Polynomial endomorphisms of the Cuntz algebras arising from permutations. II
—Branching laws of endomorphisms—

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An endomorphism ρ of the Cuntz algebra $\mathcal{O}_N$ brings a branching of representations:

$$\pi \mapsto \pi \circ \rho = \pi_1 \oplus \cdots \oplus \pi_M.$$

We show branchings of permutative representations of $\mathcal{O}_N$ by endomorphisms which are introduced in the previous our paper. Their branching laws are computed concretely and illustrated by graphs effectively.

1. Introduction

For a subgroup H of a group G, an irreducible decomposition of $\pi|_H$ for an irreducible representation π of G is one of main study in representation theory([**12**]). When such decomposition holds, the decomposition formula is called a *branching law*. This is reformulated as the branching which is brought by the inclusion map ι from H to G, that is, $\iota^*(\pi) \equiv \pi \circ \iota$ gives a map from Rep$G$ to Rep$H$. In general, any homomorphism φ from a group G_1 to other G_2 arises a transformation φ^* from RepG_2 to RepG_1. Specially, when $G = G_1 = G_2$, $\rho \in$ End$G$ gives a transformation on Rep$G$ and we can consider the branching law by ρ_* in this situation.

On the other hand, the branching law by embeddings of C*-algebras is studied in [**1**, **2**]. The difference of two theories of quantum string field is explained by the difference of representations arising from two embeddings of a pseudo Cuntz algebra in [**3**]. In the last paper [**11**], we introduce a class of endomorphisms of the Cuntz algebra $\mathcal{O}_N$ arising from permutations and show the complete reducibility of the action of them on permutative representations. For a representation π and an endomorphism ρ, if there is a decomposition $\pi \circ \rho = \pi_1 \oplus \cdots \oplus \pi_M$, we call this formula by the *branching law of ρ at π*. Because this branching is brought by ρ, the branching law of ρ shows a property of ρ. In application of branching laws for geometry, such method to study morphisms is already considered(§ 4.A. in [**12**]). In

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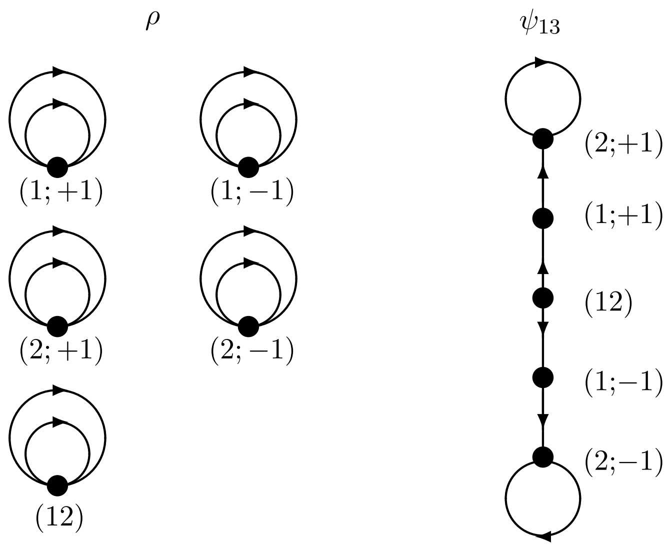
[**11**], we show the classification of endomorphisms in a class by using their branching laws. In this way, the branching law of ρ is an effective method for a study of ρ itself.

In [**12**], it is mentioned that branching laws bring us various problems and information. However, such branching laws seem complicated and difficult to understand visually in general. We develop a method to illustrate branching laws by graph. For example, consider the following two endomorphisms ρ and ψ_{13} of $\mathcal{O}_2$:

$$\begin{cases} \rho(s_1) \equiv & s_1 s_1 s_1^* + s_2 s_1 s_2^*, \\ \rho(s_2) \equiv & s_1 s_2 s_1^* + s_2 s_2 s_2^*, \end{cases} \qquad \begin{cases} \psi_{13}(s_1) \equiv & s_2 s_1 s_1^* + s_1 s_2 s_2^*, \\ \psi_{13}(s_2) \equiv & s_1 s_1 s_1^* + s_2 s_2 s_2^*. \end{cases}$$

We see that ρ is the canonical endomorphism of $\mathcal{O}_2$ and it is neither bijective nor irreducible(this notion is given in § 3). In stead of their quite concrete forms, it seems that there is no information about: 1)Whether is ψ_{13} automorphisms or not? 2)Whether is ψ_{13} irreducible or not? 3)Whether are ρ and ψ_{13} equivalent or not? We can answer these questions at once by the following two directed graphs with label which are associated with ρ and ψ_{13}:



The difference between ρ and ψ_{13} are clear by appearance of graphs(more precise explanation is given in § 3.2 and § 4.1).

In § 2, we review of branching function systems and permutative representations of $\mathcal{O}_N$. In § 3, we review permutative endomorphisms, and introduce graphs associated with branchings of endomorphisms. In § 4, we show examples of branching laws and their graphs. In § 5, we show smarter statements about branching laws by using module of endomorphism semigroups.

2. Branching function systems and permutative representations

2.1. Invariants of representations. We introduce several sets of multi indices which consist of numbers $1,\ldots,N$ for $N\geq 2$ in order to describe invariants of representations of $\mathcal{O}_N$. Their meanings are shown in § 2.4.

Put

$$\{1,\ldots,N\}^{\#}\equiv\{1,\ldots,N\}^{*}_{1}\sqcup\{1,\ldots,N\}^{\infty},$$

$$\{1,\ldots,N\}^{*}\equiv\coprod_{k\geq 0}\{1,\ldots,N\}^{k},\quad \{1,\ldots,N\}^{*}_{1}\equiv\coprod_{k\geq 1}\{1,\ldots,N\}^{k},$$

$$\{1,\ldots,N\}^{\infty}\equiv\{(j_n)_{n\in\mathbf{N}}:j_n\in\{1,\ldots,N\},\,n\in\mathbf{N}\},$$

$\{1,\ldots,N\}^{0}\equiv\{0\}$, $\{1,\ldots,N\}^{k}\equiv\{(j_l)_{l=1}^{k}:j_l=1,\ldots,N,\,l=1,\ldots,k\}$ for $k\geq 1$. For $J\in\{1,\ldots,N\}^{\#}$, the *length* $|J|$ of J is defined by $|J|\equiv k$ when $J\in\{1,\ldots,N\}^{k}$, $k\geq 0$. For $J_1,J_2\in\{1,\ldots,N\}^{*}$ and $J_3\in\{1,\ldots,N\}^{\infty}$ $J_1\cup J_2\equiv(j_1,\ldots,j_k,j_1^{'},\ldots,j_l^{'})$, $J_1\cup J_3\equiv(j_1,\ldots,j_k,j_1^{''},j_2^{''},\ldots)$ when $J_1=(j_1,\ldots,j_k)$, $J_2=(j_1^{'},\ldots,j_l^{'})$ and $J_3=(j_n^{''})_{n\in\mathbf{N}}$. Specially, we define $J\cup\{0\}=\{0\}\cup J=J$ for $J\in\{1,\ldots,N\}^{\#}$ and $(i,J)\equiv(i)\cup J$ for convention. For $J\in\{1,\ldots,N\}^{*}$ and $k\geq 2$, $J^k\equiv\underbrace{J\cup\cdots\cup J}_{k}$. For $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}^{k}$ and $\tau\in\mathbf{Z}_k$, denote $\tau(J)=(j_{\tau(1)},\ldots,j_{\tau(k)})$.

Definition 2.1. (i) $J\in\{1,\ldots,N\}^{*}_{1}$ *is periodic if there are* $m\geq 2$ *and* $J_0\in\{1,\ldots,N\}^{*}_{1}$ *such that* $J=J_0^m$.

(ii) *For* $J_1,J_2\in\{1,\ldots,N\}^{*}_{1}$, $J_1\sim J_2$ *if there are* $k\geq 1$ *and* $\tau\in\mathbf{Z}_k$ *such that* $J_1,J_2\in\{1,\ldots,N\}^{k}$ *and* $\tau(J_1)=J_2$.

(iii) *For* $(J,z),(J^{'},z^{'})\in\{1,\ldots,N\}^{*}_{1}\times U(1)$, $(J,z)\sim(J^{'},z^{'})$ *if* $J\sim J^{'}$ *and* $z=z^{'}$ *where* $U(1)\equiv\{z\in\mathbf{C}:|z|=1\}$.

(iv) *For* $J_1=(j_1,\ldots,j_k),J_2=(j_1^{'},\ldots,j_k^{'})\in\{1,\ldots,N\}^{k}$, $k\geq 1$, $J_1\prec J_2$ *if* $\sum_{l=1}^{k}(j_l^{'}-j_l)N^{k-l}\geq 0$.

(v) $J\in\{1,\ldots,N\}^{*}_{1}$ *is minimal if* $J\prec J^{'}$ *for each* $J^{'}\in\{1,\ldots,N\}^{*}_{1}$ *such that* $J\sim J^{'}$.

Specially, any element in $\{1,\ldots,N\}$ is non periodic and minimal. Put

$$[1,\ldots,N]^{*}\equiv\{J\in\{1,\ldots,N\}^{*}_{1}:J\text{ is minimal and non periodic}\}.\tag{2.1}$$

Note that $[1,\ldots,N]^{*}$ is in one-to-one correspondence with the set of all equivalence classes of non periodic elements in $\{1,\ldots,N\}^{*}_{1}$. For example,

$$[1,2]^{*}=\{(1),(2),(12),(112),(122),(1112),(1122),(1222),(11112),\ldots\},$$

$$[1,2,3]^{*}=\left\{\begin{array}{l}(1),(2),(3),(12),(13),(23),\\(112),(113),(122),(123),(132),(133),(223),(233),\\(1112),(1113),(1122),(1123),(1132),(1133),(1213),\\(1222),(1223),(1232),(1233),(1322),(1323),(1333),\\(2223),(2233),(2333),\ldots\end{array}\right\}.$$

Definition 2.2. (i) $J \in \{1,\ldots,N\}^\infty$ *is eventually periodic if there are* $J_0, J_1 \in \{1,\ldots,N\}_1^*$ *such that* $J = J_0 \cup J_1^\infty$.

(ii) *For* $J_1, J_2 \in \{1,\ldots,N\}^\infty$, $J_1 \sim J_2$ *if there are* $J_3, J_4 \in \{1,\ldots,N\}^*$ *and* $J_5 \in \{1,\ldots,N\}^\infty$ *such that* $J_1 = J_3 \cup J_5$ *and* $J_2 = J_4 \cup J_5$.

Put the set $[1,\ldots,N]^\infty$ of equivalence classes of multi indices by

$$[1,\ldots,N]^\infty \equiv \{J \in \{1,\ldots,N\}^\infty : J \text{ is non eventually periodic}\}/\sim . \tag{2.2}$$

The definition of $[1,\ldots,N]^\infty$ seems inconsistent with $[1,\ldots,N]^*$, but this definition makes sense as the set of invariants of representations of the Cuntz algebras in later. Furthermore we put

$$[1,\ldots,N]^\# \equiv [1,\ldots,N]^* \sqcup [1,\ldots,N]^\infty. \tag{2.3}$$

While the definition of several notions of multi indices are changed by comparison with [**8**, **10**, **11**], their meanings are equivalent.

2.2. Branching function systems. Let Λ be an infinite set and $N \geq 2$. $f = \{f_i\}_{i=1}^N$ is a *branching function system on* Λ if f_i is an injective transformation on Λ for $i = 1,\ldots,N$ such that a family of their images coincides a partition of Λ. For $N \geq 2$, $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_1)$ and $g = \{g_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_2)$ are *equivalent* if there is a bijection φ from Λ_1 to Λ_2 such that $\varphi \circ f_i \circ \varphi^{-1} = g_i$ for $i = 1,\ldots,N$. Put $\mathrm{BFS}_N(\Lambda)$ the set of all branching function systems on Λ. For $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$, we denote $f_J \equiv f_{j_1} \circ \cdots \circ f_{j_k}$ when $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$, $k \geq 1$, and define $f_0 \equiv id$. For $x, y \in \Lambda$, $x \sim y$(with respect to f) if there are $J_1, J_2 \in \{1,\ldots,N\}^*$ and $z \in \Lambda$ such that $f_{J_1}(z) = x$ and $f_{J_2}(z) = y$. For $x \in \Lambda$, denote $A_f(x) \equiv \{y \in \Lambda : x \sim y\}$.

Definition 2.3. *Let* $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$.

(i) f *is cyclic if there is an element* $x \in \Lambda$ *such that* $\Lambda = A_f(x)$.

(ii) *For* $k \geq 1$, $R = \{n_1,\ldots,n_k\} \subset \Lambda$ *is a* k*-cycle of* f *if* $n_l \neq n_{l'}$ *when* $l \neq l'$ *and there is* $J \in \{1,\ldots,N\}^k$ *such that* $f_{j_l}(n_l) = n_{\tau(l)}$ *for* $l = 1,\ldots,k$ *where* τ *is a shift on* $\mathbf{Z}_k$.

(iii) $R = \{n_l\}_{l\in\mathbf{N}} \subset \Lambda$ *is a chain of* f *if* $n_l \neq n_{l'}$ *when* $l \neq l'$ *and there is* $\{j_l \in \{1,\ldots,N\} : l \in \mathbf{N}\}$ *such that* $f_{j_l}^{-1}(n_l) = n_{l+1}$ *for each* $l \in \mathbf{N} \equiv \{1,2,3,\ldots\}$.

