Dual Greedy Polyhedra, Choice Functions, and Abstract Convex Geometries

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Abstract

We consider a system of linear inequalities and its associated polyhedron for which we can maximize any linear objective function by finding tight inequalities at an optimal solution in a greedy way. We call such a system of inequalities a dual greedy system and its associated polyhedron a dual greedy polyhedron. Such dual greedy systems have been considered by Faigle and Kern, and Krüger for antichains of partially ordered sets, and by Kashiwabara and Okamoto for extreme points of abstract convex geometries. Faigle and Kern also considered dual greedy systems in a more general framework than antichains. A related dual greedy algorithm was proposed by Frank for a class of lattice polyhedra.

In the present paper we show relationships among dual greedy systems, substitutable choice functions, and abstract convex geometries. We also examine the submodularity and facial structures of the dual greedy polyhedra determined by dual greedy systems. Furthermore, we consider an extension of the class of dual greedy polyhedra.

Key words: Dual greedy algorithm, choice function, convex geometry, submodularity

1. Introduction

We consider a system of linear inequalities and its associated polyhedron for which we can maximize any linear objective function by finding tight inequalities at an optimal solution

in a greedy way. We call such a system of inequalities a *dual greedy system* and its associated polyhedron a *dual greedy polyhedron*. A polymatroid [3] is a typical classic example of such a dual greedy polyhedron. Furthermore, dual greedy systems have recently been considered by Faigle and Kern [4, 5], and Krüger [13] for antichains of partially ordered sets (also see [16]), and by Kashiwabara and Okamoto [11] for extreme points of abstract convex geometries ([2]). Faigle and Kern [6] also considered dual greedy systems in a more general framework than antichains. A related dual greedy algorithm was proposed by Frank [7] for a class of lattice polyhedra [10].

In the present paper we show relationships among dual greedy systems, substitutable choice functions, and abstract convex geometries. We also examine the submodularity and facial structures of the dual greedy polyhedra determined by dual greedy systems. Furthermore, we consider an extension of the class of dual greedy polyhedra.

2.   Dual Greedy Polyhedra

The dual greedy systems considered in [3, 4, 5, 6, 11, 13, 16] have the following common features.

Let E be a finite nonempty set with $n = |E|$. Consider

 (i) a nonempty family $\mathcal{A} \subseteq 2^E$,

 (ii) a function $f : \mathcal{A} \to \mathbf{R}$,

 (iii) a system of linear inequalities

$$x(X) \leq f(X) \qquad (X \in \mathcal{A}), \tag{2.1}$$

where x is a variable vector in $\mathbf{R}^E$ and for any $X \in \mathcal{A}$ we define $x(X) = \sum_{e \in X} x(e)$.

Note that (2.1) has only $\{0, 1\}$-coefficients in the left-hand side. Define the polyhedron

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{A} : x(X) \leq f(X)\} \tag{2.2}$$

determined by (2.1).

For any nonnegative vector $w \in \mathbf{R}_+^E$ consider a linear programming problem:

$$(P_w)\ \text{Maximize} \quad \sum_{e \in E} w(e)x(e)$$
$$\text{subject to} \quad x \in \mathrm{P}(f) \tag{2.3}$$

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and its dual linear programing problem:

$$(P_w^*) \quad \text{Minimize} \quad \sum_{X \in \mathcal{A}} f(X)\lambda_X$$

$$\text{subject to} \quad \sum_{X:\, e \in X \in \mathcal{A}} \lambda_X = w(e) \quad (e \in E),$$

$$\lambda_X \geq 0 \qquad (X \in \mathcal{A}). \tag{2.4}$$

Now, suppose that we are given a function $C : 2^E \to \mathcal{A}$ such that for any $X \subseteq E$ we have (i) $C(X) \subseteq X$ and (ii) $C(X) \neq \emptyset$ if $X \neq \emptyset$. We assume that $\emptyset \in \mathcal{F}$ and $C(\emptyset) = \emptyset$. Such a function C is called a *choice function* in the literature (see, e.g., [14]).

Then, consider Procedure Dual_Greedy_Algorithm described as follows.

