

A Note on the Equivalence Between Substitutability and M^{\natural} -convexity

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Abstract

The property of “substitutability” plays a key role in guaranteeing the existence of a stable solution in the stable marriage problem and its generalizations. On the other hand, the concept of M^{\natural} -convexity, introduced by Murota–Shioura (1999) for functions defined over the integer lattice, enjoys a number of nice properties that are expected of “discrete convexity” and provides with a natural model of utility functions. In this note, we show that M^{\natural} -convexity is characterized by two variants of substitutability.

Keywords: stable marriage, discrete optimization, convex function, matroid, sub-modular function

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1 Introduction

Since the pioneering work on the stable marriage problem by Gale–Shapley [7], various generalizations and extensions of the stable marriage model have been proposed in the literature (see [1, 2, 3, 4, 6, 14, 15], etc.), where the property of “substitutability” for preferences plays a key role in guaranteeing the existence of a stable solution. On the other hand, the concept of M-convexity, introduced by Murota [8, 9] for functions defined over the integer lattice, enjoys a number of nice properties that are expected of “discrete convexity;” subsequently, its variant called M^h-convexity was introduced by Murota–Shioura [11]. Whereas M^h-convex functions are conceptually equivalent to M-convex functions, the class of M^h-convex functions is strictly larger than that of M-convex functions. Furthermore, M^h-concave functions provide with a natural model of utility functions [10, 13, 16]. In particular, it is known that M^h-concavity is equivalent to the gross substitutes property, and that M^h-concavity implies submodularity. In this note, we discuss the close relationship between substitutability and M^h-convexity/M^h-concavity.

Recently, Eguchi–Fujishige–Tamura [3] extended the stable marriage model to the framework with preferences represented by M^h-concave utility functions, and showed the existence of a stable solution in their model (see also [2]). Their proof is based on the fact that M^h-convex functions $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfy the following properties:

- (SC¹) $\forall z_1, z_2 \in \mathbf{Z}^V$ with $z_1 \geq z_2$ and $\arg \min\{f(x') \mid x' \leq z_2\} \neq \emptyset$,
 $\forall x_1 \in \arg \min\{f(x') \mid x' \leq z_1\}, \exists x_2 \in \arg \min\{f(x') \mid x' \leq z_2\}$ such that $z_2 \wedge x_1 \leq x_2$,
- (SC²) $\forall z_1, z_2 \in \mathbf{Z}^V$ with $z_1 \geq z_2$ and $\arg \min\{f(x') \mid x' \leq z_1\} \neq \emptyset$,
 $\forall x_2 \in \arg \min\{f(x') \mid x' \leq z_2\}, \exists x_1 \in \arg \min\{f(x') \mid x' \leq z_1\}$ such that $z_2 \wedge x_1 \leq x_2$,

where for $x, y \in \mathbf{R}^V$ the vector $x \wedge y \in \mathbf{R}^V$ is given by $(x \wedge y)(w) = \min\{x(w), y(w)\}$ ($w \in V$). These properties can be regarded as substitutability for utility functions f ; indeed, (SC¹) and (SC²) can be seen as generalizations of substitutability (persistence) in the sense of Alkan–Gale [1] for the choice function $C(z) = \arg \min\{f(y) \mid y \leq z\}$.

Following the work by Eguchi–Fujishige–Tamura [3], Fujishige–Tamura [6] presented a common generalization of the stable marriage model and the assignment game model with M^h-concave utility functions. It is shown in [6] that the following properties of M^h-convex functions

- (SC¹_G) $\forall p \in \mathbf{R}^V, f[p]$ satisfies (SC¹),
 (SC²_G) $\forall p \in \mathbf{R}^V, f[p]$ satisfies (SC²),

which can be seen as stronger versions of substitutability (SC¹) and (SC²), play a key role in the proof of the existence of a stable solution in this model, where for $p \in \mathbf{R}^V$ the function $f[p] : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ is defined by

$$f[p](x) = f(x) + \sum_{w \in V} p(w)x(w) \quad (x \in \mathbf{Z}^V).$$

The main aim of this note is to prove that each of (SC¹_G) and (SC²_G) characterizes M^h-convexity of a function.

Theorem 1.1. *Let $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ be a function such that the effective domain $\text{dom } f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}$ is bounded. Then,*

$$f \text{ is } M^{\natural}\text{-convex} \iff (\text{SC}_{\mathbf{G}}^1) \iff (\text{SC}_{\mathbf{G}}^2).$$

This theorem shows that M^{\natural} -concavity of utility functions is an essential assumption in the model of Fujishige–Tamura [6]. Combining Theorem 1.1 and the previous result [13, Theorem 11] clarifies the relationship between substitutability and the gross substitute property for utility functions. The equivalence in Theorem 1.1 was proven by Farooq–Tamura [5] in the special case where $\text{dom } f \subseteq \{0, 1\}^V$, i.e., f is a set function. In this note, we give a proof for a more general case where $\text{dom } f$ is bounded.

2 Preliminaries on M^{\natural} -convexity

In this section, we review the definition and fundamental properties of M^{\natural} -convex functions.

