A NOTE ON TREES AND LINEAR ALGEBRAIC GROUPS OVER THE POLYNOMIAL RINGS

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Introduction

In [2], Serre showed us a way how to describe the structure of a group when it acts on a tree: suppose that a group Γ acts on a tree X, and if we know the quotient graph $\Gamma\backslash X$ as well as the stabilizers of the edges and vertices, then we can describe Γ in terms of these data. An interesting example of trees with a group action arises from lattices and the arithmetic subgroup of algebraic groups: Let C be a connected smooth projective curve over a field with the function field K and let $p \in C$ be a closed point. The open subvariety $C \setminus \{p\}$ is an affine curve, say $\operatorname{Spec} A$. Then we can construct a tree X of which vertices are the $K^{\times}$-orbits of $\mathcal{O}_{C,p}$-lattices in $K^{\oplus 2}$, and $\mathrm{GL}_2(A)$ acts on it naturally. The structure of the quotient graph Y and the stabilizers can be described in terms of the vector bundles on C of rank 2 that is trivial on $C \setminus \{p\}$: the vertices of Y are the classes of such vector bundles modulo $\mathcal{O}_C(mP)$ ($m \in \mathbb{Z}$) and the edges are strongly related to elementary transformation of vector bundles.

We should remark here that the quotient graph is controllable in some extent. For a vector bundle E on C of rank 2, we put

$$N(E) := \sup\{\deg L - \deg E/L \mid L : \text{subbundle of rank } 1\}.$$

Let m be a sufficiently large positive integer and let Y_m be the subgraph of Y generated by the vertices $[E]$ with $N(E) \leq m$. Then the complement of Y_m is a disjoint union of half lines, and they corresponds to the line bundles of $C \setminus \{p\}$. (c.f. [2, II.2.3 Theorem 9]). One of the keys to obtain such a structure theorem is the vanishing of the first cohomology of higher degree. That implies the decomposability of E with large $N(E)$, and we can control how such bundles behave under elementary transformations.

In this note, we shall consider them over higher dimensional variety. Then our objects are not only vector bundles but reflexive modules. In section 1, we will construct the tree of the classes of "lattices" and make clear the relation with the reflexive modules. Although it can be proceeded in the same way as the case of curves except for some technical problem, the quotient graphs are much more complicated than the case of curves. We can also define $N(E)$ as well (c.f. 1.2), but E is not necessarily decomposable even if $N(E)$ is large, and it is quite difficult to control the behavior of reflexive sheaves under elementary transformation. It is accordingly impossible to expect that the quotient graph has such a simple structure as the case of curve. Nevertheless, we can describe it in the case of the projective spaces. In section 2, preparing some results on reflexive modules on the projective space, we will

Date: May 13, 2004.
*Partially supported by JSPS Research Fellowships for Young Scientists.

describe the structure of the quotient graph Y and apply it to see the structure of the groups $\mathrm{GL}_2(k[x_1, \dots x_d])$ etc.

In this note, combinatorial notions (graphs, amalgams and so on) often appear. We follow [2] on notations and terminologies on them.

Finally, the author would like to express sincere gratitude to Prof. S. Kondo. He had a useful conversation with the author, which led him to the starting point of this note.

1. Lattices, trees and reflexive sheaves

1.1. **A tree associated with the classes of lattices.** The purpose of this subsection is to define the tree X. Let us fix our notations here. Let R be a regular local ring and let $u \in R$ be a prime element. Let $\bar{R}$ always denote the integral domain $R/(u)$. A *reflexive R-module* is an R-module M of finite type such that the canonical homomorphism $M \to M^{**}$, where $M^* := \mathrm{Hom}_R(M, R)$, is an isomorphism.

Lemma 1.1. *Let L and L' be reflexive R-modules such that $L' \subsetneq L$ and $u^l(L/L') = 0$ for some $l \in \mathbb{N}$. Then there exist reflexive R-modules $L_0, L_1, \dots, L_n$ such that*

$$L' = L_n \subsetneq L_{n-1} \subsetneq \cdots \subsetneq L_0 = L$$

and that L_{i-1}/L_i is a torsion-free $\bar{R}$-module. Moreover, if we put $r_i = \mathrm{rk}_{\bar{R}}(L_{i-1}/L_i)$, then $\sum_{i=1}^n r_i$ is independent of the choice of such sequences.

Proof. If such an sequence of reflexive R-modules exists, then we can easily see $\sum_{i=1}^n r_i = \mathrm{length}_{R_{(u)}}(L_{(u)}/L'_{(u)})$ and hence $\sum_{i=1}^n r_i$ is independent of the choice of the sequences. Let us show the existence of such a sequence by induction on $l \in \mathbb{N}$, where l is the minimal integer with $u^l \left(L_{(u)}/L'_{(u)}\right) = 0$. If $l = 1$, then L/L' is a torsion-free $\bar{R}$-module by [1, Corollary 1.5], hence there is nothing to prove. Suppose $l > 1$. Then we can draw a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow & & \\ 0 & \longrightarrow & L_1 & \longrightarrow & L & \longrightarrow & T' & \longrightarrow & 0, \end{array}$$

in which $T' := \mathrm{Hom}_R(\mathrm{Hom}_R(T, \bar{R}), \bar{R})$. Since $T'_{(u)} \neq 0$, we have

$$\mathrm{length}_{R_{(u)}}\left((L_1)_{(u)}/L'_{(u)}\right) < \mathrm{length}_{R_{(u)}}\left(L_{(u)}/L'_{(u)}\right) = l$$

and hence we obtain our assertion by the induction hypothesis. □

Let K denote the subring $R[u^{-1}]$ of the quotient field of R. A sub-R-module L of $K^{\oplus 2}$ is called an *R-lattice*, or simply a *lattice*, if it is a reflexive R-module of rank 2 with $L \otimes_R K = K^{\oplus 2}$. If R is of dimension 1, an R-lattice is nothing but a lattice in the usual sense. We can restate Lemma 1.1 in terms of lattices:

Corollary 1.2. *Let L and L' be lattices with $L' \subsetneq L$. Then there exist lattices $L_0, L_1, \dots, L_n$ such that*

$$L' = L_n \subsetneq L_{n-1} \subsetneq \cdots \subsetneq L_0 = L$$

and that L_{i-1}/L_i is a torsion-free $\bar{R}$-module. Moreover, if we put $r_i = \mathrm{rk}_{\bar{R}}(L_{i-1}/L_i)$, then $\sum_{i=1}^n r_i$ is independent of the choice of such sequences.

For two lattices L and L' with $L' \subset L$, we assign an integer $d(L', L)$ by

$$d(L', L) := \begin{cases} \sum_{i=1}^{n} r_i \text{ in Corollary 1.2} & \text{if } L \neq L', \\ 0 & \text{if } L = L', \end{cases}$$

which we call the *distance* from L to L'.

Remark 1.3. For lattices L and L' with $L' \subset L$, suppose L/L' is a torsion-free $\bar{R}$-module of rank 2. Since L is reflexive, L/uL is a torsion-free $\bar{R}$-module. That implies the induced surjection $L/uL \to L/L'$ is injective, and hence $L' = uL$. Accordingly, if $uL_{i-1} \neq L_i$ in Corollary 1.2, then $r_i = 1$, and if $uL_{i-1} \neq L_i$ hold for all i, then $d(L', L) = n$.

The group $K^\times$ acts on the set of lattices by $a : L \mapsto aL$. Each orbit $K^\times L$ is a totally ordered set with respect to "$\subset$", and $aL = bL$ if and only if $a^{-1}b \in R^\times$. We put $\operatorname{Vert} X := \{\text{lattices}\}/K^\times$, and let $[L]$ denote the class in $\operatorname{Vert} X$ of a lattice L. We define

$$d(\Lambda', \Lambda) := \inf_{[L]=\Lambda, [L']=\Lambda', L' \subset L} d(L', L).$$

Lemma 1.4. *Let L and L' be representatives of distinct $\Lambda \in \operatorname{Vert} X$ and $\Lambda' \in \operatorname{Vert} X$ respectively with $L' \subset L$.*

(1) *Suppose $d(\Lambda'; \Lambda) = d(L'; L)$. Then for any sequence $L' = L_n \subset \cdots \subset L_0 = L$ as in Corollary 1.2 between L and L', and for any i with $1 \leq i \leq n$, we have $\operatorname{rk}_{\bar{R}}(L_{i-1}/L_i) = 1$ and hence $n = d(\Lambda'; \Lambda)$.*

(2) *$d(\Lambda', \Lambda) = d(\Lambda, \Lambda')$.*

(3) *$d(\Lambda', \Lambda) = d(L', L)$ if and only if $L' \not\subseteq uL$.*

Proof. (1) If L_{i-1}/L_i is of rank 2 as an $\bar{R}$-module, then $L_i = uL_{i-1}$ (c.f. Remark 1.3). Thus we have such a sequence

$$L' = L_n \subsetneq \cdots L_i \subsetneq uL_{i-2} \subsetneq \cdots \subsetneq uL_0 = uL$$

as in Corollary 1.2. That implies

$$d(\Lambda'; \Lambda) \leq d(L'; uL) < d(L'; L) = d(\Lambda'; \Lambda),$$

which is a contradiction. Accordingly $\operatorname{rk}_{\bar{R}}(L_{i-1}/L_i) = 1$ for all i.