Dual_Greedy_Algorithm
Put $w' \leftarrow w$ and $X \leftarrow E$.
For each $i = 1, 2, \cdots, n$ do the following:
    Put $X_i \leftarrow C(X)$.
    Find $e_i \in X_i$ such that $w'(e_i) = \min\{w'(e) \mid e \in X_i\}$.
    Put $\lambda_{X_i} \leftarrow w'(e_i)$, $X \leftarrow X \setminus \{e_i\}$, and $w'(e) \leftarrow w'(e) - \lambda_{X_i}$ for each $e \in X_i$.

Through Dual_Greedy_Algorithm we get $X_i \in \mathcal{A}$, $\lambda_{X_i} \geq 0$ $(i = 1, 2, \cdots, n)$ such that

$$w = \sum_{i=1}^{n} \lambda_{X_i} \chi_{X_i}. \tag{2.5}$$

Note that we get a dual feasible solution λ_{X_i} $(i = 1, 2, \cdots, n)$ together with $\lambda_X = 0$ for any other $X \in \mathcal{A}$.

We assume

(A0) *Each $X \in \mathcal{A}$ arises as an X_i by* Dual_Greedy_Algorithm *for some $w \in \mathbf{R}_+^E$.*

(A1) *The expression of w in (2.5) is unique up to terms of zero coefficients, independently of the choice of e_i's in* Dual_Greedy_Algorithm.

The sequence of X_i $(i = 1, 2, \cdots, n)$ obtained by Procedure Dual_Greedy_Algorithm defines a system of equations

$$x(X_i) = f(X_i) \qquad (i = 1, 2, \cdots, n). \tag{2.6}$$

We call the coefficient matrix of (2.6) a *dual greedy basis matrix* and $(X_i \mid i = 1, 2, \cdots, n)$ a *dual greedy basis*. We also assume

(A2) *After appropriately rearranging the columns of the dual greedy basis matrix $A = [a_{ij}]$ of (2.6), A satisfies the following properties:*

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(a) *A is an upper triangular matrix,*

(b) $a_{ii} = 1$ *for all* $i = 1, 2, \cdots, n,$

(c) *each column of A has 1's consecutively, i.e., if $a_{ij} = 1 = a_{i'j}$ for $i < i'$, then $a_{i''j} = 1$ for any i'' with $i \leq i'' \leq i'$.*

Note that the sequence of elements $e_1, e_2, \cdots, e_n$ found by Dual_Greedy_Algorithm gives the ordering of the columns of the matrix A such that properties (a), (b), and (c) hold. In particular, we have $\{e_i\} = X_i \setminus X_{i+1}$ $(i = 1, 2, \cdots, n)$ where $X_{n+1} = \emptyset$. The dual greedy basis determines a primal solution x. If such a solution x is always primal feasible, we say that the dual greedy algorithm works.

We call the system (2.1) of inequalities a *dual greedy system* and the polyhedron $\mathrm{P}(f)$ a *dual greedy polyhedron* associated with it.

Remark 1: When the dual greedy algorithm works, the optimal objective function value $\mu(w)$ of the dual problems (P_w) and (P_w^*) is given by

$$\mu(w) = \sum_{i=1}^{n} \lambda_{X_i} f(X_i) \tag{2.7}$$

according to (2.5). The function $\mu : \mathbf{R}_+^E \to \mathbf{R}$ is what is called the *support function* of $\mathrm{P}(f)$, which is convex. $\qquad\square$

Conversely, without assuming that the dual greedy algorithm works, we can define a function $\hat{\mu} : \mathbf{R}_+^E \to \mathbf{R}$ by (2.7) according to (2.5), where note that the expression (2.5) is unique up to terms with zero coefficients. Also we put $\hat{\mu}(w) = +\infty$ for $w \in \mathbf{R}^E \setminus \mathbf{R}_+^E$. The function $\hat{\mu}$ thus defined is convex only if the dual greedy algorithm works, as shown below.

