HYPERGEOMETRIC GENERATING FUNCTION OF L-FUNCTION, SLATER'S IDENTITIES, AND QUANTUM INVARIANT

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Dedicated to Ludwig Dmitrievich Faddeev on the occasion of his seventieth birthday

ABSTRACT. We study certain connections between the quantum invariants of the torus knots $\mathcal{T}_{3,2^k}$ and some q-series identities. In particular, we obtain new generalizations of Slater's identities (83) and (86).

1. INTRODUCTION

Recent studies reveal an intimate connection between the quantum invariants for 3-manifold and the modular forms. This remarkable observation originates from Refs. 18, 19. In a slightly different context [26], it was discussed that the generating function of Stoimenow's upper bound of the number of the Vassiliev invariant coincides with the half-differential of the Dedekind η-function. From a viewpoint of the quantum invariant, this generating function happens to coincide with Kashaev's invariant [17] for trefoil. This quantum knot invariant was originally defined by use of the quantum dilogarithm function [10], and is a specific value of the colored Jones polynomial [21]. Motivated by this coincidence, it was clarified [11, 13–15] that Kashaev's invariant for torus knot $\mathcal{T}_{s,t}$ and torus link $\mathcal{T}_{2,2m}$ respectively coincides with the Eichler integral of the Virasoro character for the minimal model $\mathcal{M}(s,t)$ and that of the $\widehat{su}(2)_{m-2}$ character.

One of the benefits of the correspondence between the character and the quantum invariant is that we can find new q-series identities. For example, it is well known that the Virasoro character for $\mathcal{M}(2, 2m+1)$ is related to the Gordon–Andrews identity (generalization of the famous Rogers–Ramanujan identity). Motivated from an explicit form of Kashaev's invariant for torus knot $\mathcal{T}_{2,2m+1}$, we obtained a new q-series which may be regarded as a one-parameter extension of the Gordon–Andrews identity [13]. A case of $m = 1$ corresponds to Zagier's q-series identity [26]. See also Refs. 3, 9, 20, 22 for other generalizations of Zagier's identity. Therein constructed were the q-hypergeometric type generating functions of the L-function at the negative integers. Purpose of this article is to propose a q-series identity which is related to Slater's identities.

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We define the Santos polynomial [4, 5] by

$$S_n(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{2k^2} \begin{bmatrix} n \\ 2\,k \end{bmatrix}, \tag{1.1a}$$

$$T_n(q) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} q^{2k(k+1)} \begin{bmatrix} n \\ 2\,k+1 \end{bmatrix}. \tag{1.1b}$$

Definitions of our notation are summarized in Section 2. These polynomials can also be defined recursively by

$$\begin{pmatrix} S_{n+1}(q) \\ T_{n+1}(q) \end{pmatrix} = \begin{pmatrix} 1 & q^{n+1} \\ q^n & 1 \end{pmatrix} \begin{pmatrix} S_n(q) \\ T_n(q) \end{pmatrix}, \tag{1.2}$$

with an initial condition

$$\begin{pmatrix} S_0(q) \\ T_0(q) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

By use of the Santos polynomials, we define the formal q-series $X^{(0)}(q)$ and $X^{(1)}(q)$ by

$$X^{(0)}(q) = \sum_{n=0}^{\infty} (q)_n \, \left(T_n(q) + T_{n+1}(q)\right), \tag{1.3a}$$

$$X^{(1)}(q) = \sum_{n=0}^{\infty} (q)_n \, \left(S_n(q) + S_{n+1}(q)\right). \tag{1.3b}$$

To state one of our main theorems, we introduce the periodic functions as follows;

(1.4a)

$n \mod 24$	5	11	13	19	others
$\chi_{24}^{(0)}(n)$	1	-1	-1	1	0

(1.4b)

$n \mod 24$	1	7	17	23	others
$\chi_{24}^{(1)}(n)$	1	-1	-1	1	0

Theorem 1. *Let $X^{(a)}(q)$ for $a = 0, 1$ be defined by eqs.* (1.3)*. We have an asymptotic expansion in $z \searrow 0$ as*

$$X^{(0)}(\mathrm{e}^{-z}) = \mathrm{e}^{25\,z/48} \sum_{n=0}^{\infty} \frac{t_n^{(0)}}{n!} \left(\frac{z}{48}\right)^n, \tag{1.5a}$$

$$X^{(1)}(\mathrm{e}^{-z}) = \mathrm{e}^{z/48} \sum_{n=0}^{\infty} \frac{t_n^{(1)}}{n!} \left(\frac{z}{48}\right)^n. \tag{1.5b}$$

Here t-series is given in terms of the L-function associated with $\chi_{24}^{(a)}(n)$;

$$\begin{aligned} t_n^{(a)} &= \frac{1}{2}(-1)^{n+1} L(-2n-1, \chi_{24}^{(a)}) \\ &= \frac{1}{2}(-1)^n \frac{24^{2n+1}}{2n+2} \sum_{k=1}^{24} \chi_{24}^{(a)}(k)\, B_{2n+2}(k/24), \end{aligned} \tag{1.6}$$

where $B_n(x)$ is the n-th Bernoulli polynomial.

We note that generating functions of the t-series are written as

$$\frac{\mathrm{sh}(3x)\ \mathrm{sh}(4x)}{\mathrm{sh}(12x)} = \frac{1}{2}\sum_{n=0}^{\infty} \chi_{24}^{(0)}(n)\,\mathrm{e}^{-nx} = \sum_{n=0}^{\infty}(-1)^n \frac{t_n^{(0)}}{(2n+1)!} x^{2n+1},$$

$$\frac{\mathrm{sh}(3x)\ \mathrm{sh}(8x)}{\mathrm{sh}(12x)} = \frac{1}{2}\sum_{n=0}^{\infty} \chi_{24}^{(1)}(n)\,\mathrm{e}^{-nx} = \sum_{n=0}^{\infty}(-1)^n \frac{t_n^{(1)}}{(2n+1)!} x^{2n+1}.$$

Some of the t-series are explicitly given as follows;

n	0	1	2	3	4	5	6
$t_n^{(0)}$	1	119	37201	23181479	24453497761	39286795847639	89443016674567921
$t_n^{(1)}$	2	142	38882	23439022	24521135042	39313934084302	89458458867741602
$t_n^{(2)}$	2	184	53792	32965504	34630287872	55579108685824	126502446478794752

See eq. (4.8) for definition of the t-series $t_n^{(2)}$.

