}} \Gamma(\Omega X, \mathsf{Aut}_{\mathbf{C}}(\mathrm{P})_{\Omega X}) \simeq \mathsf{Aut}_{\mathbf{C}_X}(\boldsymbol{\eta}_{\Omega X,\mathbf{C}}(\mathrm{P})).$$

Then the 2-monodromy $\mu^2_{X,\mathbf{C}}(\mathfrak{S})$ is equivalent to a pair (α, ν) where $\alpha\colon \boldsymbol{\eta}_{\Omega X,\mathbf{C}}(\mathrm{P}) \xrightarrow{\sim} \boldsymbol{\eta}_{\Omega X,\mathbf{C}}(\mathrm{P})$ is an equivalence of constant stacks on ΩX and

$$\nu\colon q_1^{-1}\alpha \circ q_2^{-1}\alpha \xrightarrow{\sim} m^{-1}\alpha$$

is an isomorphism of functors of stacks on $(\Omega X)^2$ such that the following diagram of invertible transformations of functors of stacks on $(\Omega X)^3$ commutes

$$\begin{tikzcd}
 & q_1^{-1}\alpha \circ q_2^{-1}\alpha \circ q_3^{-1}\alpha \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
q_{12}^{-1}(q_1^{-1}\alpha \circ q_2^{-1}\alpha) \circ q_3^{-1}\alpha \arrow[d, "\nu"] & & q_1^{-1}\alpha \circ q_{23}^{-1}(q_1^{-1}\alpha \circ q_2^{-1}\alpha) \arrow[d, "\nu"] \\
q_{12}^{-1}m^{-1}\alpha \circ q_3^{-1}\alpha \arrow[d, "\sim"] & & q_1^{-1}\alpha \circ q_{23}^{-1}m^{-1}\alpha \arrow[d, "\sim"] \\
(m \times id)^{-1}(q_1^{-1}\alpha \circ q_2^{-1}\alpha) \arrow[dr, "\nu"] & & (id \times m)^{-1}(q_1^{-1}\alpha \circ q_2^{-1}\alpha) \arrow[dl, "\nu"'] \\
 & (m \times id)^{-1}m^{-1}\alpha \arrow[r, "\sim"'] & (id \times m)^{-1}m^{-1}\alpha\,.
\end{tikzcd} \tag{2.2.1}$$

Roughly speaking, ν is given by a family of functorial invertible transformations $\nu_{12}\colon \alpha_{\gamma_1} \circ \alpha_{\gamma_2} \simeq \alpha_{\gamma_1\gamma_2}$ for any $\gamma_1, \gamma_2 \in \Omega X$, such that for $\gamma_1, \gamma_2, \gamma_3 \in \Omega X$ the following diagram commutes

$$\begin{tikzcd}
\alpha_{\gamma_1} \circ \alpha_{\gamma_2} \circ \alpha_{\gamma_3} \arrow[r, "\nu_{23}"] \arrow[d, "\nu_{12}"] & \alpha_{\gamma_1} \circ \alpha_{\gamma_2\gamma_3} \arrow[d, "\nu_{1,23}"] \\
\alpha_{\gamma_1\gamma_2} \circ \alpha_{\gamma_3} \arrow[r, "\nu_{12,3}"] & \alpha_{\gamma_1\gamma_2\cdot\gamma_3} \simeq \alpha_{\gamma_1\cdot\gamma_2\gamma_3}.
\end{tikzcd}$$

Definition 2.2.6. We call the triplet $(\mathrm{P}, \alpha, \nu)$ a descent datum for the locally constant stack $\mathfrak{S}$ on X.

Let us analize a particular case, for which the descent datum admits a more familiar description. Let G be a (not necesserly commutative) group. Assume that X is connected and locally relatively 2-connected, and let $\mathfrak{G}$ be a G_X-gerbes (see [7]), *i.e.* a stack locally equivalent to the stack of torsors $\mathfrak{Tors}(G_X)$. Since $\mathfrak{Tors}(G_X) \simeq G[1]_X$, $\mathfrak{G}$ is a locally constant stacks on X with stalk the groupoid $G[1]$. By Morita theorem for torsors (see *loc. cit.* cap. IV), an equivalence $\alpha\colon \mathfrak{Tors}(G_{\Omega X}) \xrightarrow{\sim} \mathfrak{Tors}(G_{\Omega X})$ is given by $\mathcal{N} \mapsto \mathcal{P} \wedge \mathcal{N}$ for a $G_{\Omega X}$-bitorsor $\mathcal{P}$, where $\cdot \wedge \cdot$ denotes the contracted product. Then, the descent datum for $\mathfrak{G}$ is equivalent to the datum of $(G[1], \mathcal{P}, \nu)$ where

$$\nu\colon q_1^{-1}\mathcal{P} \wedge q_2^{-1}\mathcal{P} \xrightarrow{\sim} m^{-1}\mathcal{P}$$

is an isomorphism of $G_{\Omega X}$-bitorsors on $(\Omega X)^2$ satisfying a commutative contraint similar to that of diagram (2.2.1). See [6] cap. 6, for related constructions of line bundles on the free loop space of a manifold.

2.3 Degree 2 non abelian cohomology with constant coefficients

Let D be a monoidal category. Denote by $\mathsf{D}[1]$ the 2-category with 1 as single object and $\mathsf{End}_{\mathsf{D}[1]}(1) = \mathsf{D}$. Note that if D is a groupoid whose monoidal structure is rigid[18], then $\mathsf{D}[1]$ is a 2-groupoid. It is easy to see that we get a fully faithful 2-functor

$$[1] : \mathbf{Mon} \longrightarrow 2\mathbf{Cat},$$

where we denote by **Mon** the 2-category of monoidal categories with monoidal functors and monoidal transformations. This functor sends rigid monoidal groupoids, called *gr*-categories, to 2-groupoids.