(iv) f *has a* k*-cycle(chain) if there is a* k*-cycle(resp. chain) of* f *in* Λ. *Specially, we call simply that* f *has a cycle if* f *has a* k*-cycle some* $k \geq 1$.

Let Ξ be a set. For a branching function system $f^{[\omega]} = \{f_i^{[\omega]}\}_{i=1}^N$ on an infinite set Λ_ω for $\omega \in \Xi$, f is the *direct sum* of $\{f^{[\omega]}\}_{\omega\in\Xi}$ if $f = \{f_i\}_{i=1}^N$ is a branching function system on a set $\Lambda \equiv \coprod_{\omega\in\Xi} \Lambda_\omega$ which is defined by $f_i(n) \equiv f_i^{[\omega]}(n)$ when $n \in \Lambda_\omega$ for $i = 1,\ldots,N$ and $\omega \in \Xi$. For a branching function system $f \in \mathrm{BFS}_N(\Lambda)$, $f = \bigoplus_{\omega\in\Xi} f^{[\omega]}$ is a *decomposition* of f into a family $\{f^{[\omega]}\}_{\omega\in\Xi}$ if there is a family $\{\Lambda_\omega\}_{\omega\in\Xi}$ of subsets of Λ such that f is the direct sum of $\{f^{[\omega]}\}_{\omega\in\Xi}$.

Proposition 2.4. *Let $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$.*

(i) *There is a decomposition $\Lambda = \coprod_{\omega\in\Xi} \Lambda_\omega$ such that $\#\Lambda_\omega = \infty$, $f|_{\Lambda_\omega} \equiv \{f_i|_{\Lambda_\omega}\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda_\omega)$ and $f|_{\Lambda_\omega}$ is cyclic for each $\omega \in \Xi$.*

(ii) *Assume that f is cyclic. Then there is only one case in the followings: a) f has just one cycle. b) f has just one chain where we identify two chains $R = \{n_l \in \Lambda : l \in \mathbf{N}\}$ and $R' = \{m_l \in \Lambda : l \in \mathbf{N}\}$ when there are $M, L \geq 0$ such that $n_{l+L} = m_l$ for each $l > M$.*

Proof. See Proposition 2.5 in [**11**]. □

Definition 2.5. (i) *For $J \in \{1,\ldots,N\}^k$, $k \geq 1$, $f \in \mathrm{BFS}_N(\Lambda)$ is $P(J)$ if f is cyclic and has a cycle $R = \{n_1,\ldots,n_k\}$ such that $f_J(n_k) = n_k$.*

(ii) *For $J \in \{1,\ldots,N\}^\infty$, $f \in \mathrm{BFS}_N(\Lambda)$ is $P(J)$ if f is cyclic and has a chain $L = \{n_j\}_{j\in\mathbf{N}}$ such that $f_{J_k}^{-1}(n_1) = n_k$ where $J_k \equiv (j_1,\ldots,j_k)$ for each $k \geq 1$ when $J = (j_l)_{l\in\mathbf{N}}$.*

By Proposition 2.4, Definition 2.5 (i) and (ii) make sense.

2.3. Transformation of branching function systems. Let $\mathfrak{S}_{N,l}$ be the set of all bijective transformations on $\{1,\ldots,N\}^l$ for $l \geq 1$. Put a bijective map κ from $\{1,\ldots,N\}^l$ to a set $\Sigma_{N^l} \equiv \{1,2,3,\ldots,N^l-1,N^l\}$ by $\kappa(i_1,\ldots,i_l) \equiv \sum_{j=1}^l N^{l-j}(i_j-1)+1$. We often identify $\mathfrak{S}_{N,l}$ and the (symmetric)group $\mathfrak{S}_{N^l}$ of all permutations on Σ_{N^l} by corresponding between $\sigma \in \mathfrak{S}_{N,l}$ and $\kappa\circ\sigma\circ\kappa^{-1} \in \mathfrak{S}_{N^l}$. Specially, $\kappa = id$ on $\{1,\ldots,N\} = \Sigma_N$. By a natural identification $\mathfrak{S}_{N,l}$ and a subset $\mathfrak{S}_{N,l}\times\{id\}$ of $\mathfrak{S}_{N,l+1}$, $l \geq 1$, we can consider $\mathfrak{S}_{N,*} \equiv \lim_{\to l} \mathfrak{S}_{N,l}$.

For $\sigma \in \mathfrak{S}_{N,l}$ and $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$, put $f^{(\sigma)} = \{f_i^{(\sigma)}\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$ by

$$f_i^{(\sigma)} \equiv f_{\sigma(i)} \quad (l=1), \quad f_i^{(\sigma)}(f_J(n)) \equiv f_{\sigma(i,J)}(n) \quad (l \geq 2) \tag{2.4}$$

for $n \in \Lambda$, $i = 1,\ldots,N$ and $J \in \{1,\ldots,N\}^{l-1}$.

Lemma 2.6. *Let $J \in \{1,\ldots,N\}^\#$ and $\sigma \in \mathfrak{S}_N = \mathfrak{S}_{N,1}$. If $f \in \mathrm{BFS}_N(\Lambda)$ is $P(J)$ in Definition 2.5, then $f^{(\sigma)}$ is $P(J_{\sigma^{-1}})$ where*

$$J_{\sigma^{-1}} \equiv \begin{cases} \left(\sigma^{-1}(j_1),\ldots,\sigma^{-1}(j_k)\right) & (J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k,\ k \geq 1), \\ \left(\sigma^{-1}(j_n)\right)_{n\in\mathbf{N}} & (J = (j_n)_{n\in\mathbf{N}} \in \{1,\ldots,N\}^\infty). \end{cases}$$

Next we show concrete examples of branching function systems on $\mathbf{N}$ and its transformation by permutations.

Lemma 2.7. *For $J \in \{1,\ldots,N\}^*_1$, define a branching function system $f = \{f_i\}_{i=1}^N$ on* $\mathbf{N}$ *defined as follows: When $J = j \in \{1,\ldots,N\}$, put*

$$f_i(1) \equiv \begin{cases} i+1 & (1 \le i < j), \\ 1 & (i = j), \\ i & (j \le i \le N), \end{cases} \qquad f_i(n) \equiv N(n-1)+i \quad (n \ge 2)$$

for $i = 1,\ldots,N$. When $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$, $k \ge 2$, put

$$f_i(1) \equiv \begin{cases} k+i & (1 \le i < j_1), \\ k & (i = j_1), \\ k+i-1 & (j_1 \le i \le N), \end{cases}$$

$$f_i(l) \equiv \begin{cases} k+(N-1)(l-1)+i & (1 \le i < j_l), \\ l-1 & (i = j_l), \\ k+(N-1)(l-1)+i-1 & (j_l \le i \le N), \end{cases}$$

$$f_i(n) \equiv \ N(n-1)+i$$

for $l = 2,\ldots,k$, $n \ge k+1$ and $i = 1,\ldots,N$. Then the followings hold:

(i) *f is $P(J)$.*

(ii) *For $\sigma \in \mathfrak{S}_{N,l}$, $l \ge 1$, $f^{(\sigma)}$ has no chain.*

(iii) *For $\sigma \in \mathfrak{S}_{N,l}$, $l \ge 1$, there is $1 \le M \le N^{l-1}$ such that $f^{(\sigma)}$ is decomposed into a direct sum of M number of cycles. Furthermore the length of each cycle is a multiple of that of J.*

Proof. See Lemma 2.12 in [**11**] except (iii). We show (iii). Assume $J \in \{1,\ldots,N\}^k$, $k \ge 1$. By (ii) and Proposition 2.4, $f^{(\sigma)}$ has only cycles and does one cycle at least. If $f_i^{(\sigma)}(f_J(m)) = f_{J'}(m)$ for $i = 1,\ldots,N$, $m \in \mathbf{N}$ and $J, J' \in \{1,\ldots,N\}^*_1$, then $|J'| = |J|+1$ by definition of $f^{(\sigma)}$. If $f_{J_0}^{(\sigma)}(n) = n$ for some $n \in \mathbf{N}$ and $J_0 \in \{1,\ldots,N\}^*_1$, then

$$f_{J_2}(m) = m \tag{2.5}$$

for suitable J_2, $|J_2| = |J_0|$ and $m \in \mathbf{N}$. (2.5) means a cycle of f and it holds only when $|J_2| = ka$ for some $1 \le a < \infty$ by Proposition 2.4 (ii). Hence $|J_0| = ka$. Therefore we have the assertion about the length of a component.

Since $f^{(\sigma)}$ has cycles in only $D \equiv \{1,\ldots,N^{l-1}k\} \subset \mathbf{N}$ and $\#D = N^{l-1}k$, the number of cycles in D is N^{l-1} at most by the results of lengths of cycles of $f^{(\sigma)}$. $\square$

Next we consider transformations of branching function systems with chain.

Lemma 2.8. *For $J = (j_n)_{n\in\mathbf{N}} \in \{1,\ldots,N\}^\infty$, put a family $\{p_n\}_{n\in\mathbf{Z}}$ of transpositions $p_n \in \mathfrak{S}_N$ by $p_n(1) \equiv j_n$ for $n \geq 1$ and $p_n \equiv id$ for $n \leq 0$. Define $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\mathbf{Z}\times\mathbf{N})$ by*

$$(2.6) \qquad f_i(n,m) \equiv \begin{cases} (n-1,\, N(m-1)+i) & (m \geq 2), \\ (n-1,\, p_n(i)) & (m = 1) \end{cases}$$

for $n \in \mathbf{Z}$. Then the followings hold:

(i) *f is $P(J)$.*

(ii) *For each $(n,m) \in \mathbf{Z}\times\mathbf{N}$, there is $m' \in \mathbf{N}$ such that $f_i^{(\sigma)}(n,m) = (n-1,m')$ for $i = 1,\ldots,N$.*

(iii) *For $\sigma \in \mathfrak{S}_{N,l}$, $f^{(\sigma)}$ has no cycle in $\mathbf{Z}\times\mathbf{N}$.*

Proof. (i) The cyclicity follows by definition of representations. f has a chain $\{(n,1) : n \in \mathbf{N}\}$ in $\mathbf{Z}\times\mathbf{N}$ which satisfy the chain condition with respect to J.
(ii) For $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 2$ and $f \in \mathrm{BFS}_N(\mathbf{Z}\times\mathbf{N})$, we have $f_i^{(\sigma)}(f_J(n,m)) = f_{\sigma(i,J)}(n,m)$ for $J \in \{1,\ldots,N\}^{l-1}$, $i = 1,\ldots,N$ and $(n,m) \in \mathbf{Z}\times\mathbf{N}$. Note $J \in \{1,\ldots,N\}^{l-1}$. We see $f_i^{(\sigma)}(n,m) = f_i^{(\sigma)}(f_J(n+l-1,m_0)) = (n-1,m')$ for suitable m_0. Therefore the statement holds.
(iii) If $f^{(\sigma)}$ has a cycle $\{x_1,\ldots,x_M\} \subset \mathbf{Z}\times\mathbf{N}$, then there are $1 \leq M < \infty$ and $J \in \{1,\ldots,N\}^M$ such that $f_J^{(\sigma)}(x_1) = x_1$. If $x_1 = (n,m)$, then $f_J^{(\sigma)}(x_1) = (n-M,m') \neq x_1$ for some $m' \in \mathbf{N}$ by (ii). Hence this is a contradiction. Therefore the statement holds. $\square$

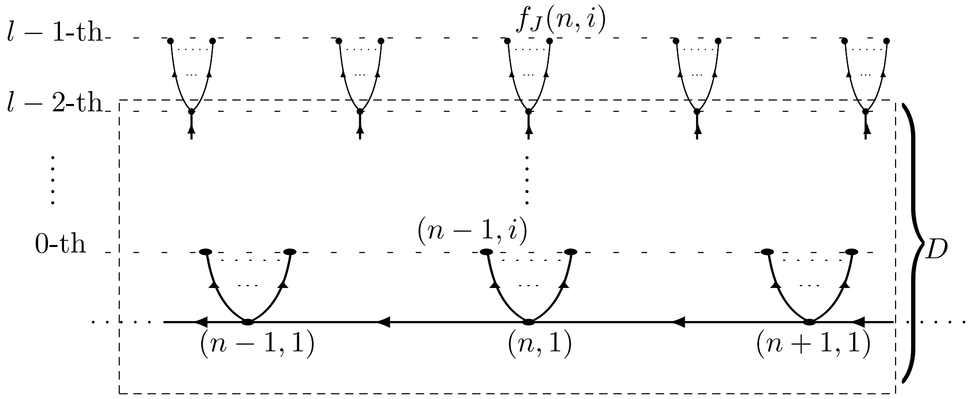
Lemma 2.9. *Let f be in (2.6) and $\sigma \in \mathfrak{S}_{N,l}$. Put*

$$C \equiv \{f_J(n,j) : j = 2,\ldots,N,\ J \in \{1,\ldots,N\}^k,\ k \geq l-1\}.$$

Then $f_i^{(\sigma)}(C) \subset C$ and $f_i^{(\sigma)}$ has neither chain nor cycle in C for $i = 1,\ldots,N$.