This paper is organized as follows. In Section 2 we collect notations and identities of q-series used in this paper. See *e.g.* Ref. 1. In Section 3 we prove Theorem 1. In Section 4 we study a (nearly) modular property of our q-series. We show that $X^{(a)}(q)$ is regarded as the Eichler integral of the modular form with weight $1/2$, which corresponds to the character of the Virasoro minimal model $\mathcal{M}(3,4)$. We further discuss on a relationship with the quantum knot invariant for the torus knot $\mathcal{T}_{3,4}$. In Section 5 we show several q-hypergeometric type expression of the character of the Virasoro minimal model $\mathcal{M}(3,k)$.

2. Notation and Identities

For our later convention, we give a list of notations and useful identities (see, *e.g.*, Ref. 1);

- q-product and q-binomial coefficient (the Gaussian polynomial),

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} (1 - a\, q^{k-1}),$$

$$(a_1, a_2, \dots, a_k; q)_n = (a_1; q)_n\, (a_2; q)_n \cdots (a_k; q)_n,$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q)_n}{(q)_k\, (q)_{n-k}}, & \text{for } n \geq k \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

- q-binomial formula,

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(a\,z)_\infty}{(z)_\infty}, \tag{2.1}$$

- q-expansion,

$$(z)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-z)^k\, q^{k(k-1)/2}, \tag{2.2}$$

$$\frac{1}{(z)_n} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} z^k, \tag{2.3}$$

- the Euler identity (a limit $n \to \infty$ of eq. (2.2)),

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q)_n} z^n = (-z)_\infty, \tag{2.4}$$

- the Jacobi triple product identity,

$$\sum_{k \in \mathbb{Z}} (-1)^k\, q^{k^2/2}\, x^k = (q, x^{-1} q^{\frac{1}{2}}, x\, q^{\frac{1}{2}}; q)_\infty, \tag{2.5}$$

- the Watson quintuple identity,

$$\sum_{k \in \mathbb{Z}} q^{k(3k-1)/2}\, x^{3k}\, (1 - x\, q^k) = (q, x, q\, x^{-1}; q)_\infty\, (q\, x^2, q\, x^{-2}; q^2)_\infty \tag{2.6}$$

We note that the Watson identity can be proved by use of eq. (2.5) (see *e.g.*, Ref 6).

Hereafter we use properties of the L-function associated with the periodic function.

Lemma 2. *Let $C_f(n)$ be a periodic function with mean value* 0 *and modulus f. Then asymptotic expansion in a limit $t \searrow 0$ is as follows;*

$$\sum_{n=0}^{\infty} n\, C_f(n)\, \mathrm{e}^{-n^2 t} \simeq \sum_{k=0}^{\infty} L(-2\,k-1, C_f)\, \frac{(-t)^k}{k!}, \tag{2.7}$$

where $L(k, C_f)$ is the L-function associated with $C_f(n)$, and is given by

$$L(-k, C_f) = -\frac{f^k}{k+1} \sum_{n=1}^{f} C_f(n)\, B_{k+1}\left(\frac{n}{f}\right).$$

Proof. It is a standard result using the Mellin transformation. See, *e.g.*, Ref. 19. □

3. Proof of Theorem 1

We define functions $H^{(a)}(x) \equiv H^{(a)}(x; q)$ by

$$\begin{aligned} H^{(0)}(x) &= \sum_{n=0}^{\infty} \chi_{24}^{(0)}(n)\, q^{\frac{n^2-25}{48}}\, x^{\frac{n-5}{2}} \qquad (3.1a)\\ &= 1 - q^2 x^3 - q^3 x^4 + q^7 x^7 + q^{17} x^{12} - q^{25} x^{15} - q^{28} x^{16} + \cdots, \end{aligned}$$

$$\begin{aligned} H^{(1)}(x) &= \sum_{n=0}^{\infty} \chi_{24}^{(1)}(n)\, q^{\frac{n^2-1}{48}}\, x^{\frac{n-1}{2}} \qquad (3.1b)\\ &= 1 - q\, x^3 - q^6 x^8 + q^{11} x^{11} + q^{13} x^{12} - q^{20} x^{15} - q^{35} x^{20} + \cdots, \end{aligned}$$

where the periodic function $\chi_{24}^{(a)}(n)$ is defined in eq. (1.4). We easily see that these q-series solve the following q-difference equations;

$$\begin{aligned} H^{(0)}(x) &= 1 - q^2 x^3 - q^3 x^4 H^{(1)}(q\, x),\\ H^{(1)}(x) &= 1 - q\, x^3 - q^6 x^8 H^{(0)}(q\, x). \end{aligned} \qquad (3.2)$$

Proposition 3. *Let the functions $H^{(a)}(x)$ be defined by eqs.* (3.1). *Then for $a = 0, 1$ we have*

$$\begin{aligned} H^{(1-a)}(x) &= \sum_{n=0}^{\infty} (x)_{n+1}\, x^{2n}\\ &\times \left(\sum_{k=0}^{\lfloor (n-a)/2 \rfloor} x^{2k-a} q^{2k(k+a)} \begin{bmatrix} n \\ 2k+a \end{bmatrix} + \sum_{k=0}^{\lfloor (n+1-a)/2 \rfloor} x^{2k+1-a} q^{2k(k+a)} \begin{bmatrix} n+1 \\ 2k+a \end{bmatrix} \right). \end{aligned} \qquad (3.3)$$