Note that if M is a monoid, then $M[1]$ is monoidal if and only if M is commutative, and that if M is also a group, then $M[1]$ is a *gr*-category. Hence we get (essentially fully faithful) functors

$$[2] = [1] \circ [1] \colon \mathsf{Mon}^{\mathsf{c}} \longrightarrow \mathbf{Mon} \longrightarrow 2\mathbf{Cat} \qquad , \qquad [2] \colon \mathsf{Gr}^{\mathsf{c}} \longrightarrow 2\mathbf{Gr}$$

where the uppercase c means commutative structures. Conversely, if $\mathbf{G}$ is a connected 2-groupoid, for each object $\mathrm{P} \in \mathrm{Ob}\,\mathbf{G}$, the inclusion 2-functor $\mathsf{Aut}_{\mathbf{G}}(\mathrm{P})[1] \longrightarrow \mathbf{G}$ is a 2-equivalence. If $\mathbf{G}$ is even 2-connected (*i.e.* moreover $\mathsf{Aut}_{\mathbf{G}}(\mathrm{P})$ is a connected groupoid for some (hence all) P), then $\mathbf{G} \simeq \mathrm{Z}_{\mathbf{G}}(\mathrm{P})[2]$.

For a not necessarily commutative group G, we can consider the strict *gr*-category $\mathsf{Aut}\,(G[1])$ which gives rise to the 2-groupoid

$$G[\![2]\!] = \mathsf{Aut}\,(G[1])[1]$$

Recall that $\mathsf{Aut}\,(G[1])$ is equivalent to $G \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G)$. Hence if G is commutative, then $\mathsf{Aut}\,(G[1])$ is completely disconnected but only $\mathrm{id} \in \mathrm{Ob}\,\mathsf{Aut}\,(G[1])$ is G-linear, so we get a monoidal functor

$$\mathsf{Aut}_G(G[1])[1] = G[2] \to G[\![2]\!] = \mathsf{Aut}\,(G[1])[1]$$

[18]Recall that a monoidal category $(\mathsf{D}, \otimes)$ is rigid if for each $P \in \mathrm{Ob}\,\mathsf{D}$ there exists an object P^* and natural ismorphisms $P \otimes P^* \simeq I$ and $P^* \otimes P \simeq I$, where I denotes the unit object in D.

that identifies $G[2]$ to a sub-2-category of $G[\![2]\!]$ which has only one morphism but the same transformations.

Consider an object C of the 2-category $\mathbf{Cat}$ (resp. $\mathbf{Cat}_G$). Then $\mathsf{End}\,(\mathsf{C})[1]$ (resp. $\mathsf{End}_G(\mathsf{C})[1]$) is just the full sub-2-category of $\mathbf{Cat}$ (resp. $\mathbf{Cat}_G$) with the single object C. Hence, $\mathsf{End}\,(\mathsf{C})[1]_X$ (resp. $\mathsf{End}_G(\mathsf{C})[1]_X$) is the 2-stack of locally constant stacks (resp. G_X-linear stacks) on X with stalk C.
If X is a locally relatively 2-connected space, by Theorem 2.2.5 equivalence classes of such stacks are classified by the set

$$\boldsymbol{\pi}_0(\mathbf{Hom}\,(\Pi_2(X), \mathsf{Aut}\,(\mathsf{C})[1]). \tag{2.3.1}$$

Assume moreover that X is connected. Then the 2-groupoid $\Pi_2(X)$ is connected, hence it is equivalent to $\Pi_1(\Omega X)[1]$ for some base point $x_0 \in X$. We thus get that (2.3.1) is isomorphic to

$$\pi_0(\mathsf{Hom}_{\otimes}(\Pi_1(\Omega X), \mathsf{Aut}\,(\mathsf{C}))/\,\mathrm{Pic}(\mathsf{C}), \tag{2.3.2}$$

where $\mathsf{Hom}_{\otimes}(\cdot,\cdot)$ denotes the category of monoidal functors and the group $\mathrm{Pic}(\mathsf{C})$ acts by conjugation. A similar reslut holds, replacing $\mathbf{Cat}$ by $\mathbf{Cat}_G$.

Let us analyze more in detail the case of gerbes. We start with the abelian case. Let G be a commutative group and take $\mathsf{C} = G[1]$. Since there is an obvious equivalence of strict gr-categories $G[1] \simeq \mathsf{End}_G(G[1])$, we get that $G[2]$ is just the full sub-2-category of $\mathbf{Cat}_G$ with the single object $G[1]$. This means that the 2-category $\mathbf{\Gamma}(X, G[2]_X)$ is equivalent to $\mathbf{Ger}_{ab}(G_X)$, the strict 2-category of abelian G_X-gerbes (see [7, 2] and [6] for a complete introduction to abelian gerbes), *i.e.* G_X-linear stacks on X locally G_X-equivalent to the G_X-linear stack of torsors $\mathfrak{Tors}(G_X) \simeq G[1]_X$.
By some cocycle arguments (see for example [6]), one shows that there is an isomorphism of groups

$$\boldsymbol{\pi}_0(\mathbf{Ger}_{ab}(G_X)) \simeq H^2(X; G_X).$$

Assume that X is locally relatively 2-connected. By Theorem 2.2.5, there is an equivalence of monoidal 2-categories

$$\mathbf{Ger}_{ab}(G_X) \simeq \mathbf{Hom}\,(\Pi_2(X), G[2]).$$

If X is connected, (2.3.2) gives the group $\pi_0(\mathsf{Hom}_{\otimes}(\Pi_1(\Omega X), G[1])$, since $\mathrm{Pic}_G(G[1]) \simeq 1$. Hence we get

Proposition 2.3.1. *Let X be connected and locally relatively 2-connected. Then for any commutative group G there is an isomorphism of groups*

$$H^2(X; G_X) \simeq \pi_0(\mathsf{Hom}_{\otimes}(\Pi_1(\Omega X), G[1])).$$

Consider a gr-category H. Then there exists an "essentially exact"[19] sequence of gr-categories

$$1 \longrightarrow \mathrm{A}_{\mathsf{H}}[1] \xrightarrow{i} \mathsf{H} \xrightarrow{\pi_0} \pi_0(\mathsf{H})[0] \longrightarrow 1,$$

[19]This means that the monoidal functor i (resp. π_0) is fully faithfull (resp. essentially surjective) and that the essential image of i is equivalent to the kernel of π_0 as monoidal categories.