Theorem 2.1: *Under Assumptions* (A0)*,* (A1)*, and* (A2) *the function* $\hat{\mu} : \mathbf{R}^E \to \mathbf{R} \cup \{+\infty\}$ *is convex if and only if the dual greedy algorithm works.*
(Proof) If the dual greedy algorithm works, then we have $\hat{\mu} = \mu$ (the support function of $\mathrm{P}(f)$) and hence $\hat{\mu}$ is convex. Conversely, suppose that $\hat{\mu}$ is convex. Note that $\hat{\mu}$ is positively homogeneous by definition and is continuous on $\mathbf{R}_+^E$ by Assumption (A1). Hence, it is a support function of a convex set $P \subseteq \mathbf{R}^E$ defined by

$$P = \{x \mid x \in \mathbf{R}^E, \ \forall w \in \mathbf{R}_+^E : \sum_{e \in E} w(e)x(e) \leq \hat{\mu}(w)\} \tag{2.8}$$

(see [17, Cor. 13.2.1]). It follows from the definition of $\hat{\mu}$ (by (2.5) and (2.7)) that we have $P = \mathrm{P}(f)$. Now, for any $w \in \mathbf{R}_+^E$ let $(X_i \mid i = 1, 2, \cdots, n)$ be the dual greedy

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basis determined by Dual_Greedy_Algorithm. For any $\bar{\lambda}_{X_i} > 0$ $(i = 1, 2, \cdots, n)$ define $\bar{w} \in \mathbf{R}_+^E$ by (2.5). Then we have

$$\hat{\mu}(\bar{w}) = \sum_{i=1}^{n} \bar{\lambda}_{X_i} f(X_i). \tag{2.9}$$

Let $\hat{x}$ be a vector in $P(= \mathrm{P}(f))$ such that

$$\sum_{e \in E} \bar{w}(e)\hat{x}(e) = \hat{\mu}(\bar{w}), \tag{2.10}$$

where recall that $\hat{\mu}$ is the support function of P, so that such a vector $\hat{x}$ exists. Since $\hat{x}(X_i) \leq f(X_i)$ and $\bar{\lambda}_{X_i} > 0$ $(i = 1, 2, \cdots, n)$, it follows from (2.5), (2.9), and (2.10) that we have $\hat{x}(X_i) = f(X_i)$ $(i = 1, 2, \cdots, n)$. That is, $\hat{x}$ is the dual greedy solution associated with the dual greedy basis $(X_i \mid i = 1, 2, \cdots, n)$. $\qquad\square$

In the following we assume

(A3) *For any* $w \in \mathbf{R}_+^E$ Dual_Greedy_Algorithm *works.*

Remark 2: The function $\hat{\mu}$ is an extension of the set function $f : \mathcal{A} \to \mathbf{R}$, which is a generalization of the so-called Lovász extension of a set function on 2^E. As is the case for the Lovász extension of a submodular function on 2^E, the convexity of the extension $\hat{\mu}$ completely characterizes the primal feasibility of dual greedy solutions. Moreover, if w is an integral vector, the coefficients λ_{X_i} in (2.5) are integers, so that under Assumptions (A0)$\sim$(A3) the system (2.1) of inequalities is totally dual integral. $\qquad\square$

We shall also investigate the primal feasiblity of the dual greedy solution in Section 4. In the next section we shall examine properties of the choice function C.

3.   Choice Functions and Abstract Convex Geometries

Let us call an ordering $(e_1, e_2, \cdots, e_n)$ generated by Dual_Greedy_Algorithm an *admissible ordering*. It follows from Dual_Greedy_Algorithm that the set of admissible orderings $(e_1, e_2, \cdots, e_n)$ for all nonnegative weight functions w coincides with the set of orderings $(e_1, e_2, \cdots, e_n)$ that can be generated by the following procedure:

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Admissible_Ordering
Put $X \leftarrow E$.

For each $i = 1, 2, \cdots, n$ do the following:

Choose $e_i \in C(X)$ and put $X \leftarrow X \setminus \{e_i\}$.

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Define $\mathcal{F} \subseteq 2^E$ by

$$\mathcal{F} = \{\{e_i, e_{i+1}, \cdots, e_n\} \mid i = 1, 2, \cdots, n, \ (e_1, e_2, \cdots, e_n) : \text{ an admissible ordering}\}. \tag{3.1}$$

By restricting the choice function C to $\mathcal{F}$ we regard C as a function from $\mathcal{F}$ to $\mathcal{A}$.