Proof. We find that the periodic function (1.4) is related to the Dirichlet character

$$-\chi_{24}^{(0)}(n) + \chi_{24}^{(1)}(n) = \left(\frac{12}{n}\right) \equiv \chi_{12}(n),$$

where we have used the Legendre symbol. We further introduce a q-series $H(x; q)$ as

$$\begin{aligned} -q^{\frac{1}{2}} x^2 H^{(0)}(x) + H^{(1)}(x) &= \sum_{n=0}^{\infty} \chi_{12}(n)\, q^{\frac{n^2-1}{48}}\, x^{\frac{n-1}{2}}\\ &\equiv H(x; q^{\frac{1}{2}}). \end{aligned} \qquad (3.4)$$

As the functions $H^{(a)}(x)$ have integral powers of q, the q-hypergeometric expression may be derived if we know that of $H(x;q^{\frac{1}{2}})$. Fortunately it is known from Refs. 1, 26 that

$$H(x;q^{\frac{1}{2}})=\sum_{n=0}^{\infty}(x;q^{\frac{1}{2}})_{n+1}\,x^n. \tag{3.5}$$

To divide this expression into a sum of $H^{(a)}(x)$, we compute as follows;

$$\begin{aligned}
&\sum_{n=0}^{\infty}(x;q^{\frac{1}{2}})_{n+1}\,x^n\\
&=\sum_{n=0}^{\infty}(x)_{n+1}(x\,q^{\frac{1}{2}})_n\,x^{2n}+\sum_{n=0}^{\infty}(x)_{n+1}(x\,q^{\frac{1}{2}})_{n+1}\,x^{2n+1}\\
&=\sum_{n=0}^{\infty}(x)_{n+1}\,x^{2n}\left(\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}(-1)^j\,x^j\,q^{j^2/2}+\sum_{j=0}^{n+1}\begin{bmatrix}n+1\\j\end{bmatrix}(-1)^j\,x^{j+1}\,q^{j^2/2}\right)\\
&=\sum_{n=0}^{\infty}(x)_{n+1}\,x^{2n}\sum_{k=0}^{\lfloor(n+1)/2\rfloor}\left(\begin{bmatrix}n\\2k\end{bmatrix}+x\begin{bmatrix}n+1\\2k\end{bmatrix}\right)x^{2k}\,q^{2k^2}\\
&\qquad-q^{\frac{1}{2}}\,x^2\sum_{n=0}^{\infty}(x)_{n+1}\,x^{2n-1}\sum_{k=0}^{\lfloor n/2\rfloor}\left(\begin{bmatrix}n\\2k+1\end{bmatrix}+x\begin{bmatrix}n+1\\2k+1\end{bmatrix}\right)x^{2k}\,q^{2k(k+1)}.
\end{aligned}$$

In the second equality, we have used eq. (2.2). We have separated a sum into even and odd parts in both the first and the last equalities. This proves a statement of the proposition. □

Using the q-binomial formula (2.1), we can rewrite eq. (3.3) into

$$\begin{aligned}
H^{(1-a)}(x)&=(q\,x)_\infty\sum_{n=0}^{\infty}\frac{q^{2n(n+a)}}{(x^2\,q)_{2n+a}}\,x^{6n+a-1}\\
&+(1-x)\sum_{n=0}^{\infty}\left((q\,x)_n-(q\,x)_\infty\right)x^{2n}\sum_{k\geq 0}x^{2k-a}\,q^{2k(k+a)}\left(\begin{bmatrix}n\\2k+a\end{bmatrix}+x\begin{bmatrix}n+1\\2k+a\end{bmatrix}\right).
\end{aligned}\tag{3.6}$$

In a limit $x\to 1$, functions $H^{(a)}(x)$ defined by eqs. (3.1) can be written in an infinite product form with a help of the Watson quintuple identity (2.6). Combining with a result (3.6) in a limit $x\to 1$, we recover the following identities.

Corollary 4 (Slater's identities, (83) and (86) [24])**.**

$$(q)_\infty\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q)_{2n+1}}=(q^3,q^5,q^8;q^8)_\infty\cdot(q^2,q^{14};q^{16})_\infty, \tag{3.7a}$$

$$(q)_\infty\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q)_{2n}}=(q,q^7,q^8;q^8)_\infty\cdot(q^6,q^{10};q^{16})_\infty. \tag{3.7b}$$

We obtain the following formulae which come from the next order of $x-1$ of eq. (3.6).

Proposition 5. *We have the following q-series identities;*

$$\frac{1}{2}\sum_{n=0}^{\infty} n\,\chi_{24}^{(0)}(n)\,q^{\frac{n^2-25}{48}}$$
$$= (q^3,q^5,q^8;q^8)_\infty\,(q^2,q^{14};q^{16})_\infty\left(\sum_{k=1}^{\infty}\frac{-q^k}{1-q^k}\right)+(q)_\infty\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q)_{2n+1}}\left(6\,n+\sum_{k=1}^{2n+1}\frac{2\,q^k}{1-q^k}\right)$$
$$-\sum_{n=0}^{\infty}\left((q)_n-(q)_\infty\right)\,\left(T_n(q)+T_{n+1}(q)\right), \quad (3.8a)$$

$$\frac{1}{2}\sum_{n=0}^{\infty} n\,\chi_{24}^{(1)}(n)\,q^{\frac{n^2-1}{48}}$$
$$= (q,q^7,q^8;q^8)_\infty\,(q^6,q^{10};q^{16})_\infty\left(-1-\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}\right)+(q)_\infty\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q)_{2n}}\left(6\,n+\sum_{k=1}^{2n}\frac{2\,q^k}{1-q^k}\right)$$
$$-\sum_{n=0}^{\infty}\left((q)_n-(q)_\infty\right)\,\left(S_n(q)+S_{n+1}(q)\right). \quad (3.8b)$$

