WKB analysis of higher order Painlevé equations with a large parameter — Local reduction of 0-parameter solutions for Painlevé hierarchies \( (P_J) \) \( (J = I, II-1 \text{ or } II-2) \)

*To be dedicated to Professor H. Komatsu on his seventieth birthday*

by

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§0. Introduction

This paper is the second of a series of articles on WKB analysis of higher order Painlevé equations with a large parameter. In the first of the series ([KKNT]) we studied the geometric aspect of the Painlevé hierarchy \((P_J)\) \((J = I, \text{II}-1 \text{ or II}-2)\) with a large parameter, and in this article we begin to analyze the WKB-theoretic structure of each member \((P_J)_m\) \((J = I, \text{II}-1 \text{ or II}-2; m = 1, 2, 3, \ldots)\) of the hierarchy. To be concrete, we show that a 0-parameter solution of \((P_J)_m\) \((J = I, \text{II}-1 \text{ or II}-2; m = 1, 2, 3, \ldots)\) constructed in [KKNT] and [N] can be reduced to a 0-parameter solution of \((P_J)_1\), the traditional (i.e., second order) Painlevé-I equation \((P_1)\) with a large parameter, i.e.,

\[
\frac{d^2 \lambda}{dt^2} = \eta^2 (6\lambda^2 + t),
\]

with the aid of a formal transformation defined near a turning point of \((P_J)_m\) of the first kind in the sense of [KKNT]. (See Theorem 3.2.1 in Section 3 for the precise statement.) Throughout this paper we use the same notions and notations used in [KKNT]. A résumé of this paper is given in [KT2].

An important step of our reasoning in this paper is to derive a pair of Schrödinger equation \((SL_J)_m\) and its deformation equation \((D_J)_m\) from the Lax pair \((L_J)_m\) associated with the \(m\)-th member \((P_J)_m\) of the Painlevé hierarchy in question. Here we make essential use of the fact that \((L_J)_m\) consists of \(2 \times 2\) systems. (See Section 1 for the details.) Once we obtain the simultaneous equations \((SL_J)_m\) and \((D_J)_m\) for one unknown function, we can employ the techniques used in [KT1]; we first establish some analyticity properties of the odd part of a solution of the Riccati equation attached to \((SL_J)_m\) (Theorem 2.1) and then in Proposition 3.2.1 we construct a semi-global transformation that brings \((SL_J)_m\) to \((SL_1)\), the Schrödinger equation underlying \((P_1)\) (cf. [KT1]). In constructing the semi-global transformation we need some matching conditions, and the constructed semi-global transformation together with the matching conditions is used to reduce the 0-parameter solution in question to a 0-parameter solution of \((P_1)\). (Theorem 3.2.1.) We note that the actual reduction is divided into two steps: we first solve an algebraic equation of degree \(m\) whose coefficients are defined in terms of a 0-parameter solution of \((P_J)_m\) to find some formal series \(b_j(t, \eta)\) \((j = 1, \ldots, m)\), and we then employ the analytic machinery of semi-globally transforming \((SL_J)_m\) to \((SL_1)\) so that we may reduce \(b_j\) that is relevant to the turning point of
$(P_j)_m$ in question to a 0-parameter solution of $(P_1)$, with the help of the constructed semi-global transformation. We discuss the geometric meaning of $b_j$ in Section 1.

In ending this introduction we want to repeat the same comment as that given in [KT1]: it is probably worth emphasizing that the above reduction is attained through the study of $(SLJ)_m$, a differential equation on the extended $(x, t)$-space, despite the fact that the required relation is relevant only to the $t$-variable.

§1. Derivation of a Schrödinger equation $(SLJ)_m$
and its deformation equation $(DJ)_m$

§1.1. The case $J = 1$

For the convenience of the reader, we first recall the definition of $(P_1)_m$ and the underlying Lax pair $(L_1)_m$. See [KKNT] and [S] for their backgrounds.

Definition 1.1.1. The $m$-th member of $P_1$-hierarchy with a large parameter $\eta$ is the following system of non-linear differential equations:

\[
\begin{align*}
\frac{du_j}{dt} &= 2\eta v_j \quad (j = 1, \ldots, m) \\
\frac{dv_j}{dt} &= 2\eta (u_{j+1} + u_j + w_j) \quad (j = 1, \ldots, m) \\
u_{m+1} &= 0
\end{align*}
\]  

(1.1.1)

where $w_j$ is a polynomial of $u_k$ and $v_l$ ($1 \leq k, l \leq j$) that is determined by the following recursive relation:

\[
w_j = \frac{1}{2} \left( \sum_{k=1}^{j} u_k u_{j+1-k} \right) + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2} \left( \sum_{k=1}^{j-1} u_k v_{j-k} \right) + c_j + \delta_{jm}t \quad (j = 1, \ldots, m).
\]  

(1.1.2)

Here $c_j$ is a constant and $\delta_{jm}$ stands for Kronecker’s delta.
**Definition 1.1.2.** The Lax pair \((L_1)_m\) underlying \((P_1)_m\) is the following pair of linear differential equations on \((x, t)\)-space:

\[
(L_1)_m \begin{cases}
    \left( \frac{\partial}{\partial x} - \eta A \right) \vec{\psi} = 0, \\
    \left( \frac{\partial}{\partial t} - \eta B \right) \vec{\psi} = 0,
\end{cases}
\]

where \(\vec{\psi} = \vec{t}(\psi_1, \psi_2),\)

\[
A = \begin{pmatrix} V(x)/2 & U(x) \\ (2x^{m+1} - xU(x) + 2W(x))/4 & -V(x)/2 \end{pmatrix}
\]

and

\[
B = \begin{pmatrix} 0 & 2 \\ u_1 + x/2 & 0 \end{pmatrix}
\]

with

\[
U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j},
\]

\[
V(x) = \sum_{j=1}^{m} v_j x^{m-j}
\]

and

\[
W(x) = \sum_{j=1}^{m} w_j x^{m-j}.
\]

**Remark 1.1.1.** As is proved in Proposition 1.1.1 of [KKNT], \((P_1)_m\) states the compatibility condition for \((L_1)_m\).

**Remark 1.1.2.** Combining (1.1.1a), (1.1.1b) and (1.1.2), we find that \(u_{j+1}\) \((j \leq m - 1)\) is a polynomial of \(u_1, \ldots, u_j, du_1/dt, \ldots, du_j/dt\) and \(d^2 u_j/dt^2\). Hence \(u_{j+1}\) \((j \leq m - 1)\) is a polynomial of \(u_1, du_1/dt, \ldots, d^2u_1/dt^2\). Substituting these polynomials into

\[
\frac{d^2 u_m}{dt^2} = 4\eta^2 (u_1 u_m + w_m),
\]
we obtain a $2m$-th order differential equation for $u_1$. It is also clear that, once a solution $u_1$ of the $2m$-th order differential equation is given, we can find $(u_1, \ldots, u_m; v_1, \ldots, v_m; w_1, \ldots w_m)$ so that they satisfy (1.1.1) and (1.1.2). Thus $(P_1)_m$ is equivalent to a single $2m$-th order differential equation. The explicit form of the resulting equation for $m = 1$ is

\begin{equation}
(1.1.10) \quad d^2u_1/dt^2 = \eta^2(6u_1^2 + 4c_1 + 4t),
\end{equation}

and that for $m = 2$ is

\begin{equation}
(1.1.11) \quad d^4u_1/dt^4 = \eta^2(20u_1d^2u_1/dt^2 + 10(du_1/dt)^2)
+ \eta^4(-40u_1^3 - 16c_1u_1 + 16c_2 + 16t).
\end{equation}

It is clear that the scaling

\begin{equation}
(1.1.12) \quad \tilde{t} = \alpha(t + c_1) \quad \text{and} \quad \lambda = \alpha^3u_1/4 \quad \text{with} \quad \alpha = 4^{1/5}
\end{equation}

brings (1.1.10) into

\begin{equation}
(1.1.13) \quad d^2\lambda/dt^2 = \eta^2(6\lambda^2 + \tilde{t}),
\end{equation}

the traditional Painlevé-I equation $(P_1)$ with a large parameter $\eta$. These facts explain why (1.1.1) is called $(P_1)$-hierarchy, or often with some abuse of language, a higher order Painlevé-I equation.

Let us first write down the equation that the first component $\psi_1$ of a solution $\psi$ of (1.1.3a) satisfies:

\begin{equation}
(1.1.14) \quad \left( \frac{\partial^2}{\partial x^2} - \frac{U_x}{U} \frac{\partial}{\partial x} - \frac{\eta^2}{4}((2x^{m+1} - xU + 2W)U + V^2) + \frac{\eta}{2}(U_xV/U - V_x) \right) \psi_1 = 0.
\end{equation}

Next we eliminate the term $-U_xU^{-1}\partial\psi_1/\partial x$ by introducing $\psi$ by

\begin{equation}
(1.1.15) \quad \exp\left(-\int_x^U \frac{U_x}{2U} dx\right) \psi_1 = \frac{1}{\sqrt{U}}\psi_1;
\end{equation}

the resulting equation for $\psi$ is

\begin{equation}
(1.1.16) \quad \frac{\partial^2 \psi}{\partial x^2} = \eta^2Q_{(1,m)}\psi,
\end{equation}

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where

\[
Q_{(1,m)} = \frac{1}{4}(2x^{n+1} - xU + 2W)U + \frac{1}{4}V^2 - \frac{\eta^{-1}U_x V}{2U} + \frac{\eta^{-1}V_x}{2} + \frac{3\eta^{-2}U_x^2}{4U^2} - \frac{\eta^{-2}U_{xx}}{2U}.
\]

On the other hand, (1.1.3.b) implies

\[
\frac{\partial \psi_1}{\partial t} = 2\eta \psi_2,
\]

and it also follows from (1.1.3.a) that

\[
\frac{\partial \psi_1}{\partial x} = \frac{\eta V}{2} \psi_1 + \eta U \psi_2.
\]

Hence we find

\[
\frac{\partial \psi_1}{\partial x} = \frac{\eta V}{2} \psi_1 + \frac{U}{2} \frac{\partial \psi_1}{\partial t}.
\]

Therefore we obtain

\[
\psi_x = \frac{1}{2}U \psi_t + \left(\frac{1}{4}U_t + \frac{1}{2} \eta V - \frac{1}{2}U_x U^{-1}\right) \psi.
\]

It also follows from (1.1.1), (1.1.6) and (1.1.7) that

\[
U_t = -2\eta V.
\]

Thus we conclude

\[
\frac{\partial \psi}{\partial t} = a_{(1,m)} \frac{\partial \psi}{\partial x} - \frac{1}{2} \frac{\partial a_{(1,m)}}{\partial x} \psi,
\]

where

\[
a_{(1,m)} = \frac{2}{U(x)}.
\]

Thus we have arrived at simultaneous equations (1.1.16) and (1.1.23) for one unknown function \(\psi\). This is the setting that [KT1] used to establish a reduction theorem for 0-parameter solutions of the traditional Painlevé equations. In what follows, the equation (1.1.16) (resp., (1.1.23)) is referred
to as \( (SL_1)_m \) (resp., \( (D_1)_m \)), and we analyze these equations by substituting a 0-parameter solution \( (\hat{u}_j, \hat{v}_j) \) \( 1 \leq j \leq m \) of \( (P_1)_m \) into their coefficients. The existence and basic properties of a 0-parameter solution of \( (P_1)_m \) are shown in [KKNT]; it is a formal series in \( \eta^{-1} \) of the following form:

\[
(1.1.25) \quad \hat{u}_j(t, \eta) = \hat{u}_{j,0}(t) + \hat{u}_{j,1}(t) \eta^{-1} + \cdots , \\
(1.1.26) \quad \hat{v}_j(t, \eta) = \hat{v}_{j,0}(t) + \hat{v}_{j,1}(t) \eta^{-1} + \cdots .
\]

