Representations of the Cuntz-Krieger algebras. II —Permutative representations—

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We generalize permutative representations of the Cuntz algebras for the Cuntz-Krieger algebra $\mathcal{O}_A$ for any A. We characterize cyclic permutative representations by notions of cycle and chain, and show their existence and uniqueness. We show necessary and sufficient conditions for their irreducibility and equivalence. In consequence, we have a complete classification of permutative representations of $\mathcal{O}_A$ for any A. Furthermore we show that the uniqueness of irreducible decomposition holds for permutative representation and decomposition formulae.

1. Introduction

Permutative representations of the Cuntz algebras are completely classified by [**1**, **3**, **4**]. We generalize their works to the Cuntz-Krieger algebra $\mathcal{O}_A$ for any A in this paper. Remarkable points is that *the uniqueness of irreducible decomposition holds* for permutative representations of $\mathcal{O}_A$ for any A. Therefore the decomposition formulae make sense.

Let $N \geq 2$ and A be an $N \times N$ matrix which has entries in $\{0, 1\}$ and has no rows or columns identically equal to zero.

Theorem 1.1. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_A$ and $s_1, \ldots, s_N$ be canonical generators of $\mathcal{O}_A$. Assume that there are a complete orthonormal basis $\{e_n\}_{n\in\Lambda}$ of $\mathcal{H}$ and a family $\{\Lambda_i\}_{i=1}^N$ of subsets of Λ such that $^\forall i \in \{1, \ldots, N\}$, $^\forall n \in \Lambda_i$, $^\exists(z_{i,n}, m_{i,n}) \in U(1) \times \Lambda$ s.t.*

$$\pi(s_i)e_n = \begin{cases} z_{i,n}e_{m_{i,n}} & (n \in \Lambda_i), \\ 0 & (\textit{otherwise}). \end{cases} \tag{1.1}$$

Then the followings hold:

(i) *$(\mathcal{H}, \pi)$ is uniquely decomposed into the direct sum of cyclic representations which satisfy (1.1).*

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(ii) *If $(\mathcal{H}, \pi)$ is cyclic, then there is a unit cyclic vector $\Omega \in \mathcal{H}$ such that either of the followings holds:*
   a) *There are $(j_1, \ldots, j_p) \in \{1, \ldots, N\}^p$ and $c \in U(1)$ such that $\pi(s_{j_1} \cdots s_{j_p})\Omega = c\Omega$.*
   b) *There is $(k_n)_{n \in \mathbf{N}} \in \{1, \ldots, N\}^\infty$ such that $\{\pi(s_{k_n}^* \cdots s_{k_1}^*)\Omega\}_{n \in \mathbf{N}}$ is an orthonormal family in $\mathcal{H}$ where $\mathbf{N} \equiv \{1, 2, 3, \ldots\}$.*

   We denote cases a) and b) by $P((j_n)_{n=1}^p; c)$ and $P((k_n)_{n \in \mathbf{N}})$, respectively.

(iii) *$P((j_n)_{n=1}^p; c)$ (resp. $P((k_n)_{n \in \mathbf{N}})$) is irreducible if and only if there is no $\sigma \in \mathbf{Z}_p \setminus \{id\}$ such that $(j_{\sigma(1)}, \ldots, j_{\sigma(p)}) = (j_1, \ldots, j_p)$(resp. there is no $(q, n_0) \in \mathbf{N} \times \mathbf{N}$ such that $k_{n+q} = k_n$ for each $n \geq n_0$.)*

(iv) *$P((j_n)_{n=1}^p; c) \not\sim P((k_n)_{n \in \mathbf{N}})$. $P((j_n)_{n=1}^p; c) \sim P((j_n^{'})_{n=1}^{p^{'}}; c^{'})$ if and only if $p = p^{'}$, $c = c^{'}$ and there is $\sigma \in \mathbf{Z}_p$ such that $j_{\sigma(n)}^{'} = j_n$ for each $n = 1, \ldots, p$. $P((k_n)_{n \in \mathbf{N}}) \sim P((k_n^{'})_{n \in \mathbf{N}})$ if and only if there is $(q, n_0) \in \mathbf{Z} \times \mathbf{N}$ such that $k_{n+q} = k_n^{'}$ for each $n \geq n_0$.*

Specially, a representation of $\mathcal{O}_A$ in Theorem 1.1 such that $z_{i,n} = 1$ for every $(i, n) \in \{1, \ldots, N\} \times \Lambda$ in (1.1) is called a *permutative representation* of $\mathcal{O}_A$.

In § 2, we prepare multiindices associated with a matrix A and introduce A-branching function systems and show their properties. In § 3, we give another definition of permutative representation and show their properties by multiindices. The existence of cyclic representations appearing in Theorem 1.1 (ii) is shown for each multiindex in § 2. We show the construction of the canonical basis of a given permutative representation. In § 4, we show uniqueness, irreducibility and equivalence of them. In § 5, we show decomposition formulae of permutative representations. Theorem 1.1 is shown here. In § 6, we show states and spectrums of $\mathcal{O}_A$ associated with permutative representations. In § 7, we show decomposition formulae of standard representations of the Cuntz-Krieger algebras. In § 8, we show examples.

2. A-branching function systems

2.1. Multiindices. We introduce several sets of multiindices which consist of numbers $1, \ldots, N$ for $N \geq 2$ in order to describe invariants of representations of $\mathcal{O}_A$.

Put $\{1, \ldots, N\}^0 \equiv \{0\}$, $\{1, \ldots, N\}^k \equiv \{(j_l)_{l=1}^k : j_l = 1, \ldots, N, \, l = 1, \ldots, k\}$ for $k \geq 1$ and $\{1, \ldots, N\}^\infty \equiv \{(j_n)_{n \in \mathbf{N}} : j_n \in \{1, \ldots, N\}, \, n \in \mathbf{N}\}$. Denote $\{1, \ldots, N\}^* \equiv \coprod_{k \geq 0} \{1, \ldots, N\}^k$, $\{1, \ldots, N\}_1^* \equiv \coprod_{k \geq 1} \{1, \ldots, N\}^k$, $\{1, \ldots, N\}^\# \equiv \{1, \ldots, N\}_1^* \sqcup \{1, \ldots, N\}^\infty$. For $J \in \{1, \ldots, N\}^\#$, the *length* $|J|$ of J is defined by $|J| \equiv k$ when $J \in \{1, \ldots, N\}^k$. For $J_1, J_2 \in \{1, \ldots, N\}^*$ and $J_3 \in \{1, \ldots, N\}^\infty$, $J_1 \cup J_2 \equiv (j_1, \ldots, j_k, j_1^{'}, \ldots, j_l^{'})$, $J_1 \cup J_3 \equiv (j_1, \ldots, j_k, j_1^{''}, j_2^{''}, \ldots)$ when $J_1 = (j_1, \ldots, j_k)$, $J_2 = (j_1^{'}, \ldots, j_l^{'})$ and $J_3 = (j_n^{''})_{n \in \mathbf{N}}$. Specially, we define $J\cup\{0\}=\{0\}\cup J=J$ for $J\in\{1,\ldots,N\}^{\#}$ and $(i,J)\equiv(i)\cup J$ for convenience. For $J\in\{1,\ldots,N\}^*$ and $k\geq 2$, $J^k\equiv\underbrace{J\cup\cdots\cup J}_{k}$ and $J^\infty\equiv J\cup J\cup J\cup\cdots\in\{1,\ldots,N\}^\infty$. For $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}^k$ and $\tau\in\mathbf{Z}_k$, denote $\tau(J)=(j_{\tau(1)},\ldots,j_{\tau(k)})$.

For $N\geq 2$, let $M_N(\{0,1\})$ be the set of all $N\times N$ matrices in which have entries in $\{0,1\}$ and has no rows or columns identically equal to zero. $A=(a_{ij})$ is *full* if $a_{ij}=1$ for each $i,j=1,\ldots,N$. For $A=(a_{ij})\in M_N(\{0,1\})$, define

$$\{1,\ldots,N\}^*_A\equiv\coprod_{k\geq 0}\{1,\ldots,N\}^k_A,$$

$$\{1,\ldots,N\}^0_A\equiv\{0\},\quad \{1,\ldots,N\}^1_A\equiv\{1,\ldots,N\},$$

$$\{1,\ldots,N\}^k_A\equiv\{(j_i)_{i=1}^k\in\{1,\ldots,N\}^k : a_{j_{i-1}j_i}=1,\, i=2,\ldots,k\}\quad (k\geq 2),$$

$$\{1,\ldots,N\}^*_{A,c}\equiv\coprod_{k\geq 1}\{1,\ldots,N\}^k_{A,c},$$

$$\{1,\ldots,N\}^k_{A,c}\equiv\{(j_i)_{i=1}^k\in\{1,\ldots,N\}^k_A : a_{j_kj_1}=1\},$$

$$\{1,\ldots,N\}^\infty_A\equiv\{(j_n)_{n\in\mathbf{N}}\in\{1,\ldots,N\}^\infty : a_{j_{n-1}j_n}=1, n\geq 2\},$$

$$\{1,\ldots,N\}^{\#}_{A,c}\equiv\{1,\ldots,N\}^*_{A,c}\sqcup\{1,\ldots,N\}^\infty_A.$$

$J\in\{1,\ldots,N\}^*_1$ is *periodic* if there are $m\geq 2$ and $J_0\in\{1,\ldots,N\}^*_1$ such that $J=J_0^m$. For $J_1,J_2\in\{1,\ldots,N\}^*_1$, $J_1\sim J_2$ if there are $k\geq 1$ and $\tau\in\mathbf{Z}_k$ such that $|J_1|=|J_2|=k$ and $\tau(J_1)=J_2$. For $(J,z),(J',z')\in\{1,\ldots,N\}^*_1\times U(1)$, $(J,z)\sim(J',z')$ if $J\sim J'$ and $z=z'$ where $U(1)\equiv\{z\in\mathbf{C}:|z|=1\}$. $J\in\{1,\ldots,N\}^\infty$ is *eventually periodic* if there are $J_0,J_1\in\{1,\ldots,N\}^*_1$ such that $J=J_0\cup J_1^\infty$. Specially, if $J\in\{1,\ldots,N\}^\infty_A$, then $J_0\in\{1,\ldots,N\}^*_A$ and $J_1\in\{1,\ldots,N\}^*_{A,c}$ in the above. For $J_1,J_2\in\{1,\ldots,N\}^\infty$, $J_1\sim J_2$ if there are $J_3,J_4\in\{1,\ldots,N\}^*$ and $J_5\in\{1,\ldots,N\}^\infty$ such that $J_1=J_3\cup J_5$ and $J_2=J_4\cup J_5$. If $J\in\{1,\ldots,N\}^\infty_A$ is eventually periodic, then there is $J_1\in\{1,\ldots,N\}^*_{A,c}$ such that $J\sim J_1^\infty$. For $J,J'\in\{1,\ldots,N\}^{\#}$, $J\sim J'$ if $J,J\in\{1,\ldots,N\}^*_1$ and $J\sim J'$, or $J,J\in\{1,\ldots,N\}^\infty$ and $J\sim J'$.

For $J_1=(j_1,\ldots,j_k),J_2=(j_1',\ldots,j_k')\in\{1,\ldots,N\}^k$, $k\geq 1$, $J_1\prec J_2$ if $\sum_{l=1}^k(j_l'-j_l)N^{k-l}\geq 0$. $J\in\{1,\ldots,N\}^*_1$ is *minimal* if $J\prec J'$ for each $J'\in\{1,\ldots,N\}^*_1$ such that $J\sim J'$. Specially, any element in $\{1,\ldots,N\}$ is non periodic and minimal. Put

$$<1,\ldots,N>^*_A\equiv\{J\in\{1,\ldots,N\}^*_{A,c} : J\text{ is minimal}\},$$

$$<1,\ldots,N>^\infty_A\equiv\{1,\ldots,N\}^\infty_A/\sim,$$

$$[1,\ldots,N]^*_A\equiv\{J\in<1,\ldots,N>^*_A : J\text{ is non periodic}\}.$$

$$[1,\ldots,N]^\infty_A\equiv\{[J]\in<1,\ldots,N>^\infty_A : J\text{ is non eventually periodic}\}$$

where $[J] \equiv \{J' \in \{1,\ldots,N\}_A^\infty : J \sim J'\}$. Then $[1,\ldots,N]_A^*$ is in one-to-one correspondence with the set of all equivalence classes of non periodic elements in $\{1,\ldots,N\}_{A,c}^*$. We denote an element $[K]$ of both $< 1,\ldots,N >_A^\infty$ and $[1,\ldots,N]_A^\infty$ by a representative element K if there is no ambiguity. Put

$$
\tag{2.1}
\begin{cases}
< 1,\ldots,N >_A^{\#} \equiv < 1,\ldots,N >_A^* \sqcup < 1,\ldots,N >_A^\infty, \\
\\
[1,\ldots,N]_A^{\#} \equiv [1,\ldots,N]_A^* \sqcup [1,\ldots,N]_A^\infty.
\end{cases}
$$

We show a systematic construction of non eventually periodic element in $\{1,\ldots,N\}_A^\infty$. For $J_1, J_2 \in \{1,\ldots,N\}_A^{\#}$, we denote $J_1J_2 \equiv J_1 \cup J_2$ simply.

Definition 2.1. *Let $A = (a_{ij}) \in M_N(\{0,1\})$.*

- (i) *A family $\{J_1,\ldots,J_l\} \subset \{1,\ldots,N\}_{A,c}^*$ is freely jointable if $J_aJ_b \in \{1,\ldots,N\}_{A,c}^*$ for each $a,b = 1,\ldots,l$.*
- (ii) *J_1 and J_2 in $\{1,\ldots,N\}_{A,c}^*$ are strongly inequivalent if there are no $a,b \in \mathbf{N}$ such that $J_1^a \sim J_2^b$.*
- (iii) *For a freely jointable family $\{J_i\}_{i=1}^l \subset \{1,\ldots,N\}_{A,c}^*$ and $K = (k_n)_{n\in\mathbf{N}} \in \{1,\ldots,l\}^\infty$, $J_K \in \{1,\ldots,N\}_A^\infty$ is defined by $J_K \equiv J_{k_1}J_{k_2}J_{k_3}\cdots$.*

By these preparations, we have the following proposition:

Proposition 2.2. *Assume that $J_1,\ldots,J_l \in \{1,\ldots,N\}_{A,c}^*$, $l \geq 2$, are freely jointable and J_a and J_b are strongly inequivalent for any $1 \leq a < b \leq l$. Then if $K = (k_n) \in \{1,\ldots,l\}^\infty$ is non eventually periodic, then J_K is non eventually periodic.*

Fix $J_1, J_2 \in \{1,\ldots,N\}_{A,c}^*$. Assume that both $J_1 = (j_1,\ldots,j_k)$ and $J_2 = (j_1',\ldots,j_m')$ are non periodic, they are inequivalent and $a_{j_k j_1'} = a_{j_m' j_1} = 1$. From this, $J_1J_2, J_2J_1 \in \{1,\ldots,N\}_{A,c}^*$. For a non eventually periodic element $K = (12112211122211112222\cdots) \in \{1,2\}^\infty$,

$$J_K = J_1J_2J_1J_1J_2J_2J_1J_1J_1J_2J_2J_2\cdots.$$

Then $J_K \in \{1,\ldots,N\}_A^\infty$ is non eventually periodic.

2.2. A-branching function systems. In [**7**], we introduce the A-branching function system on a measure space in order to define a representation of $\mathcal{O}_A$. Let X be a possibly uncountably infinite set. We consider an atomic measure μ on X by $\mu(\{x\}) \equiv 1$ for each $x \in X$. Then $L_2(X,\mu) = l_2(X)$. In this paper, we state about A-branching function systems on an atomic measure space with the normalized measure at each point and associated representations of the Cuntz-Krieger algebras for more detail.

We denote the set of injective maps from X to Y by $RN(X,Y)$ and put $RN_{loc}(X,Y) \equiv \bigcup_{X_0 \subset X} RN(X_0,Y)$. We simply denote $RN(X) \equiv RN(X,X)$. For $f \in RN_{loc}(X)$, we denote the domain and the range of f by $D(f)$ and

$R(f)$, respectively. $RN_{loc}(X)$ and $RN(X)$ are a groupoid and a semigroup by composition of maps, respectively. We denote $X \times Y$ and $X \cup Y$, the direct product and the direct sum of X and Y as sets, respectively. For $f \in RN(X_1, Y_1)$ and $g \in RN(X_2, Y_2)$, $f \oplus g \in RN(X_1 \cup X_2, Y_1 \cup Y_2)$ is defined by $(f \oplus g)|_{X_1} \equiv f$, $(f \oplus g)|_{X_2} \equiv g$.

