On the logarithmic Kodaira dimension of affine threefolds

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In the theory of affine varieties, it is important to investigate the structure associated to the log Kodaira dimension. In this article, we shall consider how to analyze smooth affine threefolds associated to the log Kodaira dimension and make the framework for this purpose under a certain geometric condition. As a consequence of our result, under this geometric condition, we can describe the construction of affine threefolds X with $\overline{\kappa}(X) = -\infty$ fairly explicitly, and show that an affine threefold X with log Kodaira dimension $\overline{\kappa}(X) = 2$ has the structure of a $\mathbb{C}^*$-fibration.

1 Introduction

Throughout the present article we work over the field of complex numbers $\mathbb{C}$. In the theory on smooth affine surfaces, it is important and essential to investigate the structure associated to the log Kodaira dimension (see [Ii77] for the definition and the relevant results on log Kodaira dimension). The following results due to Miyanishi-Sugie and Kawamata are very important in this direction in the theory of affine surfaces:

Theorem 1.1 (cf. [Mi-Su80, Kaw79, Miy01]) *Let Y be a smooth affine surface. Then:*

(1) $\overline{\kappa}(Y) = -\infty \Longrightarrow$ *Y has a structure of an $\mathbb{A}^1$-fibration. (Miyanishi-Sugie)*

(2) $\overline{\kappa}(Y) = 1 \Longrightarrow$ *Y has a structure of a $\mathbb{C}^*$-fibration. (Kawamata)*

As the three-dimensional generalization of the above mentioned result Theorem 1.1, our main interest in this article lies in the investigation of the structure on smooth affine threefolds associated to the log Kodaira dimension. Since the understanding of the minimal model theory on surfaces was indispensable in order to obtain Theorem 1.1 (cf. [Mi-Su80, Kaw79]), it seems to be natural that the theory of three-dimensional Minimal Model Program gives good tools for our purpose. Namely, the rough idea of our consideration is stated as follows:

Let X be a smooth affine threefold. We embed X into a smooth projective threefold T in such a way that the boundary divisor D is simple normal crossing (= SNC, for short). Starting with T, we run Minimal Model Program (= MMP, for short), say $g : T \cdots \rightarrow T'$, to reach the situation where T' has the structure of Mori fiber space (= Mfs, for short) or T' is a minimal model, i.e., the canonical divisor $K_{T'}$ is nef (cf. [FA], [Ko-Mo98]). Let D' be the proper transform of D on T' and put $X' := T' - \mathrm{Supp}\,(D')$. Since T' is a simple object in the set of normal projective threefolds with only $\mathbb{Q}$-factorial terminal singularities such that the function fields equal $\mathbb{C}(X)$, we expect that it is possible to analyze the structure of X'. Hereafter we wish to investigate the original affine threefold X from the data on X'.

But the following obstacles arise in this above mentioned argument.

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OBSTACLE. The birational map $g : T \cdots \to T'$ is factorized, say $g = g^{r-1} \circ \cdots \circ g^0$, where each $g^i : T^i \cdots \to T^{i+1}$ is either a divisorial contraction or a flip, and $T^0 := T$, $T^r := T'$. Let D^i denote the proper transform of D on T^i, and let $X^i := T^i - \mathrm{Supp}\,(D^i)$ denote the complement for $0 \le i \le r$.

(1) In the case where g^i is divisorial, the exceptional divisor $E^i := \mathrm{Exc}(g^i) \subset T^i$ may not be contained in the boundary Supp (D^i). Hence X^{i+1} may be smaller than X^i strictly.

(2) In the case where g^i is a flip, let $\gamma_1, \cdots, \gamma_a \subset T^i$ exhaust all the flipping curves and let $\gamma_1^+, \cdots, \gamma_a^+ \subset T^{i+1}$ denote the corresponding flipped curves. Then, in general, we cannot specify the location of these curves on T^i and T^{i+1}, respectively. For example, some of flipping curves (resp. flipped curves) may not be contained in the boundary Supp (D^i) (resp. Supp (D^{i+1})).

This is why we can not analyze the structure on X^i from that on X^{i+1} in general. Thus, even if we can analyze X' concretely, it is usually impossible to recover the data on the original X from that on X'.

Now we wish to investigate how to analyze the structure of smooth affine threefolds associated to the log Kodaira dimension. Let X be a smooth affine threefold. As usual we embed X into a smooth projective threefold T in such a way that the reduced boundary divisor D with respect to $X \hookrightarrow T$ is SNC. In this article, we assume the following additional condition (†). Our condition is geometric in nature:

(†) : *The complete linear system $\mathcal{H} := |D|$ is nef and contains a smooth member, say $S \in \mathcal{H}$.*

We can see, under the condition (†), that the above mentioned obstacles do not have a bad influence upon our consideration, and we are able to describe X fairly explicitly. Namely, our main result is stated as follows:

Theorem 1.2 *Let X be a smooth affine threefold. Assume that X is embedded into a smooth projective threefold T satisfying the condition (†). Then we can construct the following diagram (∗) consisting of normal projective threefolds T^i with only $\mathbb{Q}$-factorial terminal singularities and the birational maps among them:*

$$(*)\quad T =: T^0 \overset{g^0}{\cdots\to} T^1 \overset{g^1}{\cdots\to} T^2 \cdots \to \cdots\cdots \to T^{s-1} \overset{g^{s-1}}{\cdots\to} T^s \overset{g^s}{-}$$
$$\to T^{s+1} \xrightarrow{g^{s+1}} T^{s+2} \to \cdots\cdots \to T^{r-1} \xrightarrow{g^{r-1}} T^r =: T^\sharp,$$

satisfying the following conditions:

(1) *Let $\mathcal{H}^i$ (resp. D^i, S^i) be the proper transform of $\mathcal{H}$ (resp. D, S) on T^i, and let $X^i := T^i - \mathrm{Supp}(D^i)$ denote the complement for $0 \le i \le r$. Then $\mathcal{H}^i$ is a nef, Cartier linear system on T^i containing D^i and S^i as members. Moreover, S^i is smooth.*

(2) *The exceptional set (which is either an exceptional divisor or a union of the flipping curves) of the birational map $g^i : T^i \cdots \to T^{i+1}$ is contained in Supp (D^i) for $0 \le i < s$.*

(3) *The remaining birational maps $g^i : T^i \to T^{i+1}$ for $s \le i < r$ are described as follows: $g^i : T^i \to T^{i+1}$ contracts the exceptional divisor E^i to a smooth point, say $p^{i+1} := g^i(E^i) \in T^{i+1}$, and g^i is realized as the weighted blow-up at the point $p^{i+1} \in T^{i+1}$ with the weights $\mathrm{wt}(x, y, z) = (1, 1, b^i)$, where (x, y, z) are the suitable local analytic coordinates at $p^{i+1} \in T^{i+1}$ and $b^i \in \mathbb{N}$. It follows that $\mathcal{H}^i \cdot l^i = 1$, $(-K_{T^i} \cdot l^i) = 1 + (1/b^i)$, where l^i is a ruling on the exceptional divisor $E^i \cong \mathbb{P}(1, 1, b^i) \cong \mathbb{F}_{b^i,0}$. Furthermore, the inequalities $1 \le b^s \le b^{s+1} \le \cdots \le b^{r-1}$ hold true.*

(4) *For $0 \le i \le s$, we have $X^i \cong X$. For the remaining $s \le i < r$, X^i is obtained as the half-point attachmenmt to X^{i+1} of (b^i, k^i)-type for some $1 \le \exists\, k^i \le b^i$ (cf. Definition 1.1) unless $X^i \cong X^{i+1}$. In particular, X^{i+1} is an open affine subset of X^i such that $X^i - X^{i+1} \cong \mathbb{C}^{(k^i-1)*} \times \mathbb{A}^1$ for some $1 \le \exists\, k^i \le b^i$ unless $X^i \cong X^{i+1}$.*

(5) *The right terminal* $T^\sharp$ *in the diagram* $(*)$ *satisfies one of the following properties according to the value of the log Kodaira dimension* $\overline{\kappa}(X)$*:*

(i) *If* $\overline{\kappa}(X) = -\infty$ *then* $T^\sharp$ *has the structure of a Mfs. (See Theorem* 1.3 *for more detailed description of Mfs on* $T^\sharp$*.)*

(ii) *If* $\overline{\kappa}(X) \geq 0$ *then* $K_{T^\sharp} + D^\sharp$ *is nef and* $\kappa(T^\sharp; K_{T^\sharp} + D^\sharp) = \overline{\kappa}(X)$*, where* $D^\sharp := D^r$ *is the proper transform of* D *on* $T^\sharp$*.*

We prepare the definiton of the half-point attachments used in Theorem 1.2. This is a slight generalization in three-dimensional setting of the original definition given in [Miy01, 4.4.1. Definition].

Definition 1.1 Let $\overline{Z}$ be a normal quasi-projective threefold and let $\overline{Z} \hookrightarrow \overline{V}$ be a compactification into a normal projective threefold $\overline{V}$ with the boundary $\overline{V} - \overline{Z} = \mathrm{Supp}\,(\overline{B})$. Let $p \in \mathrm{Supp}\,(\overline{B})$ be a point where $\overline{V}$ is smooth. Let $f : V \to \overline{V}$ be the weighted blow-up at the point $p \in \overline{V}$ with the weights $\mathrm{wt}(x, y, z) = (1, 1, b)$, where (x, y, z) are the suitable local analytic coordinates at $p \in \overline{V}$ and $b \in \mathbb{N}$. Let $E \cong \mathbb{P}(1, 1, b) \cong \mathbb{F}_{b,0}$ be the exceptional divisor of f. *Assume* that the proper transform B of $\overline{B}$ by f meets E in such a way that $B|_E = \sum_{j=1}^{k} m_j l_j$, where l_j's are the mutually distinct rulings on $E \cong \mathbb{F}_{b,0}$ and $m_j \in \mathbb{N}$ with $\sum_{j=1}^{k} m_j = b$ in case $b \geq 2$, and $B|_E \in |\mathcal{O}_{\mathbb{P}^2}(1)|$ in case $b = 1$. Then the complement $Z := V - \mathrm{Supp}\,(B)$ is said to be a *half-point attachment to* $\overline{Z}$ *of* (b, k)*-type*. It is easy to see, by the definition, that $Z - \overline{Z} \cong \mathbb{C}^{(k-1)*} \times \mathbb{A}^1$.

