Galois Sections in Absolute Anabelian Geometry

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Introduction

In this paper, we continue our study of the *absolute anabelian geometry* of hyperbolic curves over p-adic local fields [i.e., finite extensions of the field of p-adic numbers, for some prime number p], begun in [Mzk2], [Mzk3]. In [Mzk3], Theorem 2.4, it was shown, as a consequence of the main theorem of [Mzk1], that certain *categories of finite étale correspondences* associated to a hyperbolic curve X_K over a p-adic local field K may be recovered from the *profinite group structure* of the étale fundamental group Π_{X_K} of X_K. In the present paper, we generalize this result to show [again as a consequence of the main theorem of [Mzk1]] that certain *categories of arbitrary dominant* [i.e., not necessarily finite étale] *correspondences* associated to X_K may be recovered from the profinite group structure of Π_{X_K} [cf. Theorem 2.3]. We then apply this result to study the extent to which the *decomposition groups* associated to closed points of X_K may be recovered from the profinite group structure of Π_{X_K} [cf. Corollaries 2.5, 2.6, 3.2]. One result that is representative of these techniques is the following special case of Corollary 3.2:

Theorem A. *Let K be a finite extension of $\mathbb{Q}_p$; X_K a* **hyperbolic curve of genus zero** *over K which is, in fact,* **defined over a number field**. *Write Π_{X_K} for the étale fundamental group of X_K. Then any automorphism of the profinite group Π_{X_K}* **preserves the decomposition groups** $\subseteq \Pi_{X_K}$ *associated to the closed points of X_K.*

This result may be regarded as a sort of [very] *weak version of the "Section Conjecture"* [cf., e.g., [Mzk1], §19 for more on the "Section Conjecture"]. Finally, in §4, we show, in the notation of Theorem A, that *various canonical auxiliary structures* associated to the decomposition groups of cusps of X_K are also preserved by arbitrary automorphisms of Π_{X_K} [cf. Corollary 4.11].

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Section 0: Notations and Conventions

Numbers:

If p is a *prime number*, then we shall denote by $\mathbb{Q}_p$ the *field of p-adic numbers*, i.e., the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic valuation of $\mathbb{Q}$. We shall refer to a field which is isomorphic to a finite extension of $\mathbb{Q}_p$ for some p as a *local field*. [In particular, in this paper, all "local fields" are *nonarchimedean*.] A *number field* is defined to be a finite extension of the field of rational numbers $\mathbb{Q}$.

Topological Groups:

Let G be a *Hausdorff topological group*, and $H \subseteq G$ a *closed subgroup*. Let us write

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot h = h \cdot g, \ \forall \ h \in H\}$$

for the *centralizer* of H in G;

$$N_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$

for the *normalizer* of H in G; and

$$C_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid (g \cdot H \cdot g^{-1}) \bigcap H \text{ has finite index in } H, \ g \cdot H \cdot g^{-1}\}$$

for the *commensurator* of H in G. Note that: (i) $Z_G(H)$, $N_G(H)$ and $C_G(H)$ are *subgroups of* G; (ii) we have *inclusions*

$$H, \ Z_G(H) \subseteq N_G(H) \subseteq C_G(H)$$

and (iii) H is *normal* in $N_G(H)$. If $H = C_G(H)$, then we shall say that H is *commensurably terminal* in G. Note that $Z_G(H)$, $N_G(H)$ are *always closed in* G, while $C_G(H)$ is *not necessarily closed in* G.

If G_1, G_2 are *Hausdorff topological groups*, then an *outer homomorphism* $G_1 \to G_2$ is defined to be an equivalence class of continuous homomorphisms $G_1 \to G_2$, where two such homomorphisms are considered equivalent if they differ by composition with an inner automorphism of G_2.

Categories:

Let $\mathcal{C}$ be a *category*. We shall denote the collections of *objects* and *arrows* of $\mathcal{C}$ by

$$\mathrm{Ob}(\mathcal{C}); \quad \mathrm{Arr}(\mathcal{C})$$

respectively. If $A \in \mathrm{Ob}(\mathcal{C})$ is an *object* of $\mathcal{C}$, then we shall denote by

$$\mathcal{C}_A$$

the category whose *objects* are morphisms $B \to A$ of $\mathcal{C}$ and whose morphisms (from an object $B_1 \to A$ to an object $B_2 \to A$) are A-morphisms $B_1 \to B_2$ in $\mathcal{C}$.

We shall refer to a *natural transformation* between functors [from one category to another] all of whose component morphisms are *isomorphisms* as an *isomorphism between the functors* in question. A functor $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ between categories $\mathcal{C}_1$, $\mathcal{C}_2$ will be called *rigid* if ϕ has no nontrivial automorphisms. A category $\mathcal{C}$ will be called *slim* if the natural functor $\mathcal{C}_A \to \mathcal{C}$ is *rigid*, for every $A \in \mathrm{Ob}(\mathcal{C})$.