Definition 2.3. *For $A = (a_{ij}) \in M_N(\{0,1\})$, $f = \{f_i\}_{i=1}^N$ is an A-branching function system on a set X if f satisfies the followings:*

(i) *There is a family $\{D(f_i)\}_{i=1}^N$ of subsets of X such that f_i is an injective map from $D(f_i)$ to X with the image $R(f_i)$ for each $i = 1, \ldots, N$.*
(ii) *$R(f_i) \cap R(f_j) = \emptyset$ when $i \neq j$.*
(iii) *$D(f_i) = \coprod_{j:a_{ij}=1} R(f_j)$ for each $i = 1, \ldots, N$.*
(iv) *$X = \coprod_{i=1}^N R(f_i)$.*

Specially, if A is full, then we call A-branching function system by (N-)branching function system simply. We denote the set of all A-branching function systems, branching function systems on X by $\mathrm{BFS}_A(X)$, $\mathrm{BFS}_N(X)$, respectively.

By definition, $\mathrm{BFS}_A(X) \neq \emptyset$ if and only if $\#X = \infty$. The notion of original branching function system was introduced in order to construct a representation of $\mathcal{O}_N$ from a family of transformations in [**1**]. Definition 2.3 coincides with originals when A is full.

Let X and Y be sets. F is the *coding map* of $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_A(X)$ if F is a map on X such that $(F \circ f_i)(x) = x$ for each $x \in X$ and $i = 1, \ldots, N$. For $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_A(X)$ and $g = \{g_i\}_{i=1}^N \in \mathrm{BFS}_A(Y)$, $f \sim g$ if there is a bijection φ from X to Y such that $\varphi \circ f_i \circ \varphi^{-1} = g_i$ for $i = 1, \ldots, N$. For a bijection φ on X and $g = \{g_i\}_{i=1}^N \in \mathrm{BFS}_A(Y)$, we denote $\varphi \boxtimes g \equiv \{\varphi \times g_i\}_{i=1}^N \in \mathrm{BFS}_A(X \times Y)$. For $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_A(X)$ and $g = \{g_i\}_{i=1}^N \in \mathrm{BFS}_A(Y)$, we denote $f \oplus g \equiv \{f_i \oplus g_i\}_{i=1}^N \in \mathrm{BFS}_A(X \cup Y)$. Let $\{X_\omega\}_{\omega \in \Xi}$ be a family of sets. For $f^{[\omega]} = \{f_i^{[\omega]}\}_{i=1}^N \in \mathrm{BFS}_A(X_\omega)$ for $\omega \in \Xi$, f is the *direct sum* of $\{f^{[\omega]}\}_{\omega \in \Xi}$ if $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_A(X)$ for a set $X \equiv \coprod_{\omega \in \Xi} X_\omega$ which is defined by $f_i(n) \equiv f_i^{[\omega]}(n)$ when $n \in X_\omega$ for $i = 1, \ldots, N$ and $\omega \in \Xi$. For $f \in \mathrm{BFS}_A(X)$, $f = \bigoplus_{\omega \in \Xi} f^{[\omega]}$ is a *decomposition* of f into a family $\{f^{[\omega]}\}_{\omega \in \Xi}$ if there is a family $\{X_\omega\}_{\omega \in \Xi}$ of subsets of X such that f is the direct sum of $\{f^{[\omega]}\}_{\omega \in \Xi}$.

For $f = \{f_i\}_{i=1}^N \in \mathrm{BFS}_A(X)$, denote $f_J \equiv f_{j_1} \circ \cdots \circ f_{j_k}$ when $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}_A^k$, $k \geq 1$, and define $f_0 \equiv id$. When we denote $f_i(x)$, we assume $x \in D(f_i)$ automatically. Define

$$\mathcal{C}_x \equiv \{f_J(x) \in X : J \in \{1, \ldots, N\}_A^* \; s.t.\; x \in D(f_J)\} \cup \{F^n(x) \in X : n \in \mathbf{N}\}$$

where F is the coding map of f.

Definition 2.4. *For $A \in M_N(\{0,1\})$, let $f \in \mathrm{BFS}_A(X)$.*

(i) f *is cyclic if there is* $x \in X$ *such that* $\mathcal{C}_x = X$.
(ii) *For* $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k_{A,c}$, $k \geq 1$, $\{x_i\}_{i=1}^k$ *is a cycle of* f *by* J *if* $f_{j_l}(x_{l+1}) = x_l$ *for* $l = 1, \ldots, k-1$ *and* $f_{j_k}(x_1) = x_k$.
(iii) *For* $J = (j_n)_{n \in \mathbf{N}} \in \{1, \ldots, N\}^\infty_A$, $\{x_n\}_{n \in \mathbf{N}}$ *is a chain of* f *by* J *if* $f_{j_{n-1}}(x_n) = x_{n-1}$ *for each* $n \geq 2$.

For $x \in X$, if $y \in \mathcal{C}_x$, then $\mathcal{C}_y = \mathcal{C}_x$. For each $x \in X$ and $f \in \mathrm{BFS}_A(X)$, $f|_{\mathcal{C}_x} \in \mathrm{BFS}_A(\mathcal{C}_x)$ and $f|_{\mathcal{C}_x}$ is cyclic.

Lemma 2.5. *For* $A \in M_N(\{0,1\})$, *let* $f \in \mathrm{BFS}_A(X)$.
(i) *If* f *is cyclic, then* f *has either only a cycle or a chain.*
(ii) *If* f *is cyclic and has two chains* $\{x_n\}_{n \in \mathbf{N}}$ *and* $\{y_n\}_{n \in \mathbf{N}}$, *then there are* p *and* $M \geq 0$ *such that* $x_{p+n} = y_n$ *or* $x_n = y_{n+p}$ *for each* $n > M$.
(iii) *For any* $f \in \mathrm{BFS}_A(X)$, *there is a decomposition* $X = \coprod_{\lambda \in \Lambda} X_\lambda$ *such that* $f|_{X_\lambda}$ *is cyclic for each* $\lambda \in \Lambda$.

Proof. Let F be the coding map of f.
(i) For $x \in X$, consider $\Omega_x \equiv \{F^n(x) : n \in \mathbf{N}\}$. If there is $x \in X$ such that $\#\Omega_x < \infty$, then Ω_x contains a cycle C. If there is other cycle $C^{'}$ in X, then there is no path from C and $C^{'}$ by f. Therefore such $C^{'}$ must not exist by cyclicity. Hence f has only one cycle. If $\#\Omega_x = \infty$ for each $x \in X$, then there is no cycle in X. Ω_x itself is a chain.
(ii) We see that $\{y_n\}_{n \in \mathbf{N}} \subset \mathcal{C}_{y_1} = X = \mathcal{C}_{x_1}$. Hence either $y_1 = f_J(x_1)$ for $|J| = k$ or $y_1 = F^m(x_1)$ for $m \geq 0$. If $y_1 = f_J(x_1)$, then $y_{k+1} = F^k(y_1) = x_1$ and $y_{k+n} = x_n$ for each $n \geq 1$. If $y_1 = F^m(x_1)$, $y_1 = x_{m+1}$ and and $y_n = x_{n+m}$ for each $n \geq 1$. In consequence, the statement holds.
(iii) Put $\Lambda \equiv X/\!\sim$ where $x \sim y$ if and only if $\mathcal{C}_x = \mathcal{C}_y$. Then we have the statement for $X_\lambda \equiv \lambda \in \Lambda$. □

Definition 2.6. *For* $A \in M_N(\{0,1\})$, *let* $f \in \mathrm{BFS}_A(X)$.
(i) *For* $J \in \{1, \ldots, N\}^*_{A,c}$*(resp.* $J \in \{1, \ldots, N\}^\infty_A$*),* f *has a* $P(J)$*-component if* f *has a cycle(resp. a chain) by* J.
(ii) *For* $J \in \{1, \ldots, N\}^\#_{A,c}$, f *is* $P(J)$ *if* f *is cyclic and has a* $P(J)$*-component.*

For $J, J^{'} \in \{1, \ldots, N\}^\#_{A,c}$, assume that f and $f^{'}$ are $P(J)$ and $P(J^{'})$, respectively. Then $f \sim f^{'}$ if and only if $J \sim J^{'}$. This follows from the uniqueness of cycle and chain up to equivalences. From this and Lemma 2.5, the following holds:

Theorem 2.7. *Let* X *be a set. For any* $A \in M_N(\{0,1\})$ *and* $f \in \mathrm{BFS}_A(X)$, *there is decomposition* $X = \coprod_{\lambda \in \Lambda} X_\lambda$ *where* $f|_{X_\lambda}$ *is* $P(J_\lambda)$ *for some* $J_\lambda \in \{1, \ldots, N\}^\#_{A,c}$ *for each* $\lambda \in \Lambda$. *This decomposition is unique up to equivalence of branching function systems.*

We can simply describe the statement in Theorem 2.7 as follows:

$$f \sim \bigoplus_{J \in <1,\ldots,N>_A^{\#}} P(J)^{\oplus \nu_J}$$

where ν_J is the multiplicity of $P(J)$ in f for $J \in < 1, \ldots, N >_A^{\#}$.

2.3. Construction of A-branching function system. In this subsection, we construct an A-branching function system which is $P(J)$ for a given $J \in \{1, \ldots, N\}_{A,c}^{\#}$ for any $A \in M_N(\{0,1\})$.

Fix $A = (a_{ij}) \in M_N(\{0,1\})$. For $k \geq 1$, denote $\mathbf{Z}_k \equiv \{1, \ldots, k\}$ and σ is the shift on $\mathbf{Z}_k$. Let

$$\text{(2.2)} \qquad \begin{cases} \mathcal{T}(A;j) \equiv & \coprod_{k\geq 1} \mathcal{T}^{(k)}(A;j), \\ \mathcal{T}^{(k)}(A;j) \equiv & \{(j_1, \ldots, j_k) \in \{1, \ldots, N\}_A^k : a_{j_k j} = 1\}, \\ \mathcal{T}(j;A) \equiv & \coprod_{k\geq 1} \mathcal{T}^{(k)}(j;A), \\ \mathcal{T}^{(k)}(j;A) \equiv & \{(j_1, \ldots, j_k) \in \{1, \ldots, N\}_A^k : a_{j j_1} = 1\}. \end{cases}$$

For $J \in \{1, \ldots, N\}_{A,c}^k$, $k \geq 1$, put $J_l \equiv (j_l, \ldots, j_k)$ for $l = 1, \ldots, k$,

$$\text{(2.3)} \qquad \Lambda(A,J) \equiv \Lambda_1(A,J) \sqcup \Lambda_2(A,J) \sqcup \Lambda_3(A,J),$$

$$\Lambda_1(A,J) \equiv \{J_l : 1 \leq l \leq k\}, \quad \Lambda_2(A,J) \equiv \coprod_{l=1}^{k} \Lambda_{2,l}(A,J),$$

$$\Lambda_{2,l}(A,J) \equiv \{(j, J_l) : j \in \mathcal{T}^{(1)}(A; j_l),\ j \neq j_{\sigma^{-1}(l)}\},$$

$$\Lambda_3(A,J) \equiv \coprod_{(j,J_l) \in \Lambda_2(A,J)} \mathcal{T}(A;j) \times \{(j, J_l)\}.$$

Lemma 2.8. *Let a family $\{D(f_i)\}_{i=1}^N$ of subsets of $\Lambda(A,J)$ by*

$$D(f_i) \equiv \mathcal{T}(i;A) \cap \Lambda(A,J) \quad (i = 1, \ldots, N)$$

and a family $f = \{f_i\}_{i=1}^N$ of maps by $f_i : D(f_i) \to \Lambda(A,J)$

$$f_i(J^{'}) \equiv \begin{cases} J_k & (J^{'} = J \text{ and } i = j_k), \\ \\ (i, J^{'}) & (\text{otherwise}) \end{cases}$$

for $i = 1, \ldots, N$. Then f is an A-branching function system on $\Lambda(A,J)$ and f is $P(J)$.

Proof. We see that f_i is injective on $D(f_i)$ for $i = 1, \ldots, N$ and

$$R(f_i) = \{(j_1^{'}, \ldots, j_m^{'}) \in \Lambda(A,J) : j_1^{'} = i\} \quad (i = 1, \ldots, N).$$

From this, we can verify the axiom in Definition 2.3 for f. □

For $J = (j_n)_{n \in \mathbf{N}} \in \{1, \ldots, N\}_A^{\infty}$, put

$$\text{(2.4)} \qquad \Lambda(A,J) \equiv \Lambda_1(A,J) \sqcup \Lambda_2(A,J) \sqcup \Lambda_3(A,J),$$

$$\Lambda_1(A,J)\equiv \mathbf{N},\quad \Lambda_2(A,J)\equiv \coprod_{m\in\mathbf{N}}\Lambda_{2,m}(A,J),$$
$$\Lambda_{2,1}(A,J)\equiv \{(j,1): j\in \mathcal{T}^{(1)}(A;j_1)\},$$
$$\Lambda_{2,m}(A,J)\equiv \{(j,m): j\in \mathcal{T}^{(1)}(A;j_m),\, j\neq j_{m-1}\}\quad (m\geq 2),$$
$$\Lambda_3(A,J)\equiv \coprod_{(j,m)\in\Lambda_2(A,J)}\mathcal{T}(A;j)\times\{(j,m)\}.$$

Lemma 2.9. *Let a family* $\{D(f_i)\}_{i=1}^N$ *of subsets of* $\Lambda(A,J)$ *by*
$$D(f_i)\equiv \{m\in\mathbf{N}: a_{ij_m}=1\}\sqcup(\mathcal{T}(i;A)\times\mathbf{N})\cap\Lambda(A,J)$$
and a family $f=\{f_i\}_{i=1}^N$ *of maps by* $f_i: D(f_i)\to\Lambda(A,J)$,
$$\begin{cases} f_i(m)\equiv\begin{cases} m-1 & (i=j_{m-1} \text{ and } m\geq 2),\\ (i,m) & (\text{otherwise})\end{cases} & (m\in\Lambda_1(A,J)\cap D(f_i)),\\ f_i(J^{'},m)\equiv(\{i\}\cup J^{'},\, m) & ((J^{'},m)\in(\Lambda_2(A,J)\sqcup\Lambda_3(A,J))\cap D(f_i)).\end{cases}$$
Then f *is an* A*-branching function system on* $\Lambda(A,J)$ *and* f *is* $P(J)$.

Proof. We see that f_i is injective on $D(f_i)$ for $i=1,\ldots,N$ and $R(f_i)=\{m\in\Lambda_1(A,J): j_m=i\}\sqcup\{((j_1^{'},\ldots,j_k^{'}),m)\in\Lambda_2(A,J)\sqcup\Lambda_3(A,J): j_1^{'}=i\}$ for $i=1,\ldots,N$. From this, we can verify the axiom in Definition 2.3 for f. □

Theorem 2.10. *For each* $A\in M_N(\{0,1\})$ *and* $J\in\{1,\ldots,N\}^{\#}_{A,c}$, *there is an element in* $\mathrm{BFS}_A(\mathbf{N})$ *which is* $P(J)$.

Proof. Because both $\Lambda(A,J)$ in Lemma 2.8 and Lemma 2.9 are countably infinite, hence there is a natural bijection φ from $\Lambda(A,J)$ to $\mathbf{N}$. By using φ, we can define $g\equiv\{\varphi\circ f_i\circ\varphi^{-1}\}_{i=1}^N\in\mathrm{BFS}_A(\mathbf{N})$ which is $P(J)$. □

3. Definition and existence of permutative representation

For $A=(a_{ij})\in M_N(\{0,1\})$, $\mathcal{O}_A$ is *the Cuntz-Krieger algebra by* A if $\mathcal{O}_A$([**2**]) is a C*-algebra which is universally generated by partial isometries $s_1,\ldots,s_N$ satisfying:

(3.1) $$s_i^*s_i=\sum_{j=1}^N a_{ij}s_js_j^*\quad(i=1,\ldots,N),\quad \sum_{i=1}^N s_is_i^*=I.$$

Specially, $\mathcal{O}_A$ is the Cuntz algebra $\mathcal{O}_N$ when A is full.