In case $\overline{\kappa}(X) = -\infty$, we can describe the construction of X more concretely. Namely, the following result holds true. Once we have Theorem 1.2, almost all parts of this result (Theorem 1.3) follow by the same argument as in [Me02] and [C-F93] essentially.

Theorem 1.3 *Let* X *be a smooth affine threefold with log Kodaira dimension* $\overline{\kappa}(X) = -\infty$*. Assume that* X *can be embedded into a smooth projective threefold* T *satisfying the condition* $(\dagger)$*. Then:*

(1) *The Mfs on* $T^\sharp$ *(cf. Theorem* 1.2 *(5)-(i)), say* $\pi : T^\sharp \to W$*, and the proper transform* $D^\sharp \in \mathcal{H}^\sharp$ *on* $T^\sharp$ *of* $D \in \mathcal{H}$ *are described according to the type of* π *as one of the following:*

– C_2-type: $\pi : T^\sharp \cong \mathbb{P}(\mathcal{E}) \to W$ *is a* $\mathbb{P}^1$*-bundle structure over a smooth projective surface* W*, where* $\mathcal{E} := \pi_*\mathcal{O}_{T^\sharp}(D^\sharp)$ *is a rank* 2 *vector bundle on* W*, and* $D^\sharp \sim \mathcal{O}(1)$*.*

– D_2'-type: $\pi : T^\sharp \to W$ *is a quadric bundle over a smooth curve* W *with a general fiber* $(F, D^\sharp|_F) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))$*, and with at most finitely many singular fibers* $G \cong \mathbb{Q}_0^2$ *and the vertex of each* G *sits in a hypersurface singularity of analytic type* $o \in (xy + z^2 + t^k = 0) \subset \mathbb{C}^4 : (x, y, z, t)$ *for* $k \geq 1$*, where* $\mathbb{Q}_0^2 \hookrightarrow \mathbb{P}^3$ *is a quadric cone.*

– D_3-type: $\pi : T^\sharp \cong \mathbb{P}(\mathcal{E}) \to W$ *is a* $\mathbb{P}^2$*-bundle structure over a smooth curve* W *with a fiber* $(F, D^\sharp|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$*, where* $\mathcal{E} := \pi_*\mathcal{O}_{T^\sharp}(D^\sharp)$ *is a rank* 3 *vector bundle on* W*.*

– D_3'-type: $\pi : T^\sharp \to W$ *is a* $\mathbb{P}^2$*-fibration over a smooth curve* W *with a general fiber* $(F, D^\sharp|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *and with at most finitely many singular fibers* $G \cong \mathbb{S}_4$ *and the vertex of* G *sits in a hyper-quotient singularity of analytic type* $o \in (xy + z^2 + t^k = 0) \subset \mathbb{C}^4 : (x, y, z, t)/\mathbb{Z}_2(1, 1, 1, 0)$ *for* $k \geq 1$*, where* $\mathbb{S}_4 \subset \mathbb{P}^5$ *is a cone over the quartic rational curve* $\subset \mathbb{P}^4$*.*

– $\mathbb{Q}$-Fano: $T^\sharp$ *is a* $\mathbb{Q}$*-Fano threefold with the Picard number* $\varrho(T^\sharp) = 1$*. More precisely, the classification of the pair* $(T^\sharp, \mathcal{H}^\sharp)$ *up to deformation is given as one of the following:*

(i) $(\mathbb{P}(1, 1, 2, 3), \mathcal{O}(6))$*;*

(ii) $(T_6 \subset \mathbb{P}(1, 1, 2, 3, a), \{x_4 = 0\} \cap T_6)$ *with* $a \in \{3, 4, 5\}$*;*

(iii) $(T_6 \subset \mathbb{P}(1, 1, 2, 2, 3), \{x_3 = 0\} \cap T_6)$*;*

(iv) $(T_6 \subset \mathbb{P}(1, 1, 1, 2, 3), \{x_0 = 0\} \cap T_6)$*;*

(v) $(\mathbb{P}(1, 1, 1, 2), \mathcal{O}(b))$ *with* $b \in \{2, 4\}$*;*

(vi) $(T_4 \subset \mathbb{P}(1, 1, 1, 1, 2), \{x_0 = 0\} \cap T_4)$*;*

(vii) $(T_4 \subset \mathbb{P}(1,1,1,2,a), \{x_4 = 0\} \cap T_4)$ *with* $a \in \{2,3\}$;

(viii) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(b))$ *with* $b \in \{1,2,3\}$, $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}}(b))$ *with* $b \in \{1,2\}$;

(ix) $(T_3 \subset \mathbb{P}(1,1,1,1,2), \{x_4 = 0\} \cap T_3)$;

(x) $(T_3 \subset \mathbb{P}^4, \mathcal{O}(1))$;

(xi) $(T_{2,2} \subset \mathbb{P}^5, \mathcal{O}(1))$;

(xii) $(V_5, \mathcal{O}(1))$, *where* $V_5 \hookrightarrow \mathbb{P}^6$ *is a linear section of the Grassmann variety* $\mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}^9$ *parametrizing lines in* $\mathbb{P}^4$;

(2) *The original affine threefold* X *is obtained from* $X^\sharp = T^\sharp - \mathrm{Supp}(D^\sharp)$ *(where* $T^\sharp$ *and* $D^\sharp$ *are the one described above) via the composite of several half-point attachments of suitable types (cf. Definition* 1.1*) unless* $X \cong X^\sharp$. *More precisely, each of the appearing half-point attachments is described as in the following fashion according to Mfs on* $T^\sharp$:

– C_2-type: *An appearing half-point attachment is of* $(1,1)$*-type and* $X - X^\sharp$ *is a union of several affine planes* $\mathbb{A}^2$ *unless* $X \cong X^\sharp$.

– D_2'-type: *An appearing half-point attachment is of* $(1,1)$*-type and* $X - X^\sharp$ *is a union of several affine planes* $\mathbb{A}^2$ *unless* $X \cong X^\sharp$.

– D_3-type: *Then* $X \cong X^\sharp$.

– D_3'-type: *An appearing half-point attachment is of* (b,k)*-type with* $1 \le b \le 2$, $1 \le k \le b$, *and the corresponding irreducible component of* $X - X^\sharp$ *is isomorphic to* $\mathbb{C}^{(k-1)*} \times \mathbb{A}^1$ *unless* $X \cong X^\sharp$.

– $\mathbb{Q}$-Fano:
- *In case of* (v) *with* $b = 2$ *and* (viii) *with* $b = 1$, $X \cong X^\sharp$.
- *In case of* (iv), (vi), (viii) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$, (x), (xi) *and* (xii), *an appearing half-point attachment is of* $(1,1)$*-type, and* $X - X^\sharp$ *is a union of several affine planes* $\mathbb{A}^2$ *unless* $X \cong X^\sharp$.
- *In case of* (iii), (vii) *with* $a = 2$, (viii) $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}}(2))$ *and* (ix), *an appearing half-point attachment is of* (b,k)*-type with* $1 \le b \le 2$, $1 \le k \le b$ *and the corresponding irreducible component of* $X - X^\sharp$ *is isomorphic to* $\mathbb{C}^{(k-1)*} \times \mathbb{A}^1$ *unless* $X \cong X^\sharp$.
- *In case* (ii) *with* $a = 3$, (vii) *with* $a = 3$ *and* (viii) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$, *an appearing half-point attachment is of* (b,k)*-type with* $1 \le b \le 3$, $1 \le k \le b$ *and the corresponding irreducible component of* $X - X^\sharp$ *is isomorphic to* $\mathbb{C}^{(k-1)*} \times \mathbb{A}^1$ *unless* $X \cong X^\sharp$.
- *In case of* (ii) *with* $a = 4$ *and* (v) *with* $b = 4$, *an appearing half-point attachment is of* (b,k)*-type with* $1 \le b \le 4$, $1 \le k \le b$ *and the corresponding irreducible component of* $X - X^\sharp$ *is isomorphic to* $\mathbb{C}^{(k-1)*} \times \mathbb{A}^1$ *unless* $X \cong X^\sharp$.
- *In case of* (ii) *with* $a = 5$, *an appearing half-point attachment is of* (b,k)*-type with* $1 \le b \le 5$, $1 \le k \le b$ *and the corresponding irreducible component of* $X - X^\sharp$ *is isomorphic to* $\mathbb{C}^{(k-1)*} \times \mathbb{A}^1$ *unless* $X \cong X^\sharp$.
- *In case of* (i), *an appearing half-point attachment is of* (b,k)*-type with* $1 \le b \le 6$, $1 \le k \le b$ *and the corresponding irreducible component of* $X - X^\sharp$ *is isomorphic to* $\mathbb{C}^{(k-1)*} \times \mathbb{A}^1$ *unless* $X \cong X^\sharp$.

On the other hand, in case $\overline{\kappa}(X) = 2$, we obtain the following result as a three-dimensional version of the result due to Kawamata (cf. Theorem 1.1 (2)) by making use of Theorem 1.2.