Given two arrows $f_i : A_i \to B_i$ (where $i = 1, 2$) in a category $\mathcal{C}$, we shall refer to a commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\sim} & A_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{\sim} & B_2 \end{array}$$

— where the horizontal arrows are isomorphisms in $\mathcal{C}$ — as an *abstract equivalence* from f_1 to f_2. If there exists an abstract equivalence from f_1 to f_2, then we shall say that f_1, f_2 are *abstractly equivalent* and write $f_1 \overset{\mathrm{abs}}{\approx} f_2$.

Let G be a profinite group. Then we recall that the category $\mathcal{B}(G)$ of finite sets with continuous G-action and morphisms of G-sets is *slim* if and only if $Z_G(H) = \{1\}$ for all open subgroups $H \subseteq G$.

Curves:

Suppose that $g \geq 0$ is an *integer*. Then if S is a scheme, a *family of curves of genus* g

$$X \to S$$

is defined to be a smooth, proper, geometrically connected morphism of schemes $X \to S$ whose geometric fibers are curves of genus g.

Suppose that $g, r \geq 0$ are *integers* such that $2g - 2 + r > 0$. We shall denote the *moduli stack of r-pointed stable curves of genus g* (where we assume the points to be *unordered*) by $\overline{\mathcal{M}}_{g,r}$ [cf. [DM], [Knud] for an exposition of the theory of such curves; strictly speaking, [Knud] treats the finite étale covering of $\overline{\mathcal{M}}_{g,r}$ determined by *ordering* the marked points]. The open substack $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ of smooth curves will be referred to as the *moduli stack of smooth r-pointed stable curves of genus g* or, alternatively, as the *moduli stack of hyperbolic curves of type* (g, r). The *divisor at infinity* $\overline{\mathcal{M}}_{g,r} \backslash \mathcal{M}_{g,r}$ of $\overline{\mathcal{M}}_{g,r}$ determines a *log structure* on $\overline{\mathcal{M}}_{g,r}$; denote the resulting log stack by $\overline{\mathcal{M}}_{g,r}^{\log}$.

A *family of hyperbolic curves of type* (g, r)

$$X \to S$$

is defined to be a morphism which factors $X \hookrightarrow Y \to S$ as the composite of an open immersion $X \hookrightarrow Y$ onto the complement $Y \backslash D$ of a relative divisor $D \subseteq Y$

which is finite étale over S of relative degree r, and a family $Y \to S$ of curves of genus g. One checks easily that, if S is *normal*, then the pair (Y, D) is *unique up to canonical isomorphism*. (Indeed, when S is the spectrum of a field, this fact is well-known from the elementary theory of algebraic curves. Next, we consider an arbitrary *connected normal* S on which a prime l is *invertible* (which, by Zariski localization, we may assume without loss of generality). Denote by $S' \to S$ the finite étale covering parametrizing *orderings of the marked points* and *trivializations of the l-torsion points of the Jacobian of Y*. Note that $S' \to S$ is *independent* of the choice of (Y, D), since (by the normality of S), S' may be constructed as the *normalization* of S in the function field of S' (which is independent of the choice of (Y, D) since the restriction of (Y, D) to the generic point of S has already been shown to be unique). Thus, the uniqueness of (Y, D) follows by considering the classifying morphism (associated to (Y, D)) from S' to the finite étale covering of $(\mathcal{M}_{g,r})_{\mathbb{Z}[\frac{1}{l}]}$ parametrizing orderings of the marked points and trivializations of the l-torsion points of the Jacobian [since this covering is well-known to be a scheme, for l sufficiently large].) We shall refer to Y (respectively, D; D; D) as the *compactification* (respectively, *divisor at infinity*; *divisor of cusps*; *divisor of marked points*) of X. A *family of hyperbolic curves* $X \to S$ is defined to be a morphism $X \to S$ such that the restriction of this morphism to each connected component of S is a *family of hyperbolic curves of type* (g, r) for some integers (g, r) as above. If the divisor of cusps of a family of hyperbolic curves $X \to S$ forms a *split* finite étale covering over S, then we shall say that this family of hyperbolic curves is *cuspidally split*. A family of hyperbolic curves $X \to S$ of type $(0, 3)$ (respectively, $(1, 1)$) will be referred to as a *tripod* (respectively, *once-punctured elliptic curve*).

If X_K (respectively, Y_L) is a *hyperbolic curve over a field* K (respectively, L), then we shall say that X_K is *isogenous* to Y_L if there exists a hyperbolic curve Z_M over a field M together with *finite étale morphisms* $Z_M \to X_K$, $Z_M \to Y_L$.

Section 1: Brief Review of Anabelian Geometry

Let K, L be *local fields* [cf. §0]; X_K (respectively, Y_L) a *hyperbolic curve* [cf. §0] over K (respectively, L). Any choice of basepoint for X_K determines, up to inner automorphism, the *étale fundamental group* $\Pi_{X_K} \stackrel{\text{def}}{=} \pi_1(X_K)$ of X_K. Moreover, Π_{X_K} fits into a *natural exact sequence*

$$1 \to \Delta_X \to \Pi_{X_K} \to G_K \to 1$$

where G_K is the *absolute Galois group* of K; Δ_X, which is often referred to as the *geometric fundamental group* of X_K, is defined so as to make the sequence exact. Any choice of basepoint for Y_L determines a similar exact sequence for Y_L.

