Representations of the Cuntz-Krieger algebras. I
—General theory—

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We show several systematic construction of representations of the Cuntz-Krieger algebras from transformations on measure spaces as a generalization of permutative representation of the Cuntz algebras. We introduce these examples of them and their properties.

1. Introduction

Representation theory of the Cuntz algebras is studied by [**4**, **6**, **7**, **8**, **9**, **16**, **17**]. It is remarkable that representations of the Cuntz algebras in some class satisfy the uniqueness of irreducible decomposition. Furthermore these representations are related to quantum filed theory([**1**, **2**, **3**]), dynamical systems([**11**, **13**, **14**, **15**]) and fractals([**12**]), and their branching laws are computed by automata([**18**]). We generalize these results for the Cuntz-Krieger algebras.

In this paper, we start to show general properties and systematic constructions of representations of the Cuntz-Krieger algebras by embedding of the Cuntz-Krieger algebras in [**10**].

Let $N \geq 2$ and (X, μ) be a measure space. Assume that there are a family $\{D_i\}_{i=1}^N$ of non μ-null subsets of X and a family $f = \{f_i\}_{i=1}^N$ of measurable maps such that f_i is an injective map from D_i to $R_i \equiv f(D_i) \subset X$ and the Radon-Nikodým derivative Φ_i of $\mu \circ f_i$ with respect to μ is non zero for each $i = 1, \ldots, N$. Define a partial isometry $S(f_i)$ on $L_2(X, \mu)$ by

$$(1.1) \qquad (S(f_i)\phi)(x) \equiv \begin{cases} \left\{\Phi_i\left(f_i^{-1}(x)\right)\right\}^{-1/2} \phi(f_i^{-1}(x)) & (x \in R_i), \\ \\ 0 & (\text{otherwise}) \end{cases}$$

for $\phi \in L_2(X, \mu)$ and $x \in X$. We consider a C*-algebra $C^* < \{S(f_i)\}_{i=1}^N >$ generated by operators $S(f_1), \ldots, S(f_N)$.

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Theorem 1.1. *Let $A = (a_{ij})$ be an $N \times N$ matrix which has entries in $\{0,1\}$ and has no rows or columns identically equal to zero. For a family $f = \{f_i\}_{i=1}^N$ of maps on a measure space (X,μ) in the above,*

$$C^* < \{S(f_i)\}_{i=1}^N > \cong \mathcal{O}_A$$

if all the followings are μ-null subsets of X:
$X \setminus R_1 \cup \cdots \cup R_N,\ D_i \setminus \bigcup_{j:a_{ij}=1} R_j\ (i = 1,\ldots,N),\ R_i \cap R_j\ (i \neq j).$

Theorem 1.1 is shown by checking that $S(f_1),\ldots,S(f_N)$ satisfy relations of canonical generators of $\mathcal{O}_A$ in § 3. f in Theorem 1.1 is called an *A-branching function system on* (X,μ). Although we do not know *what* $C^* < \{S(f_i)\}_{i=1}^N >$ *is* for general $f_1,\ldots,f_N$, our aim is *not* to create a new example of C^*-algebra *but* to study representation $(L_2(X,\mu),\pi_f)$ of $\mathcal{O}_A$ arising from $f = \{f_i\}_{i=1}^N$ in Theorem 1.1. Therefore problems are i) what the condition for f is so that $(L_2(X,\mu),\pi_f)$ is irreducible, and ii) what the condition for f and g is so that $(L_2(X,\mu),\pi_f) \sim (L_2(Y,\nu),\pi_g)$.

In § 2, we show general theory of representations of $\mathcal{O}_A$. We treat construction and decomposition of representation of $\mathcal{O}_A$, and review results about the Cuntz algebras. In § 3, we show properties of partial isometries in (1.1) and a general construction of representations of $\mathcal{O}_A$ from branching function systems. In § 4, we show the standard constructions of representation of $\mathcal{O}_A$ on $l_2(\mathbf{N})$, $l_2(\mathbf{Z})$, $L_2[0,1]$, $L_2(\mathbf{T}^1)$ and $L_2(\mathbf{R})$ by using representations of the Cuntz algebras. In § 5, we show examples branching function systems and representations of the Cuntz-Krieger algebras.

2. General theory of representations of $\mathcal{O}_A$

2.1. Multiindices. We introduce several sets of multiindices which consist of numbers $1,\ldots,N$ for $N \geq 2$.

Put $\{1,\ldots,N\}^0 \equiv \{0\}$, $\{1,\ldots,N\}^k \equiv \{(j_l)_{l=1}^k : j_l = 1,\ldots,N,\ l = 1,\ldots,k\}$ for $k \geq 1$ and $\{1,\ldots,N\}^\infty \equiv \{(j_n)_{n\in\mathbf{N}} : j_n \in \{1,\ldots,N\},\ n \in \mathbf{N}\}$. Denote $\{1,\ldots,N\}^* \equiv \coprod_{k\geq 0}\{1,\ldots,N\}^k$, $\{1,\ldots,N\}_1^* \equiv \coprod_{k\geq 1}\{1,\ldots,N\}^k$, $\{1,\ldots,N\}^\# \equiv \{1,\ldots,N\}_1^* \sqcup \{1,\ldots,N\}^\infty$. For $J \in \{1,\ldots,N\}^\#$, the *length* $|J|$ of J is defined by $|J| \equiv k$ when $J \in \{1,\ldots,N\}^k$. For $J_1, J_2 \in \{1,\ldots,N\}^*$ and $J_3 \in \{1,\ldots,N\}^\infty$ $J_1 \cup J_2 \equiv (j_1,\ldots,j_k,j_1^{'},\ldots,j_l^{'})$, $J_1 \cup J_3 \equiv (j_1,\ldots,j_k,j_1^{''},j_2^{''},\ldots)$ when $J_1 = (j_1,\ldots,j_k)$, $J_2 = (j_1^{'},\ldots,j_l^{'})$ and $J_3 = (j_n^{''})_{n\in\mathbf{N}}$. Specially, we define $J \cup \{0\} = \{0\} \cup J = J$ for $J \in \{1,\ldots,N\}^\#$ and $(i,J) \equiv (i) \cup J$ for convenience. For $J \in \{1,\ldots,N\}^*$ and $k \geq 2$, $J^k \equiv \underbrace{J \cup \cdots \cup J}_{k}$ and $J^\infty = J \cup \cdots \cup J \cup \cdots \in \{1,\ldots,N\}^\infty$. For $J = (j_1,\ldots,j_k) \in \{1,\ldots,N\}^k$ and $\tau \in \mathbf{Z}_k$, denote $\tau(J) = (j_{\tau(1)},\ldots,j_{\tau(k)})$.

In order to treat representations of $\mathcal{O}_A$, we modify multiindices with respect to A. Let $M_N(\{0,1\})$ be the set of all $N \times N$ matrices in which have entries in $\{0,1\}$ and have no rows or columns identically equal to zero. $A=(a_{ij})\in M_N(\{0,1\})$ is *full* if $a_{ij}=1$ for each $i,j=1,\ldots,N$. For $A=(a_{ij})\in M_N(\{0,1\})$, define

$$\{1,\ldots,N\}^*_A\equiv\coprod_{k\geq 0}\{1,\ldots,N\}^k_A,$$

$$\{1,\ldots,N\}^0_A\equiv\{0\},\quad \{1,\ldots,N\}^1_A\equiv\{1,\ldots,N\},$$

$$\{1,\ldots,N\}^k_A\equiv\{(j_i)^k_{i=1}\in\{1,\ldots,N\}^k : a_{j_{i-1}j_i}=1,\, i=2,\ldots,k\}\quad (k\geq 2),$$

$$\{1,\ldots,N\}^*_{A,c}\equiv\coprod_{k\geq 1}\{1,\ldots,N\}^k_{A,c},$$

$$\{1,\ldots,N\}^k_{A,c}\equiv\{(j_i)^k_{i=1}\in\{1,\ldots,N\}^k_A : a_{j_kj_1}=1\},$$

$$\{1,\ldots,N\}^\infty_A\equiv\{(j_n)_{n\in\mathbf{N}}\in\{1,\ldots,N\}^\infty : a_{j_{n-1}j_n}=1, n\geq 2\},$$

$$\{1,\ldots,N\}^\#_{A,c}\equiv\{1,\ldots,N\}^*_{A,c}\sqcup\{1,\ldots,N\}^\infty_A.$$

For example, if $A=\begin{pmatrix}1&1\\1&0\end{pmatrix}$, then $\{1,2\}_A=\{1,2\}$, $\{1,2\}^2_A=\{(11),(21),(12)\}$, $\{1,2\}^3_A=\{(111),(211),(121),(112),(212)\}$, $\{1,2\}^4_A=\{(1111),(2111),(1211),(1121),(2121),(1112),(2112),(1212)\}$.

$J\in\{1,\ldots,N\}^*_1$ is *periodic* if there are $m\geq 2$ and $J_0\in\{1,\ldots,N\}^*_1$ such that $J=J_0^m$. For $J_1,J_2\in\{1,\ldots,N\}^*_1$, $J_1\sim J_2$ if there are $k\geq 1$ and $\tau\in\mathbf{Z}_k$ such that $|J_1|=|J_2|=k$ and $\tau(J_1)=J_2$. For $(J,z),(J',z')\in\{1,\ldots,N\}^*_1\times U(1)$, $(J,z)\sim(J',z')$ if $J\sim J'$ and $z=z'$ where $U(1)\equiv\{z\in\mathbf{C}:|z|=1\}$. Specially, any element in $\{1,\ldots,N\}$ is non periodic. $J\in\{1,\ldots,N\}^\infty$ is *eventually periodic* if there are $J_0,J_1\in\{1,\ldots,N\}^*_1$ such that $J=J_0\cup J_1^\infty$. For $J_1,J_2\in\{1,\ldots,N\}^\infty$, $J_1\sim J_2$ if there are $J_3,J_4\in\{1,\ldots,N\}^*$ and $J_5\in\{1,\ldots,N\}^\infty$ such that $J_1=J_3\cup J_5$ and $J_2=J_4\cup J_5$.