Theorem 1.4 *Let* X *be a smooth affine threefold with log Kodaira dimension* $\overline{\kappa}(X) = 2$. *Assume that* X *can be embedded into a smooth projective threefold* T *satisfying the condition* (†). *Then* X *has the structure of a* $\mathbb{C}^*$*-fibration over a normal surface.*

We shall state the scheme of this article. Let X be a smooth affine threefold which is embedded into a smooth projective threefold T satisfying (†). In order to prove Theorem 1.2, we need the theory on $\sharp$-Minimal Model Program (= $\sharp$-MMP, for short) which is developed by M. Mella, recently (cf. [Me02]). In the original context of $\sharp$-MMP in [Me02], the varieties treated there are assumed to be uniruled. But the varieties which we need to consider in this paper are not necessarily uniruled. In Section 2, we apply the $\sharp$-MMP to our present situation with a suitable modification for the proof of Theorem 1.2. In general there are many ways to choose the intermediate process to reach either a Mfs or a minimal model starting with T. Then the $\sharp$-MMP is useful to choose the right MMP with respect to the linear system $\mathcal{H} := |D|$. We shall investigate the process of $\sharp$-MMP associated to the pair $(T, \mathcal{H})$ around the proper transforms of D and S concretely, and analyze how the inside affine threefold X changes via $\sharp$-MMP carefully. Although we have to argue according to the value of log Kodaira dimension $\overline{\kappa}(X)$ (namely, $\overline{\kappa}(X) = -\infty$ or $\overline{\kappa}(X) \geq 0$), the intermediate process are very similar to each other. Therefore, we shall demonstrate the proof for both cases simultaneously. In the process of $\sharp$-MMP to prove Theorem 1.2, we often encounter the situation where a $(2,0)$-type divisorial contraction occurs in such a way that the proper transform of the smooth member $S \in \mathcal{H}$ intersects the exceptional divisor E. Then, in [Me02, Proposition 3.6, Case 3.9], M. Mella asserts that E satisfies $(E, E|_E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$ which is contracted to a smooth point. But his statement is incorrect. We shall state the correct statement and give the proof for it in Proposition 2.1.

In the following Sections 3 and 4, we shall prove Theorems 1.3 and 1.4, respectively, by making use of Theorem 1.2 and the proof of it.

We employ the following notation in this article.

Notation.

- $\sim$: linear equivalence;
- $\equiv$: numerical equivalence;
- $\mathbb{A}^n$: the n-dimensional affine space;
- $\mathbb{P}^n$: the n-dimensional projective space;
- $\mathbb{P}(a,b,c)$: the weighted projective plane with weights $\mathrm{wt}(x,y,z) = (a,b,c)$, i.e., $\mathrm{Proj}(\mathbb{C}[x,y,z])$ with $\mathrm{wt}(x,y,z) = (a,b,c)$;
- $\mathbb{F}_b$: the Hirzebruch surface of degree b $(b \geq 0)$;
- $\mathbb{F}_{b,0}$: the normal surface obtained from $\mathbb{F}_b$ by contracting the minimal section;
- $\mathbb{C}^{(k)*}$: the affine line with k-point(s) punctured. We write $\mathbb{C}^{(0)*} = \mathbb{A}^1$ and $\mathbb{C}^{(1)*} = \mathbb{C}^*$ usually.
- $\mathrm{Exc}(f)$: the exceptional set of a given birational morphism f;
- $\overline{\mathrm{NE}}(V)$: the closure of the cone of effective 1-cycles on V.
- $\mathrm{index}(V)$: the Gorenstein index of V, i.e., the least positive integer such that $\mathrm{index}(V)K_V$ is a Cartier divisor.

The projective birational morphism $f : V \to W$ from a normal threefold V with only $\mathbb{Q}$-factorial terminal singularities is said to be a *divisorial contraction* if it is obtained as the contraction of a K_V-negative extremal ray and the exceptional set $E := \mathrm{Exc}(f)$ is a prime divisor. f is said to be of $(2,0)$-*type* (resp. $(2,1)$-*type*) if E is contracted to a point (resp. to a curve).

2 The Proof of Theorem 1.2

In this section, we shall give the proof of Theorem 1.2. The most important and essential part in our proof is an adequate modification of the theory on $\sharp$-MMP developed by M. Mella (cf. [Me02]) to our present situation. Although we need to argue according to the value of log Kodaira dimension $\overline{\kappa}(X)$ ($\overline{\kappa}(X) = -\infty$ or $\overline{\kappa}(X) \geq 0$), the intermediate process to reach the right terminal $T^\sharp$ in Theorem 1.2 are very similar to each other. Hence we shall treat both cases $\overline{\kappa}(X) = -\infty$ and $\overline{\kappa}(X) \geq 0$ simultaneously otherwise mentioned.

Let X be a smooth affine threefold. We embed X into a smooth projective threefold T in such a way that the reduced boundary divisor D is SNC and satisfies the condition (†) in Section 1. First of all, we have the following:

Lemma 2.1 *$\mathcal{H} = |D|$ is nef and big.*

Proof. By virtue of the result [Go69, Theorem 1], there exists a closed subscheme $F \subset \mathrm{Supp}\,(D)$ and the blowing-up $b : \overline{T} \to T$ along the ideal sheaf $\mathcal{I}_F \subset \mathcal{O}_T$ of $F \subset T$ such that the isomorphic image $X \cong b^{-1}(X) \subset \overline{T}$ is the complement of an effective ample Cartier divisor on $\overline{T}$. Hence the reduced divisor $\overline{D}$ on $\overline{T}$ supported by $\overline{T} - X$ is big (cf. [Ko-Mo98, Lemma 2.60]). We can write $b^*(D) = \overline{D} + \overline{G}$, where $\overline{G}$ is an effective divisor such that $\mathrm{Supp}\,(\overline{G}) \subseteq \mathrm{Exc}(b)$. Thus $b^*(D)$ is nef and big. (Note that D is nef by the assumption.) Then we can see that $(D^3) = b^*(D)^3 > 0$ and D is nef and big (cf. [*ibid*; Proposition 2.61]). □

In the case where $\overline{\kappa}(X) = -\infty$, we have:

Lemma 2.2 *If $\overline{\kappa}(X) = -\infty$, then $K_T + D$ is not nef.*

Proof. Assume to the contrary that $K_T + D$ is nef. Then, by the Non-vanishing Theorem (cf. [Ko-Mo98, Theorem 3.4]), it follows that $|m(K_T + D)| \neq \emptyset$ for $m >> 0$. This is a contradiction to $\overline{\kappa}(X) = \kappa(T; K_T + D) = -\infty$. □

On the other hand, in the case $\overline{\kappa}(X) \geq 0$, we ask whether or not $K_T + D$ is nef. If $K_T + D$ is already nef, then we have nothing to do in order to obtain Theorem 1.2. Hence we may and shall assume that $K_T + D$ is not nef.

We put $t^0 := \sup\{\mu \in \mathbb{Q}\ ; D + \mu K_T \text{ is nef}\}$. By noting that D is nef (the assumption (†)) and that $K_T + D$ is not nef, we have $0 \leq t^0 < 1$. Then there exists an extremal ray $R^0 = \mathbb{R}_+[l^0] \subset \overline{\mathrm{NE}}\,(T)$ on the hyperplane $\{D + t^0 K_T\}^{\perp}$ (cf. [Ke-M-M94]). Let $g^0 : T^0 := T \cdots \to T^1$ be the rational map associated with this ray R^0. (Note that since T is a smooth projective threefold, g^0 can not be a flip, see [C-K-M88]. But, for the inductive argument performed in the subsequent, we do not exclude the case that g^0 is a flip in Lemmas 2.3, 2.4, 2.5, 2.6.) Then we have the following:

Lemma 2.3 (1) *In the case $\overline{\kappa}(X) = -\infty$, g^0 is birational unless it does not yield a Mfs.*

(2) *In the case $\overline{\kappa}(X) \geq 0$, g^0 is birational.*

Proof. The assertion (1) is clear. We shall show (2). Assume to the contrary that g^0 is not birational. Then g^0 gives rise to a Mfs. By the construction it follows that $D + t^0 K_T = (g^0)^*(A)$ for some $\mathbb{Q}$-Cartier divisor A on T^1 (cf. [Ko-Mo98]). Then we have $(D + K_T \cdot l^0) = \left((g^0)^*(A) + (1 - t^0)K_T\right) \cdot l^0 = (1 - t^0)(K_T \cdot l^0) < 0$ as $t^0 < 1$. On the other hand, since g^0 is a Mfs and $\overline{\kappa}(X) = \kappa(T; K_T + D) \geq 0$, we have $(K_T + D \cdot l^0) \geq 0$. This is a contradiction. □

Thus we may and shall assume that $g^0 : T \cdots \to T^1$ is birational. (If g^0 yields a Mfs, then there is nothing to do for the proof of Theorem 1.2. Note that this situation can occur only for the case $\overline{\kappa}(X) = -\infty$, cf. Lemma 2.3.) Let $\mathcal{H}^1 := (g^0)_*(\mathcal{H})$ be the proper transform of $\mathcal{H}$ by g^0. Let $D^1 := (g^0)_*(D)$ and $S^1 := (g^0)_*(S)$ be the members of $\mathcal{H}^1$ corresponding to D and S, respectively, where $S \in \mathcal{H}$ is a smooth member (see the assumption (†) in Section 1). We have the following:

Lemma 2.4 (1) *T^1 is a normal projective threefold with only $\mathbb{Q}$-factorial, terminal singularities.*

(2) *If $t^0 = 0$, then the exceptional set E^0 of $T \cdots \to T^1$ (E^0 is an exceptional divisor (resp. a union of the flipping curves) if g^0 is a divisorial contraction (resp. a flip)) is contained in the boundary* $\mathrm{Supp}\,(D)$.