Throughout this paper, we assume that V is a nonempty finite set. The sets of reals and integers are denoted by \mathbf{R} and by \mathbf{Z} , respectively. For a vector $x = (x(w) \mid w \in V) \in \mathbf{Z}^V$, we define

$$\begin{aligned} \text{supp}^+(x) &= \{w \in V \mid x(w) > 0\}, & \text{supp}^-(x) &= \{w \in V \mid x(w) < 0\}, \\ \text{supp}(x) &= \{w \in V \mid x(w) \neq 0\}, \\ \langle p, x \rangle &= \sum_{w \in V} p(w)x(w) \quad (p \in \mathbf{R}^V), & x(S) &= \sum_{w \in S} x(w) \quad (S \subseteq V). \end{aligned}$$

For any $u \in V$, the characteristic vector of u is denoted by $\chi_u \in \{0, 1\}^V$, i.e., $\chi_u(w) = 1$ if $w = u$ and $\chi_u(w) = 0$ otherwise. We also denote by χ_0 the zero vector. For $x, y \in \mathbf{Z}^V$ with $x \leq y$, we denote $[x, y]_{\mathbf{Z}} = \{z \in \mathbf{Z}^V \mid x \leq z \leq y\}$.

Let $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ be a function. We denote the set of minimizers of f by $\arg \min f = \{x \in \mathbf{Z}^V \mid f(x) \leq f(y) \ (\forall y \in \mathbf{Z}^V)\}$, which can be the empty set. For a vector $z \in \mathbf{Z}^V$, we denote

$$X^*(f, z) = \arg \min \{f(x) \mid x \leq z\} (= \{x \in \mathbf{Z}^V \mid x \leq z, f(x) \leq f(y) \ (\forall y \in \mathbf{Z}^V \text{ with } y \leq z)\}).$$

We call a function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ M^{\natural} -convex if it satisfies $\text{dom } f \neq \emptyset$ and (M^{\natural} -EXC):

$$(\mathbf{M}^{\natural}\text{-EXC}) \quad \forall x, y \in \text{dom } f, \forall u \in \text{supp}^+(x-y), \exists v \in \text{supp}^-(x-y) \cup \{0\}:$$

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

See [11] for the original definition.

We also define the set version of M^{\natural} -convexity. A nonempty set $B \subseteq \mathbf{Z}^V$ is said to be M^{\natural} -convex if its indicator function $\delta_B : \mathbf{Z}^V \rightarrow \{0, +\infty\}$ defined by

$$\delta_B(x) = \begin{cases} 0 & \text{if } x \in B, \\ +\infty & \text{otherwise} \end{cases}$$

is M^{\natural} -convex. Equivalently, an M^{\natural} -convex set is defined as a nonempty set satisfying the exchange property (B^{\natural} -EXC $_{\pm}$):

(B^h-EXC_±) $\forall x, y \in B, \forall u \in \text{supp}^+(x - y), \exists v, w \in \text{supp}^-(x - y) \cup \{0\}$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_w \in B$.

Theorem 2.1 ([11, 17]). *A nonempty set $B \subseteq \mathbf{Z}^V$ is M^h -convex if and only if it satisfies (B^h-EXC_±).*

An M^h -convex function with bounded effective domain can be characterized by the sets of minimizers.

Theorem 2.2 ([10, Theorem 6.30]). *Let $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ be a function such that $\text{dom } f$ is bounded. Then, f is M^h -convex if and only if for each $p \in \mathbf{R}^V$ the set $\arg \min f[p]$ is M^h -convex.*

3 Proofs

The implications “ f is M^h -convex \implies (SC_G¹)” and “ f is M^h -convex \implies (SC_G²)” are shown in [3, 5, 6] (see also Section 4).

Theorem 3.1. *An M^h -convex function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies (SC_G¹) and (SC_G²).*

In this section, we prove the implications “(SC_G²) \implies (SC_G¹)” and “(SC_G¹) \implies f is M^h -convex.”

Theorem 3.2. *Let $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$.*

(i) *If f satisfies (SC_G²), then f also satisfies (SC_G¹).*

(ii) *Suppose that $\text{dom } f$ is bounded. If f satisfies (SC_G¹), then f is M^h -convex.*

Combining Theorems 3.1 and 3.2 yields Theorem 1.1, our main result.

3.1 Proof of “(SC_G²) \implies (SC_G¹)”

We prove Theorem 3.2 (i).

Suppose that f satisfies (SC_G²). Let $p \in \mathbf{R}^V, z_1, z_2 \in \mathbf{Z}^V$ be any vectors satisfying $z_1 \geq z_2$ and $X^*(f[p], z_2) \neq \emptyset$, and $x_1^* \in X^*(f[p], z_1)$. Also, let $x_2^* \in X^*(f[p], z_2)$ be a vector minimizing the cardinality of the set $\text{supp}^+(x_2^* - x_1^*)$, and put $S^+ = \text{supp}^+(x_2^* - x_1^*)$. We assume that x_2^* maximizes the value $x_2^*(V \setminus S^+)$ among all vectors $y \in X^*(f[p], z_2)$ with $\text{supp}^+(y - x_1^*) = S^+$. We show that x_2^* satisfies the inequality $z_2 \wedge x_1^* \leq x_2^*$.