Proof. Any cycle does not exists by Lemma 2.8. If there is a chain $L = \{x_n\}_{n\in\mathbf{N}}$ in C, then there is $J = (j_n) \in \{1,\ldots,N\}^\infty$ such that $f_{J_k}^{-1}(x_1) = x_n$. By definition of f_i in (2.6), the component of f with respect to $\mathbf{N}$ is monotone increasing. Therefore there is a lower bound x for $L = \{x_n\}_{n\in\mathbf{N}}$ with respect to $\mathbf{N}$-component. However $f_i^{-1}(x) \notin C$. This contradicts the assumption $L \subset C$. Hence there is no chain in C. $\square$

$D \equiv (\mathbf{Z} \times \mathbf{N}) \setminus C$ is the following:



Lemma 2.10. *Let f be in* (2.6) *and $\sigma \in \mathfrak{S}_{N,l}$. Then there is $1 \leq M \leq N^{l-1}$ such that $f^{(\sigma)}$ decomposes into just M chains.*

Proof. By Lemma 2.8 (ii), Lemma 2.9 and Proposition 2.4, f^{σ} has chains in $D \equiv \mathbf{Z} \times \mathbf{N}$. Let $L = \{x_n\}_{n\in\mathbf{N}} \subset D$ be a chain. Denote $x_n = (y_n, z_n)$ for $n \in \mathbf{N}$. $y_{n+1} = y_n + 1$ for $n \in \mathbf{N}$ by Lemma 2.8 (ii). Hence the cut of L by the set $E_m \equiv \{(m, m') \in D : m' \in \mathbf{N}\}$ is just one point. On the other hand, $\#E_m = N^{l-1}$ for each $m \in \mathbf{Z}$. Hence the number of chains in D is N^{l-1} at most. □

2.4. Permutative representations. For $N \geq 2$, let $\mathcal{O}_N$ be the Cuntz algebra([**5**]), that is, it is a C*-algebra with generators $s_1, \ldots, s_N$ which satisfy

$$s_i^* s_j = \delta_{ij} I \quad (i,j = 1,\ldots,N), \quad s_1 s_1^* + \cdots + s_N s_N^* = I. \tag{2.7}$$

In this paper, any representation and endomorphism are assumed unital and $*$-preserving. By simplicity and uniqueness of $\mathcal{O}_N$, it is sufficient to define operators $S_1, \ldots, S_N$ on an infinite dimensional Hilbert space which satisfy (2.7) in order to construct a representation of $\mathcal{O}_N$. In the same reason, it is sufficient to define elements $T_1, \ldots, T_N$ in $\mathcal{O}_N$ which satisfy (2.7) in order to construct an endomorphism of $\mathcal{O}_N$.

Put α an action of a unitary group $U(N)$ on $\mathcal{O}_N$ defined by $\alpha_g(s_i) \equiv \sum_{j=1}^{N} g_{ji} s_j$ for $i = 1, \ldots, N$. Specially we denote $\gamma_w \equiv \alpha_{g(w)}$ when $g(w) = w \cdot I \subset U(N)$ for $w \in U(1) \equiv \{z \in \mathbf{C} : |z| = 1\}$. For multiindices $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k$, we denote $s_J = s_{j_1} \cdots s_{j_k}$ and $s_J^* = s_{j_k}^* \cdots s_{j_1}^*$.

Definition 2.11. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$.*

(i) $(\mathcal{H}, \pi)$ *is a permutative representation of $\mathcal{O}_N$ if there are a complete orthonormal basis $\{e_n\}_{n\in\Lambda}$ of $\mathcal{H}$ and a branching function system $f =$*

$\{f_i\}_{i=1}^N$ *on* Λ *such that* $\pi(s_i)e_n = e_{f_i(n)}$ *for each* $n \in \Lambda$ *and* $i = 1,\ldots,N$.

(ii) $(\mathcal{H},\pi,\Omega)$ *is a generalized permutative(=GP) representation of* $\mathcal{O}_N$ *with cycle by* $J \in \{1,\ldots,N\}^k$, $k \geq 1$ *and phase* $z \in U(1)$ *if* $\Omega \in \mathcal{H}$ *is a cyclic unit vector such that* $\pi(s_J)\Omega = z\Omega$ *and* $\{\pi(s_{j_1}\cdots s_{j_l})\Omega : l = 1,\ldots,k\}$ *is an orthonormal family in* $\mathcal{H}$. *We denote* $P(J;z) = (\mathcal{H},\pi,\Omega)$ *and* $P(J) \equiv P(J;1)$ *simply.*

(iii) $(\mathcal{H},\pi,\Omega)$ *is a GP representation of* $\mathcal{O}_N$ *with chain by* $J \in \{1,\ldots,N\}^\infty$ *if* $\Omega \in \mathcal{H}$ *is a cyclic unit vector such that* $\{\pi(s_{J_n})^*\Omega\}_{n\in\mathbf{N}}$ *is an orthonormal family where* $J_n \equiv (j_1,\ldots,j_n)$ *when* $J = (j_m)_{m\in\mathbf{N}}$. *We denote* $P(J) = (\mathcal{H},\pi,\Omega)$ *simply.*

(iv) $(l_2(\Lambda),\pi_f)$ *is the permutative representation of* $\mathcal{O}_N$ *by* $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\Lambda)$ *if* $\pi_f(s_i)e_n \equiv e_{f_i(n)}$ *for* $n \in \Lambda$ *and* $i = 1,\ldots,N$.

Branching function system was introduced in [**4**, **6**, **7**]. Recall Definition 2.1 about multiindices. Here $\sim$ means the unitary equivalence of representations.

Theorem 2.12. (i) *Any permutative representation is completely reducible.*

(ii) *Any cyclic(resp. irreducible)permutative representation is equivalent to* $P(J)$ *for some* $J \in \{1,\ldots,N\}^\#$ *(resp. some* $J \in [1,\ldots,N]^*$ *or some non eventually periodic* $J \in \{1,\ldots,N\}^\infty$*).*

(iii) *For each* $J \in \{1,\ldots,N\}^\#$, $P(J)$ *exists and unique up to unitary equivalences.* $P(J)$ *is equivalent to a cyclic permutative representation.*

(iv) *If* $J \in \{1,\ldots,N\}^k$, $k \geq 1$ *and* $z \in U(1)$, *then* $P(J;1)\circ\gamma_z = P(J;z^k)$. *If* $J \in \{1,\ldots,N\}^\infty$ *and* $z \in U(1)$, *then* $P(J)\circ\gamma_z = P(J)$.

(v) *For* $J \in \{1,\ldots,N\}_1^*$ *and* $z \in U(1)$, $P(J;z)$ *is irreducible if and only if* J *is non periodic.*

(vi) *For* $J \in \{1,\ldots,N\}^\infty$, $P(J)$ *is irreducible if and only if* J *is non eventually periodic.*

(vii) *For* $J_1,J_2 \in \{1,\ldots,N\}_1^*$ *and* $z_1,z_2 \in U(1)$, $P(J_1;z_1) \sim P(J_2;z_2)$ *if and only if* $(J_1,z_1) \sim (J_2;z_2)$ *where* $P(J_1;z_1) \sim P(J_2;z_2)$ *means the unitary equivalence of two representations which satisfy the condition* $P(J_1;z_1)$ *and* $P(J_2;z_2)$, *respectively.*

(viii) *For* $J_1,J_2 \in \{1,\ldots,N\}^\infty$, $P(J_1) \sim P(J_2)$ *if and only if* $J_1 \sim J_2$.

(ix) *For* $J \in \{1,\ldots,N\}_1^*$ *and* $l \geq 1$,

$$P(J^l;1) = \bigoplus_{j=1}^{l} P(J;\xi^{j-1})$$

where $\xi \equiv e^{2\pi\sqrt{-1}/l}$. *This decomposition is unique up to unitary equivalences.*

Proof. Note $P(J;z) = GP(\bar{z}\varepsilon_J)$ in [**8**] where $\varepsilon_J = \varepsilon_{j_1}\otimes\cdots\otimes\varepsilon_{j_k}$ when $J = (j_1,\ldots,j_k)$, and $P(J) = GP(\varepsilon_J)$ where $\varepsilon_J = (\varepsilon_{j_n})_{n\in\mathbf{N}}$ when

$J = (j_n)_{n\in\mathbf{N}} \in \{1,\ldots,N\}^\infty$ in [**10**], and $\{\varepsilon_j\}_{j=1}^N$ is the canonical basis of $\mathbf{C}^N$. Hence the statement holds from [**8**, **9**, **10**]. $\square$

We omit the decomposition of chain in this article(see [**10**]). By Theorem 2.12 (ii), it is sufficient for a statement about $P(J_1)$ to show by a suitable concrete representation which is $P(J)$ for $J \in \{1,\ldots,N\}^\#$. By combining Theorem 2.12 (iii) and (viii), we have the following:

$$P(J^l; z) = \bigoplus_{j=1}^{l} P(J; \xi^{j-1} z^{1/l})$$

where $\xi \equiv e^{2\pi\sqrt{-1}/l}$. In consequence, we have the following:

Theorem 2.13. (i) *A set $\{P(J;z); J \in \{1,\ldots,N\}_1^*,\ z \in U(1)\}$ of representations of $\mathcal{O}_N$ is closed under irreducible decomposition, and the number of components of decomposition is always finite.*

(ii) *$[1,\ldots,N]^\#$ is in one-to-one correspondence with the set of equivalence classes of irreducible permutative representations of $\mathcal{O}_N$.*

Characterizations of permutative representations are given by terminology of branching function systems. The followings hold from definition of branching function system and Definition 2.11 (iv) immediately:

Proposition 2.14. *Let f be a branching function system on an infinite set Λ. Recall $(l_2(\Lambda), \pi_f)$ in Definition 2.11 (iv).*

(i) *If g is a branching function system on an infinite set Λ' such that $f \sim g$, then $(l_2(\Lambda), \pi_f) \sim (l_2(\Lambda'), \pi_g)$.*

(ii) *If f is cyclic, then $(l_2(\Lambda), \pi_f)$ is cyclic.*

(iii) *For $J \in \{1,\ldots,N\}^\#$, if f is $P(J)$, then $(l_2(\Lambda), \pi_f)$ is $P(J)$, too.*

(iv) *If $f = f^{(1)} \oplus f^{(2)}$ and $\Lambda = \Lambda_1 \sqcup \Lambda_2$ is the associated decomposition of f, then $(l_2(\Lambda), \pi_f) \sim (l_2(\Lambda_1), \pi_{f^{(1)}}) \oplus (l_2(\Lambda_2), \pi_{f^{(2)}})$.*

3. Permutative endomorphisms and their graph invariants

3.1. Permutative endomorphisms. We review endomorphisms of $\mathcal{O}_N$ arising from permutations in [**11**] and refine their results.

Assume that End$\mathcal{A}$ is the set of all unital $*$-endomorphisms of a unital $*$-algebra $\mathcal{A}$ and $\rho, \rho' \in \mathrm{End}\mathcal{A}$ in this subsection. ρ is *proper* if $\rho(\mathcal{A}) \neq \mathcal{A}$. ρ is *irreducible* if $\rho(\mathcal{A})' \cap \mathcal{A} = \mathbf{C}I$ where $\rho(\mathcal{A})' \cap \mathcal{A} \equiv \{x \in \mathcal{A} : \rho(a)x = x\rho(a)$ for each $a \in \mathcal{A}\}$. ρ is *reducible* if ρ is not irreducible. ρ and ρ' are *equivalent* if there is a unitary $u \in \mathcal{A}$ such that $\rho' = \mathrm{Ad}u \circ \rho$. In this case, we denote $\rho \sim \rho'$.

Let Rep$\mathcal{A}$(*resp.* IrrRep$\mathcal{A}$) be the set of all unital (*resp.* irreducible)$*$-representations of $\mathcal{A}$. We simply denote π for $(\mathcal{H}, \pi) \in \mathrm{Rep}\mathcal{A}$.

Lemma 3.1. (i) *If* $\rho, \rho' \in \mathrm{End}\mathcal{A}$ *and* $\pi, \pi' \in \mathrm{Rep}\mathcal{A}$ *satisfy* $\rho \sim \rho'$ *and* $\pi \sim \pi'$, *then* $\pi \circ \rho \sim \pi' \circ \rho'$.

(ii) *Assume that* $\mathcal{A}$ *is simple. If there is* $\pi \in \mathrm{IrrRep}\mathcal{A}$ *such that* $\pi \circ \rho \in \mathrm{IrrRep}\mathcal{A}$, *too, then* ρ *is irreducible. Specially, if* $\pi \circ \rho \sim \pi \in \mathrm{IrrRep}\mathcal{A}$, *then* $\rho^n \equiv \underbrace{\rho \circ \cdots \circ \rho}_{n}$ *is irreducible for each* $n \geq 1$.

(iii) *If there is* $\pi \in \mathrm{Rep}\mathcal{A}$ *such that* $\pi \circ \rho \not\sim \pi \circ \rho'$, *then* $\rho \not\sim \rho'$.