Example 1 (Antichains of a poset) [4, 5, 6], [13], [1]

For any partially ordered set (poset) $\mathcal{P} = (E, \preceq)$ let $\mathcal{F} \subseteq 2^E$ be the set of all the (lower) ideals of $\mathcal{P}$, where $I \subseteq E$ is a (lower) ideal of $\mathcal{P}$ if and only if $e_1 \preceq e_2 \in I$ implies $e_1 \in I$. For each ideal $I \in \mathcal{F}$ define $C(I)$ to be the set of all maximal elements of I in poset $\mathcal{P}$. Note that the set of $C(I)$ $(I \in \mathcal{F})$ coincides with that of antichains of $\mathcal{P}$. $\square$

Example 2 (Extreme points of an abstract convex geometry) [2], [11]

Let $(E, \mathcal{F})$ be an abstract convex geometry on E with a family $\mathcal{F}$ of closed sets, i.e., (1) $\emptyset, E \in \mathcal{F}$, (2) $\mathcal{F}$ is closed with respect to set intersection, and (3) the length of each maximal chain of $\mathcal{F}$, considered as a lattice, is equal to $|E|$. For each $X \in \mathcal{F}$ let $C(X)$ be the set of extreme points of X. Recall that $e \in X(\in \mathcal{F})$ is an extreme point of X if and only if $X \setminus \{e\} \in \mathcal{F}$. $\square$

We can easily see the following properties of C.

Lemma 3.1: *Function $C : \mathcal{F} \to \mathcal{A}$ satisfies the following two*:
(C1) *For any $X \in \mathcal{F}$ with $X \neq \emptyset$ we have $C(X) \neq \emptyset$.*
(C2) *For any nonempty $X \in \mathcal{F}$ and any $e \in C(X)$,*

$$C(X) \setminus \{e\} \subseteq C(X \setminus \{e\}). \tag{3.2}$$

(Proof) (C1) follows from the definition of C or (b) of Assumption (A2). Also (C2) follows from the consecutive 1's property (c) of (A2). $\square$

It should be noted that the choice function $C : \mathcal{F} \to \mathcal{A}$ having properties (C1) and (C2) completely characterizes the collection of basis matrices A with properties (a)∼(c) where we assume that X_i is determined by $\cup\{X_k \mid k = i, i+1, \cdots, n\}$. Property (C2) shows a kind of *substitutability* of choice function C ([14]).

Theorem 3.2: *Consider a choice function $C : \mathcal{F} \to \mathcal{A}$ satisfying (C1) and (C2), where $\mathcal{F}$ is defined by (3.1). Then, the pair $(E, \mathcal{F})$ is an abstract convex geometry with a family $\mathcal{F}$ of closed sets.*

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(Proof) First note that the length of any maximal chain of $\mathcal{F}$ is equal to $|E|$. Suppose that $X \in \mathcal{F}$ and $e, e' \in C(X)$ where $e \neq e'$. It suffices to show that $X \setminus \{e, e'\} \in \mathcal{F}$ (see, e.g., [12, Lemma 1.2]). Let $(\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_n)$ be an admissible ordering such that $X = E \setminus \{\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_k\}$ for $k = |E \setminus X|$. Note that from the assumption we have $e \in C(X)$ and furthermore, $e' \in C(X \setminus \{e\})$ due to (C2). It follows from Procedure Admissible_Ordering that there exists an admissible ordering of the form $(\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_k, e, e', \cdots)$. Hence we have $X \setminus \{e, e'\} \in \mathcal{F}$. $\qquad\square$

Theorem 3.2 implies

Theorem 3.3: *The class of dual greedy systems (or dual greedy polyhedra) coincides with the one considered by Kashiwabara and Okamoto* [11] *for abstract convex geometries.* $\qquad\square$

4. Adjacency in Dual Greedy Polyhedra

There is a one-to-one correspondence between the set of admissible orderings and that of dual greedy bases. Let $(e_1, e_2, \cdots, e_n)$ be an admissible ordering. Then we have a corresponding dual greedy basis formed by

$$X_i = C(\{e_i, e_{i+1}, \cdots, e_n\}) \in \mathcal{A} \qquad (i = 1, 2, \cdots, n) \tag{4.1}$$

that determines the basis matrix $A = [a_{ij}]$ and a vertex, say v, in $\mathrm{P}(f)$.