Proof. We differentiate eq. (3.6) with respect to x, and then substitute $x\to 1$. □

Proof of Theorem 1. We substitute $q=\mathrm{e}^{-z}$ for eqs. (3.8). As terms which include infinite product terms such as $(q)_\infty$ vanish in a limit $z\searrow 0$, we get formal q-series identities;

$$X^{(0)}(q) = -\frac{1}{2}\sum_{n=0}^{\infty} n\,\chi_{24}^{(0)}(n)\,q^{\frac{n^2-25}{48}}, \quad (3.9a)$$

$$X^{(1)}(q) = -\frac{1}{2}\sum_{n=0}^{\infty} n\,\chi_{24}^{(1)}(n)\,q^{\frac{n^2-1}{48}}. \quad (3.9b)$$

Applying Lemma 2, we obtain the statement of Theorem. □

We see that

$$\chi_{24}^{(0)}(n)+\chi_{24}^{(1)}(n)=\left(\frac{24}{n}\right)\equiv\chi_{24}(n),$$

where we have used the Legendre symbol. Using Zagier's result, we can see that, when the q-series $X(q)$ is defined by

$$X(q)=\sum_{n=0}^{\infty}(-q^{\frac{1}{2}};-q^{\frac{1}{2}})_n, \quad (3.10)$$

we have an asymptotic expansion

$$X(\mathrm{e}^{-z}) = \mathrm{e}^{z/48} \sum_{n=0}^{\infty} \frac{t_n}{n!} \left(\frac{z}{48}\right)^n , \tag{3.11}$$

where

$$\begin{aligned} t_n &= \frac{1}{2}(-1)^{n+1} L(-2n-1, \chi_{24}) \\ &= t_n^{(0)} + t_n^{(1)}. \end{aligned}$$

4. Modularity and Knot Invariant

The q-series which we have studied in preceding sections is related to the modular form as follows. We define

$$\mathbf{\Phi}(\tau) \equiv \begin{pmatrix} \Phi^{(0)}(\tau) \\ \Phi^{(1)}(\tau) \\ \Phi^{(2)}(\tau) \end{pmatrix} = \begin{pmatrix} q^{\frac{25}{48}} H^{(0)}(x=1) \\ q^{\frac{1}{48}} H^{(1)}(x=1) \\ \eta(2\tau) \end{pmatrix}. \tag{4.1}$$

Here $q = \exp(2\pi\,\mathrm{i}\,\tau)$ with τ in the upper half plane $\mathbb{H}$, and we have used the Dedekind η-function,

$$\eta(\tau) = q^{\frac{1}{24}} (q)_\infty = \sum_{n=0}^{\infty} \chi_{12}(n)\, q^{\frac{n^2}{24}}. \tag{4.2}$$

We can find by use of the Poisson summation formula that the vector $\mathbf{\Phi}(\tau)$ is modular with weight $1/2$; the modular S- and T-transformations are respectively written as

$$\mathbf{\Phi}(\tau) = \sqrt{\frac{\mathrm{i}}{\tau}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \cdot \mathbf{\Phi}(-1/\tau) \equiv \sqrt{\frac{\mathrm{i}}{\tau}}\, \mathbf{M} \cdot \mathbf{\Phi}(-1/\tau), \tag{4.3}$$

$$\mathbf{\Phi}(\tau+1) = \begin{pmatrix} \mathrm{e}^{\frac{25}{24}\pi\mathrm{i}} & & \\ & \mathrm{e}^{\frac{1}{24}\pi\mathrm{i}} & \\ & & \mathrm{e}^{\frac{1}{6}\pi\mathrm{i}} \end{pmatrix} \mathbf{\Phi}(\tau). \tag{4.4}$$

These modular forms represent the character of the minimal model $\mathcal{M}(3,4)$, *i.e.*, the Ising model (see, *e.g.*, Refs. 16, 23.).

We consider an asymptotic behavior of $X^{(a)}(q)$ when q is near at the N-th primitive root of unity. Hereafter we use

$$\omega = \mathrm{e}^{2\pi\mathrm{i}/N}.$$

We define the Eichler integral of $\mathbf{\Phi}(\alpha)$ for $\alpha \in \mathbb{Q}$ by

$$\widetilde{\mathbf{\Phi}}(\alpha) \equiv \begin{pmatrix} \widetilde{\Phi}^{(0)}(\alpha) \\ \widetilde{\Phi}^{(1)}(\alpha) \\ \widetilde{\eta}(2\,\alpha) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\frac{25}{24}\pi\mathrm{i}\alpha}\, X^{(0)}(\mathrm{e}^{2\pi\mathrm{i}\alpha}) \\ \mathrm{e}^{\frac{1}{24}\pi\mathrm{i}\alpha}\, X^{(1)}(\mathrm{e}^{2\pi\mathrm{i}\alpha}) \\ \mathrm{e}^{\frac{1}{6}\pi\mathrm{i}\alpha}\, X^{(2)}(\mathrm{e}^{2\pi\mathrm{i}\alpha}) \end{pmatrix}, \tag{4.5}$$

where $X^{(0)}(q)$ and $X^{(1)}(q)$ are defined in eqs. (1.3), and

$$X^{(2)}(q) = 2\sum_{n=0}^{\infty}(q^2;q^2)_n. \tag{4.6}$$

One sees that $\widetilde{\mathbf{\Phi}}(\alpha)$ converges to finite value for $\alpha \in \mathbb{Q}$ as an infinite sum terminates at finite order due to $(q)_n$.