(iv) *If there is* $\pi \in \mathrm{IrrRep}\mathcal{A}$ *such that* $\pi \circ \rho \notin \mathrm{IrrRep}\mathcal{A}$, *then* ρ *is proper.*

We identify the symmetric group $\mathfrak{S}_{N^k}$ and $\mathfrak{S}_{N,k}$ the set of all bijective transformations on $\{1,\ldots,N\}^k$ for $k \geq 1$ by the method in § 2.3. Define

$$\sigma \mapsto u_\sigma = \sum_{J \in \{1,\ldots,N\}^k} s_{\sigma(J)} s_J^*. \tag{3.1}$$

Then we have $u_\sigma u_{\sigma'} = u_{\sigma\circ\sigma'}$ for $\sigma, \sigma' \in \mathfrak{S}_{N,k}$.

Definition 3.2. *For* $\sigma \in \mathfrak{S}_{N,k}$, $\psi_\sigma \in \mathrm{End}\mathcal{O}_N$ *is defined by*

$$\psi_\sigma(s_i) \equiv u_\sigma s_i \quad (i = 1,\ldots,N).$$

ψ_σ *is called the permutative endomorphism of* $\mathcal{O}_N$ *by* σ *where* u_σ *is in (*3.1*).*

Put the following sets:

$$E_{N,k} \equiv \{\psi_\sigma \in \mathrm{End}\mathcal{O}_N : \sigma \in \mathfrak{S}_{N,k}\} \quad (k \geq 1). \tag{3.2}$$

Proposition 3.3. (i) *If* $\sigma \in \mathfrak{S}_N$, *then* ψ_σ *is an automorphism of* $\mathcal{O}_N$ *which satisfies* $\psi_\sigma(s_i) = s_{\sigma(i)}$ *for* $i = 1,\ldots,N$. *Specially, if* $\sigma = id$, *then* $\psi_{id} = id$.

(ii) *If* $\sigma \in \mathfrak{S}_{N,2}$ *is defined by* $\sigma(i,j) \equiv (j,i)$ *for* $i,j = 1,\ldots,N$, *then* ψ_σ *is the canonical endomorphism of* $\mathcal{O}_N$.

(iii) $\gamma_z \circ \psi_\sigma = \psi_\sigma \circ \gamma_z$ *for each* $z \in U(1)$ *and* $\sigma \in \mathfrak{S}_{N,*} \equiv \coprod_{l \geq 1} \mathfrak{S}_{N,l}$.

(iv) *If* $\rho \in E_{N,k}$ *and* $\rho' \in E_{N,k'}$, *then* $\rho \circ \rho' \in E_{N,k+k'-1}$ *for each* $k, k' \geq 1$.

Proof. Immediately, we see (i)∼(iii) by definition. (iv) follows from Proposition 4.5 in [**11**]. □

Theorem 3.4. (i) *Let* Λ *be an infinite set. For* $\sigma \in \mathfrak{S}_{N,k}$, $k \geq 1$, *and* $f \in \mathrm{BFS}_N(\Lambda)$, *let* $(l_2(\Lambda), \pi_f)$ *be in Definition* 2.11 *(iv) and* $f^{(\sigma)}$ *in (*2.4*). Then we have* $\pi_f \circ \psi_\sigma = \pi_{f^{(\sigma)}}$.

(ii) *If* ρ *is a permutative endomorphism and* $(\mathcal{H}, \pi)$ *is a permutative representation of* $\mathcal{O}_N$, *then* $\pi \circ \rho$ *is a permutative representation, too.*

(iii) *If* $(\mathcal{H}, \pi)$ *is* $P(J)$ *for* $J \in \{1,\ldots,N\}^\#$ *and* $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$, *then there are* $1 \leq M \leq N^{l-1}$, *a family* $\{J_i\}_{i=1}^M \subset \{1,\ldots,N\}^\#$ *and a family*

$\{(\mathcal{H}_i,\pi_i)\}_{i=1}^M$ *of subrepresentations of* $(\mathcal{H},\pi\circ\psi_\sigma)$ *such that*

$$(3.3)\qquad (\mathcal{H},\pi\circ\psi_\sigma)=\bigoplus_{i=1}^M(\mathcal{H}_i,\pi_i)$$

and $(\mathcal{H}_i,\pi_i)$ *is* $P(J_i)$ *for* $i=1,\ldots,M$. *Furthermore if* $J\in\{1,\ldots,N\}^k$, $k\geq 1$, *then* $\{J_i\}_{i=1}^M\subset\coprod_{a=1}^{N^{l-1}}\{1,\ldots,N\}^{ak}$, *and if* $J\in\{1,\ldots,N\}^\infty$, *then* $\{J_i\}_{i=1}^M\subset\{1,\ldots,N\}^\infty$.

(iv) *The rhs in* (3.3) *is unique up to unitary equivalences.*

Proof. See Lemma 4.9 and Theorem 4.10 in [**11**] for (i) and (ii). By (i) and Proposition 2.14, it is sufficient for the statement about permutative representations to check properties of branching function systems associated with them. Statements in (iii) follows from Lemma 2.7 (iii) and Lemma 2.10. (iv) follows from Theorem 2.12 (i) and (ix). □

By uniqueness of $P(J)$, we simply denote (3.3) as

$$(3.4)\qquad P(J)\circ\psi_\sigma=\bigoplus_{i=1}^M P(J_i).$$

Specially, if $\sigma\in\mathfrak{S}_N=\mathfrak{S}_{N,1}$, then $P(J)\circ\psi_\sigma=P(J_{\sigma^{-1}})$ by Lemma 2.6. Roughly speaking, we can say that a permutative endomorphism transforms cycles(*resp.*chains) to cycles(*resp.*chains).

Theorem 3.5. *For each* $\sigma\in\mathfrak{S}_{N,l}$, $l\geq 1$, $J\in\{1,\ldots,N\}_1^*$ *and* $z\in U(1)$, *there are* $1\leq M\leq N^{l-1}$, $\{J_i\}_{i=1}^M\subset\{1,\ldots,N\}_1^*$ *and* $\{z_i\}_{i=1}^M\subset U(1)$ *such that*

$$P(J;z)\circ\psi_\sigma=\bigoplus_{i=1}^M P(J_i;z_i).$$

Proof. Applying Theorem 2.12 (vi) and Proposition 3.3 (iii) for (3.4), we have

$$P(J;z)\circ\psi_\sigma=P(J;1)\circ\psi_\sigma\circ\gamma_{z^{1/l}}=\bigoplus_{i=1}^M P(J_i;1)\circ\gamma_{z^{1/l}}=\bigoplus_{i=1}^M P(J_i;z^{l_i/l})$$

where $J_i\in\{1,\ldots,N\}^{l_i}$ for each $i=1,\ldots,M$. Putting $z_i\equiv z^{l_i/l}$, we have the statement. □

Furthermore we extend our results. For $\sigma\in\mathfrak{S}_{N,l}$ and $z\in U(1)$, put $\psi_{\sigma,z}\equiv\psi_\sigma\circ\gamma_z$. Then

$$P(J;z)\circ\psi_{\sigma,y}=\bigoplus_{i=1}^M P(J_i;z_i)\circ\gamma_y=\bigoplus_{i=1}^M P(J_i;z_iy^{l_i})$$

where $J_i \in \{1,\ldots,N\}^{l_i}$ for $i=1,\ldots,M$.

Theorem 3.6. *Let $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 1$ and $y \in U(1)$.*

(i) *For $J \in \{1,\ldots,N\}^k$, $k \geq 1$, and $z \in U(1)$, we have $P(J;z)\circ\psi_{\sigma,y} = P(J;zy^k)\circ\psi_\sigma$.*

(ii) *For $J \in \{1,\ldots,N\}_1^*$ and $z \in U(1)$, there are $1 \leq M \leq N^{l-1}$, $\{J_i\}_{i=1}^M \subset \{1,\ldots,N\}_1^*$ and $\{z_i\}_{i=1}^M \subset U(1)$ such that*

$$P(J;z)\circ\psi_{\sigma,y} = \bigoplus_{i=1}^{M} P(J_i;z_i).$$

This decomposition is unique up to unitary equivalences.

(iii) *For each $J \in \{1,\ldots,N\}^\infty$, $P(J)\circ\psi_{\sigma,y} = P(J)\circ\psi_\sigma$.*

3.2. Graph invariants of endomorphisms. In order to classify endomorphisms of $\mathcal{O}_N$, we introduce a graph from branching laws of an endomorphism and show examples. A *graph* (V,E) in this subsection means a pair of sets V and $E \subset V\times V$. V and E are the set of vertices and that of edges. An element $(x,y)\in E$ is an edge of (V,E) with direction from x to y, respectively.

Let $\mathcal{S}$ be a set and $\{e_x\}_{x\in\mathcal{S}}$ the canonical basis of a Hilbert space $l_2(\mathcal{S})$. Put

$$W_+(\mathcal{S}) \equiv \{v \in l_2(S)\setminus\{0\} : < e_x|v > \in \mathbf{Z}_{\geq 0} \text{ for each } x \in \mathcal{S}\}$$

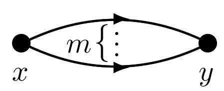
where $\mathbf{Z}_{\geq 0}$ is the set of all non-negative integers. Then $W_+(\mathcal{S})$ is an abelian semigroup with respect to addition in $l_2(\mathcal{S})$.

Definition 3.7. *Let φ be a transformation on $W_+(\mathcal{S})$.*

(i) *(V_φ, E_φ) is the graph of φ with the label set $\mathcal{S}$ if $V_\varphi \equiv \mathcal{S}$ is the set of vertices and $E_\varphi \equiv \{(x,y)\in V_\varphi\times V_\varphi : < \varphi(e_x)|e_y > \neq 0\}$ is the set of directed edges on V_φ.*

(ii) *For a subset $\mathcal{S}_0 \subset \mathcal{S}$, a subgraph $(V_\varphi(\mathcal{S}_0), E_\varphi(\mathcal{S}_0))$ of (V_φ,E_φ) is defined by $V_\varphi(\mathcal{S}_0) \equiv \mathcal{S}_0$ and $E_\varphi(\mathcal{S}_0) \equiv \{(x,y)\in E_\varphi : x,y \in V_\varphi(\mathcal{S}_0)\}$.*

If $< \varphi(e_x)|e_y > = m \geq 1$, then we draw m-directed arrows from x to y:



Remark $< \varphi(e_x)|e_y > < \infty$ for each $x,y \in \mathcal{S}$ by definition of $W_+(\mathcal{S})$. The graph (V_φ,E_φ) of φ explains the property of φ effectively by illustration. We prepare notions about graphs.

Definition 3.8. (i) *(V,E) and (V',E') are strongly equivalent if $V=V'$ and $E=E'$. In this case, we denote $(V,E)=(V',E')$.*

(ii) (V,E) *has a (non trivial)branch if there is* $v \in V$ *such that* $\#E_{v,out} \geq 2$ *where* $E_{v,out} \equiv \{w \in V : (v,w) \in E\}$.

(iii) $n \equiv \max\{\#E_{v,out} : v \in V\}$ *is called the branching number of* (V,E).

(iv) (V,E) *is* n*-regular if* $n = \#E_{v,out}$ *for each* $v \in V$.

(v) (V,E) *is finitely branched if the branching number of* (V,E) *is finite.*

(vi) C *is a cycle of* (V,E) *if* $C = \{x_j\}_{j=1}^M \subset V$ *and* $(x_j, x_{j+1}), (x_M, x_1) \in E$ *for each* $j = 1, \ldots, M-1$, $1 \leq M < \infty$. *In this case, we call* C *by* M*-cycle, too.*

For any transformation φ on $W_+(\mathcal{S})$, (V_φ, E_φ) is finitely branched by definition of $W_+(\mathcal{S})$.

We construct a graph from the branching of an endomorphism of $\mathcal{O}_N$. Let $\mathcal{S} \equiv [1,\ldots,N]^* \times U(1)$. Then an element in $W_+(S)$ is just an equivalence class of a finite direct sum of irreducible representations with parameters in $\mathcal{S}$ by Theorem 2.12 (v).

Put

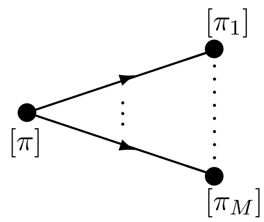
$$E_{N,*} \equiv \{\psi_\sigma : \sigma \in \mathfrak{S}_{N,*}\}, \quad \widehat{E}_{N,*} \equiv \{\rho \circ \gamma_z : \rho \in E_{N,*},\ z \in U(1)\}. \tag{3.5}$$

We see that $\rho \in \widehat{E}_{N,*}$ is a transformation on $W_+(\mathcal{S})$ by Theorem 3.4 and Theorem 3.6.

Definition 3.9. (V_ρ, E_ρ) *is called the branching graph of* $\rho \in \hat{E}_{N,*}$ *on* $\mathcal{S}$. *For a subset* $\mathcal{S}_0 \subset \mathcal{S}$, $(V_\rho(\mathcal{S}_0), E_\rho(\mathcal{S}_0))$ *is, too.*

A branching law of an endomorphism ρ at π is illustrated as follows:

$$\pi \circ \rho \sim \bigoplus_{i=1}^{M} \pi_i \qquad \Leftrightarrow$$



where we assume that $\pi, \pi_1, \ldots, \pi_M$ are mutually inequivalent.