For $g=(z_1,\ldots,z_N)\in T^N(\equiv U(1)^N)$, define $\alpha_g\in\mathrm{Aut}\mathcal{O}_A$ by $\alpha_g(s_i)\equiv z_is_i$ for $i=1,\ldots,N$. We denote the canonical $U(1)$-action(=gauge action) on $\mathcal{O}_A$ by γ. For a multiindex $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}^k$ and canonical generators $s_1,\ldots,s_N$ of $\mathcal{O}_A$, we denote $s_J=s_{j_1}\cdots s_{j_k}$ and $s_J^*=s_{j_k}^*\cdots s_{j_1}^*$. When $J\in\{1,\ldots,N\}^*$, $s_J\neq 0$ if and only if $J\in\{1,\ldots,N\}^*_A$.

In this paper, a representation always means a unital *-representation on a complex Hilbert space. $(\mathcal{H}_1,\pi_1)\sim(\mathcal{H}_2,\pi_2)$ means the unitary equivalence between two representations $(\mathcal{H}_1,\pi_1)$ and $(\mathcal{H}_2,\pi_2)$ of $\mathcal{O}_A$.

3.1. Definition. For $f=\{f_i\}_{i=1}^N\in \mathrm{BFS}_A(X)$, a representation $(l_2(X),\pi_f)$ of $\mathcal{O}_A$ is given by

$$\pi_f(s_i)e_n=\chi_{D(f_i)}(n)\cdot e_{f_i(n)}\quad (i=1,\ldots,N,\, n\in X) \tag{3.2}$$

where $\chi_{D(f_i)}$ is the characteristic function on $D(f_i)$. By the following proposition, we see that (3.2) is a generalization of permutative representation of $\mathcal{O}_N$ by [**1**].

Proposition 3.1. *For a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_A$, the followings are equivalent:*

(i) *There are a complete orthonormal basis $\{e_n\}_{n\in X}$ of $\mathcal{H}$ and a family $\{X_i\}_{i=1}^N$ of subsets of X which satisfy: $^\forall i\in\{1,\ldots,N\}$, $^\forall n\in X_i$, $^\exists m_{i,n}\in X$ s.t.*

$$\pi(s_i)e_n=\chi_{X_i}(n)\cdot e_{m_{i,n}}\quad (n\in X).$$

(ii) *There are a complete orthonormal basis $\{e_n\}_{n\in X}$ of $\mathcal{H}$ and $f=\{f_i\}_{i=1}^N\in \mathrm{BFS}_A(X)$ such that $\pi=\pi_f$ in (3.2) under identification $\mathcal{H}\cong l_2(X)$.*

Proof. (ii)⇒(i) is trivial. Assume (i) for $(\mathcal{H},\pi)$. Then we have a family $f=\{f_i\}_{i=1}^N$ of maps on X such that $\pi(s_i)e_n=e_{f_i(n)}$ by assumption. We can verify axioms in Definition 2.3 for f from conditions $\pi(s_i)^*\pi(s_i)=\sum_{j=1}^N a_{ij}\pi(s_j)\pi(s_j)^*$ and $\sum_{j=1}^N \pi(s_j)\pi(s_j)^*=I$. Hence we obtain (i)⇒(ii). □

Definition 3.2. *$(\mathcal{H},\pi)$ is a permutative representation of $\mathcal{O}_A$ if $(\mathcal{H},\pi)$ satisfies the statement (i) or (ii) in Proposition* 3.1.

In [**7**], we define an A-branching function system f on a measure space (X,μ) and define a representation $(L_2(X,\mu),\pi_f)$ associated with f. Assume that (X,μ) is an atomic measure space, that is, $\mu(\{x\})>0$ for each $x\in X$ so that μ is possibly not normalized at each point. If $f\in \mathrm{BFS}_A(X)$ and X is countably infinite, then there is $f'\in \mathrm{BFS}_A(\mathbf{N})$ such that $(L_2(X,\mu),\pi_f)$ is unitarily equivalent to $(l_2(\mathbf{N}),\pi_{f'})$. Therefore it is sufficient to consider a permutative representation on (direct sum of)$l_2(\mathbf{N})$ for a representation associated with A-branching function system on an atomic measure space.

For a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_A$ and a unitary operator U on a Hilbert space $\mathcal{K}$, we have a new representation $(\mathcal{K}\otimes\mathcal{H},\, U\boxtimes\pi)$ of $\mathcal{O}_A$ which is defined by

$$(U\boxtimes\pi)(s_i)\equiv U\otimes\pi(s_i)\quad (i=1,\ldots,N). \tag{3.3}$$

Let X and Y be sets. For $f \in \mathrm{BFS}_A(X)$ and $g \in \mathrm{BFS}_A(Y)$, if $f \sim g$, then $\pi_f \sim \pi_g$. For any bijection φ on X, $f \in \mathrm{BFS}_A(X)$ and $g \in \mathrm{BFS}_A(Y)$, the followings hold:

$$\pi_{\varphi \boxtimes g} \sim S(\varphi) \boxtimes \pi_g, \quad \pi_{f \oplus g} \sim \pi_f \oplus \pi_g \tag{3.4}$$

where $S(\varphi)$ is a unitary operator on $l_2(X)$ defined by $S(\varphi)e_n \equiv e_{\varphi(n)}$ for $n \in X$.

Definition 3.3. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_A$.*

(i) *$(\mathcal{H}, \pi)$ is $P(J; z)$ for $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k_{A,c}$, $k \geq 1$ and $z \in U(1)$ if there is a cyclic unit vector $\Omega \in \mathcal{H}$ such that $\pi(s_J)\Omega = z\Omega$ and $\{\pi(s_{j_l} \cdots s_{j_k})\Omega : l = 1, \ldots, k\}$ is an orthonormal family. $\{\pi(s_{j_l} \cdots s_{j_k})\Omega : l = 1, \ldots, k\}$ is called a cycle of π by J. Specially, we denote $P(J) \equiv P(J; 1)$.*

(ii) *$(\mathcal{H}, \pi)$ is $P(J)$ for $J = (j_n)_{n \in \mathbf{N}} \in \{1, \ldots, N\}^\infty_A$ if there is a cyclic unit vector $\Omega \in \mathcal{H}$ such that $\{\pi(s^*_{j_n} \cdots s^*_{j_1})\Omega\}_{n \in \mathbf{N}}$ is an orthonormal family. $\{\pi(s^*_{j_n} \cdots s^*_{j_1})\Omega\}_{n \in \mathbf{N}}$ is called a chain of π by J.*

Ω in (i) and (ii) is called the GP vector of $(\mathcal{H}, \pi)$.

We denote $(l_2(X), \pi_f)$ by π_f simply.

Theorem 3.4. *Let $f \in \mathrm{BFS}_A(X)$.*

(i) *If σ_r is the shift on $\mathbf{Z}$ for $r \in \mathbf{Z}$ which is defined by $\sigma_r(n) \equiv n - r$ for $n \in \mathbf{Z}$, then the following holds:*

$$\pi_{\sigma_r \boxtimes f} \sim \begin{cases} \displaystyle\int^{\oplus}_{U(1)} \pi_f \circ \gamma_{w^r} \, d\eta(w) & (r \neq 0), \\ (\pi_f)^{\oplus\infty} & (r = 0). \end{cases}$$

(ii) *If σ is the shift of $\mathbf{Z}_p$ for $p \geq 1$, then*

$$\pi_{\sigma \boxtimes f} \sim \textstyle\bigoplus_{j=1}^p \pi_f \circ \gamma_{\xi^j}$$

where $\xi \equiv e^{2\pi\sqrt{-1}/p}$.

(iii) *If f is cyclic, then $(l_2(X), \pi_f)$ is cyclic.*

(iv) *If f contains a $P(J)$-component for $J \in \{1, \ldots, N\}^\#_{A,c}$, then $(l_2(X), \pi_f)$ contains a $P(J)$-component, too.*

Proof. About (i) and (ii), see Proposition 3.6 in [**7**]. About (iii) and (iv), see Theorem 3.7 in [**7**]. □

In this way, characterizations of permutative representations are given by terminology of branching function systems.

Lemma 3.5. (i) *For $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k_{A,c}$, $g = (z_i)_{i=1}^N \in T^N$ and $w \in U(1)$, $P(J; w) \circ \alpha_g = P(J; wz_J)$ where $z_J \equiv z_{j_1} \cdots z_{j_k}$. Specially, for $z, w \in U(1)$, $P(J; w) \circ \gamma_z = P(J; wz^k)$ and $P(J) \circ \gamma_z = P(J; z^k)$.*

(ii) *For each $J \in \{1, \ldots, N\}^\infty_A$ and $g \in T^N$, $P(J) \circ \alpha_g = P(J)$.*

Proof. (i) Let $(\mathcal{H}, \pi)$ be $P(J; w)$ with GP vector Ω. Because $\alpha_g(s_J) = z_J s_J$, $(\pi \circ \alpha_g)(s_J)\Omega = z_J w\Omega$. Because $(\mathcal{H}, \pi \circ \alpha_g)$ is cyclic, too, the statement holds.
(ii) Let $(\mathcal{H}, \pi)$ be $P(J)$ with GP vector Ω, $g = (z_i)_{i=1}^N$, $J = (j_n)_{n \in \mathbf{N}}$ and $z_{J_n} \equiv z_{j_1} \cdots z_{j_n}$ for $n \geq 1$. Then $\{\bar{z}_{J_n} \pi(s^*_{J_n})\Omega\}_{n \in \mathbf{N}}$ is a chain of π by J and Ω is a cyclic vector. Hence the statement holds. □

Proposition 3.6. *Let $A \in M_N(\{0, 1\})$.*

(i) *For an infinite set Λ, $f \in \mathrm{BFS}_A(\Lambda)$ and $J \in \{1, \ldots, N\}^\#_{A,c}$, if f is $P(J)$, then $(l_2(\Lambda), \pi_f)$ is $P(J)$, too.*

(ii) *For each $J \in \{1, \ldots, N\}^\#_{A,c}$, there exists a representation $(\mathcal{H}, \pi)$ which is $P(J)$.*

(iii) *For each $J \in \{1, \ldots, N\}^*_{A,c}$ and $z \in U(1)$, there exists a representation $(\mathcal{H}, \pi)$ which is $P(J; z)$.*

Proof. (i) This holds from definition of branching function system immediately.
(ii) By (i), Lemma 2.8 and Lemma 2.9, the statement holds.
(iii) By (i) and Lemma 3.5, the statement holds. □

Proposition 3.7. *For any permutative representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_A$, there is a family $\{(\mathcal{H}_\lambda, \pi_\lambda)\}_{\lambda \in \Lambda}$ of cyclic permutative representations of $\mathcal{O}_A$ such that $(\mathcal{H}, \pi) = \bigoplus_{\lambda \in \Lambda} (\mathcal{H}_\lambda, \pi_\lambda)$. Furthermore $(\mathcal{H}_\lambda, \pi_\lambda)$ is $P(J_\lambda)$ for some $J_\lambda \in \{1, \ldots, N\}^\#_{A,c}$ for each $\lambda \in \Lambda$.*

Proof. By Theorem 2.7, Proposition 3.1, (3.4) and Proposition 3.6, it holds. □

Lemma 3.8. *For $A \in M_N(\{0, 1\})$ and $J \in \{1, \ldots, N\}^\#_{A,c}$, let $(\mathcal{H}, \pi)$ be $P(J)$ with the GP vector Ω. Then the followings hold:*

(i) *When $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k_{A,c}$, $k \geq 1$,*

$$\pi(s_{J'})^*\Omega = \delta_{J', J[1, ak+p]} \pi(s_{J[p+1,k]})\Omega \quad (J^{'} \in \{1, \ldots, N\}^{ak+p}_A).$$

where $J[m, \ldots, n] \equiv (j_m, \ldots, j_n)$ for $1 \leq m \leq n \leq k$ and

$$J[m, \ldots, ak+p] \equiv (j_m, \ldots, j_k) \cup J^{a-1} \cup (j_1, \ldots, j_p)$$

for $a \geq 1$, $1 \leq m, p \leq k-1$.

(ii) *When* $J = (j_n)_{n \in \mathbf{N}} \in \{1, \ldots, N\}_A^\infty$,

$$\pi(s_{J'})^* \Omega = \delta_{J', J[1,n]} e_{n+1} \quad (J' \in \{1, \ldots, N\}_A^n)$$

where $J[1,n] \equiv (j_1, \ldots, j_n)$ *and* $e_n \equiv \pi(s^*_{J[1,n]})\Omega$ *for* $n \in \mathbf{N}$.

Proof. (i) Recall $\pi(s_J)\Omega = \Omega$. When $J' \in \{1, \ldots, N\}_A^{ak+p}$, $\pi(s^*_{J'})\Omega = \pi(s^*_{J'} s_{J^{a+1}})\Omega = \delta_{J', J[1,ak+p]} \pi(s_{J[p+1,k]})\Omega$.
(ii) Because $\Omega = \pi(s_{J[1,n]}) e_{n+1}$ for each $n \in \mathbf{N}$, the statement holds. □

3.2. Canonical basis of permutative representation. For a given permutative representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_A$ which is $P(J)$ for $J \in \{1, \ldots, N\}_{A,c}^{\#}$, we construct a complete orthonormal basis of $\mathcal{H}$ in a canonical way.

For $J \in \{1, \ldots, N\}_{A,c}^*$, let $(\mathcal{H}, \pi)$ be $P(J)$ with the GP vector Ω. Put

$$e_x \equiv \pi(s_x)\Omega \quad (x \in \Lambda(A, J))$$

where $\Lambda(A, J)$ is in (2.3).

Lemma 3.9. *For* $J \in \{1, \ldots, N\}_{A,c}^*$, $\{e_x : x \in \Lambda(A, J)\}$ *is a complete orthonormal basis of* $\mathcal{H}$.

Proof. Recall notation $J_l = (j_l, \ldots, j_k)$ for $l = 1, \ldots, k$ and note that $J_1 = J$, $e_{J_1} = \Omega$ and $\pi(s_{J_1}) e_{J_1} = e_{J_1}$. We simply denote $\pi(s_i)$ by s_i. For J' and $J'' \in \Lambda(A, J)$ such that $J' = J_1' \cup J_2'$, $|J_1'| = |J''|$, $< e_{J'} | e_{J''} >= \delta_{J_1', J''} < e_{J_2'} | \Omega >$. Therefore it is sufficient to show that $< \Omega | e_x >= 0$ for each $x \in \Lambda(A, J) \setminus \{J\}$.

(i)When $x \in \Lambda_1(A, J)$, by definition of $P(J)$, $\{e_{J_l}\}_{J_l \in \Lambda_1(A,J)}$ is orthonormal. (ii)When $x \in \Lambda_2(A, J)$, $x = (j, J_l)$ for some $l \in \{1, \ldots, k\}$ and $j \in \mathcal{T}^{(1)}(A; j_l) \setminus \{j_{l-1}\}$ where $\mathcal{T}^{(1)}(A; j_l)$ is in (2.2). Hence $< e_x | \Omega >=< s_j s_{J_l} \Omega | s_{J_1} \Omega >= \delta_{j, j_1} < s_{J_l} \Omega | s_{J_2} \Omega >= \delta_{j, j_1} \delta_{l,2}$. If $j = j_1$, then, $l \neq 2$ by the choice of x. Hence $< e_x | \Omega >= 0$ for each $x \in \Lambda_2(A, J)$. In the same way, we see that $< e_x | s_{J_l} \Omega >= 0$ for each $x \in \Lambda_2(A, J)$ and $l = 1, \ldots, k$. (iii)When $x \in \Lambda_3(A, J)$, there are $J' \in \{1, \ldots, N\}_A^*$ and $y \in \Lambda_2(A, J)$ such that $x = J' \cup y$, $|J'| = mk + l - 1$. Then $< e_x | \Omega >=< s_{J'} e_y | \Omega >= \delta_{J', J_1^m \cup (j_1, \ldots, j_{l-1})} < e_y | s_{J_l} \Omega >= 0$ by (ii). By (i),(ii),(iii), $\{e_x : x \in \Lambda(A, J)\}$ is an orthonormal family in $\mathcal{H}$.