As was proved in Ref. 15 (see also Refs. 19,26), we have an asymptotic behavior of the Eichler integral of the modular form of weight $1/2$.

Theorem 6 ([15]). *For $N \in \mathbb{Z}_{>0}$, we have an asymptotic expansion in $N \to \infty$ as*

$$\widetilde{\mathbf{\Phi}}(1/N) + (-\mathrm{i}\,N)^{\frac{3}{2}}\ \mathbf{M}\cdot\widetilde{\mathbf{\Phi}}(-N) \simeq \sum_{n=0}^{\infty}\frac{\mathbf{t}_n}{n!}\left(\frac{\pi}{24\,\mathrm{i}\,N}\right)^n, \tag{4.7}$$

where $\mathbf{M}$ is the 3×3 matrix defined in eq. (4.3). We mean that t-series is

$$\mathbf{t}_n = \begin{pmatrix} t_n^{(0)} \\ t_n^{(1)} \\ t_n^{(2)} \end{pmatrix},$$

where $t_n^{(0)}$ and $t_n^{(1)}$ are given in eq. (1.6), and

$$\begin{aligned} t_n^{(2)} &= -(-4)^n\, L(-2\,n-1,\,\chi_{12}) \\ &= \frac{1}{2}\,(-1)^n\,\frac{24^{2n+1}}{2\,n+2}\sum_{k=1}^{12}\chi_{12}(k)\,B_{2n+2}(k/12). \end{aligned} \tag{4.8}$$

It is noted that the Eichler integrals $\widetilde{\mathbf{\Phi}}(N)$ at $N \in \mathbb{Z}$ in the left hand side of eq. (4.7) are computed as

$$\widetilde{\mathbf{\Phi}}(N) = \begin{pmatrix} \mathrm{e}^{\frac{25}{24}\pi\mathrm{i}N} \\ 2\,\mathrm{e}^{\frac{1}{24}\pi\mathrm{i}N} \\ 2\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}N} \end{pmatrix}. \tag{4.9}$$

Proposition 7 ([15]). *Kashaev's invariant $\langle\mathcal{K}\rangle_N$ for torus knot $\mathcal{K} = \mathcal{T}_{3,4}$ (see Fig. 1) is proportional to $X^{(0)}(\omega)$;*

$$\omega^3\, X^{(0)}(\omega) = \langle\mathcal{T}_{3,4}\rangle_N. \tag{4.10}$$

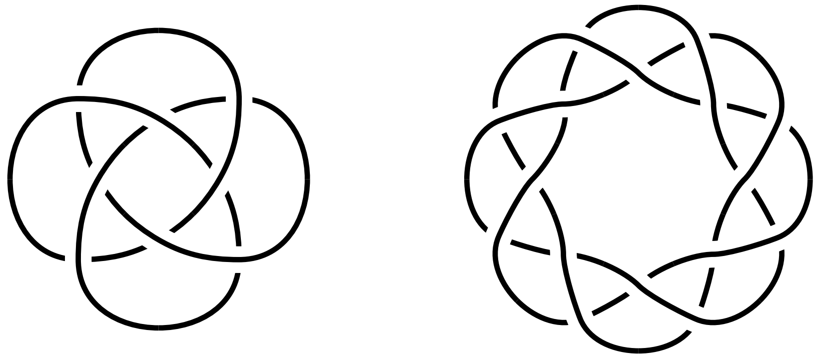

FIGURE 1. Torus knots, $\mathcal{T}_{3,4}$ and $\mathcal{T}_{3,8}$.

See also Ref. 12 for a connection with the colored Jones polynomial with generic q, the Alexander polynomial, A-polynomial, for the torus knot.

The quantum invariant such as the colored Jones polynomial was identified with the Chern–Simons path integral [25]. The asymptotic behavior thereof is known to be related to the classical topological invariants such as the Reidemeister torsion and the Chern–Simons invariant. Looking at the nearly modular property (4.7) with eq. (4.10), the limiting value of the Eichler integral (4.9) at $\tau \to N \in \mathbb{Z}$ can be identified with the Chern–Simons invariant of the torus knot $\mathcal{T}_{3,4}$ [15].

5. APPLICATIONS

In our previous paper [15], we have demonstrated that Kashaev's invariant for the torus knot $\mathcal{T}_{s,t}$ where s and t are relatively prime positive integers (see *e.g.* Fig. 1) is regarded as the Eichler integral of the Virasoro character of the minimal model $\mathcal{M}(s,t)$. Prop. 7 is for a case of $(s,t) = (3,4)$. The Virasoro character $\mathrm{ch}^{s,t}_{n,m}(\tau)$ of $\mathcal{M}(s,t)$ for an irreducible highest weight with conformal weight $\Delta^{s,t}_{n,m} = \frac{(nt-ms)^2-(s-t)^2}{4st}$ for $1 \leq n \leq s-1$ and $1 \leq m \leq t-1$, is known to be [23]

$$\mathrm{ch}^{s,t}_{n,m}(\tau) = \frac{\Phi^{(n,m)}_{s,t}(\tau)}{\eta(\tau)}. \tag{5.1}$$

Here we have

$$\Phi^{(n,m)}_{s,t}(\tau) = \sum_{k=0}^{\infty} \chi^{(n,m)}_{2st}(k)\, q^{\frac{k^2}{4st}}, \tag{5.2}$$

with a periodic function

$k \mod 2st$	$nt-ms$	$nt+ms$	$2st-(nt+ms)$	$2st-(nt-ms)$	others
$\chi^{(n,m)}_{2st}(k)$	1	-1	-1	1	0