We note that

\[
(1.1.27) \quad \hat{u}_{j+1,0} + \hat{u}_{1,0} \hat{u}_{j,0} + \hat{w}_{j,0} = 0, \quad j = 1, \ldots, m, \\
(1.1.28) \quad \hat{v}_{j,0} = 0, \quad j = 1, \ldots, m,
\]

and

\[
(1.1.29) \quad \hat{w}_{j,0} = \frac{1}{2} \left( \sum_{k=1}^{j} \hat{u}_{k,0} \hat{u}_{j+1-k,0} \right) + \sum_{k=1}^{j-1} \hat{u}_{k,0} \hat{w}_{j-k,0} + c_j + \delta_{j,0} t, \quad j = 1, \ldots, m
\]

follow from (1.1.1) and (1.1.2) and that these relations together with (1.1.1.c) determine \( \hat{u}_{j,0} \) algebraically. (See [KKNT, §2.1] for the details.) If we substitute the expansions (1.1.25) and (1.1.26) into the coefficients of \( U, V \) and \( W \), they are accordingly expanded in powers of \( \eta^{-1} \); we let \( U, V, W \) respectively denote the coefficient of \( \eta^{-1} \) in the expansion. Using the 0-parameter solution we define series \( b_j(t, \eta) \) \( j = 1, \ldots, m \) as solutions of the equation

\[
(1.1.30) \quad U(b_j) = 0, \quad j = 1, 2, \ldots, m,
\]

that is,

\[
(1.1.31) \quad b_j^m - \sum_{j=1}^{m} \hat{u}_j b_j^{m-j} = 0.
\]

It is clear that \( b_j(t, \eta) \) is also expanded as

\[
(1.1.32) \quad b_j = b_{j,0}(t) + b_{j,1}(t) \eta^{-1} + \cdots .
\]

Although we have started our discussion with the equation (1.1.1) with unknown functions \( (u_j, v_j = (du_j/dt)/2\eta)) \), the quantities \( (u_1, \ldots, u_m) \) were
first introduced as the elementary symmetric polynomials of \((b_1, \ldots, b_m)\) in [S].

Since \(b_j (j = 1, \ldots, m)\) is determined by \((\dot{u}_1, \ldots, \dot{u}_m)\) through the algebraic equation (1.1.31), we try to find a transformation that brings \(b_j\) to a 0-parameter solution of \((R)\), i.e., (1.1.13), in a neighborhood of a turning point of \((P_1)_m\) that is relevant to \(b_j\). This task is accomplished in Section 3 with the essential use of the results in Section 2. Before proceeding further, we note two important geometric properties of the function \(b_{j,0}(t) (j = 1, \ldots, m)\), the top order term of the expansion of \(b_j\) in \(\eta^{-1}\).

First, \(x = b_{j,0}(t)\) is, as a zero of \(U_0(x)\), a singular point of \(Q_{(1,m)}\) and \(a_{(1,m)}\). Hence their expansions in \(\eta^{-1}\) are considered outside the point; their coefficients of \(\eta^{-i}\) are denoted respectively by \(Q_{(1,m),i}\) and \(a_{(1,m),i}\). Second, \(x = b_{j,0}(t)\) is a double turning point of \((SL)_m\). In fact, (1.1.27) together with the definition of \(U_0, V_0\) and \(W_0\) entails
\[
(1.1.33) \quad 2x^{m+1} - xU_0(x) + 2W_0(x) \\
= x^{m+1} + \sum_{j=1}^{m} \dot{u}_{j,0}x^{m+1-j} - 2\sum_{j=1}^{m} (\ddot{u}_{j+1,0} + \dot{u}_{1,0} \dot{u}_{j,0})x^{m-j} \\
= x^{m+1} + 2\dot{u}_{1,0}x^m - \dot{u}_{1,0}x^m - \sum_{j=2}^{m} \dot{u}_{j,0}x^{m+1-j} \\
- 2\dot{u}_{1,0} \sum_{j=1}^{m} \dot{u}_{j,0}x^{m-j} \\
= (x + 2\dot{u}_{1,0})U_0(x),
\]
and hence (1.1.28) and (1.1.33) imply
\[
(1.1.34) \quad Q_{(1,m),0} = \frac{1}{4}(x + 2\dot{u}_{1,0})U_0(x)^2.
\]

As we will see below, similar facts are observed for \((P_J)_m (J = I-1 \text{ or II-2})\), and they play critically important roles in Section 2 and Section 3.

\[ \textbf{§1.2. The case } J = \text{II-1} \]

Let us begin our discussions by briefly recalling the definition of \((P_{I-1})_m\) and the underlying Lax pair \((L_{I-1})_m\). See [KKNT] and [GP] for the details. We refer the reader to [GP] for their relevance to the non-isospectral scattering problem.
**Definition 1.2.1.** The $m$-th member of $P_{1,1}$-hierarchy with a large parameter $\eta$ is, by definition, the following differential equation for $v$:

\[(P_{1,1})_m : (\eta^{-1} \frac{\partial}{\partial t} + 2v) K_m + g(2tv + \eta^{-1}) + c = 0,\]

where $g$ and $c$ are constants and $K_m$ is a polynomial of $v$ and its derivatives defined by the following recursive relation:

\[\eta^{-1} \frac{\partial}{\partial t} K_{p+1} = (\eta^{-3} \frac{\partial}{\partial x} + 4\eta^{-1}(v^2 - \eta^{-1}v_t) \frac{\partial}{\partial t} + 2(2vv_t - \eta^{-1}v_{tt}))K_p, \quad p = 0, 1, 2, \ldots\]

with

\[(1.2.3) \quad K_0 = 1/2.\]

**Remark 1.2.1.** The above recursive relation allows $K_p$ to contain integrated terms like $\frac{1}{\eta}v$. However, we can choose $K_p$ so that it is a polynomial of $v$ and its derivatives. (See [KKNT, Appendix A] for the proof.) One can then easily confirm that such preferred $K_p$ has the form

\[(1.2.4) \quad \frac{(-1)^p2^{p-1}(2p-1)!}{p!} v^{2p} + \sum_{l=1}^{2(p-1)} \eta^{-l} K_{p,l} + \eta^{-(2p-1)} \frac{d^{2p-1}v}{dt^{2p-1}}.\]

Hence $(P_{1,1})_m$ is a $2m$-th order non-linear differential equation. The explicit form of the first two preferred $K_p$ is as follows:

\[K_1 = -v^2 + \eta^{-1}v_t,\]

\[K_2 = 3v^4 - 6\eta^{-1}v^2v_t + \eta^{-2}(2v_t)^2 - 2vv_{tt} + \eta^{-3}v_{ttt}.\]

Hence we find

\[(1.2.7) \quad (P_{1,1})_1 : \eta^{-2} \frac{d^2v}{dt^2} = v^3 - 2gtv - (c + gn^{-1}),\]

and

\[(1.2.8) \quad (P_{1,1})_2 : \eta^{-3} \frac{d^4v}{dt^4} = \eta^{-2}(10v^2 \frac{d^2v}{dt^2} + 10v \left(\frac{dv}{dt}\right)^2)\]

\[- 6v^5 - 2gtv - (c + gn^{-1}).\]

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As (1.2.7) is the traditional Painlevé-II equation \( P_{11} \) with a large parameter \( \eta \), it is reasonable to call the totality of these equations the Painlevé-II hierarchy with a large parameter \( \eta \). As we will see in Section 1.3 another hierarchy whose first member is \( P_{11} \), in order to distinguish these two hierarchies, we coin the terminology \( P_{11,1} \)-hierarchy to call the equations discussed in this section. The equations discussed in Section 1.3 will be called \( P_{11,2} \)-hierarchy.

**Remark 1.2.2.** We sometimes allow constants \( c \) and \( g \) to contain powers of \( \eta^{-1} \) like \( c = c_0 + c_1 \eta^{-1} \). For example, we usually assume that \( g \) is a genuine constant (i.e., free from \( \eta \)) and that

\[
(1.2.9) \quad c = c_0 - g \eta^{-1}
\]

so that \( c + g \eta^{-1} \) is free from \( \eta^{-1} \). In what follows we also assume that \( g \) is different from 0.

**Definition 1.2.2.** The Lax pair \( (L_{11,1})_m \) underlying \( (P_{11,1})_m \) is the following pair of linear differential equations on \((x,t)\)-space:

\[
(1.2.10) \quad (L_{11,1})_m : \begin{cases}
\left( \frac{\partial}{\partial x} - \eta A \right) \tilde{\psi} = 0, \\
\left( \frac{\partial}{\partial t} - \eta B \right) \tilde{\psi} = 0,
\end{cases}
\]

where \( \tilde{\psi} = \{\psi_1, \psi_2\}, \)

\[
(1.2.11) \quad A = \frac{1}{4g} \begin{pmatrix} -\eta^{-1} \partial_x T_m & 2T_m \\ 2q T_m - \eta^{-2} \partial^2_x T_m & \eta^{-1} \partial_x T_m \end{pmatrix}
\]

and

\[
(1.2.12) \quad B = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}
\]

with \( T_m \) and \( q \) being given respectively by

\[
(1.2.13) \quad T_m = gt + \sum_{k=0}^{m} (4x)^k K_{m-k}
\]

and

\[
(1.2.14) \quad q = x - K_1
\]
**Remark 1.2.3.** As is discussed in [KKNT, §1.2] and [GP], \((P_{11})_m\) states the compatibility condition of \((L_{11})_m\).

As in Section 1.1, we begin our discussion by writing down the equation that the first component \(\psi_1\) of \(\psi\) should satisfy:

\[
\begin{align*}
(1.2.15) \quad \left( \frac{\partial^2}{\partial x^2} + \left( \frac{1}{x} - \frac{T_{m,x}}{T_m} \right) \frac{\partial}{\partial x} - \frac{1}{16g^2x^2} \left( T_{m,t}^2 \right)^2 + 4\eta^2 qT_m^2 - 2T_mT_{m,tt} \right) \\
\quad \quad \quad \quad \quad \quad \quad + \frac{1}{4gx} \left( T_{m,tx} - \frac{T_{m,t}T_{m,t}}{T_m} \right) \right) \psi_1 = 0.
\end{align*}
\]

Here \(T_{m,tx}\) etc. designate \(\partial^2 T_m/\partial t \partial x\) etc. By introducing \(\psi\) by

\[
\exp\left( \frac{1}{2} \int^x \left( \frac{1}{x} - \frac{T_{m,x}}{T_m} \right) dx \right) \psi_1 = x^{1/2} T^{-1/2}_m \psi_1,
\]

we find the required Schrödinger equation for \(\psi:\)

\[
(1.2.16) \quad \frac{\partial^2}{\partial x^2} \psi = \eta^2 Q_{[11,1,m]} \psi,
\]

where

\[
(1.2.17) \quad Q_{[11,1,m]} = \frac{1}{4g^2x^2} qT_m^2 + \frac{\eta^2}{16g^2x^2} \left( T_{m,t}^2 - 2T_mT_{m,tt} \right) \\
\quad \quad \quad \quad \quad \quad \quad + \frac{\eta^2}{4gx} \left( \frac{T_{m,xt}}{T_m} - \frac{T_{m,tx}}{T_m} \right) + \frac{3\eta^2 T_{m,x}^2}{4T_m} - \frac{\eta^2 T_{m,xx}}{2T_m} \\
\quad \quad \quad \quad \quad \quad \quad - \frac{\eta^2 T_{m,xx}}{4x^2}.
\]