By cyclicity of Ω, $X \equiv \{s_J s^*_{J'} \Omega : J, J' \in \{1, \ldots, N\}^*\}$ spans $\mathcal{H}$. On the other hand, $X = \{e_x : x \in \Lambda(A, J)\}$. Hence $\{e_x : x \in \Lambda(A, J)\}$ is complete. □

For $J = (j_n)_{n \in \mathbf{N}} \in \{1, \ldots, N\}_A^\infty$, let $(\mathcal{H}, \pi)$ be $P(J)$ with the GP vector Ω. Let

$$e_n \equiv \pi(s_{J[1,n]})^* \Omega \quad (n \in \mathbf{N})$$

where $J[1,n] \equiv (j_1,\ldots,j_n)$. Then $\{e_n\}_{n\in\mathbf{N}}$ is an orthonormal family by definition. Put

$$e_{J',n} \equiv \pi(s_{J'})e_n \quad (J' \in \mathcal{T}(A;j_n)\setminus\{j_{n-1}\}).$$

Define

$$\widetilde{\Lambda}(A,J) \equiv \mathcal{T}(A;j_1) \sqcup \coprod_{n\geq 1} \mathcal{T}_n^{out}(A;J), \quad \mathcal{T}_n^{out}(A;J) \equiv \mathcal{T}(A;j_n)\setminus\{j_{n-1}\}$$

where $\mathcal{T}(A;j_n)$ is in (2.2).

Lemma 3.10. *$\{e_x : x \in \widetilde{\Lambda}(A,J)\}$ is a complete orthonormal basis of $\mathcal{H}$.*

Proof. For $x,y \in \widetilde{\Lambda}(A,J)$, assume that $x=(J',n)$ and $y=(J'',m)$. If $|J'|=|J''|$, then $<e_x|e_y>=\delta_{J',J''}\delta_{n,m}$. If $|J'|>|J''|$, then there are $l\in\mathbf{N}$ and J''' such that $<e_x|e_y>=\delta_{J',J''\cup J'''}<e_n|e_l>$. If $J'=J''\cup J'''$, then $n\neq l$ if and only if $x\neq y$. Hence $\{e_x\}_{x\in\widetilde{\Lambda}(A,J)}$ is an orthonormal family. By cyclicity of Ω, $X\equiv\{\pi(s_{J_1}s_{J_2}^*)\Omega : J_1,J_2\in\{1,\ldots,N\}_A^*\}$ spans $\mathcal{H}$. On the other hand, $X=\{e_x : x\in\widetilde{\Lambda}(A,J)\}$. Hence the statement holds. $\square$

4. Uniqueness, irreducibility and equivalence

Let $A\in M_N(\{0,1\})$.

4.1. Uniqueness up to unitary equivalences.

Lemma 4.1. *For $J\in\{1,\ldots,N\}_{A,c}^{\#}$, if both $(\mathcal{H},\pi)$ and $(\mathcal{H}',\pi')$ are $P(J)$, then $(\mathcal{H},\pi)\sim(\mathcal{H}',\pi')$.*

Proof. By Lemma 3.9 and Lemma 3.10, there is the canonical basis of $P(J)$ and the action of $\mathcal{O}_A$ on them is always same. Therefore the correspondence among canonical basis of $(\mathcal{H},\pi)$ and that of $(\mathcal{H}',\pi')$ gives a unitary U from $\mathcal{H}$ to $\mathcal{H}'$ such that $\mathrm{Ad}U\circ\pi=\pi'$. $\square$

Theorem 4.2. (i) *For $J\in\{1,\ldots,N\}_{A,c}^{\#}$, $P(J)$ exists uniquely up to unitary equivalences.*

(ii) *For $J\in\{1,\ldots,N\}_{A,c}^{*}$ and $z\in U(1)$, $P(J;z)$ exists uniquely up to unitary equivalences.*

Proof. (i) By Proposition 3.6 and Lemma 4.1, the statement holds. (ii) By Proposition 3.6, the existence follows. Assume that $J\in\{1,\ldots,N\}_A^k$. If both $(\mathcal{H},\pi)$ and $(\mathcal{H}',\pi')$ are $P(J;z)$, then both $(\mathcal{H},\pi\circ\gamma_{\bar{z}^{1/k}})$ and $(\mathcal{H}',\pi'\circ\gamma_{\bar{z}^{1/k}})$ are $P(J)$ by Lemma 3.5. By Lemma 4.1, $(\mathcal{H},\pi\circ\gamma_{\bar{z}^{1/k}})\sim(\mathcal{H}',\pi'\circ\gamma_{\bar{z}^{1/k}})$. Therefore the statement holds. $\square$

By Theorem 4.2, symbols $P(J)$ and $P(J;z)$ make sense as equivalence classes of representation.

4.2. Sufficient condition of irreducibility.

Lemma 4.3. *Let $(\mathcal{H},\pi)$ be $P(J)$ with the GP vector Ω for $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}^k_{A,c}$, $k\geq 1$ and $\Omega_l\equiv\pi(s_{j_l}\cdots s_{j_k})\Omega$ for $l=1,\ldots,k$. Assume that J is non periodic. Then the followings hold:*

(i) *If $J^{'}\in\{1,\ldots,N\}^*_A$ is not in $\{J^n : n\geq 1\}$, then there is $n_0\in\mathbf{N}$ such that $\pi(s_J^*)^n\pi(s_{J^{'}})\Omega=0$ for some $n\geq n_0$.*

(ii) *If $v\in\mathcal{H}$ satisfies $<v|\Omega>=0$, then $\lim_{n\to\infty}\pi(s_J^*)^n v=0$.*

Proof. We simply denote $\pi(s_i)$ by s_i for $i=1,\ldots,N$.
(i) If $J^{'}\in\{1,\ldots,N\}^l_A$ for $1\leq l<k$, then the non-periodicity of J implies $s_J^*\pi(s_{J^{'}})\Omega=\delta_{(j_1,\ldots,j_l),J^{'}}\cdot s^*_{j_{l+1},\ldots,j_k}\Omega=0$. If $J^{'}=J_1^{'}\cup J_2^{'}$ and $|J_1^{'}|=nk$ and $|J_2^{'}|=l$ for $l=1,\ldots,k-1$, then $(s_J^*)^{n+1}s_{J^{'}}\Omega=\delta_{J^n,J_1^{'}}s_J^*s_{J_2^{'}}\Omega=0$ by the last case.
(ii) By Lemma 3.9, there is a family $\{J_m^{'}\in\{1,\ldots,N\}^*_A : m\in\mathbf{N}\}$ such that $\{s_{J_m^{'}}\Omega\}_{m\in\mathbf{N}}$ is a complete orthonormal basis of $\mathcal{H}$ and $J_1^{'}=J$. When $<v|\Omega>=0$, we can denote $v=\sum_{n=2}^{\infty}a_n s_{J_n^{'}}\Omega$. If $m\geq 2$, then $J_m^{'}\not\in\{J^n : n\geq 1\}$. Therefore there is $n_0\in\mathbf{N}$ such that $(s_J^*)^n s_{J_m^{'}}\Omega=0$ for $n\geq n_0$ by (i). Hence $\|(s_J^*)^n v\|$ is monotone decreasing and the statement holds. $\square$

Lemma 4.4. *Let $(\mathcal{H},\pi)$ be $P(J)$ for $J\in\{1,\ldots,N\}^*_{A,c}$ and $\Omega,\Omega^{'}$ be vectors of $\mathcal{H}$ such that $\pi(s_J)\Omega=\Omega$ and $\pi(s_J)\Omega^{'}=\Omega^{'}$. If J is non periodic, then $\Omega^{'}=c\Omega$ for some $c\in\mathbf{C}$.*

Proof. By assumption and Lemma 3.9, there is a set $\Lambda\subset\{1,\ldots,N\}^*_A$ such that $\Omega^{'}$ is written as $c\Omega+\sum_{J^{''}\in\Lambda}a_{J^{''}}\pi(s_{J^{''}})\Omega$ where $<\pi(s_{J^{''}})\Omega|\Omega>=0$ for each $J^{''}\in\Lambda$, and $\pi(s_J)^*\Omega^{'}=\Omega^{'}$. By Lemma 4.3, $\Omega^{'}=c\Omega$. $\square$

Theorem 4.5. *If $J\in\{1,\ldots,N\}^*_{A,c}$ is non periodic, then $P(J;z)$ is irreducible for any $z\in U(1)$. Specially, if J is non periodic, then $P(J)$ is irreducible.*

Proof. Assume that J is non periodic and $(\mathcal{H},\pi)$ is $P(J)$ with the GP vector Ω. For $v\in\mathcal{H}$, $v\neq 0$, there is $J^{'}\in\{1,\ldots,N\}^*_A$ such that $<\pi(s^*_{J^{'}})v|\Omega>\neq 0$. Therefore we can always replace v and $\pi(s^*_{J^{'}})v$. Assume that $v=\Omega+y$ such that $<y|\Omega>=0$. Then $\lim_{n\to\infty}\pi((s_J^*)^n)y=0$ by Lemma 4.3. Hence $\lim_{n\to\infty}\pi((s_J^*)^n)v=\Omega$ and $\Omega\in\pi(\mathcal{O}_A)v$. Because Ω is a cyclic vector, $\pi(\mathcal{O}_A)v=\mathcal{H}$. Therefore $(\mathcal{H},\pi)$ is irreducible. By this and Lemma 3.5, $P(J;z)$ is, too for each $z\in U(1)$. $\square$

Lemma 4.6. *Let $(\mathcal{H},\pi)$ be $P(J)$ with the GP vector Ω for $J=(j_n)_{n\in\mathbf{N}}\in\{1,\ldots,N\}_A^\infty$. Assume that J is non eventually periodic and put $J_n\equiv(j_1,\ldots,j_n)$ for $n\in\mathbf{N}$. Then the followings hold:*

(i) *For each $m,k\geq 1$ and $J^{'}\in\{1,\ldots,N\}_A^k$ so that $J^{'}\neq J_k$, there is $n_0\in\mathbf{N}$ such that $\pi(s_{J_n}s_{J_n}^*s_{J'}s_{J_m}^*)\Omega=0$ for each $n\geq n_0$.*

(ii) *For each $k\geq 1$ and $J^{'}\in\{1,\ldots,N\}_A^k$ so that $J^{'}\neq J_k$, there is $n_0\in\mathbf{N}$ such that $\pi(s_{J_n}s_{J_n}^*s_{J'})\Omega=0$ for each $n\geq n_0$.*

(iii) *For each $m\geq 1$, there is $n_0\in\mathbf{N}$ such that $\pi(s_{J_n}s_{J_n}^*s_{J_m}^*)\Omega=0$ for each $n\geq n_0$.*

(iv) *If $y\in\mathcal{H}$ satisfies $<y|\Omega>=0$, then $\lim_{n\to\infty}\pi(s_{J_n}s_{J_n}^*)y=0$.*

Proof. We simply denote $\pi(s_i)$ by s_i.
(i) and (ii) follow by assumption for $J^{'}$.
(iii) Since J is non eventually periodic, there is $n_0\in\mathbf{N}$ such that $s_{J_n}^*s_{J_m}^*s_{J_{n+m}}=0$ for each $n\geq n_0$. Therefore $s_{J_n}^*s_{J_m}^*\Omega=s_{J_n}^*s_{J_m}^*s_{J_{n+m}}s_{J_{n+m}}^*\Omega=0$ for each $n\geq n_0$. Hence the statement holds.
(iv) Denote $e_1\equiv\Omega$ and $e_m\equiv s_{J_{m-1}}^*\Omega$ for $m\geq 2$. By Lemma 3.10, there is a family $\{K_{n,m}\}_{n,m\in\mathbf{N}}\subset\{1,\ldots,N\}_A^*$ such that $\{s_{K_{n,m}}e_m\}_{n,m\in\mathbf{N}}$ is a complete orthonormal basis of $\mathcal{H}$. If $y\in\mathcal{H}$ satisfies $<y|\Omega>=0$, then we can denote $y=\sum_{m\geq 2}\sum_{n\geq 1}a_{n,m}s_{K_{n,m}}e_m$. Therefore $\|s_{J_n}s_{J_n}^*y\|$ is monotone decreasing by (i),(ii),(iii) and the statement holds. □

Theorem 4.7. *If $J\in\{1,\ldots,N\}_A^\infty$ is non eventually periodic, then $P(J)$ is irreducible.*

Proof. Let $(\mathcal{H},\pi)$ be $P(J)$ with the GP vector Ω for $J=(j_n)_{n\in\mathbf{N}}\in\{1,\ldots,N\}_A^\infty$. Denote $J_n\equiv(j_1,\ldots,j_n)$. Put $v\in\mathcal{H}$, $v\neq 0$. We can assume that $<v|\Omega>\neq 0$ by replacing v by $\pi(s_{J'}s_{J''}^*)v$ for suitable $J^{'},J^{''}\in\{1,\ldots,N\}_A^*$ if it is necessary. We can assume that $v=\Omega+y$ for some $y\in\mathcal{H}$ such that $<y|\Omega>=0$. By Lemma 4.6, $\Omega=\lim_{n\to\infty}\pi(s_{J_n}s_{J_n}^*)y\in\pi(\mathcal{O}_A)y$. Because Ω is a cyclic vector, $\pi(\mathcal{O}_A)y=\mathcal{H}$ and the statement holds. □

The necessary condition of irreducibility of permutative representation is given in § 5.

4.3. Equivalence. Recall the equivalence among multiindices in § 2.1.

Lemma 4.8. *Let $(\mathcal{H},\pi)$ be $P(J)$ with the GP vector Ω for $J\in\{1,\ldots,N\}_{A,c}^*$. Assume that J is non periodic and choose $J^{'}\in\{1,\ldots,N\}^*$ such that $J^{'}\not\sim J$.*

(i) *If $\Omega^{'}\in\mathcal{H}$ such that $\pi(s_{J'})\Omega^{'}=\Omega^{'}$, then $<\Omega|\Omega^{'}>=0$.*

(ii) *If $v\in\mathcal{H}$, then $\lim_{n\to\infty}\pi(s_{J'}^*)^nv=0$.*

Proof. (i) Assume that $|J| = k$ and $|J^{'}| = l$. Because of the non-periodicity of J, $J^l \neq (J^{'})^k$. Hence $< \Omega|\Omega^{'} >=< \pi(s_J^k)\Omega|\pi(s_{J^{'}}^l)\Omega^{'} >= \delta_{J^l,(J^{'})^k} < \Omega|\Omega^{'} >= 0$.
(ii) If $v = \pi(s_{J^{''}})\Omega$, then $\pi((s_{J^{'}}^*)^{n_0+n})v = \delta_{(J^{'})^m,J^{''}} \cdot \pi((s_{J^{'}}^*)^n)\Omega \to 0$ when $n \to \infty$ by Lemma 4.3 and (i). Because any $v \in \mathcal{H}$ is a limit of linear combination of $\{\pi(s_{J^{''}})\Omega : J^{''} \in \{1,\dots,N\}_A^*\}$, the statement holds. □

Lemma 4.9. *Let $J, J^{'} \in \{1,\dots,N\}_{A,c}^*$.*

(i) *If $J, J^{'}$ are non periodic and $J \not\sim J^{'}$, then $P(J) \not\sim P(J^{'})$.*
(ii) *For $z, z^{'} \in U(1)$, if $(J,z) \sim (J^{'},z^{'})$, then $P(J;z) \sim P(J^{'};z^{'})$. Specially, if $J \sim J^{'}$, then $P(J) \sim P(J^{'})$.*

Proof. (i) Assume that $P(J) \sim P(J^{'})$. Then there is a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_A$ which is $P(J)$ and $P(J^{'})$. Assume that $\Omega,\Omega^{'} \in \mathcal{H}$ are GP vectors with respect to $P(J)$ and $P(J^{'})$, respectively. Then $< \Omega^{'}|v >=< \pi(s_{J^{'}}^n)\Omega^{'}|v >=< \Omega^{'}|\pi((s_{J^{'}}^*)^n)v >\to 0$ when $n \to \infty$ by Lemma 4.8. Hence $\Omega^{'} = 0$. Therefore this is contradiction. Hence the statement holds.
(ii) Assume that $J \sim J^{'}$ and $J = (j_1,\dots,j_k) \in \{1,\dots,N\}_{A,c}^k$ for $k \geq 1$. Let $(\mathcal{H},\pi)$ be $P(J)$ with the GP vector Ω. Then $\pi(s_J)\Omega = \Omega$. By assumption there is $\sigma \in \mathbf{Z}_k$ such that $J^{'} = \sigma(J) = (j_{\sigma(1)},\dots,j_{\sigma(k)})$. Put $\Omega^{'} \equiv \pi(s_{j_{\sigma(1)}}\cdots s_k)\Omega$. Then $\pi(s_{J^{'}})\Omega^{'} = \Omega^{'}$ and $\Omega^{'}$ is a cyclic vector of $(\mathcal{H},\pi)$. By Lemma 4.1, $P(J) \sim (\mathcal{H},\pi) \sim P(J^{'})$.