The function $\Phi^{(n,m)}_{s,t}(\tau)$ is modular covariant with weight $1/2$ [8,16], and spans a $(s-1)(t-1)/2$-dimensional space due to a symmetry $\mathrm{ch}^{s,t}_{n,m}(\tau) = \mathrm{ch}^{s,t}_{s-n,t-m}(\tau)$. Then Kashaev's invariant $\langle\mathcal{K}\rangle_N$

for a torus knot $\mathcal{K} = \mathcal{T}_{s,t}$ is written in terms of the Eichler integral $\widetilde{\Phi}_{s,t}^{(1,t-1)}(1/N)$, which is defined by the half-differential of the modular form as

$$\widetilde{\Phi}_{s,t}^{(n,m)}(\tau) = -\frac{1}{2}\sum_{k=0}^{\infty} k\, \chi_{2st}^{(n,m)}(k)\, q^{\frac{k^2}{4st}}, \tag{5.3}$$

where $\tau \in \mathbb{H}$. The limiting value at $\tau \to 1/N \in \mathbb{Q}$ is computed as

$$\widetilde{\Phi}_{s,t}^{(n,m)}(1/N) = \frac{s\,t\,N}{2}\sum_{k=1}^{2stN} \chi_{2st}^{(n,m)}(k)\, \mathrm{e}^{\frac{k^2}{2stN}\pi \mathrm{i}}\, B_2\left(\frac{k}{2\,s\,t\,N}\right). \tag{5.4}$$

We consider the q-series identities associated with (the Eichler integral of) the minimal model $\mathcal{M}(3,k)$. As we have discussed in preceding sections, the character of the minimal model $\mathcal{M}(3,4)$ is related to Slater's identities, and we can expect that such identities could be constructed. In fact there are multi-variable generalizations of Slater's identities due to G. Andrews [2]. These q-series identities may be interpreted as fermionic formulae for the characters of the Virasoro algebra. The q-series identities which come from the computation of the quantum invariants of the corresponding torus knots $\mathcal{T}_{3,k}$ are rather messy.

To study q-series identities associated to the character of the minimal model $\mathcal{M}(s,t)$, it will be convenient to define a function [12, 15]

$$H_{s,t}^{(n,m)}(x) \equiv H_{s,t}^{(n,m)}(x;q) = \sum_{k=0}^{\infty} \chi_{2st}^{(n,m)}(k)\, q^{\frac{k^2-(nt-ms)^2}{4st}}\, x^{\frac{k-|nt-ms|}{2}}. \tag{5.5}$$

See that

$$\Phi_{s,t}^{(n,m)}(\tau) = q^{\frac{(nt-ms)^2}{4st}}\, H_{s,t}^{(n,m)}(1),$$

and that $H^{(a)}(x)$ for $a = 0, 1$ defined in eq. (3.1) is nothing but $H_{s,t}^{(n,m)}(x)$ with $(s,t) = (3,4)$ and $(n,m) = (1,3), (1,1)$, respectively. We demonstrate below how our function (5.5) does work for investigations of the q-hypergeometric type functions associated to the minimal Virasoro characters. It is noted that linear combinations among characters for minimal Virasoro model was studied in Ref. 7 in a different method.

5.1 $\mathcal{M}(3, 2^p)$

First we pay attention to a case of $(s,t) = (3, 2\,k)$. The periodic function satisfies

$$\chi_{12}(2\,k+3)\,\chi_{12}(n) = \sum_{a=0}^{k-1} (-1)^{a-1}\, \chi_{12k}^{(1,2a+1)}(n). \tag{5.6}$$

Using this identity, we have the following proposition;

Proposition 8. *Let the function $H(x;q)$ be defined by eq.* (3.4)*. We have*

$$\chi_{12}(2k+3)\, H(x;q^{\frac{1}{k}}) = \sum_{a=0}^{k-1} (-1)^{a-1}\, q^{\ell_k(a)} x^{\frac{|2k-6a-3|-1}{2}}\, H_{3,2k}^{(1,2a+1)}(x;q), \tag{5.7}$$

where

$$\ell_k(a) = \frac{(k-3a-1)(k-3a-2)}{6k}.$$

When $k=2^p$, we see that $\ell_k(a)-\ell_k(b)\notin\mathbb{Z}$ for all a and b satisfying $0<a\neq b\leq 2k-1$. Thus we can extract $H_{3,2^{p+1}}^{(1,2a+1)}(x;q)$ from above lemma for a case of $\mathcal{M}(3,2^{p+1})$ following a method of Prop. 3.

We take an example $\mathcal{M}(3,8)$. We have

$$H(x;q^{\frac{1}{4}}) = H_{3,8}^{(1,3)}(x) - q^{\frac{1}{4}}\, x^2\, H_{3,8}^{(1,1)}(x) - q^{\frac{1}{2}} x^3\, H_{3,8}^{(1,5)}(x) + q^{\frac{7}{4}}\, x^6\, H_{3,8}^{(1,7)}(x), \tag{5.8a}$$

$$H_{3,8}^{(1,2)}(x) = H^{(1)}(x^2;q^2), \tag{5.8b}$$

$$H_{3,8}^{(1,6)}(x) = H^{(0)}(x^2;q^2), \tag{5.8c}$$

$$H_{3,8}^{(1,4)}(x) = H(x^4;q^4). \tag{5.8d}$$

From a result in Section 3 we already have a q-hypergeometric expression of $H_{3,8}^{(1,a)}(x)$ for even a cases. For odd a cases, we can derive q-hypergeometric functions from eq. (5.8a) by the same method with previous section. We have

$$\begin{aligned}
H_{3,8}^{(1,1)}(x) &= -q^{-\frac{1}{4}}\, x^{-2}\, I^{(1)}(x),\\
H_{3,8}^{(1,3)}(x) &= I^{(0)}(x),\\
H_{3,8}^{(1,5)}(x) &= -q^{-\frac{1}{2}}\, x^{-3}\, I^{(2)}(x),\\
H_{3,8}^{(1,7)}(x) &= q^{-\frac{7}{4}}\, x^{-6}\, I^{(3)}(x).
\end{aligned}$$