(2) Let $L' = L_n \subsetneq \cdots \subsetneq L_0 = L$ be such a sequence that attains $d(\Lambda, \Lambda')$. Then $u^n L = u^n L_0 \subsetneq u^{n-1} L_1 \subsetneq \cdots \subsetneq L_n = L'$, is also such a sequence between L' and $u^n L$, and $\operatorname{rk}_{\bar{R}}(u^{i-1}L_{n-i+1}/u^i L_{n-i}) = 1$. That implies $d(\Lambda', \Lambda) = d(u^n L, L') \geq d(\Lambda, \Lambda')$. The same argument also shows $d(\Lambda', \Lambda) \leq d(\Lambda, \Lambda')$, and hence we obtain (2).

(3) The "only if" part is immediate. To show the other part, let, for $i = 1, 2$, L_i be a representative of Λ and L'_i be that of Λ' such that $L'_i \subset L_i$ and $L'_i \not\subseteq uL_i$. Let a, a' be elements of $K^\times$ with $L_2 = aL_1$ and $L'_2 = a'L'_1$. Then $L'_1 \subset (a')^{-1}aL_1$ and $L'_1 \not\subseteq u(a')^{-1}aL_1$. That implies $(a')^{-1}aL_1$ is the smallest element of $\{L \in K^\times L_1 \mid L'_1 \subset L\}$ as well as L_1, hence $(a')^{-1}aL_1 = L_1$. Therefore $L'_2 = aL'_1$ as well, and the multiplication a gives a one-to-one correspondence from a sequence between L'_1 and L_1 to that between L'_2 and L_2. Accordingly, the length of such a sequence is uniquely determined and gives the distance. □

Two $\Lambda, \Lambda' \in \operatorname{Vert} X$ is said to be *adjacent* if $d(\Lambda, \Lambda') = 1$. In this way we define a combinatorial graph X with $\operatorname{Vert} X$ as the set of vertices (c.f. [2, I.2.1]). For $\Lambda, \Lambda' \in \operatorname{Vert} X$, we call a sequence $(\Lambda_0 = \Lambda, \Lambda_1, \ldots, \Lambda_n = \Lambda')$ of $\operatorname{Vert} X$ with $d(\Lambda_{i-1}, \Lambda_i) = 1$ a *path* between Λ and $\Lambda'$of length n. For any path $(\Lambda_0, \Lambda_1, \ldots, \Lambda_n)$, we can take a sequence of lattices

$L_n \subset \cdots \subset L_0$ satisfying the condition of Lemma 1.4 (3) with $[L_i] = \Lambda_i$. In this situation, Lemma 1.4 (3) says that $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ is a path of the minimal length if and only if $L_n \not\subseteq uL_0$.

Proposition 1.5. *The graph X is a tree.*

Proof. For any distinct $\Lambda, \Lambda' \in \operatorname{Vert} X$, we can take representatives L of Λ and L' of Λ' such that $L' \subset L$ and $L' \not\subseteq uL$. Therefore the connectedness follows from Corollary 1.2 and Lemma 1.4. To see that X is a tree, it is sufficient to show that any path without backtracking is a path of minimal length. Let $L' = L_n \subset L_{n-1} \subset \cdots \subset \cdots \subset L_0 = L$ be a sequence that gives a path between $[L]$ and $[L']$ without backtracking. Let us prove it by induction on n. Suppose it true up to $n-1$. By the assumption that it has no backtracking, we can see $L_n \neq uL_{n-2}$. Moreover, since L_n and uL_{n-2} are lattices contained in L_{n-1} such that $d(L_n, L_{n-1}) = d(uL_{n-2}, L_{n-1}) = 1$, we have $uL_{n-2} \not\subseteq L_n$. Therefore the image of uL_{n-2} in L_{n-1}/L_n is non-trivial, and hence

$$\operatorname{codim}_{\operatorname{Spec} R}(\operatorname{Supp}(L_{n-1}/(uL_{n-2} + L_n))) \geq 2. \tag{1.5.1}$$

Now suppose $L_n \subset uL$. Then there is a natural inclusion $uL_{n-2} + L_n \hookrightarrow uL$, and since we have (1.5.1) and uL is reflexive, it extends to $L_{n-1} \hookrightarrow uL$, i.e., $L_{n-1} \subset uL$. That implies a path $([L_{n-1}], \ldots, [L_0] = [L])$ without backtracking of length $n-1$ is not of minimal length by Lemma 1.4 (3), which contradicts the induction hypothesis. □

1.2. **A tree with a group-action and reflexive sheaves.** Let S be a normal projective variety over a field k and let H be an effective irreducible reduced ample Cartier divisor on S. An open subscheme $U := S \setminus \operatorname{Supp} H$ is an affine scheme, say $\operatorname{Spec} A$. Let $p \in S$ be a closed point with $p \in \operatorname{Supp} H$. We assume S is regular at p. Let $u \in \mathcal{O}_{S,p}$ be an element defining H around p. Put $R := \mathcal{O}_{S,p}$ and $K := R[u^{-1}]$, and let Q denote the function field. Then, we can naturally regard R as a subring of K_p and A as that of $Q|_U$. For an R-lattice L, let E_L be a subsheaf of $Q^{\oplus 2}$ satisfying the following conditions.

(a) E_L is a reflexive $\mathcal{O}_S$-module.
(b) $E_L|_U = A^2$.
(c) $E_{L,p} = L$.

It is not difficult to see that there exists an open subscheme V containing U and p such that there exists uniquely a reflexive $\mathcal{O}_V$-module $E_{L,V}$ satisfying the above conditions (b) and (c). Since $\operatorname{codim}_S(V) \geq 2$, it can be uniquely extended to a reflexive sheaf on S by the following lemma, hence E_L with the above three properties uniquely exists.

Lemma 1.6. *Let S be a locally noetherian integral normal scheme and let $V \subset S$ be an open subscheme with* $\operatorname{codim}_S(S \setminus V) \geq 2$. *Let E be a reflexive $\mathcal{O}_S$-module on V. Then there exists a unique reflexive sheaf $\tilde{E}$ on S with $\tilde{E}|_S = E$.*

Proof. The uniqueness follows from [1, Proposition 1.6 (iii)]. To show the existence, we may replace S by its open subscheme. Since E is coherent, it is the cokernel of a homomorphism ϕ between free $\mathcal{O}_V$-modules of finite rank. Since S is normal, $i_*\phi$, where $i : V \hookrightarrow S$ is the canonical inclusion, is also a homomorphism between free $\mathcal{O}_S$-modules of finite rank. Its cokernel is coherent and coincides with E on V, and hence the double dual $\tilde{E}$ is a required reflexive $\mathcal{O}_S$-module. □

If $\alpha \in K^\times$ is of form $\alpha = \lambda u^n +$ (higher in u) with $0 \neq \lambda \in k$, then $E_{\alpha L} = E_L(-nH)$. Let Γ denote $\mathrm{GL}_2(A)$ or $\mathrm{PGL}_2(A)$. Then Γ acts on the set of lattices and on $\operatorname{Vert} X$. Two

$\mathcal{O}_S$-modules E_L and $E_{L'}$ are isomorphic if and only if there is an element $g \in \Gamma$ with $L' = gL$. In particular, if $gL = u^n L$, then $E_L \cong E_{gL} = E_L(-nH)$, and hence $n = 0$.