For $w \in S^+$, we have $\min\{z_2(w), x_1^*(w)\} = x_1^*(w) < x_2^*(w)$ since $x_1^*(w) < x_2^*(w) \leq z_2(w)$. Hence, it suffices to prove that

$$\min\{z_2(w), x_1^*(w)\} \leq x_2^*(w) \quad (w \in V \setminus S^+). \quad (3.1)$$

To show this, we define $\tilde{z}_1, \tilde{z}_2 \in \mathbf{Z}^V$ by

$$\tilde{z}_1 = x_1^* \vee x_2^*, \quad \tilde{z}_2 = (x_1^* \vee x_2^*) \wedge z_2.$$

For $i = 1, 2$, $x_i^* \in X^*(f[p], \tilde{z}_i) \subseteq X^*(f[p], z_i)$ holds since $x_i^* \leq \tilde{z}_i \leq z_i$. As shown below, there exists a vector $q \in \mathbf{R}^V$ satisfying the following conditions:

$$X^*(f[q], \tilde{z}_1) \neq \emptyset, \text{ and } x(w) = x_1^*(w) \text{ (} w \in V \setminus S^+ \text{) for all } x \in X^*(f[q], \tilde{z}_1), \quad (3.2)$$

$$x_2^* \in X^*(f[q], \tilde{z}_2). \quad (3.3)$$

Then, it follows from (SC_G^2) that there exists some $x \in X^*(f[q], \tilde{z}_1)$ such that $x \wedge \tilde{z}_2 \leq x_2^*$, implying

$$\min\{x_1^*(w), z_2(w)\} = \min\{x(w), \tilde{z}_2(w)\} \leq x_2^*(w) \quad (w \in V \setminus S^+),$$

where the equality is by (3.2) and the definition of \tilde{z}_2 . Hence, we have the desired inequality (3.1).

We now show that there exists a vector $q \in \mathbf{R}^V$ satisfying (3.2) and (3.3). Let k be a sufficiently large positive number such that $k > \tilde{z}_1(w) - x_1^*(w)$ ($w \in S^+$). Define $d \in \mathbf{R}^V$ by

$$d(w) = \begin{cases} \frac{1}{k|S^+|} & (w \in S^+), \\ 1 & (w \in V \setminus S^+). \end{cases}$$

For $i = 1, 2$, we define a value $\eta_i \in \mathbf{R}$ by

$$\eta_i = \max\{\langle d, x \rangle \mid x \in X^*(f[p], \tilde{z}_i)\}.$$

Since the set $\hat{Y}_i = \{y \in \mathbf{Z}^V \mid \langle d, y \rangle > \eta_i, y \leq \tilde{z}_i\}$ is finite and satisfies $f[p](y) > f[p](x_i^*)$ ($y \in \hat{Y}_i$), we have

$$X^*(f[q], \tilde{z}_i) = \{x \mid x \in X^*(f[p], \tilde{z}_i), \langle d, x \rangle = \eta_i\} \quad (i = 1, 2) \quad (3.4)$$

by putting $q = p - \varepsilon d$ with a sufficiently small positive number ε .

To show that the condition (3.2) holds, let $x \in X^*(f[q], \tilde{z}_1)$. For $w \in V \setminus S^+$, we have $x(w) \leq \tilde{z}_1(w) = x_1^*(w)$, implying $x(V \setminus S^+) - x_1^*(V \setminus S^+) \leq 0$. By (3.4), we have

$$\begin{aligned} 0 \leq \langle d, x \rangle - \langle d, x_1^* \rangle &= \frac{1}{k|S^+|} \sum_{w \in S^+} \{x(w) - x_1^*(w)\} + x(V \setminus S^+) - x_1^*(V \setminus S^+) \\ &\leq \frac{1}{k|S^+|} \sum_{w \in S^+} \{\tilde{z}_1(w) - x_1^*(w)\} + x(V \setminus S^+) - x_1^*(V \setminus S^+). \end{aligned}$$

Since $(1/k|S^+|) \sum_{w \in S^+} \{\tilde{z}_1(w) - x_1^*(w)\} < 1$ and $x(V \setminus S^+) - x_1^*(V \setminus S^+)$ is a nonpositive integer, we have $x(V \setminus S^+) - x_1^*(V \setminus S^+) = 0$, implying (3.2).

We next prove that the condition (3.3) holds. It suffices to show that $\langle d, y \rangle \leq \langle d, x_2^* \rangle$ for all $y \in X^*(f[p], \tilde{z}_2)$. By the definition of \tilde{z}_2 , we have $y(S^+) \leq \tilde{z}_2(S^+) = x_2^*(S^+)$ and $y(w) \leq \tilde{z}_2(w) \leq x_1^*(w)$ ($w \in V \setminus S^+$), where the latter implies $\text{supp}^+(y - x_1^*) \subseteq S^+$. By the choice of x_2^* , it holds that $\text{supp}^+(y - x_1^*) = S^+$ and $y(V \setminus S^+) \leq x_2^*(V \setminus S^+)$. Therefore,

$$\langle d, y \rangle - \langle d, x_2^* \rangle = \frac{y(S^+) - x_2^*(S^+)}{k|S^+|} + \{y(V \setminus S^+) - x_2^*(V \setminus S^+)\} \leq 0.$$

This concludes the proof of Theorem 3.2 (i).