On the other hand, (1.2.10.b) implies

\[
(1.2.18) \quad \frac{\partial \psi_1}{\partial t} = \eta \psi_2,
\]

and (1.2.10.a) entails

\[
(1.2.19) \quad \frac{\partial \psi_1}{\partial x} = \frac{1}{4gx} (-T_{m,t}\psi_1 + 2\eta T_m \psi_2).
\]

Hence we find

\[
(1.2.20) \quad \frac{\partial \psi_1}{\partial x} = \frac{1}{4gx} (-T_{m,t}\psi_1 + 2T_m \frac{\partial \psi_1}{\partial t}).
\]
Then, combining (1.2.16) and (1.2.21), we obtain

\[
\begin{align*}
(1.2.22) \quad 4g & x(-\frac{1}{2}x^{-3/2}T_m^{1/2}\psi + \frac{1}{2}x^{-1/2}T_m^{-1/2}T_m, x\psi \\
& + \frac{1}{2}x^{-1/2}T_m^{1/2}\psi_x) = -T_m, x, x^{-1/2}T_m^{1/2}\psi \\
& + 2T_m x^{-1/2}(\frac{1}{2}T_m^{-1/2}T_m, x\psi + T_m^{1/2}\psi_t),
\end{align*}
\]

that is,

\[
(1.2.23) \quad \psi_t = \frac{2gx}{T_m}\psi_x + \frac{1}{T_m}(gxT_m^{-1}T_m, x - g)\psi.
\]

Therefore, by setting

\[
(1.2.24) \quad a_{[II, m]} = \frac{2gx}{T_m},
\]

we find

\[
(1.2.25) \quad \frac{\partial\psi}{\partial t} = a_{[II, m]} \frac{\partial\psi}{\partial x} - \frac{1}{2} \frac{\partial a_{[II, m]}}{\partial x} \psi.
\]

Thus we obtain simultaneous equations (1.2.17) and (1.2.25) for the unknown function \(\psi\); equation (1.2.17) (resp., (1.2.25)) is referred to as \((SL_{II, m})\) (resp., \((D_{II, m})\)). In parallel with the case of \((SL_{I})\) and \((D_{I})\), we first construct a 0-parameter solution

\[
(1.2.26) \quad \hat{\psi}(t, \eta) = \hat{\psi}_0(t) + \hat{\psi}_1(t)\eta^{-1} + \cdots
\]

of \((RI, m)\), and then substitute it into the coefficients of \((SL_{II, m})\) and \((D_{II, m})\) to analyze their structure. We note that (1.2.4) implies

\[
(1.2.27) \quad \frac{(-1)^m 2^m (2m - 1)!!}{m!} \hat{\psi}_0^{2m+1} + 2gt\hat{\psi}_0 + c_0 = 0.
\]

In what follows we let \(T_{m, 0}, K_{m, 0}, q_0\), etc. respectively denote the top order term of the expansion obtained by substituting the 0-parameter solution into the coefficients of \(T_m, K_m, q\), etc; for example, \(q_0 = x + \hat{v}_0^2\).

Using the 0-parameter solution \(\hat{\psi}\), we introduce another set of formal series

\[
(1.2.28) \quad b_j(t, \eta) = b_{j, 0}(t) + \eta^{-1}b_{j, 1}(t) + \cdots \quad (j = 1, \ldots, m)
\]
as solutions of the equation

\[(1.2.29) \quad T_m(x, t, \eta) \bigg|_{x = b_j(t, \eta)} = 0.\]

We then immediately find that \(x = b_{j,0}(t)\) is a singular point of \(Q_{(\Pi-1,m)}\) and \(a_{(\Pi-1,m)}\). It is also clear from (1.2.18) that \(x = b_{j,0}(t)\) is a double turning point of \((SL_{\Pi-1})_m\). These observations are exactly the same as those for the series \(b_j(t, \eta)\) introduced in the previous subsection.

§1.3. The case \(J = \Pi-2\)

Let us first recall the definition of \(P_{\Pi-2}\)-hierarchy with a large parameter \(\eta\) and its underlying Lax pair \((L_{\Pi-2})\). We refer the reader to [GJP] and [N] for the detailed discussions concerning \(P_{\Pi-2}\)-hierarchy.

**Definition 1.3.1.** The \(m\)-th member of \(P_{\Pi-2}\)-hierarchy with a large parameter \(\eta\) is, by definition, the following differential equations for the unknown functions \(u\) and \(v\):

\[
(1.3.1) \quad (P_{\Pi-2})_m : \begin{cases} 
K_{m+1} + \sum_{j=1}^{m-1} c_j K_j + g t = 0 \\
L_{m+1} + \sum_{j=1}^{m-1} c_j L_j = \delta
\end{cases}
\]

Here \(c_j, g\) and \(\delta\) are constants, and \(K_j\) and \(L_j\) are polynomials of \(u, v\) and their derivatives, which are defined by the following recursive relations:

\[
(1.3.2) \quad \eta^{-1} \partial_t \begin{pmatrix} K_{j+1} \\ L_{j+1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \eta^{-1} \partial_t u - \eta^{-2} \partial_t^2 & 2\eta^{-1} \partial_t \\ 2\eta^{-1} v \partial_t + \eta^{-1} v_t & \eta^{-1} u \partial_t + \eta^{-2} \partial_t^2 \end{pmatrix} \begin{pmatrix} K_j \\ L_j \end{pmatrix}
\]

\((j \geq 0)\)

with \(K_0 = 2\) and \(L_0 = 0\).

**Remark 1.3.1.** See [N] for the proof of the existence of such preferred \(K_j\) and \(L_j\), that is, those which are polynomials of \(u, v\) and their derivatives. The
first three terms of such preferred \( K_j \) and \( L_j \) are as follows:

\[
\begin{align*}
(1.3.3) \quad \begin{pmatrix} K_1 \\ L_1 \end{pmatrix} &= \begin{pmatrix} u \\ v \end{pmatrix} \\
(1.3.4) \quad \begin{pmatrix} K_2 \\ L_2 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} u^2 + 2v - \eta^{-1}u_t \\ 2uv + \eta^{-1}v_t \end{pmatrix} \\
(1.3.5) \quad \begin{pmatrix} K_3 \\ L_3 \end{pmatrix} &= \left( \frac{1}{2} \right)^2 \begin{pmatrix} u^3 + 6uv - 3\eta^{-1}u u_t + \eta^{-2}u u_{tt} \\ 3u^2v + 3v^2 + 3\eta^{-1}u v_t + \eta^{-2}v u_{tt} \end{pmatrix}
\end{align*}
\]

**Remark 1.3.2.** In what follows we assume

\[
(1.3.6) \quad c_j = 0, \quad j = 1, 2, \ldots, m - 1.
\]

We also assume that \( g \) is a non-zero genuine constant and that \( \delta \) has the form \( \delta_0 + \eta^{-1}\delta_1 \) with

\[
(1.3.7) \quad \delta_1 = -g/2.
\]

**Remark 1.3.3.** (i) \((P_{1,2})_1\) is reduced to

\[
(1.3.8) \quad \eta^{-2} \frac{d^2 u}{dt^2} = 2u^3 + 2g(2tu + \eta^{-1}) + 4\delta.
\]

(ii) \((P_{1,2})_2\) is reduced to

\[
(1.3.9) \quad \eta^{-4} \frac{d^4 u}{dt^4} = \frac{1}{2u^2}(\eta^{-4} \left( -4 \left( \frac{du}{dt} \right)^2 \frac{d^2 u}{dt^2} + 3u \left( \frac{d^2 u}{dt^2} \right)^2 \right) \\
+ 4u \frac{du}{dt} \frac{d^3 u}{dt^3} + \eta^{-2} (16g \frac{du}{dt} - 16gt \left( \frac{du}{dt} \right)^2 \\
+ 5u^3 \left( \frac{du}{dt} \right)^2 + 16gtu \frac{d^2 u}{dt^2} + 10u^4 \frac{d^2 u}{dt^2} \\
- 24\eta^{-1}u^3 + (16g^2t^2u - 48\delta u^3 - 16gtu^4 - 5u^7))
\]

**Definition 1.3.2.** The Lax pair \((L_{1,2})_m\) underlying \((P_{1,2})_m\) is the following pair of linear differential equations on \((x,t)\)-space:

\[
(1.3.10) \quad (L_{1,2})_m : \begin{align*}
\left( \frac{\partial}{\partial x} - \eta A \right) \psi &= 0, \quad (1.3.10.a) \\
\left( \frac{\partial}{\partial t} - \eta B \right) \psi &= 0, \quad (1.3.10.b)
\end{align*}
\]
where $\bar{\psi} = \psi_1, \psi_2$,

\[(1.3.11)\quad A = \frac{1}{g} \left( - (2x - u) T_m - \eta^{-1} T_{m,t} \cdot \frac{2T_m}{(2x - u) T_m + \eta^{-1} T_{m,t} + K_{m+1}} \right) \]

with

\[(1.3.12)\quad T_m = \frac{1}{2} \sum_{j=0}^{m} x^{m-j} K_j,\]

and

\[(1.3.13)\quad B = \begin{pmatrix} -x + u/2 & 1 \\ -v & x - u/2 \end{pmatrix}.\]

In parallel with the discussions in the preceding subsections, we first write down the differential equation that the first component $\psi_1$ of the solution $\bar{\psi}$ of the equation (1.3.10) should satisfy:

\[(1.3.14)\quad \left[ \frac{\partial^2}{\partial x^2} - \frac{T_{m,x}}{T_m} \frac{\partial}{\partial x} + \frac{\eta^2}{g^2} \left( - (2x - u)^2 T_m^2 + 4v T_m^2 \right) + \frac{\eta}{g^2} (-2(2x - u) T_m T_{m,t} + 2T_m \partial_t ((2x - u) T_m + \eta^{-1} T_{m,t} + K_{m+1}) + 2g T_m) \right. \]

\[\left. - \frac{T_{m,t}^2}{g T_m} - \frac{T_{m,t} T_{m,x}}{g T_m} + \frac{T_{m,t} x}{g} \right] \psi_1 = 0.\]

To eliminate the first order differential operator part, we introduce

\[(1.3.15)\quad \psi = \exp \left( - \frac{1}{2} \int x \frac{T_{m,x}}{T_m} dx \right) \psi_1 = T_m^{-1/2} \psi_1\]

and find the Schrödinger equation for $\psi$:

\[(1.3.16)\quad \frac{\partial^2 \psi}{\partial x^2} = \eta^2 Q_{(11,2;m)} \psi,\]
where

\[(1.3.17) \quad Q_{(11-2,m)} = \frac{1}{g^2}((2x - u)^2 - 4v)T_m^2
\]

\[+ \frac{\eta^{-1}}{g^2}(2u_tT_m^2 - 2T_{m,K,m+1,t} - 2gT_m)
\]

\[+ \eta^{-2}\left(\frac{3}{4} \frac{T_{m,x}^2}{T_m^2} - \frac{T_{m,xx}}{2T_m} + \frac{T_{m,t}^2}{g^2} - 2\frac{T_mT_{m,tt}}{g^2} - \frac{T_{m,t,T_m,x}}{g} - \frac{T_{m,xx}}{g}\right).
\]

On the other hand, (1.3.10.b) implies

\[(1.3.18) \quad \frac{\partial \psi_1}{\partial t} = \eta(-x + \frac{u}{2})\psi_1 + \eta\psi_2,
\]

and (1.3.10.a) implies

\[(1.3.19) \quad \frac{\partial \psi_3}{\partial x} = -\frac{\eta}{g}((2x - u)T_m + \eta^{-1}T_{m,t})\psi_1 + \frac{2\eta}{g}T_m\psi_2.
\]