where A_{H} denotes the commutative group $\mathrm{Aut}_{\mathsf{H}}(I)$ of automorphims of the unit object, and the group $\pi_0(\mathsf{H})$ is view as discrete category, which acts on A_{H} by conjugation. If G is another *gr*-category, we get an exact sequence of groups

$$1 \longrightarrow \pi_0(\mathsf{Hom}_{\otimes}(\pi_0(\mathsf{H})[0], \mathsf{G})) \longrightarrow \pi_0(\mathsf{Hom}_{\otimes}(\mathsf{H}, \mathsf{G})) \longrightarrow \pi_0(\mathsf{Hom}_{\otimes}(\mathrm{A}_{\mathsf{H}}[1], \mathsf{G})). \quad (2.3.3)$$

Lemma 2.3.2. *Let $\mathsf{G} = G[1]$, for G an abelian group. Then 2.3.3 gives an exact sequence of abelian groups*

$$1 \longrightarrow H^2(\pi_0(\mathsf{H}); G) \longrightarrow \pi_0(\mathsf{Hom}_{\otimes}(\mathsf{H}, G[1])) \longrightarrow \mathrm{Hom}_{\mathsf{Gr}}(A_{\mathsf{H}}, G),$$

where G is view as a $\pi_0(\mathsf{H})$-module with trivial action.

Proof. Denote by H the group $\pi_0(\mathsf{H})$. It is easy to see that a monoidal functor $H[0] \longrightarrow G[1]$ is given by a set-theoretic function $\lambda\colon H \times H \longrightarrow G$ such that

$$\lambda(h_1, h_2)\lambda(h_1h_2, h_3) = \lambda(h_2, h_3)\lambda(h_1, h_2h_3).$$

Two monoidal functors λ, λ' are isomorphic if and only if there exists a function $\nu\colon H \to G$ such that

$$\lambda(h_1, h_2)\nu(h_1h_2) = \lambda'(h_1, h_2)\nu(h_1)\nu(h_2).$$

Hence we get an isomorphism of groups

$$\pi_0(\mathsf{Hom}_{\otimes}(H[0], G[1]) \simeq H^2(H; G),$$

where G is view as a H-module with trivial action. Similarly, one easily checks that $\pi_0(\mathsf{Hom}_{\otimes}(\mathrm{A}_{\mathsf{H}}[1], G[1]))$ is isomorphic to $\mathrm{Hom}_{\mathsf{Gr}}(\mathrm{A}_{\mathsf{H}}, G)$. ☐

Taking $\mathsf{H} = \Pi_1(\Omega X)$ in Lemma 2.3.2 and using Proposition 2.3.1, we get

Corollary 2.3.3 (Hopf's theorem for 2-cohomolgy). *Let X be connected and locally relatively 2-connected. Then for any commutative group G, there is an exact sequence of abelian groups*

$$1 \longrightarrow H^2(\pi_1(X); G) \longrightarrow H^2(X; G_X) \longrightarrow \mathrm{Hom}_{\mathsf{Gr}}(\pi_2(X), G),$$

where G is view as a $\pi_1(X)$-module with trivial action.

One can be more precise as follow. Let $\mathrm{Hom}_{\pi_1(X)}(\pi_2(X), G)$ denote the subgroup of morphisms in $\mathrm{Hom}_{\mathsf{Gr}}(\pi_2(X), G)$ which are $\pi_1(X)$-linear. One easily checks that the morphism $H^2(X; G_X) \to \mathrm{Hom}_{\mathsf{Gr}}(\pi_2(X), G)$ factors through $\mathrm{Hom}_{\pi_1(X)}(\pi_2(X), G)$ and that one gets an exact sequence

$$1 \longrightarrow H^2(\pi_1(X); G) \longrightarrow H^2(X; G_X) \longrightarrow \mathrm{Hom}_{\pi_1(X)}(\pi_2(X), G) \longrightarrow H^3(\pi_1(X); G).$$

Let us describe the last morphism. Recall that to each gr-category H one associates a cohomology class $c(\mathsf{H})$ in $H^3(\pi_0(\mathsf{H}), \mathrm{A}_{\mathsf{H}})$, where A_{H} is endowed with the conjugation action of $\pi_0(\mathsf{H})$ (if H is strict, *i.e.* $\mathsf{H} \simeq H^{-1} \xrightarrow{d} H^0$, the class $c(\mathsf{H})$ coincides with the usual class attached to the crossed module $H^{-1} \xrightarrow{d} H^0$ in $H^3(\operatorname{coker} d, \ker d)$. See for example [4] cap V).
For $\mathsf{H} = \Pi_1(\Omega X)$, we get a class $c(\Pi_1(\Omega X))$ in $H^3(\pi_1(X), \pi_2(X))$. Hence, to each $\pi_1(X)$-linear morphism $f \colon \pi_2(X) \to G$, one associates the image of $c(\Pi_1(\Omega X))$ by the induced morphism $\hat{f} \colon H^3(\pi_1(X), \pi_2(X)) \to H^3(\pi_1(X), G)$.

Moreover, we have

Corollary 2.3.4. *Let X be connected and locally relatively 2-connected. Suppose that $c(\Pi_1(\Omega X)) = 1$ in $H^3(\pi_1(X), \pi_2(X))$. Then for any commutative group G, there is a split exact sequence of abelian groups*

$$1 \longrightarrow H^2(\pi_1(X); G) \longrightarrow H^2(X; G_X) \longrightarrow \operatorname{Hom}_{\pi_1(X)}(\pi_2(X), G) \longrightarrow 1. \tag{2.3.4}$$

Proof. One possible way to show that the sequence splits is the following. Since $c(\Pi_1(\Omega X)) = 1$ in $H^3(\pi_1(X), \pi_2(X))$, the "essentially exact" sequence of gr-categories

$$1 \longrightarrow \pi_2(X)[1] \longrightarrow \Pi_1(\Omega X) \longrightarrow \pi_1(X)[0] \longrightarrow 1,$$

splits. This means that there is an equivalence of monoidal categories $\Pi_1(\Omega X) \xrightarrow{\sim} \left(\pi_2(X) \xrightarrow{1} \pi_1(X)\right)$. Hence, a direct calculation as in 2.3.2 shows that there is an isomorphism of groups

$$\pi_0(\mathsf{Hom}_{\otimes}(\pi_2(X) \xrightarrow{1} \pi_1(X), G[1]) \simeq H^2(\pi_1(X); G) \times \operatorname{Hom}_{\pi_1(X)}(\pi_2(X), G).$$

$\square$

Since $H^2(\pi_1(X); G)$ classifies (central) extensions of $\pi_1(X)$ by G, the sequence 2.3.4 is an homotopical version of the Universal Coefficient Theorem.