3.2 Proof of “(SC_G¹) ⇒ f is M^h-convex”

We prove Theorem 3.2 (ii).

Let $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ be a function such that $\text{dom } f$ is bounded, and suppose that f satisfies (SC_G¹). We prove the M^h-convexity of f by using Theorem 2.2, a characterization of M^h-convex functions by the sets of minimizers. Since $f[p]$ satisfies (SC_G¹) for all $p \in \mathbf{R}^V$, it suffices to show that $\arg \min f$ is an M^h-convex set. To prove the M^h-convexity of $\arg \min f$, we use Theorem 2.1; we first consider the case where $x \leq y$ or $x \geq y$ (Lemma 3.3), then the case where $x - y = \chi_s + \chi_u - \chi_r - \chi_t$ for some $r, s, t, u \in V \cup \{0\}$ (Lemmas 3.4, 3.6, 3.7), and finally the general case (Lemma 3.9).

Lemma 3.3. *For any $x, y \in \arg \min f$ with $x \leq y$, we have $[x, y]_{\mathbf{Z}} \subseteq \arg \min f$.*

Proof. We show that any $\tilde{x} \in [x, y]_{\mathbf{Z}}$ is contained in $\arg \min f$. Since $y \in X^*(f, y)$ and $\tilde{x} \leq y$, (SC_G¹) implies that there exists some $x_2 \in X^*(f, \tilde{x})$ ($\subseteq \arg \min f$) such that $\tilde{x} = \tilde{x} \wedge y \leq x_2 \leq \tilde{x}$, i.e., $x_2 = \tilde{x}$. \square

Lemma 3.4. *For any $x, y \in \arg \min f$ with $x - y = 2\chi_u - \chi_v$ for some distinct $u, v \in V$, we have $x - \chi_u, x - \chi_u + \chi_v \in \arg \min f$.*

Proof. We firstly prove that $x - \chi_u + \chi_v \in \arg \min f$. If $x + \chi_v \in \arg \min f$, then Lemma 3.3 implies $x - \chi_u + \chi_v \in \arg \min f$ since $x - \chi_u + \chi_v \in [y, x + \chi_v]_{\mathbf{Z}}$. Hence, we assume $x + \chi_v \notin \arg \min f$. Let M be a sufficiently large positive number, and ε be a sufficiently small positive number. We define $p \in \mathbf{R}^V$ by

$$p(w) = \begin{cases} -2\varepsilon & \text{if } w = u, \\ -3\varepsilon & \text{if } w = v, \\ -M & \text{otherwise.} \end{cases}$$

Assume, to the contrary, that $x - \chi_u + \chi_v \notin \arg \min f$. Then, we have $X^*(f[p], x - \chi_u + \chi_v) = \{y\}$ and $X^*(f[p], x + \chi_v) = \{x\}$. Since $x - \chi_u + \chi_v \leq x + \chi_v$, it follows from (SC_G¹) that $x - \chi_u = (x - \chi_u + \chi_v) \wedge x \leq y$, a contradiction since $x(u) - 1 > y(u)$. Hence, $x - \chi_u + \chi_v \in \arg \min f$ holds.

We then prove that $x - \chi_u \in \arg \min f$. If there exists some $x' \in \arg \min f$ with $x' \leq x - \chi_u$, then Lemma 3.3 implies $x - \chi_u \in \arg \min f$ since $x - \chi_u \in [x', x]_{\mathbf{Z}}$. Hence, we assume that there exists no such $x' \in \arg \min f$, and derive a contradiction. Put $x_* = x + \chi_v - \alpha_*\chi_v$ and $y_* = x + \chi_v - \beta_*\chi_u$, where

$$\alpha_* = \max\{\alpha \mid x + \chi_v - \alpha\chi_v \in \arg \min f\}, \quad \beta_* = \max\{\beta \mid x + \chi_v - \beta\chi_u \in \arg \min f\}.$$

We define $\hat{p} \in \mathbf{R}^V$ by

$$\hat{p}(w) = \begin{cases} \varepsilon\alpha_* & \text{if } w = u, \\ \varepsilon(\beta_* + 1) & \text{if } w = v, \\ -M & \text{otherwise.} \end{cases}$$

Then, we have $X^*(f[\hat{p}], x + \chi_v) = \{x_*\}$ and $X^*(f[\hat{p}], x - \chi_u + \chi_v) = \{y_*\}$. By (SC_G¹), we have $x_* - \chi_u = (x - \chi_u + \chi_v) \wedge x_* \leq y_*$, a contradiction since $x_*(u) - 1 = x(u) - 1 > y(u) \geq y_*(u)$. \square