(3) *If $t^0 > 0$, then $t^0 = 1/2$, $\mathcal{H} \cdot l^0 = 1$ and g^0 is an E2-type divisorial contraction, namely, the exceptional divisor E^0 satisfies $(E^0, E^0|_{E^0}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$ which is contracted to a smooth point, where l^0 is a line on $E^0 \cong \mathbb{P}^2$.*

(4) $\mathcal{H}^1$ *is a nef, Cartier linear system on* T^1*, and* S^1 *is a smooth member of* $\mathcal{H}^1$*.*

(5) $\mathcal{H}^1 + t^0 K_{T^1}$ *is nef.*

Proof. The assertion (1) is obvious. Assume that $t^0 = 0$. Then $D \cdot l^0 = 0$. Note that l^0 can not be contained in $X = T - \mathrm{Supp}\,(D)$ since X is affine. Therefore, $D \cdot l^0 = 0$ implies that $l^0 \subset \mathrm{Supp}\,(D)$. Thus (2) is proved. Assume that $t^0 > 0$. Then we have $D \cdot l^0 = t^0(-K_T \cdot l^0) < (-K_T \cdot l^0)$. Hence $(-K_T \cdot l^0) > 1$, and this implies that $g^0 : T \cdots \to T^1$ is a divisorial contraction of $(2,0)$-type, i.e., the exceptional divisor E^0 is contracted to a point, say $p^1 := g^0(E^0)$ (cf. [C-K-M88], [C-F93, (2.1)Lemma]). Let $B := S \cap E^0$ be the intersection curve. Note that B is connected as B is ample on E^0. We have $(K_S \cdot B_0)_S = (K_T \cdot B_0) + (D \cdot B_0) < 0$ for any irreducible component $B_0 \subset B$, hence $B_0 = B$ is a (-1)-curve on S. By Proposition 2.1, it follows that g^0 is realized as the weighted blow-up with $\mathrm{wt}(x, y, z) = (1, 1, b)$, where (x, y, z) are the local analytic coordinates at $p^1 \in T^1$ and $b \in \mathbb{N}$. If $b \geq 2$, then T has a terminal singularity of type $\frac{1}{b}(1, 1, -1)$, which is a contradiction. Thus $b = 1$, i.e., g^0 is an $E2$-type divisorial contraction. Since $D \cdot l^0 = t^0(-K_T \cdot l^0) = 2t^0$ with $0 < t^0 < 1$, it follows that $t^0 = 1/2$ and $D \cdot l^0 = 1$ as stated in (3). We shall prove the assertion (4). We consider according to the value t^0, separately. Note that since S is a smooth Cartier divisor, it is disjoint from the singular loci $\mathrm{Sing}\,(T)$. At first, we consider:

$\boxed{\text{CASE: } t^0 = 0}$

First we consider the case $t^0 = 0$. Then g^0 is either a divisorial contraction or a flip. If g^0 is a divisorial contraction of $(2,0)$-type, $S \cdot l^0 = 0$ means that S is disjoint from the exceptional divisor. Hence $g^0|_S : S \to S^1$ is an isomorphism and S^1 is a smooth Cartier divisor. If g^0 is divisorial of $(2,1)$-type, the equality $S \cdot l^0 = 0$ means that $S \cap E^0$ is either empty or a union of several fibers of $g^0|_{E^0} : E^0 \to g^0(E^0)$, where $E^0 := \mathrm{Exc}(g^0)$. Thus $g^0|_S : S \to S^1$ is either an isomorphism or a contraction of several disjoint (-1)-curves, so S^1 is a smooth Cartier divisor on T^1. In any case, by the construction, we have $(g^0)^*(S^1) = S$. Let C be an irreducible curve on T^1, and let $\widetilde{C}$ be a curve on T such that $(g^0)_*(\widetilde{C}) = C$. Then we have $S^1 \cdot C = ((g^0)^*(S^1) \cdot \widetilde{C}) = S \cdot \widetilde{C} \geq 0$ as $S \in \mathcal{H}$ is nef. On the other hand, if g^0 is a flip, then the equality $S \cdot l^0 = 0$ and $S \cap \mathrm{Sing}(T) = \emptyset$ implies that S is disjoint from the flipping curves because each flipping curve has to pass through some of the singularities. Therefore $g^0|_S : S \to S^1$ is an isomorphism and S^1 is a smooth Cartier divisor on T^1. We shall show that $S^1 \in \mathcal{H}^1$ is nef. For this purpose, we consider the common resolution as in Figure 1:

$$\begin{tikzcd} & U \arrow[dl, "p"'] \arrow[dr, "q"] & \\ T \arrow[rr, dashed, "g^0"] & & T^1 \end{tikzcd}$$

Figure 1

Let $\{F_j\}$ be the set of all exceptional divisors. Note that since g^0 is an isomorphism in codimension one, each F_j is p-exceptional and q-exceptional. As usual, we write:

$$p^*(S) = S_U + \sum \alpha_j F_j,$$

$$q^*(S^1) = S_U + \sum \beta_j F_j,$$

where S_U is the proper transform on U of S and S^1. Hence we have:

$$p^*(S) - q^*(S^1) \equiv \sum (\alpha_j - \beta_j) F_j,$$

which is q-nef by noting that $S \in \mathcal{H}$ is nef. Therefore it follows that $\alpha_j \leq \beta_j$ $(\forall j)$ by the Negativity Lemma (cf. [FA]). Let $C \subset T^1$ be an irreducible curve. If C is distinct from a flipped curve, we take a curve $\widetilde{C} \subset U$ so that $q_*(\widetilde{C}) = C$. Then we have:

$$S^1 \cdot C = q^*(S^1) \cdot \widetilde{C} = \Big(p^*(S) + \sum(\beta_j - \alpha_j)F_j\Big) \cdot \widetilde{C} \geq S \cdot p_*(\widetilde{C}).$$

Note that $S \cdot p_*(\widetilde{C}) \geq 0$ as $S \in \mathcal{H}$ is nef. On the other hand, if C is one of the flipped curve, then we have $S^1 \cdot C = 0$ since S is disjoint from the flipping curves as seen before. Hence $S^1 \in \mathcal{H}^1$ is nef as desired.

Next we consider:

$\boxed{\text{CASE: } t^0 > 0}$

Then we have $t^0 = 1/2$, $S \cdot l^0 = 1$ and g^0 is an $E2$-type divisorial contraction by the argument in the proof of (3), where l^0 is a line on the exceptional divisor E^0 with $(E^0, E^0|_{E^0}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$. Hence S meets E^0 along a line on it. Thus $g^0|_S : S \to S^1$ is a contraction of a (-1)-curve $S \cap E^0$, so S^1 is a smooth Cartier divisor. By the construction of the contraction g^0, we have $(g^0)^*(S^1) = S + E^0$. Let C be an irreducible curve on T^1 and let $\widetilde{C} \subset T$ be such that $(g^0)_*(\widetilde{C}) = C$. Then $S^1 \cdot C = \big((g^0)^*(S^1) \cdot \widetilde{C}\big) = (S + E^0 \cdot \widetilde{C}) \geq (S \cdot \widetilde{C}) \geq 0$. Hence $S^1 \in \mathcal{H}^1$ is nef.

Finally we prove (5). In the case $t^0 = 0$, the assertion is obvious as $\mathcal{H}^1$ is nef. In the case $t^0 > 0$, it follows that g^0 is divisorial and $(g^0)^*(\mathcal{H}^1 + t^0 K_{T^1}) = \mathcal{H} + t^0 K_T$ by the construction. Let C be an irreducible curve on T^1 and let $\widetilde{C} \subset T$ be such that $(g^0)_*(\widetilde{C}) = C$. Then $(\mathcal{H}^1 + t^0 K_{T^1}) \cdot C = (g^0)^*(\mathcal{H}^1 + t^0 K_{T^1}) \cdot \widetilde{C} = (\mathcal{H} + t^0 K_T) \cdot \widetilde{C} \geq 0$ as $\mathcal{H} + t^0 K_T$ is nef. □

Lemma 2.5 *With the above notation, we put $X^1 := T^1 - \mathrm{Supp}\,(D^1)$. Then:*

(1) *If $t^0 = 0$, then $X^1 \cong X$.*

(2) *If $t^0 > 0$ (in fact $t^0 = 1/2$), then X^1 is an open affine subset of X. More precisely, X is obtained as a half-point attachment to X^1 of $(1,1)$-type (cf. Definition 1.1) and $X - X^1 \cong \mathbb{A}^2$ unless $X \cong X^1$.*

Proof. (1) Let us assume that $t^0 = 0$. Then the exceptional set E^0 of $T \cdots \to T^1$ is contained in the boundary $\mathrm{Supp}(D)$ as seen in the proof of Lemma 2.4. If g^0 is divisorial, then the assertion is obvious to see. On the other hand, if g^0 is a flip, then E^0 is composed of the flipping curves, say $E^0 = \gamma_1 \cup \cdots \cup \gamma_a$, where each γ_j is a flipping curve. Since $D \cdot \gamma_j = 0$, we have $D^1 \cdot \gamma_j^+ = 0$, where γ_j^+ is the flipped curve corresponding to γ_j. Hence either $\gamma_j^+ \cap D^1 = \emptyset$ or $\gamma_j^+ \subset \mathrm{Supp}\,(D^1)$ occurs. Assume that some of the flipped curves, say $\gamma_{j_1}^+, \cdots, \gamma_{j_t}^+$, are not contained in $\mathrm{Supp}\,(D^1)$. Then X is embedded into T^1 in such a way that $T^1 - X = \mathrm{Supp}\,(D_1) \cup \gamma_{j_1}^+ \cup \cdots \cup \gamma_{j_t}^+$, which is not of pure codimension one. This is a contradiction. Thus all flipped curves are contained in $\mathrm{Supp}\,(D^1)$, so we have $X^1 \cong X$ as desired.

(2) Assume that $t^0 > 0$. Then $t^0 = 1/2$, $D \cdot l^0 = 1$ and g^0 is a divisorial contraction with the exceptional divisor E^0 satisfying $(E^0, E^0|_{E^0}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$. If E^0 is contained in $\mathrm{Supp}\,(D)$, then $X^1 \cong X$ obviously. On the other hand, if E^0 is not contained in $\mathrm{Supp}\,(D)$, the equality $D \cdot l^0 = 1$ means that D meets E^0 in such a way that $D|_{E^0} \in |\mathcal{O}_{\mathbb{P}^2}(1)|$. Then it is not difficult to see that X is obtained as a half-point attachment to X^1 of $(1,1)$-type and X^1 is an open affine subset of X such that $X - X^1 \cong \mathbb{A}^2$. □

Concerning the log Kodaira dimension, we obtain the following:

Lemma 2.6 $\kappa(T^1; K_{T^1} + D^1) = \kappa(T; K_T + D) = \overline{\kappa}(X)$.