Proposition 1.1. (First Properties)

(i) Π_{X_K} *is slim [cf. §0].*

(ii) Every isomorphism of profinite groups $\alpha : \Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_L}$ *fits into a unique commutative diagram*

$$\begin{array}{ccc} \Pi_{X_K} & \xrightarrow{\alpha} & \Pi_{Y_L} \\ \downarrow & & \downarrow \\ G_K & \longrightarrow & G_L \end{array}$$

where the vertical arrows are the surjections of the natural exact sequence(s) discussed above; the horizontal arrows are isomorphisms.

Proof. Assertion (i) (respectively, (ii)) follows from [Mzk2], Lemma 1.3.1 (respectively, [Mzk2], Lemma 1.3.8). ◯

Theorem 1.2. (Anabelian Theorem for Hyperbolic Curves over Local Fields) *The étale fundamental group functor determines a bijection between the set of* **dominant morphisms of schemes**

$$X_K \to Y_L$$

and the set of **open outer homomorphisms** $\phi : \Pi_{X_K} \to \Pi_{Y_L}$ *that fit into a commutative diagram*

$$\begin{array}{ccc} \Pi_{X_K} & \xrightarrow{\phi} & \Pi_{Y_L} \\ \downarrow & & \downarrow \\ G_K & \longrightarrow & G_L \end{array}$$

for which the induced morphism $G_K \to G_L$ *is an open immersion [i.e., an isomorphism onto an open subgroup of* G_L*] which arises from an embedding of fields* $L \hookrightarrow K$*.*

Proof. Recall that given a local field M, the topology of M may be always be recovered solely from the field structure of M by observing that the ring of integers $\mathcal{O}_M$ of M is additively generated by $\mathcal{O}_M^\times$, and that $\mathcal{O}_M^\times \subseteq M$ is equal to the subgroup of elements of $M^\times$ that are *infinitely divisible* by powers of some prime number. In particular, the $\mathbb{Q}_p$-algebra structure of M [for some suitable prime number p], as well as the prime number p itself [i.e., the unique prime number l such that $\mathcal{O}_M$ is not infinitely divisible by powers of l], may be recovered from the *field structure* of M. In a similar vein, given a function field in one variable M' over M, consideration of the discrete valuations on M' with trivial restriction to M reveals that the subfield $M \subseteq M'$ may be recovered — solely from the *field structure* of M' — as the subfield generated by the elements of $(M')^\times$ that are *infinitely divisible* by powers of some prime number. In light of these remarks, Theorem 1.2 follows formally from [Mzk1], Theorem A. ◯

Next, let us write $X_K \hookrightarrow \overline{X}_K$ for the *compactification* [cf. §0] of X_K. Let

$$x \in \overline{X}_K$$

be a *closed point*. Thus, x determines, up to conjugation by an element of Π_{X_K}, a *decomposition group*:

$$D_x \subseteq \Pi_{X_K}$$

We shall refer to a closed subgroup of Π_{X_K} which arises in this way as a *decomposition group* of Π_{X_K}. If x is a *cusp*, then we shall refer to the decomposition group D_x as *cuspidal.* Note that D_x always *surjects* onto an open subgroup of G_K. Moreover, the subgroup

$$I_x \stackrel{\text{def}}{=} D_x \bigcap \Delta_X$$

is isomorphic to $\widehat{\mathbb{Z}}(1)$ [i.e., the profinite completion of $\mathbb{Z}$, Tate twisted once] (respectively, $\{1\}$) if x is (respectively, is not) a *cusp.* We shall refer to a closed subgroup of Π_{X_K} which is equal to "I_x" for some *cusp* x as a *cuspidal geometric decomposition group.*

Theorem 1.3. (Decomposition Groups)

(i) **(Determination of the Point)** *The closed point x is completely determined by the conjugacy class of the closed subgroup $D_x \subseteq \Pi_{X_K}$. If x is a* **cusp**, *then x is completely determined by the conjugacy class of the closed subgroup $I_x \subseteq \Pi_{X_K}$.*

(ii) **(Commensurable Terminality)** *The subgroup D_x is commensurably terminal in Π_{X_K}. If x is a* **cusp**, *then $D_x = C_{\Pi_{X_K}}(H)$ for any open subgroup $H \subseteq I_x$.*

(iii) **(Absoluteness of Cuspidal Decomposition Groups)** *Every isomorphism of profinite groups*

$$\alpha : \Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_L}$$

preserves cuspidal decomposition groups and cuspidal geometric decomposition groups.

(iv) **(Cuspidal and Noncuspidal Decomposition Groups)** *No noncuspidal decomposition group of Π_{X_K} is contained in a cuspidal decomposition group of Π_{X_K}.*

Proof. The first half of assertion (i) follows, for instance, formally from [Mzk1], Theorem C; the second half of assertion (i) follows from elementary facts about fundamental groups of topological surfaces. Assertion (ii) follows formally from assertion (i) and the definition of a "decomposition group". Assertion (iii) follows from assertion (ii) and [Mzk2], Lemma 1.3.9. As for assertion (iv), we may assume, by passing to a finite étale covering of X_K, that X_K is of genus ≥ 2, so that $\overline{X}_K$ is still hyperbolic. Then assertion (iv) follows from assertion (i). ◯

Section 2: Categories of Dominant Morphisms

Let X_K be a *hyperbolic curve* over a field K. Write $X_K \hookrightarrow \overline{X}_K$ for the *compactification* of X_K.