For any $k \in \{1, 2, \cdots, n\}$ remove the kth row from A and consider the following system of equations.

$$x(X_i) = f(X_i) \qquad (i = 1, \cdots, k-1, k+1, \cdots, n). \tag{4.2}$$

The set of solutions of (4.2) is a line (denoted by L_v^k) through v. Let d be a $\{0, \pm 1\}$-valued solution of (4.2) with $f(X_i)$ replaced by zero for each $i = 1, \cdots, k-1, k+1, \cdots, n$, where the consecutive 1's property of the coefficient matrix guarantees the existence of a $\{0, \pm 1\}$-valued solution. If L_v^k determines an edge vector z from v to one of its adjacent vertex u (possibly a point at infinity), z must be equal, up to a positive multiple, to the $\{0, \pm 1\}$-vector $d = \chi_{F_1} - \chi_{F_2}$ such that $F_1, F_2 \subseteq E$, $F_1 \cap F_2 = \emptyset$, $e_k \in F_2$, and

$$|F_1 \cap X_i| - |F_2 \cap X_i| = 0 \qquad (i = 1, \cdots, k-1, k+1, \cdots, n). \tag{4.3}$$

See Figure 1, where $d = \chi_{F_1} - \chi_{F_2}$ with $F_1 = \{e_1, e_4, e_5\}$ and $F_2 = \{e_2, e_6\}$.

Define

$$\hat{\alpha} = \sup\{\alpha \mid \alpha \geq 0, \ v + \alpha(\chi_{F_1} - \chi_{F_2}) \in \mathrm{P}(f)\}. \tag{4.4}$$

$$\begin{array}{c}\begin{array}{cccccccccc}e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10}\end{array}\\\begin{array}{c}X_1\\X_2\\X_3\\X_4\\X_5\\X_6>\\X_7\\X_8\\X_9\\X_{10}\end{array}\left(\begin{array}{cccccccccc}1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0\\0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1\\0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1\\0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1\\0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1\\0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{array}\right)\end{array}$$

Figure 1: An example of a dual greedy basis matrix, where $E = \{e_1, e_2, \cdots, e_{10}\}$ and X_6 is to be removed, i.e., $k = 6$.

Theorem 4.1: *The following three statements hold*:

(i) $\hat{\alpha} = +\infty$ *if and only if $F_1 = \emptyset$ and $F_2 = \{e_k\}$.*

(ii) *If $0 < \hat{\alpha} < +\infty$, then the vertex u adjacent from v in the direction of $\chi_{F_1} - \chi_{F_2}$ corresponds to the admissible ordering*

$$(e_1, \cdots, e_{k-2}, e_k, e_{k-1}, e_{k+1}, \cdots, e_n). \tag{4.5}$$

The sequence

$$X_1, \ \cdots, \ X_{k-1}, \ C(\{e_{k-1}, e_{k+1}, \cdots, e_n\}), \ X_{k+1}, \ \cdots, \ X_n \in \mathcal{A} \tag{4.6}$$

determines the vertex u adjacent to v, and we have

$$\hat{\alpha} = f(C(\{e_{k-1}, e_{k+1}, \cdots, e_n\})) - v(C(\{e_{k-1}, e_{k+1}, \cdots, e_n\})). \tag{4.7}$$

(iii) *We have $\hat{\alpha} = 0$ if and only if the sequence (4.6) gives the same vertex v, so that*

$$v(C(\{e_{k-1}, e_{k+1}, \cdots, e_n\})) = f(C(\{e_{k-1}, e_{k+1}, \cdots, e_n\})). \tag{4.8}$$

(Proof) (i) If $e_k \notin C(\{e_{k-1}, e_k, \cdots, e_n\})$, then because of (a), (b), and (c) $x(e_k)$ does not appear in (4.2) explicitly. Hence we have $F_1 = \emptyset$, $F_2 = \{e_k\}$, and $\hat{\alpha} = +\infty$. On the other hand, if $e_k \in C(\{e_{k-1}, e_k, \cdots, e_n\})$, then $\{e_{k-1}, e_{k+1}, \cdots, e_n\} \in \mathcal{F}$ and hence $C(\{e_{k-1}, e_{k+1}, \cdots, e_n\})$ is defined. It follows from (C2) that $e_{k-1} \in C(\{e_{k-1}, e_{k+1}, \cdots,$