From now on, let *"reflexive sheaf" stand for "reflexive $\mathcal{O}_S$-module of rank 2 trivial over U"* in this article. We say that two reflexive sheaves E and E' are *H-equivalent* if there is an integer n with $E' \cong E(-nH)$. Since Γ acts on X from the left without inversion, we can make the quotient graph $Y := \Gamma\backslash X$. The following proposition follows easily from those observations.

Proposition 1.7. *Let L be a lattice and let $[L]$ denote its class.*

(1) *The correspondences $L \mapsto [L]$ and $L \mapsto E_L$ induces a bijection of* $\operatorname{Vert} Y$ *onto the set of H-equivalence classes of reflexive sheaves on S (c.f.* [2, II.2, Proposition 4]*).*

(2) *We have canonically*

$$\Gamma_L \cong \begin{cases} \operatorname{Aut}(E_L) & \text{if } \Gamma = \mathrm{GL}_2(A), \\ \operatorname{Aut}(E_L)/k^{\times} & \text{if } \Gamma = \mathrm{PGL}_2(A). \end{cases}$$

(3) $\Gamma_L = \Gamma_{[L]}$.

Via the identification in Proposition 1.7 (1), we regard $\operatorname{Vert} Y$ as the set of H-classes of reflexive sheaves.

Definition 1.8. Let E be a reflexive sheaf. A reflexive subsheaf $E' \subset E$ is said to be *H-maximal* or simply *maximal* if E/E' is a torsion-free $\mathcal{O}_H$-module of rank 1.

It is easy to see that two $\mathcal{E}, \mathcal{E}' \in \operatorname{Vert} Y$ are adjacent if and only if there exist reflexive sheaves E and E' representing $\mathcal{E}$ and $\mathcal{E}'$ respectively such that E' is an H-maximal reflexive subsheaf of E. Moreover, we can say the following for the edges.

Proposition 1.9. *For an $\mathcal{E} \in \operatorname{Vert} Y$, let us fix a reflexive sheaf $E \subset Q^{\oplus 2}$ representing $\mathcal{E}$. Then, there exists a natural bijection*

$$\operatorname{Aut}(E)\backslash\{\textit{H-maximal reflexive subsheaves of } E\} \to \{\textit{the edges with } \mathcal{E} \textit{ as an extremity}\}.$$

Proof. For a maximal reflexive subsheaf $E' \subset E$, let $e(E')$ be the edge of X such that the vertices $[E_p]$ and $[E'_p]$ are the extremities of $e(E')$. Then the map is given by

$$E' \mapsto \text{the image of } e(E') \text{ in } Y.$$

The inverse is given as follows. For an edge e of Y with $\mathcal{E}$ as an extremity, there exists a vertex $\Lambda \in \operatorname{Vert} X$ such that Λ is adjacent to $[E_p]$ and that e is the image of the edge connecting Λ and $[E_p]$. For such two vertices Λ_1 and Λ_2, there exist unique maximal reflexive subsheaves $E'_1, E'_2 \subset E$ respectively with $[E'_{i,p}] = \Lambda_i$. Then by the choice of Λ_1 and Λ_2, there exists $\gamma \in \Gamma_{[E_p]}$ with $\gamma\Lambda_1 = \Lambda$. By virtue of Proposition 1.7 (3) and (2), we see therefore $E'_{1,p}$ and $E'_{2,p}$ coincide up to $\operatorname{Aut}(E)$, hence E'_1 and E'_2 also coincide up to $\operatorname{Aut}(E)$. That implies that the correspondence $e \mapsto E'_1$ gives a well-defined inverse map. □

For a reflexive sheaf E on S, we define an integer $N_H(E)$ by

$$N_H(E) := \sup_L\{\deg_H(L) - \deg_H(E/L)\},$$

where L runs through the saturated coherent subsheaves of E of rank 1. Since a saturated subsheaf of a reflexive $\mathcal{O}_S$-module is reflexive, above L is automatically an invertible sheaf. Note that N_H naturally induces a function on $\operatorname{Vert} Y$.

From the definition of $N_H(E)$, it seems that the stability of E is closely related to the negativity of $N_H(E)$. Actually, if E is stable (resp. semistable), then $N_H(E) < 0$ (resp. $N_H(E) \leq 0$) in general, and the converse also holds if k is algebraically closed. We shall show that it holds for general k when S is the projective space in the next section.

Finally in this section, let us introduce the notion of N_H-sequence of a reflexive sheaf E. We call an exact sequence

$$0 \to L \to E \to M \to 0,$$

where L is an invertible sheaf and M is a torsion-free sheaf, an *N_H-sequence* of E if the image of $L \to E$ attains $N_H(E)$, i.e. $N_H(E) = \deg_H L - \deg_H M$. We regard two N_H-sequences

$$0 \to L_1 \to E \to M_1 \to 0$$

and

$$0 \to L_2 \to E \to M_2 \to 0$$

of E to be same if they coincides up to constant: if we can draw a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & E & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & \cong\big\downarrow & & id\big\downarrow & & \cong\big\downarrow & & \\ 0 & \longrightarrow & L_2 & \longrightarrow & E & \longrightarrow & M_2 & \longrightarrow & 0. \end{array}$$

2. The case of $\mathbb{P}^d_k$

We defined in the previous section the graph X with a Γ-action and described the relation between the quotient graph Y and the reflexive sheaves. In this section, we shall investigate the structure of Y deeply and describe Γ in terms of amalgams in the case of $\mathbb{P}^d_k$.

From now on, we will consider the case of $\mathbb{P}^d_k$ only. Let us fix our notation. We write $\mathcal{O}$ for the structure sheaf for simplicity. We fix an infinite hyperplane H, a k-point p on it and inhomogeneous coordinates $x_1, \dots x_d$ of $\mathbb{P}^d_k \setminus H \cong \mathbb{A}^d_k$. We denote $N_H(\cdot)$ by $N(\cdot)$ and call an N_H-sequence an N-sequence simply.

2.1. **Properties of the reflexive sheaves.** In this subsection, we shall show some properties on reflexive sheaves that we shall need in describing the graph of group (Γ, Y).

First of all, let us make clear the relation between the stability of E and the negativity of $N(E)$ in the case of the projective spaces.

Proposition 2.1. *Let E be a reflexive sheaf. Then E is stable (resp. semistable) if and only if $N(E) < 0$ (resp. $N(E) \leq 0$).*

Proof. The "if" part for a general k is the only assertion that needs proof. For a reflexive sheaf E on $\mathbb{P}^d_k$, let a be the maximal integer such that $E(-a)$ has a nontrivial global section. Note that a is characterized by the following property: a is the maximal integer such that there exists an injective homomorphism $\mathcal{O}(a) \to E$. Therefore an N-sequence of E is of form

$$0 \to \mathcal{O}(a) \to E \to I(b) \to 0, \tag{2.1.1}$$

where I is a coherent ideal with $\operatorname{codim}(\operatorname{Supp}(\mathcal{O}/I)) \geq 2$. On the other hand, since

$$h^0(\mathbb{P}^d_k, E(-a)) = h^0(\mathbb{P}^d_{\bar{k}}, (E \otimes_k \bar{k})(-a)),$$

where $\bar{k}$ is an algebraic closure of k, this integer a is also such one for $E \otimes_k \bar{k}$. Therefore an N-sequence of $E \otimes_k \bar{k}$ is of form

$$0 \to \mathcal{O}_{\mathbb{P}^d_{\bar{k}}}(a) \to E \otimes_k \bar{k} \to J(b) \to 0, \tag{2.1.2}$$

where J is a coherent ideal of $\mathcal{O}_{\mathbb{P}^d_{\bar{k}}}$ with $\operatorname{codim}(\operatorname{Supp}(\mathcal{O}_{\mathbb{P}^d_{\bar{k}}}/I)) \geq 2$. Consequently if E is not semistable, then in (2.1.2) we must have $a > b$ hence $N(E) = a - b > 0$, and if E is not stable, then $a \geq b$ hence $N(E) = a - b \geq 0$. □

The above proof tells us more information on N-sequences. If $N(E) > 0$, then the pull-back to $\mathbb{P}^d_{\bar{k}}$ of (2.1.1) is nothing but the exact sequence arising from the Harder-Narasimhan filtration, hence unique up to constants. Suppose $N(E) = 0$. If E is not the direct sum of two copies of an invertible sheaf, then $I \neq \mathcal{O}$. That implies $h^0(E(-a)) = 1$ and an N-sequence is determined uniquely up to constant. If E is the direct sum of two copies of an invertible sheaf, then (2.1.1) is of form

$$0 \to \mathcal{O}(a) \to E \to \mathcal{O}(a) \to 0,$$

which is an N-sequence of E. In this case, therefore, an N-sequence is uniquely determined up to $\operatorname{Aut}(E) \cong \mathrm{GL}_2(k)$: if

$$0 \to L \to E \to L \to 0,$$

is an N-sequence, then there exists $\sigma \in \operatorname{Aut}(E)$ such that we can draw the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(a) \arrow[r] \arrow[d, "\cong"'] & E \arrow[r] \arrow[d, "\cong"', "\sigma"] & \mathcal{O}(a) \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & L \arrow[r] & E \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

For a non-stable reflexive sheaf E, we say E is of *type* (n, I), where n is a non-negative integer and I is a coherent ideal, if $n = N(E)$ and there exists an N-sequence of form

$$0 \to L \to E \to IL(-n) \to 0.$$

By the observation above, this definition makes sense.