Definition 2.1.

(i) We shall refer to an open immersion

$$X_K \hookrightarrow Y_K$$

as a *partial compactification*, or *PC*, for short, of X_K if the natural open immersion $X_K \hookrightarrow \overline{X}_K$ factors as the composite of the given morphism $X_K \hookrightarrow Y_K$ with some open immersion $Y_K \hookrightarrow \overline{X}_K$. By abuse of notation, we shall also often speak of "Y_K" as a PC of X_K.

(ii) If $X_K \hookrightarrow Y_K$ is a PC such that Y_K is a *hyperbolic curve*, then we shall say that $X_K \hookrightarrow Y_K$ [or Y_K] is a *hyperbolic partial compactification*, or *HPC*, of X_K.

(iii) If $X_K \hookrightarrow Y_K$ is a PC such that the arrow "$\hookrightarrow$" is an *isomorphism*, then we shall say that $X_K \hookrightarrow Y_K$ [or Y_K] is a *trivial partial compactification* of X_K.

Now we define a *"category of dominant localizations"*

$$\mathrm{DLoc}(X_K)$$

associated to the hyperbolic curve X_K as follows: The *objects* of this category are the *hyperbolic partial compactifications*

$$Y \hookrightarrow Z$$

where Y is a hyperbolic curve over some field [which is necessarily a finite separable extension of K] that arises as a *finite étale covering* $Y \to X_K$. The *morphisms* of this category from an object $Y \hookrightarrow Z$ to an object $Y' \hookrightarrow Z'$ are diagrams of the form

$$\begin{array}{ccc} Y & & Y' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z' \end{array}$$

where the vertical morphisms are the given morphisms, and the horizontal morphism is a *dominant* morphism of schemes. By abuse of notation, we shall often simply refer to the horizontal arrow $Z \to Z'$ as being a *morphism* of $\mathrm{DLoc}(X_K)$.

Similarly, by stipulating that all schemes appearing in the definition of the category $\mathrm{DLoc}(X_K)$ given in the preceding paragraph be equipped with *K-structures* [where we take the K-structure on X_K to be the given K-structure] and that all morphisms be *K-morphisms*, we obtain a category

$$\mathrm{DLoc}_K(X_K)$$

together with a *natural faithful functor* $\mathrm{DLoc}_K(X_K) \to \mathrm{DLoc}(X_K)$.

Remark 2.2.0. Thus, the category $\mathrm{DLoc}(X_K)$ is reminiscent of the category "$\mathrm{Loc}(X_K)$" of [Mzk3], §2. Indeed, there is a *natural faithful functor*

$$\mathrm{Loc}(X_K) \to \mathrm{DLoc}(X_K)$$

whose essential image consists of the objects $Y \hookrightarrow Z$ which are *trivial partial compactifications* and the dominant morphisms $Z \to Z'$ which are *finite étale*. In particular, if we denote by $\acute{\mathrm{E}}\mathrm{t}(X_K)$ the category of *finite étale coverings* of X_K and *morphisms* over X_K, then we have *natural faithful functors*:

$$\acute{\mathrm{E}}\mathrm{t}(X_K) \to \mathrm{Loc}(X_K) \to \mathrm{DLoc}(X_K)$$

Similarly, we have natural faithful functors: $\acute{\mathrm{E}}\mathrm{t}(X_K) \to \mathrm{Loc}_K(X_K) \to \mathrm{DLoc}_K(X_K)$.

Proposition 2.2. (Slimness of the Category of Dominant Localizations) *Suppose that K is a* **local field***. Then the categories* $\mathrm{DLoc}(X_K)$, $\mathrm{DLoc}_K(X_K)$ *are* **slim**.

Proof. Indeed, by using the various copies of "$\acute{\mathrm{E}}\mathrm{t}(Z)$" [where, say, $Y \hookrightarrow Z$ is an object of $\mathrm{DLoc}(X_K)$] lying inside $\mathrm{DLoc}(X_K)$, $\mathrm{DLoc}_K(X_K)$ [cf. Remark 2.2.0], the slimness of the categories $\mathrm{DLoc}(X_K)$, $\mathrm{DLoc}_K(X_K)$ follows formally from Proposition 1.1, (i) [cf. also the discussion of slimness in §0]. ○

Next, let us consider the category $\mathrm{DLoc}_{G_K}(\Pi_{X_K})$ defined as follows: An *object* of this category is a surjection of profinite groups

$$H \twoheadrightarrow J$$

where $H \subseteq \Pi_{X_K}$ is an open subgroup; J is the quotient of H by the closed normal subgroup generated by some collection of *cuspidal geometric decomposition groups*; and we assume that J is *"hyperbolic"*, in the sense that the image of $\Delta_X \bigcap H$ in J is *nonabelian*. Given two objects $H_i \twoheadrightarrow J_i$, where $i = 1, 2$, of this category, a morphism in this category is defined to be a diagram of the form

$$\begin{array}{ccc} H_1 & & H_2 \\ \downarrow & & \downarrow \\ J_1 & \longrightarrow & J_2 \end{array}$$

where the vertical morphisms are the given morphisms, and the horizontal morphism is an *open outer homomorphism* that is *compatible* with the various natural [open] outer homomorphisms from the H_i, J_i to G_K.