Now, let G be a not necessarily commutative group and consider $\mathsf{C} = G[1]$. Then the 2-category $\boldsymbol{\Gamma}(X, \mathsf{End}\,(G[1])[1]_X)$ is equivalent to $\mathbf{Ger}(G_X)$, the strict 2-category of G_X-gerbes. One may show (see for example [3]) that there is an isomorphism of pointed sets[20]

$$\boldsymbol{\pi}_0(\mathbf{Ger}(G_X)) \simeq H^2(X; G_X \xrightarrow{\mathrm{ad}} \operatorname{Aut}_{\mathsf{Gr}}(G_X)),$$

where the right hand side is the second cohomology set of X with values in the crossed module $G_X \xrightarrow{\mathrm{ad}} \operatorname{Aut}_{\mathsf{Gr}}(G_X)$.

[20] The pointed set $\pi_0(\mathbf{Ger}(G_X))$ is sometimes denoted by $H^2_g(X; G_X)$ and called the Giraud's second non abelian cohomology set of X with values in G_X.

Assume that X is locally relatively 2-connected. By Theorem 2.2.5, there is an equivalence of 2-categories

$$\mathbf{Ger}(G_X) \simeq \mathbf{Hom}\,(\Pi_2(X), \mathsf{End}\,(G[1])[1]).$$

Since $\mathrm{Pic}(G[1]) \simeq \mathrm{Out}_{\mathsf{Gr}}(G)$, by (2.3.2) we get

Proposition 2.3.5. *Let X be connected and locally relatively 2-connected. Then for any group G there is an isomorphism of pointed sets*

$$H^2(X; G_X \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G_X)) \simeq \pi_0(\mathsf{Hom}_{\otimes}(\Pi_1(\Omega X), (G \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G)))/\,\mathrm{Out}_{\mathsf{Gr}}(G).$$

With similar computations as for the commutative case, we get the following

Lemma 2.3.6. *Let $\mathsf{G} = G^{-1} \xrightarrow{d} G^0$. Then 2.3.3 gives an exact sequence of pointed sets*

$$1 \longrightarrow H^2(\pi_0(\mathsf{H}); G^{-1} \xrightarrow{d} G^0) \longrightarrow \pi_0(\mathsf{Hom}_{\otimes}(\mathsf{H}, G^{-1} \xrightarrow{d} G^0)) \longrightarrow \mathrm{Hom}_{\mathsf{Gr}}(A_{\mathsf{H}}, \ker d),$$

where the first term is the cohomolgy set of $\pi_0(\mathsf{H})$ with values in the crossed module $G^{-1} \xrightarrow{d} G^0$ (see for example [2]).

Combining Lemma 2.3.6 with Proposition 2.3.5, we get

Corollary 2.3.7 (Hopf's theorem for non abelian 2-cohomolgy). *Let X be connected and locally relatively 2-connected. Then for any group G there is an exact sequence of pointed sets*

$$1 \longrightarrow H^2(\pi_1(X); G \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G))/\,\mathrm{Out}_{\mathsf{Gr}}(G) \longrightarrow H^2(X; G_X \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G_X)) \longrightarrow$$

$$\longrightarrow \mathrm{Hom}_{\mathsf{Gr}}(\pi_2(X), \mathrm{Z}(G))/\,\mathrm{Out}_{\mathsf{Gr}}(G),$$

where the action of $\mathrm{Out}_{\mathsf{Gr}}(G)$ on $\mathrm{Hom}_{\mathsf{Gr}}(\pi_2(X), \mathrm{Z}(G))$ is induced by the natural action on $\mathrm{Z}(G)$. If moreover $\pi_1(X)$ is trivial, one gets an isomorphism

$$H^2(X; G_X \xrightarrow{\mathrm{ad}} \mathrm{Aut}_{\mathsf{Gr}}(G_X)) \simeq \mathrm{Hom}_{\mathsf{Gr}}(\pi_2(X), \mathrm{Z}(G))/\,\mathrm{Out}_{\mathsf{Gr}}(G),$$

Final comments

What's next? It seems clear that, using the same technique, one should expect for each n-category $\mathbf{C}$ and each locally relatively n-connected space X a natural n-equivalence

$$\mathbf{C}_X(X) \simeq \mathbf{Hom}\,(\Pi_n(X), \mathbf{C}), \qquad (.0.5)$$

where $\mathbf{C}_X$ denotes the constant n-stack with stalk $\mathbf{C}$ and $\Pi_n(X)$ the homotopy n-groupoid of X. However, some care has to be taken since there are several non-equivalent definition of n-categories for $n \geqslant 3$, so it may well be necessary to suppose that $\mathbf{C}$ is an "enriched" (or even strict) n-category. We will not investigate this problem any further here. An answer in this direction is partially given in [12], for $\mathbf{C} = (n-1)\mathbf{Cat}$, the strict n-category of $(n-1)$-categories and X a CW-complex.

For a commutative group G, denote by $G[n]$ the strict gr- n-category with a single element, trivial i-arrows for $i \leq n-1$ and G as n-arrows. Then one may check that there is an isomorphism of groups

$$H^n(X; G_X) \simeq \boldsymbol{\pi}_0(G[n]_X(X)).$$

Suppose that X is locally relatively n-connected. From .0.5, we have an isomorphism of groups

$$H^n(X; G_X) \simeq \boldsymbol{\pi}_0(\mathbf{Hom}\,(\Pi_n(X), G[n])). \tag{.0.6}$$

This isomorphism should be interpreted as as the categorical version of the cohomological Hopf's theorem. Indeed, if we suppose that X is connected and that $\pi_i(X) \simeq 1$ for all $2 \leq i \leq n-1$, we get an "essentially exact" sequence of gr-n-categories

$$1 \longrightarrow \pi_n(X)[n] \longrightarrow \Pi_{n-1}(\Omega X) \longrightarrow \pi_1(X)[0] \longrightarrow 1,$$

and hence an exact sequence of groups

$$1 \longrightarrow \boldsymbol{\pi}_0(\mathbf{Hom}_{\otimes}(\pi_1(X)[0], G[n])) \longrightarrow \boldsymbol{\pi}_0(\mathbf{Hom}_{\otimes}(\Pi_{n-1}(\Omega X), G[n])) \longrightarrow$$

$$\longrightarrow \boldsymbol{\pi}_0(\mathbf{Hom}_{\otimes}(\pi_n(X)[n], G[n]))$$

From the isomorphism .0.6 and a direct calculation, we finally get an exact sequence

$$1 \longrightarrow H^n(\pi_1(X); G) \longrightarrow H^n(X; G_X) \longrightarrow \mathrm{Hom}_{\mathsf{Gr}}(\pi_n(X), G),$$

where G is view as a $\pi_1(X)$-module with trivial action.