The next proposition is an assertion on the type of a maximal reflexive subsheaf. It is a key that makes it possible to control the graph Y.

Proposition 2.2. *Let E be a reflexive sheaf.*

(1) *For any maximal reflexive subsheaf $E' \subset E$, we have $|N(E') - N(E)| = 1$.*
(2) *For a maximal reflexive subsheaf $E' \subset E$, $N(E') > N(E)$ if and only if there exists an N-sequence*

$$0 \to L \to E \to IL(-N(E)) \to 0$$

such that the natural surjection $E \to E/E'$ factors through $E \to IL(-N(E))$. Moreover, if E is non-stable and E' satisfies that condition, then we can say the following.

(a) *Such E' is unique up to $\operatorname{Aut}(E)$. Furthermore, if E is not of type $(0, \mathcal{O})$, E' is unique.*
(b) *Suppose that E is of type (n, I) and E' is of type (n', I'). Then $I \subset I'$ and the equality holds if and only if $\operatorname{Ass}_{\mathcal{O}}(\mathcal{O}/I) \cap H = \emptyset$.*

Proof. Let us prove (2) first. Let

$$0 \to L' \to E' \to I'L'(-N(E')) \to 0$$

be an N-sequence of E' and let L be the saturation of L' in E. Then we have the following commutative diagram, in which any line is exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L' \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & I'L(-N(E')) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E/L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L/L' \arrow[r] \arrow[d] & E/E' \arrow[r] \arrow[d] & E/(E'+L) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Suppose $N(E') > N(E)$. Then the degree of any subsheaf of rank 1 of E cannot exceed $\deg(L')$. Therefore $L = L'$ and we can see that the middle horizontal line is an N-sequence. Furthermore, $E/E' \cong E/(E'+L)$ hence the natural homomorphism $E \to E/E'$ factors through the surjection $E \to E/L$ in the N-sequence.

Conversely let

$$0 \to L \to E \to IL(-N(E)) \to 0$$

be an N-sequence such that $E \twoheadrightarrow E/E'$ factors through $E \to IL(-N(E))$. Then the homomorphism $IL(-N(E))|_H/(\mathcal{O}_H\text{-torsion}) \twoheadrightarrow E/E'$ is an isomorphism, and we have the following commutative diagram, in which any line is exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & I'L(-(N(E)+1)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & IL(-N(E)) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & E/E' \arrow[r, "\cong"] \arrow[d] & IL(-N(E))|_H/(\mathcal{O}_H\text{-torsion}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

In this case, E' cannot have an invertible subsheaf with greater degree than L and hence the first horizontal line is an N-sequence of E'. Note in particular $N(E') = N(E) + 1$.

Under those equivalent conditions, it is easy to see, from the diagram, that $I \subset I'$ and that the equality holds if and only if $\mathrm{Ass}_{\mathcal{O}}(\mathcal{O}/I) \cap H = \emptyset$. The rest of the assertions in (2) in case that E is non-stable follows from the uniqueness of the N-sequence.

For (1), we have already shown $N(E') = N(E) + 1$ when $N(E') > N(E)$. We see that the case $N(E) = N(E')$ does not occur, for the parity of $\deg E$ and that of $\deg E'$ are different. Suppose $N(E) > N(E')$. Then $E(-H)$ is naturally a maximal subsheaf of E' with $N(E(-H)) > N(E')$, and hence it follows from the result that has been already obtained. □

Remark 2.3. In the case of $N(E') < N(E)$ above, we can draw the following commutative diagram, in which the first two horizontal sequences are N-sequences.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L(-H) \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & I'L(-N(E)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & IL(-N(E)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L|_H \arrow[r] \arrow[d] & E/E' \arrow[r] \arrow[d] & I/I' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

For a short exact sequence

$$e : 0 \longrightarrow L \longrightarrow E \xrightarrow{p} M \longrightarrow 0$$

of $\mathcal{O}$-modules, we define an injective k-linear map

$$\Phi_e : k \oplus \mathrm{Hom}(M, L) \to \mathrm{End}(E)$$

by $(\lambda, \phi) \mapsto [s \mapsto \lambda s + p^*\phi(s)]$. If we endow $k \oplus \mathrm{Hom}(M, L)$ with the k-algebra structure as the trivial extension algebra of k by $\mathrm{Hom}(M, L)$, then Φ_e is a k-algebra-homomorphism.

Lemma 2.4. *Let E be a non-stable reflexive sheaf of type (n, I) with $n > 0$, and assume $\mathrm{Ass}_{\mathcal{O}}(\mathcal{O}/I) \cap H = \emptyset$. Let $E' \subset E$ be a maximal reflexive subsheaf of type $(n-1, I)$. Then such an E' is unique up to* $\mathrm{Aut}(E)$

Proof. Tensoring an invertible sheaf to E if necessary, we may assume that $\mathcal{O}$ is the maximal invertible sheaf in E. By the assumption on the type, we then obtain the following commutative diagram (c.f. Remark 2.3).

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}(-H) \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & I(-n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & I(-n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_H \arrow[r] \arrow[d] & E/E' \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Restricting the N-sequence of E, we have an exact sequence

$$e|_H : 0 \longrightarrow \mathcal{O}_H \xrightarrow{i} E|_H \longrightarrow I|_H(-n) \longrightarrow 0,$$

where $I|_H$ is a torsion-free $\mathcal{O}_H$-module for $\mathrm{Ass}_{\mathcal{O}}(\mathcal{O}/I) \cap H = \emptyset$. The middle vertical sequence and the last horizontal one in the above diagram tell us that there exists a surjective homomorphism $E \to \mathcal{O}_H$ and E' is its kernel. Taking account of the existence of a surjection $E \to \mathcal{O}_H$, we see the sequence

$$0 \longrightarrow \mathrm{Hom}(I|_H(-n), \mathcal{O}_H) \longrightarrow \mathrm{Hom}(E|_H, \mathcal{O}_H) \xrightarrow{i^*} \mathrm{Hom}(\mathcal{O}_H, \mathcal{O}_H) \longrightarrow 0$$

is exact. We claim that $\mathrm{Aut}(E|_H)$ acts on $\mathrm{Hom}(E|_H, \mathcal{O}_H) \setminus \mathrm{Hom}(I|_H(-n), \mathcal{O}_H)$ transitively. In fact, let α_1 and α_2 be surjective homomorphism from $E|_H$ to $\mathcal{O}_H$. Then $i^*(\alpha_i)$ is a non-zero scalar, for which we write a_i. We put

$$\psi := {a_1}^{-1}\alpha_2 - a_2{a_1}^{-2}\alpha_1 \in \mathrm{Hom}(E|_H, \mathcal{O}_H).$$

Since $i^*(\psi) = 0$, it can be regarded as an element of $\mathrm{Hom}(I|_H(-n), \mathcal{O}_H)$. Put $\Psi := \Phi_{e|_H}({a_1}^{-1}a_2, \psi)$. For any local section s of $E|_H$, we then have

$$\alpha_1 \circ \Psi(s) = \alpha_1({a_1}^{-1}a_2 s + \psi(s)) = -a_1\psi(s) + \alpha_2(s) + \alpha_1(\psi(s)) = \alpha_2(s),$$

which implies the action is transitive.