2.2. Construction and decomposition of representations of $\mathcal{O}_A$. For $A=(a_{ij})\in M_N(\{0,1\})$, $\mathcal{O}_A$ is *the Cuntz-Krieger algebra by A* if $\mathcal{O}_A$([**5**]) is a C*-algebra which is universally generated by partial isometries $s_1,\ldots,s_N$ satisfying:

$$s_i^*s_i=\textstyle\sum_{j=1}^N a_{ij}s_js_j^*\quad (i=1,\ldots,N),\quad \sum_{i=1}^N s_is_i^*=I. \tag{2.1}$$

Specially, $\mathcal{O}_A$ is the Cuntz algebra $\mathcal{O}_N$ when A is full.

We denote the canonical $U(1)$-action(=gauge action) on $\mathcal{O}_A$ by γ and the canonical $U(N)$-action on $\mathcal{O}_N$ by α. For a multiindex $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}^k$ and canonical generators $s_1,\ldots,s_N$ of $\mathcal{O}_A$, we denote $s_J=s_{j_1}\cdots s_{j_k}$ and $s_J^*=s_{j_k}^*\cdots s_{j_1}^*$. When $J\in\{1,\ldots,N\}^*$, $s_J\neq 0$ if and only if $J\in\{1,\ldots,N\}^*_A$.

In this paper, a representation always means a unital *-representation on a complex Hilbert space. $(\mathcal{H}_1,\pi_1)\sim(\mathcal{H}_2,\pi_2)$ means the unitary equivalence between two representations $(\mathcal{H}_1,\pi_1)$ and $(\mathcal{H}_2,\pi_2)$ of $\mathcal{O}_A$.

For a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_A$ and a unitary operator U on a Hilbert space $\mathcal{K}$, we have a new representation $(\mathcal{K}\otimes\mathcal{H},\, U\boxtimes\pi)$ of $\mathcal{O}_A$ which is defined by

$$(U\boxtimes\pi)(s_i)\equiv U\otimes\pi(s_i)\quad (i=1,\ldots,N). \tag{2.2}$$

Lemma 2.1. *For a representation in (2.2), the followings hold:*

(i) *If U has an eigenvalue $c\in U(1)$ on $\mathcal{K}$, then $(\mathcal{K}\otimes\mathcal{H}, U\boxtimes\pi)$ has a subrepresentation which is equivalent to $(\mathcal{H},\pi\circ\gamma_c)$.*

(ii) *If a unitary V on $\mathcal{K}$ is conjugate with U by a unitary, then $U\boxtimes\pi\sim V\boxtimes\pi$.*

(iii) *If there are $p\in\mathbf{Z}$ and a complete orthonormal basis $\{e_n : n\in\mathbf{Z}\}$ of $\mathcal{K}$ such that $Ue_n=e_{n+p}$ for each $n\in\mathbf{Z}$, then $(\mathcal{K}\otimes\mathcal{H},\, U\boxtimes\pi)$ is decomposed as*

$$\begin{cases} \displaystyle\int_{U(1)}^{\oplus}(\mathcal{H},\,\pi\circ\gamma_{w^p})\,d\eta(w) & (p\neq 0),\\[2ex] (\mathcal{H},\pi)^{\oplus\infty} & (p=0)\end{cases}$$

where η is the Haar measure of $U(1)$.

(iv) *If there is $p\geq 2$ such that $U^p=I$, $U^i\neq I$ for $i=1,\ldots,p-1$, then*

$$U\boxtimes\pi\sim\left(\bigoplus_{i=1}^{p}\pi\circ\gamma_{\xi_i}\right)^{\oplus\nu}$$

where $\nu\equiv(\dim\mathcal{K})/p$ and $\xi_i\equiv e^{2\pi\sqrt{-1}(i-1)/p}$.

(v) *If $\mathcal{K}$ is decomposed into eigenspaces of U and U has eigenvalues $\{z_\lambda\}_{\lambda\in\Lambda}$ with multiplicities $\{\nu_\lambda\}_{\lambda\in\Lambda}$, then*

$$U\boxtimes\pi\sim\bigoplus_{\lambda\in\Lambda}(\pi\circ\gamma_{z_\lambda})^{\oplus\nu_\lambda}.$$

Proof. (i) Let $v\in\mathcal{K}$ be an eigenvector of U such that $Uv=cv$. Put $\mathcal{H}'\equiv\mathbf{C}v\otimes\mathcal{H}$. Then we see that $(U\boxtimes\pi)(s_i)(v\otimes\phi)=v\otimes(\pi\circ\gamma_c)(s_i)\phi$ for each $i=1,\ldots,N$. Therefore $(U\boxtimes\pi)|_{\mathcal{H}'}\sim\pi\circ\gamma_c$.
(ii) If W is a unitary on $\mathcal{K}$ such that $WUW^*=V$, then $(W\otimes I)(U\boxtimes\pi)(s_i)(W\otimes I)^*=(V\boxtimes\pi)(s_i)$ for each $i=1,\ldots,N$.
(iii) This is obtained by slightly generalizing Lemma 2.4 in [**15**].
(iv) Put $E_i\equiv\frac{1}{p}\sum_{j=1}^{p}\bar{\xi}_i^{j-1}U^{j-1}$. Then $UE_i=\xi_iE_i$, $E_i^*E_i=E_i$ and $E_i^*=E_i$. Hence $\mathcal{K}=\mathcal{K}_1\oplus\cdots\oplus\mathcal{K}_p$ where $\mathcal{K}_i\equiv E_i\mathcal{K}$ for $i=1,\ldots,p$. From this, $(\mathcal{K}_i\otimes\mathcal{H},(U\boxtimes\pi)|_{\mathcal{K}_i\otimes\mathcal{H}})\sim(\mathcal{H},\pi\circ\gamma_{\xi_i})^{\oplus\nu}$.
(v) This follows from the proof of (iv). □

Proposition 2.2. *For a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_A$, put a new representation $(L_2(\mathbf{R},\mathcal{H}),\,\hat{\pi})$ of $\mathcal{O}_A$ by*

$$(\hat{\pi}(s_i)\phi)(r)\equiv\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}e^{-\sqrt{-1}tr}\pi(s_i)\phi(t)\,dt\quad(\phi\in L_2(\mathbf{R},\mathcal{H}),\,r\in\mathbf{R})$$

for $i=1,\ldots,N$. Then

$$\hat{\pi}\sim\{\pi\oplus\pi\circ\gamma_{\sqrt{-1}}\oplus\pi\circ\gamma_{-1}\oplus\pi\circ\gamma_{-\sqrt{-1}}\}^{\oplus\infty}.$$

Proof. We see that $\hat{\pi}=\mathcal{F}\boxtimes\pi$ for the Fourie unitary operator $\mathcal{F}$ on $L_2(\mathbf{R})$. Because $\mathcal{F}^4=I$ and $\mathcal{F}^j\neq I$ for $j=1,2,3$, the statement holds by Lemma 2.1. □

We review results in [**10**].

Definition 2.3. *For $A=(a_{ij})\in M_N(\{0,1\})$, a data $\{(M_i,q_i,B_i)\}_{i=1}^N$ is called the (canonical)A-coordinate if*

$$B_i\equiv\ \{\,j\in\{1,\ldots,N\}:a_{ij}=1\,\},\quad M_i\equiv a_{i1}+\cdots+a_{iN},$$

$$q_i:\ \ B_i\to\{1,\ldots,M_i\};\quad q_i(j)\equiv\#\{k\in B_i:k\leq j\}$$

for $i=1,\ldots,N$.