If G is a not necessarily commutative group, we define the n-groupoid $G[\![n]\!]$ by induction as

$$G[\![1]\!] = G[1], \qquad G[\![n+1]\!] = \mathbf{Aut}_{n\mathbf{Cat}}(G[\![n]\!])[1],$$

where $\mathbf{Aut}_{n\mathbf{Cat}}(G[\![n]\!])$ denotes the gr-n-category of auto-n-equivalence of $G[\![n]\!]$ (note that, when G is commutative, if we require G-linearity at each step in the definition of $G[\![n]\!]$, we recover $G[n]$). Then one may define the non abelian n-cohomology set of X with coefficient in G_X as

$$H^n_g(X; G) = \boldsymbol{\pi}_0(G[\![n]\!]_X(X)).$$

If X is locally relatively n-connected, then the n-equivalence .0.5 gives an isomorphism of pointed sets

$$H^n_g(X; G) \simeq \boldsymbol{\pi}_0(\mathbf{Hom}\,(\Pi_n(X), G[\![n]\!])).$$

This is the non abelian version of the cohomological Hopf's theorem.

A The stack of sheaves with values in a complete category

We recall here the construction of the stack of sheaves with values in a complete category C, *i.e.* a category which admits all small limits.

Definition A.0.8. A presheaf on X with values in C is a functor

$$\mathsf{Op}(X)^\circ \longrightarrow \mathsf{C}.$$

A morphism between presheaves is a morphism of functors. We denote by $\mathsf{PSh}_X(\mathsf{C})$ the category of presheaves on X with values in C.
A presheaf is called a sheaf if it commutes to filtered limits indexed by coverings that are stable by finite intersection, and we denote by $\mathsf{Sh}_X(\mathsf{C})$ the full subcategory of $\mathsf{PSh}_X(\mathsf{C})$ whose objects are sheaves.

Note that if $U \subset X$ is an open subset and $\mathcal{F}$ is a sheaf on X, then its restriction $\mathcal{F}|_U$ is also a sheaf. Hence we can define the prestack of sheaves on X, denoted by $\mathfrak{Sh}_X(\mathsf{C})$, by assigning $X \supset U \mapsto \mathsf{Sh}_U(\mathsf{C})$.
Let $\mathcal{F}, \mathcal{G}$ be two presheaves on X. We have a natural bijective map of sets

$$\mathrm{Hom}_{\mathsf{PSh}_X(\mathsf{C})}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \varprojlim_{\substack{(U,V)\\ V \subset U}} \mathrm{Hom}_{\mathsf{C}}(\mathcal{F}(U), \mathcal{G}(V))$$

where (U, V) is considered as an object of $\mathsf{Op}(X)^\circ \times \mathsf{Op}(X)$. Now let $U \subset X$ be an open subset, $\mathcal{F}$ a presheaf on X and $\mathcal{G}$ a presheaf on U. Then it is easy to see that we have the isomorphism of sets

$$\mathrm{Hom}_{\mathsf{PSh}_U(\mathsf{C})}(\mathcal{F}|_U, \mathcal{G}) \xrightarrow{\sim} \varprojlim_{\substack{(V,W)\\ W \subset V \subset U}} \mathrm{Hom}_{\mathsf{C}}(\mathcal{F}(V), \mathcal{G}(W)) \xrightarrow{\sim} \varprojlim_{\substack{(V,W)\\ W \subset V \subset X}} \mathrm{Hom}_{\mathsf{C}}(\mathcal{F}(V), \mathcal{G}(W \cap U)).$$

Lemma A.0.9. *Let $\mathcal{F}$ be a presheaf and $\mathcal{G}$ be a sheaf on X.*
Then the presheaf $\mathcal{H}om_{\mathsf{PSh}_X(\mathsf{C})}(\mathcal{F}, \mathcal{G})$ defined by

$$\mathcal{H}om_{\mathsf{PSh}_X(\mathsf{C})}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathsf{PSh}_U(\mathsf{C})}(\mathcal{F}|_U, \mathcal{G}|_U)$$

is a sheaf of sets.

Proof. We have to show that $\mathcal{H}om_{\mathsf{PSh}_X(\mathsf{C})}(\mathcal{F}, \mathcal{G})$ commutes to small filtered limits indexed by coverings that are stabel by finite intersection. Let $\{U_i\}_{i\in I}$ be such a covering of an open subset $U \subset X$.

$$\begin{aligned}\mathcal{H}om_{\mathsf{PSh}_X(\mathsf{C})}(\mathcal{F},\mathcal{G})(U) = \operatorname{Hom}_{\mathsf{PSh}_U(\mathsf{C})}(\mathcal{F}|_U,\mathcal{G}|_U) &\simeq \varprojlim_{\substack{(V,W)\\ W\subset V}} \operatorname{Hom}_{\mathsf{C}}(\mathcal{F}(V),\mathcal{G}(U\cap W))\\ &\simeq \varprojlim_{\substack{(V,W)\\ W\subset V}} \operatorname{Hom}_{\mathsf{C}}(\mathcal{F}(V),\varprojlim_{i\in I}\mathcal{G}(U_i\cap W)) \simeq \varprojlim_{\substack{(V,W)\\ W\subset V}}\varprojlim_{i\in I}\operatorname{Hom}_{\mathsf{C}}(\mathcal{F}(V),\mathcal{G}(U_i\cap W))\\ &\simeq \varprojlim_{i\in I}\varprojlim_{\substack{(V,W)\\ W\subset V}} \operatorname{Hom}_{\mathsf{C}}(\mathcal{F}(V),\mathcal{G}(U_i\cap W)) \simeq \varprojlim_{i\in I}\operatorname{Hom}_{\mathsf{PSh}_{U_i}(\mathsf{C})}(\mathcal{F}|_{U_i},\mathcal{G}|_{U_i})\\ &\simeq \varprojlim_{i\in I}\mathcal{H}om_{\mathsf{PSh}_X(\mathsf{C})}(\mathcal{F},\mathcal{G})(U_i)\end{aligned}$$

□

Lemma A.0.10. *Let $\mathcal{F}$ be a presheaf on X. Then $\mathcal{F}$ is a sheaf if and only if for any object $A \in \operatorname{Ob}\mathsf{C}$ and any open subset $U \subset X$ the presheaf*

$$U \supset V \mapsto \operatorname{Hom}_{\mathsf{C}}(A,\mathcal{F}(V))$$

is a sheaf of sets.