Now let $\tilde{\psi} \in \mathrm{Hom}(I(-n), \mathcal{O})$ be the pull-back of ψ by the surjection $\mathrm{Hom}(I(-n), \mathcal{O}) \to \mathrm{Hom}(I|_H(-n), \mathcal{O}_H)$, and put $\tilde{\Psi} := \Phi_e(a_1^{-1}a_2, \tilde{\psi})$, where e is the N-sequence of E in the above diagram. Then we can make the following commutative diagram.

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "\tilde{\Psi}", "\cong"'] & E|_H \arrow[r, "\alpha_2"] \arrow[d, "\Psi"] & \mathcal{O}_H \arrow[d, "id"] \\
E \arrow[r] & E|_H \arrow[r, "\alpha_1"] & \mathcal{O}_H
\end{tikzcd}$$

That implies for any two surjective homomorphisms from E to $\mathcal{O}_H$, the kernel of one can be mapped to that of the other by an automorphism of E. Thus we obtain the uniqueness of E' as in this lemma up to $\mathrm{Aut}(E)$. $\square$

Remark 2.5. In Lemma 2.4, we did not mention the existence of such a subsheaf E'. There does not exist such one in general, but does in the case of $\mathbb{P}^2_k$. In fact, let

$$0 \to \mathcal{O} \to E \to I(-n) \to 0.$$

be the N-sequence of E. Then $E|_H \cong \mathcal{O}_H \oplus \mathcal{O}_H(-n)$ and if E' is the kernel of the projection $E \twoheadrightarrow \mathcal{O}_H$, then it is such a subsheaf.

The following lemma will help us to describe the endomorphisms.

Lemma 2.6. *Let E be a non-stable reflexive sheaf of type (n, I) with $I \neq \mathcal{O}$, and let e be its N-sequence. Then Φ_e is an isomorphism.*

Proof. We may assume $\mathcal{O}$ is the maximal invertible subsheaf of E, and hence

$$0 \longrightarrow \mathcal{O} \longrightarrow E \xrightarrow{\ p\ } I(-n) \longrightarrow 0$$

is the N-sequence. To show Φ_e surjective, let us take any $\sigma \in \mathrm{End}(E)$. By the uniqueness of N-sequences, σ induces an endomorphism $\sigma_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}$, hence there exists $\lambda \in k$ such that $(\sigma - \lambda)|_{\mathcal{O}} = 0$. Thus $\sigma - \lambda$ factors through $E \to I(-n)$, and let $\psi : I(-n) \to E$ be the factorization.

We claim $\mathrm{Image}\,\psi \subset \mathcal{O}$. Suppose contrarily $\mathrm{Image}\,\psi \not\subseteq \mathcal{O}$, or equivalently $\phi := p \circ \psi \neq 0$. Since

$$\mathrm{End}(I(-n))) = \mathrm{Hom}(I, I) \subset \mathrm{Hom}(I, \mathcal{O}) = \mathrm{Hom}(\mathcal{O}, \mathcal{O}) = k,$$

ϕ is nothing but a non-zero scalar $\lambda \in k^\times$. Therefore, the homomorphism $\lambda^{-1}\cdot\psi : I(-n) \to E$ is a section of p, and we have $E \cong \mathcal{O} \oplus I(-n)$. That contradicts to the reflexivity of E. Thus we conclude $\mathrm{Image}\,\psi \subset \mathcal{O}$.

Accordingly, we have a homomorphism $\psi : I(-n) \to \mathcal{O}$, and

$$\Phi_e(\lambda, \psi)(s) = \lambda s + p^*\psi(s) = \lambda s + (\sigma - \lambda)(s) = \sigma(s).$$

That implies that Φ_e is surjective, and hence bijective. $\square$

Now we can describe the automorphism groups. To simplify the notation, let $k[\underline{x}]$ denote the polynomial ring $k[x_1, \ldots, x_d]$.

Proposition 2.7. *Let E be a non-stable reflexive sheaf. Then*

$$\mathrm{End}(E) \cong \begin{cases} M_2(k) & \text{if } E \text{ is of type } (0, \mathcal{O}), \\ \left\{ \begin{pmatrix} a & f \\ 0 & b \end{pmatrix} \middle| \begin{array}{c} a, b \in k, f \in k[\underline{x}] \\ \deg(f) \leq n \end{array} \right\} & \text{if } E \text{ is of type } (n, \mathcal{O}) \text{ with } n \geq 1, \\ \left\{ \begin{pmatrix} a & f \\ 0 & a \end{pmatrix} \middle| \begin{array}{c} a \in k, f \in k[\underline{x}] \\ \deg(f) \leq n \end{array} \right\} & \text{if } E \text{ is of type } (n, I) \text{ with } I \neq \mathcal{O}, \end{cases}$$

and hence

$$\mathrm{Aut}(E) \cong \begin{cases} \mathrm{GL}_2(k) & \text{if } E \text{ is of type } (0,\mathcal{O}), \\ \left\{ \begin{pmatrix} a & f \\ 0 & b \end{pmatrix} \middle| \begin{matrix} a,b \in k^\times, f \in k[\underline{x}] \\ \deg(f) \le n \end{matrix} \right\} & \text{if } E \text{ is of type } (n,\mathcal{O}) \text{ with } n \ge 1, \\ \left\{ \begin{pmatrix} a & f \\ 0 & a \end{pmatrix} \middle| \begin{matrix} a \in k^\times, f \in k[\underline{x}] \\ \deg(f) \le n \end{matrix} \right\} & \text{if } E \text{ is of type } (n,I) \text{ with } I \neq \mathcal{O}. \end{cases}$$

Proof. If E is of type $(0,\mathcal{O})$, then $E \cong L^{\oplus 2}$ for an invertible sheaf L and hence $\mathrm{End}(E) \cong M_2(k)$. If E is of type $(n,\mathcal{O})$ with $n \ge 1$, we may assume $E = \mathcal{O} \oplus \mathcal{O}(-n)$. Then

$$\mathrm{End}(E) = H^0(E \otimes E^*) \cong k^{\oplus 2} \oplus H^0(\mathcal{O}(n)) \cong k^{\oplus 2} \oplus \{f \in k[\underline{x}] \mid \deg f \le n\},$$

and the isomorphism in the assertion follows. The last case follows from Lemma 2.6 and the identification

$$\mathrm{Hom}(I(-n),\mathcal{O}) = \mathrm{Hom}(\mathcal{O}(-n),\mathcal{O}) \cong \{f \in k[\underline{x}] \mid \deg f \le n\}.$$

The assertion on Aut is immediate. □

Remark 2.8. Let E be a non-stable reflexive sheaf of type $(n,I) \neq (0,\mathcal{O})$. If $E' \subset E$ is a maximal reflexive subsheaf with $N(E') > n$, then it is unique by Proposition 2.2 (2) (a). That implies we have a canonical injection $\mathrm{Aut}(E) \hookrightarrow \mathrm{Aut}(E')$. On the other hand, we have another inclusion $\mathrm{Aut}(E) \hookrightarrow \mathrm{Aut}(E')$ via the isomorphisms in Proposition 2.7. It is not difficult to see one inclusion coincides with the other.

2.2. **The structure of the graph of groups.** We describe, in this subsection, the structure of the graph $Y := \Gamma\backslash X$ and that of Γ, in the case of $\mathbb{P}^d_k$.

As have been said in the previous section, the vertices of Y correspond to the H-equivalence classes of reflexive sheaves. Let E be a representative of $\mathcal{E} \in \mathrm{Vert}\, Y$. Set $N(\mathcal{E}) := N(E)$, and it is well-defined. We define $Y_{\ge 0}$ to be the subgraph of Y generated by the vertices $\mathcal{E}$ with $N(\mathcal{E}) \ge 0$. We call an N-sequence of a representative of $\mathcal{E}$ an *N-sequence of $\mathcal{E}$*. An N-sequence of $\mathcal{E}$ is called a *normalized* N-sequence if the invertible sheaf that appears in the first term is trivial, i.e., it is of form

$$0 \to \mathcal{O} \to E \to I(-N(E)) \to 0.$$

For $\mathcal{E} \in \mathrm{Vert}\, Y$ with $N(\mathcal{E})$, we say $\mathcal{E}$ is of type (n,I) if one (hence any) representative of $\mathcal{E}$ is of type (n,I).