Proof. We shall prove the assertion only for the case $\overline{\kappa}(X) \geq 0$. The assertion for the case $\overline{\kappa}(X) = -\infty$ also is proved by the similar argument, so we shall omit the detail. As in the proofs of the previous lemmas, we consider according to the value of t^0.

$\boxed{\text{CASE: } t^0 = 0}$

In the case where g^0 is a flip, it is clear that members of $|m(K_T + D)|$ correspond to those of $|m(K_{T^1} + D^1)|$ bijectively for all $m \in \mathbb{N}$ since $g^0 : T \cdots \to T^1$ is an isomorphism in codimension one, hence we have $\kappa(T; K_T + D) = \kappa(T^1; K_{T^1} + D^1)$. In the case where g^0 is divisorial, we have $(g^0)^*(K_{T^1} + D^1) = K_T + D - aE^0$, where $a > 0$ is the discrepancy of E^0, i.e., $a := a(E^0; K_{T^1})$. Hence, for any sufficiently large positive integer $m \in \mathbb{N}$ with $ma \in \mathbb{N}$, maE^0 is contained in the fixed part of $|m(K_T + D)|$. Therefore, we have $\kappa(T; K_T + D) = \kappa(T^1; K_{T^1} + D^1)$ as desired.

$\boxed{\text{CASE: } t^0 > 0}$

Then $t^0 = 1/2$, $D \cdot l^0 = 1$ and $(g^0)^*(K_{T^1} + D^1) = K_T + D - E^0$. Hence mE^0 is contained in the fixed part of $|m(K_T + D)|$. Therefore $\kappa(T^1; K_{T^1} + D^1) = \kappa(T; K_T + D)$. Thus we obtain the assertion. □

Summarizing the above mentioned arguments, we have:

$(\mathrm{i})_1$ T^1 : a normal projective threefold with only $\mathbb{Q}$-factorial terminal singularities.

$(\mathrm{ii})_1$ $\mathcal{H}^1$: a nef, Cartier linear system on T^1.

$(\mathrm{iii})_1$ D^1 : the proper transform of D on T^1, which is a member of $\mathcal{H}^1$ and $\kappa(T^1; K_{T^1} + D^1) = \kappa(T; K_T + D) = \overline{\kappa}(X)$.

$(\mathrm{iv})_1$ S^1 : the proper transform of S on T^1, which is a smooth member of $\mathcal{H}^1$.

$(\mathrm{v})_1$ t^0 : the rational number with $0 \leq t^0 < 1$ such that $\mathcal{H}^1 + t^0 K_{T^1}$ is nef.

$(\mathrm{vi})_1$ $X^1 := T^1 - \mathrm{Supp}\,(D^1)$: an open affine subset of X. X is obtained as a half-point attachment to X^1 of $(1,1)$-type and $X - X^1 \cong \mathbb{A}^2$ unless $X \cong X^1$ (cf. Definition 1.1).

In order to proceed inductively, assume that we obtain a birational map $g^{i-1} : T^{i-1} \cdots \to T^i$ so that the following properties $(\mathrm{i})_i, (\mathrm{ii})_i, (\mathrm{iii})_i, (\mathrm{iv})_i$, $(\mathrm{v})_i$ and $(\mathrm{vi})_i$ are satisfied:

$(\mathrm{i})_i$ T^i : a normal projective threefold with only $\mathbb{Q}$-factorial terminal singularities.

$(\mathrm{ii})_i$ $\mathcal{H}^i$: a nef, Cartier linear system on T^i.

$(\mathrm{iii})_i$ D^i : the proper transform of D^{i-1} on T^i, which is a member of $\mathcal{H}^i$ and $\kappa(T^i; K_{T^i} + D^i) = \kappa(T^{i-1}; K_{T^{i-1}} + D^{i-1}) = \overline{\kappa}(X)$.

$(\mathrm{iv})_i$ S^i : the proper transform of S^{i-1} on T^i, which is a smooth member of $\mathcal{H}^i$.

$(\mathrm{v})_i$ t^{i-1} : the rational number with $0 \leq t^{i-1} < 1$ such that $\mathcal{H}^i + t^{i-1} K_{T^i}$ is nef.

$(\mathrm{vi})_i$ $X^i := T^i - \mathrm{Supp}\,(D^i)$: an open affine subset of X^{i-1}. X^{i-1} is obtained as a half-point attachment to X^i of (b^{i-1}, k^{i-1})-type for some $b^{i-1} \geq 1$, $1 \leq k^{i-1} \leq b^{i-1}$ and $X^{i-1} - X^i \cong \mathbb{C}^{(k^{i-1}-1)*} \times \mathbb{A}^1$ unless $X^{i-1} \cong X^i$ (cf. Definition 1.1).

Lemma 2.7 *Assume that the conditions* $(\mathrm{i})_i, (\mathrm{ii})_i, (\mathrm{iii})_i, (\mathrm{iv})_i$, $(\mathrm{v})_i$ *and* $(\mathrm{vi})_i$ *are satisfied. Then:*

(1) *If* $\overline{\kappa}(X) = -\infty$, *then* $K_{T^i} + D^i$ *is not nef and one of the following* (*a*) *and* (*b*) *holds:*

(a) T^i *has the structure of a Mfs,*

(b) *there exists a birational map* $g^i : T^i \cdots \to T^{i+1}$ *such that* T^{i+1} *and the proper transforms* $\mathcal{H}^{i+1} := (g^0)_*(\mathcal{H}^i)$, $D^{i+1} := (g^0)_*(D^i)$, $S^{i+1} := (g^0)_*(S^i)$ *satisfy the properties* $(\mathrm{i})_{i+1}$, $(\mathrm{ii})_{i+1}$, $(\mathrm{iii})_{i+1}$, $(\mathrm{iv})_{i+1}$ $(\mathrm{v})_{i+1}$, $(\mathrm{vi})_{i+1}$.

(2) *If* $\overline{\kappa}(X) \geq 0$ *and* $K_{T^i} + D^i$ *is not nef, then the same conclusion as in (1)-(b) holds true.*

Proof. Note that D^i is nef and big by the similar reason as in the proof of Lemma 2.1.

(1) Assume that $\overline{\kappa}(X) = -\infty$. By the inductive hypothesis $(\mathrm{iii})_i$, we have $\kappa(T^i; K_{T^i} + D^i) = -\infty$. If $K_{T^i} + D^i$ is nef, then it follows that $|m(K_{T^i} + D^i)| \neq \emptyset$ for $m >> 0$ with $r^i|m$ by the Non-vanishing Theorem (cf. [Ko-Mo98, Theorem 3.4]), where $r^i \in \mathbb{N}$ is the Gorenstein index of T^i. This is a contradiction to $\kappa(T^i; K_{T^i} + D^i) = -\infty$. Thus $K_{T^i} + D^i$ is not nef. We put $t^i := \sup\{\mu \in \mathbb{Q}; D^i + \mu K_{T^i} \text{ is nef}\}$, which satisfies $0 \leq t^i < 1$ Note that $t^{i-1} \leq t^i$. Then there exists an extremal ray $R^i = \mathbb{R}_+[l^i] \subset \overline{\mathrm{NE}}\,(T^i) \cap \{D^i + t^i K_{T^i}\}^{\perp}$ (cf. [Ke-M-M94]). Let $g^i : T^i \cdots \to T^{i+1}$ be the rational map associated to the ray R^i. Assume that g^i does not yield a Mfs. Then g^i is either a divisorial contraction or a flip. Let $\mathcal{H}^{i+1}$ (resp. D^{i+1}, S^{i+1}) be the proper transform on T^{i+1} of $\mathcal{H}^i$ (resp. D^i, S^i). We consider according to the value of t^i, separately.

$\boxed{\text{CASE: } t^i = 0}$

Then $D^i \cdot l^i = S^i \cdot l^i = 0$. Since the complement $T^i - \mathrm{Supp}\,(D^i) = X^i$ is affine, the equality $D^i \cdot l^i = 0$ implies that $l^i \subset \mathrm{Supp}\,(D^i)$. Hence the exceptional set, say E^i, of $g^i : T^i \cdots \to T^{i+1}$ is contained in the boundary $\mathrm{Supp}\,(D^i)$. The similar argument as in the proof of Lemma 2.4 ascertains that $\mathcal{H}^{i+1}$ is a nef, Cartier linear system on T^{i+1} containing S^{i+1} as a smooth member. Thus we can verify the conditions $(\mathrm{i})_{i+1}$, $(\mathrm{ii})_{i+1}$, $(\mathrm{iv})_{i+1}$ are satisfied. The property $(\mathrm{v})_{i+1}$ is obvious as $\mathcal{H}^{i+1}$ is nef. In the case g^i is a divisorial contraction, it is clear that $X^{i+1} \cong X^i$ because the exceptional divisor E^i is contained in $\mathrm{Supp}(D^i)$. Moreover, it follows that $(g^i)^*(K_{T^{i+1}} + D^{i+1}) = K_{T^i} + D^i - aE^i$, where $a := a(E^i; K_{T^{i+1}}) > 0$ is the discrepancy of E^i. Hence we can easily see that $\kappa(T^{i+1}; K_{T^{i+1}} + D^{i+1}) = \kappa(T^i; K_{T^i} + D^i) = \overline{\kappa}(X)$. Thus we obtain $(\mathrm{iii})_{i+1}$ and $(\mathrm{vi})_{i+1}$. On the other hand, in the case where g^i is a flip, it follows that all the flipping curves (resp. the flipped curves) are contained in $\mathrm{Supp}(D^i)$ (resp. $\mathrm{Supp}(D^{i+1})$) by the same reason as in the proof of Lemma 2.5. Hence $X^{i+1} \cong X^i$. Since $g^i : T^i \cdots \to T^{i+1}$ is an isomorphism in codimension one, the members in $|m(K_{T^i} + D^i)|$ correspond to those in $|m(K_{T^{i+1}} + D^{i+1})|$ bijectively. Therefore $\kappa(T^{i+1}; K_{T^{i+1}} + D^{i+1}) = \kappa(T^i; K_{T^i} + D^i) = \overline{\kappa}(X)$, and we obtain $(\mathrm{iii})_{i+1}$, $(\mathrm{vi})_{i+1}$ as desired.