Proof. Follows immediately from Yoneda's Lemma. □

Proposition A.0.11. *The prestack $\mathfrak{Sh}_X(\mathsf{C})$ of sheaves with values in C is a stack.*

Proof. By Lemma A.0.9, the prestack is seperated. Now let $(\{U_i\}_{i\in I},\{\mathcal{F}_i\}_{i\in I},\{\theta_{ij}\}_{i,j\in I})$ be a descent datum for $\mathfrak{Sh}_X(\mathsf{C})$ on open subset $U \subset X$. By taking a refinement, we can assume that the covering $\{U_i\}_{i\in I}$ is stable by finite intersections.
Let $V \subset U$. Then the cocycle condition allows us to define

$$\mathcal{F}(V) = \varprojlim_{i\in I}\mathcal{F}_i(V\cap U_i).$$

It is then obvious that $\mathcal{F}$ is a sheaf (for instance using Lemma A.0.10 and the fact that this is true if $\mathsf{C} = \mathsf{Set}$) which by construction is isomorphic to $\mathcal{F}_i$ on U_i. □

Proposition A.0.12. *The stack $\mathfrak{Sh}_X(\mathsf{C})$ admits all small limits.*

Proof. Let $\beta\colon \mathsf{I} \to \mathsf{Sh}_X(\mathsf{C})$ be a functor, with I a small category. Then, for each open subset $U \subset X$, set

$$\mathcal{F}(U) = \varprojlim_{i\in\mathsf{I}}\beta(i)(U).$$

It is immediately verified that $\mathcal{F}$ is a sheaf on X that satisfies

$$\mathcal{F} \simeq \varprojlim_{i\in\mathsf{I}}\beta(i).$$

□

Definition A.0.13. Let $\mathcal{F}$ be a presheaf. A sheaf $\widetilde{\mathcal{F}}$ together with a morphism $\mathcal{F} \to \widetilde{\mathcal{F}}$ is called the sheaf associated to $\mathcal{F}$ if it satisfies the usual universal property, *i.e.* any morphism from $\mathcal{F}$ into a sheaf $\mathcal{G}$ factors uniquely through $\widetilde{\mathcal{F}}$:

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \\ \widetilde{\mathcal{F}} \arrow[ur] & \end{tikzcd}$$

Proposition A.0.14. *Let $M \in \operatorname{Ob} \mathsf{C}$. Assume that X is locally connected. Then the sheaf associated to the constant presheaf with stalk M exists.*

Proof. Let $U \subset X$ be an open subset. Denote by $\#U$ the set of connected components of U. Set

$$M_X(U) = M^{\#U}.$$

Let $x \in U$. Then we denote by $\overline{x}_U$ its class in $\#U$.
For any inclusion $V \subset U$ of open subsets and for any $x \in V$, we have the natural morphisms

$$M_X(U) \longrightarrow M_{\overline{x}_U} \longrightarrow M_{\overline{x}_V}.$$

These morphisms define the morphism

$$M_X(U) \longrightarrow M_X(V).$$

Since we know that this is a sheaf if $\mathsf{C} = \mathsf{Set}$, Lemma A.0.10 implies that M_X is a sheaf which verifies the desired universal property. $\square$

Definition A.0.15. Let $M \in \operatorname{Ob} \mathsf{C}$. The sheaf associated to the constant presheaf with stalk M is called the constant sheaf with stalk M, and we denote it by M_X.

Remark A.0.16. The hypothesis on the connectivity of X in the Proposition A.0.14 is necessary to recover the classical definition of constant sheaf. More precisely, if M is a set and M_X is the constant sheaf defined in the usual way, there is a natural injective map (the 0-monodromy) $\mu^0 \colon M_X(X) \to \operatorname{Hom}(\#X, M) \simeq M^{\#X}$ defined by $\mu^0(s)(\overline{x}_X) = s(\overline{x}_X)$ for a section s in $M_X(X)$. Clearly, if X is locally connected, μ^0 is a bijection.

Let X be a locally connected topological space. Denote by $\mathsf{CSh}_X(\mathsf{C})$ the full subcategory of $\mathsf{Sh}_X(\mathsf{C})$ of constant sheaves. The previous construction defines a faithful functor

$$(\cdot)_X \colon \mathsf{C} \longrightarrow \mathsf{CSh}_X(\mathsf{C}),$$

which is an equivalence if X is connected (a quasi-inverse is given by the global sections functor).

Definition A.0.17. A sheaf $\mathcal{F}$ is called locally constant if there is an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is isomorphic to a constant sheaf.
We denote by $\mathfrak{LcSh}_X(\mathsf{C})$ the full substack of $\mathfrak{Sh}_X(\mathsf{C})$ whose objects are the locally constant sheaves.

B The 2-stack of stacks with values in a 2-complete 2-category

Let us recall the construction of the 2-stack of stacks with values in a 2-complete 2-category $\mathbf{C}$, *i.e.* a 2-category which admits all small 2-limits wher, for the sake of simplicity, the reader might assume that $\mathbf{C}$ is a strict 2-category. For the basic definitions about 2-stacks, we refer to [2]. In particular we get a Yoneda 2-Lemma, that states that the 2-functor

$$\mathbf{C} \longrightarrow \mathsf{Hom}\,(\mathbf{C}^\circ, \mathbf{Cat}) \quad ; \quad A \mapsto \mathsf{Hom}_{\mathbf{C}}(\,\cdot\,, A)$$

is fully faithful.

Let X be a topological space and denote by $\mathbf{Op}(X)$ the 2-category of its open subsets.