Lemma 2.9. (1) *If two $\mathcal{E},\mathcal{E}' \in \mathrm{Vert}\, Y$ are adjacent, then $|N(\mathcal{E}) - N(\mathcal{E}')| = 1$.*

(2) *For any $\mathcal{E} \in \mathrm{Vert}\, Y_{\ge 0}$, there exists a unique vertex $\mathcal{E}' \in \mathrm{Vert}\, Y$ adjacent to $\mathcal{E}$ with $N(\mathcal{E}') > N(\mathcal{E})$ (hence $N(\mathcal{E}') = N(\mathcal{E})+1$). Moreover, if $\mathcal{E}$ is of type $(N(\mathcal{E}),I)$ and $\mathcal{E}'$ is of type $(N(\mathcal{E}'),I')$, then $I \subset I'$ and the equality holds if and only if $\mathrm{Ass}_{\mathcal{O}}(\mathcal{O}/I) \cap H = \emptyset$.*

(3) *$Y_{\ge 0}$ is combinatorial.*

Proof. The assertions (1) and (2) are just restatements of Proposition 2.2. Let us prove (3). We will show that for any $\mathcal{E}'$ adjacent to $\mathcal{E} \in \operatorname{Vert} Y_{\geq 0}$ with $N(\mathcal{E}') > N(\mathcal{E})$, there exists a unique edge jointing them. Fix a representative $E \subset Q^{\oplus 2}$ of $\mathcal{E}$. By virtue of Proposition 1.9, it suffices to show the $\operatorname{Aut}(E)$-action on the set

$$\{E' \subset E \mid [E'] = \mathcal{E}' \text{ and } E' \text{ is maximal}\}$$

is transitive. It, however, follows from Proposition 2.2 (2) immediately. $\square$

For the convenience in describing the structure of $Y_{\geq 0}$, we prepare some words and notations. We call $\mathcal{E} \in \operatorname{Vert} Y$ a *city* if $\mathcal{E} \in \operatorname{Vert} Y_{\geq 0}$ and $\operatorname{Ass}_{\mathcal{O}}(\mathcal{O}/I) \cap H = \emptyset$, where I is the ideal that appears in its N-sequence. A city $\mathcal{E}$ with $N(\mathcal{E}) = n$ is called a *n-city.* We call a connected component of the subgraph of Y generated by the cities a *street.* We denote by $\mathcal{C}_n$ the set of n-cities, by $\mathcal{S}$ the set of streets and by $\pi_0(Y_{\geq 0})$ the set of connected components of $Y_{\geq 0}$. For any vertex $\mathcal{E}$ of a connected component Z of $Y_{\geq 0}$, we have a sequence $\{\mathcal{E}_i\}_{i=0}^{\infty}$ such that $\mathcal{E}_0 = \mathcal{E}$ and that $\mathcal{E}_i$ is adjacent to $\mathcal{E}_{i-1}$ with $N(\mathcal{E}_i) = N(\mathcal{E}_{i-1}) + 1$. If (n_i, I_i) denotes the type of $\mathcal{E}_i$, then the first statement of Lemma 2.9 (2) says $I_0 \subset I_1 \subset \cdots$, hence there is an integer l such that $I_l = I_{l+1} = \cdots$. Therefore by the second statement of Lemma 2.9 (2), we see that $\operatorname{Ass}_{\mathcal{O}}(\mathcal{O}/I_l) \cap H = \emptyset$, that means $\mathcal{E}_i$ for $i \geq l$ is a city. In particular, we have the following lemma:

Lemma 2.10. *Any connected component of $Y_{\geq 0}$ contains a city.*

We define maps $\nu_n : \mathcal{C}_n \to \mathcal{C}_{n+1}$ by $\mathcal{E} \mapsto$ ($\mathcal{E}'$ in Lemma 2.9 (1)), $\mu_n : \mathcal{C}_n \to \mathcal{S}$ by $\mathcal{E} \mapsto$ (the street containing $\mathcal{E}$), $\delta : \mathcal{S} \to \pi_0(Y_{\geq 0})$ by $s \mapsto$ (the connected component containing s), and $\lambda_n : \mathcal{C}_n \to \pi_0(Y_{\geq 0})$ by $\mathcal{E} \mapsto$ (the connected component $\mathcal{E}$). We have $\lambda_n = \delta \circ \mu_n$.

Proposition 2.11. *The maps ν_n, μ_n and λ_n are injective. Further $\varinjlim \mu_n : \varinjlim \mathcal{C}_n \to \mathcal{S}$, $\varinjlim \lambda_n : \varinjlim \mathcal{C}_n \to \pi_0(Y_{\geq 0})$ and hence $\delta : \mathcal{S} \to \pi_0(Y_{\geq 0})$ are bijective.*

Proof. The injectivity of ν_n follows from Lemma 2.4 immediately. The injectivity of μ_n follows from that of λ_n. Let us show λ_n injective. Suppose that $\mathcal{E}$ and $\mathcal{E}'$ are n-cities with $\mathcal{E} \neq \mathcal{E}'$ but $\lambda_n(\mathcal{E}) = \lambda_n(\mathcal{E}')$. Then there exists a chain in $Y_{\geq 0}$ connecting $\mathcal{E}$ and $\mathcal{E}'$. Let us express that chain by $(\mathcal{E} = \mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_l = \mathcal{E}')$, where $\mathcal{E}_i \in \operatorname{Vert} Y$, $\mathcal{E}_i \neq \mathcal{E}_j$ for $i \neq j$ and $\mathcal{E}_i$ and $\mathcal{E}_j$ are adjacent. Then by virtue of the uniqueness in Lemma 2.9 (2), there exists an integer m $(0 < m < l)$ such that

$$N(\mathcal{E}_0) < N(\mathcal{E}_1) < \cdots < N(\mathcal{E}_{m-1}) < N(\mathcal{E}_m) > N(\mathcal{E}_{m+1}) > \cdots > N(\mathcal{E}_l),$$

hence in particular they are cities. That implies that $\mathcal{E}_{m-1}$ and $\mathcal{E}_{m+1}$ are $(N(\mathcal{E}_m) - 1)$-cities such that $\nu_{N(\mathcal{E}_m)-1}(\mathcal{E}_{m-1}) = \nu_{N(\mathcal{E}_m)-1}(\mathcal{E}_{m+1})$, which contradicts to the injectivity of $\nu_{N(\mathcal{E}_m)-1}$.

Finally, the injectivity of $\varinjlim \mu_n$ and that of $\varinjlim \lambda_n$ follow from that of ν_n, that of μ_n and that of λ_n. The surjectivity of $\varinjlim \mu_n$ follows from the definition of the streets, and that of $\varinjlim \lambda_n$ follows from Lemma 2.10. $\square$

For each non-negative integer n, we define a set Σ_n by

$$\Sigma_n := \operatorname{Ext}^1_{\mathcal{O}}(\mathcal{O}(-n), \mathcal{O}) \amalg \left(\coprod_{I \subsetneq \mathcal{O} \text{ coherent, } \operatorname{Ass}_{\mathcal{O}}(\mathcal{O}/I) \cap H = \emptyset} \mathbb{P}(\operatorname{Ext}^1_{\mathcal{O}}(I(-n), \mathcal{O}))(k) \right),$$

where $\mathbb{P}(\mathrm{Ext}^1_{\mathcal{O}}(I(-n),\mathcal{O}))(k)$ denote the set of lines in the k-vector space $\mathrm{Ext}^1_{\mathcal{O}}(I(-n),\mathcal{O})$. We define in addition a map $\tilde{\nu}_n : \Sigma_n \to \Sigma_{n+1}$ for each n characterized by the following conditions.

(a) $\tilde{\nu}_n(\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(-n),\mathcal{O})) \subset \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(-(n+1)),\mathcal{O})$.
(b) For $I \subsetneq \mathcal{O}$, $\tilde{\nu}_n(\mathbb{P}(\mathrm{Ext}^1_{\mathcal{O}}(I(-n),\mathcal{O}))(k)) \subset \mathbb{P}(\mathrm{Ext}^1_{\mathcal{O}}(I(-(n+1)),\mathcal{O})(k))$ and it coincides with the map induced by the canonical injection

$$\mathrm{Ext}^1_{\mathcal{O}}(I(-n),\mathcal{O}) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}}(I(-(n+1)),\mathcal{O}).$$

Now we can propose the main result.