Lemma 2.4. *Let $A=(a_{ij})\in M_N(\{0,1\})$ with the A-coordinate $\{(M_i,q_i,B_i)\}_{i=1}^N$ and $M_0\equiv N$. Assume that a unital C^*-algebra $\mathcal{B}$ satisfies the following condition: $\mathcal{B}$ contains $\mathcal{O}_{M_i}$ for each $i=0,\ldots,N$ when $M_i\geq 2$ as C^*-subalgebras with common unit. Let $t_{M_i,1},\ldots,t_{M_i,M_i}$ be canonical generators of $\mathcal{O}_{M_i}$ for $i=0,\ldots,N$ as elements in $\mathcal{B}$, respectively where we put $\mathcal{O}_1=\mathbf{C}I$ and $t_{1,1}=I$. Under these assumptions, put $s_i\equiv t_{M_0,i}(\,a_{i1}t_{M_i,q_i(1)}t^*_{M_0,1}+\cdots+a_{iN}t_{M_i,q_i(N)}t^*_{M_0,N}\,)$. Then $\{s_i\}_{i=1}^N$ satisfies (2.1) with respect to A.*

By these preparation, we show a method to construct representations of $\mathcal{O}_A$ from representations of the Cuntz algebras as follows:

Lemma 2.5. *Let $A\in M_N(\{0,1\})$ with the A-coordinate $\{(M_i,q_i,B_i)\}_{i=1}^N$ and $M_0\equiv N$ and $s_1,\ldots,s_N$, $t_{M,1},\ldots,t_{M,M}$ be canonical generators of $\mathcal{O}_A$ and $\mathcal{O}_M$, respectively for $M=M_0,\ldots,M_N$. Let $\pi^{(M_i)}$ be representation of $\mathcal{O}_{M_i}$ on a Hilbert space $\mathcal{H}$ where $t_{1,1}\equiv I$, $\pi^{(1)}(I)\equiv I$ when $M_i=1$, then there is a representation $\pi^{(A)}$ of $\mathcal{O}_A$ on $\mathcal{H}$ defined by*

$$\pi^{(A)}(s_i)\equiv\sum_{j=1}^N a_{ij}\pi^{(N)}(t_{N,i})\pi^{(M_i)}(t_{M_i,q_i(j)})\pi^{(N)}(t_{N,j})^*\quad(i=1,\ldots,N).$$

Proof. By Lemma 2.4, it holds. □

2.3. Permutative representations and GP representations of $\mathcal{O}_N$. A representation $(\mathcal{H},\pi)$ of $\mathcal{O}_N$ is *permutative* if there is a complete orthonormal basis $\{e_n\}_{n\in\Lambda}$ of $\mathcal{H}$ which satisfies ${}^{\forall}(n,i)\in\Lambda\times\in\{1,\ldots,N\}$, ${}^{\exists}m\in\Lambda$ s.t. $\pi(s_i)e_n=e_m$. Any permutative representation is completely reducible. We generalize this class of representation as *generalized permutative representations =(GP representations)* in [**8**, **9**, **16**, **17**]. In order to explain easily, we show GP representations of the Cuntz algebras with a 1-cycle. Let $S(\mathbf{C}^N)\equiv\{z\in\mathbf{C}^N:\|z\|=1\}$ be the complex sphere in a complex vector space $\mathbf{C}^N$.

Definition 2.6. *Let $(\mathcal{H},\pi)$ be a representation of $\mathcal{O}_N$.*

(i) *$(\mathcal{H},\pi)$ is $P(J;z)$ for $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}^k$, $k\geq 1$ and a phase $z\in U(1)$ if there is a cyclic unit vector $\Omega\in\mathcal{H}$ such that $\pi(s_J)\Omega=z\Omega$ and $\{\pi(s_{j_l}\cdots s_{j_k})\Omega : l=1,\ldots,k\}$ is an orthonormal family in $\mathcal{H}$.*

(ii) *$(\mathcal{H},\pi)$ is $GP(z)$ for $z=(z_1,\ldots,z_N)\in S(\mathbf{C}^N)$ if there is a cyclic unit vector $\Omega\in\mathcal{H}$ such that $\pi(z_1s_1+\cdots+z_Ns_N)\Omega=\Omega$.*

(iii) *$(\mathcal{H},\pi)$ is $P(J)$ for $J=(j_n)_{n\in\mathbf{N}}\in\{1,\ldots,N\}^{\infty}$ if there is an orthonormal family $\{e_n\}_{n\in\mathbf{N}}$ in $\mathcal{H}$ such that $\pi(s_{j_n})^*e_n=e_{n+1}$ for each $n\in\mathbf{N}$.*

For $J\in\{1,\ldots,N\}^*_1$, denote $P(J)\equiv P(J;1)$. For any $J\in\{1,\ldots,N\}^{\#}$, $P(J)$ is equivalent to a cyclic permutative representation.

We review results about $P(J)$ here: For $J\in\{1,\ldots,N\}^*_1$ and $z\in U(1)$, $P(J;z)$ is irreducible if and only if J is non periodic. For $J\in\{1,\ldots,N\}^{\infty}$, $P(J)$ is irreducible if and only if J is non eventually periodic. For $J_1,J_2\in\{1,\ldots,N\}^*_1$ and $z_1,z_2\in U(1)$, $P(J_1;z_1)\sim P(J_2;z_2)$ if and only if $(J_1,z_1)\sim(J_2,z_2)$ where $P(J_1;z_1)\sim P(J_2;z_2)$ means the unitary equivalence of two representations which satisfy the condition $P(J_1;z_1)$ and $P(J_2;z_2)$, respectively. For $J_1,J_2\in\{1,\ldots,N\}^{\infty}$, $P(J_1)\sim P(J_2)$ if and only if $J_1\sim J_2$. If $J\in\{1,\ldots,N\}^k$, $k\geq 1$ and $z\in U(1)$, then $P(J;1)\circ\gamma_z=P(J;z^k)$. If $J\in\{1,\ldots,N\}^{\infty}$ and $z\in U(1)$, then $P(J)\circ\gamma_z=P(J)$. For $J\in\{1,\ldots,N\}^*_1$, $z\in U(1)$ and $p\geq 1$,

$$P(J^p;z)=\bigoplus_{j=1}^{p}P(J;\xi^{j-1}z^{1/p}) \tag{2.3}$$

where $\xi\equiv e^{2\pi\sqrt{-1}/p}$. (2.3) is unique up to unitary equivalences. Especially we have $P(J^p;1)=\bigoplus_{j=1}^{p}P(J;\xi^{j-1})$. For each $J\in\{1,\ldots,N\}^*_1$,

$$P(J^{\infty})=\int_{U(1)}^{\oplus}P(J;z)\,d\eta(z).$$

For any $z\in S(\mathbf{C}^N)$, $GP(z)$ exists uniquely up to unitary equivalences. For any $z\in S(\mathbf{C}^N)$, $GP(z)$ is irreducible. For $z,z'\in S(\mathbf{C}^N)$, $GP(z)\sim GP(z')$ if and only if $z=z'$. For $z=(z_1,\ldots,z_N)\in S(\mathbf{C}^N)$, $GP(z)$ is equivalent to the GNS-representation by a state ρ of $\mathcal{O}_N$ which is defined by

$\rho(s_J s_{J'}^*) \equiv \overline{z_J} z_{J'}$ where $J, J' \in \{1,\ldots,N\}^*$, $|J| + |J'| \geq 1$, $z_J \equiv z_{j_1} \cdots z_{j_k}$ when $J = (j_1,\ldots,j_k)$, and $s_J = I$, $z_J = 1$ when $J = \emptyset$.

We see that $GP(z\varepsilon_j) = P(j;\bar{z})$ for $j = 1,\ldots,N$ and $z \in U(1)$ where $\{\varepsilon_j\}_{j=1}^N$ is the canonical basis of $\mathbf{C}^N$.

Eigenequations are important to classify representations of the Cuntz-Krieger algebras. For $J \in \{1,\ldots,N\}_A^*$, there is a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_A$ such that $\pi(s_J)$ has eigenvalue if and only if $J \in \{1,\ldots,N\}_{A,c}^*$. This is proved in [**19**].

About states of GP representations of $\mathcal{O}_N$, see [**8**, **16**]. About type III representations of $\mathcal{O}_A$, see [**21**].

3. Representations of $\mathcal{O}_A$ by branching function systems

Representations of the Cuntz-Krieger algebras are constructed by partial isometries on $L_2(X,\mu)$ for a measure space (X,μ). We introduce a simple method to construct partial isometries from maps on measure spaces([**12**, **13**, **14**, **15**]).

3.1. ***A*-branching function systems.** Let (X,μ) and (Y,ν) be measure spaces and f be a measurable map from X to Y which is injective and there exists the Radon-Nikodým derivative Φ_f of $\nu \circ f$ with respect to μ and Φ_f is non zero almost everywhere in X. We denote the set of such maps by $RN(X,Y)$ and put $RN_{loc}(X,Y) \equiv \bigcup_{X_0 \subset X} RN(X_0,Y)$. We simply denote $G(X,Y) \equiv \{\varphi \in RN(X,Y) : {}^{\exists}\varphi^{-1} \in RN(Y,X)\}$, $RN(X) \equiv RN(X,X)$ and $G(X) \equiv G(X,X)$. For $f \in RN_{loc}(X)$, we denote the domain and the range of f by $D(f)$ and $R(f)$, respectively. If $f \in RN(Y)$, then $f^{-1} \in RN(R(f))$. $RN_{loc}(X)$, $RN(X)$ and $G(X)$ are a groupoid, a semigroup and a group by composition of maps, respectively. We denote $X \times Y$ and $X \cup Y$, the direct product and the direct sum of (X,μ) and (Y,ν) as measure space, respectively. For $f \in RN(X_1,Y_1)$ and $g \in RN(X_2,Y_2)$, $f \oplus g \in RN(X_1 \cup X_2, Y_1 \cup Y_2)$ is defined by $(f \oplus g)|_{X_1} \equiv f$, $(f \oplus g)|_{X_2} \equiv g$.

Definition 3.1. *For a measure space (X,μ) and $A \in M_N(\{0,1\})$, a family $f = \{f_i\}_{i=1}^N$ of measurable maps on X is an A-branching function system on (X,μ) if f satisfies the following conditions:*

(i) $f_i \in RN_{loc}(X)$ *for each* $i = 1,\ldots,N$.
(ii) $\mu(R(f_i) \cap R(f_j)) = 0$ *when* $i \neq j$.
(iii) $\mu(D(f_i) \setminus \bigcup_{j:a_{ij}=1} R(f_j)) = 0$ *for each* $i = 1,\ldots,N$.
(iv) $\mu(X \setminus \bigcup_{i=1}^N R(f_i)) = 0$.