where

$$\begin{aligned}
I^{(a)}(x) = \sum_{k=0}^{\infty} (x)_{k+1}\, x^{4k} & \sum_{1\geq\epsilon_1\geq\epsilon_2\geq\epsilon_3\geq 0} x^{\epsilon_1+\epsilon_2+\epsilon_3}\\
&\times \sum_{\substack{\ell_1,\ell_2,\ell_3\geq 0\\ \ell_1+2\ell_2+3\ell_3=a \mod 4}} (-x)^{\ell_1+\ell_2+\ell_3}\, q^{\frac{\ell_1+2\ell_2+3\ell_3}{4}} \left(\prod_{j=1}^{3} \begin{bmatrix} k+\epsilon_j\\ \ell_j\end{bmatrix} q^{\frac{1}{2}\ell_j(\ell_j-1)}\right).
\end{aligned}$$

5.2 $\mathcal{M}(3,5)$

We list some formulae concerning $H_{s,t}^{(n,m)}(x)$ for $(s,t)=(3,5)$ defined in eq. (5.5). In this case we have four independent functions $(n,m)=(1,1)$, $(1,2)$, $(1,3)$ and $(1,4)$, and they satisfy the following difference equations;

$$H_{3,5}^{(1,4)}(x)=1-q^2\,x^3-q^4\,x^5\cdot H_{3,5}^{(1,1)}(q\,x),$$

$$H_{3,5}^{(1,3)}(x)=1-q^3\,x^5-q^4\,x^6\cdot H_{3,5}^{(1,2)}(q\,x),$$

$$H_{3,5}^{(1,2)}(x)=1-q^2\,x^5-q^6\,x^9\cdot H_{3,5}^{(1,3)}(q\,x),$$

$$H_{3,5}^{(1,1)}(x)=1-q\,x^3-q^8\,x^{10}\cdot H_{3,5}^{(1,4)}(q\,x),$$

We note that

$$-q^{\frac{3}{4}}\,x^{\frac{5}{2}}\,H_{3,5}^{(1,4)}(x)+H_{3,5}^{(1,1)}(x)=H_{3,5}^{(1,3)}(x^{\frac{1}{2}};q^{\frac{1}{4}}),$$

$$-q^{\frac{1}{4}}\,x^{\frac{3}{2}}\,H_{3,5}^{(1,3)}(x)+H_{3,5}^{(1,2)}(x)=H_{3,5}^{(1,1)}(x^{\frac{1}{2}};q^{\frac{1}{4}}),$$

and that the Watson quintuple identity (2.6) proves

$$H_{3,5}^{(1,4)}(1)=(q^2,q^{18};q^{20})_\infty\,(q^4,q^6,q^{10};q^{10})_\infty,$$

$$H_{3,5}^{(1,3)}(1)=(q^4,q^{16};q^{20})_\infty\,(q^3,q^7,q^{10};q^{10})_\infty,$$

$$H_{3,5}^{(1,2)}(1)=(q^6,q^{14};q^{20})_\infty\,(q^2,q^8,q^{10};q^{10})_\infty,$$

$$H_{3,5}^{(1,1)}(1)=(q^8,q^{12};q^{20})_\infty\,(q,q^9,q^{10};q^{10})_\infty.$$

Though we do not obtain a q-series identity which appeared in Slater's 130 identities, we obtain the following formulae.

Proposition 9.

$$H_{3,5}^{(1,4)}(x)=\sum_{n=0}^{\infty}(-q^4\,x^5;q^{10})_n\,\left(-q^4\,x^5\right)^n-q^2\,x^3\sum_{n=0}^{\infty}(-q^6\,x^5;q^{10})_n\,\left(-q^6\,x^5\right)^n \tag{5.9}$$

$$\begin{aligned}&=\sum_{n=0}^{\infty}(-q^2\,x^3;q^6)_n\,\left(-q^2\,x^3\right)^{n+2c}\sum_{c=0}^{n}q^{6c^2}\begin{bmatrix}n\\c\end{bmatrix}_{q^6}\\&\qquad-q^4\,x^5\sum_{n=0}^{\infty}(-q^4\,x^3;q^6)_n\,\left(-q^4\,x^3\right)^{n+2c}\sum_{c=0}^{n}q^{6c^2}\begin{bmatrix}n\\c\end{bmatrix}_{q^6}\end{aligned} \tag{5.10}$$

$$=\sum_{n,c=0}^{\infty}(-1)^n q^{\frac{5}{2}n(n+1)+5c(c+1)-n-c}\,x^{5(n+c)}\,\left(1-q^{2(n+c+1)}\,x^3\right)\begin{bmatrix}n\\c\end{bmatrix}_{q^5} \tag{5.11}$$

$$
\begin{aligned}
H_{3,5}^{(1,3)}(x) &= -q^3\,x^5 \sum_{n,c=0}^{\infty} (-q\,x^3;\,q^6)_{n+1} \left(-q\,x^3\right)^{n+2c} q^{6c(c+1)} \begin{bmatrix} n \\ c \end{bmatrix}_{q^6} \\
&\qquad + \sum_{n,c=0}^{\infty} (-q^{-1}\,x^3;\,q^6)_{n+1} \left(-q^{-1}\,x^3\right)^{n+2c} q^{6c(c+1)} \begin{bmatrix} n \\ c \end{bmatrix}_{q^6} \qquad (5.12) \\
&= \sum_{n,c=0}^{\infty} (-1)^n q^{\frac{5}{2}n(n+1)+5c(c+1)-2(n+c)}\,x^{5(n+c)} \left(1-q^{4(n+c+1)}\,x^6\right) \begin{bmatrix} n \\ c \end{bmatrix}_{q^5} \qquad (5.13)
\end{aligned}
$$