Lemma 3.5. *Let $x, y \in \arg \min f$ be any distinct vectors with $x(V) \geq y(V)$. Suppose that there exists no $z \in \arg \min f$ satisfying $z \leq x \vee y$, $\text{supp}(x - z) \subseteq \text{supp}(x - y)$, and $z(V) > x(V)$. Then, for any $u \in \text{supp}^+(x - y)$ there exists $v \in \text{supp}^-(x - y) \cup \{0\}$ such that $x - \chi_u + \chi_v \in \arg \min f$.*

Proof. Let $u \in \text{supp}^+(x - y)$. Since $x \in X^*(f, x \vee y)$, it follows from (SC_G^1) that there exists some $x_2 \in X^*(f, (x \vee y) - \chi_u)$ ($\subseteq \arg \min f$) such that $((x \vee y) - \chi_u) \wedge x \leq x_2$. This inequality implies

$$\begin{aligned} x_2(u) &= x(u) - 1, & x_2(w) &= x(w) \quad (w \in V \setminus [\text{supp}^-(x - y) \cup \{u\}]), \\ x_2(w) &\geq x(w) \quad (w \in \text{supp}^-(x - y)), \end{aligned}$$

from which follows $x(V) \geq x_2(V) \geq x(V) - 1$. Hence, $x_2 = x - \chi_u + \chi_v$ holds for some $v \in \text{supp}^-(x - y) \cup \{0\}$. \square

Lemma 3.6. *For any $x, y \in \arg \min f$ with $x - y = \chi_s + \chi_u - \chi_v$ for some distinct $s, u, v \in V$, we have $x - \chi_s + \chi_v, x - \chi_u \in \arg \min f$ or $x - \chi_u + \chi_v, x - \chi_s \in \arg \min f$ (or both).*

Proof. It suffices to show the following claims hold:

- (a) $x - \chi_u + \chi_v \in \arg \min f$ or $x - \chi_u \in \arg \min f$,
- (b) $x - \chi_s + \chi_v \in \arg \min f$ or $x - \chi_s \in \arg \min f$,
- (c) $x - \chi_s + \chi_v \in \arg \min f$ or $x - \chi_u + \chi_v \in \arg \min f$,
- (d) $x - \chi_s \in \arg \min f$ or $x - \chi_u \in \arg \min f$.

We firstly prove the claims (a) and (b). If $x + \chi_v \in \arg \min f$, then Lemma 3.3 implies $\{x - \chi_u + \chi_v, x - \chi_s + \chi_v\} \subseteq [y, x + \chi_v]_{\mathbf{Z}} \subseteq \arg \min f$. If $x + \chi_v \notin \arg \min f$, then Lemma 3.5 for x and y implies (a) and (b) since $\text{supp}^-(x - y) = \{v\}$.

We then prove (c). Assume, to the contrary, that neither $x - \chi_s + \chi_v$ nor $x - \chi_u + \chi_v$ is in $\arg \min f$. Then, we have $x - \chi_u \in \arg \min f$ by (a). Since $x - \chi_u \leq x - \chi_u + \chi_v \leq x + \chi_v$, Lemma 3.3 implies $x + \chi_v \notin \arg \min f$. Put $z_1 = x + \chi_v$ and $z_2 = x - \chi_u + \chi_v$. Let M be a sufficiently large positive number, and ε be a sufficiently small positive number. We define $p \in \mathbf{R}^V$ by

$$p(w) = \begin{cases} -2\varepsilon & \text{if } w \in \{s, u\}, \\ -3\varepsilon & \text{if } w = v, \\ -M & \text{otherwise.} \end{cases}$$

Then, $X^*(f[p], z_1) = \{x\}$. By (SC_G^1) , there exists some $x_2 \in X^*(f[p], z_2)$ with $x - \chi_u = z_2 \wedge x \leq x_2 \leq x - \chi_u + \chi_v$, i.e., x_2 is either $x - \chi_u$ or $x - \chi_u + \chi_v$. However, we have

$$\begin{aligned} f[p](x - \chi_u) - f[p](y) &= \varepsilon + f(x - \chi_u) - f(y) > 0, \\ f[p](x - \chi_u + \chi_v) - f[p](y) &= -2\varepsilon + f(x - \chi_u + \chi_v) - f(y) > 0 \end{aligned}$$

since $y \in \arg \min f$ and $x - \chi_u + \chi_v \notin \arg \min f$. This shows that $x_2 \notin X^*(f[p], z_2)$, a contradiction. Hence, the claim (c) holds.