If $(J,z) \sim (J^{'},z^{'})$, then $J \sim J^{'}$ and $z = z^{'}$ by definition. Then $P(J) \sim P(J^{'})$ by the last result. By Lemma 3.5, the statement holds. □

Theorem 4.10. *For $J, J^{'} \in \{1,\dots,N\}_{A,c}^*$, assume that $J, J^{'}$ are non periodic. Then the followings hold:*

(i) *$P(J) \sim P(J^{'})$ if and only if $J \sim J^{'}$.*
(ii) *For $z, z^{'} \in U(1)$, $P(J;z) \sim P(J^{'};z^{'})$ if and only if $(J,z) \sim (J^{'},z^{'})$.*

Proof. (i) By Lemma 4.9, the statement holds.
(ii) If $P(J)$ and $P(J^{'};z)$ are equivalent, then $J \sim J^{'}$ by the proof of Lemma 4.9. If $P(J)$ and $P(J;z)$ are equivalent, then there is a representation $(\mathcal{H},\pi)$ which is $P(J)$ and $P(J;z)$ with GP vectors Ω and $\Omega^{'}$, respectively. Then $\pi(s_J)\Omega = \Omega$ and $\pi(s_J)\Omega^{'} = z\Omega^{'}$. Hence $< \Omega|\Omega^{'} >=< \pi(s_J)\Omega|\Omega^{'} >=< \Omega|\pi(s_J)^*\Omega^{'} >= \bar{z} < \Omega|\Omega^{'} >$. Therefore $\bar{z} = 1$ or $< \Omega|\Omega^{'} >= 0$. If $\bar{z} \neq 1$, then $< \Omega|\Omega^{'} >= 0$ implies that $\pi((s_J^*)^n)\Omega^{'} \to 0$ when $n \to \infty$. This is contradiction. Therefore $P(J) \not\sim P(J;z)$ when $z \neq 1$. On the other hand, $P(J;z) \sim P(J^{'};z^{'})$ if and only if $P(J) \sim P(J^{'};z^{'}) \circ \gamma_{\bar{z}^{1/k}} = P(J^{'};z^{'}\bar{z}^{l/k})$ when $|J| = k$ and $|J^{'}| = l$. Therefore $P(J;z) \sim P(J^{'};z^{'})$ if and only if

$J \sim J^{'}$ and $z^{'}\bar{z} = 1$. This implies the statement. $\square$

In § 5.1, we show the statement in Theorem 4.10 without assumption non periodicity.

Lemma 4.11. *For $J, J^{'} \in \{1, \ldots, N\}_A^{\infty}$, $P(J) \not\sim P(J^{'})$ if $J \not\sim J^{'}$.*

Proof. Assume that $J = (j_n)_{n\in\mathbf{N}}$, $J^{'} = (j_n^{'})_{n\in\mathbf{N}}$, $J \not\sim J^{'}$ and $P(J) \sim P(J^{'})$. Then there is a representation $(\mathcal{H}, \pi)$ which is $P(J)$ and $P(J^{'})$ with GP vectors Ω and $\Omega^{'}$, respectively. Put $e_n \equiv \pi(s_{j_n}^* \cdots s_{j_1}^*)\Omega$ and $e_n^{'} \equiv \pi(s_{j_n^{'}}^* \cdots s_{j_1^{'}}^*)\Omega^{'}$ where $J_n \equiv (j_1, \ldots, j_n)$ and $J_n^{'} \equiv (j_1^{'}, \ldots, j_n^{'})$ for $n \geq 1$. Then $< \Omega|\Omega^{'} >=< \pi(s_{J_n})e_{n+1}|\pi(s_{J_n^{'}})e_{n+1}^{'} >= 0$ for some $n \in \mathbf{N}$ because $J \not\sim J^{'}$. In the same way, we see that $< e_m|e_n^{'} >= 0$ for each $n, m \in \mathbf{N}$. From this, $< \pi(s_K)e_m|e_n^{'} >= 0$ for each $K \in \{1, \ldots, N\}_A^*$ and $n, m \in \mathbf{N}$. Therefore $< v|e_n^{'} >= 0$ for each $v \in \mathcal{H}$ and $n \in \mathbf{N}$. Hence $e_n^{'} = 0$ for each $n \in \mathbf{N}$. This contradicts with the choice of $\{e_n^{'}\}_{n\in\mathbf{N}}$. Therefore $P(J) \not\sim P(J^{'})$. $\square$

Theorem 4.12. *For $J, J^{'} \in \{1, \ldots, N\}_A^{\infty}$, $P(J) \sim P(J^{'})$ if and only if $J \sim J^{'}$.*

Proof. By Lemma 4.11, it is sufficient to show that $J \sim J^{'}$ implies $P(J) \sim P(J^{'})$. Assume that $J = (j_n)_{n\in\mathbf{N}}$ and $J^{'} = (j_n^{'})_{n\in\mathbf{N}}$ and $J \sim J^{'}$. Then there are $p \in \mathbf{Z}$ and $M \geq 1$ such that $j_n^{'} = j_{n+p}$ for each $n \geq M$. If $(\mathcal{H}, \pi)$ is $P(J)$ with the GP vector Ω, then put $J_0^{'} \equiv (j_1^{'}, \ldots, j_M^{'})$, $J_0 \equiv (j_1, \ldots, j_{M+p})$ and $\Omega^{'} \equiv \pi(s_{J_0^{'}} s_{J_0}^*)\Omega$. Then

$$\pi(s_{j_{M+n}^{'}}^* \cdots s_{j_{M+1}^{'}}^* s_{j_M^{'}}^* \cdots s_{j_1^{'}}^*)\Omega^{'} = \pi(s_{j_{M+n+p}}^* \cdots s_{j_{M+1+p}}^* s_{j_{M+p}}^* \cdots s_{j_1}^*)\Omega$$

for each $n \geq 1$. Therefore $\{\pi(s_{j_l^{'}}^* \cdots s_{j_1^{'}}^*)\Omega^{'}\}_{l\geq 1}$ is a chain of π by $J^{'}$. Because Ω is a cyclic vector, $\Omega^{'}$ is, too. Hence $P(J) \sim (\mathcal{H}, \pi) \sim P(J^{'})$. $\square$

5. Decomposition and complete reducibility

5.1. Decomposition of cycle.

Theorem 5.1. *For $(J, c) \in \{1, \ldots, N\}_{A,c}^* \times U(1)$ and $p \geq 1$,*

$$P(J^p; c) \sim \bigoplus_{j=1}^{p} P(J; c^{1/p}\xi^j) \tag{5.1}$$

where $\xi \equiv e^{2\pi\sqrt{-1}/p}$. *Specially,*

$$P(J^p) \sim \bigoplus_{j=1}^{p} P(J;\xi^j). \tag{5.2}$$

Proof. Assume that $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k_{A,c}$. Let $(\mathcal{H}_j, \pi_j)$ be $P(J;\xi^j)$ with the GP vector Ω_j for $j = 1, \ldots, p$. Put $\Omega \equiv p^{-1/2} \sum_{j=1}^p \Omega_j \in \mathcal{H} \equiv \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_p$ and $\pi \equiv \pi_1 \oplus \cdots \oplus \pi_p$. Then $\pi(s_{J^p})\Omega = \Omega$ and $\{\pi(s_{j_l} \cdots s_{j_k} s_{J^a})\Omega : a = 0, \ldots, p-1,\, l = 1, \ldots, k\}$ is an orthonormal family. Therefore $V \equiv \pi(\mathcal{O}_A)\Omega$ is a $P(J^p)$-component of $\mathcal{H}$. On the other hand, $\mathrm{Lin} < \{\pi(s_{J^q})\Omega : q = 1, \ldots, p\} >= \mathrm{Lin} < \Omega_q : q = 1, \ldots, p\} >$. Hence $\Omega_i \in V$ and $\mathcal{H}_i \subset V$ for $i = 1, \ldots, p$. Therefore $\mathcal{H} = V$ and $(\mathcal{H}, \pi)$ is $P(J^p)$. From this, we obtain (5.2). By Lemma 3.5 and (5.2), $P(J^p; c) = P(J^p) \circ \gamma_{c^{1/kp}} \sim \left(\bigoplus_{j=1}^p P(J;\xi^j)\right) \circ \gamma_{c^{1/kp}} \sim \bigoplus_{j=1}^p P(J; c^{1/p}\xi^j)$. (5.1) is obtained. $\square$

Corollary 5.2. (i) *For* $(J, z) \in \{1, \ldots, N\}^*_{A,c} \times U(1)$, $P(J;z)$ *is irreducible if and only if* J *is non periodic. Specially, for* $J \in \{1, \ldots, N\}^*_{A,c}$, $P(J)$ *is irreducible if and only if* J *is non periodic.*

(ii) *For* $J, J' \in \{1, \ldots, N\}^*_{A,c}$ *and* $z, z' \in U(1)$, $P(J;z) \sim P(J';z')$ *if and only if* $(J, z) \sim (J', z')$.

(iii) *The decomposition in* (5.1) *is multiplicity free.*

Proof. (i) By Theorem 4.5 and Theorem 5.2, the statement holds.
(ii) If $(J, z) \sim (J', z')$, then $P(J;z) \sim P(J';z')$ by Lemma 4.9.

Assume that $P(J;z) \sim P(J';z')$. If J and J' are non periodic, then the statement is shown in Theorem 4.10. If J is periodic, then $P(J;z)$ is not irreducible by (i) and decomposed into direct sum of finite irreducible components by Theorem 5.2. Therefore $P(J';z')$ must not irreducible. By (i), J' is periodic. By comparing irreducible components of $P(J;z)$ and those of $P(J';z')$ and Theorem 4.10, we see that their sets of irreducible components coincide up to unitary equivalences. From this, $(J, z) \sim (J', z')$.
(iii) By (ii), the assertion holds. $\square$

By Theorem 4.12 and Corollary 5.2, we have the following:

Theorem 5.3. *For* $J, J' \in \{1, \ldots, N\}^{\#}_{A,c}$, $P(J) \sim P(J')$ *if and only if* $J \sim J'$.

5.2. Decomposition of chain. In this subsection, an equality among representations means their unitary equivalence. Recall $U \boxtimes \pi$ in (3.3).

Lemma 5.4. *Let $\varphi \in L_\infty(U(1))$ such that $|\varphi(w)| = 1$ almost everywhere in $U(1)$, M_φ be the multiplication operator on $L_2(U(1))$ by φ, and $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_A$. Then we have the followings:*

(i)
$$M_\varphi \boxtimes \pi = \int_{U(1)}^{\oplus} \pi \circ \gamma_{\varphi(w)}\, d\eta(w)$$
where η is the Haar measure of $U(1)$. Specially, when $\varphi(w) \equiv w^{1/p}$ for $w \in U(1)$ where $w^{1/p} = e^{2\pi\sqrt{-1}\theta/p}$ for $w = e^{2\pi\sqrt{-1}\theta}$, $0 \le \theta < 1$, we denote M_φ by $M_{w^{1/p}}$. Then
$$M_{w^{1/p}} \boxtimes \pi = \int_{U(1)}^{\oplus} \pi \circ \gamma_{w^{1/p}}\, d\eta(w).$$

(ii) *$M_\varphi \boxtimes \pi = M_{\overline{\varphi}} \boxtimes \pi$ where $\bar{\varphi}(w) \equiv \overline{\varphi(w)}$ for $w \in U(1)$.*

Proof. These follow from a slight generalization of Lemma 5.7 in [**6**]. □

Lemma 5.5. *Let $(\mathcal{H}, \pi)$ be $P(J)$ with the GP vector Ω for $J = (j_1, \ldots, j_p) \in \{1, \ldots, N\}^p_{A,c}$ for $p \ge 1$. Denote $\Omega_l \equiv \pi(s_{j_l} \cdots s_{j_p})\Omega$ for $l = 1, \ldots, p$.*

(i) *$\pi(s_{j_1} \cdots s_{j_l})^*\Omega = \Omega_{l+1}$ for $l = 1, \ldots, p-1$.*

(ii) *Let $\zeta_c(w) \equiv w^c$ for $c \in \mathbf{R}$ and $w \in U(1)$, $K \equiv J^\infty$ and*
$$v_{np+l-1} \equiv \zeta_{n+(l-1)/p} \otimes \Omega_l \tag{5.3}$$
for $l = 1, \ldots, p$, $n \in \mathbf{Z}$. If $\pi' \equiv M_{\bar{w}^{1/p}} \boxtimes \pi$, then $\pi'(s_{K_n})^* v_0 = v_n$ for each $n \in \mathbf{N}$ where we denote $K = (k_n)_{n\in\mathbf{N}}$, $K_n \equiv (k_1, \ldots, k_n)$ for $n \in \mathbf{N}$.

(iii) *If $\pi' \equiv M_{\bar{w}^{1/p}} \boxtimes \pi$, then $\zeta_n \otimes \pi(s_{J'})\Omega_{l-1} = \pi'(s_{J'})v_{np+l-1}$ for $J' \in \{1, \ldots, N\}^*_A$, $n \in \mathbf{Z}$ and $l = 1, \ldots, p$.*

(iv) *$M_{\bar{w}^{1/p}} \boxtimes \pi$ is cyclic.*

Proof. (i) By direct computation, we see the statement.
(ii) Since $K_{np} = J^n$, $(\pi'(s_J)(\phi \otimes \Omega))(w) = \bar{w}\phi(w)\Omega$ for $\phi \in L_2(U(1))$ and $w \in U(1)$. Hence $\pi(s_{K_{np+l-1}})^*\Omega_1 = \pi(s_{j_1} \cdots s_{j_{l-1}})^*\pi(s_J^p)^*\Omega = \pi(s_{j_1} \cdots s_{j_{l-1}})^*\Omega = \Omega_l$ for $n \in \mathbf{N}$ and $l = 2, \ldots, p$ by (i). Therefore
$$\pi'(s_{K_{np+l-1}})^* v_0 = \pi'(s_{K_{np+l-1}})^*(1 \otimes \Omega) = \zeta_{n+(l-1)/p} \otimes \Omega_l = v_{np+l-1}.$$
From this, the statement holds.
(iii) For $w \in U(1)$, $J' \in \{1, \ldots, N\}^k_A$, $k \ge 1$, $c \in \mathbf{R}$, and $l = 1, \ldots, p$,
$$(\pi'(s_{J'})(\zeta_c \otimes \Omega_l))(w) = \bar{w}^{k/p}\zeta_c(w) \otimes \pi(s_{J'})\Omega_l = \zeta_{c-k/p}(w) \otimes \pi(s_{J'})\Omega_l.$$
From this, $\zeta_c \otimes \pi(s_{J'})\Omega_l = \pi'(s_{J'})(\zeta_{c+k/p} \otimes \Omega_l)$. Hence we have the assertion.