$\boxed{\text{CASE: } t^i > 0}$

Then we have $D^i \cdot l^i = S^i \cdot l^i = t^i(-K_{T^i} \cdot l^i) < (-K_{T^i} \cdot l^i)$, so $(-K_{T^i} \cdot l^i) > 1$ and g^i is a divisorial contraction of $(2,0)$-type (cf. [C-K-M88], [C-F93]). Let E^i denote the exceptional divisor of g^i and put $p^{i+1} := g^i(E^i)$. Let $B := S^i \cap E^i$ be the intersection curve. Note that since B is ample on E^i, B is connected. For any irreducible component $B_0 \subset B$, we have $(K_{S^i} \cdot B_0)_{S^i} = (K_{T^i} + D^i) \cdot B_0 < 0$. Hence $B_0 = B$ is a (-1)-curve on S^i. By Proposition 2.1, we know that $g^i : T^i \to T^{i+1}$ is realized as the weighted blow-up at $p^{i+1} \in T^{i+1}$ with $\mathrm{wt}(x,y,z) = (1,1,b^i)$, where (x,y,z) are the suitable local analytic coordinates around p^{i+1} and $b^i \in \mathbb{N}$. Moreover, we have $S^i \cdot l^i = 1$ and $(-K_{T^i} \cdot l^i) = 1 + (1/b^i)$, where l^i is a ruling on $E^i \cong \mathbb{P}(1,1,b^i) \cong \mathbb{F}_{b^i,0}$. Hence $t^i = b^i/(b^i+1)$. By the similar argument as in the proof of Lemma 2.4, it follows that $\mathcal{H}^{i+1}$ is a nef, Cartier linear system on T^{i+1} containing S^{i+1} as a smooth member. Thus we obtain the properties $(\mathrm{i})_{i+1}$, $(\mathrm{ii})_{i+1}$, $(\mathrm{iv})_{i+1}$. Note that $(g^i)^*(\mathcal{H}^{i+1} + t^i K_{T^i}) = \mathcal{H}^i + t^i K_{T^i}$. Hence $\mathcal{H}^{i+1} + t^i K_{T^i}$ is nef, which is the property $(\mathrm{v})_{i+1}$. By the construction, we have $(g^i)^*(K_{T^{i+1}} + D^{i+1}) = K_{T^i} + D^i - E^i$, so we can easily see that $\kappa(T^{i+1}; K_{T^{i+1}} + D^{i+1}) = \kappa(T^i; K_{T^i} + D^i) = \overline{\kappa}(X)$, the property $(\mathrm{iii})_{i+1}$. If E^i is contained in $\mathrm{Supp}\,(D^i)$, then $X^{i+1} \cong X^i$ obviously, hence we have $(\mathrm{vi})_{i+1}$. On the other hand, we claim the following:

Claim. If $E^i \not\subset \mathrm{Supp}\,(D^i)$, then X^i is obtained as a half-point attachment to X^{i+1} of (b^i, k^i)-type for some $1 \leq k^i \leq b^i$ (cf. Definition 1.1).

Proof of Claim. It is not difficult to see that $(g^i)^*(D^{i+1}) = D^i + b^i E^i$ and $D^i|_{E^i} \equiv b^i l^i$, where l^i is a ruling on $E^i \cong \mathbb{F}_{b^i,0}$. Assume that $D^i|_{E^i}$ is an irreducible curve which does not pass through the vertex $v \in E^i \cong \mathbb{F}_{b^i,0}$ for $b^i \geq 2$. Then the complement $X^i = T^i - \mathrm{Supp}\,(D^i)$ has the terminal singularity $v \in X^i$ of analytic type $\frac{1}{b^i}(1,1,-1)$. This is a contradiction as X^i is smooth by noting the conditions $(\mathrm{vi})_1, \cdots, (\mathrm{vi})_i$. Thus we can write $D^i|_{E^i} = \sum_{j=1}^{k^i} m_j l_j$ with $\sum_{j=1}^{k^i} m_j = b^i$ for some $1 \leq k^i \leq b^i$, where l_j's are the mutually distinct rulings on $E^i \cong \mathbb{F}_{b^i,0}$. Thus we obtain the assertion. □

Thus we obtain the property $(\mathrm{vi})_{i+1}$.

(2) Assume that $\overline{\kappa}(X) \geq 0$ and $K_{T^i} + D^i$ is not nef. We put $t^i := \sup\{\mu \in \mathbb{Q}; D^i + \mu K_{T^i} \text{ is nef}\}$ as above. Note that $0 \leq t^i < 1$ and $t^{i-1} \leq t^i$. Then there exists an extremal ray $R^i = \mathbb{R}_+[l^i] \subset \overline{\mathrm{NE}}(T^i) \cap \{D^i + t^i K_{T^i}\}^{\perp}$ (cf. [Ke-M-M94]) and we denote by $g^i : T^i \cdots \rightarrow T^{i+1}$ the rational map associated to this ray R^i. By the same argument as in Lemma 2.3 (2), it follows that g^i is birational. Then we can verify the assertion (2) by the same argument as in the proof of (1). □

Hence we obtain the following result by the inductive argument:

Lemma 2.8 *We obtain the following diagram:*

$$(*) \quad (T, \mathcal{H}) \overset{g^0}{\cdots\rightarrow} (T^1, \mathcal{H}^1) \overset{g^1}{\cdots\rightarrow} \cdots\cdots\cdots \rightarrow (T^{r-1}, \mathcal{H}^{r-1}) \overset{g^{r-1}}{\cdots\rightarrow} (T^r, \mathcal{H}^r) =: (T^\sharp, \mathcal{H}^\sharp),$$

where each $g^i : T^i \cdots \rightarrow T^{i+1}$ is a birational map associated to the extremal ray $R^i = \mathbb{R}_+[l^i]$ contained in $\overline{\mathrm{NE}}(T^i) \cap \{\mathcal{H}^i + t^i K_{T^i}\}^{\perp}$, and T^i, $\mathcal{H}^i$, D^i, S^i satisfy the conditions $(\mathrm{i})_i$, $(\mathrm{ii})_i$, $(\mathrm{iii})_i$, $(\mathrm{iv})_i$, $(\mathrm{v})_i$, $(\mathrm{vi})_i$. More precisely, the following assertions hold true:

(1) *Let $E^i \subset T^i$ be the exceptional set of $g^i : T^i \cdots \rightarrow T^{i+1}$. Then g^i is described as one of the following:*

  (a) *$t^i = 0$ and g^i is a flip, $E^i \subset \mathrm{Supp}\,(D^i)$ and $E^{i+} \subset \mathrm{Supp}\,(D^{i+1})$, where E^i (resp. E^{i+}) is a union of all flipping curves (resp. flipped curves).*

  (b) *$t^i = 0$ and g^i is a divisorial contraction such that $E^i \subset \mathrm{Supp}\,(D^i)$.*

  (c) *g^i is a $(2,0)$-type divisorial contraction which contracts an exceptional divisor E^i satisfying $(E^i, E^i|_{E^i}) \cong (\mathbb{F}_{b^i,0}, -l^i)$ to a smooth point $p^{i+1} := g^i(E^i)$, where l^i is a ruling on the cone $E^i \cong \mathbb{F}_{b^i,0}$. $g^i : T^i \rightarrow T^{i+1}$ is realized as the weighted blow-up at $p^{i+1} \in T^{i+1}$ with $\mathrm{wt}(x,y,z) = (1,1,b^i)$ for some $b^i \in \mathbb{N}$, where (x,y,z) are the local analytic coordinates around $p^{i+1} \in T^{i+1}$. In this case, $t^i = b^i/(b^i+1)$.*

(2) *If $\overline{\kappa}(X) = -\infty$, then $T^\sharp$ has the structure of a Mfs.*

(3) *If $\overline{\kappa}(X) \geq 0$, then $K_{T^\sharp} + D^\sharp$ is nef and $\kappa(T^\sharp; K_{T^\sharp} + D^\sharp) = \overline{\kappa}(X)$, where $D^\sharp$ is the proper transform of D on $T^\sharp$.*

Proof. By Lemma 2.7 (and the proof of it) and the fact that there is no sequence consisting of infinitely many flips (cf. [FA]), we obtain the desired diagram. □

We have the following concerning the nef value of $\mathcal{H}^\sharp$ on the right terminal object $T^\sharp$ in $(*)$.