We finally prove (d). Assume, to the contrary, that neither $x - \chi_s$ nor $x - \chi_u$ is in $\arg \min f$. Since $\{x, x - \chi_u + \chi_v, x - \chi_s + \chi_v\} \subseteq \arg \min f$ by (a) and (b), Lemma 3.4 implies $x - 2\chi_u + \chi_v, x - 2\chi_s + \chi_v, x - \chi_v \notin \arg \min f$. By Lemma 3.3, if $x' \in \mathbf{Z}^V$ satisfies at least one of the

inequalities $x' \leq x - \chi_u$, $x' \leq x - \chi_s$, $x' \leq x - \chi_v$, $x' \leq x - 2\chi_u + \chi_v$, and $x' \leq x - 2\chi_s + \chi_v$, then $x' \notin \arg \min f$. This shows that $\arg \min f \cap \{x' \mid x' \leq z_1\} \subseteq \{x, y, x - \chi_u + \chi_v, x - \chi_s + \chi_v, x + \chi_v\}$, where $z_1 = x + \chi_v$. We define $\widehat{p} \in \mathbf{R}^V$ by

$$\widehat{p}(w) = \begin{cases} \varepsilon & \text{if } w \in \{s, u\}, \\ 3\varepsilon & \text{if } w = v, \\ -M & \text{otherwise.} \end{cases}$$

Then, we have $X^*(f[\widehat{p}], z_1) = \{x\}$ and $X^*(f[\widehat{p}], z_2) = \{y\}$, where $z_2 = x - \chi_u + \chi_v$. By (SC_G¹), we have $x - \chi_u = z_2 \wedge x \leq y$, a contradiction since $x(s) > y(s)$. Hence, the claim (d) holds. \square

Lemma 3.7. *Let $x, y \in \text{dom } f$ be any vectors satisfying $\|x - y\|_1 = 4$ and $x(V) = y(V)$, and $u \in \text{supp}^+(x - y)$. Then, there exist $v, w \in \text{supp}^-(x - y) \cup \{0\}$ such that $x - \chi_u + \chi_v, y + \chi_u - \chi_w \in \arg \min f$.*

Proof. Suppose that $y = x - \chi_s - \chi_u + \chi_r + \chi_t$ for some $r, s, t, u \in V$ with $\{s, u\} \cap \{r, t\} = \emptyset$. We show that $x - \chi_u + \chi_v \in \arg \min f$ and $y + \chi_u - \chi_w \in \arg \min f$ hold for some $v, w \in \{r, t, 0\}$.

We firstly consider the case where there exists some $z \in \arg \min f$ satisfying

$$z \leq x \vee y, \quad \text{supp}(x - z) \subseteq \text{supp}(x - y), \quad z(V) > x(V). \quad (3.5)$$

This assumption implies

$$\{x + \chi_r, x + \chi_t, x + \chi_r + \chi_t, y + \chi_s, y + \chi_u\} \cap \arg \min f \neq \emptyset.$$

We first claim that $x + \chi_r \in \arg \min f$ or $x + \chi_t \in \arg \min f$ holds. If $x + \chi_r + \chi_t \in \arg \min f$, then Lemma 3.3 implies $\{x + \chi_r, x + \chi_t\} \subseteq \arg \min f$. If $y + \chi_u \in \arg \min f$, then Lemmas 3.4 and 3.6 for $y + \chi_u = x - \chi_s + \chi_r + \chi_t$ and x imply $x + \chi_r \in \arg \min f$ or $x + \chi_t \in \arg \min f$. The case where $y + \chi_s \in \arg \min f$ can be dealt with similarly.

We, w.l.o.g., assume that $x + \chi_r \in \arg \min f$. Lemmas 3.4 and 3.6 for $x + \chi_r = y + \chi_u + \chi_s - \chi_t$ and y imply $\{y + \chi_u, y + \chi_s - \chi_t\} \subseteq \arg \min f$ or $\{y + \chi_s, y + \chi_u - \chi_t\} \subseteq \arg \min f$. If the former holds, then we are done since $y + \chi_s - \chi_t = x - \chi_u + \chi_r$. If the latter holds, then we can apply Lemmas 3.4 and 3.6 to $y + \chi_s = x - \chi_u + \chi_r + \chi_t$ and x to obtain $x - \chi_u + \chi_r \in \arg \min f$ or $x - \chi_u + \chi_t \in \arg \min f$.

We then consider the case where there exists no $z \in \arg \min f$ satisfying (3.5). By Lemma 3.5, we have $x - \chi_u + \chi_v \in \arg \min f$ and $x - \chi_s + \chi_{v'} \in \arg \min f$ for some $v, v' \in \{r, t, 0\}$. If $v' \neq 0$, then we have $x - \chi_s + \chi_{v'} = y + \chi_u - \chi_w$ for some $w \in \{r, t\}$. If $v' = 0$, then we can apply Lemmas 3.4 and 3.6 to y and $x - \chi_s$ to obtain $y + \chi_u - \chi_r \in \arg \min f$ or $y + \chi_u - \chi_t \in \arg \min f$. \square

Lemma 3.8. *Let $x, y, z \in \mathbf{Z}^V$ be any distinct vectors with $z \leq x \vee y$ and $z(V) > \max\{x(V), y(V)\}$. Then, we have $\|z - x\|_1 < \|x - y\|_1$ and $\|z - y\|_1 < \|x - y\|_1$.*