(iv) Put $\pi^{'} \equiv M_{\bar{w}^{1/p}} \boxtimes \pi$. We extend $K = \{k_n\}_{n\in\mathbf{N}}$ as $K = \{k_n\}_{n\in\mathbf{Z}}$ by $k_{-np+l} \equiv j_l$ for $n \geq 1$ and $l = 1,\ldots,p$. Note $\pi^{'}(s_{K_n})v_0 = v_{-n}$ for $n \geq 1$. Hence $\{v_n\}_{n\in\mathbf{Z}} \subset \mathcal{V}$. Since Lin< $\{\pi(s_{J'})\Omega_l : J^{'} \in \{1,\ldots,N\}_A^*,\, l = 1,\ldots,p\}$ > is dense in $\mathcal{H}$, Lin< $\{\zeta_n \otimes \pi(s_{J'})\Omega_l : n \in \mathbf{Z},\, J^{'} \in \{1,\ldots,N\}_A^*,\, l = 1,\ldots,p\}$ > is dense in $L_2(U(1))\otimes\mathcal{H}$. By (iii), $\mathcal{V} = L_2(U(1))\otimes\mathcal{H}$. Therefore $\pi^{'}$ is cyclic. $\square$

Proposition 5.6. *If $J \in \{1,\ldots,N\}_{A,c}^p$, $p \geq 1$, then*

$$P(J^\infty) = \int_{U(1)}^{\oplus} P(J;w)\, d\eta(w).$$

Proof. When $(\mathcal{H},\pi) = P(J)$, we denote $U\boxtimes P(J)$ instead of $U\boxtimes\pi$ for convenience. Let v_n be in (5.3). By Lemma 5.5 (iii), $v_n \in \mathcal{V} \equiv \pi(\mathcal{O}_A)v_0$ for each $n \in \mathbf{N}$. Since $\{v_n\}_{n\geq 1}$ is an orthonormal family, $\pi^{'}$ contains $P(J^\infty)$ as a subrepresentation. By Lemma 5.5 (iv), $M_{\bar{w}^{1/p}}\boxtimes P(J) = P(J^\infty)$. By Lemma 5.4 (ii), $M_{\bar{w}^{1/p}}\boxtimes P(J) = M_{w^{1/p}}\boxtimes P(J)$. Hence $M_{w^{1/p}}\boxtimes P(J) = P(J^\infty)$. By this, Lemma 5.4 and Lemma 3.5, the statement holds. $\square$

Corollary 5.7. (i) *If $K \in \{1,\ldots,N\}_A^\infty$ is eventually periodic , then there is $J \in \{1,\ldots,N\}_{A,c}^*$ such that J is non periodic and*

$$P(K) = \int_{U(1)}^{\oplus} P(J;w)\, d\eta(w). \tag{5.4}$$

(ii) *If there is $J^{'} \in \{1,\ldots,N\}_{A,c}^*$ which satisfies the statement (i) with respect to K, then $J^{'} \sim J$.*

(iii) *The decomposition in* (5.4) *is multiplicity free.*

Proof. (i) If K is eventually periodic then there is a non periodic element $J \in \{1,\ldots,N\}_{A,c}^*$ such that $K \sim J^\infty$. Hence $P(K) \sim P(J^\infty)$ by Theorem 4.12. By Proposition 5.6, the statement holds.
(ii) By Proposition 5.6, $P((J^{'})^\infty) = \int_{U(1)}^{\oplus} P(J^{'};w)\, d\eta(w) = P(K) = P(J^\infty)$. By Theorem 4.12, $(J^{'})^\infty \sim J^\infty$. Because both J and $J^{'}$ are non periodic, $J^{'} \sim J$.
(iii) This follows from Corollary 5.2 (ii). $\square$

Theorem 5.8. *For $K \in \{1,\ldots,N\}_A^\infty$, $P(K)$ is irreducible if and only if K is non eventually periodic.*

Proof. By Theorem 4.7 and Corollary 5.7, the statement holds. $\square$

5.3. Completely reducibility and uniqueness of decomposition.

Theorem 5.9. *For $A \in M_N(\{0,1\})$, let $(\mathcal{H},\pi)$ be a permutative representation of $\mathcal{O}_A$, and $< 1,\ldots,N >_A^{\#}$ and $[1,\ldots,N]_A^{\#}$ be in* (2.1)*.*

(i) *The following decomposition into cyclic subspaces holds:*

$$(\mathcal{H},\pi) \quad \sim \bigoplus_{J\in<1,\ldots,N>_A^{\#}} P(J)^{\oplus\nu_J} \tag{5.5}$$

where ν_J is the multiplicity of $P(J)$ for $J \in< 1,\ldots,N >_A^{\#}$. Furthermore (5.5) *is unique up to unitary equivalences.*

(ii) *The following irreducible decomposition holds:*

$$\mathcal{H} \quad = \bigoplus_{J\in[1,\ldots,N]_A^{*}} \mathcal{H}_J \quad \oplus \bigoplus_{K\in[1,\ldots,N]_A^{\infty}} \mathcal{H}_K,$$

$$\mathcal{H}_J \quad = \quad \bigoplus_{p\geq 1}\left\{\bigoplus_{j=1}^{p}\mathcal{H}_{J,p,j}\right\}^{\oplus\nu_{J,p}} \quad \oplus \quad \left\{\int_{U(1)}^{\oplus}\mathcal{H}_{J,\infty,z}\,dm(z)\right\}^{\oplus\nu_{J,\infty}},$$

$$\mathcal{H}_K = \mathcal{H}_{K,0}^{\oplus\nu_K}$$

where

$$\mathcal{H}_{J,p,j} \sim P(J; e^{2\pi\sqrt{-1}j/p}), \quad \mathcal{H}_{J,\infty,z} \sim P(J;z), \quad \mathcal{H}_{K,0} \sim P(K)$$

and $\nu_{J,p}$ and ν_K are multiplicities.

Proof. (i) By Theorem 2.7, (3.4) and Theorem 5.3, the statement holds.
(ii) Theorem 5.1 and Corollary 5.7 imply the decomposition. □

Assume that there are two irreducible decompositions of a given permutative representation $(\mathcal{H},\pi)$ of $\mathcal{O}_A$. If there is no direct integral component, then the uniqueness follows. If there is a direct integral decomposition on $U(1)$ as a style in (5.4), then the uniqueness holds in a sense of Corollary 5.7 (ii). In consequence, the irreducible decomposition of permutative representation as a form in Theorem 5.9 (ii) is unique up to unitary equivalences.

Theorem 5.10. *For any $A \in M_N(\{0,1\})$, any permutative representation of $\mathcal{O}_A$ is completely reducible and irreducible decomposition as a form in Theorem* 5.9 *(ii) is unique up to unitary equivalences.*

5.4. Decomposition of permutative representation with phases. $(\mathcal{H},\pi)$ is a *permutative representation of $\mathcal{O}_A$ with phases* if there are a complete orthonormal basis $\{e_n\}_{n\in\Lambda}$ of $\mathcal{H}$ and a family $\{\Lambda_i\}_{i=1}^{N}$ of subsets of Λ such that ${}^\forall i \in \{1,\ldots,N\}$, ${}^\forall n \in \Lambda_i$, ${}^\exists(z_{i,n}, m_{i,n}) \in U(1)\times\Lambda$ s.t.

$$\pi(s_i)e_n = z_{i,n}\cdot\chi_{\Lambda_i}(n)\cdot e_{m_{i,n}} \quad (n\in\Lambda).$$

Proposition 5.11. *For $A \in M_N(\{0,1\})$, let $(\mathcal{H}, \pi)$ be a permutative representation of $\mathcal{O}_A$ with phases. Then the following unique decomposition into cyclic representations up to unitary equivalences holds:*

$$(\mathcal{H}, \pi) \sim \bigoplus_{(J,c)\in[1,\dots,N]_A^*\times U(1)} P(J;c)^{\oplus \nu_{J,c}} \oplus \bigoplus_{K\in<1,\dots,N>_A^\infty} P(K)^{\oplus \nu_K}$$

where $\nu_{J,c}$ and ν_K are multiplicities. Specially, if $(\mathcal{H}, \pi)$ is cyclic, then $(\mathcal{H}, \pi)$ is equivalent to either $P(J;c)$ or $P(K)$ for some $(J,c) \in< 1,\dots,N >_A^*$ $\times U(1)$ or $K \in< 1,\dots,N >_A^\infty$.

Proof. By assumption, there are a complete orthonormal basis $\{e_n\}_{n\in\Lambda}$ of $\mathcal{H}$, $\{\Lambda_i\}_{i=1}^N$ and $\{(z_{i,n}, m_{i,n}) \in U(1) \times \Lambda : (i,n) \in \{1,\dots,N\} \times \Lambda_i\}$ such that $\pi(s_i)e_n = z_{i,n}\chi_{\Lambda_i}(n) \cdot e_{m_{i,n}}$ for each $(i,n) \in \{1,\dots,N\} \times \Lambda$. Define a new permutative representation $(\mathcal{H}, \pi_0)$ of $\mathcal{O}_A$ by $\pi_0(s_i)e_n \equiv \chi_{\Lambda_i}(n) \cdots e_{m_{i,n}}$ for $(i,n) \in \{1,\dots,N\} \times \Lambda$. By Theorem 5.9 (i), π_0 is decomposed into the direct sum of permutative representations:

$$\pi_0 \sim \bigoplus_{J\in<1,\dots,N>_A^*} P(J)^{\oplus \nu_J} \oplus \bigoplus_{K\in<1,\dots,N>_A^\infty} P(K)^{\oplus \nu_K}.$$

Therefore $\pi_0|_V \sim P(J)$ or $\pi_0|_V \sim P(K)$ for some subspace $V \subset \mathcal{H}$. If $\pi_0|_V \sim P(J)$, then there is a cyclic unit vector $\Omega \in V$ such that $\pi_0(s_J)\Omega = \Omega$. By definition of π_0, there is $c_J \in U(1)$ such that $\pi(s_J)\Omega = c_J\Omega$. Because $(V, \pi_0|_V)$ is cyclic, $(V, \pi|_V)$ is, too. Therefore $\pi|_V \sim P(J; c_J)$. If $\pi_0|_V \sim P(K)$, then we see that $\pi|_V \sim P(K)$ by checking the condition of chain. In consequence

$$\pi \sim \bigoplus_{J\in<1,\dots,N>_A^*} P(J; c_J)^{\oplus \nu_J} \oplus \bigoplus_{K\in<1,\dots,N>_A^\infty} P(K)^{\oplus \nu_K}.$$

When J is periodic, $P(J; c_J)$ is decomposed into the direct sum of elements in $\{P(J'; c') : (J', c') \in [1,\dots,N]_A^* \times U(1)\}$ by Theorem 5.2. Hence the statement holds. ☐

By Proposition 5.11 and results of permutative representation of $\mathcal{O}_A$ in § 4, § 5, Theorem 1.1 is proved. Proposition 5.11 for $\mathcal{O}_A = \mathcal{O}_N$ is shown in [**3**, **4**].

6. States and spectrums

Fix $A \in M_N(\{0,1\})$.

6.1. States of permutative representations. Operator algebraists prefer *states* than representations. Therefore we show states of the Cuntz-Krieger algebras associated with permutative representations.

Theorem 6.1. *Let $(\mathcal{H}, \pi)$ be $P(J)$ with the GP vector Ω for $J \in \{1, \ldots, N\}^{\#}_{A,c}$. Define a state ω of $\mathcal{O}_A$ by $\omega \equiv < \Omega|\pi(\cdot)\Omega >$. Then the followings hold:*

(i) *When $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k_{A,c}$, $k \geq 1$,*

$$\omega(s_{J'} s^*_{J''}) = \begin{cases} 1 & (0 \leq^{\exists} p \leq k-1, \ s.t. \ J', J'' \in \mathcal{I}_p(J)), \\ 0 & (otherwise) \end{cases} \tag{6.1}$$

where $\mathcal{I}_p(J) \equiv \{J^a \cup (j_1, \ldots, j_p) \in \{1, \ldots, N\}^*_A : a \geq 0\}$.

(ii) *When $J = (j_n)_{n \in \mathbf{N}} \in \{1, \ldots, N\}^{\infty}_A$,*

$$\omega(s_{J'} s^*_{J''}) = \begin{cases} 1 & (^{\exists} n \in \mathbf{N} \, s.t. \ J' = J'' = (j_1, \ldots, j_n)), \\ 0 & (otherwise). \end{cases} \tag{6.2}$$

(iii) *The GNS representation of $\mathcal{O}_A$ by a state ω which satisfies (6.1) or (6.2) is equivalent to $P(J)$.*

(iv) *ω is pure if and only if J is non periodic or non eventually periodic.*

Proof. (i) Assume that $J' \in \{1, \ldots, N\}^{ak+p}_A$ and $J'' \in \{1, \ldots, N\}^{bk+q}_A$. By Lemma 3.8 and its notations,

$$\begin{aligned} \omega(s_{J'} s^*_{J''}) &= < \Omega|\pi(s_{J'} s^*_{J''})\Omega > \\ &= < \pi(s^*_{J'})\Omega|\pi(s^*_{J''})\Omega > \\ &= \delta_{J', J[1,\ldots,ak+p]} \delta_{J'', J[1,\ldots,bk+q]} < \pi(s_{J[p+1,k]})\Omega|\pi(s_{J[q+1,k]})\Omega > \\ &= \delta_{J', J[1,\ldots,ak+p]} \delta_{J'', J[1,\ldots,bk+q]} \delta_{p,q} \\ &= \begin{cases} 1 & (J', J'' \in \mathcal{I}_p(J)), \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

(ii) By Lemma 3.8 and the same way in (i), the statement holds.
(iii) The statement follows from the uniqueness of the GNS representation.
(iv) Corollary 5.2 and Theorem 5.8 imply the assertion. □

6.2. Spectrums. We consider the spectrum of $\mathcal{O}_A$ associated with permutative representations of $\mathcal{O}_A$. Spec$\mathcal{O}_A$ is the *spectrum* of $\mathcal{O}_A$ which consists of all unitary equivalence classes of irreducible representations of $\mathcal{O}_A$. Put PSpec$\mathcal{O}_A$ the subset of Spec$\mathcal{O}_A$ which consists of all unitary equivalence classes of irreducible permutative representations. By Theorem 4.2, Corollary 5.2 and Theorem 5.8, the following one-to-one correspondence holds:

$$\mathrm{PSpec}\mathcal{O}_A \cong [1, \ldots, N]^{\#}_A.$$

Specially, PSpec$\mathcal{O}_N \cong [1, \ldots, N]^{\#}$. By regarding phase factor, $\{[1, \ldots, N]^*_A \times U(1)\} \sqcup [1, \ldots, N]^{\infty}_A$ is identified with a subset of Spec$\mathcal{O}_A$.

When $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we see that $\{1,2\}^*_{A,c} = \{(1)^n, (2)^n : n \geq 1\}$ and $\{1,2\}^\infty_A = \{(1)^\infty, (1)^{n-1} \cup (2)^\infty : n \geq 1\}$. Hence $[1,2]^*_A = \{(1), (2)\}$, $[1,2]^\infty_A = \emptyset$ and $\#\mathrm{PSpec}\mathcal{O}_A = 2$.

When $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\{(1)^n \cup (2) : n \geq 1\}$ is a proper subset of $[1,2]^*_{A,c}$. From this, $\#\mathrm{PSpec}\mathcal{O}_A = \infty$.

7. Decomposition of standard representation

We introduced the standard representation of $\mathcal{O}_A$ for each A in [**7**], which is a kind of permutative representation. In this section, we show decomposition formulae of them.

7.1. Definition and decomposition formula. We review the standard A-branching function system and the standard representation of $\mathcal{O}_A$ for a given $A \in M_N(\{0,1\})$.

Definition 7.1. *Let $A = (a_{ij}) \in M_N(\{0,1\})$.*

(i) *A data $\{(M_i, q_i, B_i)\}_{i=1}^N$ is called the (canonical)A-coordinate if*

$$B_i \equiv \{\, j \in \{1,\ldots,N\} : a_{ij} = 1\,\}, \quad M_i \equiv a_{i1} + \cdots + a_{iN},$$

$$q_i : \ B_i \to \{1,\ldots,M_i\}; \quad q_i(j) \equiv \#\{k \in B_i : k \leq j\}$$

for $i = 1,\ldots,N$.

(ii) *An A-branching function system $f^{(A)} = \{f_i^{(A)}\}_{i=1}^N$ on $\mathbf{N}$ defined by*

$$f_i^{(A)}(N(m-1)+j) \equiv N(M_i(m-1)+q_i(j)-1)+i \quad (m \in \mathbf{N},\ j \in B_i),$$

$$R(f_i^{(A)}) \equiv \{N(n-1)+i : n \in \mathbf{N}\},\ D(f_i^{(A)}) \equiv \textstyle\coprod_{j\in B_i} R(f_j^{(A)}) \ (i = 1,\ldots,N)$$

is called the standard A-branching function system. (iii)

(iii) *$(l_2(\mathbf{N}), \pi_S^{(A)})$ is the standard representation of $\mathcal{O}_A$ if $(l_2(\mathbf{N}), \pi_S^{(A)})$ is a representation of $\mathcal{O}_A$ defined by*

$$\pi_S^{(A)}(s_i)e_n \equiv \chi_{D(f_i^{(A)})}(n)e_{f_i^{(A)}(n)} \quad (n \in \mathbf{N},\ i = 1,\ldots,N)$$

where $f^{(A)} = \{f_i^{(A)}\}_{i=1}^N$ is the standard A-branching function system.