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$e_n\}$), while $e_k \notin C(\{e_{k-1}, e_{k+1}, \cdots, e_n\})$. Since we have $F_1 \cup F_2 \subseteq \{e_1, e_2, \cdots, e_k\}$ and $(F_1 \cup F_2) \cap C(\{e_{k-1}, e_{k+1}, \cdots, e_n\}) = \{e_{k-1}\}$, the difference

$$f(C(\{e_{k-1}, e_{k+1}, \cdots, e_n\})) - v(C(\{e_{k-1}, e_{k+1}, \cdots, e_n\}))$$

gives an upper bound for $\hat{\alpha}$. Hence $\hat{\alpha} < +\infty$. Note that in this case we have $e_{k-1} \in F_1$.

(ii) As shown in the proof of (i), if $\hat{\alpha} < +\infty$, there is an admissible ordering (4.5) and the corresponding dual greedy basis matrix is determined by (4.6), where X_k has been replaced by $C(\{e_{k-1}, e_{k+1}, \cdots, e_n\})$. Since these two are the only possible dual greedy basis matrices that satisfy (a)$\sim$(c) and have $X_1, \cdots, X_{k-1}, X_{k+1}, \cdots, X_n$ in common, the statement (ii) holds due to Assumption (A3) (also see Theorem 3.2).

(iii) The present statement follows from the proof of (ii). $\qquad\square$

We say that the dual greedy basis given by (4.6) is *adjacent to* the dual greedy basis $(X_1, X_2, \cdots, X_n)$. It should be noted that the set of all the dual greedy bases is connected with respect to the adjacency.

Remark 3: The collection of all the dual greedy bases forms a *shape*, which was introduced and examined by Sohoni [18]. Related arguments were also made by Narayanan [15]. As shown by Sohoni [18], a shape, the collection of all the dual greedy bases considered here, determines a simplicial division of the intersection of the unit sphere in $\mathbf{R}^E$ and the nonnegative orthant $\mathbf{R}_+^E$. The adjacency of the simplices in the division coincides with the adjacency of dual greedy bases. $\qquad\square$

Lemma 4.2: *Without Assumption* (A3)*, suppose that there exists at least one dual greedy solution belonging to* $\mathrm{P}(f)$*. Then, for any nonnegative weight vector* $w \in \mathbf{R}_+^E$ *the dual greedy algorithm finds an optimal solution if and only if for each admissible ordering* $(e_1, e_2, \cdots, e_n)$ *and its associated dual greedy solution* v *we have*

$$v(C(\{e_{k-1}, e_{k+1}, \cdots, e_n\})) \leq f(C(\{e_{k-1}, e_{k+1}, \cdots, e_n\})) \qquad (4.9)$$

for each $k \in \{1, 2, \cdots, n\}$ *such that* $e_k \in C(\{e_{k-1}, e_k, \cdots, e_n\})$.
(Proof) The 'only if' part is trivial. Hence we prove the 'if' part. It follows from Theorem 4.1 that there is no hyperplane $x(C(X)) = f(C(X))$ with $X \in \mathcal{F}$ that separates any two adjacent dual greedy solutions. Hence we see from the assumption and the connectedness of the set of all the dual greedy bases that any dual greedy solution is primal feasible. $\quad\square$

Remark 4: Kashiwabara and Okamoto [11] gave a kind of submodularity condition on f for the primal feasibility of dual greedy solutions. $\qquad\square$

Remark 5: In the case of a polymatroid, since $C(X) = X$ for any $X \subseteq E$, inequality (4.9) can be rewritten as

$$
\begin{aligned}
& f(\{e_{k-1}, e_{k+1}, \cdots, e_n\}) \\
& \geq v(C(\{e_{k-1}, e_{k+1}, \cdots, e_n\})) \\
& = v(\{e_{k-1}, e_{k+1}, \cdots, e_n\}) \\
& = v(e_{k-1}) + v(\{e_{k+1}, \cdots, e_n\}) \\
& = f(\{e_{k-1}, e_k, \cdots, e_n\}) - f(\{e_k, \cdots, e_n\}) + f(\{e_{k+1}, \cdots, e_n\}). \quad (4.10)
\end{aligned}
$$

As is well-known, this is equivalent to the submodularity of f on 2^E. $\qquad \square$

5.   Submodularity in Dual Greedy Polyhedra

In this section we examine the structure of the set of edge vectors, which will reveal a submodularity structure behind dual greedy polyhedra.