Proof. We prove $\|z - x\|_1 < \|x - y\|_1$ only. Put $S^+ = \text{supp}^+(x - y)$, $C = \text{supp}^-(x - z) (\subseteq \text{supp}^-(x - y))$, $D = \text{supp}^-(x - y) \setminus C$, and $E = V \setminus \text{supp}(x - y)$. Then,

$$\begin{aligned} \|x - y\|_1 - \|x - z\|_1 &= z(S^+ \cup D \cup E) + y(C \cup D) - y(S^+) - z(C) - 2x(D) - x(E) \\ &> 2[y(C) - z(C)] + 2[y(D) - x(D)] \geq 0, \end{aligned}$$

where the first inequality is by $z(V) > y(V)$ and $y(E) = x(E)$, and the second by $y(C) \geq z(C)$ and $y(D) \geq x(D)$. \square

Lemma 3.9. *$\arg \min f$ satisfies $(B^1\text{-EXC}_\pm)$, i.e., $\arg \min f$ is an M^\sharp -convex set if it is nonempty.*

Proof. Let $x, y \in \arg \min f$ and $u \in \text{supp}^+(x - y)$. We show by induction on $\|x - y\|_1$ that

$$x - \chi_u + \chi_v \in \arg \min f \quad (\exists v \in \text{supp}^-(x - y) \cup \{0\}), \quad (3.6)$$

$$y + \chi_u - \chi_w \in \arg \min f \quad (\exists w \in \text{supp}^-(x - y) \cup \{0\}). \quad (3.7)$$

By Lemmas 3.3, 3.4, and 3.6, we may assume $\text{supp}^+(x - y) \neq \emptyset$, $\text{supp}^-(x - y) \neq \emptyset$, and $\|x - y\|_1 \geq 4$.

We first claim that the following (3.8) or (3.9) holds:

$$x' = x - \chi_s + \chi_t \in \arg \min f \quad (\exists s \in \text{supp}^+(x - y), \exists t \in \text{supp}^-(x - y) \cup \{0\}), \quad (3.8)$$

$$y' = y + \chi_i - \chi_j \in \arg \min f \quad (\exists i \in \text{supp}^+(x - y) \cup \{0\}, \exists j \in \text{supp}^-(x - y)). \quad (3.9)$$

If there exists no $z \in \arg \min f$ satisfying $z \leq x \vee y$, $\text{supp}(x - z) \subseteq \text{supp}(x - y)$, and $z(V) > \max\{x(V), y(V)\}$, then Lemma 3.5 implies (3.8) or (3.9) according as $x(V) \geq y(V)$ or $x(V) < y(V)$. Hence, we assume that such $z \in \arg \min f$ exists. We may also assume $z \neq x \vee y$, since otherwise $(x \vee y) - \chi_w \in \arg \min f$ ($\forall w \in \text{supp}(x - y)$) holds by Lemma 3.3. Therefore, we have $\text{supp}^+(x - z) \cap \text{supp}^+(x - y) \neq \emptyset$ or $\text{supp}^-(z - y) \cap \text{supp}^-(x - y) \neq \emptyset$. Note that $\|x - z\|_1 < \|x - y\|_1$ and $\|y - z\|_1 < \|x - y\|_1$ by Lemma 3.8. If $\text{supp}^+(x - z) \cap \text{supp}^+(x - y) \neq \emptyset$, then the induction hypothesis for x and z implies $x - \chi_s + \chi_t \in \arg \min f$ for some $s \in \text{supp}^+(x - z) \cap \text{supp}^+(x - y)$ and $t \in \text{supp}^-(x - z) \cup \{0\} \subseteq \text{supp}^-(x - y) \cup \{0\}$, i.e., (3.8) holds. Similarly, (3.9) holds if $\text{supp}^-(z - y) \cap \text{supp}^-(x - y) \neq \emptyset$.

In the following, we assume that (3.8) holds; the case where (3.9) holds can be dealt with similarly and therefore the proof is omitted.

(Case 1: $\text{supp}^+(x' - y) = \emptyset$) We have $\text{supp}^+(x - y) = \{u\}$, implying $x' = x - \chi_u + \chi_t$ ($\exists t \in \text{supp}^-(x - y) \cup \{0\}$), i.e., (3.6) holds. Since $x' \leq y$, it follows from Lemma 3.3 that $y - \chi_j \in \arg \min f$ for $j \in \text{supp}^-(x' - y) \subseteq \text{supp}^-(x - y)$. Since $\|x - (y - \chi_j)\|_1 < \|x - y\|_1$ and $\text{supp}^+(x - (y - \chi_j)) = \{u\}$, the induction hypothesis implies $(y - \chi_j) + \chi_u - \chi_h \in \arg \min f$ for some $h \in \text{supp}^-(x - (y - \chi_j)) \cup \{0\} \subseteq \text{supp}^-(x - y) \cup \{0\}$. If $h \neq 0$ then we apply Lemma 3.4 or 3.6 to $y - \chi_j + \chi_u - \chi_h$ and y to obtain $\{y + \chi_u - \chi_j, y + \chi_u - \chi_h\} \cap \arg \min f \neq \emptyset$, i.e., (3.7) holds.