In order to show the decomposition formula of the standard representation, we define cycles arising from some finite dynamical system associated with A. For $A = (a_{ij}) \in M_N(\{0,1\})$, put a map φ_A on $\{1,\ldots,N\}$ by

$$\varphi_A(i) \equiv \min\{j \in \{1,\ldots,N\} : a_{ij} = 1\} \quad (i = 1,\ldots,N). \tag{7.1}$$

Then $\{1,\ldots,N\}$ contains cycles by φ_A, that is, $C = \{n_i \in \{1,\ldots,N\} : i = 1,\ldots,m\}$ is a *cycle* in $\{1,\ldots,N\}$ by φ_A if $\varphi_A(n_i) = n_{i+1}$ for $i = 1,\ldots,m-1$ and $\varphi_A(n_m) = n_1$.

Definition 7.2. (i) $\{C_i\}_{i=1}^k$ *is the A-cycle set if* $\{C_i\}_{i=1}^k$ *is the set of all cycles in* $\{1,\ldots,N\}$ *by* φ_A. *Put* $m_i \equiv \#C_i$ *and* $j_{i,1} \equiv \min C_i$ *for* $i=1,\ldots,k$.

(ii) $\mathcal{J}_A \equiv \{J_i\}_{i=1}^k \subset \{1,\ldots,N\}^*$ *is the A-cyclic index set if* $J_i \equiv (j_{i,c})_{c=1}^{m_i} \in \{1,\ldots,N\}^{m_i}$ *and* $j_{i,c} \equiv \varphi_A^{c-1}(j_{i,1})$ *for* $c=1,\ldots,m_i$ *where* $j_{i,1} \in \{1,\ldots,N\}$ *and* m_i *are in (i).*

(iii) *An element* $J=(j_1,\ldots,j_m) \in \mathcal{J}_A$ *is isolated if*

$$a_{j_i,l} = \delta_{l,j_{i+1}} \quad (i=1,\ldots,m-1), \quad a_{j_m,l} = \delta_{l,j_1}$$

for $l=1,\ldots,N$. *We denote the set of all isolated elements in* $\mathcal{J}_A$ *by* $\mathcal{J}_{A,\infty}$ *and* $\mathcal{J}_{A,1} \equiv \mathcal{J}_A \setminus \mathcal{J}_{A,\infty}$.

Theorem 7.3. *For* $A \in M_N(\{0,1\})$, *let* $(l_2(\mathbf{N}), \pi_S^{(A)})$ *be the standard representation of* $\mathcal{O}_A$. *Then the followings hold:*

(i)

$$(l_2(\mathbf{N}), \pi_S^{(A)}) \sim \bigoplus_{J\in\mathcal{J}_{A,1}} P(J) \oplus \left\{ \bigoplus_{K\in\mathcal{J}_{A,\infty}} P(K) \right\}^{\oplus\infty}$$

where $\mathcal{J}_{A,1}$ *and* $\mathcal{J}_{A,\infty}$ *are in Definition* 7.1.

(ii) $\pi_S^{(A)}$ *is multiplicity free if and only if* $\mathcal{J}_{A,\infty} = \emptyset$. *Under this condition,* $\pi_S^{(A)}$ *is irreducible if and only if* $\#\mathcal{J}_{A,1} = 1$.

If A is full, that is, $\mathcal{O}_A = \mathcal{O}_N$, then $\pi_S^{(A)} \sim P(1)$ for each $N \geq 2$. We prove Theorem 7.3 in § 7.2.

7.2. Proof of Theorem 7.3. In Definition 7.2, we see $C_i = \{\varphi_A^{l-1}(j_{i,1}) : l=1,\ldots,m_i\}$. Because $C_i \cap C_j = \emptyset$ when $i \neq j$, any two elements in $\mathcal{J}_A$ are inequivalent.

Lemma 7.4. *Let* $A \in M_N(\{0,1\})$, $\mathcal{J}_A$ *be the A-cyclic index set and* $f^{(A)}$ *be the standard A-branching function system. Then the followings hold:*

(i) $\mathcal{J}_A \subset \{1,\ldots,N\}^*_{A,c}$.

(ii) $\bigoplus_{J\in\mathcal{J}_A} P(J)$ *is a component of* $f^{(A)}$.

(iii) *If* $J \in \mathcal{J}_{A,\infty}$, *then* $f^{(A)}$ *contains a* $P(J)^{\oplus\infty}$*-component.*

Proof. We simply denote $f^{(A)} = \{f_i^{(A)}\}_{i=1}^N$ by $f = \{f_i\}_{i=1}^N$. Put $\mathcal{J}_A = \{J_i\}_{i=1}^k$ and $J_i = (j_{i,1},\ldots,j_{i,m_i})$ for $i=1,\ldots,k$. Let $\{(M_i,q_i,B_i)\}_{i=1}^N$ be the A-coordinate. We see that $q_i(\varphi_A(i)) = 1$ for each $i=1,\ldots,N$.
(i) By definition of φ_A, $a_{j_{i,1},j_{i,2}} = \cdots = a_{j_{i,m_i-1},j_{i,m_i}} = 1$. Because C_i is a cycle, $\varphi_A(j_{i,m_i}) = \varphi_A^{m_i}(j_{i,1}) = j_{i,1}$ and $a_{j_{i,m_i},j_{i,1}} = 1$. Hence the statement holds.
(ii) Note that $q_{j_{i,c}}(j_{i,c+1}) = 1$ for $c=1,\ldots,m_i-1$ and $q_{j_{i,m_i}}(j_{i,1}) = 1$. From this, $f_{j_{i,c}}(j_{i,c+1}) = N(q_{j_{i,c}}(j_{i,c+1})-1)+j_{i,c} = j_{i,c}$ and $f_{j_{i,m_i}}(j_{i,1}) = j_{i,m_i}$.

Then $f_{J_i}(j_{i,1}) = (f_{j_{i,1}} \circ \cdots \circ f_{j_{i,m_i}})(j_{i,1}) = \cdots = f_{j_{i,1}}(j_{i,2}) = j_{i,1}$. Therefore $P(J_i)$ is a component of f. Because each two elements in the A-cycle set $\{C_i\}_{i=1}^k$ are disjoint, $P(J_1) \oplus \cdots \oplus P(J_k)$ is a component of f.
(iii) Note that $J = (j_1, \ldots, j_m) \in \mathcal{J}_{A,\infty}$ if and only if $M_{j_c} = 1$ for each $c = 1, \ldots, m$. $D(f_{j_{i,c}}) = \{N(m-1) + j_{i,c+1} : m \geq 1\}$, $D(f_{j_{i,m_l}}) = \{N(m-1)+j_{i,1} : m \geq 1\}$ and $f_{j_{i,c}}(N(m-1)+j_{i,c+1}) = N(m-1)+j_{i,c}$, $f_{j_{i,m_i}}(N(m-1) + j_{i,1}) = N(m-1) + i_{l,m_l}$. Hence $f_{J_i}(N(m-1) + j_{i,1}) = (f_{j_{i,1}} \circ \cdots \circ f_{j_{i,m_i}})(N(m-1) + j_{i,1}) = N(m-1) + j_{i,1}$. In consequence, $f_{J_i}(n) = n$ for each $n \in \{N(m-1) + j_{i,1} : m \geq 1\}$. Therefore $C_i^{(m)} \equiv \{N(m-1) + j_{i,c} : c = 1, \ldots, m_i\}$ is a cycle of f by J_i for each $m \geq 1$. Hence the statement holds. $\square$

In order to decompose the permutative representation associated with the standard A-branching function system, we show decomposition formula of the standard A-branching function system. We denote $f^{(A)}$ by f simply.

Lemma 7.5. *For $A \in M_N(\{0,1\})$, let $\{(M_i, q_i, B_i)\}_{i=1}^N$ be the A-coordinate, f be the standard A-branching function system and $J = (j_1, \ldots, j_k) \in \{1, \ldots, N\}_{A,c}^k$. Then the followings hold:*

(i) *For $m \geq 1$ and $j_0 \in \{1, \ldots, N\} \cap D(f_{j_k})$,*

$$f_J(N(m-1) + j_0) = N(L_k(m-1) + \alpha - 1) + j_1 \tag{7.2}$$

where $L_i \equiv M_{j_1} \cdots M_{j_i}$ for $i = 1, \ldots, k$ and

$$\alpha \equiv \begin{cases} q_{j_1}(j_0) & (k=1), \\ q_{j_1}(j_2) + M_{j_1}(q_{j_2}(j_0) - 1) & (k=2), \\ q_{j_1}(j_2) + \sum_{i=1}^{k-2} L_i(q_{j_{i+1}}(j_{i+2}) - 1) + L_{k-1}(q_{j_k}(j_0) - 1) & (k \geq 3). \end{cases}$$

(ii) *If there is $n_0 \in \mathbf{N}$ such that $f_J(n_0) = n_0$, then there is $m \geq 1$ such that $n_0 = N(m-1) + j_1$ and*

$$q_{j_i}(j_{i+1}) = 1 \quad (i = 1, \ldots, k-1), \quad q_{j_k}(j_1) = 1, \tag{7.3}$$

and

$$m = L_k(m-1) + 1. \tag{7.4}$$

(iii) *Assume that there is a cycle C of f by J. If $C \not\subset \{1, \ldots, N\}$, then $M_{j_i} = 1$ for each $i = 1, \ldots, k$.*

(iv) *Assume that there is a cycle C of f by J. If $M_{j_i} \geq 2$ for some $i \in \{1, \ldots, k\}$, then $C \subset \{1, \ldots, N\}$.*

Proof. (i) By direct computation, we have the statement.
(ii) By definition of f_1, the first statement holds. By (i), $N(m-1) + j_1 = N(L_k(m-1) + \alpha - 1) + j_1$. From this, we have $m = L_k(m-1) + \alpha$.

If $m=1$, then $\alpha=1$. By definition of α, (7.3) holds. If $m\geq 2$, then $L_k=1$ and $\alpha=1$. Therefore (7.3) holds, too. In consequence, we have (7.4).
(iii) By assumption, there is $n_0\in C\subset \mathbf{N}$ such that $n_0=N(m-1)+i$ for some $m\geq 2$. Then there is $J^{'}=(j_1^{'},\ldots,j_k^{'})\in\{1,\ldots,N\}^k_{A,c}$ such that $J^{'}\sim J$ and $f_{J^{'}}(n_0)=n_0$. By (7.4) and proof of (ii), $L_k=1$ and the statement holds.
(iv) By (iii), the statement holds. □

Lemma 7.6. *Let $A\in M_N(\{0,1\})$ with the A-coordinate $\{(M_i,q_i,B_i)\}_{i=1}^N$. For $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}^k_{A,c}$, the standard A-branching function system has a $P(J)$-component if and only if $q_{j_i}(j_{i+1})=1$ for each $i=1,\ldots,k-1$ and $q_{j_k}(j_1)=1$.*

Proof. By Lemma 7.4 and Lemma 7.5, the statement holds. □

Lemma 7.7. *For any $A\in M_N(\{0,1\})$, the standard A-branching function system has no chain.*

Proof. Let $f=\{f_i\}_{i=1}^N$ be the standard A-branching function system. Assume that there is a chain $C=\{m_n\}_{n=1}^\infty\subset\mathbf{N}$ of f by $J=(j_n)_{n\in\mathbf{N}}\in\{1,\ldots,N\}^\infty_A$. Denote $J_k\equiv(j_1,\ldots,j_k)$ for $k\geq 1$. Put $x=N(m-1)+j_0\in C$. By Lemma 7.5, $f_{J_k}(x)=L_k(x-j_0)+N(\alpha-1)+j_1$. From this, for each $y\in C$ and $k\geq 1$, there are $j_{y,k}\in\{1,\ldots,N\}$ and $\alpha_{y,k}\geq 1$ such that

$$z_k(y)\equiv f_{J_k}^{-1}(y)=L_k^{-1}(y-(N(\alpha_{y,k}-1)+j_1))+j_{y,k}.$$

Note $\#\{z_k(y):k\in\mathbf{N}\}=\infty$. Because $z_k(y)\in C\subset\mathbf{N}$ for each $k\geq 1$ and $L_{k+1}\geq L_k\geq 1$ for each $k\geq 1$, there is $k_0\in\mathbf{N}$ such that $L_k=1$ for each $k\geq k_0$. Put $J^{'}\equiv(j_n)_{n=k_0}^\infty$. Then $J^{'}\sim J$. By replacing J and $J^{'}$, we can assume that $z_k(y)=y-(N(\alpha_{y,k}-1)+j_1)+j_{y,k}$ for $k\geq 1$. In this case,

$$\alpha_{y,k}=q_{j_1}(j_2)+\textstyle\sum_{i=1}^{k-2}(q_{j_{i+1}}(j_{i+2})-1)+(q_{j_k}(j_{y,k})-1)\quad(k\geq 3).$$

Because $\alpha_{y,k+1}\geq\alpha_{y,k}\geq 1$ for each $k\geq 1$ and $z_k(y)\in C\subset\mathbf{N}$ for each $k\geq 1$, there is $k_0\geq 1$ such that $q_{j_k}(j_{k+1})=1$ for each $k\geq k_0$. By replacing J and $\{j_n\}_{n=k_0}^\infty$, we can assume that $q_{j_k}(j_{k+1})=1$ for each $k\geq 1$. Then $\alpha_{y,k}=1$ for each $k\geq 1$. In consequence, $z_k(y)=y-j_1+j_{y,k}$ for $k\geq 1$. Therefore $\{z_k(y):k\in\mathbf{N}\}\subset\{y-j_1\pm n:n=0,\ldots,N\}$ and $\#\{z_k(y):k\in\mathbf{N}\}\leq 2N<\infty$. This contradicts the choice of y and J. Therefore there is no chain of f. □

Theorem 7.8. *For $A\in M_N(\{0,1\})$, if $\mathcal{J}_A$ is the A-cyclic index set and $\mathcal{J}_{A,1}$, $\mathcal{J}_{A,\infty}$ are in Definition 7.1, then the standard A-branching function* *system is decomposed as*

$$\bigoplus_{J\in\mathcal{J}_{A,1}} P(J)\oplus\left\{\bigoplus_{K\in\mathcal{J}_{A,\infty}} P(K)\right\}^{\oplus\infty}.$$

Proof. Let f be the standard A-branching function system. By Theorem 2.7 and Lemma 7.7, f is decomposed into only cycles. On the other hand, any cycle component of f is one of $\{P(J) : J \in \mathcal{J}_A\}$ by Lemma 7.6. Therefore f is decomposed as a direct sum of $\{P(J) : J \in \mathcal{J}_A\}$ with multiplicities. If $J \in \mathcal{J}_{A,\infty}$, then f has a $P(J)^{\oplus\infty}$-component by Lemma 7.4. If $J \in \mathcal{J}_{A,1}$, then the cycle of f by J is a subset of $\{1,\ldots,N\}$. By definition of $\mathcal{J}_A$, $P(J)$ appears in $\{1,\ldots,N\}$ at only once. In consequence, the statement holds. $\square$

Proof of Theorem 7.3*:* (i) By Theorem 7.8 and (3.4), the statement holds. (ii) By (i), the first statement holds immediately. If $J = (j_1,\ldots,j_m) \in \mathcal{J}_{A,1}$, then $j_i \neq j_{i'}$ when $i \neq i'$. Therefore, any element in $\mathcal{J}_{A,1}$ is non periodic. Hence the second statement holds. $\square$

8. Examples

8.1. Examples by naive observation. We compute two permutative representations directly. Put matrices $A_1, A_2 \in M_3(\{0,1\})$ by

$$A_1 \equiv \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 1\\ 1 & 1 & 1\end{pmatrix},\quad A_2 \equiv \begin{pmatrix} 0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 1\end{pmatrix}.$$