Lemma 5.1: *Consider an edge vector $\chi_{F_1} - \chi_{F_2}$ determined by (4.2). Then,*

$$
|F_1 \cap X_i| = |F_2 \cap X_i| = 0 \text{ or } 1 \qquad (i = 1, 2, \cdots, k-1). \qquad (5.1)
$$

(Proof) The present lemma easily follows from properties (a), (b), and (c). $\qquad \square$

Here recall that $F_1 \cup F_2 \subseteq \{e_1, e_2, \cdots, e_k\}$. Put $Z_i = \{e_{i+1}, \cdots, e_n\}$ $(i = 1, 2, \cdots, n)$ and define a face $\mathbf{F}(Z_k)$ of $\mathrm{P}(f)$ by

$$
\mathbf{F}(Z_k) = \{x \mid x \in \mathrm{P}(f), \forall i \in \{k+1, k+2, \cdots, n\} : x(X_i) = f(X_i)\}. \qquad (5.2)
$$

Also define a face, determined by face $\mathbf{F}(Z_k)$ and a supporting hyperplane $x(X_l) = f(X_l)$ for a positive integer l with $1 \leq l \leq k$, by

$$
\mathbf{F}(Z_k, l) = \{x \mid x \in \mathbf{F}(Z_k), \ x(X_l) = f(X_l)\}. \qquad (5.3)
$$

From Lemma 5.1 we have

Theorem 5.2: *Let k and l be positive integers such that $1 \leq l < k \leq n$. For a face $\mathbf{F}(Z_k, l)$ of $\mathbf{F}(Z_k)$, the projection of $\mathbf{F}(Z_k)$ into the subspace $\mathbf{R}^{X_l}$ along $\mathbf{R}^{E \setminus (X_l \cup Z_k)}$ is a base polyhedron associated with a submodular function on 2^{X_l}.*
(Proof) After the projection of $\mathbf{F}(Z_k)$ into $\mathbf{R}^{X_l}$ we have $x(X_l) = f(X_l)$. It follows from Lemma 5.1 that each edge vector of the projected face is one of the forms $\chi_e - \chi_{e'}$ $(e, e' \in X_l, \ e \neq e')$. Hence the projected face is a base polyhedron associated with a submodular function on 2^{X_l}, due to Tomizawa (see, e.g., [8, Theorem 3.26], [9, Appendix]). $\qquad \square$

6. Abstract Convex Geometries Associated with Faces

Given a nonnegative weight vector $w \in \mathbf{R}_+^E$, consider the LP problem (P_w) in (2.3). In the description of Dual_Greedy_Algorithm in Section 2 define

$$C^w(C(X)) = \{e' \mid e' \in X_i, \ w'(e') = \min\{w'(e) \mid e \in X_i\}\}, \tag{6.1}$$

where $X_i = C(X)$. Note that C^w and the composition $C^w C$ are choice functions.

Lemma 6.1: *The choice function $C^w C$ satisfies* (C1) *and* (C2) *in Lemma* 3.1.
(Proof) Property (C1) is immediate. We prove (C2). If $|C^w(C(X))| = 1$, then (C2) holds. If $|C^w(C(X))| \geq 2$, then for any chosen $e_i \in C^w(C(X))$ and an updated new w' we have

$$e \in C^w(C(X)) \setminus \{e_i\} \implies w'(e) = 0. \tag{6.2}$$

It follows that if $e \in C^w(C(X)) \setminus \{e_i\}$, then we have $e \in \arg\min\{w'(e') \mid e' \in C(X \setminus \{e_i\})\} = C^w(C(X \setminus \{e_i\}))$. $\qquad\square$

We see from this lemma and Theorem 3.2 that $C^w C$ defined by (6.1) determines an abstract convex geometry associated with the face of optimal primal solutions of (P_w). It should be noted that the abstract convex geometry is determined by w but not by the face of optimal primal solutions of (P_w). Distinct weight vectors w_1 and w_2 may determine the same face of optimal solutions and distinct choice functions $C^{w_1} C$ and $C^{w_2} C$, which occurs only if $\mathrm{P}(f)$ is degenerate.