Specially, if A is full, then we call A-branching function system by (N-)branching function system simply. We denote the set of all A-branching function systems, branching function systems on (X,μ) by $\mathrm{BFS}_A(X)$, $\mathrm{BFS}_N(X)$, *respectively.*

The notion of original branching function system was introduced in order to construct a representation of $\mathcal{O}_N$ from a family of transformations by [**4**]. Definition 3.1 coincides with originals when A is full.

Definition 3.2. *Let (X,μ) and (Y,ν) be measure spaces.*

(i) *F is the coding map of $f=\{f_i\}_{i=1}^N \in \mathrm{BFS}_A(X)$ if F is a map on X such that $(F\circ f_i)(x)=x$ almost everywhere in X and $i=1,\ldots,N$.*

(ii) *For $f=\{f_i\}_{i=1}^N \in \mathrm{BFS}_A(X)$ and $g=\{g_i\}_{i=1}^N \in \mathrm{BFS}_A(Y)$, $f\sim g$ if there is $\varphi\in G(X,Y)$ such that $\varphi\circ f_i\circ\varphi^{-1}=g_i$ for $i=1,\ldots,N$.*

(iii) *For $\varphi\in G(X)$ and $g=\{g_i\}_{i=1}^N\in\mathrm{BFS}_A(Y)$, we denote $\varphi\boxtimes g\equiv\{\varphi\times g_i\}_{i=1}^N\in\mathrm{BFS}_A(X\times Y)$.*

(iv) *For $f=\{f_i\}_{i=1}^N\in\mathrm{BFS}_A(X)$ and $g=\{g_i\}_{i=1}^N\in\mathrm{BFS}_A(Y)$, we denote $f\oplus g\equiv\{f_i\oplus g_i\}_{i=1}^N\in\mathrm{BFS}_A(X\cup Y)$.*

The following are easily proved by checking the axiom in 3.1:

Lemma 3.3. *Let (X,μ) be a measure space and $A\in M_N(\{0,1\})$ with the A-coordinate $\{(M_i,q_i,B_i)\}_{i=1}^N$ and $M_0\equiv N$. If there is $f^{(M_i)}=\{f_j^{(M_i)}\}_{j=1}^{M_i}\in\mathrm{BFS}_{M_i}(X)$ for each $i=0,\ldots,N$, then a family $f^{(A)}\equiv\{f_i^{(A)}\}_{i=1}^N$ of maps on X defined as follows is an A-branching function system on X:*

$$f_i^{(A)}(x)\equiv\left\{f_i^{(N)}\circ f_{q_i(j)}^{(M_i)}\circ(f_j^{(N)})^{-1}\right\}(x)\quad(\textit{when } x\in f_j^{(N)}(X),\ j\in B_i)$$

for $i=1,\ldots,N$ where we put $\mathrm{BFS}_1(X)\equiv\{id_X\}$ for convenience.

By Lemma 3.3, if we find sufficiently many branching function systems on a measure space, we can construct an A-branching function system from them.

For $f=\{f_i\}_{i=1}^N\in\mathrm{BFS}_A(X)$, denote $f_J\equiv f_{j_1}\circ\cdots\circ f_{j_k}$ when $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}_A^k$, $k\geq 1$, and define $f_0\equiv id$. For $X_0\subset X$, put $<X_0>_f\equiv\{f_J(x),F^n(x)\in X: J\in\{1,\ldots,N\}_A^*,\ n\in\mathbf{N},\ x\in X_0\}$ where F is the coding map of f.

Definition 3.4. *For $A\in M_N(\{0,1\})$, let $f\in\mathrm{BFS}_A(X)$.*

(i) *For $X_0\subset X$, f is X_0-cyclic if $\mu(X\setminus<X_0>_f)=0$. Specially, we call that f is cyclic if f is $\{x_0\}$-cyclic for some $x_0\in X$.*

(ii) *For $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}_{A,c}^k$, $\{Y_i\}_{i=1}^k$ is a μ-cycle by J if $f_J(x)=x$ almost everywhere in Y_1, Y_i is a non μ-null subset of X, $\mu(Y_i\cap Y_{i'})=0$ when $i\neq i'$ and $\mu(f_{j_{i-1}}(Y_i)\setminus Y_{i-1})=0$ for $i=2,\ldots,k$ and $\mu(f_{j_k}(Y_1)\setminus Y_k)=0$.*

(iii) *For $J=(j_n)_{n\in\mathbf{N}}\in\{1,\ldots,N\}_A^\infty$, $\{Y_n\}_{n\in\mathbf{N}}$ is a μ-chain by J if Y_n is a non μ-null subset of X, $\mu(Y_n\cap Y_m)=0$ when $n\neq m$ and $\mu(f_{j_{n-1}}(Y_n)\setminus Y_{n-1})=0$ for $n\geq 2$.*

(iv) *For $J\in\{1,\ldots,N\}_{A,c}^*$(resp. $J\in\{1,\ldots,N\}_A^\infty$), f has a $MP(J)$-component if f has a μ-cycle(resp. a μ-chain) by J.*

(v) *For* $J=(j_i)_{i=1}^k\in\{1,\ldots,N\}_{A,c}^k$ *(resp.* $J=(j_n)_{n\in\mathbf{N}}\in\{1,\ldots,N\}_A^\infty$*),* f *is* $MP(J)$ *if there is a subset* $Y\subset X$ *such that* f *is* Y*-cyclic and* $\{Y_i\}_{i=1}^k$ *is a* μ*-cycle by* J *where* $Y_i\equiv(f_{j_i}\circ\cdots\circ f_{j_k})(Y)$ *for* $i=1,\ldots,k$ *(resp.* $\{Y_n\}_{n\in\mathbf{N}}$ *is a* μ*-chain by* J *where* $Y_n\equiv(f_{j_1}\circ\cdots\circ f_{j_n})^{-1}(Y)$ *for* $n\geq 1$*).*

3.2. Representations of $\mathcal{O}_A$ by A-branching function systems. For $f\in RN_{loc}(X,Y)$, define an operator $S(f)$ from $L_2(X,\mu)$ to $L_2(Y,\nu)$ by

$$(3.1)\qquad (S(f)\phi)(x)\equiv\begin{cases}\left\{\Phi_f\left(f^{-1}(x)\right)\right\}^{-1/2}\phi(f^{-1}(x)) & (x\in R(f)),\\ \\ 0 & (\text{otherwise})\end{cases}$$

for $\phi\in L_2(X,\mu)$ and $x\in X$. $S(f)$ is a partial isometry from $L_2(X,\mu)$ to $L_2(Y,\nu)$ with the range projection $M_{\chi_{R(f)}}$ and the domain projection $M_{\chi_{D(f)}}$ where M_g is the multiplication operator of $g\in L_\infty(X,\mu)$ and χ_W is the characteristic function on $W\subset X$. Furthermore we see that $S(f)^*=S(f^{-1})$, $S(id_X)=I$ and $S(f)L_2(\Omega)=L_2(f(\Omega))$ for $\Omega\subset X$.

Let $\mathrm{PIso}(\mathcal{H})$ be the groupoid of all partial isometries on a Hilbert space $\mathcal{H}$ by the ordinary product of operators. Let (X_i,μ_i) be measure spaces for $i=1,2,3,4$. Let $f\in RN_{loc}(X_1,X_2)$ and $g\in RN_{loc}(X_2,X_3)$. If $\mu(D(g)\cap R(f))\neq 0$, then $g\circ f\in RN_{loc}(X_1,X_3)$ and

$$(3.2)\qquad S(g)S(f)=S(g\circ f).$$

Specially, a map S from $RN_{loc}(X_i)$ to $\mathrm{PIso}(L_2(X_i,\mu_i))$ is a groupoid homomorphism for $i=1,2,3,4$. For $f\in RN(X_1,X_2)$ and $g\in RN(X_3,X_4)$,

$$S(f\times g)=S(f)\otimes S(g),\quad S(f\oplus g)=S(f)\oplus S(g)$$

where we identify $L_2(X_i\times X_j,\mu_i\times\mu_j)$ and $L_2(X_i,\mu_i)\otimes L_2(X_j,\mu_j)$, $L_2(X_i\cup X_j,\mu_i\cup\mu_j)$ and $L_2(X_i,\mu_i)\oplus L_2(X_j,\mu_j)$ for $i,j=1,2,3,4$, respectively.

Theorem 3.5. *Let* $A\in M_N(\{0,1\})$*. For a family* $f=\{f_i\}_{i=1}^N$ *of maps on a measure space* (X,μ)*,* $C^*<\{S(f_i)\}_{i=1}^N>\cong\mathcal{O}_A$ *if* $f\in\mathrm{BFS}_A(X)$*.*

Proof. We can easily verify that $S(f_1),\ldots,S(f_N)$ satisfy (2.1) by using (3.2). □

By Theorem 3.5, Theorem 1.1 is shown and we see that

$$\pi_f(s_i)\equiv S(f_i)\quad(i=1,\ldots,N)$$

defines a representation $(L_2(X,\mu),\pi_f)$ of $\mathcal{O}_A$.

Let (X,μ) and (Y,ν) be measure spaces. For $f\in\mathrm{BFS}_A(X)$ and $g\in\mathrm{BFS}_A(Y)$, if $f\sim g$, then $\pi_f\sim\pi_g$. For $\varphi\in G(X)$, $f\in\mathrm{BFS}_A(X)$ and $g=\{g_i\}_{i=1}^N\in\mathrm{BFS}_A(Y)$, the followings hold:

$$(3.3)\qquad \pi_{\varphi\boxtimes g}\sim S(\varphi)\boxtimes\pi_g,\quad \pi_{f\oplus g}\sim\pi_f\oplus\pi_g$$

where $S(\varphi)\boxtimes\pi_g$ is in (2.2).