Combining these relations, we obtain

\[(1.3.20) \quad \frac{\partial \psi_1}{\partial x} = -\frac{1}{g}T_{m,t}\psi_1 + \frac{2T_m}{g} \frac{\partial \psi_1}{\partial t}.
\]

Substituting (1.3.15) into (1.3.20), we find

\[(1.3.21) \quad \frac{\partial \psi}{\partial t} = \frac{g}{2T_m} \frac{\partial \psi}{\partial x} + \frac{gT_{m,x}}{4T_m^2}\psi.
\]

Therefore, by setting

\[(1.3.22) \quad a_{(11-2,m)} = \frac{g}{2T_m},
\]

we arrive at

\[(1.3.23) \quad \frac{\partial \psi}{\partial t} = a_{(11-2,m)} \frac{\partial \psi}{\partial x} - \frac{1}{2} \frac{\partial a_{(11-2,m)}}{\partial x}\psi.
\]

In what follows, (1.3.17) (resp., (1.3.23)) is referred to as $SL_{11-2,m}$ and $(D_{11-2,m})$, and our aim is to analyze them by substituting a 0-parameter solution $(\hat{u}, \hat{v})$ of $(P_{11-2,m})$ in their coefficients. We refer the reader to [N]
concerning the existence proof of a 0-parameter solution and its basic properties. In parallel with the preceding subsections, we introduce another set of formal series

\[(1.3.24) \quad b_j(t, \eta) = b_{j,0}(t) + b_{j,1}(t)\eta^{-1} + \cdots \quad (j = 1, \ldots, m)\]

as solutions of the equation

\[(1.3.25) \quad T_m(x, t, \eta) \big|_{(u, v) = (\alpha, \beta), x = b_j(t, \eta)} = 0.\]

It is then clear that \(x = b_{j,0}(t)\) is a singular point of \(Q_{12,m}\) and \(a_{12,m}\), and that it is a double turning point of \((SL_{12})_m\). These facts are completely in parallel with the results we obtained in the preceding subsections.

\[\S 2. \text{Regularity of } S_{\text{odd}} \text{ near } x = b_{j,0}(t)\]

In Section 1 we have derived a pair of Schrödinger equation \((SL_J)_m\) and its deformation equation \((D_J)_m\) from the Lax pair \((L_J)_m(J = I, II-1, II-2)\). We have also confirmed that all of them share the following important property: the point \(x = b_{j,0}(t)\) \((j = 1, \ldots, m)\) is a double turning point of the Schrödinger equation we obtained, where \(b_{j,0}(t)\) is the top order term of the formal series \(b_j(t, \eta)\) which is determined algebraically by a 0-parameter solution of \((P_J)_m\) \((J = I, II-1, II-2)\). In the subsequent section (Section 3) we will construct a formal transformation that reduces \(b_j(t, \eta)\) to a 0-parameter solution of the traditional Painlevé-I equation (i.e., \((P_1)\)) near an appropriate turning point of \((P_J)_m\) and in this section we prepare some results needed for the construction. As our reasoning in this section applies uniformly to every \((SL_J)_m(J = I, II-1, II-2)\), we omit the suffix \((J, m)\) of \(Q_{J,m}\) and \(a_{J,m}\). In what follows we let \(S^\pm\) denote the solution of the Riccati equation associated with \((SL_J)_m\), i.e.,

\[(2.1) \quad (S^\pm)^2 + \frac{\partial S^\pm}{\partial x} = \eta^2 Q,\]

that begins with \(\pm \eta \sqrt{Q_0}\) (with an appropriate choice of the branch of \(\sqrt{Q_0}\)). We also use the symbol \(S_{\text{odd}}\) to denote

\[(2.2) \quad \frac{1}{2}(S^+ - S^-).\]

We note that the definition of \(S_{\text{odd}}\) given here is different from that given in [KT1], although they coincide in the situation discussed in [KT1]. As a
matter of fact, they are also coincident for $(SL_1)_m$ by a result on the structure of a 0-parameter solution (cf. Appendix); in general, if $Q$ has the form
\[(2.3) \quad \sum_{l \geq 0} \eta^{-2l} Q_{2l},\]
then the two definitions coincide. When $Q$ contains odd degree terms in $\eta$, the definition given in [KT1] does not work; then we should use the definition (2.2). Making use of the reasoning in [AKT, §2], we can readily deduce the following relation (2.4) from $(D_J)_m$:
\[(2.4) \quad \frac{\partial S_{\text{odd}}}{\partial t} = \frac{\partial}{\partial x} (a S_{\text{odd}})\]
for $S_{\text{odd}}$ thus defined.

Remark 2.1. We note that the denominator of $a$ is a polynomial of degree $m$ in $x$; in the analysis of the traditional Painlevé equations ([O], [KT1]), the corresponding function was linear in $x$.

Now, using the relation (2.4) we prove the following.

**Theorem 2.1.** Assume that $x = b_{j,0}(t)$ is an exactly double zero of $Q_0(x,t)$ near $(x,t) = (b_{j,0}(t_0),t_0)$. Then the series $S_{\text{odd}}$ and $a S_{\text{odd}}$ are holomorphic on a neighborhood of $x = b_{j,0}(t)$ in the sense that each of their coefficients as formal power series in $\eta^{-1}$ is holomorphic on the neighborhood of $x = b_{j,0}(t)$.

**Proof.** For the sake of the uniformity of the presentation we use the symbol $U$ also to denote $T_m/(g x)$ if $J = \Pi-1$ and $4T_m/g$ if $J = \Pi-2$. Let us substitute a 0-parameter solution into the coefficients of $a$ and $U$ and expand them in powers of $\eta^{-1}$ as follows:
\[(2.5) \quad a = \sum_{j \geq 0} a_j(x,t) \eta^{-j},\]
\[(2.6) \quad U = \sum_{j \geq 0} U_j(x,t) \eta^{-j}.\]

To simplify the notation we let $R$ denote $S_{\text{odd}}$; in accordance with this convention, $R_l$ stands for the coefficient of $\eta^{-l}$ in the expansion of $S_{\text{odd}}$. It then follows from (2.4) that
\[(2.7) \quad \frac{\partial R_{m-1}}{\partial t} = \frac{\partial}{\partial x} \left( \sum_{k=0}^{m} a_k R_{m-1-k} \right).\]
It also follows from the definition of $a$ that

\[ (2.8) \quad U_0 a_k + \sum_{l=1}^{k} U_l a_{k-l} = 0 \text{ for } k \geq 1. \]

Since $U_l(l \geq 0)$ is a polynomial in $x$, (2.8) shows that $a_k$ has the form $N_k U_0^{-k-1}$ with some polynomial $N_k$ in $x$.

Now, combining (2.7) and (2.8), we find

\[ (2.9) \quad \frac{\partial R_{m-1}}{\partial t} = \frac{\partial}{\partial x} (a_0 R_{m-1}) - \frac{\partial}{\partial x} \left[ \frac{1}{U_0} \sum_{k=1}^{m} \left( \sum_{l=1}^{k} U_l \left( \sum_{k-l}^{m} a_{k-l} R_{m-1-k} \right) \right) \right] \]

Making use of (2.9) we show that there exists an open neighborhood $\omega$ of $x = b_{j,0}(t)$ on which the following assertion $(\mathcal{A})_n$ is validated for $n = 0, 1, 2, \ldots$:

\[ (\mathcal{A})_n : \quad \begin{cases} 
(i) \quad R_{n-1} \text{ is holomorphic}, \\
(ii) \quad \sum_{l=0}^{n} a_l R_{n-1-l} \text{ is holomorphic}. 
\end{cases} \]

We prove this by the induction on $n$. But, before embarking on proving this, we make some preparatory study on the structure of the function $R_l = (S_l^+ - S_l^-)/2$. By solving the Riccati equation (2.1) we can find a neighborhood $\omega$ of $x = b_{j,0}(t)$ on which $S_l^\pm$ has the following form:

\[ (2.10) \quad \frac{C_l^\pm P_l^\pm}{(S_{l+1}^\pm)^{p_l^\pm} U_0^{q_l^\pm x}}, \]

where $p_l^\pm$ and $q_l^\pm$ are some non-negative integers, $C_l^\pm$ is an analytic function that does not vanish on $\omega$ and $P_l^\pm$ is a polynomial in $x$ that depends analytically on $t$ on $\omega$. Since $S_{l+1}^\pm$ has the form $\pm \alpha U_0$ with a non-vanishing analytic factor $\alpha$ on $\omega$, we may assume every $S_l^\pm$, in particular $S_n^\pm$, has the form $\tilde{C}^\pm P_l^\pm U_0^{p_l^\pm x}$ with an integer $p_l^\pm$, a polynomial $P_l^\pm$ in $x$ and a non-vanishing
analytic factor $\widetilde{C}$ on $\omega$. Hence we find $R_n$ has the form $\widetilde{C}PU_0^p$ with an integer $p$, a polynomial $P$ in $x$ and a non-vanishing analytic factor $\widetilde{C}$ on $\omega$. Here $P$ is assumed not to vanish identically on $\{(x, t); x = b_{j, 0}(t)\}$. Having this structure of $R_n$ in mind, we embark on the confirmation of $(A)_n$ by the induction on $n$. First of all, $(A)_0$ is clear, because $R_{-1}$ has the form $\alpha U_0$ with an analytic factor $\alpha$ on $\omega$ and $a_0 = 2/U_0$ (resp., $2x/U_0$) for $J = I$ or $J = \Pi-2$ (resp., $J = \Pi-1$). Let us next assume that $(A)_m$ is validated for $m = 0, 1, \cdots, n$. Then this induction hypothesis guarantees that

$$
(2.11) \quad \sum_{s=0}^{n} U_{n+1-s} \left( \sum_{r=0}^{s} a_r R_{s-1-r} \right)
$$

is holomorphic on $\omega$, and hence the second term in the right-hand side of (2.9) with $m = n + 1$, namely,

$$
(2.12) \quad -\frac{\partial}{\partial x} \left[ \frac{1}{U_0} \left( \sum_{s=0}^{n} U_{n+1-s} \left( \sum_{r=0}^{s} a_r R_{s-1-r} \right) \right) \right],
$$

has an at most double pole at $x = b_{j, 0}(t)$ that originates from the simple pole factor $U_0^{-1}$. On the other hand, our preparatory study on the structure of $R_n$ shows that $\partial R_n/\partial t$ has the form $\beta U_0^{p-1}$ with an analytic factor $\beta$ on $\omega$ and that $\partial(a_0 R_n)/\partial x = \partial(2\widetilde{C}PU_0^{p-1})/\partial x = \beta U_0^{p-2}$ with another non-vanishing analytic factor $\widetilde{\beta}$ on $\omega$. Therefore (2.9) with $m = n + 1$ implies $p \geq 0$, i.e., $R_n$ should be holomorphic. This validates the first part of the assertion $(A)_{n+1}$. It also entails that $\partial R_n/\partial t$ is holomorphic on $\omega$, and hence the relation (2.7) with $m = n + 1$ shows that

$$
(2.13) \quad \frac{\partial}{\partial x} \left( \sum_{k=0}^{n+1} a_k R_{n-k} \right)
$$

is holomorphic on $\omega$. But, then, in view of the structure of $a_k$ and $R_{n-k}$, that is, the fact that their singularities, if any, are of the form $U_0^{-r}$ for some non-negative integer $r$, we conclude that

$$
(2.14) \quad \sum_{r=0}^{n+1} a_k R_{n-k}
$$

should be holomorphic on $\omega$. This is nothing but the second part of the assertion $(A)_{n+1}$. Thus the induction proceeds.
It is clear that the validity of $(A)_n$ for every $n(\geq 1, 2, \ldots)$ means that $R = S_{\text{odd}}$ and $aS_{\text{odd}}$ are holomorphic on $\omega$. This completes the proof of the theorem.