(i) Define a representation $(l_2(\mathbf{N}),\pi)$ of $\mathcal{O}_{A_1}$ by

$$\pi(s_1)e_{4(n-1)+i} \equiv \delta_{2,i}e_{4(n-1)+1},\quad \pi(s_2)e_{4(n-1)+i} \equiv \delta_{1,i}e_{4(n-1)+4} + \delta_{2,i}e_{4(n-1)+3},$$

$$\pi(s_3)e_n \equiv e_{4(n-1)+2}$$

for $n \in \mathbf{N}$ and $i = 1,2,3,4$. Then

$$(l_2(\mathbf{N}),\pi) \sim P(13).$$

Proof. Put $D_1 \equiv \{4(n-1)+2 : n \in \mathbf{N}\}$, $D_2 \equiv \{4(n-1)+1, 4(n-1)+2 : n \in \mathbf{N}\}$, $D_3 \equiv \mathbf{N}$ and $f = \{f_1, f_2, f_3\}$ by $\pi(s_i)e_n \equiv e_{f_i(n)}$ for $i = 1,2,3$ and $n \in D_i$. Then $f \in \mathrm{BFS}_{A_1}(\mathbf{N})$. Note that $f_i(n) > n$ for any $n \in X \equiv \{n \in \mathbf{N} : n \geq 3\}$ $i = 2,3$. $f_1(n) = n-1$ for $n \in D(f_1)$, $R(f_1) \subset D(f_i)$ for $i = 2,3$ and $R(f_1) \cap D(f_1) = \emptyset$. $(f_2\circ f_1)(4(n-1)+2) = f_2(4(n-1)+1) = 4(n-1)+4 > 4(n-1)+2$. $(f_3\circ f_1)(4(n-1)+2) = f_3(4(n-1)+1) = 4(4(n-1))+2$. Therefore $(f_i\circ f_1)(n) > n$ for $n \in D(f_1)\cap X$ and $i = 2,3$. From these, f has neither chain nor cycle in X. Therefore f has only cycles in $\mathbf{N} \setminus X = \{1,2\}$ and f is cyclic. We see that $(f_1 \circ f_3)(1) = 1$. This implies that $\pi(s_1 s_3)e_1 = e_1$. Therefore $(l_2(\mathbf{N}), \pi)$ has one cycle $\{e_1, e_2\}$. $\square$

(ii) Define a representation $(l_2(\mathbf{N} \times \{1,2\}), \pi)$ of $\mathcal{O}_{A_2}$ by

$$\begin{cases} \pi(s_1)e_{n,i} \equiv & \delta_{2,i}e_{n,1}, \\ \pi(s_2)e_{5(n-1)+m,i} \equiv & \delta_{1,i}e_{5(5(n-1)+m-1)+1,2} \\ & +\delta_{2,i}(\delta_{4,m}e_{5(n-1)+2,2} + \delta_{5,m}e_{5(n-1)+3,2}), \\ \pi(s_3)e_{n,i} \equiv & \delta_{1,i}e_{5(n-1)+4,2} + \delta_{2,i}e_{5(n-1)+5,2} \end{cases}$$

for $i = 1,2$, $m = 1,\ldots,5$ and $n \in \mathbf{N}$ where $e_{n,i} \equiv e_n^{'} \otimes e_i^{''}$ and $e_n^{'}$, $e_i^{''}$ are canonical basis of $l_2(\mathbf{N})$ and $\mathbf{C}^2$, respectively. Then

$$(l_2(\mathbf{N} \times \{1,2\}), \pi) \sim P(12).$$

Proof. We see that $\pi(s_1 s_2)e_{1,1} = e_{1,1}$ and there is no cycle except $X \equiv \{e_{1,1}, e_{1,2}\}$. Hence π has only a cycle in X. Furthermore $\{\pi(s_J)e_{1,1} : J \in \{1,\ldots,N\}_A^*\} = \{e_{n,i} : n \in \mathbf{N},\, i = 1,2\}$. Therefore the statement holds. $\square$

8.2. Examples of standard representation. We show examples of the standard representation $\pi_S^{(A)}$ of $\mathcal{O}_A$ for $A \in M_N(\{0,1\})$ by Theorem 7.3. In order to this aim, we use $s_1,\ldots,s_N$ as canonical generators of $\mathcal{O}_A$ and define operators $t_1, t_2, k_1, k_2, k_3, u_1, u_2, u_3, u_4$ on $l_2(\mathbf{N})$ by

$$t_i e_n \equiv e_{2(n-1)+i}, \quad k_i e_n \equiv e_{3(n-1)+i}, \quad u_i e_n \equiv e_{4(n-1)+i}$$

where $\{e_n : n \in \mathbf{N}\}$ is the canonical basis of $l_2(\mathbf{N})$. Then the followings hold:

(i) Put $A_3 \equiv \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Then the standard representation $(l_2(\mathbf{N}), \pi_S^{(A_3)})$ of $\mathcal{O}_{A_3}$ is given by

$$\pi_S^{(A_3)}(s_1) \equiv k_1(t_1 k_2^* + t_2 k_3^*), \quad \pi_S^{(A_3)}(s_2) \equiv k_2(t_1 k_1^* + t_2 k_3^*),$$
$$\pi_S^{(A_3)}(s_3) \equiv k_3(t_1 k_1^* + t_2 k_2^*).$$

Then φ_{A_3} in (7.1) is given by

$$\varphi_{A_3} : \{1,2,3\} \to \{1,2,3\}; \quad \varphi_{A_3}(1) = 2, \quad \varphi_{A_3}(2) = 1, \quad \varphi_{A_3}(3) = 1.$$

From this, we see that $\varphi_{A_3}^2(1) = 1$ and $\mathcal{J}_{A_3} = \mathcal{J}_{A_3,1} = \{(12)\}$. Therefore

$$(l_2(\mathbf{N}), \pi_S^{(A_3)}) \sim P(12).$$

In fact, $\pi_S^{(A_3)}(s_1 s_2)e_1 = e_1$ is the only one cycle on $\{e_n : n \in \mathbf{N}\}$. This implies the statement.

(ii) Put $A_4 \equiv \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. The standard representation of $\mathcal{O}_{A_4}$ is as follows:

$$\pi_S^{(A_4)}(s_1) \equiv k_1(t_1 k_1^* + t_2 k_3^*), \quad \pi_S^{(A_4)}(s_2) \equiv k_2(t_1 k_2^* + t_2 k_3^*), \quad \pi_S^{(A_4)}(s_3) \equiv k_3.$$

Then φ_{A_4} is given by

$$\varphi_{A_4} : \{1,2,3\} \to \{1,2,3\}; \quad \varphi_{A_4}(1) = 1, \quad \varphi_{A_4}(2) = 2, \quad \varphi_{A_4}(3) = 1.$$

From this, we see that $\varphi_{A_4}(1) = 1$ and $\varphi_{A_4}(2) = 2$. Hence $\mathcal{J}_{A_4,1} = \{(1),(2)\}$. These are related that $\pi_S^{(A_4)}(s_1)e_1 = e_1$ and $\pi_S^{(A_4)}(s_2)e_2 = e_2$. Therefore

$$(l_2(\mathbf{N}), \pi_S^{(A_4)}) \sim P(1) \oplus P(2).$$

Next, we show the decomposition of the standard representation $\pi_S^{(A)}$ of $\mathcal{O}_A$ for every 2×2 matrices without proof as follows:

A	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
$\pi_S^{(A)}$	$P(1)$	$P(12)$	$P(1)^{\oplus\infty}$	$P(1) \oplus P(2)^{\oplus\infty}$	$P(1)$

A	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\pi_S^{(A)}$	$(P(1) \oplus P(2))^{\oplus\infty}$	$P(12)^{\oplus\infty}$

In this way, the standard representation of $\mathcal{O}_A$ depends on the form of A.

We show other examples as follows:

A	$\pi_S^{(A)}$
$\begin{pmatrix} 0&1&0&1\\ 0&1&0&1\\ 1&1&0&1\\ 0&1&1&1 \end{pmatrix}$	$\pi_S^{(A)}(s_1)=u_1(t_1u_2^*+t_2u_4^*)$, $\pi_S^{(A)}(s_2)=u_2(t_1u_2^*+t_2u_4^*)$, $\pi_S^{(A)}(s_3)=u_3(k_1u_1^*+k_2u_2^*+k_3u_4^*)$, $\pi_S^{(A)}(s_4)=u_4(k_1u_2^*+k_2u_3^*+k_3u_4^*)$, $\pi_S^{(A)}\sim P(2)$.
$\begin{pmatrix} 0&1&0&1\\ 0&0&1&1\\ 1&1&0&1\\ 0&1&1&1 \end{pmatrix}$	$\pi_S^{(A)}(s_1)=u_1(t_1u_2^*+t_2u_4^*)$, $\pi_S^{(A)}(s_2)=u_2(t_1u_3^*+t_2u_4^*)$, $\pi_S^{(A)}(s_3)=u_3(k_1u_1^*+k_2u_2^*+k_3u_4^*)$, $\pi_S^{(A)}(s_4)=u_4(k_1u_2^*+k_2u_3^*+k_3u_4^*)$, $\pi_S^{(A)}\sim P(123)$.
$\begin{pmatrix} 0&0&1&1\\ 1&0&1&1\\ 0&1&0&1\\ 0&1&1&1 \end{pmatrix}$	$\pi_S^{(A)}(s_1)=u_1(t_1u_3^*+t_2u_4^*)$, $\pi_S^{(A)}(s_2)=u_2(k_1u_1^*+k_2u_3^*+k_3u_4^*)$, $\pi_S^{(A)}(s_3)=u_3(t_1u_2^*+t_2u_4^*)$, $\pi_S^{(A)}(s_4)=u_4(k_1u_2^*+k_2u_3^*+k_3u_4^*)$, $\pi_S^{(A)}\sim P(132)$.

For $N\geq 4$,

$$A_1^{(N)}=\begin{bmatrix} 0&1&1&1&\cdots&1&1\\ 0&0&1&1&\cdots&1&1\\ 0&0&0&1&\cdots&1&1\\ \vdots&&&\ddots&\ddots&&\vdots\\ 0&0&0&0&\cdots&0&1\\ 1&1&1&1&\cdots&1&1 \end{bmatrix},\quad A_2^{(N)}=\begin{bmatrix} 0&1&1&1&\cdots&1&1\\ 0&0&1&1&\cdots&1&1\\ 0&0&0&1&\cdots&1&1\\ \vdots&&&\ddots&\ddots&&\vdots\\ 0&0&0&0&\cdots&0&1\\ 0&1&1&1&\cdots&1&1 \end{bmatrix},$$

$$A_3^{(N)}=\begin{bmatrix} 0&1&0&0&\cdots&0&0\\ 1&0&0&0&\cdots&0&0\\ 1&1&1&1&\cdots&1&1\\ \vdots&&&\vdots&&\vdots&\vdots\\ 1&1&1&1&\cdots&1&1\\ 0&0&0&0&\cdots&0&1 \end{bmatrix},\qquad \begin{aligned} &\pi_S^{(A_1^{(N)})}\sim P(12\cdots N),\\ &\pi_S^{(A_2^{(N)})}\sim P(23\cdots N),\\ &\pi_S^{(A_3^{(N)})}\sim (P(12)\oplus P(N))^{\oplus\infty}. \end{aligned}$$

8.3. Shift representation. We show the shift representation of $\mathcal{O}_A$ as an example of permutative representation which is multiplicity free, and prove its decomposition formula.

For $A = (a_{ij}) \in M_N(\{0,1\})$, let

$$X_A \equiv \{1,\ldots,N\}_A^{\infty}.$$

Define an A-branching function system $f = \{f_i\}_{i=1}^N$ on X_A by

$$f_i : D(f_i) \to R(f_i); \quad f_i(j_1, j_2, \ldots) \equiv (i, j_1, j_2, \ldots), \tag{8.1}$$

$$R(f_i) \equiv \{(j_1, j_2, \ldots) \in X_A : j_1 = i\}, \quad D(f_i) \equiv \coprod_{j; a_{ij}=1} R(f_j)$$

for $i = 1, \ldots, N$.

The permutative representation $(l_2(X_A), \pi_f)$ of $\mathcal{O}_A$ associated with f in (8.1) is called the *shift representation* of $\mathcal{O}_A$.

Proposition 8.1. *There is the following irreducible decomposition of $l_2(X_A)$ by π_f*

$$l_2(X_A) = \bigoplus_{J \in [1,\ldots,N]_A^*} \mathcal{H}_J \quad \oplus \quad \bigoplus_{K \in [1,\ldots,N]_A^{\infty}} \mathcal{K}_K$$

where

$$\mathcal{H}_J \equiv \overline{\mathrm{Lin}\!< \{e_x : x \in Y_{J^\infty}\} >}, \quad \mathcal{K}_K \equiv \overline{\mathrm{Lin}\!< \{e_x : x \in Y_K\} >},$$

$\{e_x : x \in X_A\}$ is the canonical basis of $l_2(X_A)$, and $Y_L \equiv \{L' \in X_A : L \sim L'\}$ for $L \in X_A$. Furthermore

$$\mathcal{H}_J \sim P(J), \quad \mathcal{K}_K \sim P(K).$$

That is, any irreducible permutative representation appears as a component of $(l_2(X_A), \pi_f)$ once for all. Specially, $(l_2(X_A), \pi_f)$ is multiplicity free.

Proof. Let $K \in X_A$. Then K is either eventually periodic or not. Denote $X_{A,ep}$ by the set of all eventually periodic elements in X_A and $X_{A,nep} \equiv X_A \setminus X_{A,ep}$.

If $K \in X_{A,ep}$, then there are $J_0 \in \{1,\ldots,N\}_A^*$ and $J \in \{1,\ldots,N\}_{A,c}^*$ such that J is non periodic and $J = J_0 \cup J^\infty$. Therefore $Y_K = Y_{J^\infty}$ and $Y_{J^\infty} = \{J_0 \cup J^\infty : J_0 \in \{1,\ldots,N\}_A^*, J_0 \cup J \in \{1,\ldots,N\}_A^*\}$. From this, $X_{A,ep} = \bigoplus_{J \in [1,\ldots,N]_A^*} Y_{J^\infty}$, $f|_{Y_{J^\infty}}$ is a cyclic A-branching function system on Y_{J^∞} and $f_J(J^\infty) = J^\infty$. Therefore $f|_{Y_{J^\infty}}$ is $P(J)$. We have the following decomposition

$$f|_{X_{A,ep}} = \bigoplus_{J \in [1,\ldots,N]_A^*} f|_{Y_{J^\infty}} \quad \sim \bigoplus_{J \in [1,\ldots,N]_A^*} P(J).$$

Assume that $K = (k_n)_{n \in \mathbf{N}} \in X_{A,nep}$ and $x_n \equiv (k_n, k_{n+1}, k_{n+2}, \ldots) \in X_A$ for $n \geq 1$. Then $x_n \in Y_K$ and $Y_K = \bigcup_{n \geq 1} \{J_0 \cup x_n : J_0 \in \{1,\ldots,N\}_A^*,\ J_0 \cup \{k_n\} \in \{1,\ldots,N\}_A^*\}$. Then $X_{A,nep} = \bigoplus_{K \in [1,\ldots,N]_A^{\infty}} Y_K$, $f|_{Y_K}$ is a cyclic A-branching function system on Y_K and $\{x_n\}_{n \in \mathbf{N}}$ is a chain of $f|_{Y_K}$ by K in

Y_K. In consequence,

$$f|_{X_{A,nep}} = \bigoplus_{K\in[1,\ldots,N]_A^\infty} f|_{Y_K} \quad \sim \bigoplus_{K\in[1,\ldots,N]_A^\infty} P(K).$$

Because $X_A = X_{A,ep} \sqcup X_{A,nep}$, we have a cyclic decomposition of f into the direct sum of $\{P(J) : J \in [1,\ldots,N]_A^\#\}$. From this and (3.4), the statement holds. □

By Proposition 8.1, we obtain

$$(l_2(X_A), \pi_f) \sim \bigoplus_{J\in[1,\ldots,N]_A^\#} P(J).$$

By this and § 6.2, $(l_2(X_A), \pi_f)$ is just the atomic representation([**5**]) of $\mathcal{O}_A$ within the compass of a class of permutative representations.

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