Now we have the following analogue of [Mzk3], Theorem 2.4:

Theorem 2.3. (Group-theoreticity of the Category of Dominant Localizations) *Let K, L be* **local fields***; X_K (respectively, Y_L) a* **hyperbolic curve** *over K (respectively, L). Then:*

(i) The étale fundamental group functor determines **equivalences of categories**

$$\mathrm{DLoc}_K(X_K) \xrightarrow{\sim} \mathrm{DLoc}_{G_K}(\Pi_{X_K}); \quad \mathrm{DLoc}_L(Y_L) \xrightarrow{\sim} \mathrm{DLoc}_{G_L}(\Pi_{Y_L})$$

(ii) Every isomorphism of profinite groups

$$\alpha : \Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_L}$$

induces an **equivalence of categories**

$$\mathrm{DLoc}_{G_K}(\Pi_{X_K}) \xrightarrow{\sim} \mathrm{DLoc}_{G_L}(\Pi_{Y_L})$$

hence also [by applying the equivalences of (i)] an **equivalence of categories**

$$\mathrm{DLoc}_K(X_K) \xrightarrow{\sim} \mathrm{DLoc}_L(Y_L)$$

in a fashion that is **functorial***, up to unique isomorphisms of equivalences of categories, with respect to* α*.*

Proof. Indeed, assertion (i) follows formally from Theorem 1.2, while assertion (ii) follows, in light of Proposition 1.1, (ii); Theorem 1.3, (iii), formally from the definition of the categories "$\mathrm{DLoc}_{G_K}(\Pi_{X_K})$", "$\mathrm{DLoc}_{G_L}(\Pi_{Y_L})$". [Here, we note that the uniqueness of the isomorphisms of equivalences of categories involved follows from Proposition 2.2.] ○

Next, let

$$D_x \subseteq \Pi_{X_K}$$

be a *decomposition group* associated to some closed point $x \in \overline{X}_K$.

Definition 2.4. We shall say that x or D_x is *of* DLoc*-type* if D_x admits an open subgroup that arises as the image via a morphism $Z \to X_K$ of $\mathrm{DLoc}_K(X_K)$ of some *cuspidal* decomposition group of Π_Z.

Corollary 2.5. (Group-theoreticity of Decomposition Groups of DLoc-type) *In the notation of Theorem 2.3, the isomorphism*

$$\alpha : \Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_L}$$

preserves the decomposition groups of DLoc*-type.*

Proof. This follows immediately from the definitions; Theorem 2.3 [and its proof]; Theorem 1.3, (ii), (iii). ○

Corollary 2.6. **(The Case of Once-punctured Elliptic Curves)** *In the notation of Theorem 2.3, let us suppose further that* X_K*,* Y_L *are* **once-punctured elliptic curves***. Then the isomorphism*

$$\alpha : \Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_L}$$

preserves the decomposition groups of the "torsion closed points" — i.e., the closed points that arise from **torsion points** *of the underlying elliptic curve. Moreover, the resulting bijection between torsion closed points of* X_K*,* Y_L *is* **compatible** *with the isomorphism on abelianizations of geometric fundamental groups* $\Delta_X^{\mathrm{ab}} \xrightarrow{\sim} \Delta_Y^{\mathrm{ab}}$ *— i.e., "Tate modules" — induced by* α*.*

Proof. Indeed, if $n \geq 1$ is an integer, write

$$\phi : Z_K \to X_K$$

for the finite étale covering determined by "multiplication by n". Note that this covering may also be described more group-theoretically as the covering associated to the open subgroup $H \subseteq \Pi_{X_K}$ [which is easily verified to be unique, up to conjugation in Π_{X_K}] such that: (i) H contains a cuspidal decomposition group of Π_{X_K}; (ii) $H \bigcap \Delta_X$ is equal to the inverse image in Δ_X of the subgroup $n \cdot \Delta_X^{\mathrm{ab}} \subseteq \Delta_X^{\mathrm{ab}}$.