§3. Reduction of $b_j(t, \eta)$ ($j = 1, \ldots, m$) to a 0-parameter solution of $(P_1)_1$

§3.1. Some preparation of notions and notations about the Stokes geometry of $(P_J)_m$ and that of $(SL_J)_m$ ($J = I, II-I, II-I$).

Before entering the analysis of $(SL_J)_m$ we clarify the geometric setting on which we consider the problem. To begin with, let us fix a turning point $t = \tau$ of the first kind of $(P_J)_m$ ($J = I, II-I, II-I$) in the sense of [KKNT, §2], that is, there exist two solutions $\nu(t)$ of the characteristic equation of the linearization of $(P_J)_m$ at a 0-parameter solution (often called the Fréchet derivative of $(P_J)_m$) which merge at $t = \tau$ and whose values $\nu(\tau)$ are 0. Then it follows from the explicit form of the characteristic equation of the Fréchet derivative (cf. [KKNT, (2.1.23), (2.2.13), (2.3.8)]) that some $b_j(t)$, a double turning point of $(SL_J)_m$, and a simple turning point, say $a(t)$, of $(SL_J)_m$ merge at $t = \tau$. Note that every turning point of $(P_J)_m$ is of the first kind if $m = 1$. This explains why the turning point is not assumed to be of the first kind in [KT1]. We further assume, as in [KT1], that the turning point is simple: unlike the situation discussed in [KT1], we want to impose the condition without using the explicit form of the equation and employ the general definition given in [AKKT]. However, the characteristic equation written in $t$-variable has singularities at turning points and an immediate application of [AKKT] is not possible. Hence we use a local parameter $u$ of the Riemann surface $\mathcal{R}$ associated with the 0-parameter solution as the independent variable that replaces $t$. Note that the Stokes geometry of $(P_J)_m$ is described on $\mathcal{R}$ (cf. [KKNT] and [NT]). Thus we require that the characteristic polynomial $P(u, \nu)$ of the Fréchet derivative of $(P_J)_m$ should satisfy the following conditions at $\hat{u}_0 = u(\tau)$:

\begin{align}
(3.1.1) \quad P(\hat{u}_0, 0) &= \frac{\partial P_0}{\partial \nu}(\hat{u}_0, 0) = 0 \\
(3.1.2) \quad \frac{\partial P}{\partial u}(\hat{u}_0, 0) &= 0, \quad \frac{\partial^2 P}{\partial \nu^2}(\hat{u}_0, 0) \neq 0.
\end{align}
These conditions guarantee that \( \tau \) is a square-root branch point of \( \mathcal{R} \), and hence they imply that
\[
(3.1.3) \quad \nu_{\pm}(t) \quad \text{is of exactly order} \quad (t - \tau)^{1/4}.
\]
The results in [KKNT, §2] tell us then that
\[
(3.1.4) \quad \nu_{-} = -\nu_{+}
\]
and
\[
(3.1.5) \quad \int_{\tau}^{t} \nu_{+}(s) ds = 2 \int_{a(t)}^{b_{j,0}(t)} \sqrt{Q_{[,m],0}(x,t)} dx
\]
hold. Note that a Stokes curve of \((P_{j})_{m}\) that emanates from \( \tau \) is, by definition, given by
\[
(3.1.6) \quad \text{Im} \int_{\tau}^{t} \nu_{+}(s) ds = 0.
\]
Since \( a(\tau) \) and \( b_{j,0}(\tau) \) coincide by their definition, (3.1.5) guarantees that \( a(t) \) and \( b_{j,0}(t) \) are connected by a Stokes curve (or, rather a Stokes segment) of \((SL_{j})_{m}\) if \( t \) is a point in a Stokes curve of \((P_{j})_{m}\) that is sufficiently close to \( \tau \). Note, however, that Stokes curves of \((P_{j})_{m}\) cross for \( m \geq 2 \), and that the so-called Nishikawa phenomena ([N]) are observed at crossing points. Hence we cannot expect, in general, that \( a(t) \) and \( b_{j,0}(t) \) are connected by a Stokes curve of \((SL_{j})_{m}\) even if \( t \) lies in a Stokes curve of \((P_{j})_{m}\). Thus we consider the problem near a point \( \sigma(\neq \tau) \) in a Stokes curve of \((P_{j})_{m}\) that emanates from \( \tau \) and that satisfies the following condition:
\[
(3.1.7) \quad a(\sigma) \text{ and } b_{j,0}(\sigma) \text{ are connected by a Stokes curve of } (SL_{j})_{m}.
\]
In this geometric setting we try to reduce \( b_{j}(t, \eta) \) to a 0-parameter solution of \((P_{j})_{1}\) on a neighborhood of \( \sigma \). This is what we will achieve in the next subsection.

¶§3.2. Construction of formal transformations

In the setting described in Section 3.1 we construct appropriate formal transformations \( \bar{x}(x, t, \eta) \) and \( \bar{t}(t, \eta) \) for which the following relation holds:
\[
(3.2.1) \quad \bar{x}(x, t, \eta) \mid_{x=b_{j}(t, \eta)} = \lambda_{t}(\bar{t}(t, \eta), \eta),
\]
where $\lambda_1(\tilde{t}, \eta)$ stands for a 0-parameter solution of the traditional Painlevé-I equation, that is,

$$
\frac{d^2 \lambda_1}{d\tilde{t}^2} = \eta^2 (6\lambda_1^2 + \tilde{t}).
$$

(3.2.2)

Note that a 0-parameter solution is uniquely fixed once we fix the branch of its highest degree term $\lambda_0(t) = \sqrt{-t/6}$. In what follows we also use symbols $\nu_1(\tilde{t}, \eta)$ and $\tilde{Q}(\tilde{x}, \tilde{t}, \eta)$ to denote respectively

$$
\eta^{-1} d\lambda_1 / d\tilde{t}
$$

(3.2.3)

and

$$
4\tilde{x}^3 + 2\tilde{t} \tilde{x} + \nu_1^2 - 4\lambda_1^3 - 2\tilde{t} \lambda_1 - \eta^{-1} \frac{\nu_1}{\tilde{x} - \lambda_1} + \eta^{-2} \frac{3}{4(x - \lambda_1)^2}.
$$

(3.2.4)

We note that $\tilde{Q}$ is the potential of the Schrödinger equation $(SL_1)$ that is associated with the traditional Painlevé-I equation in the notation of [KT1]. Hence we use the symbol $\tilde{S}_{1, \text{odd}}(\tilde{x}, \tilde{t})$ to denote the odd part of a solution $\tilde{S}$ of the Riccati equation associated with $(SL_1)$, that is,

$$
\tilde{S}^2 + \frac{\partial \tilde{S}}{\partial \tilde{x}} = \eta^2 \tilde{Q}.
$$

(3.2.5)

Using these symbols we first prove the following

**Proposition 3.2.1.** Let $\tau$ be a simple turning point of the first kind of $(P_{\tau})_m$ $(J = I, I\!I-1, I\!I-2; m = 1, 2, 3, \cdots)$, and let $\sigma(\neq \tau)$ be a point that is sufficiently close to $\tau$ (that is, $\sigma$ satisfies the assumption (3.1.7)) and that lies in a Stokes curve of $(P_{\tau})_m$ which emanates from $\tau$. Let $\gamma$ denote the Stokes segment which connects turning points $b_{j,0}(t)$ and $a(t)$ of $(SL_j)_m$ that are fixed in terms of $\tau$ in Section 3.1. Then there exist a neighborhood $\Omega$ of $\gamma$, a neighborhood $\omega$ of $\sigma$ and holomorphic functions $\tilde{x}_j(x, t)$ $(j = 0, 1, 2, \cdots)$ on $\Omega \times \omega$ and $\tilde{t}_j(t)$ $(j = 0, 1, 2, \cdots)$ on $\omega$ so that they satisfy the following relations:

1. The function $\tilde{t}_0(t)$ satisfies the following relation

$$
\int_{\tau}^{\tilde{t}} \nu_+(s) ds = \left( \int_{0}^{\tilde{t}} \sqrt{12\lambda_0(s)} ds \right)_{|_{\tilde{t}=\tilde{t}_0(t)}},
$$

(3.2.6)
where \( \nu_+ \) denotes the solution of the characteristic equation of the Fréchet derivative of \((P_J)_m\) which is fixed in terms of \( \tau \) in Section 3.1.

(ii) \( \tilde{x}_0(b_j, 0(t), t) = \lambda(t_0(t)) \) and \( \tilde{x}_0(a(t), t) = -2\lambda(t_0(t)) \).

(iii) \( d\tilde{x}_0/dt \neq 0 \) with \( \partial \tilde{x}_0/\partial x \neq 0 \) on \( \Omega \times \omega \).

(iv) Letting \( \tilde{x}(x, t, \eta) \) and \( \tilde{t}(t, \eta) \) respectively denote \( \sum_{j \geq 0} \tilde{x}_j(x, t, \eta^{-j}) \) and \( \sum_{j \geq 0} \tilde{t}_j(t, \eta^{-j}) \), we find the following relation:

\[
Q_{(j,m)}(x, t, \eta) = \left( \frac{\partial \tilde{x}}{\partial x} \right)^2 \tilde{Q}(\tilde{x}(x, t, \eta), \tilde{t}(t, \eta), \eta) - \frac{1}{2} \eta^2 \{\tilde{x}(x, t, \eta); x\},
\]

where \( \{\tilde{x}; x\} \) denotes the Schwarzian derivative

\[
\frac{\partial^3 \tilde{x}}{\partial x^3} - \frac{3}{2} \left( \frac{\partial^2 \tilde{x}}{\partial x^2} \right)^2.
\]

**Proof.** To begin with, we note that the relation (3.2.7) follows from the following relation (3.2.9) together with the relevant Riccati equations (cf. [AKT]):

\[
S_{(j,m), odd}(x, t, \eta) = \left( \frac{\partial \tilde{x}}{\partial x} \right) \tilde{S}_{l, odd}(\tilde{x}(x, t, \eta), \tilde{t}(t, \eta), \eta),
\]

where \( S_{(j,m), odd} \) stands for the odd part of a solution of the Riccati equation (2.1) with \( Q = Q_{(j,m)} \). To simplify the notations, we use the symbol \( R \) and \( \tilde{R} \) respectively to denote \( S_{(j,m), odd} \) and \( \tilde{S}_{l, odd} \); accordingly \( R_l \) and \( \tilde{R}_l \) respectively stand for the coefficient of \( \eta^{-l} \) \( (l = -1, 0, 1, 2, \cdots) \) of \( R \) and \( \tilde{R} \).

In constructing \( \tilde{x}_j(x, t) \) and \( \tilde{t}_j(t) \) in an inductive manner, we make use of the following assertion \((C)_n \) \( (n = 0, 1, 2, \cdots) \) to make the argument run smoothly:

\((C)_n\)

We can construct \( \{\tilde{x}_j(x, t)\}_{0 \leq j \leq n} \) and \( \{\tilde{t}_j(t)\}_{0 \leq j \leq n} \) so that 

(3.2.9) holds modulo terms of order equal to or at most \( \eta^{-n} \).