Remark that $g \circ f$ in rhs of (3.2) is the ordinary composition of two transformations f and g but not special product of them. By (3.2), we see that the map S realizes the iteration of transformations on a measure space as the product of operators on a Hilbert space naturally. In fact, if F is the coding map of f in Definition 3.2, then

$$(\pi_f(s_i)\phi)(x) = \chi_{R(f_i)}(x)\sqrt{\Phi_F(x)}\phi(F(x)) \quad (i = 1, \ldots, N)$$

for $\phi \in L_2(X, \mu)$ and $x \in X$. We denote $(L_2(X, \mu), \pi_f)$ by π_f simply. From this, $\pi_f(s_J) = S(f_J)$ for each $J \in \{1, \ldots, N\}_A^*$ and

$$\pi_f(s_J)\phi = \chi_{R(f_J)} \cdot \sqrt{\Phi_{F^k}} \cdot \phi \circ F^k \quad (|J| = k).$$

In this sense, π_f realizes the action of a semigroup $\{F^n : n \geq 1\}$ generated by F.

Proposition 3.6. *Let $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_A(X)$.*

(i) *Let σ_r be the shift on $\mathbf{Z}$ for $r \in \mathbf{Z}$ which is defined by $\sigma_r(n) \equiv n - r$ for $n \in \mathbf{Z}$. Then the following holds:*

$$\pi_{\sigma_r \boxtimes f} \sim \begin{cases} \displaystyle\int_{U(1)}^{\oplus} \pi_f \circ \gamma_{w^r} \, d\eta(w) & (r \neq 0), \\ (\pi_f)^{\oplus\infty} & (r = 0). \end{cases}$$

(ii) *If σ is the shift of $\mathbf{Z}_p$ for $p \geq 1$, then*

$$\pi_{\sigma \boxtimes f} \sim \bigoplus_{j=1}^{p} \pi_f \circ \gamma_{\xi^j}$$

where $\xi \equiv e^{2\pi\sqrt{-1}/p}$.

Proof. By Lemma 2.1, (3.3) and a slightly generalization of Proposition 3.9 in [**15**], they hold. □

Theorem 3.7. *Let (X, μ) be a measure space and $f \in \mathrm{BFS}_A(X)$.*

(i) *If f is X_0-cyclic for $X_0 \subset X$, then $\pi_f(\mathcal{O}_A)L_2(X_0) = L_2(X, \mu)$. Specially, if f is cyclic, then $(L_2(X, \mu), \pi_f)$ is cyclic.*

(ii) *If there is a μ-cycle $\{Y_n\}_{n=1}^k$ by $J \in \{1, \ldots, N\}_{A,c}^k$, then $(L_2(X, \mu), \pi_f)$ contains a $P(J)^{\oplus\nu}$-component where $\nu \equiv \mathrm{dim} L_2(Y_1)$.*

(iii) *If there is a μ-chain $\{Y_n\}_{n\in\mathbf{N}}$ by $J \in \{1, \ldots, N\}_A^\infty$, then $(L_2(X, \mu), \pi_f)$ contains a $P(J)^{\oplus\nu}$-component where $\nu \equiv \mathrm{dim} L_2(Y_1)$.*

Proof. (i) Since $\pi_f(s_J)L_2(X_0) = L_2(f_J(X_0))$ for each $J \in \{1, \ldots, N\}_A^*$, $\pi_f(\mathcal{O}_A)L_2(X_0) \supset L_2(< X_0 >_f)$. By the choice of X_0, the statement holds. (ii) By assumption, $\pi_f(s_J)\phi = \phi$ for each $\phi \in L_2(Y_1)$. Let $\{e_a\}_{a\in\Lambda}$ be a complete orthonormal basis of $L_2(Y_1)$ such that $\#\Lambda = \nu$. Then $V_a \equiv \pi_f(\mathcal{O}_A)e_a$ is $P(J)$ and $\{V_a\}_{a\in\Lambda}$ is a mutually orthogonal family. Hence $L_2(X, \mu) \supset \bigoplus_{a\in\Lambda} V_a \sim P(J)^{\oplus\nu}$.

(iii) By assumption, $S(f_{j_n})^* L_2(Y_n) = L_2(Y_{n+1})$ for each $n \in \mathbf{N}$. Let $\{e_a^{(1)}\}_{a\in\Lambda}$ be a complete orthonormal basis of $L_2(Y_1)$. Put $e_a^{(m)} \equiv S(f_{j_{m-1}})^* \cdots S(f_{j_1})^* e_a^{(1)} \in L_2(Y_m)$ for $m \geq 2$. Then $\{e_a^{(m)}\}_{a\in\Lambda}$ be a complete orthonormal basis of $L_2(Y_m)$. Therefore $V_a \equiv \pi_f(\mathcal{O}_A)e_a^{(1)}$ is $P(J)$. In the same way as the case (ii), we have the statement. $\square$

Examples of Theorem 3.7 is shown in [**20**].

When (X,μ) is atomic, then $(L_2(X,\mu),\pi_f)$ is well-studied. We treat these as *permutative representations of the Cuntz-Krieger algebras* in [**19**].

4. Standard constructions of representations of the Cuntz-Krieger algebras

We show the standard construction of A-branching function system on measure spaces $X = \mathbf{N}, \mathbf{Z}, [0,1], \mathbf{T}^1, \mathbf{R}$ for any $A \in M_N(\{0,1\})$. By Lemma 3.3, it is sufficient to give a family $\{f^{(M)}\}_{M\geq 1}$ of branching function systems on a measure space for each $M \geq 2$ in order to construct an A-branching function system $f^{(A)}$ on X. The meaning of "standardness" of $f^{(A)}$ is understood from that of $\{f^{(M)}\}_{M\geq 1}$.

In this section, we fix $N \geq 2$ and $A = (a_{ij}) \in M_N(\{0,1\})$ with the A-coordinate $\{(M_i, q_i, B_i)\}_{i=1}^N$.

4.1. Standard representations on $l_2(\mathbf{N})$. For $M \geq 1$, define $f^{(M)} = \{f_i^{(M)}\}_{i=1}^M \in \mathrm{BFS}_M(\mathbf{N})$ by

$$f_i^{(M)}(n) \equiv M(n-1)+i \quad (i=1,\ldots,M,\ n\in\mathbf{N}). \tag{4.1}$$

Then

$$(l_2(\mathbf{N}), \pi_{f^{(M)}}) \sim P(1) \quad (M \geq 2).$$

Specially, the permutative representation of $\mathcal{O}_M$ by $f^{(M)}$ is called *the standard representation of* $\mathcal{O}_M$ in [**1**, **2**]. We denote $(l_2(\mathbf{N}), \pi_{f^{(M)}})$ by $(l_2(\mathbf{N}), \pi_S)$. The standard representation $\mathcal{O}_M$ is irreducible for each $N \geq 2$. This is well-known in [**4**, **6**, **7**, **8**]. The restriction $(l_2(\mathbf{N}), \pi_{f^{(M)}}|_{UHF_M})$ is irreducible, too where $UHF_M \equiv \mathcal{O}_M^{U(1)}$. When $M = 2$, $(l_2(\mathbf{N}), \pi_{f^{(M)}}|_{CAR})$ is equivalent to the Fock representation of $CAR = \mathcal{O}_2^{U(1)}$ under the standard embedding of CAR into $\mathcal{O}_2$([**1**]).

$f^{(A)} = \{f_i^{(A)}\}_{i=1}^N \in \mathrm{BFS}_A(\mathbf{N})$ in Lemma 3.3 is given by

$$f_i^{(A)}(N(m-1)+j) = N(M_i(m-1)+q_i(j)-1)+i \quad (m\in\mathbf{N},\ j\in B_i)$$

where $R(f_i^{(A)}) = \{N(n-1)+i : n\in\mathbf{N}\}$ and $D(f_i^{(A)}) = \coprod_{j\in B_i} R(f_j^{(A)})$ for $i = 1,\ldots,N$. $f^{(A)}$ is a permutative representation of $\mathcal{O}_A$([**19**]).

4.2. Representations on $l_2(\mathbf{Z})$. For $M \geq 1$, define $f^{(M)} = \{f_i^{(M)}\}_{i=1}^M \in \mathrm{BFS}_M(\mathbf{Z})$ by

$$f_i^{(M)}(n) \equiv Mn + i - 1 \quad (i = 1, \ldots, M,\, n \in \mathbf{Z}). \tag{4.2}$$

Then $f^{(A)} = \{f_i^{(A)}\}_{i=1}^N \in \mathrm{BFS}_A(\mathbf{Z})$ in Lemma 3.3 is given by

$$f_i^{(A)}(Nm + j - 1) = NM_i m + N(q_i(j) - 1) + i - 1$$

where $R(f_i^{(A)}) = \{Nn + i - 1 : n \in \mathbf{Z}\}$ and $D(f_i^{(A)}) = \coprod_{j \in B_i} R(f_j^{(A)})$ for $i = 1, \ldots, N$. There is no general theory of classification of π_f for f in the above.

Next, we show a classification of some representations of $\mathcal{O}_N$ on $l_2(\mathbf{Z})$.