Definition B.0.18. A prestack on X with values in $\mathbf{C}$ is a 2-functor

$$\mathbf{Op}(X)^\circ \longrightarrow \mathbf{C}.$$

A functor between prestacks is a 2-transformation of 2-functors and transformations of functors of prestacks are modifications of 2-transformations of 2-functors. We denote by $\mathbf{PSt}_X(\mathbf{C})$ the 2-category of prestacks on X with values in $\mathbf{C}$.
A prestack is called a stack if it commutes to filtered 2-limits indexed by coverings that are stable by finite intersection, and we denote by $\mathbf{St}_X(\mathbf{C})$ the full subcategory of $\mathbf{PSt}_X(\mathbf{C})$ whose objects are stacks.

Note that if $U \subset X$ is an open subset and $\mathfrak{S}$ is a stack on X, then its restriction $\mathfrak{S}|_U$ is also a stack. Hence the assignment $X \supset U \mapsto \mathbf{St}_U(\mathbf{C})$ defines the pre-2-stack of stacks on X with values in $\mathbf{C}$, which we denote by $\mathfrak{St}_X(\mathbf{C})$.
Let $\mathfrak{S}, \mathfrak{T}$ be two prestacks on X. We have a natural equivalence of categories

$$\mathsf{Hom}_{\mathbf{PSt}_X(\mathbf{C})}(\mathfrak{S}, \mathfrak{T}) \xrightarrow{\sim} 2\varprojlim_{\substack{(U,V)\\ V\subset U}} \mathsf{Hom}_{\mathbf{C}}(\mathfrak{S}(U), \mathfrak{T}(V))$$

where (U, V) is considered as an object of $\mathbf{Op}(X)^\circ \times \mathbf{Op}(X)$. Now let $U \subset X$ be an open subset, $\mathfrak{S}$ a prestack on X and $\mathfrak{T}$ a prestack on U. Then it is easy to see that we have the equivalence of categories

$$\mathsf{Hom}_{\mathbf{PSt}_U(\mathbf{C})}(\mathfrak{S}|_U, \mathfrak{T}) \xrightarrow{\sim} 2\varprojlim_{\substack{(V,W)\\ W\subset V\subset U}} \mathsf{Hom}_{\mathbf{C}}(\mathfrak{S}(V), \mathfrak{T}(W)) \xrightarrow{\sim} 2\varprojlim_{\substack{(V,W)\\ W\subset V\subset X}} \mathsf{Hom}_{\mathbf{C}}(\mathfrak{S}(V), \mathfrak{T}(W\cap U)).$$

Lemma B.0.19. *Let $\mathfrak{S}$ be a prestack and $\mathfrak{T}$ be a stack on X.*
Then the prestack $\mathfrak{Hom}_{\mathbf{PSt}_X(\mathbf{C})}(\mathfrak{S}, \mathfrak{T})$ defined by

$$\mathfrak{Hom}_{\mathbf{PSt}_X(\mathbf{C})}(\mathfrak{S}, \mathfrak{T})(U) = \mathsf{Hom}_{\mathbf{PSt}_U(\mathbf{C})}(\mathfrak{S}|_U, \mathfrak{T}|_U)$$

is a stack of categories.

Proof. We have to show that $\mathfrak{Hom}_{\mathbf{PSt}_X(\mathbf{C})}(\mathfrak{S}, \mathfrak{T})$ commutes to small filtered limits indexed by coverings that are stable by finite intersection. Let $\{U_i\}_{i\in I}$ be such a covering of an open subset $U \subset X$.

$$\begin{aligned}
\mathfrak{Hom}_{\mathbf{PSt}_X(\mathbf{C})}(\mathfrak{S}, \mathfrak{T})(U) &= \mathsf{Hom}_{\mathbf{PSt}_U(\mathbf{C})}(\mathfrak{S}|_U, \mathfrak{T}|_U) \simeq 2\varprojlim_{\substack{(V,W)\\ W\subset V}} \mathsf{Hom}_{\mathbf{C}}(\mathfrak{S}(V), \mathfrak{T}(U\cap W)) \\
&\simeq 2\varprojlim_{\substack{(V,W)\\ W\subset V}} \mathsf{Hom}_{\mathbf{C}}(\mathfrak{S}(V), 2\varprojlim_{i\in I} \mathfrak{T}(U_i\cap W)) \simeq 2\varprojlim_{\substack{(V,W)\\ W\subset V}} 2\varprojlim_{i\in I} \mathsf{Hom}_{\mathbf{C}}(\mathfrak{S}(V), \mathfrak{T}(U_i\cap W)) \\
&\simeq 2\varprojlim_{i\in I} 2\varprojlim_{\substack{(V,W)\\ W\subset V}} \mathsf{Hom}_{\mathbf{C}}(\mathfrak{S}(V), \mathfrak{T}(U_i\cap W)) \simeq 2\varprojlim_{i\in I} \mathsf{Hom}_{\mathbf{PSt}_{U_i}(\mathbf{C})}(\mathfrak{S}|_{U_i}, \mathfrak{T}|_{U_i}) \\
&\simeq 2\varprojlim_{i\in I} \mathfrak{Hom}_{\mathbf{PSt}_X(\mathbf{C})}(\mathfrak{S}, \mathfrak{T})(U_i)
\end{aligned}$$

□

Lemma B.0.20. *Let $\mathfrak{S}$ be a prestack on X. Then $\mathfrak{S}$ is a stack if and only if for any object* $\mathrm{P} \in \mathrm{Ob}\,\mathbf{C}$ *and any open subset $U \subset X$ the prestack*

$$U \supset V \mapsto \mathsf{Hom}_{\mathbf{C}}(\mathrm{P}, \mathfrak{S}(V))$$

is a stack of categories.