Let us first show \((C)_0\); the way of our reasoning is exactly the same as that used in [KT1], but for the sake of completeness we repeat it here. [The only difference is the usage of \( \sim \) in the rotations \((x, t)\) etc. and \((\tilde{x}, \tilde{t})\) etc.; it is reversed here.] The construction of the function \( \tilde{t}_0(t) \) is attained by solving the implicit relation (3.2.6); we readily find it is a constant multiple of

\[
\left( \int_{\tau}^t \nu_+(s) ds \right)^{4/5},
\]

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which is holomorphic on a neighborhood of $\tau$ by the relation (3.1.3). If we define $\tilde{\sigma}$ by $\tilde{t}_0(\sigma)$, the relation (3.2.6) implies that $\tilde{\sigma}$ lies on a Stokes curve of $(P_1)$, and hence a double turning point $\tilde{x} = \lambda_0(\tilde{\sigma})$ and a simple turning point $x = \tilde{a}(\tilde{\sigma}) = -2\lambda_0(\tilde{\sigma})$ of $(SL_1)$ are connected by a Stokes segment $\tilde{\gamma}$ of $(SL_1)$. Here we note that $(SL_1)$ has one double turning point and one simple turning point if $\tilde{t} \neq 0$; in fact we know

\begin{equation}
\tilde{Q}_0 = 4(\tilde{x} - \lambda_0(\tilde{t}))^2(\tilde{x} + 2\lambda_0(\tilde{t})).
\end{equation}

Now we note

\begin{equation}
\int_0^{\tilde{t}} \sqrt{12\lambda_0(s)}d\tilde{s} = 2 \int_{-2\lambda_0(\tilde{t})}^{\lambda_0(\tilde{t})} \sqrt{\tilde{Q}_0(\tilde{x}, \tilde{t})}d\tilde{x}
\end{equation}

holds as a special case of (3.1.5). Hence combining (3.1.5), (3.2.6) and (3.2.12) we find

\begin{equation}
\int_{a(t)}^{b_1(t)} \sqrt{Q_{(J_m),0}(x, t)}dx = \int_{-2\lambda_0(\tilde{t}_0(t))}^{\lambda_0(\tilde{t}_0(t))} \sqrt{\tilde{Q}_0(\tilde{x}, \tilde{t})}d\tilde{x}.
\end{equation}

Furthermore it is a real number when $t$ lies in the Stokes curve of $(P_\nu)_m$ in question; we may assume without loss of generality that the number is negative. We let $\rho = \rho(t)$ denote the number multiplied by $(-1)$. Let us now introduce the following functions $z_1(x, t)$ and $z_2(\tilde{x}, t)$:

\begin{equation}
z_1(x, t) = \int_{b_1(t)}^{x} \sqrt{Q_{(J_m),0}(y, t)}dy,
\end{equation}

\begin{equation}
z_2(\tilde{x}, t) = 2 \int_{\lambda_0(\tilde{t}_0(t))}^{\tilde{x}} (\tilde{y} - \lambda_0(\tilde{t}_0(t))) \sqrt{\tilde{y} + 2\lambda_0(\tilde{t}_0(t))}d\tilde{y}.
\end{equation}

We then try to construct $\tilde{x}_0(x, t)$ that satisfies

\begin{equation}
z_1(x, t) = z_2(\tilde{x}_0(x, t), t).
\end{equation}

It is clear that (3.2.16) guarantees (3.2.9) at the level of $\eta^{-1}$. Hence the construction of $\tilde{x}_0(x, t)$ satisfying (3.2.16) will show $(C)_0$.

Now, the following assertions immediately follow from the definitions of
$z_1, z_2$ and $\rho$:

(3.2.17) $z_1(\gamma, t)$, i.e., the image of the segment $\gamma$ by the map $z$, is a closed interval $[0, \rho]$,

(3.2.18) $\partial z_1 / \partial x \neq 0$ on $\gamma$ except for its endpoints,

(3.2.19) $z_1^{1/2}$ is holomorphic at $x = b_{j,0}(t)$ and $(\partial z_1^{1/2} / \partial x) |_{x=b_{j,0}(t)} \neq 0$,

(3.2.20) $(z_1 - \rho)^{2/3}$ is holomorphic at $x = a(t)$ and $\frac{\partial}{\partial x} (z_1 - \rho)^{2/3} |_{x=a(t)} \neq 0$,

(3.2.21) $z_2(\gamma, t) = [0, \rho]$,

(3.2.22) $\partial z_2 / \partial x \neq 0$ on $\gamma$ except for its endpoints,

(3.2.23) $z_2^{1/2}$ is holomorphic at $\bar{x} = \lambda_0(\bar{t}_0(t))$ and $\frac{\partial}{\partial x} z_2^{1/2} \big|_{\bar{x} = \lambda_0(\bar{t}_0(t))} \neq 0$,

(3.2.24) $(z_2 - \rho)^{2/3}$ is holomorphic at $\bar{x} = -2\lambda_0(\bar{t}_0(t))$.

We next consider the composition of maps $z_1$ and $z_2^{-1}$, the inverse map of $z_2$, and we denote it by $x_0$, that is,

(3.2.25) $x_0 = z_2^{-1} \circ z_1 : \gamma \rightarrow \gamma$.

It is then clear that

(3.2.26) $x_0(b_{j,0}(t), t) = \lambda_0(\bar{t}_0(t))$

and

(3.2.27) $x_0(a(t), t) = -2\lambda_0(\bar{t}_0(t))$

hold. It also follows from (3.2.18) and (3.2.22) that $x_0$ is holomorphic on $\gamma$ except for its endpoints and that $\partial x_0 / \partial x \neq 0$ holds there. To confirm its analyticity at $b_{j,0}(t)$ and $a(t)$, first say at $b_{j,0}(t)$, let us consider the following equation for $\bar{x}_0^1(x, t)$ near $x = b_{j,0}(t)$:

(3.2.28) $z_1(x, t)^{1/2} = z_2(\bar{x}_0^1(x, t), t)^{1/2}$,

where the branch of $z_1^{1/2}$ (resp., $z_2^{1/2}$) is chosen so that it may be positive in $\gamma$ (resp., $\gamma$). It then follows from (3.2.19) and (3.2.23) that (3.2.28) has a unique holomorphic solution $\bar{x}_0^1(x, t)$ near $x = b_{j,0}(t)$ that satisfies

(3.2.29) $\bar{x}_0^1(b_{j,0}(t), t) = \lambda_0(\bar{t}_0(t))$ and $\frac{\partial \bar{x}_0^1}{\partial x}(b_{j,0}(t), t) \neq 0$. 

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It is clear that $\bar{x}_0^1$ and $x_0$ coincide on their common domain of definition. Hence $\bar{x}_0$ is holomorphic at $x = b_{j,0}(t)$ and $\partial \bar{x}_0 / \partial x$ does not vanish there. The holomorphy of $\bar{x}_0(x, t)$ at $x = a(t)$ is also confirmed by a similar reasoning if we start with the following equation (3.2.30) instead of (3.2.28):

$$ (z_1(x, t) - \rho)^{2/3} = (z_2(\bar{x}(x, t), t) - \rho)^{2/3}. $$

Thus we have proved (C)$_0$. In the course of the proof we have also confirmed properties (i), (ii) and (iii) in the statement of the proposition.

We now embark on the proof of (C)$_n$ $(n \geq 1)$. Our method of the proof is essentially the same as that given in [KT1]. There is, however, one important difference: we have to construct non-zero $\bar{x}_j$ and $\bar{t}_j$ even for odd $j$. (As we show in Appendix, a 0-parameter solution of $(P_1)_m$ enjoys a nice property which guarantees (2.3); in this case we may assume $\bar{x}_j = \bar{t}_j = 0$ for odd $j$.

But a 0-parameter solution of $(P_{1\ell})_m$ or $(P_{12})_m$ does not have the property.) Our strategy of the proof is to construct a solution of the equation (3.2.31.2) below globally on $\Omega \times \omega$ by matching a solution holomorphic near $x = b_{j,0}(t)$ with another solution holomorphic near $x = a(t)$ with an appropriate choice of the "parameter" $\bar{t}_n(t)$. One technical problem in putting this idea into practice is the non-analyticity of the coefficients of (3.2.32.1) at $x = a(t)$; we circumvent this problem by considering another defining equation (3.2.32.1) of $x_n$ as a replacement of (3.2.32.1).

Now the actual task in proceeding from (C)$_{n-1}$ to (C)$_n$ $(n \geq 1)$ is to construct $\bar{x}_n(x, t)$ and $\bar{t}_n(t)$, the coefficients of $\eta^{1-n}$ of (3.2.9), so that the following relation (3.2.31.1) may be satisfied globally on $\Omega \times \omega$:

$$ R_{n-1}(x, t) = \bar{R}_{n-1}(\bar{x}_0(x, t), \bar{t}_0(x, t)) \frac{\partial \bar{x}_n}{\partial x}(x, t) 
+ \frac{\partial \bar{x}_0}{\partial x}(x, t) \left\{ \frac{\partial \bar{R}_{n-1}}{\partial \bar{x}}(\bar{x}_0(x, t), \bar{t}_0(x, t)) \bar{x}_n(x, t) 
+ \frac{\partial \bar{R}_{n-1}}{\partial \bar{t}}(\bar{x}_0(x, t), \bar{t}_0(x, t)) \bar{t}_n(x, t) \right\} + \hat{\rho}_n \quad (n \geq 1), $$

where $\hat{\rho}_n$ is a function of $\{\bar{x}_j, \bar{t}_k\}_{0 \leq j, k \leq n-1}$. Note that Theorem 2.1 guarantees that (3.2.31.1) is a differential equation for $\bar{x}_n(x, t)$ with analytic coefficients near $x = b_{j,0}(t)$. To make the computation run smoothly we introduce a new variable $z$ by defining it to be $\bar{x}_0(x, t)$. Then (3.2.31.1) can be rewritten as
follows:

\[(3.2.32.\, n) \quad \left( \tilde{R}_n \frac{\partial}{\partial z} + \frac{\partial \tilde{R}_n}{\partial \tilde{x}} \right) \tilde{x}_n = \left( \frac{\partial \tilde{x}_0}{\partial x} \right)^{-1} (R_{n-1} - \tilde{\rho}_n) - \frac{\partial \tilde{R}_n}{\partial \tilde{t}} \tilde{t}_n. \]

We also find the following relation \((3.2.33.\, n)\) through the comparison of the coefficients of \(\eta^{-n}\) of \((3.2.7)\) (divided by \((\partial \tilde{x}_0/\partial x)^2\)):

\[(3.2.33.\, n) \quad \left( 2\tilde{Q}_0 \frac{\partial}{\partial z} + \frac{\partial \tilde{Q}_0}{\partial \tilde{x}} \right) \tilde{x}_n = \left( \frac{\partial \tilde{x}_0}{\partial x} \right)^{-2} (Q_n - \tilde{r}_n) - \frac{\partial \tilde{Q}_0}{\partial \tilde{t}} \tilde{t}_n, \]

where \(\tilde{r}_n\) is a holomorphic function of \(\{\tilde{x}_j, \tilde{t}_k\}_{0 \leq j, k \leq n-1}\). In what follows we let \(L_{\tilde{R}}\) and \(L_{\tilde{Q}}\) denote respectively the differential operator

\[(3.2.34) \quad \tilde{R}_n \frac{\partial}{\partial z} + \frac{\partial \tilde{R}_n}{\partial \tilde{x}} \]

and another differential operator

\[(3.2.35) \quad 2\tilde{Q}_0 \frac{\partial}{\partial z} + \frac{\partial \tilde{Q}_0}{\partial \tilde{x}} \]

Clearly they satisfy

\[(3.2.36) \quad 2\sqrt{\tilde{Q}_0} L_{\tilde{R}} = L_{\tilde{Q}}. \]