Proposition 4.1. *For $M \geq 2$ and $j \in \mathbf{Z}$, let $g^{[j]} = \{g_i^{[j]}\}_{i=1}^M \in \mathrm{BFS}_M(\mathbf{Z})$ by*

$$g_i^{[j]}(n) \equiv Mn + i + j \quad (n \in \mathbf{Z},\, i = 1, \ldots, M).$$

For the representation $(l_2(\mathbf{Z}), \pi_{g^{[j]}})$ of $\mathcal{O}_M$ by $g^{[j]}$, the followings hold:

(i) *When $M = 2$, $(l_2(\mathbf{Z}), \pi_{g^{[j]}}) \sim P(1) \oplus P(2)$ for each $j \in \mathbf{Z}$.*

(ii) *When $M \geq 3$ and $j \equiv r \bmod M - 1$ for $r = 0, \ldots, M - 2$,*

$$(l_2(\mathbf{Z}), \pi_{g^{[j]}}) \sim \begin{cases} P(1) \oplus P(M) & (r = M - 2), \\ \\ P(N - 1 - r) & (r \neq M - 2). \end{cases}$$

Proof. $g_i^{[j]}$ is monotone increasing(*resp.* decreasing) on $\{n \in \mathbf{Z} : n > -(j+1)/(M-1)\}$ (*resp.* $\{n \in \mathbf{Z} : n < -(j+M)/(M-1)\}$.) Therefore $g^{[j]}$ has neither cycle nor chain in $\mathbf{Z} \setminus W$. From these, $g^{[j]}$ has cycles in $W \equiv \{n \in \mathbf{Z} : \alpha \geq n \geq \alpha - 1\} = \{[\alpha], [\alpha] - 1\}$ where $\alpha \equiv -(j+1)/(M-1)$.
(i) If $M = 2$, then $\alpha = -(j+1)$ and $g_i^{[j]}(\alpha) = -j - 2 + i$. From these, we see that $g_1^{[j]}(\alpha) = \alpha$. $g_i^{[j]}(\alpha - 1) = -j - 4 + i$. Hence $g_2^{[j]}(\alpha - 1) = \alpha - 1$. $(l_2(\mathbf{Z}), \pi_{g^{[j]}}) \sim P(1) \oplus P(2)$ for each $j \in \mathbf{Z}$.
(ii) If $M \geq 3$, then put $j = (M-1)k - l - 1$ where $0 \leq l \leq M - 2$. Then $\alpha = -k + l/(M-1)$ and $[\alpha] = -k$. $g_i^{[j]}([\alpha]) = [\alpha] - l - 1 + i$. Hence $i = 1 + l \in \{1, \ldots, M-1\}$ if and only if $g_{1+l}^{[j]}([\alpha]) = [\alpha]$. By taking $r = M - 2 - l$, $(l_2(\mathbf{Z}), \pi_{g^{[j]}})$ always has a component $P(l+1) = P(M-1-r)$. Furthermore $g_i^{[j]}([\alpha] - 1) = -M + [\alpha] - l - 1 + i$. Hence $M + l = i$ if and only if $g_M^{[j]}([\alpha] - 1) = [\alpha] - 1$. Therefore $(l_2(\mathbf{Z}), \pi_{g^{[j]}})$ has a $P(M)$-component only when $l = 0$. In consequence, $(l_2(\mathbf{Z}), \pi_{g^{[j]}}) \sim P(1) \oplus P(M)$ when $r = M - 2$. $\square$

For example,

$\underline{M=3}$ $\underline{M=4}$

j	*odd*	*even*
$\pi_{g^{[j]}}$	$P(1)\oplus P(3)$	$P(2)$

j	$3k$	$3k-1$	$3k-2$
$\pi_{g^{[j]}}$	$P(3)$	$P(1)\oplus P(4)$	$P(2)$

Corollary 4.2. *For $f^{(M)} \in \mathrm{BFS}_M(\mathbf{Z})$ in (4.2),*

$$\pi_{f^{(M)}} \sim P(1)\oplus P(M) \quad (M\geq 2).$$

Proof. Because $f^{(M)} = g^{[-1]}$ and $-1\equiv M-2 \bmod M-1$, the statement holds. $\square$

By Corollary 4.2, the irreducible decomposition of $\pi_{f^{(M)}}$ can be described as a same style. This shows that the definition in (4.2) seems standard.

4.3. Representations on $L_2[0,1]$. For $M\geq 1$, put $f^{(M)} \equiv \{f_i^{(M)}\}_{i=1}^M \in \mathrm{BFS}_M([0,1])$ by

$$f_i^{(M)}(x) \equiv (x+i-1)/M \quad (i=1,\ldots,M,\ x\in[0,1]). \tag{4.3}$$

Then $f^{(A)} = \{f_i^{(A)}\}_{i=1}^N \in \mathrm{BFS}_A([0,1])$ in Lemma 3.3 is given by

$$f_i^{(A)}(x) = \frac{1}{M_i}x + \frac{M_i(i-1)+q_i(j)-j}{M_iN} \quad (x\in R(f_j^{(A)}),\ j\in B_i) \tag{4.4}$$

where $R(f_i^{(A)}) = [(i-1)/N,\ i/N]$ and $D(f_i^{(A)}) = \bigcup_{j\in B_i} R(f_j^{(A)})$ for $i=1,\ldots,N$ and $j\in B_i$. Note that $f_i^{(A)}$ is defined on $D(f_i^{(A)})$ up to measure-zero subset in $[0,1]$. That is, $f_i^{(A)}$ is well-defined as a function in $L_\infty[0,1]$. We see that the representation $(L_2[0,1],\pi_{f^{(A)}})$ of $\mathcal{O}_A$ on $L_2[0,1]$ is given by

$$(\pi_{f^{(A)}}(s_i)\phi)(x) = \chi_{R(f_i^{(A)})}(x)\sqrt{M_i}\,\phi((f_i^{(A)})^{-1}(x))$$

for $i=1,\ldots,N$ and $\phi\in L_2[0,1]$. Hence $\pi_{f^{(A)}}(s_i)\mathbf{1} = \sqrt{M_i}\chi_{R(f_i^{(A)})}$ for $i=1,\ldots,N$ where $\mathbf{1}$ is the constant function on $[0,1]$ with value 1.

Proposition 4.3. *For $M\geq 2$ and $f^{(M)}$ in (4.3),*

$$(L_2[0,1],\pi_{f^{(M)}}) \sim GP(M^{-1/2},\ldots,M^{-1/2}).$$

Proof. Let a unit vector $z\equiv (M^{-1/2},\ldots,M^{-1/2})\in\mathbf{C}^M$. Then $\mathbf{1}$ is a cyclic vector of $(L_2[0,1],\pi_{f^{(M)}})$ and $\pi_{f^{(M)}}(s(z))\mathbf{1} = \mathbf{1}$. $\square$

Proposition 4.3 is a special case of Theorem 2.8 in [**12**].

4.4. Representations on $L_2(\mathbf{T}^1)$. We show two kinds of representations of $\mathcal{O}_A$ on $L_2(\mathbf{T}^1)$ for $\mathbf{T}^1 \equiv \{z \in \mathbf{C} : |z| = 1\}$.

(i) Let R_M and V be operators on $L_2(\mathbf{T}^1)$ by

$$(R_M\phi)(z) \equiv \phi(z^M), \quad (V\phi)(z) \equiv z\phi(z) \quad (\phi \in L_2(\mathbf{T}^1),\, z \in \mathbf{T}^1)$$

and $T_{M,i} \equiv V^{i-1}R_M$ for $i = 1, \ldots, M$. Put

$$\pi^{(M)}(s_i) \equiv T_{M,i} \quad (i = 1, \ldots, M).$$

Then $(L_2(\mathbf{T}^1), \pi^{(M)}) \sim P(1) \oplus P(M) \sim (l_2(\mathbf{Z}), \pi_{f^{(M)}})$ in (4.2). Then the representation $(L_2(\mathbf{T}^1), \pi^{(A)})$ of $\mathcal{O}_A$ in Lemma 2.5 from $\{\pi^{(M_i)}\}_{i=0}^N$ is as follows:

$$\pi^{(A)}(s_i) = T_{N,i} \sum_{j \in B_i} T_{M_i, q_i(j)} T_{N,j}^* \quad (i = 1, \ldots, N).$$

(ii) Put $f^{(M)} = \{f_i^{(M)}\}_{i=1}^M \in \mathrm{BFS}_M(\mathbf{T}^1)$ by

$$f_i^{(M)}(z) \equiv z^{1/M} e^{2\pi\sqrt{-1}(i-1)/M} \quad (z \in \mathbf{T}^1,\, i = 1, \ldots, M).$$

Then $\pi_{f^{(M)}} \sim GP(M^{-1/2}, \ldots, M^{-1/2})$ and this is equivalent to that by (4.3) for each $M \geq 2$. Then $f^{(A)} = \{f_i^{(A)}\}_{i=1}^N \in \mathrm{BFS}_A(\mathbf{T}^1)$ in Lemma 3.3 is given by

$$f_i^{(A)}(z) = z^{1/M_i} e^{2\pi\sqrt{-1}\{(q_i(j)-1)/NM_i + (i-1)/N\}} \quad (z \in \mathbf{T}^1,\, i = 1, \ldots, N).$$

4.5. Representations on $L_2(\mathbf{R})$. For $M \geq 1$, put $f^{(M)} \equiv \{f_i^{(M)}\}_{i=1}^M \in \mathrm{BFS}_M(\mathbf{R})$ by

$$f_i^{(M)}(x) \equiv x + (M-1)[x] + i - 1 \quad (i = 1, \ldots, M,\, x \in \mathbf{R}) \tag{4.5}$$

where $[x]$ is the Gauss symbol. Then $f^{(A)} = \{f_i^{(A)}\}_{i=1}^N \in \mathrm{BFS}_A(\mathbf{R})$ in Lemma 3.3 is given by

$$f_i^{(A)}(x + (N-1)[x] + j - 1) = x + (NM_i - 1)[x] + N(q_i(j) - 1) + i - 1$$

for $i = 1, \ldots, N$, $x \in R(f_j^{(A)})$, $j \in B_i$ where $D(f_i^{(A)}) \equiv \bigcup_{j \in B_i} R(f_j^{(A)})$ and $R(f_i^{(A)}) \equiv \coprod_{k \in \mathbf{Z}} [Nk + i - 1,\, Nk + i]$ for $i = 1, \ldots, N$. For example, when $N = 2$, $R(f_1^{(2)}) = \coprod_{k \in \mathbf{Z}} [2k, 2k+1]$, $R(f_2^{(2)}) = \coprod_{k \in \mathbf{Z}} [2k+1, 2k+2]$. $f_1^{(2)}(x) = x + [x]$, $f_2^{(2)}(x) = x + [x] + 1$. Then $f_1^{(2)}(x) = x$ for $x \in [0, 1]$ and $f_2^{(2)}(x) = x$ for $x \in [-1, 0]$.