Theorem 2.12 (Structure of Y). (1) *$Y_{\geq 0}$ is a disjoint union of trees, and any street is a chain.*
(2) *Naturally we have $\varinjlim \mathcal{C}_n \cong \mathcal{S} \cong \pi_0(Y_{\geq 0})$.*
(3) *There is a canonical injection $\Upsilon_n : \mathcal{C}_n \to \Sigma_n$, and a diagram*

$$\begin{array}{ccc} \mathcal{C}_{n-1} & \xrightarrow{\Upsilon_{n-1}} & \Sigma_{n-1} \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \tilde{\nu}} \\ \mathcal{C}_n & \xrightarrow{\Upsilon_n} & \Sigma_n \end{array}$$

is commutative. Consequently, we have

$$\varinjlim \mathcal{C}_n \hookrightarrow \varinjlim \Sigma_n.$$

Proof. The assertion on $Y_{\geq 0}$ in (1) follows from Lemma 2.9 (2) immediately. Proposition 2.11 says that for any vertex $\mathcal{E}$ of a street s, there exists a unique vertex $\mathcal{E}'$ vertex of s adjacent to $\mathcal{E}$ with $N(\mathcal{E}') = N(\mathcal{E}) + 1$ and there exists at most one vertex with lower N than $\mathcal{E}$. In particular any vertex of a street has at most two adjacent vertices in it, which implies that it is a chain.

The assertion (2) is nothing but a restatement of Proposition 2.11.

Let us prove (3). Let us construct Υ_n. For an n-city $\mathcal{E}$, let

$$0 \to \mathcal{O} \to E \to I(-n) \to 0$$

be its normalized N-sequence. If $I = \mathcal{O}$, then $E = \mathcal{O} \oplus \mathcal{O}(-n)$ and $[E]$ gives the unique point of $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(-n),\mathcal{O})$. Otherwise, normalized N-sequence is unique up to constant, and hence gives a point in $\mathbb{P}(\mathrm{Ext}^1_{\mathcal{O}}(I(-n),\mathcal{O}))(k)$. Thus the map Υ_n is defined. Its injectivity is immediate from the construction. The commutativity of the diagram follows from the construction of the maps ν_n and $\tilde{\nu}_n$. □

Remark 2.13. We give remark of the structure of Y in the case of $\mathbb{P}^2_k$. Remark 2.5 tells us the map $\nu_n : \mathcal{C}_n \to \mathcal{C}_{n+1}$ is also surjective. Therefore, there exists a bijection between the following sets:

(a) $\pi_0(Y_{\geq 0})$: the set of connected components of $Y_{\geq 0}$.
(b) $\mathcal{S}$: the set of main streets
(c) $\mathcal{C}_n$: the set of n-cities for each $n = 0, 1, 2, \ldots$.

Moreover, it is not difficult to see the canonical homomorphism

$$\mathrm{Ext}^1_{\mathcal{O}}(I(-n),\mathcal{O}) \to \mathrm{Ext}^1_{\mathcal{O}}(I(-(n+1)),\mathcal{O})$$

is an isomorphism and actually they are isomorphic to $H^0(\mathbb{P}^2_k, \mathcal{E}xt^1_{\mathcal{O}}(I, \mathcal{O}))$. In other words, Σ_n for any n and hence $\varinjlim \Sigma_n$ are canonically isomorphic to

$$H^0(\mathbb{P}^2_k, \mathcal{E}xt^1_{\mathcal{O}}(\mathcal{O}, \mathcal{O})) \amalg \left(\coprod_{I \subsetneq \mathcal{O} \text{ coherent, } \operatorname{Supp} \mathcal{O}/I \cap H = \emptyset} \mathbb{P}(H^0(\mathbb{P}^2_k, \mathcal{E}xt^1_{\mathcal{O}}(I, \mathcal{O})))(k) \right).$$

By virtue of the structure theorem, we can describe Γ in terms of amalgam. Here let us recall the graph of groups associated with a tree with a group-action (see [2] for detail). A *graph of groups* is the following data:

(a) a graph Z,
(b) for each $z \in \operatorname{Vert} Z$, a group G_z, and for each $e \in \operatorname{Ed} Z$(:= the set of edges of Z), a group G_e.
(c) for each $e \in \operatorname{Ed} Z$, group homomorphisms $G_e \to G_{o(e)}$ and $G_e \to G_{t(e)}$, where $o(e)$ and $t(e)$ is the origin and the terminus respectively.

Generally, let W be a tree and let G be a group acting on W without inversion. Then we can make the quotient graph $Z := G\backslash W$. Let T be a maximal tree of Z. Then we can obtain a lift $j : T \hookrightarrow W$. In [2, I.5.4], the graph of groups (G, Z) associated with a G-tree W can be constructed as follows (see [2] for detail): after extending the map $j : \operatorname{Ed} T \to \operatorname{Ed} W$ induced by the section, to $j : \operatorname{Ed} Z \to \operatorname{Ed} W$ in a suitable way, we put $G_z = G_{j(z)}$ and $G_e = G_{j(e)}$ for $y \in \operatorname{Vert} T = \operatorname{Vert} Z$ and $e \in \operatorname{Ed} Z$, where $G_{j(z)}$ and $G_{j(e)}$ are the stabilizers, and the necessary group homomorphisms are also constructed appropriately. In our case, they can be described as follows. (The case of GL_2 only. We can obtain the results in the case of PGL_2 if we divide them by $k^\times$.) Suppose that we have chosen a maximal tree T so that $Y_{\geq 0} \subset T$ and a lift $j : T \to X$ so that $j([\mathcal{O} \oplus \mathcal{O}(-n)]) = [\mathcal{O}_p \oplus {x_0}^n \mathcal{O}_p]$, where x_0 defines the infinite hyperplane around p. For any vertex $[E] \in \operatorname{Vert} Y$, we know $\Gamma_{[E]} \cong \operatorname{Aut}(E)$. If $N(E) < 0$, then E is stable by Proposition 2.1 and hence $\Gamma_{[E]} = k^\times$. From the choice of j, we have

$$\Gamma_{[\mathcal{O} \oplus \mathcal{O}(-n)]} = \left\{ \begin{pmatrix} a & f \\ 0 & b \end{pmatrix} \middle| \begin{array}{c} a, b \in k^\times, f \in k[\underline{x}] \\ \deg(f) \leq n \end{array} \right\}.$$

Let e be an edge of Y. If an extremity of e has negative N, then $\Gamma_e = k^\times$ and the homomorphisms from Γ_e to the group at the extremities are the canonical ones. If e is the edge such that $o(e) = [\mathcal{O}^{\oplus 2}]$ and $t(e) = [\mathcal{O} \oplus \mathcal{O}(-1)]$, then $\Gamma_e = B_2(k)$, where B_2 indicates the subgroup of GL_2 consisting of the upper triangular matrices, and the homomorphisms $\Gamma_e \to \Gamma_{o(e)}$ and $\Gamma_e \to \Gamma_{t(e)}$ are given by the canonical maps

$$B_2(k) \hookrightarrow \mathrm{GL}_2(k)$$

and

$$B_2(k) \hookrightarrow \left\{ \begin{pmatrix} a & f \\ 0 & b \end{pmatrix} \middle| \begin{array}{c} a, b \in k^\times, f \in k[\underline{x}] \\ \deg(f) \leq 1 \end{array} \right\}$$

respectively (c.f. Proposition 2.7). If e is the edge jointing the vertex $o(e) = [\mathcal{O} \oplus \mathcal{O}(-n)]$ and $t(e) = [\mathcal{O} \oplus \mathcal{O}(-(n+1))]$, then the homomorphisms $\Gamma_e \to \Gamma_{o(e)}$ and $\Gamma_e \to \Gamma_{t(e)}$ respectively are described as

$$id : (\Gamma_e =) \left\{ \begin{pmatrix} a & f \\ 0 & b \end{pmatrix} \middle| \begin{matrix} a, b \in k^\times, f \in k[\underline{x}] \\ \deg(f) \le n \end{matrix} \right\} \to \left\{ \begin{pmatrix} a & f \\ 0 & b \end{pmatrix} \middle| \begin{matrix} a, b \in k^\times, f \in k[\underline{x}] \\ \deg(f) \le n \end{matrix} \right\}$$

and the canonical injection

$$\left\{ \begin{pmatrix} a & f \\ 0 & b \end{pmatrix} \middle| \begin{matrix} a, b \in k^\times, f \in k[\underline{x}] \\ \deg(f) \le n \end{matrix} \right\} \hookrightarrow \left\{ \begin{pmatrix} a & f \\ 0 & b \end{pmatrix} \middle| \begin{matrix} a, b \in k^\times, f \in k[\underline{x}] \\ \deg(f) \le n+1 \end{matrix} \right\}.$$