An automorphism is a special endomorphism and its branching is trivial as follows:

Lemma 3.10. *If* $\rho \in \hat{E}_{N,*}$ *is an automorphism of* $\mathcal{O}_N$, *then the branching graph* (E_ρ, V_ρ) *of* ρ *has no branch. Specially, if* ρ *is inner, then* (E_ρ, V_ρ) *is the collection of* 1*-cycles.*

Proof. If $\rho \in \mathrm{Aut}\mathcal{O}_N$, then any irreducible representation is transformed to irreducibles. Therefore there is no branch. Because an inner automorphism preserves any equivalence class of representation, the assertion holds. $\square$

Proposition 3.11. *Let $\rho, \rho^{'} \in \widehat{E}_{N,*}$ and (V_ρ, E_ρ), $(V_{\rho'}, E_{\rho'})$ be branching graphs of ρ and $\rho^{'}$ on $\mathcal{S} \equiv [1, \ldots, N]^* \times U(1)$, respectively. Then the followings hold:*

(i) *If $\rho \sim \rho^{'}$, then $(V_\rho, E_\rho) = (V_{\rho'}, E_{\rho'})$.*
(ii) *If $(V_\rho(\mathcal{S}_0), E_\rho(\mathcal{S}_0)) \neq (V_{\rho'}(\mathcal{S}_0), E_{\rho'}(\mathcal{S}_0))$ for some $\mathcal{S}_0 \subset \mathcal{S}$, then $\rho \not\sim \rho^{'}$.*
(iii) *If (V_ρ, E_ρ) has a branch, then ρ is proper.*
(iv) *If there is a vertex in (V_ρ, E_ρ) with only one outgoing edge, then ρ is irreducible. Specially, if there is a 1-cycle in (V_ρ, E_ρ), then ρ^n is irreducible for each $n \geq 1$.*

Proof. (i) By Lemma 3.1 (i) and Theorem 3.6 (i), it holds.
(ii) This follows from (i) and the definition of subgraph.
(iii) By Lemma 3.10 (i), it holds.
(iv) By Lemma 3.1 (ii), the statement holds. □

By Proposition 3.11 (i), we see that the branching graph of an endomorphism ρ is an invariant of ρ up to equivalences. By Proposition 3.11 (ii), this invariant is effective to distinguish two endomorphisms. We show examples in § 4.

4. Examples of branching graphs

4.1. Branching graphs of $E_{2,2}$. We show the branching graphs of elements in $E_{2,2}$ in (3.2). By Lemma 5.9 in in [**11**], the set of all equivalence classes in $E_{2,2}$ is

$$(4.1) \qquad SE_{2,2} \equiv \left\{ [\psi_\sigma] : \sigma = \begin{array}{l} id, (12), (13), (14), (23), (24), (34), \\ (123), (132), (124), (142), (143), (234), \\ (1243), (1342), (12)(34) \end{array} \right\}$$

where $[\psi_\sigma] = \{\rho \in E_{2,2} : \rho \sim \psi_\sigma\}$ and we use the labeling $\mathfrak{S}_{2,2} \cong \mathfrak{S}_4$ in § 2.3. By Proposition 3.11 (i), it is sufficient to show graphs for elements in $SE_{2,2}$. In the following, we identify $SE_{2,2}$ and the set of representatives of elements in $SE_{2,2}$. First, we review the definition of elements in $SE_{2,2}$ as follows:

Table 4.1. *(Elements in $SE_{2,2}$)*

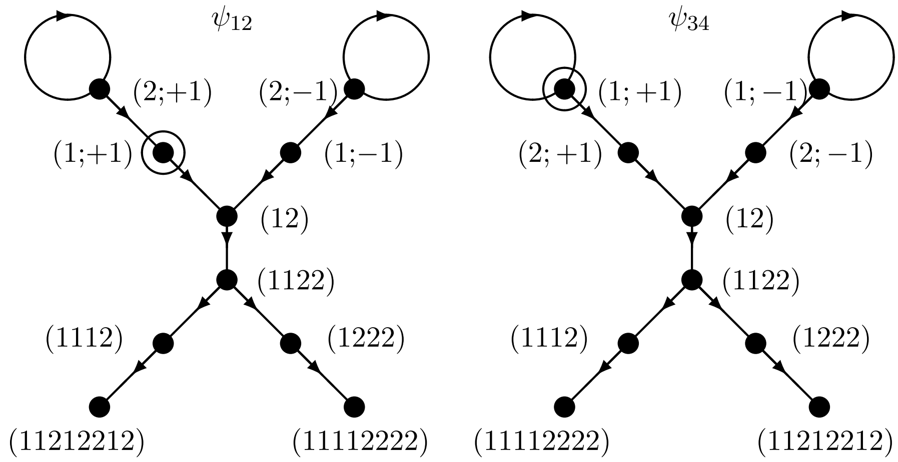
ψ_σ	$\psi_\sigma(s_1)$	$\psi_\sigma(s_2)$	property
ψ_{id}	s_1	s_2	$inn.aut$
ψ_{12}	$s_{12,1}+s_{11,2}$	s_2	$irr.end$
ψ_{13}	$s_{21,1}+s_{12,2}$	$s_{11,1}+s_{22,2}$	$irr.end$
ψ_{14}	$s_{22,1}+s_{12,2}$	$s_{21,1}+s_{11,2}$	$red.end$
ψ_{23}	$s_{11,1}+s_{21,2}$	$s_{12,1}+s_{22,2}$	$red.end$
ψ_{24}	$s_{11,1}+s_{22,2}$	$s_{21,1}+s_{12,2}$	$irr.end$
ψ_{34}	s_1	$s_{22,1}+s_{21,2}$	$irr.end$
ψ_{123}	$s_{12,1}+s_{21,2}$	$s_{11,1}+s_{22,2}$	$red.end$
ψ_{132}	$s_{21,1}+s_{11,2}$	$s_{12,1}+s_{22,2}$	$red.end$
ψ_{124}	$s_{12,1}+s_{22,2}$	$s_{21,1}+s_{11,2}$	$red.end$
ψ_{142}	$s_{22,1}+s_{11,2}$	$s_{21,1}+s_{12,2}$	$irr.end$
ψ_{143}	$s_{22,1}+s_{12,2}$	$s_{11,1}+s_{21,2}$	$red.end$
ψ_{234}	$s_{11,1}+s_{21,2}$	$s_{22,1}+s_{12,2}$	$red.end$
ψ_{1243}	$s_{12,1}+s_{22,2}$	$s_{11,1}+s_{21,2}$	$red.end$
ψ_{1342}	$s_{21,1}+s_{11,2}$	$s_{22,1}+s_{12,2}$	$red.end$
$\psi_{(12)(34)}$	$s_{12,1}+s_{11,2}$	$s_{22,1}+s_{21,2}$	$out.aut$

where "inn.aut", "out.aut", "irr.end" and "red.end" mean an inner automorphism, an outer automorphism, a proper irreducible endomorphism and a proper reducible endomorphism, respectively, $u \equiv s_1 s_2^* + s_2 s_1^*$ and $s_{ij,k} \equiv s_i s_j s_k^*$ for $i,j,k = 1,2$. The branching law of an automorphism of $\mathcal{O}_2$ is shown in the above Theorem 3.5. Hence $P(J) \circ \psi_{id} = P(J)$ for $J \in \{1,\ldots,N\}^{\#}$ and $P(1) \circ \psi_{(12)(34)} = P(2)$, $P(2) \circ \psi_{(12)(34)} = P(1)$, $P(12) \circ \psi_{(12)(34)} = P(12)$.

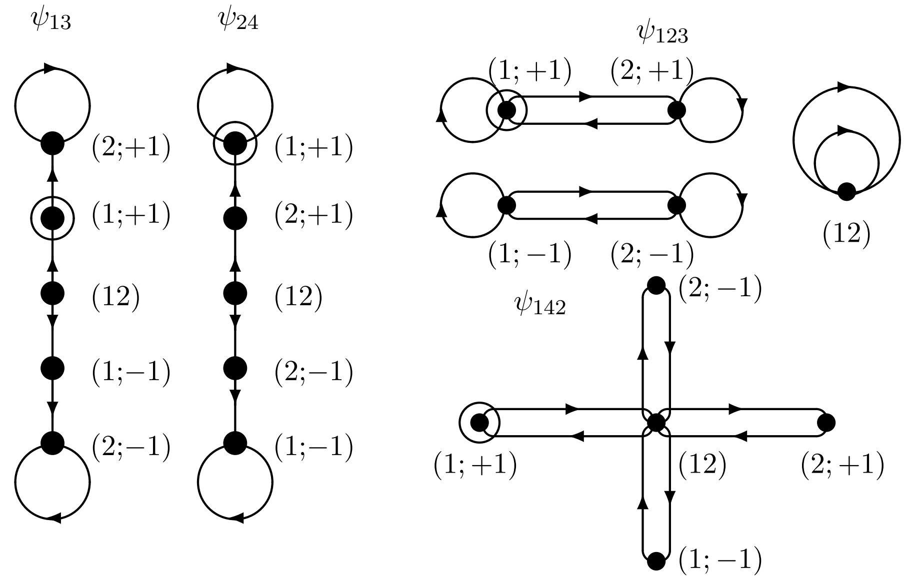
We consider other 14-endomorphisms in $SE_{2,2}$. For

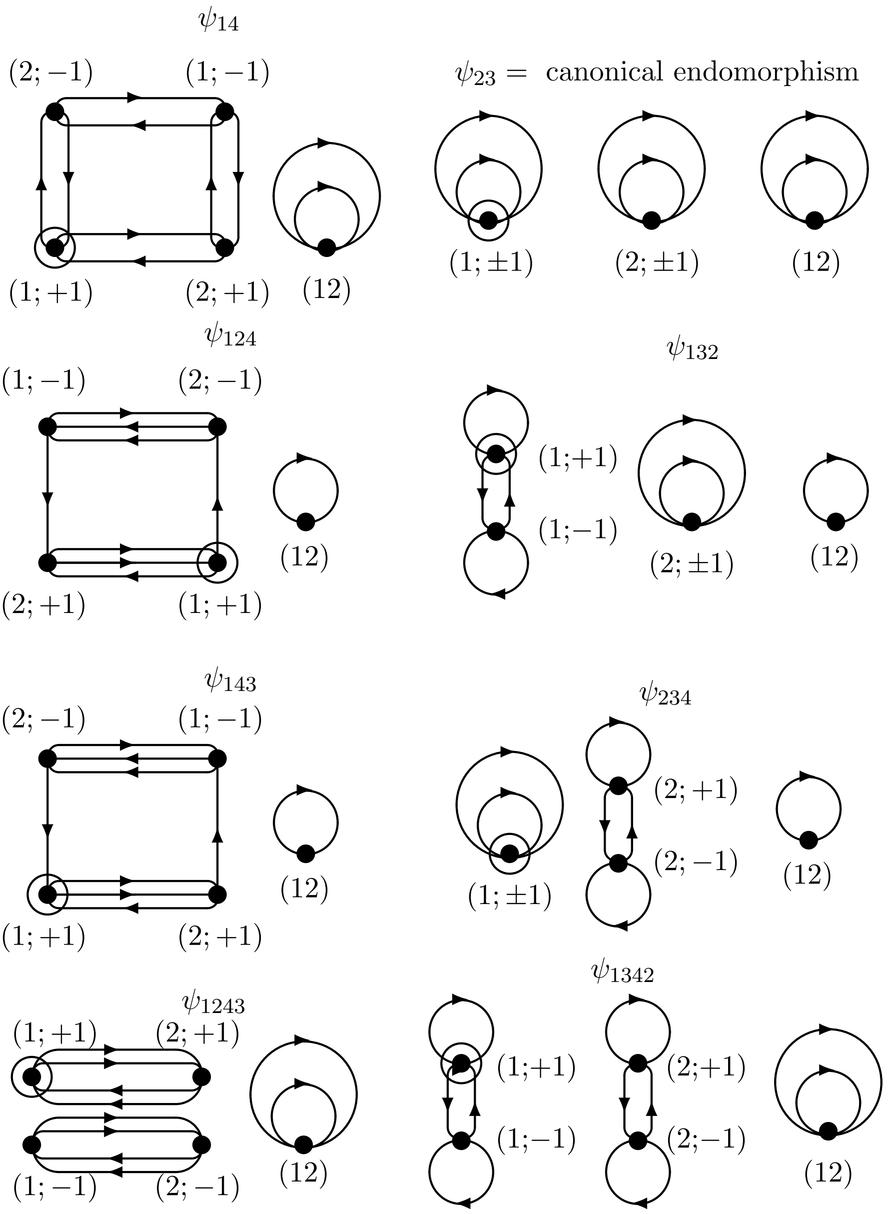
$$(4.2) \quad \mathcal{S}_1 \equiv \left\{ \begin{array}{l} P(1;\pm 1), P(2;\pm 1), P(12), P(1122), P(1112), P(1222), \\ P(11212212), P(11112222) \end{array} \right\},$$

the branching graph $(V_\rho(\mathcal{S}_1), E_\rho(\mathcal{S}_1))$ of $\rho = \psi_{12}, \psi_{34}$ are followings:



where we denote $P(J;z)$ by $(J;z)$ simply and a vertex with a small circle means $P(1)$. The difference between ψ_{12} and ψ_{34} is the position of $(1;\pm 1)$ and $(2;\pm 1)$. In the same way, we show branching graphs for other endomorphisms in $E_{2,2}$ on $\mathcal{S}_2 \equiv \{P(1;\pm 1), P(2;\pm 1), P(12)\}$:

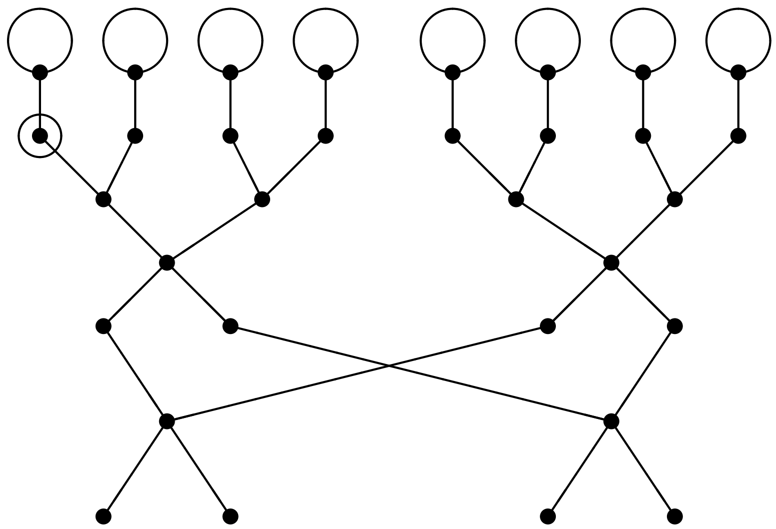




These branching laws are computed by checking branching function systems associated with them(see Table 5.7 in [**11**]). Branching graphs of ψ_{12} and ψ_{13} on $\mathcal{S}_2$ are same except direction of their edges. The branching graph of ψ_{12} on $\mathcal{S}_0 \equiv \mathcal{S}_{0,1} \cup \mathcal{S}_{0,2} \cup \mathcal{S}_{0,3}$, $\mathcal{S}_{0,1} \equiv \{P(n;z) : n = 1,2,\ z = e^{2\pi l\sqrt{-1}/8},\ l = 1,\ldots,8\}$, $\mathcal{S}_{0,2} \equiv \{P(12;z), P(J;\pm 1) : J = (1112),(1122),(1222),\ z =$

$\pm 1, \pm\sqrt{-1}\}$, $\mathcal{S}_{0,3} \equiv \{P(11112222), P(11212212), P(12122222), P(11111212),$ $P(11211222),\ P(11122122)\}$, is the following:

ψ_{12}



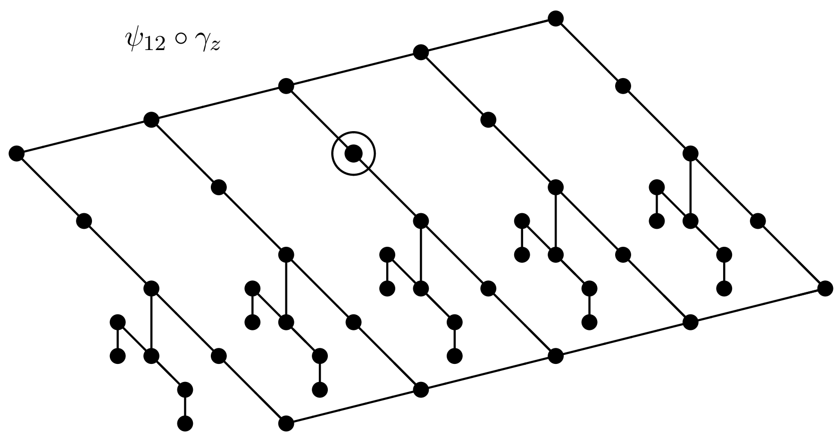
where we omit labels of vertices and arrows of the graph. The direction of any edge is from top to down.

Question *Find the general rule to draw this graph.*

This question is equivalent to find the general branching law of ψ_{12} on a given representation.

The branching graph of $\psi_{12}\circ\gamma_z$, $z = e^{2\pi\theta\sqrt{-1}}$, $\theta \in \mathbf{R}\backslash\mathbf{Q}$ is the following:

$\psi_{12} \circ \gamma_z$



where we omit other labels of vertices and arrows of the graph.

Because $\#E_{2,3} = 8!$, it may be difficult to classify elements in $E_{2,3}$ by only their branching graphs. We show examples in $E_{2,3}$.

Example 4.2. (i) Put $\rho \equiv \psi_{12} \circ \psi_{12}$. Then the branching laws of ρ are given by

$$\begin{aligned} P(1) \circ \rho &= P(12) \circ \psi_{12} = P(1122), \\ P(2) \circ \rho &= P(1) \oplus P(2) \oplus (12), \\ P(12) \circ \rho &= P(1112) \oplus P(1222), \\ P(1122) \circ \rho &= P(11212212) \oplus P(11112222). \end{aligned}$$

By Proposition 4.6 in [**11**], $\rho \in E_{2,3}$. By the first equation in the above and Lemma 3.1 (i), we see that ρ is irreducible. In the same way, we have

$$\begin{aligned} P(2) \circ (\psi_{12})^3 = &\ P(1) \oplus P(2) \oplus P(12) \oplus P(1122), \\ P(2) \circ (\psi_{12})^4 = &\ P(1) \oplus P(2) \oplus P(12) \oplus P(1122) \oplus P(1112) \oplus P(1222). \end{aligned}$$

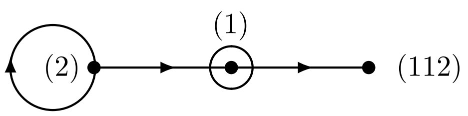
(ii) Define $\rho \in E_{2,3}$ by

$$\rho(s_1) \equiv s_1 s_1 s_2 s_{11}^* + s_1 s_1 s_1 s_2^* s_1^* + s_1 s_2 s_2^*, \quad \rho(s_2) \equiv s_2.$$

Then ρ is irreducible and proper. We show the sketch of proof. Put a branching function system $f_i(n) \equiv 3(n-1) + i$ for $n \in \mathbf{N}$ and $i = 1, 2, 3$. Then we have the following unique cycle: $1 \stackrel{h_2}{\to} 2 \stackrel{h_1}{\to} 3 \stackrel{h_1}{\to} 1$ where $h_i \equiv f_i^{(\sigma)}$, $i = 1, 2, 3$, and σ is the permutation associated with ρ. Hence $P(1) \circ \rho = P(112)$. Therefore ρ is irreducible by Lemma 3.1 (ii).

On the other hand, we see $P(2) \circ \rho = P(1) \oplus P(2)$ in the same way. Hence ρ is proper. □

The branching graph of ρ on $\{P(1), P(2), P(112)\}$ is the following:



By Table 5.7 in [**11**], there is no path from $P(1)$ to $P(112)$ in branching graphs of elements in $E_{2,2}$. Hence ρ is inequivalent to any elements in $E_{2,2}$.

4.2. Branching graph of ρ_ν. Let $\rho_\nu \in E_{3,2}$ by

$$(4.3)\qquad \begin{cases} \rho_\nu(s_1) \equiv & s_{23,1} + s_{31,2} + s_{12,3}, \\ \rho_\nu(s_2) \equiv & s_{32,1} + s_{13,2} + s_{21,3}, \\ \rho_\nu(s_3) \equiv & s_{11,1} + s_{22,2} + s_{33,3} \end{cases}$$

where $s_{ij,k} \equiv s_i s_j s_k^*$ for $i,j,k = 1,2,3$. ρ_ν is proper and irreducible(Theorem 1.2 in [**11**]). We show the branching law of ρ_ν.

Theorem 4.3. $P(1) \circ \rho_\nu = P(12) \oplus P(3)$.

Proof. Put σ_0 a transformation on $\{1,2,3\}^2$ defined by the following:

$$(4.4)\qquad \sigma_0 : \begin{pmatrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{pmatrix} \mapsto \begin{pmatrix} (2,3) & (3,1) & (1,2) \\ (3,2) & (1,3) & (2,1) \\ (1,1) & (2,2) & (3,3) \end{pmatrix}.$$

Then we see $\rho_\nu = \psi_{\sigma_0}$. Put a branching function system $h \equiv f^{(\sigma_0)}$ for $f_i(n) \equiv 3(n-1)+i$ for $i = 1,2,3$ and $n \in \mathbf{N}$. Then we have the list of values of h_1, h_2, h_3:

n	$h_1(n)$	$h_2(n)$	$h_3(n)$
1	8	6	1
2	3	7	5
3	4	2	9

From this, we have two cycles $1 \overset{h_3}{\to} 1$, $2 \overset{h_1}{\to} 3 \overset{h_2}{\to} 2$. On the other hand, $(l_2(\mathbf{N}), \pi_f)$ and $(l_2(\mathbf{N}), \pi_h)$ are $P(1)$ and $(l_2(\mathbf{N}), \pi_f \circ \psi_{\sigma_0})$, respectively. Hence we see that $(l_2(\mathbf{N}), \pi_h)$ contains $P(3)$ and $P(12)$. We see $\mathbf{N} = A_1 \sqcup A_2$ where $A_1 \equiv \{f_J(1) : J \in \{1,\ldots,N\}^*\}$ and $A_2 \equiv \{f_J(2) : J \in \{1,\ldots,N\}^*\}$. We see that there is no cycle in $\{n \in \mathbf{N} : n \geq 4\}$. From this, we have the assertion. □

In the same way, we have the following branching laws of ρ_ν:

$$P(2) \circ \rho_\nu = P(3) \circ \rho_\nu = P(12) \oplus P(3),$$

$$P(12) \circ \rho_\nu = P(13) \circ \rho_\nu = P(23) \circ \rho_\nu = P(113223).$$

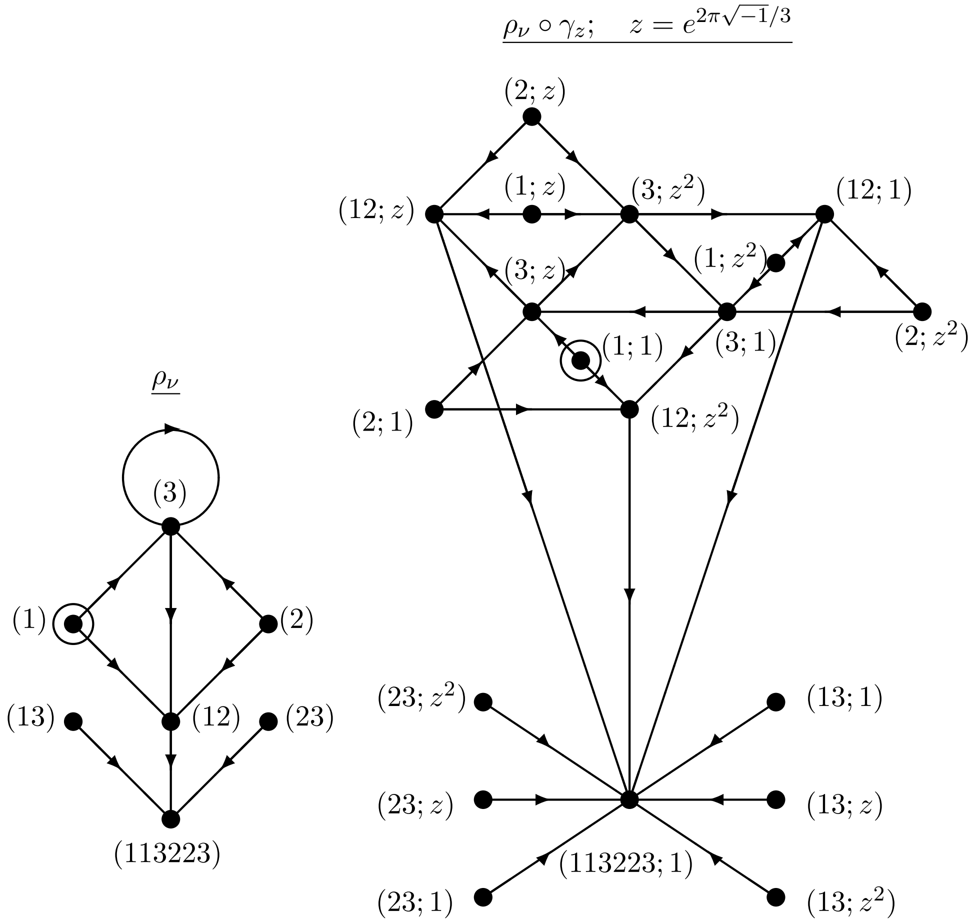
For $z \in U(1)$, put $\rho' \equiv \rho_\nu \circ \gamma_z$. Then $P(J) \circ \rho' = P(J; z^k) \circ \rho_\nu$ when $J \in \{1,\ldots,N\}^k$ for $k \geq 1$ by Theorem 3.6 (i). For example,

$$P(i) \circ \rho' = P(i) \circ \rho_\nu \circ \gamma_z = (P(12) \oplus P(3)) \circ \gamma_z = P(12; z^2) \oplus P(3; z)$$

for $i = 1,2,3$. In the same way, we have

$$P(12) \circ \rho' = P(13) \circ \rho' = P(23) \circ \rho' = P(113223; z^6).$$

In consequence, we have the following branching graphs of ρ_ν on $\mathcal{S}_0 \equiv \{(1),(2),(3),(12),(13),(23),(113223)\}$ and $\rho_\nu \circ \gamma_z$ when $z = e^{2\pi\sqrt{-1}/3}$ on $\mathcal{S}_1 \equiv \{(J;\xi),(113223;1) : J = (1),(2),(3),(12),(13),(23),\ \xi = 1, z, z^2\}$:



We see that the deformation of graphs in the above is arisen by the action of γ_z. The canonical $U(1)$-action γ is often called the *gauge action* on $\mathcal{O}_N$. In this sense, a transformation $\rho \mapsto \rho \circ \gamma_z$ is the *gauge transformation of the endomorphism* ρ, the transformation of branching law associated with this transformation of endomorphism is the *gauge transformation of branching law*, and the transformation of graph associated with this in the above is the *gauge transformation of graph*.