Theorem 4.4. *For $M \geq 2$ and $\lambda \in \mathbf{R}$, put*

$$g_i^{[\lambda]}(x) \equiv x + (M-1)[x] + i + \lambda \quad (x \in \mathbf{R},\, i = 1, \ldots, M).$$

Then $g^{[\lambda]} = \{g_i^{[\lambda]}\}_{i=1}^M \in \mathrm{BFS}_M(\mathbf{R})$ and the followings hold:

(i) *When* $M=2$,

$$(L_2(\mathbf{R}),\pi_{g^{[\lambda]}}) \sim \begin{cases} (P(1)\oplus P(2))^{\oplus\infty} & (\lambda\in\mathbf{Z}),\\ \\ (P(11)\oplus P(22))^{\oplus\infty} & (\lambda\in\mathbf{R}\setminus\mathbf{Z}). \end{cases}$$

(ii) *When* $M\geq 3$,

$$(L_2(\mathbf{R}),\pi_{g^{[\lambda]}}) \sim \begin{cases} (P(1)\oplus P(M))^{\oplus\infty} & (\lambda\in\mathbf{Z},\, r=M-2),\\ \\ (P(M-1-r))^{\oplus\infty} & (\lambda\in\mathbf{Z},\, r\neq M-2),\\ \\ (P(11)\oplus P(MM))^{\oplus\infty} & (\lambda\in\mathbf{R}\setminus\mathbf{Z},\, r=M-2),\\ \\ (P(M-1-r,M-1-r))^{\oplus\infty} & (\lambda\in\mathbf{R}\setminus\mathbf{Z},\, r\neq M-2) \end{cases}$$

where $r\in\{0,\ldots,M-2\}$ *such that* $[\lambda]\equiv r \bmod M-1$.

Proof. Assume $M\geq 3$. When $\lambda=j\in\mathbf{Z}$, then $g_i^{[\lambda]}(x)=x+(M-1)[x]+i+j$. Put a map Ψ from $\mathbf{R}$ to $\mathbf{Z}\times[0,1)$ by $\Psi(x)\equiv([x],x-[x])$. Then $\Psi\circ g_i^{[\lambda]}\circ\Psi^{-1}=h_i\times id$ where $h_i(n)\equiv Nn+i+j$. Hence $\pi_{g^{[\lambda]}}\sim\pi_h^{\oplus\infty}$ and $\pi_h=P(1)\oplus P(M)$ when $j\equiv M-2 \bmod M-1$, $\pi_h=P(M-1-r)$ when $j\equiv r \bmod M-1$ and $r\neq M-2$ by Proposition 4.1. When $\lambda\in\mathbf{R}\setminus\mathbf{Z}$, put $\theta\equiv\lambda-[\lambda]$ and a map Ψ from $\mathbf{R}$ to $\mathbf{Z}\times\mathbf{Z}_2\times[0,\theta)$ by

$$\Psi(x)\equiv\begin{cases} ([x],1,x-[x]) & (\text{when } x-[x]\in[0,\theta)),\\ \\ ([x],2,x-[x]-\theta) & (\text{when } x-[x]\in[\theta,1)). \end{cases}$$

Then $\Psi\circ g_i^{[\lambda]}\circ\Psi^{-1}=h_i\times\sigma\times id$ where σ is a shift on $\mathbf{Z}_2\equiv\{1,2\}$. From this, $\pi_{g^{[\lambda]}}\sim(S(\sigma)\boxtimes\pi_h)^{\oplus\infty}$, $S(\sigma)\boxtimes\pi_h\sim\pi_h\oplus(\pi_h\circ\gamma_{-1})$. $\pi_h\sim P(M-1-r)$ by Proposition 4.1. By Proposition 3.6 and (2.3), the statement holds. The case $M=2$ follows from the proof of that of $M\geq 3$. □

This is an example of Theorem 3.7 (ii) when A is full and $\nu=\infty$.

Corollary 4.5. *For* $f^{(M)}$ *in* (4.5)*,*

$$(L_2(\mathbf{R}),\pi_{f^{(M)}})\sim P(1)^{\oplus\infty}\oplus P(M)^{\oplus\infty}\quad (M\geq 2).$$

5. Examples

We show examples of representations of $\mathcal{O}_A$ for matrices in p 268, [**5**] and their open problems. In this section, $s_1,\ldots,s_N$ are canonical generators of $\mathcal{O}_A$ for $A\in M_N(\{0,1\})$.

5.1. Example 1. Put a matrix $A_1 \in M_3(\{0,1\})$ by

$$A_1 \equiv \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

There is an isomorphism φ from $\mathcal{O}_4$ to $\mathcal{O}_{A_1}$ as follows:

$$\varphi(v_1) \equiv s_1 s_3, \quad \varphi(v_2) \equiv s_3, \quad \varphi(v_3) \equiv s_2 s_3, \quad \varphi(v_4) \equiv s_2 s_1 s_3 \tag{5.1}$$

where $v_1, \ldots, v_4$ are canonical generators of $\mathcal{O}_4$([**10**]). We see that $\varphi^{-1}(s_1) = v_1 v_2^*$, $\varphi^{-1}(s_2) = v_4 v_1^* + v_3 v_2^*$, $\varphi^{-1}(s_3) = v_2$.

Example 5.1. Define operators T_1, T_2, T_3 on $l_2(\mathbf{N})$ by

$$T_1 e_{4(n-1)+i} \equiv \delta_{2,i} e_{4(n-1)+1}, \quad T_2 e_{4(n-1)+i} \equiv \delta_{1,i} e_{4(n-1)+4} + \delta_{2,i} e_{4(n-1)+3},$$

$$T_3 e_n \equiv e_{4(n-1)+2}$$

for $i = 1, 2, 3, 4$ and $n \in \mathbf{N}$. Then the followings hold:

(i) Put $\pi_0(s_i) \equiv T_i$, $i = 1, 2, 3$. Then a representation $(l_2(\mathbf{N}), \pi_0)$ of $\mathcal{O}_{A_1}$ is irreducible.

(ii) Any representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_{A_1}$ with a cyclic vector Ω which satisfies

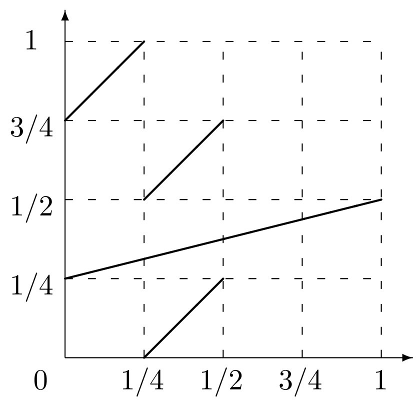
$$\pi(s_1 s_3)\Omega = \Omega \tag{5.2}$$

is equivalent to $(l_2(\mathbf{N}), \pi_0)$ in (i). Specially, a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_{A_1}$ with a cyclic vector Ω which satisfies (5.2) is unique up to unitary equivalences.

Proof. For the standard representation $(l_2(\mathbf{N}), \pi_S)$ of $\mathcal{O}_4$ in § 4.1 and φ is in (5.1), we see that $\pi_0 = \pi_S \circ \varphi^{-1}$. Since π_S is irreducible, π_0 is, too. Since π_S is uniquely characterized by $\pi_S(v_1)\Omega = \Omega$, π_0 is, too by $\varphi(v_1) = s_1 s_3$ for $\Omega \equiv e_1$. $\square$

Example 5.2. Let g_1, g_2, g_3 be functions on $[0,1]$ as follows:

$$\begin{cases} g_1(x) = \chi_{[1/4,1/2]}(x) \cdot (x - 1/4), \\ g_2(x) = \chi_{[0,1/4]}(x) \cdot (x + 3/4) \\ \qquad\qquad + \chi_{[1/4,1/2]}(x) \cdot (x + 1/4), \\ g_3(x) = (x+1)/4. \end{cases}$$



Then $g = \{g_1, g_2, g_3\}$ is an A_1-branching function system on $[0,1]$ and the followings hold:

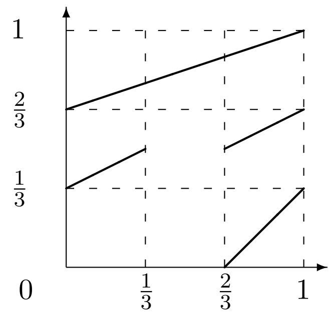
(i) A representation $(L_2[0,1], \pi_g)$ of $\mathcal{O}_{A_1}$ is irreducible and it is equivalent to a representation $(\mathcal{H}, \pi)$ with a cyclic vector Ω which satisfies $\pi(s_1s_3 + s_3 + s_2s_3 + s_2s_1s_3)\Omega = 2\Omega$.