Lemma 2.9 *Let $t^\sharp := t^r := \sup\{\mu \in \mathbb{Q}; \mathcal{H}^\sharp + \mu K_{T^\sharp} \text{ is nef}\}$ be the nef value. Then:*

(1) *If $\overline{\kappa}(X) = -\infty$, then $0 < t^\sharp < 1$.*

(2) *If $\overline{\kappa}(X) \geq 0$, then $t^\sharp \geq 1$.*

Proof. (1) Assume that $\overline{\kappa}(X) = -\infty$. Then $T^\sharp$ has a structure of a Mfs, say $\pi : T^\sharp \rightarrow W$. More precisely, this Mfs π is obtained as the contraction of the extremal ray, say $R^\sharp = \mathbb{R}_+[l^\sharp]$, contained in $\overline{\mathrm{NE}}(T^\sharp) \cap \{\mathcal{H}^\sharp + t^\sharp K_{T^\sharp}\}^{\perp}$. If $t^\sharp \geq 1$, then $K_{T^\sharp} + D^\sharp$ is nef and we have $|m(K_{T^\sharp} + D^\sharp)| \neq \emptyset$ for $m >> 0$ with $\mathrm{index}(T^\sharp)|m$ by the Non-vanishing Theorem (cf. [Ko-Mo98, Theorem 3.4]). This is absurd as $\kappa(T^\sharp; K_{T^\sharp} + \mathcal{H}^\sharp) = -\infty$ (the property $(\mathrm{iii})_r$). On the other hand, if $t^\sharp = 0$ then we have $\mathcal{H}^\sharp \cdot l^\sharp = D^\sharp \cdot l^\sharp = 0$ by the choice of the extremal ray $R^\sharp$. Since $T^\sharp - \mathrm{Supp}\,(D^\sharp) = X^r$ is affine, the equality $D^\sharp \cdot l^\sharp = 0$ means that $l^\sharp$ is contained in $\mathrm{Supp}(D^\sharp)$. Since π yields a Mfs, this is absurd obviously.

(2) The assertion (2) is clear because $K_{T^\sharp} + \mathcal{H}^\sharp$ is nef. □

Remark 2.1 Concerning the nef value $t^i = \sup\{\mu \in \mathbb{Q}; \mathcal{H}^i + \mu K_{T^i} \text{ is nef}\}$ for $0 \leq i < r$, the inequalities $0 \leq t^0 \leq t^1 \leq \cdots \leq t^{r-1} < 1$ hold true by the construction of the diagram $(*)$. Let $s \in \{0, 1, \cdots, r\}$ be the least integer such that $t^s > 0$. In case $s < r$, $g^i : T^i \rightarrow T^{i+1}$ is realized as the weighted blow-up at the smooth point $p^{i+1} := g^i(E^i) \in T^{i+1}$ with the weights $\mathrm{wt}(x,y,z) = (1,1,b^i)$, where (x,y,z) are the suitable local analytic coordinates around $p^{i+1} \in T^{i+1}$ and $b^i \in \mathbb{N}$. Note that $t^i = b^i/(b^i+1)$. Hence the inequalities $1 \le b^s \le b^{s+1} \le \cdots \le b^{r-1}$ hold true. Furthermore, we have $X \cong X^1 \cong \cdots \cong X^s \supset X^{s+1} \supset \cdots \supset X^r$ so that X^i is obtained as a half-point attachment to X^{i+1} of (b^i, k^i)-type for some $1 \le k^i \le b^i$ (cf. Definition 1.1) and $X^i - X^{i+1} \cong \mathbb{C}^{(k^i-1)*} \times \mathbb{A}^1$ unless $X^i \cong X^{i+1}$ for $s \le i < r$. If $\overline{\kappa}(X) = -\infty$, then we reach $T^\sharp$ with the structure of Mfs. We can, in fact, restrict the possibility of the weights $\mathrm{wt} = (1, 1, b^i)$ of the weighted blow-up $g^i : T^i \to T^{i+1}$ more concretely according to the type of Mfs on $T^\sharp$ (cf. Lemmas 3.2 and 3.3).

By combining Lemma 2.8 with Remark 2.1, we obtain Theorem 1.2.

When we prove Lemmas 2.4 and 2.7, we make use of the following result. This result may be well known to the experts but we were not able to find a reference. Hence we shall give the proof for the convenience of the reader.

Proposition 2.1 *Let $f : U \to \overline{U}$ be the divisorial contraction from a normal projective threefold U with only $\mathbb{Q}$-factorial terminal singularities, E the exceptional divisor of f. Assume that $(-K_U \cdot C) > 1$ for any irreducible curve C on E contracted to a point via f, and there exists an irreducible surface A on U meeting E in such a way that $B := A \cap E$ is a (-1)-curve on A. Then E is contracted to a smooth point, say $p := f(E) \in \overline{U}$ and f is realized as the weighted blow-up at p with weights $\mathrm{wt}(x, y, z) = (1, 1, b)$, where (x, y, z) are the suitable local analytic coordinates at the point $p \in \overline{U}$ and $b \in \mathbb{N}$. In particular, $A \cdot E = B$ (scheme-theoretically), $E \cong \mathbb{P}(1, 1, b) \cong \mathbb{F}_{b,0}$, $(-K_U \cdot l) = 1 + (1/b)$ and $(A \cdot l) = 1$, where l is a ruling on the cone $E \cong \mathbb{F}_{b,0}$.*

Proof. First of all, note that f is of $(2, 0)$-type as $(-K_U \cdot C) > 1$ for any irreducible curve contracted to a point by f (cf. [C-F93, (2.1)Lemma]). Put $p := f(E) \in \overline{U}$ and $\overline{A} := f(A)$. By the assumption, $f|_A : A \to \overline{A}$ is the contraction of the (-1)-curve, say $B := A \cap E$. Hence $\overline{A}$ is smooth at p. Then, by [Me97, Lemma 1.7] (see also [FA, Lemma 5.3]), p is a smooth point of $\overline{U}$. Hence $f : U \to \overline{U}$ is realized as the weighted blow-up at the point $p \in \overline{U}$ with weights $\mathbf{w} := \mathrm{wt}(x, y, z) = (1, a, b)$, where (x, y, z) are the suitable local analytic coordinates at $p \in \overline{U}$ and a, b are the positive integers such that $\gcd(a, b) = 1$ by the result due to Kawakita [Ka01]. We may and shall assume that $a \le b$. We can write $K_U = f^*(K_{\overline{U}}) + (a + b)E$, where $E \cong \mathbb{P}(1, a, b)$. Let $h \in \widehat{\mathcal{O}_p} \cong \mathbb{C}[[x, y, z]]$ be the local equation of $\overline{A}$ in the neighbourhood of $p \in \overline{U}$. Since $\overline{A}$ is smooth at p, h contains a linear term. Hence the multiplicity $m := \mathbf{w}\text{-mult}_p(h) \in \mathbb{N}$ of $\overline{A}$ at p with respect to the weights $\mathbf{w} = (1, a, b)$ satisfies $m \le b$. We can write $f^*(\overline{A}) = A + mE$ and $K_A = (K_U + A)|_A = (f|_A)^*(K_{\overline{A}}) + (a + b - m)E|_A$ by the adjunction. Hence it follows that $a + b - m = 1$ and $E \cdot A = B$ scheme-theoretically. Thus we have $m = a + b - 1 \le b$. This implies that $a = 1$, and $f : U \to \overline{U}$ is the weighted blow-up at p with weights $\mathrm{wt}(x, y, z) = (1, 1, b)$ as desired. The remaining assertions are easy to see. □

Remark 2.2 Concerning Proposition 2.1, the statement stated in [Me02, Proposition 3.6, Case 3.9] is not correct. In fact, M. Mella asserted, under the same conditions as in Proposition 2.1, that $f : U \to \overline{U}$ is the ordinary blow-up of $\overline{U}$ at the smooth point $p \in \overline{U}$.

3 The Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3. Once we have obtained Theorem 1.2, the essential part of the proof is contained in [Me02, Section 5] (see also [C-F93]). Let X be a smooth affine threefold with log Kodaira dimension $\overline{\kappa}(X) = -\infty$. Assume that X is embedded into a smooth projective threefold T so that the reduced boundary divisor D is SNC and satisfies the condition (†). Then, by Theorem 1.2, we can construct the following diagram (∗):

$$(*) \quad T =: T^0 \overset{g^0}{\cdots\to} T^1 \overset{g^1}{\cdots\to} T^2 \cdots \to \cdots\cdots \to T^{s-1} \overset{g^{s-1}}{\cdots\to} T^s \overset{g^s}{-}$$

$$\to T^{s+1} \overset{g^{s+1}}{\to} T^{s+2} \to \cdots\cdots \to T^{r-1} \overset{g^{r-1}}{\to} T^r =: T^\sharp,$$

such that $T^\sharp$ has a structure of a Mfs, say $\pi : T^\sharp \to W$ which is obtained as the contraction of an extremal ray $R^\sharp = \mathbb{R}_+[l^\sharp] \subset \overline{\mathrm{NE}}(T^\sharp)$ contained in $\{\mathcal{H}^\sharp + t^\sharp K_{T^\sharp}\}^\perp$, where $\mathcal{H}^\sharp$ is the proper transform of $\mathcal{H}$ on $T^\sharp$ and $t^\sharp$ is the nef value associated to $\mathcal{H}^\sharp$ (cf. Lemma 2.9).

Lemma 3.1 *The pair $(T^\sharp, \mathcal{H}^\sharp)$ is described as one of the pairs stated in Theorem* 1.3 *(1).*

Proof. First of all, note that the nef value $t^\sharp$ satisfies $0 < t^\sharp < 1$ and $\mathcal{H}^\sharp$ contains a smooth member $S^\sharp$ (cf. Theorem 1.2, Lemma 2.9). Then we can obtain the assertion by the same argument as in the proof of [Me02, Theorem 5.3]. □

Thus we obtain Theorem 1.3 (1). We shall prove the second assertion.