It also follows immediately from the induction hypothesis that

\[(3.2.37) \quad 2\sqrt{\tilde{Q}_0} \left( \frac{\partial \tilde{x}_0}{\partial x} \right)^{-1} (R_{n-1} - \tilde{\rho}_n) = \left( \frac{\partial \tilde{x}_0}{\partial x} \right)^{-2} (Q_{n-1} - \tilde{r}_n). \]

Therefore \((3.2.32.\, n)\) and \((3.2.33.\, n)\) are equivalent; in what follows we make full use of this fact. Let us first note that the differential equation \(L_{\tilde{R}} u = f\) (resp., \(L_{\tilde{Q}} v = g\)) has a unique holomorphic solution \(u\) (resp., \(v\)) near \(z = \lambda_0(\tilde{t}_0(t))\) (resp., \(z = -2\lambda_0(\tilde{t}_0(t))\)) if \(f\) (resp., \(g\)) is holomorphic there, because the characteristic exponent of \(L_{\tilde{R}}\) (resp., \(L_{\tilde{Q}}\)) at \(z = \lambda_0\) (resp., \(z = -2\lambda_0\)) is equal to \(-1\) (resp., \(-1/2\)). Now let \(f_1\) and \(f_2\) respectively denote \((\partial \tilde{x}_0/\partial x)^{-1}(R_{n-1} - \tilde{\rho}_n)\) and \(\partial \tilde{R}_n/\partial \tilde{t}\). Then Theorem 2.1 together with the
induction hypothesis guarantees that \( f_1 \) and \( f_2 \) are holomorphic near \( z = \lambda_0 \). Hence we find a unique holomorphic solution \( \phi_j \) of the equation

\[
L_R \phi_j = f_j \quad (j = 1, 2)
\]

near \( z = \lambda_0 \). Since (3.2.37) entails the holomorphy of \( 2\sqrt{Q_0}f_1 \) at \( z = -2\lambda_0 \) and since \( 2\sqrt{Q_0}f_2 = \partial Q_0 / \partial \ell \) is clearly holomorphic at \( z = -2\lambda_0 \), we find a unique holomorphic solution \( \hat{\phi}_j \) of the equation

\[
L_{\bar{Q}} \hat{\phi}_j = 2\sqrt{Q_0}f_j \quad (j = 1, 2)
\]

near \( z = -2\lambda_0 \). Let now \( \phi \) denote a non-zero multi-valued analytic solution of \( L_R \phi = 0 \) on a neighborhood of \( \gamma \); it is unique up to a constant multiple. On the other hand, (3.2.36) implies

\[
L_{\bar{Q}} \phi_j = 2\sqrt{Q_0}f_j \quad (j = 1, 2)
\]

near \( z = -2\lambda_0 \) after the analytic continuation of \( \phi_j \) along \( \gamma \). Therefore we find

\[
\phi_j - \hat{\phi}_j = c_j \phi \quad (j = 1, 2)
\]

for some constants \( c_j \) \((j = 1, 2)\). If we can choose a constant \( \ell_n \) so that

\[
c_1 - \ell_n c_2 = 0
\]

holds, then, by choosing

\[
\bar{x}_n = \phi_1 - \ell_n \phi_2,
\]

we find that all the required conditions are satisfied. Thus what remains to be done is the confirmation of the non-vanishing of the constant \( c_2 \). It follows from the definition of the operator \( L_R \) and the function \( f_2 \) together with the explicit form of \((SL_1)\) (cf. (3.2.11)) that \( \phi_2 \) satisfies the following relation near \( z = \lambda_0 \):

\[
2(z + 2\lambda_0)^{1/2}(z - \lambda_0) \frac{\partial \phi_2}{\partial z} + \{(z - \lambda_0)(z + 2\lambda_0)^{-1/2} + 2(z + 2\lambda_0)^{1/2}\} \phi_2
\]

\[
= -2(z + 2\lambda_0)^{1/2} \frac{d\lambda_0}{dt} + 2(z - \lambda_0)(z + 2\lambda_0)^{-1/2} \frac{d\lambda_0}{dt},
\]

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that is,

\[(3.2.45) \quad (z + 2\lambda_0)(z - \lambda_0) \frac{\partial \phi_2}{\partial z} + \frac{3}{2}(z + \lambda_0) \phi_2 = -3\lambda_0 \frac{d\lambda_0}{dt}.
\]

Since we know (cf. (3.2.2))

\[(3.2.46) \quad 6\lambda_0 (\dot{t})^2 + \ddot{t} = 0,
\]

the right-hand side of (3.2.45) is equal to 1/4. Hence by integrating (3.2.45) we find

\[(3.2.47) \quad \phi_2 = \frac{1}{4(z - \lambda_0) \sqrt{(z + 2\lambda_0)}} \int^{z}_{\lambda_0} \frac{dw}{\sqrt{w + 2\lambda_0}}
\]

Then we analytically continue \(\phi_2\) near \(z = -2\lambda_0\) to find

\[(3.2.48) \quad \frac{1}{4(z - \lambda_0) \sqrt{(z + 2\lambda_0)}} \left( \int^{z}_{\lambda_0} \frac{dw}{\sqrt{w + 2\lambda_0}} + \int^{z}_{-2\lambda_0} \frac{dw}{\sqrt{w + 2\lambda_0}} \right).
\]

As it is evident that

\[(3.2.49) \quad L_R \left( \frac{1}{4(z - \lambda_0) \sqrt{(z + 2\lambda_0)}} \right) = 0,
\]

the expression (3.2.48) implies

\[(3.2.50) \quad c_2 \phi = \frac{1}{4(z - \lambda_0) \sqrt{(z + 2\lambda_0)}} \int^{z}_{-2\lambda_0} \frac{dw}{\sqrt{w + 2\lambda_0}}
\]

\[= -\frac{1}{2(z - \lambda_0) \sqrt{(z + 2\lambda_0)}}
\]

Thus we see that \(c_2\) is different from 0 on the condition that \(\lambda_0\) is different from 0. Since we are considering the problem near \(\sigma \neq \tau\), we may assume that \(\lambda_0(\ddot{t}(t))\) is different from 0 for \(t\) in \(\omega\). Thus we have constructed \((\bar{x}_n, \bar{t}_n)\) that satisfy (3.2.31.n), that is, the induction proceeds, completing the proof of Proposition 3.2.1.

Using the formal series \(\bar{x}(x, t, \eta)\) and \(\bar{t}(t, \eta)\) which satisfy (3.2.7), and hence (3.2.9), we obtain the following reduction theorem.
\textbf{Theorem 3.2.1.} In the geometric setting of Proposition 3.2.1, the series \( \tilde{x}(x,t,\eta) \) and \( \tilde{t}(t,\eta) \) constructed there satisfy the following relation:

\begin{equation}
\tilde{x}(x,t,\eta) \big|_{x=b_j(t,\eta)} = \lambda_t(\tilde{t}(t,\eta),\eta),
\end{equation}

where \( \lambda_t(\tilde{t},\eta) \) designates a 0-parameter solution of the traditional Painlevé-I equation, namely,

\begin{equation}
\frac{d^2\lambda_t}{dt^2} = \eta^2 (6\lambda_t^2 + \tilde{t}).
\end{equation}

\textit{Proof.} First we note that every \( \mathbf{a}_{(J,m)} \) \((J = I, II-1, II-2; m = 1, 2, \cdots)\) has the form

\begin{equation}
\mathbf{c}_{(J,m)}(x,t,\eta) = \frac{c_t(x,t,\eta)}{(x-b_j(t,\eta))},
\end{equation}

where \( \mathbf{c}_{(J,m)} \) has the form

\begin{equation}
\sum_{l \geq 0} c_l(x,t) \eta^{-l}
\end{equation}

with

\begin{equation}
c_0(x,t) \big|_{x=b_j,0(t)} \neq 0.
\end{equation}

In what follows we say, as is always the case in this paper, that a series in \( \eta^{-1} \) is holomorphic if the coefficient of \( \eta^{-1} \) is holomorphic on a fixed open set for every \( l \). Using this wording, we know by Theorem 2.1 that

\begin{equation}
\mathbf{c}(x,t,\eta) \big|_{x=b_j,0(t)} = 0.
\end{equation}

Since \( c_0 \) is different from 0 at \( x = b_j,0(t) \), the series \( \mathbf{c} \) is invertible as a formal series in \( \eta^{-1} \). Hence (3.2.56) implies

\begin{equation}
\mathbf{S}_{(J,m),odd} \big|_{x=b_j,0(t)} \big|_{x-b_j(t,\eta)} \text{ is holomorphic on a neighborhood of } x = b_j,0(t).
\end{equation}

On the other hand, (3.2.9) implies

\begin{equation}
\mathbf{S}_{(J,m),odd} \big|_{x-b_j(t,\eta)} \big|_{x-b_j(t,\eta)} = \frac{x-b_j(t,\eta)}{\lambda_t(\tilde{t}(t,\eta),\eta)} \frac{\partial \tilde{x}(x,t,\eta)}{\partial x} \mathbf{S}_{(J,m),odd} \big|_{x-b_j(t,\eta)} \big|_{x-b_j(t,\eta)}
\end{equation}

\begin{equation}
= \frac{x-b_j(t,\eta)}{\lambda_t(\tilde{t}(t,\eta),\eta)} \frac{\partial \tilde{x}(x,t,\eta)}{\partial x} \mathbf{S}_{(J,m),odd} \big|_{x-b_j(t,\eta)} \big|_{x-b_j(t,\eta)}.
\end{equation}
Since $\partial \tilde{x}_0 / \partial x$ is different from 0 at $x = b_{j,0}(t)$, the series $\partial \tilde{x} / \partial x$ is invertible there. We also find by an explicit computation that

$$
(3.2.59) \quad \frac{\tilde{S}_{l-1}(\tilde{x}_0(x, t), \tilde{t}_0(t))}{\tilde{x}_0(x, t) - \lambda_{1,0}(t_0(t))} = 2\sqrt{\tilde{x}_0(x, t) + 2\lambda_{1,0}(t_0(t))},
$$

which is clearly different from 0 at $x = b_{j,0}(t)$. Hence

$$
(3.2.60) \quad \frac{\tilde{S}_{l,0}(\tilde{x}(x, t, \eta), \tilde{t}(t, \eta), \eta)}{\tilde{x}(x, t, \eta) - \lambda_l(t(t, \eta), \eta)}
$$

is also invertible near $x = b_{j,0}(t)$. Therefore (3.2.57) and (3.2.58) imply that

$$
(3.2.61) \quad \tilde{x}(x, t, \eta) - \lambda_l(t(t, \eta), \eta) = (x - b_j(t, \eta))d(x, t, \eta)
$$

holds for some holomorphic series $d(x, t, \eta)$ near $x = b_{j,0}(t)$. Setting $x = b_j(t, \eta)$ in (3.2.61), we obtain the required relation (3.2.51).

**Appendix**

The purpose of this Appendix is to prove the following Proposition A.1 concerning the structure of a 0-parameter solution of $(P)_m$, which guarantees that $Q_{1,m}$ satisfies the condition (2.3).

**Proposition A.1.** Let $(\hat{u}, \hat{v}) = (\hat{u}_1, \cdots, \hat{u}_m, \hat{v}_1, \cdots, \hat{v}_m)$ be a 0-parameter solution of $(P)_m$ defined near $t = t_0$ and $\hat{w} = (\hat{w}_1, \cdots, \hat{w}_m)$ be the formal series determined by $(\hat{u}, \hat{v})$ through the relation (1.1.2). Assume that the simple turning point of $(SL)_m$, namely $x = -2\hat{u}_{1,0}(t)$, does not coincide with any double turning point of $(SL)_m$ at $t = t_0$. Then all the odd degree (in $\eta^{-1}$) terms of $\hat{u}, \hat{v}$ and $\hat{w}$ vanish.