If e is the edge jointing the vertex $\mathcal{E}$ of type (n, I) and $\mathcal{E}'$ of type $(n+1, J)$, then $\Gamma_e = \Gamma_{\mathcal{E}} \hookrightarrow \Gamma_{\mathcal{E}'}$. We know

$$\Gamma_{\mathcal{E}} \cong \left\{ \begin{pmatrix} a & f \\ 0 & a \end{pmatrix} \middle| \begin{matrix} a \in k^\times, f \in k[\underline{x}] \\ \deg(f) \le n \end{matrix} \right\}$$

only abstractly, but the homomorphism $\Gamma_e \hookrightarrow \Gamma_{\mathcal{E}'}$ looks like the canonical injection

$$\left\{ \begin{pmatrix} a & f \\ 0 & a \end{pmatrix} \middle| \begin{matrix} a \in k^\times, f \in k[\underline{x}] \\ \deg(f) \le n \end{matrix} \right\} \hookrightarrow \left\{ \begin{pmatrix} a & f \\ 0 & a \end{pmatrix} \middle| \begin{matrix} a \in k^\times, f \in k[\underline{x}] \\ \deg(f) \le n+1 \end{matrix} \right\}$$

(c.f. Proposition 2.7 and Remark 2.8).

For a commutative ring R, let $B_2(R)$ be the subgroup of $\mathrm{GL}_2(R)$ consisting of the upper triangular matrices as above, and let $PB_2(R)$ be the image of $B_2(R)$ in $\mathrm{PGL}_2(R)$. From the above observation and Proposition 2.7, we obtain the following by using the calculation on [2, I.5.1].

Proposition 2.14. *Let Z be a connected component of $Y_{\ge 0}$ and let (Γ, Z) be the graph of group obtained by restricting (Γ, Y) to Z. Let $\pi_1(\Gamma, Z, Z)$ be the fundamental group of a graph of group (Γ, Z) at Z (c.f. [2, I.5.1]). If $[\mathcal{O}^{\oplus 2}] \in Z$, then*

$$\pi_1(\Gamma, Z, Z) \cong \begin{cases} \mathrm{GL}_2(k) *_{B_2(k)} B_2(k[\underline{x}]) & \text{if } \Gamma = \mathrm{GL}_2, \\ \mathrm{PGL}_2(k) *_{PB_2(k)} PB_2(k[\underline{x}]) & \text{if } \Gamma = \mathrm{PGL}_2. \end{cases}$$

Otherwise,

$$\pi_1(\Gamma, Z, Z) \cong \begin{cases} \left\{ \begin{pmatrix} \lambda & f \\ 0 & \lambda \end{pmatrix} \in \mathrm{GL}_2(k[\underline{x}]) \right\} & \text{if } \Gamma = \mathrm{GL}_2, \\ \\ k[\underline{x}] & \text{if } \Gamma = \mathrm{PGL}_2. \end{cases}$$

We can regard the image of $\varinjlim \mathcal{C}_n \hookrightarrow \varinjlim \Sigma_n$ in Theorem 2.12 (3) as an index set of $\pi_0(Y_{\ge 0})$ or $\mathcal{S}$. We denote this index set by Π. In Π, there is an element o corresponding to the connected component containing the class $[\mathcal{O}^{\oplus 2}]$. We put $\Pi^\circ := \Pi \setminus \{o\}$. Now we obtain the following theorem.

Theorem 2.15. *Let F be the fundamental group of the topological realization of the graph Y. Then we have the following.*

(1) $\mathrm{GL}_2(k[\underline{x}])$ *is isomorphic to*

$$\left(\mathrm{GL}_2(k) *_{B_2(k)} B_2(k[\underline{x}])\right) *_{k^\times} \left(\left\{ \begin{pmatrix} \lambda & f \\ 0 & \lambda \end{pmatrix} \in \mathrm{GL}_2(k[\underline{x}]) \right\}^{*_{k^\times} \Pi^\circ} \right) * F \Big/ \mathcal{R},$$

where $\mathcal{R}$ is the relation generated by $\{\lambda\gamma\lambda^{-1}\gamma^{-1}\}_{\lambda\in k^\times,\gamma\in F}$.

(2) $\mathrm{PGL}_2(k[\underline{x}]) \cong \big(\mathrm{PGL}_2(k) *_{PB_2(k)} PB_2(k[\underline{x}])\big) * ((k[\underline{x}])^{*\Pi^\circ}) * F$.

Proof. Taking account of Proposition 2.14 and the observation above it, we can obtain our assertion along the calculation on [2, I.5.1]. □

Unfortunately, the author does not have ideas how to determine the free part F.

Remark 2.16. Let us give remark on the case of $\Gamma = \mathrm{SL}_2$. In general, the vertices of the quotient $\mathrm{SL}_2(k[\underline{x}])\backslash X$ correspond to the classes of reflexive sheaves with an identification $\det(E) \cong \mathcal{O}(\deg E)$. *Under the assumption that any element of k is a square*, however, the datum of the identification is trivial and the vertices are the classes of reflexive sheaves too. Therefore we can describe it: under that assumption, $\mathrm{SL}_2(k[\underline{x}])$ is isomorphic to

$$\big(\mathrm{SL}_2(k) *_{B\,\mathrm{SL}_2(k)} B\,\mathrm{SL}_2(k[\underline{x}])\big) *_{\{\pm1\}} \left(\left\{\begin{pmatrix}\lambda & f\\ 0 & \lambda\end{pmatrix} \in \mathrm{SL}_2(k[\underline{x}])\right\}^{*_{\{\pm1\}}\Pi^\circ}\right) * F \Big/ \mathcal{R},$$

where $B\,\mathrm{SL}_2 := \mathrm{GL}_2 \cap B_2$ and $\mathcal{R}$ is the relation generated by $\{(-1)\gamma(-1)\gamma^{-1}\}_{\gamma\in F}$.

Finally, as an easy application of Theorem 2.15, we can show the following.

Proposition 2.17. *Assume that k is algebraically closed. If $a \in \mathrm{GL}_2(k[\underline{x}])$ is a torsion-element, then there exists $g \in \mathrm{GL}_2(k[\underline{x}])$ such that gag^{-1} is an upper triangular matrix.*

Proof. Suppose $a \in \mathrm{GL}_2(k[\underline{x}])$ is a torsion. Then so is the class $\bar{a} \in \mathrm{PGL}_2(k[\underline{x}])$. Let us fix an isomorphism

$$\alpha : \big(\mathrm{PGL}_2(k) *_{PB_2(k)} PB_2(k[\underline{x}])\big) * ((k[\underline{x}])^{*\mathcal{S}}) * F \to \mathrm{PGL}_2(k[\underline{x}]),$$

such that $\alpha|_{\mathrm{PGL}_2(k)}$ and $\alpha|_{PB_2(k[\underline{x}])}$ is just the canonical inclusion. By [2, I.1.3 Corollary 1], there exists an element $b \in \mathrm{GL}_2(k[\underline{x}])$ such that $\alpha^{-1}(\bar{b}\bar{a}\bar{b}^{-1})$ sits in one of the amalgam factors, and since it is a torsion, we have $\alpha^{-1}(\bar{b}\bar{a}\bar{b}^{-1}) \in \mathrm{PGL}_2(k) *_{PB_2(k)} PB_2(k[\underline{x}])$. Again by the assumption and [2, I.1.3 Corollary 1], there exists $c \in \mathrm{GL}_2(k[\underline{x}])$ such that $\alpha^{-1}(\bar{c}\bar{a}\bar{c}^{-1}) \in \mathrm{PGL}_2(k)$ or $\alpha^{-1}(\bar{c}\bar{a}\bar{c}^{-1}) \in PB_2(k[\underline{x}])$, and equivalently $cac^{-1} \in \mathrm{GL}_2(k)$ or $cac^{-1} \in B_2(k[\underline{x}])$. In the latter case, it is already upper triangular matrix, and in the other case, it can be triangulated since k is algebraically closed. □

References

[1] R. Hartshorne, Stable Reflexive Sheaves, Math. Ann. 254. 121–176 (1980).
[2] J-P. Serre, Trees, Springer Monographs in Mathematics.

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