Observe that Z_K admits X_K as an HPC, by "filling in" all of the cusps other that the "origin". Thus, we obtain an *open immersion*

$$\psi : Z_K \hookrightarrow X_K$$

— i.e., an object of $\mathrm{DLoc}_K(X_K)$, which exhibits the closed points of X_K that arise from n-torsion points of the underlying elliptic curve as *closed points of* DLoc-*type type*. Thus, by *transporting* ϕ, ψ via the equivalences of Theorem 2.3, (i), and applying Theorem 1.3, (ii), (iii) [as in the proof of Corollary 2.5], we conclude that α preserves the decomposition groups of the torsion closed points. Finally, the *compatibility* with the induced morphism on Tate modules follows by considering the *automorphisms* of Z_K over [i.e., relative to ϕ] X_K, after possibly enlarging K. ○

Definition 2.7. We shall say that a closed point $x \in \overline{X}_K$ is *algebraic* if, for some finite extension L of K, some hyperbolic curve Y_F over a *number field* $F \subseteq L$, and some L-isomorphism $X_L \xrightarrow{\sim} Y_L$ [where $X_L \stackrel{\mathrm{def}}{=} X_K \times_K L$, $Y_L \stackrel{\mathrm{def}}{=} Y_F \times_F L$], x lies under a closed point $x_L \in \overline{X}_L$ which maps to a *closed point* of $\overline{Y}_F$ under the composite $\overline{X}_L \xrightarrow{\sim} \overline{Y}_L \to \overline{Y}_F$.

Remark 2.7.1. One verifies immediately that if a closed point $x \in \overline{X}_K$ is *algebraic*, then given *any* L'-isomorphism

$$X_{L'} \xrightarrow{\sim} Y'_{L'}$$

[where $X_{L'} \stackrel{\text{def}}{=} X_K \times_K L'$; $Y'_{L'} \stackrel{\text{def}}{=} Y'_{F'} \times_{F'} L'$; L' is a finite extension of K; $Y'_{F'}$ is a hyperbolic curve over a number field $F' \subseteq L'$], it holds that any point $x_{L'} \in \overline{X}_{L'}$ lying over x maps to a *closed point* of $\overline{Y}'_{F'}$ under the composite $\overline{X}_{L'} \xrightarrow{\sim} \overline{Y}'_{L'} \to \overline{Y}'_{F'}$.

Corollary 2.8. **(The Case of Genus Zero)** *In the notation of Theorem 2.3, let us suppose further that X_K (respectively, Y_L) is* **isogenous** *[cf. §0] to a hyperbolic curve of* **genus zero**. *Then the isomorphism*

$$\alpha : \Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_L}$$

preserves the decomposition groups of the **algebraic** *closed points. In particular, X_K is* **defined over a number field** *[or, equivalently: X_K has at least one algebraic point] if and only if Y_L is.*

Proof. By Theorem 1.3, (ii), and [the "$\mathrm{Loc}_K(-)$ portion" — already contained in [Mzk3], Theorem 2.4 — of] Theorem 2.3, (ii), one reduces immediately to the case where both X_K and Y_L are of *genus zero*. Also, by Theorem 1.3, (ii), we may always enlarge K, L without loss of generality; in particular, we may assume that X_K, Y_L are *cuspidally split*, so that both curves admit a [cuspidally split] *tripod* as an HPC. Then we argue as in the proof of Corollary 2.6: That is to say, given any *algebraic* $x \in \overline{X}_K$, we observe that [after possibly enlarging K] there exists, by the definition of "algebraic" and the famous main result of [Belyi], a *"Belyi map"*

$$\beta : X_K \to \overline{X}_K$$

that maps x, as well as all of the cusps of X_K, to *cusps* of $\overline{X}_K$, and, moreover, is *unramified* over the open subscheme of $\overline{X}_K$ determined by the *tripod* that forms an HPC for X_K. In particular, β is unramified over the open subscheme $X_K \subseteq \overline{X}_K$. Put another way, there exists an *open immersion* $\phi : Z_K \hookrightarrow X_K$ [i.e., an HPC] such that the composite $\beta \circ \phi$ factors through $X_K \subseteq \overline{X}_K$ in such a way that the resulting morphism $\beta_Z : Z_K \to X_K$ is *finite étale*. In particular, β_Z exhibits ϕ as an *object* of $\mathrm{DLoc}_K(X_K)$, and so ϕ exhibits x as a *closed point of* DLoc*-type*. Thus, by *transporting* ϕ, β_Z via the equivalences of Theorem 2.3, (i), and applying Theorem 1.3, (ii), (iii) [as in the proof of Corollary 2.5], we conclude that α preserves the decomposition groups of algebraic closed points, as desired. ○

Remark 2.8.1. In fact, tracing through the proofs of Corollaries 2.6, 2.8 shows that in these proofs, we did not actually need to use the full "Hom" version of Theorem 1.2. That is to say, for these proofs, in fact the *"isomorphism version"* of Theorem 1.2 [i.e., the bijection between *isomorphisms* "$X_K \xrightarrow{\sim} Y_L$" and certain *isomorphisms* "$\Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_L}$"], applied in combination with Theorem 1.3, (iii), is *sufficient*. Indeed, if we use the natural faithful functor discussed in Remark 2.2.0 to think of $\mathrm{Loc}_K(X_K)$ as a [not necessarily full!] subcategory of $\mathrm{DLoc}_K(X_K)$, then let us denote by

$$\mathrm{Arr}(\mathrm{Loc}_K(X_K)) \subseteq \mathrm{OFLoc}_K(X_K) \subseteq \mathrm{Arr}(\mathrm{DLoc}_K(X_K))$$