**Remark A.1.** It is evident from (1.1.33) that the above assumption of non-coincidence of the simple turning point and a double turning point can be summarized as follows:

$$
(A.1) \quad U_0(-2\hat{u}_{1,0}(t_0)) \neq 0.
$$

To make the logical structure of the proof of Proposition A.1 lucid, we divide the proof into several steps; each step is summarized as a sublemma, and the proof of the Proposition is completed after Sublemma A.3.
Sublemma A.1. We find

\[ w_{j,1} = \hat{u}_{1,0} \hat{u}_{j,1} \]  \hspace{1cm} (A.2)

holds for \( j = 1,2,\ldots,m \).

Proof. As it follows from the definition of \( \hat{w}_j \) (cf. (1.1.2)) that

\[ \hat{w}_1 = \frac{1}{2} \hat{u}_1^2 + c_1 + \delta_{1m} t, \]  \hspace{1cm} (A.3)

we find

\[ \hat{w}_{1,1} = \hat{u}_{1,0} \hat{u}_{1,1}. \]  \hspace{1cm} (A.4)

Thus (A.2) holds for \( j = 1 \). We now use the induction on \( j \); let us suppose that (A.2) holds for \( j = 1,2,\ldots,j_0 \). The definition of \( \hat{w}_j \) implies

\[ \hat{w}_{j_0+1,1} = \frac{1}{2} \left( \sum_{k=1}^{j_0+1} \hat{u}_{k,0} \hat{w}_{j_0+2-k,1} + \sum_{k=1}^{j_0+1} \hat{u}_{k,1} \hat{w}_{j_0+2-k,0} \right) + \sum_{k=1}^{j_0} \left( \hat{u}_{k,0} \hat{w}_{j_0+1-k,1} + \hat{u}_{k,1} \hat{w}_{j_0+1-k,0} \right) \]  \hspace{1cm} (A.5)

because we know by (1.1.28)

\[ \hat{v}_{j,0} = 0, \hspace{0.5cm} j = 1,\ldots,m. \]  \hspace{1cm} (A.6)

Then the induction hypothesis entails

\[ \hat{w}_{j_0+1,1} = \sum_{l=1}^{j_0+1} \hat{u}_{j_0+2-l,0} \hat{u}_{l,1} + \sum_{k=1}^{j_0} \hat{u}_{k,0} \hat{u}_{1,0} \hat{w}_{j_0+1-k,1} + \sum_{k=1}^{j_0} \hat{u}_{k,1} \hat{w}_{j_0+1-k,0} \]  \hspace{1cm} (A.7)

\[ = \hat{u}_{1,0} \hat{u}_{j_0+1,1} + \sum_{l=1}^{j_0} (\hat{u}_{j_0+2-l,0} + \hat{u}_{1,0} \hat{w}_{j_0+1-l,0} + \hat{w}_{j_0+1-l,0}) \hat{u}_{l,1}. \]

Hence (1.1.27) with \( j = j_0 + 1 - l \) proves

\[ \hat{w}_{j_0+1,1} = \hat{u}_{1,0} \hat{w}_{j_0+1,1}. \]  \hspace{1cm} (A.8)

Thus the induction proceeds, completing the proof of Sublemma A.1.
**Sublemma A.2.** The coefficient of the degree one (in \( \eta^{-1} \)) part of \( \hat{u} = (\hat{u}_1, \cdots, \hat{u}_m) \), i.e., \( (\hat{u}_{1,1}, \hat{u}_{2,1}, \cdots, \hat{u}_{m,1}) \), is zero.

**Proof.** First, the comparison of the coefficients of \( \eta^0 \) in (1.1.1.1.b) with the help of (A.6) entails

(A.9) \[ \hat{u}_{j+1,1} + \hat{u}_{1,0} \hat{u}_{j,1} + \hat{u}_{1,1} \hat{u}_{j,0} + \hat{w}_{j,1} = 0, \quad j = 1, \cdots, m. \]

Then Sublemma A.1 implies

(A.10) \[ \hat{u}_{j+1,1} + 2\hat{u}_{1,0} \hat{u}_{j,1} + \hat{u}_{j,0}, \hat{u}_{1,1} = 0, \quad j = 1, \cdots, m. \]

Since \( \hat{u}_{m+1} \), and in particular \( \hat{u}_{m+1,1} \), vanishes by its definition, the relation (A.10) can be re-written as a matrix equation for the unknown vector \( \hat{t}(\hat{u}_{1,1}, \hat{u}_{2,1}, \cdots, \hat{u}_{m,1}) \):

\[
\begin{pmatrix}
3\hat{u}_{1,0} & 1 & 0 \\
\hat{u}_{2,0} & 2\hat{u}_{1,0} & 1 \\
\hat{u}_{3,0} & 0 & 2\hat{u}_{1,0} & 1 \\
\vdots & \ddots & \ddots & \ddots \\
\hat{u}_{m-1,0} & & & \vdots \\
\hat{u}_{m,0} & 0 & & \hat{u}_{m-1,1} \\
\end{pmatrix}
= 0.
\]

Then the determinant \( \Delta \) of the matrix in the left-hand side of (A.11) is

(A.12)

\[
3\hat{u}_{1,0}(2\hat{u}_{1,0})^{m-1} - \{ \hat{u}_{2,0}(2\hat{u}_{1,0})^{m-2} - \hat{u}_{3,0}(2\hat{u}_{1,0})^{m-3} + \hat{u}_{4,0}(2\hat{u}_{1,0})^{m-4} - \cdots + (-1)^{m-1}(\hat{u}_{m-1,0}(2\hat{u}_{1,0}) - \hat{u}_{m,0}) \}
\]

\[
= (-1)^{m} \{ (-2\hat{u}_{1,0})^{m} - \hat{u}_{1,0}(2\hat{u}_{1,0})^{m-1} - \hat{u}_{2,0}(-2\hat{u}_{1,0})^{m-2} - \hat{u}_{3,0}(-2\hat{u}_{1,0})^{m-3} - \hat{u}_{4,0}(-2\hat{u}_{1,0})^{m-4} - \cdots - \hat{u}_{m-1,0}(-2\hat{u}_{1,0}) - \hat{u}_{m,0} \}
\]

\[
= (-1)^{m} U_{0}(-2\hat{u}_{1,0}).
\]

Hence Remark A.1 guarantees that the determinant \( \Delta \) does not vanish at \( t = t_{0} \). Therefore the solution \( \hat{t}(\hat{u}_{1,1}, \hat{u}_{2,1}, \cdots, \hat{u}_{m,1}) \) of the homogeneous equation (A.11) should be 0. This completes the proof of Sublemma A.2.

**Sublemma A.3.** Suppose that

(A.13.p_{0}) \[ \hat{u}_{j,2p-1} = \hat{w}_{j,2p-1} = \hat{v}_{j,2p-2} = 0 \quad (j = 1, 2, \cdots, m) \]

*hold for* \( p = 1, 2, \cdots, p_{0} \).
Then we find

\[(A.14.p_0 + 1) \quad \hat{w}_{j,2p_0+1} = \hat{u}_{1,0}\hat{u}_{j,2p_0+1} \quad (j = 1, 2, \cdots, m).\]

**Proof.** We can use essentially the same reasoning as in the proof of Sublemma A.1. First we note that (A.3) together with (A.13.p_0) entails

\[(A.15) \quad \hat{w}_{1,2p_0+1} = \hat{u}_{1,0}\hat{u}_{1,2p_0+1}.\]

Hence we use the induction on \(j\) to prove (A.14.p_0 + 1), starting with (A.15): let us suppose

\[(A.16) \quad \hat{w}_{j,2p_0+1} = \hat{u}_{1,0}\hat{u}_{j,2p_0+1}\]

holds for \(j = 1, 2, \cdots, j_0\). Since

\[(A.17) \quad \hat{v}_{j,2p_0} = \frac{1}{2} \frac{d\hat{u}_{j,2p_0-1}}{dt}\]

holds by (1.1.1.a), (A.13.p_0) implies

\[(A.18) \quad \hat{v}_{j,2p_0} = 0 \quad (j = 1, 2, \cdots, m).\]

Then, in parallel with (A.5), we find

\[(A.19) \quad \hat{w}_{j_0+1,2p_0+1} = \frac{1}{2} \left( \sum_{k=1}^{j_0+1} \hat{u}_{k,0}\hat{u}_{j_0+2-k,2p_0+1} + \sum_{k=1}^{j_0+1} \hat{u}_{k,2p_0+1}\hat{u}_{j_0+2-k,0} \right) \]

\[+ \sum_{k=1}^{j_0} \left( \hat{u}_{k,0}\hat{w}_{j_0+1-k,2p_0+1} + \hat{u}_{k,2p_0+1}\hat{w}_{j_0+1-k,0} \right) .\]

Hence the induction hypothesis together with (1.1.27) (with \(j = j_0 + 1 - k\)) proves

\[(A.20) \quad \hat{w}_{j_0+1,2p_0+1} = \hat{u}_{1,0}\hat{u}_{j_0+1,2p_0+1}.\]

Thus the induction on \(j\) proceeds, proving (A.14.p_0 + 1).
Proof of Proposition A.1. Sublemma A.2, Sublemma A.1 and (A.6) imply that \((A.13,p_0)\) is true for \(p_0 = 1\). We now prove by induction on \(p_0\) that \((A.13,p_0)\) holds for every \(p_0 = 1, 2, \cdots\); it clearly proves Proposition A.1. In view of Sublemma A.3 and (A.18), it suffices to prove that \((A.13,p_0)\) implies

\[
\hat{u}_{j,2p_0+1} = 0 \text{ for } j = 1, \cdots, m.
\]

Now, with the help of the induction hypothesis supplemented by (A.18), the comparison of the coefficients of \(\eta^{-2p_0}\) in (1.1.1b) gives us

\[
\hat{u}_{j+1,2p_0+1} + (\hat{u}_{1,0}\hat{u}_{j,2p_0+1} + \hat{u}_{1,2p_0+1}\hat{u}_{j,0}) + \hat{u}_{j,2p_0+1} = 0
\]

for every \(j = 1, 2, \cdots, m\). Then, applying Sublemma A.3 to (A.22), we find

\[
\hat{u}_{j+1,2p_0+1} + 2\hat{u}_{1,0}\hat{u}_{j,2p_0+1} + \hat{u}_{j,0}\hat{u}_{1,2p_0+1} = 0, \quad j = 1, 2, \cdots, m.
\]

Since \(\hat{u}_{m+1,2p_0+1} = 0\) by (1.1.1c), (A.23) leads to the same matrix equation as (A.11) with the replacement of the unknown vector \(i(\hat{u}_{1,1}, \hat{u}_{2,1}, \cdots, \hat{u}_{m,1})\) by \(i(\hat{u}_{1,2p_0+1}, \hat{u}_{2,2p_0+1}, \cdots, \hat{u}_{m,2p_0+1})\), in exactly the same manner as (A.10) has led to (A.11). We have already confirmed in the proof of Sublemma A.2 that the determinant \(\Delta\) of the coefficient matrix in (A.11) is different from 0 at \(t = t_0\) by the assumption of Proposition A.1. Therefore we conclude that \(i(\hat{u}_{1,2p_0+1}, \hat{u}_{2,2p_0+1}, \cdots, \hat{u}_{m,2p_0+1})\) should vanish. Thus the induction on \(p_0\) proceeds, and we have completed the proof of Proposition A.1.

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