(ii) $(L_2[0,1], \pi_g)$ is inequivalent to $(l_2(\mathbf{N}), \pi_0)$ in Example 5.1.

Proof. Let $f = \{f_i\}_{i=1}^4 \in \mathrm{BFS}_4([0,1])$ by $f_i(x) \equiv (x+i-1)/4$ for $i = 1, \ldots, 4$, $\Omega \equiv \mathbf{1}$ and φ in (5.1), then $\pi_f(v_i) = \pi_g(\varphi(v_i))$ for $i = 1, \ldots, 4$. Since $\pi_f \not\sim \pi_0 \circ \varphi$, $\pi_g \not\sim \pi_0$. Because $(L_2[0,1], \pi_f)$ is an irreducible representation of $\mathcal{O}_4$, $(L_2[0,1], \pi_g)$ is irreducible, too. Furthermore $\pi_g(s_1s_3 + s_3 + s_2s_3 + s_2s_1s_3)\Omega = \pi_f(v_1 + v_2 + v_3 + v_4)\Omega = 2\Omega$. $\square$

Example 5.3. We have the following $f = \{f_1, f_2, f_3\} \in \mathrm{BFS}_{A_1}([0,1])$ in § 4.3:

$$\begin{cases} f_1(x) = x - 2/3 & (x \in [2/3, 1]), \\ f_2(x) = \begin{cases} x/2 + 1/3 & (x \in [0, 1/3]), \\ x/2 + 1/6 & (x \in [2/3, 1]), \end{cases} \\ f_3(x) = x/3 + 2/3. \end{cases}$$



From this, we have the following representation $(L_2[0,1], \pi_f)$ of $\mathcal{O}_{A_1}$:

$$\begin{cases} (\pi_f(s_1)\phi)(x) = & \chi_{[0,1/3]}(x)\phi(x + 2/3), \\ (\pi_f(s_2)\phi)(x) = & \sqrt{2}\{\chi_{[1/3,1/2]}(x)\phi(2x - 2/3) + \chi_{[1/2,2/3]}(x)\phi(2x - 1/3)\}, \\ (\pi_f(s_3)\phi)(x) = & \sqrt{3}\chi_{[2/3,1]}(x)\phi(3x - 2) \end{cases}$$

for $\phi \in L_2[0,1]$ and $x \in [0,1]$.

Question 5.4. *Show the property of* π_f*, whether* π_f *is irreducible or not, whether* π_f *is equivalent to representations in Example* 5.1 *or Example* 5.2.

5.2. Example 2. Put a matrix $A_2 \in M_3(\{0,1\})$ by

$$A_2 \equiv \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

There is an isomorphism ψ from $\mathcal{O}_5 \otimes M_2(\mathbf{C})$ to $\mathcal{O}_{A_2}$ as follows:

$$(5.3) \qquad \begin{cases} \psi(t_1 \otimes I) \equiv & s_1 s_2 s_1 s_1^* + s_2 s_1, \\ \psi(t_2 \otimes I) \equiv & s_1 s_2 s_3 s_1 s_1^* + s_2 s_3 s_1, \\ \psi(t_3 \otimes I) \equiv & s_1 s_2 s_3 s_1^* + s_2 s_3 s_1^* s_1, \\ \psi(t_4 \otimes I) \equiv & s_1 s_3 s_1 s_1^* + s_3 s_1, \\ \psi(t_5 \otimes I) \equiv & s_1 s_3 s_1^* + s_3 s_1^* s_1, \\ \psi(I \otimes E_{12}) \equiv & s_1 \end{cases}$$

where $t_1, \ldots, t_5$ are canonical generators of $\mathcal{O}_5$ and $\{E_{ij}\}_{i,j=1,2}$ is the matrix unit of $M_2(\mathbf{C})$([**10**]). On the contrary, we see that $\psi^{-1}(s_1) = I \otimes E_{12}$, $\psi^{-1}(s_2) = t_1 \otimes E_{21} + (t_2 t_4^* + t_3 t_5^*) \otimes E_{22}$, $\psi^{-1}(s_3) = t_4 \otimes E_{21} + t_5 \otimes E_{22}$.

Example 5.5. Define operators $\pi(s_1), \pi(s_2), \pi(s_3)$ on $l_2(\mathbf{N} \times \{1,2\})$ by

$$\begin{cases} \pi(s_1) e_{n,i} \equiv & \delta_{2,i} e_{n,1}, \\ \pi(s_2) e_{5(n-1)+m,i} \equiv & \delta_{1,i} e_{5(5(n-1)+m-1)+1,2} \\ & +\delta_{2,i}(\delta_{4,m} e_{5(n-1)+2,2} + \delta_{5,m} e_{5(n-1)+3,2}), \\ \pi(s_3) e_{n,i} \equiv & \delta_{1,i} e_{5(n-1)+4,2} + \delta_{2,i} e_{5(n-1)+5,2} \end{cases}$$

for $i = 1,2$, $m = 1, \ldots, 5$ and $n \in \mathbf{N}$ where $e_{n,i} \equiv e_n^{'} \otimes e_i^{''}$ and $e_n^{'}$, $e_i^{''}$ are canonical basis of $l_2(\mathbf{N})$ and $\mathbf{C}^2$, respectively. Then the followings hold:

(i) $(l_2(\mathbf{N} \times \{1,2\}), \pi)$ is an irreducible representation of $\mathcal{O}_{A_2}$.

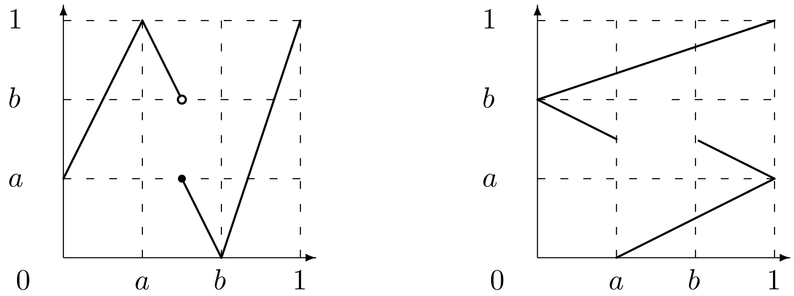
(ii) Any cyclic representation $(\mathcal{H}, \pi^{'})$ of $\mathcal{O}_{A_2}$ with a cyclic vector Ω which satisfies

$$\pi^{'}(s_1 s_2 s_1 s_1^* + s_2 s_1)\Omega = \Omega$$

is equivalent to $(l_2(\mathbf{N} \times \{1,2\}), \pi)$.

Proof. (i) Let π_0 be the standard representation of $\mathcal{O}_5$ on $l_2(\mathbf{N})$ and ι is the canonical representation of $M_2(\mathbf{C})$ on $\mathbf{C}^2$. Then $((\pi_0 \otimes \iota) \circ \psi^{-1})(s_i) = \pi(s_i)$ on $l_2(\mathbf{N} \times \{1,2\}) \cong l_2(\mathbf{N}) \otimes \mathbf{C}^2$ for $i = 1,2,3$ where ψ is in (5.3). Because $\pi_0 \otimes \iota$ is an irreducible representation of $\mathcal{O}_5 \otimes M_2(\mathbf{C})$, π is irreducible, too. (ii) By (i), the characterization of π is uniquely given by the equation $(\pi_0 \otimes \iota)(t_1 \otimes I)e_{1,1} = e_{1,1}$. By (5.3), $\pi(s_1 s_2 s_1 s_1^* + s_2 s_1) = (\pi_0 \otimes \iota)(t_1 \otimes I)$. Hence the statement holds. □

Example 5.6. For $0 < a < b < 1$, consider a map $F^{(a,b)}$ on $X = [0,1]$ which graph is given as follows:



$F^{(a,b)}$ is the coding map of $f^{(a,b)} = \{f_1^{(a,b)}, f_2^{(a,b)}, f_3^{(a,b)}\} \in \mathrm{BFS}_{A_2}([0,1])$ given as follows: $f_i^{(a,b)} : D_i \to R_i$, $i = 1,2,3$,

$$\begin{cases} f_1^{(a,b)}(x) = & a(x-a)/(1-a) \quad (x \in D_1), \\ f_2^{(a,b)}(x) = & \begin{cases} -\alpha x + b, & (x \in R_1), \\ -\alpha(x-1) + a & (x \in R_2), \end{cases} \\ f_3^{(a,b)}(x) = & (1-b)x + b \quad (x \in [0,1]) \end{cases}$$

where $\alpha \equiv (b-a)/(1-b+a)$, $R_1 \equiv [0,a]$, $R_2 \equiv [a,b]$, $R_3 \equiv [b,1]$, $D_1 \equiv [a,1]$, $D_2 \equiv [b,1]$, $D_3 \equiv [0,1]$. Let $\pi^{(a,b)} \equiv \pi_{f^{(a,b)}}$.

Question 5.7. *Classify a representation $\pi^{(a,b)}$ of $\mathcal{O}_{A_2}$ by a, b.*

This is not so simple as its appearance. For example, a family of slope parameters of a branching function system on a closed interval is the complete invariant(up to unitary equivalence) of representations in Theorem 2.8 in [**12**].

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