7. An Extension

In the previous sections any dual greedy polyhedron $\mathrm{P}(f)$ has its characteristic cone (or recession cone) $\mathbf{R}_-^E$. We extend the class of dual greedy polyhedra to that of polyhedra having more general characteristic cones, which has not been considered in the literature.

Consider a choice function C_1 that satisfies Properties (C1) and (C2) in Lemma 3.1. Also let C_2 be a choice function such that the composition $C_2 C_1$ satisfies Properties (C1) and (C2), and let $\mathcal{F}$ be the family of closed sets of the abstract convex geometry associated with $C_2 C_1$. Then, consider the following system of inequalities

$$x(C_1(X)) \leq f(C_1(X)) \qquad (X \in \mathcal{F}), \tag{7.1}$$

where $f : \mathcal{F} \to \mathbf{R}$ is a function such that the dual greedy algorithm based on the choice function $C_2 C_1$ works for any nonnegative weight function w satisfying $C_2(C_1(X)) \cap C_1^w(C_1(X)) \neq \emptyset$ $(X \in \mathcal{F})$. Here, C_1^w is the choice function defined by (6.1) with C replaced by C_1. Denote by $\mathrm{P}(f)$ the set of all the feasible solutions of (7.1).

DG Systems on Convex Geometries
with possibly *unbounded* faces
of maximal vectors

↗   ↖

DG Systems on Convex Geometries $(\mathcal{D}, f)$
of Kashiwabara and Okamoto   Submodular Systems
with *bounded* faces of maximal vectors   with possibly *unbounded* base polyhedra

↖   ↗

$(2^E, f)$
Submodular Systems
with *bounded* base polyhedra

Figure 2: A generalization diagram (DG stands for Dual Greedy).

Remark 6: Note that $\mathcal{F}$ in (7.1) is defined from $C_2 C_1$ but not from C_1. This makes a great difference between (7.1) considered here and (2.1) in Section 2. An admissible ordering for $C_2 C_1$ is admissible for C_1 but the converse is not true in general. □

The following two examples show dual greedy polyhedra with unbounded faces of maximal vectors (also see Figure 2).

Example 3: For a poset $(E, \preceq)$ suppose that $C_1(X) = X$ $(X \subseteq E)$ and let $C_2(X)$ be the set of all the maximal elements of $X \subseteq E$. Then the family of closed sets associated with $C_2 C_1$ is the family, denoted by $\mathcal{D}$, of all the (lower) ideals of poset $(E, \preceq)$. The set $\mathrm{P}(f)$ of all the feasible solutions of (7.1) is the so-called *submodular polyhedron* associated with a submodular system $(\mathcal{D}, f)$, where f is a submodular function on $\mathcal{D}$ (see [8]). Note that in this example $\mathcal{D}$ is closed with respect to set union as well as set intersection. Also note that the characteristic cone of $\mathrm{P}(f)$ is generated by vectors $\chi_e - \chi_{e'}$ for all arcs (e, e') of the Hasse diagram of poset $(E, \preceq)$ and vectors $-\chi_e$ for all minimal elements e of $(E, \preceq)$. It is different from $\mathbf{R}_-^E$ in general. □

Example 4: Let $(E, \mathcal{F})$ be an abstract convex geometry with a family $\mathcal{F}$ of closed sets. Then consider $C_1, C_2 : \mathcal{F} \to 2^E$ given by

$$C_1(X) = X, \quad C_2(X) = \mathrm{ex}(X) \qquad (X \in \mathcal{F}), \tag{7.2}$$

where $\mathrm{ex}(X)$ denotes the set of extreme points of X in the abstract convex geometry $(E, \mathcal{F})$. □

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It should be noted that the choice function C_2^w defined by (6.1) for any nonnegative weight vector w satisfies the condition for C_2. Hence each face of $\mathrm{P}(f)$ for (2.1) gives an example for (7.1).

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