$$
\begin{aligned}
H_{3,5}^{(1,2)}(x) &= \sum_{n=0}^{\infty}(-q^2\,x^5;\,q^{10})_n \left(-q^2\,x^5\right)^n - q^6\,x^9 \sum_{n=0}^{\infty}(-q^8\,x^5;\,q^{10})_n \left(-q^8\,x^5\right)^n \qquad (5.14) \\
&= \sum_{n,c=0}^{\infty} (-1)^n q^{\frac{5}{2}n(n+1)+5c(c+1)-3(n+c)}\,x^{5(n+c)} \left(1-q^{6(n+c+1)}\,x^9\right) \begin{bmatrix} n \\ c \end{bmatrix}_{q^5} \qquad (5.15)
\end{aligned}
$$

$$
\begin{aligned}
H_{3,5}^{(1,1)}(x) &= 1 - x^{-2} + x^{-2} \sum_{n,c=0}^{\infty} (-1)^n q^{\frac{5}{2}n(n+1)+5c(c+1)-4(n+c)}\,x^{5(n+c)} \left(1-q^{8(n+c+1)}\,x^{12}\right) \begin{bmatrix} n \\ c \end{bmatrix}_{q^5} \qquad (5.16) \\
&= -q\,x^3 \sum_{n=0}^{\infty}(-q\,x^5;\,q^{10})_{n+1} \left(-q\,x^5\right)^n + \sum_{n=0}^{\infty}(-q^{-1}\,x^5;\,q^{10})_{n+1} \left(-q^{-1}\,x^5\right)^n \qquad (5.17)
\end{aligned}
$$

Proof. We recall results from Refs. 13, 26;

$$
\sum_{n=0}^{\infty} \chi_6(n)\,q^{\frac{n^2-1}{24}}\,x^{\frac{n-1}{2}} = \sum_{n=0}^{\infty}(-x)_{n+1}\,(-x)^n, \qquad (5.18)
$$

$$
\sum_{n=0}^{\infty} \chi_{10}^{(0)}(n)\,q^{\frac{n^2-9}{40}}\,x^{\frac{n-3}{2}} = \sum_{n=0}^{\infty}(-x)_{n+1}\,(-x)^n \sum_{c=0}^{n} q^{c(c+1)}\,x^{2c} \begin{bmatrix} n \\ c \end{bmatrix}, \qquad (5.19)
$$

$$
\sum_{n=0}^{\infty} \chi_{10}^{(1)}(n)\,q^{\frac{n^2-1}{40}}\,x^{\frac{n-1}{2}} = \sum_{n=0}^{\infty}(-x)_{n+1}\,(-x)^n \sum_{c=0}^{n+1} q^{c^2}\,x^{2c} \begin{bmatrix} n+1 \\ c \end{bmatrix}. \qquad (5.20)
$$

Here the periodic functions are meant to be

$n \mod 6$	1	5	others
$\chi_6(n)$	1	-1	0

$n \mod 10$	3	7	others
$\chi_{10}^{(0)}(n)$	1	-1	0

$n \mod 10$	1	9	others
$\chi_{10}^{(1)}(n)$	1	-1	0

Using eq. (5.18), $\chi_{30}^{(1,1)}(n) = -\chi_6(\frac{n-3}{5}) + \chi_6(\frac{n+3}{5})$, and $\chi_{30}^{(1,2)}(n) = -\chi_6(\frac{n-6}{5}) + \chi_6(\frac{n+6}{5})$, we obtain eqs. (5.17) and (5.14). Eq. (5.9) also follows in the same manner. Eqs. (5.10) and (5.12) follow from eqs. (5.19), (5.20), $\chi_{30}^{(1,3)}(n) = -\chi_{10}^{(0)}(\frac{n-5}{3}) + \chi_{10}^{(0)}(\frac{n+5}{3})$, and $\chi_{30}^{(1,4)}(n) = -\chi_{10}^{(1)}(\frac{n-10}{3}) + \chi_{10}^{(1)}(\frac{n+10}{3})$.

To prove remaining identities, we use a formula [14],

$$\sum_{n=0}^{\infty} \widetilde{\chi}_6(n)\, q^{\frac{n^2-4}{12}}\, x^{\frac{n-2}{2}} = \sum_{n=0}^{\infty} (-1)^n\, q^{\frac{1}{2}n(n+1)} \sum_{c=0}^{n} x^{n+c}\, q^{c(c+1)} \begin{bmatrix} n \\ c \end{bmatrix}, \tag{5.21}$$

with

$n \mod 6$	2	4	others
$\widetilde{\chi}_6(n)$	1	-1	0

Combining with a fact that $\chi_{30}^{(1,5-a)}(n) = -\widetilde{\chi}_6(\frac{n-3a}{5}) + \widetilde{\chi}_6(\frac{n+3a}{5})$ for $a = 1, 2, 3$, we can complete the proof. □

We should remark that an identity (5.21) gives

$$H_{3,t}^{(1,m)}(x) = \begin{cases} I_t^{(m)}(x), & \text{for } t - 3m < 0, \\ 1 - x^{3m-t} + x^{3m-t}\, I_t^{(m)}(x), & \text{for } t - 3m > 0, \end{cases} \tag{5.22}$$

where

$$I_t^{(m)}(x) = \sum_{n,c=0}^{\infty} (-1)^n\, q^{\frac{t}{2}n(n+1)+tc(c+1)-(t-m)(n+c)}\, x^{t(n+c)} \left(1 - q^{2(t-m)(c+n+1)}\, x^{3(t-m)}\right) \begin{bmatrix} n \\ c \end{bmatrix}_{q^t}.$$

Here we have set $t > 0$ and $1 \le m < t$ such that $(3, t) = 1$, and $H_{3,t}^{(1,m)}(x = 1)$ is a $(t-1)$-dimensional representation of the modular group.

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