the collection of arrows $Z \to Z'$ of $\mathrm{DLoc}_K(X_K)$ which factor as the composite of an arrow $Z \to Z''$ [of $\mathrm{DLoc}_K(X_K)$] which is an *open immersion* [i.e., an HPC] with an arrow $Z'' \to Z'$ [of $\mathrm{DLoc}_K(X_K)$] which is *finite étale*. We shall refer to the arrows of $\mathrm{OFLoc}_K(X_K)$ as *arrows of OF-type* [i.e., "open immersion + finite étale" type]. Similarly, we define

$$\mathrm{OFLoc}_K(\Pi_{X_K}) \subseteq \mathrm{Arr}(\mathrm{DLoc}_{G_K}(\Pi_{X_K}))$$

to be the collection of arrows $J_1 \to J_2$ of $\mathrm{DLoc}_{G_K}(\Pi_{X_K})$ that factor as the composite of a surjection $J_1 \twoheadrightarrow J_3$ [in $\mathrm{DLoc}_{G_K}(\Pi_{X_K})$] whose kernel is normally topologically generated by some collection of cuspidal geometric decomposition groups, with an open immersion $J_3 \hookrightarrow J_2$ [in $\mathrm{DLoc}_{G_K}(\Pi_{X_K})$]. Then [cf. Theorem 2.3, (ii), and its proof] we obtain an *equivalence of categories*

$$\mathrm{DLoc}_{G_K}(\Pi_{X_K}) \xrightarrow{\sim} \mathrm{DLoc}_{G_L}(\Pi_{Y_L})$$

whose induced map on "Arr(−)'s" maps $\mathrm{OFLoc}_K(\Pi_{X_K})$ into $\mathrm{OFLoc}_L(\Pi_{Y_L})$ by applying Proposition 1.1, (ii); Theorem 1.3, (iii) [i.e., without using Theorem 1.2 *at all*!]. Moreover, the *isomorphism portion* of Theorem 1.2 implies that the étale fundamental group functor induces a natural commutative diagram

$$\begin{array}{ccc} \mathrm{OFLoc}_K(X_K) & \subseteq & \mathrm{Arr}(\mathrm{DLoc}_K(X_K)) \\ \downarrow & & \downarrow \\ \mathrm{OFLoc}_K(\Pi_{X_K}) & \subseteq & \mathrm{Arr}(\mathrm{DLoc}_{G_K}(\Pi_{X_K})) \end{array}$$

such that the vertical arrow on the left is *"essentially surjective"* — i.e., more precisely: induces a bijection on *abstract equivalence* [cf. §0] classes [defined relative to the category structures of $\mathrm{DLoc}_K(X_K)$, $\mathrm{DLoc}_{G_K}(\Pi_{X_K})$] lying in $\mathrm{OFLoc}_K(X_K)$, $\mathrm{OFLoc}_K(\Pi_{X_K})$. Since the proofs of Corollaries 2.6, 2.8 only make use of *arrows of OF-type*, the bijection of abstract equivalence classes just observed, together with the equivalence $\mathrm{DLoc}_{G_K}(\Pi_{X_K}) \xrightarrow{\sim} \mathrm{DLoc}_{G_L}(\Pi_{Y_L})$ — all of which involves only the *isomorphism portion* of Theorem 1.2 — are sufficient for the proofs of these categories, as claimed.

Section 3: Limits of Galois Sections

Let X_K be a *hyperbolic curve* over a *local field* K. As in §1, 2, we have an exact sequence:

$$1 \to \Delta_X \to \Pi_{X_K} \to G_K \to 1$$

Since Δ_X is *topologically finitely generated*, it follows that there exists a sequence of *characteristic open subgroups*

$$\ldots \subseteq \Delta_X[j+1] \subseteq \Delta_X[j] \subseteq \ldots \subseteq \Delta_X$$

[where j ranges over the positive integers] of Δ_X such that $\bigcap_j \Delta_X[j] = \{1\}$. In particular, given any *section*

$$\sigma : G_K \to \Pi_{X_K}$$

we obtain open subgroups

$$\Pi_{X_K[j,\sigma]} \stackrel{\mathrm{def}}{=} \mathrm{Im}(\sigma) \cdot \Delta_X[j] \subseteq \Pi_{X_K}$$

[where $\mathrm{Im}(\sigma)$ denotes the image of σ in Π_{X_K}] corresponding to a tower of *finite étale coverings*

$$\ldots \to X_K[j+1,\sigma] \to X_K[j,\sigma] \to \ldots \to X_K$$

of X_K by hyperbolic curves over K.

The following lemma is reminiscent of the techniques of [Tama], [Mzk1]:

Lemma 3.1. (Criterion for Galois Sections Associated to Rational Points) *Suppose that X_K is defined over a* **number field***, i.e., there exists a hyperbolic curve X_K over a number field $F \subseteq K$ such that $X_K = X_F \times_F K$. Let $\sigma : G_K \to \Pi_{X_K}$ be a section such that* $\mathrm{Im}(\sigma)$ *is* **not** *contained in any cuspidal decomposition group of Π_{X_K}. Then the following conditions on σ are* **equivalent***:*

(i) σ arises from a point $x \in X_K(K)$ [i.e., "$\mathrm{Im}(\sigma) = D_x$"].