Proof. Follows immediately from Yoneda's 2-Lemma (see for example [1] cap. 7). □

Proposition B.0.21. *The pre-2-stack $\mathfrak{St}_X(\mathbf{C})$ of stacks with values in $\mathbf{C}$ is a 2-stack.*

Proof. By Lemma B.0.19, the pre-2-stack is seperated. Now let

$$(\{U_i\}_{i\in I}, \{\mathfrak{S}_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I}, \{\varphi_{ijk}\}_{i,j,k\in I})$$

be a descent datum for $\mathfrak{St}_X(\mathbf{C})$ on open subset $U \subset X$. This means that $\{U_i\}_{i\in I}$ is an open covering of U, $\mathfrak{S}_i$ are stacks on U_i, $F_{ij}\colon \mathfrak{S}_j|_{U_{ij}} \xrightarrow{\sim} \mathfrak{S}_i|_{U_{ij}}$ are equivalences of stacks, and $\varphi_{ijk}\colon \varphi_{ij}\circ\varphi_{jk} \to \varphi_{ik}$ are invertible transformations of functors from $\mathfrak{S}_k|_{U_{ijk}}$ to $\mathfrak{S}_i|_{U_{ijk}}$, such that for any $i,j,k,l \in I$, the following diagram of transformations of functors from $\mathfrak{S}_l|_{U_{ijkl}}$ to $\mathfrak{S}_i|_{U_{ijkl}}$ commutes

$$\begin{tikzcd}
F_{ij}\circ F_{jk}\circ F_{kl} \arrow[r, "\varphi_{ijk}"] \arrow[d, "\varphi_{jkl}"] & F_{ik}\circ F_{kl} \arrow[d, "\varphi_{ikl}"] \\
F_{ij}\circ F_{jl} \arrow[r, "\varphi_{ijl}"] & F_{il}.
\end{tikzcd} \tag{B.0.7}$$

By taking a refinement, we can assume that the covering $\{U_i\}_{i\in I}$ is stable by finite intersections.
Let $V \subset U$. Then the cocycle condition B.0.7 allows us to define

$$\mathfrak{S}(V) = 2\varprojlim_{i\in I} \mathfrak{S}_i(V\cap U_i).$$

It is then obvious that $\mathfrak{S}$ is a stack (for instance using Lemma B.0.20 and the fact that this is true if $\mathbf{C} = \mathbf{Cat}$) and that, by construction, there are equivalences of stacks $F_i \colon \mathfrak{S}|_{U_i} \xrightarrow{\sim} \mathfrak{S}_i$, Moreover, one checks that there exist invertible transformations of functors $\varphi_{ij} \colon F_{ij} \circ F_j|_{U_{ij}} \xrightarrow{\sim} F_i|_{U_{ij}}$ such that $\varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}} \circ \varphi_{ijk}$. ☐

Proposition B.0.22. *The 2-stack $\mathfrak{St}_X(\mathbf{C})$ admits all small 2-limits.*

Proof. Let $\beta \colon \mathbf{I} \to \mathbf{St}_X(\mathbf{C})$ be a 2-functor, with $\mathbf{I}$ a small 2-category. Then, for each open subset $U \subset X$, set

$$\mathfrak{S}(U) = 2\varprojlim_{i \in \mathbf{I}} \beta(i)(U).$$

It is immediately verified that $\mathfrak{S}$ is a stack on X that satisfies

$$\mathfrak{S} \simeq 2\varprojlim_{i \in \mathbf{I}} \beta(i).$$

☐

Definition B.0.23. Let $\mathfrak{S}$ be a prestack. A stack $\widetilde{\mathfrak{S}}$ together with a functor $\mathfrak{S} \to \widetilde{\mathfrak{S}}$ is called the stack associated to $\mathfrak{S}$ if it satisfies the usual universal property, *i.e.* any functor from $\mathfrak{S}$ into a stack $\mathfrak{T}$ factors through $\widetilde{\mathfrak{S}}$ up to unique equivalence:

$$\begin{tikzcd} \mathfrak{S} \arrow[r] \arrow[d] & \mathfrak{T} \\ \widetilde{\mathfrak{S}} \arrow[ur] & \end{tikzcd}$$

Proposition B.0.24. *Let* $\mathrm{P} \in \mathrm{Ob}\,\mathbf{C}$. *Assume that X is locally 1-connected. Then the stack associated to the constant prestack with stalk* P *exists.*

Proof. Let $U \subset X$ be an open subset. Set

$$\mathrm{P}_X(U) = \mathrm{P}^{\Pi_1(U)},$$

where $\mathrm{P}^{\Pi_1(U)}$ denotes the 2-limit of the constant 2-functor $\Delta(P) \colon \Pi_1(U) \to \mathbf{C}$ at P. Let $x \in V \subset U$ and denote by x_U the corresponding object in $\Pi_1(U)$. Then we have the natural 1-arrows

$$\mathrm{P}_X(U) \longrightarrow \mathrm{P}_{x_U} \longrightarrow \mathrm{P}_{x_V}\,.$$

These 1-arrows define the 1-arrow

$$\mathrm{P}_X(U) \longrightarrow \mathrm{P}_X(V).$$

Since we know that this construction gives a stack if $\mathbf{C} = \mathbf{Cat}$ (in fact, if $\mathrm{P} = \mathsf{C}$ is a category, then $\mathsf{C}^{\Pi_1(U)}$ is equivalent to $\mathsf{Hom}\,(\Pi_1(U), \mathsf{C}) \underset{\mu}{\xleftarrow{\sim}} \mathsf{C}_U(U)$, hence it is the stack of locally constant sheaves on U with values in C), Lemma B.0.20 implies that P_X is a stack which verifies the desired universal property. ☐

Definition B.0.25. Let $\mathrm{P} \in \mathrm{Ob}\,\mathbf{C}$. The stack associated to the constant prestack with stalk P is called the constant stack with stalk P, and we denote it by P_X.

Let X be a locally 1-connected topological space. Denote by $\mathbf{CSt}_X(\mathbf{C})$ the full sub-2-category of $\mathbf{St}_X(\mathbf{C})$ of constant stacks. The previous construction defines a faithful 2-functor

$$(\cdot)_X \colon \mathbf{C} \longrightarrow \mathbf{CSt}_X(\mathbf{C}),$$

which is an equivalence if X is connected (a quasi-2-inverse is given by the global sections 2-functor).

Definition B.0.26. A stack $\mathfrak{S}$ is called locally constant if there exists an open covering $X = \bigcup U_i$ such that $\mathfrak{S}|_{U_i}$ is isomorphic to a constant stack.
We denote by $\boldsymbol{\mathfrak{LcSh}}_X(\mathbf{C})$ the full sub-2-stack of $\boldsymbol{\mathfrak{St}}_X(\mathbf{C})$ whose objects are the locally constant stacks.

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