(Case 2: $\text{supp}^+(x' - y) \neq \emptyset$, $u \notin \text{supp}^+(x' - y)$) Since $u \in \text{supp}^+(x - y)$, we have $x' = x - \chi_u + \chi_t$ for some $t \in \text{supp}^-(x - y) \cup \{0\}$, i.e., (3.6) holds. Since $\|x' - y\|_1 < \|x - y\|_1$, the induction hypothesis for x' and y implies $\tilde{y} = y + \chi_i - \chi_j \in \arg \min f$ for some $i \in \text{supp}^+(x' - y) \subseteq \text{supp}^+(x - y) \setminus \{u\}$ and $j \in \text{supp}^-(x' - y) \cup \{0\} \subseteq \text{supp}^-(x - y) \cup \{0\}$. Since $\|x - \tilde{y}\|_1 < \|x - y\|_1$, the induction hypothesis for x , \tilde{y} , and $u \in \text{supp}^+(x - \tilde{y})$ implies $\tilde{y} + \chi_u - \chi_h \in \arg \min f$ for some $h \in \text{supp}^-(x - \tilde{y}) \cup \{0\} \subseteq \text{supp}^-(x - y) \cup \{0\}$. Applying Lemma 3.3, 3.4, 3.6, or 3.7 to $\tilde{y} + \chi_u - \chi_h = y + \chi_i + \chi_u - \chi_j - \chi_h$ and y , we have $\{y + \chi_u - \chi_j, y + \chi_u - \chi_h\} \cap \arg \min f \neq \emptyset$, i.e., (3.7) holds.

(Case 3: $u \in \text{supp}^+(x' - y)$) Since $\|x' - y\|_1 < \|x - y\|_1$, the induction hypothesis for x' , y , and $u \in \text{supp}^+(x' - y)$ implies $y + \chi_u - \chi_w \in \arg \min f$ for some $w \in \text{supp}^-(x' - y) \cup \{0\} \subseteq \text{supp}^-(x - y) \cup \{0\}$, i.e., (3.7) holds. By using this fact we can show (3.6) in a similar way as in Case 2. \square

4 Concluding Remarks

It is shown in [3, 5, 6] that M^{\natural} -convexity of a function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ implies the properties (SC¹) and (SC²). Theorem 3.1 is an immediate consequence of this fact since $f[p]$ is M^{\natural} -convex for any $p \in \mathbf{R}^V$ if f is M^{\natural} -convex. In fact, the properties (SC¹) and (SC²) hold true under a weaker assumption than M^{\natural} -convexity. We call a function f *semistrictly quasi M^{\natural} -convex* if $\text{dom } f \neq \emptyset$ and it satisfies (SSQM^h):

$$\begin{aligned} & \text{(SSQM}^{\text{h}}\text{)} \quad \forall x, y \in \text{dom } f, \forall u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y) \cup \{0\}: \\ & \quad \text{(i)} \quad f(x - \chi_u + \chi_v) \geq f(x) \implies f(y + \chi_u - \chi_v) \leq f(y), \quad \text{and} \\ & \quad \text{(ii)} \quad f(y + \chi_u - \chi_v) \geq f(y) \implies f(x - \chi_u + \chi_v) \leq f(x). \end{aligned}$$

It is easy to see that any M^{\natural} -convex function satisfies (SSQM^h). See [12] for more accounts on semistrictly quasi M^{\natural} -convex functions.

Theorem 4.1. *A function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ with (SSQM^h) satisfies (SC¹) and (SC²).*

Proof. We prove (SC¹) only; (SC²) can be shown similarly and the proof is omitted.

Let $z_1, z_2 \in \mathbf{Z}^V$ be any vectors with $z_1 \geq z_2$ and $X^*(f, z_2) \neq \emptyset$. Also, let $x_1 \in X^*(f, z_1)$. We choose $x_2 \in X^*(f, z_2)$ minimizing the value $\sum \{x_1(w) - x_2(w) \mid w \in \text{supp}^+((x_1 \wedge z_2) - x_2)\}$. Assume, to the contrary, that $\text{supp}^+((x_1 \wedge z_2) - x_2) \neq \emptyset$. Let $u \in \text{supp}^+((x_1 \wedge z_2) - x_2) (\subseteq \text{supp}^+(x_1 - x_2))$. By (SSQM^h), there exists $v \in \text{supp}^-(x_1 - x_2) \cup \{0\}$ such that if $f(x_1 - \chi_u + \chi_v) \geq f(x_1)$ then $f(x_2 + \chi_u - \chi_v) \leq f(x_2)$. Since $x_1 - \chi_u + \chi_v \leq x_1 \vee x_2 \leq z_1$, we have $f(x_1 - \chi_u + \chi_v) \geq f(x_1)$. Hence, $f(x_2 + \chi_u - \chi_v) \leq f(x_2)$ follows. By the choice of u we have $x_2 + \chi_u - \chi_v \leq z_2$. This implies that $x_2 + \chi_u - \chi_v \in X^*(f, z_2)$, which contradicts the choice of x_2 . Hence we have $x_1 \wedge z_2 \leq x_2$. \square

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