4.3. Endomorphisms by transpositions. We consider permutative endomorphisms ψ_σ such that $\sigma \in \mathfrak{S}_{N,2}$ is a transposition, that is, there are $x, y \in \{1,\ldots,N\}^2$ and $\sigma(x) = y$, $\sigma(y) = x$, $\sigma(z) = z$ for $z \in \{1,\ldots,N\}^2$,

$z \neq x, y$. Both ψ_{12} and ψ_{13} in $E_{2,2}$ are examples of such endomorphisms of $\mathcal{O}_2$ and they are proper irreducible. We prepare the formula of branching function systems associated with endomorphisms of transpositions.

Let $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_N(\mathbf{N})$ be defined by $f_i(n) \equiv N(n-1)+i$ for $i = 1, \ldots, N$, $n \in \mathbf{N}$.

Lemma 4.4. *Let $\sigma \in \mathfrak{S}_{N,2}$.*

(i) *For $n \in \mathbf{N}$ and $i, j = 1, \ldots, N$, we have*

$$f_i^{(\sigma)}(f_j(n)) = N^2(n-1) + N(\sigma_2(i,j)-1) + \sigma_1(i,j)$$

where σ_1 and σ_2 are taken by $\sigma(i,j) = (\sigma_1(i,j), \sigma_2(i,j))$ for $(i,j) \in \{1, \ldots, N\}^2$.

(ii) *(Transposition) If σ is a transposition which is defined by $\sigma(i_1, j_1) = (i_2, j_2)$, then*

$$f_i^{(\sigma)}(f_j(n)) = \begin{cases} N^2(n-1) + N(j_2-1) + i_2 & ((i,j) = (i_1, j_1)), \\ N^2(n-1) + N(j_1-1) + i_1 & ((i,j) = (i_2, j_2)), \\ N^2(n-1) + N(j-1) + i & (\textit{otherwise}) \end{cases}$$

for $n \in \mathbf{N}$ and $i, j = 1, \ldots, N$. Specially,

$$f_i^{(\sigma)}(j) = \begin{cases} N(j_2-1) + i_2 & ((i,j) = (i_1, j_1)), \\ N(j_1-1) + i_1 & ((i,j) = (i_2, j_2)), \\ N(j-1) + i & (\textit{otherwise}) \end{cases}$$

for $i, j = 1, \ldots, N$.

We know that $f^{(\sigma)}$ has cycles in only a set $\{1, \ldots, N\}$ for each $\sigma \in \mathfrak{S}_{N,2}$ by Lemma 2.11 in [**11**]. Hence it is sufficient to check the behavior of $f^{(\sigma)}$ on $\{1, \ldots, N\}$. As applications of Lemma 4.4, we have the following results:

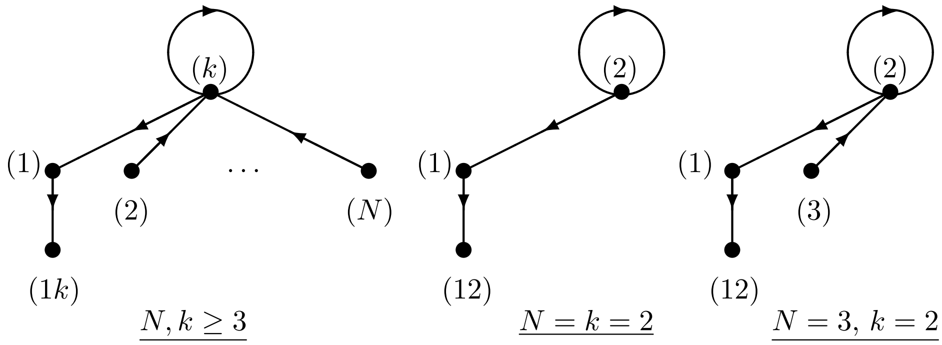
Example 4.5. Let $N \geq 2$. Put ψ_{1k} is the permutative endomorphism of $\mathcal{O}_N$ which is associated with a transposition $\sigma \in \mathfrak{S}_{N,2}$ defined by $\sigma(1,1) = (1,k)$. ψ_{1k} is given as follows:

$$\psi_{1k}(s_i) = \begin{cases} s_1 s_k s_1^* + s_1 s_1 s_k^* + \sum_{j \neq 1,k} s_1 s_j s_j^* & (i = 1), \\ s_i & (i \neq 1). \end{cases}$$

When $k = 2, \ldots, N$, we have

$$P(l) \circ \psi_{1k} = \begin{cases} P(1k) & (l = 1), \\ P(1) \oplus P(k) & (l = k), \\ P(k) & (otherwise). \end{cases}$$

From these, we see that ψ_{1k} is irreducible and proper by Proposition 3.11. The branching graph of ψ_{1k} is following:



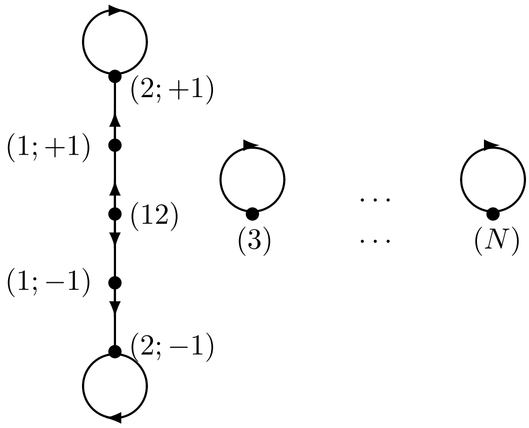
where we denote $P(J)$ by J.

Example 4.6. Let $\psi_{1,N+1}$ be a permutative endomorphism of $\mathcal{O}_N$ by a transposition between $(1,1)$ and $(2,1)$. When $N \geq 3$, we have the following:

$$P(k) \circ \psi_{1,N+1} = \begin{cases} P(2) & (k = 1, 2), \\ P(k) & (3 \leq k \leq N), \end{cases}$$

$$P(12) \circ \psi_{1,N+1} = P(1; +1) \oplus P(1; -1).$$

Specially, $\psi_{1,N+1}$ is irreducible and proper when $N \geq 2$. From this, $\psi^n_{1,N+1} \equiv \underbrace{\psi^n_{1,N+1} \circ \cdots \circ \psi_{1,N+1}}_{n}$ is irreducible and proper for $N \geq 2$ and each $n \geq 1$ by Proposition 3.11. The branching graph of $\psi_{1,N+1}$ is following:



4.4. $\mathbf{Z}_N$-invariant endomorphisms. ρ_ν in (4.3) is invariant under the left $\mathbf{Z}_3$-action, that is, $\alpha_{\tau'} \circ \rho_\nu = \rho_\nu$ for each $\tau' \in \mathbf{Z}_3 \subset U(3)$. We generalize this to an endomorphism of $\mathcal{O}_N$ with $\mathbf{Z}_N$-invariance for $N \geq 4$.

For $N \geq 3$, we define an endomorphism $\rho \in E_{N,2}$ which is invariant under $\mathbf{Z}_N$-action. Put τ is a shift on a set $\{1, \ldots, N\}$ which is defined by $\tau(n) \equiv n + 1$ for $n = 1, \ldots, N-1$ and $\tau(N) = 1$. Define ρ by

$$\rho(s_i) \equiv \begin{cases} \sum_{j=1}^{N} s_j s_j s_j^* & (i = 1), \\ \sum_{j=1}^{N} s_{\tau^{j-1}(1)} s_{\tau^{j+i-2}(1)} s^*_{\tau^{j+i-1}(1)} & (i = 2, \ldots, N-1), \\ \sum_{j=1}^{N} s_{\tau^{j-1}(1)} s_{\tau^{j-2}(1)} s^*_{\tau^{j}(1)} & (i = N). \end{cases}$$

When $N = 3$, $\rho(s_1), \rho(s_2), \rho(s_3)$ are given by

$$s_{11,1} + s_{22,2} + s_{33,3}, \quad s_{12,3} + s_{23,1} + s_{31,2}, \quad s_{13,2} + s_{21,3} + s_{32,1}.$$

Note $\rho = \rho_\nu \circ \alpha_\sigma$ for $\sigma \in \mathbf{Z}_3$, $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ where ρ_ν is in § 4.2. By definition, we see

$$\alpha_{\tau'} \circ \rho = \rho \quad (\text{ for each } \tau' \in \mathbf{Z}_N). \tag{4.5}$$

Proposition 4.7. *For each $N \geq 3$, we have the followings*

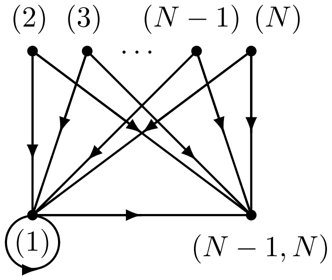
(i) $P(i) \circ \rho = P(1) \oplus P(N-1, N)$ *for each* $i = 1, \ldots, N$.

(ii) ρ *is proper.*

(iii) $\rho|_{\mathcal{O}_N^{\mathbf{Z}_N}}$ *is an endomorphism of* $\mathcal{O}_N^{\mathbf{Z}_N}$*, too, where* $\mathcal{O}_N^{\mathbf{Z}_N} \equiv \{x \in \mathcal{O}_N : \alpha_{\tau'}(x) = x$ *for each* $\tau' \in \mathbf{Z}_N\}$.

Proof. (i) For a branching function system $f = \{f_i\}_{i=1}^N$ on $\mathbf{N}$ which is defined by $f_i(n) \equiv N(n-1)+i$, put $h_i \equiv f_i^{(\sigma)}$ for $\sigma \in \mathfrak{S}_{N,2}$ which satisfies $\rho = \psi_\sigma$. Then we have $1 \overset{h_1}{\mapsto} 1$ and $2 \overset{h_{N-1}}{\mapsto} 3 \overset{h_N}{\mapsto} 2$. Hence we see that h is $P(1) \oplus P(N, N-1)$. Hence $P(1) \circ \rho = P(1) \oplus P(N, N-1)$. By (4.5) and this, the assertion holds.
(ii) The number of branching components is $2(< N)$ on (1). By (4.5), ρ is proper.
(iii) By (4.5), the statement holds. □

The branching graph of ρ is the following:



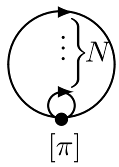
4.5. Canonical endomorphism. The most famous non trivial polynomial endomorphism is the canonical endomorphism by Proposition 3.3 (ii). We show that its branching law is quite simple.

Proposition 4.8. *Let* ρ *be the canonical endomorphism of* $\mathcal{O}_N$*. For any representation* π *of* $\mathcal{O}_N$*, we have* $\pi \circ \rho \sim \pi^{\oplus N}$*. Furthermore* $\pi \circ \rho^l \sim \pi^{\oplus N^l}$ *where* $\rho^l \equiv \underbrace{\rho \circ \cdots \circ \rho}_{l}$ *and* $\pi^{\oplus l} \equiv \underbrace{\pi \oplus \cdots \oplus \pi}_{l}$ *for* $l \geq 1$.

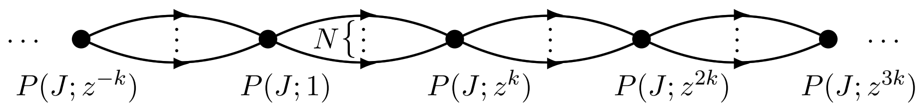
Proof. Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$. Put $\mathcal{H}_N \equiv \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{N}$ and a unitary U from $\mathcal{H}$ to $\mathcal{H}_N$ by $Ux \equiv (\pi(s_1)^*x, \ldots, \pi(s_N)^*x)$. Then $\mathrm{Ad}U \circ \pi \circ \rho = \pi^{\oplus N}$. From this, we have the second statement immediately. □

That is, ρ acts on representations as N-times copy of them. Note $\rho^l \in E_{N,l+1}$ for each $l \geq 1$ by Proposition 3.3 (iv). Therefore the power of canonical endomorphism is the case with possible maximal branching number in Theorem 3.4 (iii).