Lemma 3.2 *We have the following concerning the process of the weighted blow-up $g^i : T^i \to T^{i+1}$ and the weights* $\mathrm{wt} = (1, 1, b^i)$ *of g^i for $s \le i < r$:*

(1) *If $\pi : T^\sharp \to W$ is of C_2-type, then $b^i = 1$, i.e., g^i is the ordinary blow-up at a smooth point. X is obtained from $X^\sharp$ via the composite of half-point attachments of $(1,1)$-type and $X - X^\sharp$ is a union of several affine planes $\mathbb{A}^2$ unless $X \cong X^\sharp$.*

(2) *If $\pi : T^\sharp \to W$ is of D_2'-type, then $b^i = 1$, i.e., g^i is the ordinary blow-up at a smooth point. X is obtained from $X^\sharp$ via the composite of half-point attachments of $(1,1)$-type and $X - X^\sharp$ is a union of several affine planes $\mathbb{A}^2$ unless $X \cong X^\sharp$.*

(3) *If $\pi : T^\sharp \to W$ is of D_3-type, then $s = r$, i.e., $X \cong X^\sharp$.*

(4) *If $\pi : T^\sharp \to W$ is of D_3'-type, then either $b^i = 1$ or $b^i = 2$ holds true. X is obtained from $X^\sharp$ via the composite of half-point attachments of $(1,1)$-type and $(2,k)$-type $(k = 1, 2)$ and each irreducible component of $X - X^\sharp$ is isomorphic to either $\mathbb{A}^2$ or $\mathbb{C}^* \times \mathbb{A}^1$ unless $X \cong X^\sharp$.*

Proof. Note that $\phi^i : T^i \to T^{i+1}$ is the weighted blow-up at smooth points with weights $\mathrm{wt} = (1, 1, b^i)$ for some $b^i \in \mathbb{N}$ and $t^i = b^i/(b^i + 1) \ge 1/2$ for $s \le i < r$. Furthermore, the inequalities $0 < t^s \le \cdots \le t^{r-1} \le t^r < 1$ hold true (cf. Remark 2.1).

(1) Let us consider the case where π is of C_2-type. Then $t^r = 1/2$. Hence we have $t^s = \cdots = t^{r-1} = t^r = 1/2$, i.e., $b^i = 1$ for $s \le \forall i < r$.

(2) We can obtain the assertion (2) by the same argument as in (1).

(3) Let us assume that π is of D_3-type, i.e., a $\mathbb{P}^2$-bundle structure over a smooth curve W. Then $t^r = \rho = 1/3$. If $s < r$, then $t^s = b^s/(b^s + 1) \ge 1/2$. But since $t^s \le t^r = 1/3$, this is a contradiction. Hence $s = r$ and $X \cong X^\sharp$ as desired.

(4) If π is of D_3'-type, we have $t^r = \rho = 2/3$. Since $t^i = b^i/(b^i + 1) \le t^r = 2/3$, it follows that b^i is equal to 1 or 2. □

We, furthermore, have to consider the case where $T^\sharp$ is a $\mathbb{Q}$-Fano threefold with $\varrho(T^\sharp) = 1$ to complete the proof of Theorem 1.3 (2). Since the arguments are very similar to each other (i) $\sim$ (xii) in Theorem 1.3, we shall consider the case (i) $(T^\sharp, \mathcal{H}^\sharp) \cong (\mathbb{P}(1,1,2,3), \mathcal{O}(6))$ only. Then we have the following:

Lemma 3.3 *Assume that $(T^\sharp, \mathcal{H}^\sharp) \cong (\mathbb{P}(1,1,2,3), \mathcal{O}(6))$. Then the weights* $\mathrm{wt} = (1, 1, b^i)$ *of the weighted blow-ups $g^i : T^i \to T^{i+1}$ for $s \le i < r$ satisfy the inequalities $1 \le b^s \le \cdots \le b^{r-1} \le 6$. X is obtained from $X^\sharp$ via the composite of half-point attachments of (b,k)-type with $1 \le b \le 6$, $1 \le k \le b$ and the corresponding irreducible component of $X - X^\sharp$ is isomorphic to $\mathbb{C}^{(k-1)*} \times \mathbb{A}^1$ unless $X \cong X^\sharp$.*

Proof. Since $(T^\sharp, \mathcal{H}^\sharp) \cong (\mathbb{P}(1,1,2,3), \mathcal{O}(6))$, we have $t^r = 6/7$. On the other hand, the inequalities $0 \le t^s \le \cdots \le t^{r-1} \le t^r = 6/7$ and $t^i = b^i/(b^i + 1)$ $(s \le i < r)$ hold true by the construction (cf. Theorem 1.2). Hence we have $b^i \le 6$. The remaining assertion is easy to see. □

By performing the similar argument as above to the other situations (ii) $\sim$ (xii) in Theorem 1.3, we finally complete the proof of Theorem 1.3 (2).

4 The Proof of Theorem 1.4

In this section, we prove Theorem 1.4 as a three-dimensional generalization of Kawamata's result (cf. Theorem 1.1 (2)). Let X be a smooth affine threefold with log Kodaira dimension $\overline{\kappa}(X)=2$. Assume that X is embedded into a smooth projective threefold T in such a way that the reduced boundary divisor D is an SNC divisor satisfying (†). Then, by Theorem 1.2, we obtain the following diagram $(*)$:

$$(*)\quad T=:T^0 \overset{g^0}{\dashrightarrow} T^1 \overset{g^1}{\dashrightarrow} T^2 \dashrightarrow \cdots\cdots \dashrightarrow T^{s-1} \overset{g^{s-1}}{\dashrightarrow} T^s \overset{g^s}{-}$$

$$\to T^{s+1} \xrightarrow{g^{s+1}} T^{s+2} \longrightarrow \cdots\cdots \longrightarrow T^{r-1} \xrightarrow{g^{r-1}} T^r =: T^\sharp,$$

so that $K_{T^\sharp}+D^\sharp$ is nef with $\kappa(T^\sharp; K_{T^\sharp}+D^\sharp)=2$, where $D^\sharp$ is the proper transform of D on $T^\sharp$. Concerning the complement $X^i := T^i - \mathrm{Supp}\,(D^i)$, we have $X=:X^0\cong X^1\cong\cdots\cong X^s\supset X^{s+1}\supset\cdots\supset X^{r-1}\supset X^r=:X^\sharp$, and $X^i - X^{i+1}\cong \mathbb{C}^{(k^i-1)*}\times\mathbb{A}^1$ for some $1\le\exists\, k^i\le b^i$ unless $X^i\cong X^{i+1}$ for $s\le i<r$ (cf. Theorem 1.2, Remark 2.1).

Lemma 4.1 $\mathrm{Bs}\,|m(K_{T^\sharp}+D^\sharp)|=\emptyset$ *for* $m>>0$ *with* $t|m$, *where* $t:=\mathrm{index}(T^\sharp)$ *is the Gorenstein index of* $T^\sharp$.

Proof. Note that $D^\sharp$ is nef and big by the same reason as in the proof of Lemma 2.1. Since $t(K_{T^\sharp}+D^\sharp)$ is a nef, Cartier divisor and $t(K_{T^\sharp}+D^\sharp)-K_{T^\sharp}=(t-1)(K_{T^\sharp}+D^\sharp)+D^\sharp$ is nef and big, it follows that $\mathrm{Bs}\,|m(K_{T^\sharp}+D^\sharp)|=\emptyset$ for $m>>0$ with $t|m$ by the Base Point Freeness Theorem (cf. [Ko-Mo98, Theorem 3.3]). □

Let $\Phi: T^\sharp\to W$ be the Stein factorization of the morphism defined by the base point free linear system $|m(K_{T^\sharp}+D^\sharp)|$. Since $\overline{\kappa}(X)=\kappa(T^\sharp;K_{T^\sharp}+D^\sharp)=2$, it follows that W is a normal projective surface. The general fibers of φ are one-dimensional.

Lemma 4.2 *The restriction of* $\Phi: T^\sharp\to W$ *onto the open affine subset* $X^\sharp(\subset T^\sharp)$ *yields a* $\mathbb{C}^*$*-fibration.*

Proof. Let $C\subset W$ be a general smooth curve and let $G:=\Phi^*(C)$ be the pull-back. Let l be a general fiber of $\Phi|_G: G\to C$. Then $(-K_G\cdot l)_G=(-K_{T^\sharp}\cdot l)=(D^\sharp\cdot l)\ge 0$ and $(l^2)_G=0$. If $(D^\sharp\cdot l)=0$ for a general fiber l, then l must be contained in $\mathrm{Supp}\,(D^\sharp)$ by noting that the complement $X^\sharp=T^\sharp-\mathrm{Supp}\,(D^\sharp)$ is affine. But this is obviously absurd. Therefore, we have $l\cong\mathbb{P}^1$ and $(D^\sharp\cdot l)=2$. This implies that a general fiber l meets $\mathrm{Supp}\,(D^\sharp)$ in the distinct two points transversally. (Note that as we work over the field of complex numbers $\mathbb{C}$, the case where a general fiber l of Φ meets $\mathrm{Supp}\,(D^\sharp)$ in a single point of order two does not occur.) Hence we have the assertion. □

Let $\varphi:=\Phi|_{X^\sharp}$ denote the restriction of Φ, which is a $\mathbb{C}^*$-fibration by Lemma 4.2.

Lemma 4.3 φ *can be extended to a* $\mathbb{C}^*$*-fibration on* X.

Proof. It is enough to show that φ is extended to a $\mathbb{C}^*$-fibration on $X^s=T^s-\mathrm{Supp}\,(D^s)$ by noting that $X\cong X^s$ (cf. Theorem 1.2, Remark 2.1). We put $g:=g^{r-1}\circ\cdots\circ g^s: T^s\to T^\sharp$, where $g^i: T^i\to T^{i+1}$ is the divisorial contraction with the exceptional divisor E^i so that $(E^i, E^i|_{E^i})\cong(\mathbb{F}_{b^i,0},-l^i)$ for $s\le i<r$, where l^i is a ruling on the cone $E^i\cong\mathbb{F}_{b^i,0}$. Let $\Phi':=\Phi\circ g: T^s\to W$ be the composite which gives rise to a $\mathbb{P}^1$-fibration induced by a $\mathbb{P}^1$-fibration Φ on $T^\sharp$. It is easy to see that $D^s\cdot l'=D^\sharp\cdot l=2$, where l' is the proper transform of a general fiber $l\cong\mathbb{P}^1$ on T^s. Hence the restriction $\varphi':=\Phi'|_{X^s}$ yields a $\mathbb{C}^*$-fibration on $X^s\cong X$. □

Thus we complete the proof of Theorem 1.4.

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