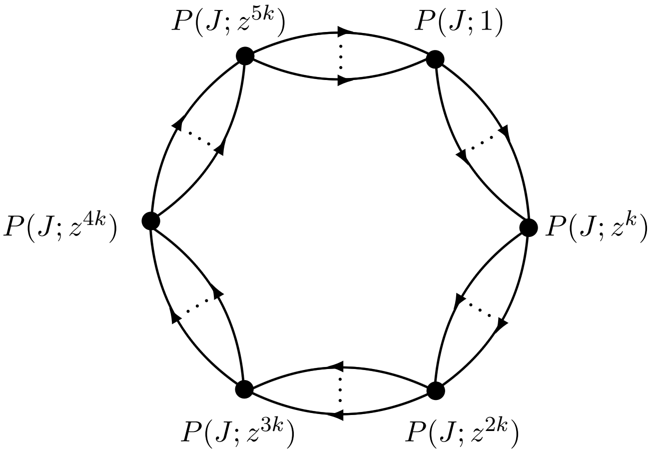
By Proposition 4.8, the branching graph $(E_\rho(\mathcal{S}_0), V_\rho(\mathcal{S}_0))$ on $\mathcal{S}_0 \equiv \{[\pi]\}$ is the following for each equivalence class $[\pi]$ of representations of $\mathcal{O}_N$:



Put $\rho_z \equiv \rho \circ \gamma_z$ and $J \in \{1,\ldots,N\}^k$, $k \geq 1$. If $z = e^{2\pi\sqrt{-1}\theta}$, $\theta \in \mathbf{R} \setminus \mathbf{Q}$, then the branching graph of ρ_z is following:



If $z = e^{2\pi\sqrt{-1}/p}$ and $(p,k) = 1$, then the branching graph of ρ_z is a finite regular graph with N-outgoing and N-incoming edges. For example, when $p = 6$, it is the following:



5. Branching laws and spectrum modules

We interpret results of branching laws of endomorphisms to smarter statements about modules of the endomorphism semigroup $\mathrm{End}\mathcal{O}_N$ of $\mathcal{O}_N$.

5.1. Spectrum semigroup. Let $\mathrm{BSpec}\mathcal{A}$ be the set of all unitary equivalence classes of unital $*$-representations of a unital $*$-algebra $\mathcal{A}$. $\mathrm{BSpec}\mathcal{A}$ is closed under direct integral and it is an abelian semigroup with respect to direct sum:

$$\mathrm{BSpec}\mathcal{A} \times \mathrm{BSpec}\mathcal{A} \ni ([\pi],[\pi^{'}]) \mapsto [\pi] \oplus [\pi^{'}] \equiv [\pi \oplus \pi^{'}] \in \mathrm{BSpec}\mathcal{A}.$$

We call $\mathrm{BSpec}\mathcal{A}$ the *spectrum semigroup of* $\mathcal{A}$.

For $\mathcal{S} \subset \mathrm{BSpec}\mathcal{A}$, define

$$\begin{array}{lcl}
<\mathcal{S}> & : & \text{the set of all finite direct sums of elements in } \mathcal{S}, \\
<\mathcal{S}>_\infty & : & \text{the set of all countable infinite direct sums of elements in } \mathcal{S}, \\
<\mathcal{S}>_\int & : & \text{the set of all direct integrals of elements in } \mathcal{S}.
\end{array}$$

$<\mathcal{S}>, <\mathcal{S}>_\infty, <\mathcal{S}>_\int$ are subsemigroups of $\mathrm{BSpec}\mathcal{A}$ and $<\mathcal{S}> \subset <\mathcal{S}>_\infty \subset <\mathcal{S}>_\int$.

We introduce several subsemigroups of $\mathrm{BSpec}\mathcal{O}_N$. Let $BP(\mathcal{O}_N)$ (*resp.* $P(\mathcal{O}_N)$) be the set of all unitary equivalence classes of (*resp.* irreducible) permutative representations of $\mathcal{O}_N$, and $BP_*(\mathcal{O}_N)$(*resp.* $BP_\infty(\mathcal{O}_N)$) the subset of $BP(\mathcal{O}_N)$ which consists of all cyclic representations with a cycle (*resp.* a chain). Put

$$BP_\#(\mathcal{O}_N) \equiv BP_*(\mathcal{O}_N) \sqcup BP_\infty(\mathcal{O}_N),$$
$$P_{c,*}(\mathcal{O}_N) \equiv \{P(J;z) : J \in [1,\ldots,N]^*,\ z \in U(1)\},$$
$$P_*(\mathcal{O}_N) \equiv BP_*(\mathcal{O}_N) \cap P(\mathcal{O}_N), \quad P_\infty(\mathcal{O}_N) \equiv BP_\infty(\mathcal{O}_N) \cap P(\mathcal{O}_N),$$
$$P_c(\mathcal{O}_N) \equiv P_{c,*}(\mathcal{O}_N) \sqcup P_\infty(\mathcal{O}_N).$$

In this section, we identify a representation and its unitary equivalence class. We see $P(\mathcal{O}_N) = \{P(J) : J \in [1,\ldots,N]^\#\}$, $P_\infty(\mathcal{O}_N) = \{P(J) : J \in [1,\ldots,N]^\infty\}$, $P_*(\mathcal{O}_N) = \{P(J) : J \in [1,\ldots,N]^*\}$ by Theorem 2.12.

By Proposition 4.7 and Corollary 5.12 in [**10**], the following inclusions of semigroups hold:

$$\begin{array}{ccccc}
<P_\infty(\mathcal{O}_N)> & & & & \\
\cap & & & & \\
<BP_\infty(\mathcal{O}_N)> & & & & \\
\cap & & & & \\
<BP_\infty(\mathcal{O}_N)>_\infty & & & & \\
\cap & & & & \\
BP(\mathcal{O}_N) & & \subset & & <P_c(\mathcal{O}_N)>_\int \\
\cup & & & & \cup \\
<BP_*(\mathcal{O}_N)>_\infty & \subset & <P_{c,*}(\mathcal{O}_N)>_\infty & \subset & <P_c(\mathcal{O}_N)>_\infty \\
\cup & & \cup & & \\
<BP_*(\mathcal{O}_N)> & \subset & <P_{c,*}(\mathcal{O}_N)> & & \\
\cup & & & & \\
<P_*(\mathcal{O}_N)> & & & &
\end{array}$$

where any inclusion is proper. These inclusions show relations among classes of representations. For example, $<BP_*(\mathcal{O}_N)> \subset <P_{c,*}(\mathcal{O}_N)>$ means that any element in $BP_*(\mathcal{O}_N)$ can be expressed as a finite direct sum of elements in $P_{c,*}(\mathcal{O}_N)$. Since $P_{c,*}(\mathcal{O}_N)$ is the set of equivalence classes of irreducible representations, this inclusion shows irreducible decomposition of elements in $BP_*(\mathcal{O}_N)$.

Furthermore the followings hold:

$$< BP_{\#}(\mathcal{O}_N) >_\infty = BP(\mathcal{O}_N) = < BP(\mathcal{O}_N) >_\infty .$$

5.2. Spectrum modules of endomorphism semigroups. Let $\mathrm{End}\mathcal{A}$ be the set of all unital $*$-endomorphisms of a unital $*$-algebra $\mathcal{A}$. Then $\mathrm{End}\mathcal{A}$ is a unital semigroups with respect to composition:

$$\mathrm{End}\mathcal{A} \times \mathrm{End}\mathcal{A} \ni (\rho, \rho^{'}) \mapsto \rho \circ \rho^{'} \in \mathrm{End}\mathcal{A}.$$

For $\rho \in \mathrm{End}\mathcal{A}$, define a right transformation R_ρ on $\mathrm{BSpec}\mathcal{A}$ by

$$[\pi]R_\rho \equiv [\pi \circ \rho] \quad ([\pi] \in \mathrm{BSpec}\mathcal{A}).$$

R_ρ is defined without ambiguity by Lemma 3.1 (i). Then R is a right action of $\mathrm{End}\mathcal{A}$ on $\mathrm{BSpec}\mathcal{A}$, that is,

$$([\pi] \oplus [\pi^{'}])R_\rho = [\pi]R_\rho \oplus [\pi^{'}]R_\rho,$$

$$([\pi]R_\rho)R_{\rho'} = [\pi](R_\rho R_{\rho'}) = [\pi]R_{\rho\circ\rho'}, \quad [\pi]R_{id} = [\pi]$$

for each $[\pi], [\pi^{'}] \in \mathrm{BSpec}\mathcal{A}$ and $\rho, \rho^{'} \in \mathrm{End}\mathcal{A}$. In other words, $(\mathrm{BSpec}\mathcal{A}, R)$ is a right $\mathrm{End}\mathcal{A}$-module.

Definition 5.1. *Let G be a subsemigroup of* $\mathrm{End}\mathcal{A}$.

(i) $(\mathrm{BSpec}\mathcal{A}, R|_G)$ *is called the (right)spectrum module of G.*
(ii) *V is a G-submodule of* $\mathrm{BSpec}\mathcal{A}$ *if V is a subsemigroup of* $\mathrm{BSpec}\mathcal{A}$ *and $VR_g \subset V$ for each $g \in G$.*

Recall $E_{N,*}$ and $\widehat{E}_{N,*}$ in (3.5). For each $\sigma, \sigma^{'} \in \mathfrak{S}_{N,*}$, there is $\sigma^{''} \in \mathfrak{S}_{N,*}$ such that $\psi_\sigma \circ \psi_{\sigma'} = \psi_{\sigma''}$ and $(\psi_\sigma \circ \gamma_z) \circ (\psi_{\sigma'} \circ \gamma_{z'}) = \psi_{\sigma''} \circ \gamma_{zz'}$ for each $z, z^{'} \in U(1)$ by Proposition 3.3 (ii) and (iv). Therefore both $E_{N,*}$ and $\widehat{E}_{N,*}$ are subsemigroups of $\mathrm{End}\mathcal{O}_N$. According to these subsemigroups and their spectrum modules, Theorem 3.4, Theorem 3.5 and Theorem 3.6 are interpreted as follows:

Proposition 5.2. (i) *The followings are proper inclusions of $E_{N,*}$-submodules of* $(\mathrm{BSpec}\mathcal{O}_N, R|_{E_{N,*}})$:

$$\begin{array}{ccccc} < BP_\infty(\mathcal{O}_N) > & & & & \\ \cap & & & & \\ < BP_\infty(\mathcal{O}_N) >_\infty & \subset & BP(\mathcal{O}_N) & \subset & < P_c(\mathcal{O}_N) >_\int \\ & & \cup & & \cup \\ & & < BP_*(\mathcal{O}_N) >_\infty & \subset & < P_{c,*}(\mathcal{O}_N) >_\infty \\ & & \cup & & \cup \\ & & < BP_*(\mathcal{O}_N) > & \subset & < P_{c,*}(\mathcal{O}_N) > . \end{array}$$

(ii) *The followings are proper inclusions of* $\widehat{E}_{N,*}$*-submodules of* $(\mathrm{BSpec}\mathcal{O}_N, R|_{\widehat{E}_{N,*}})$*:*

$$\begin{array}{ccccc} < BP_\infty(\mathcal{O}_N) > & \subset & < BP_\infty(\mathcal{O}_N) >_\infty & \subset & < P_c(\mathcal{O}_N) >_\int \\ & & & & \cup \\ & & < P_{c,*}(\mathcal{O}_N) > & \subset & < P_{c,*}(\mathcal{O}_N) >_\infty . \end{array}$$

(iii) *Put* $BP_k(\mathcal{O}_N) \equiv \{P(J) : J \in \{1,\ldots,N\}^k_{min}\}$ *where* $\{1,\ldots,N\}^k_{min}$ *is the set of all minimal elements in* $\{1,\ldots,N\}^k$ *for* $k \geq 1$*. For the following grading*

$$< BP_*(\mathcal{O}_N) >= \bigoplus_{k\geq 1} < BP_k(\mathcal{O}_N) >,$$

we have

$$< BP_k(\mathcal{O}_N) > R_\rho \subset \bigoplus_{a=1}^{N^{l-1}} < BP_{ak}(\mathcal{O}_N) >$$

when $\rho \in E_{N,l}$*,* $l \geq 1$*.*

References

[1] M.Abe and K.Kawamura, *Recursive Fermion System in Cuntz Algebra. I —Embeddings of Fermion Algebra into Cuntz Algebra—*, Comm. Math. Phys. **228**, 85-101 (2002).

[2] M.Abe and K.Kawamura, *Recursive Fermion System in Cuntz Algebra. II —Endomorphism, Automorphism and Branching of Representation—*, preprint RIMS-1362 (2002).

[3] M.Abe and K.Kawamura, *Pseudo Cuntz Algebra and Recursive FP Ghost System in String Theory*, Int. J. Mod. Phys. **A18**, No. 4 (2003) 607-625.

[4] O.Bratteli and P.E.T.Jorgensen, *Iterated function Systems and Permutation Representations of the Cuntz algebra*, Memories Amer. Math. Soc. No.663 (1999).

[5] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57**, 173-185 (1977).

[6] K.R.Davidson and D.R.Pitts, *The algebraic structure of non-commutative analytic Toeplitz algebras*, Math.Ann. 311, 275-303 (1998).

[7] K.R.Davidson and D.R.Pitts, *Invariant subspaces and hyper-reflexivity for free semigroup algebras*, Proc. London Math. Soc. (3) 78 (1999) 401-430.

[8] K.Kawamura, *Generalized permutative representations of the Cuntz algebras. I —Generalization of cycle type—*, preprint RIMS-1380 (2002).

[9] K.Kawamura, *Generalized permutative representations of the Cuntz algebras. II —Irreducible decomposition of periodic cycle—*, preprint RIMS-1388 (2002).

[10] K.Kawamura, *Generalized permutative representations of the Cuntz algebras. III —Generalization of chain type—*, preprint RIMS-1423 (2003).

[11] K. Kawamura, *Polynomial endomorphisms of the Cuntz algebras arising from permutations. I —General theory—*, preprint RIMS-1427 (2003).

[12] T. Kobayashi, *Theory of discretely decomposable restrictions of unitary representations of semisimple Lie groups and some applications*, preprint RIMS-1387 (2002).