(ii) For every integer $j \geq 1$, $X_K[j,\sigma](K) \neq \emptyset$.

(iii) For every integer $j \geq 1$, $X_K[j,\sigma](K)^{\mathrm{alg}} \neq \emptyset$ [where the superscript "alg" denotes the subset of algebraic [K-rational] closed points].

(iv) For every integer $j \geq 1$, $\Pi_{X_K[j,\sigma]}$ contains a decomposition group [i.e., relative to Π_{X_K}] of an **algebraic** *closed point of X_K that* **surjects** *onto G_K.*

Proof. (i) $\Longrightarrow$ (ii): It follows from the definitions that $x \in X_K(K)$ lifts to a point of $\in X_K[j,\sigma](K)$, for all $j \geq 1$, which implies (ii).

(iii) $\Longrightarrow$ (ii), (iv); (iv) $\Longrightarrow$ (iii): Immediate from the definitions.

(ii) $\Longrightarrow$ (i): For $j \geq 1$, choose points $x_j \in X_K[j,\sigma](K)$. Since the topological space

$$\prod_{j \geq 1} \overline{X}_K[j,\sigma](K)$$

is *compact*, it follows that there exists some *infinite set of positive integers* J' such that for any $j \geq 1$, the images of the $x_{j'}$, where $j' \geq j$, in

$$\overline{X}_K[j,\sigma](K)$$

converge to a point $y_j \in \overline{X}_K[j,\sigma](K)$. Moreover, note that, by the *definition* of y_j, it follows that if $j_1 > j_2$, then y_{j_1} maps to y_{j_2} in $\overline{X}_K[j_2,\sigma](K)$. In particular, if we write $y \in \overline{X}_K(K)$ for the image of the y_j in $\overline{X}_K(K)$, then it follows formally from the fact that the y_j form a *compatible sequence* of points of the sets $\overline{X}_K[j,\sigma](K)$ that $\mathrm{Im}(\sigma)$ is *contained* in the decomposition group [well-defined up to conjugation] D_y. On the other hand, by our assumption that $\mathrm{Im}(\sigma)$ is *not* contained in any cuspidal decomposition group of Π_{X_K}, we conclude that y is *not* a cusp, hence that "$\mathrm{Im}(\sigma) = D_y$", as desired.

(ii) $\implies$ (iii): Given a point $x_j \in X_K[j,\sigma](K)$ with image $x \in X_K(K) = X_F(K)$, it follows from *"Krasner's lemma"* [cf., e.g., [Kobl], p. 69-70] that one may *approximate* x by a point $x' \in X_F(F') \subseteq X_F(K) = X_K(K)$, where $F' \subseteq K$ is a finite extension of F, which is sufficiently close to x that [just like x] it lifts to a point $x'_j \in X_K[j,\sigma](K)$, which is necessarily *algebraic*, as desired. $\bigcirc$

Corollary 3.2. (Absoluteness of Decomposition Groups for Genus Zero) *Let K, L be* **local fields***; X_K (respectively, Y_L) a* **hyperbolic curve** *over K (respectively, L), which is, in fact,* **defined over a number field***. Suppose, moreover, that X_K (respectively, Y_L) is* **isogenous** *[cf. §0] to a hyperbolic curve of* **genus zero***. Then every isomorphism of profinite groups*

$$\alpha : \Pi_{X_K} \xrightarrow{\sim} \Pi_{Y_L}$$

preserves the decomposition groups of the closed points.

Proof. Indeed, Corollary 3.2 follows formally from Corollary 2.8; Theorem 1.3, (iii), (iv); and the equivalence (i) $\iff$ (iv) of Lemma 3.1. $\bigcirc$

Remark 3.2.1. Since any once-punctured elliptic curve is isogenous to a hyperbolic curve of genus zero, one might think, at first glance, that Corollary 2.6 is [essentially] a *"special case"* of Corollary 3.2. In fact, however, this is *false*, since Corollary 2.6 applies even to curves which are *not necessarily defined over a number field.*

Section 4: Discrete and Integral Structures at Cusps

Let X_K be a *hyperbolic curve* over a *local field* K; write $X_K \hookrightarrow \overline{X}_K$ for the *compactification* of X_K. Also, if p is the *residue characteristic* of K, then we shall write $\widehat{\mathbb{Z}}' \stackrel{\mathrm{def}}{=} \widehat{\mathbb{Z}}/\mathbb{Z}_p$. Let

$$D_x \subseteq \Pi_{X_K}$$

be a *decomposition group* associated to some *cusp* $x \in \overline{X}_K(K)$. Then we have an *exact sequence*

$$1 \to I_x \ (\cong \widehat{\mathbb{Z}}(1)) \to D_x \to G_K \to 1$$

whose *splittings* form a *torsor* over

$$H^1(G_K, \widehat{\mathbb{Z}}(1)) \cong (K^\times)^\wedge$$

[where the "$\wedge$" denotes the profinite completion]. If ω_x denotes the *cotangent space* to $\overline{X}_K$ at x, then any *choice of a nonzero* $\theta \in \omega_x$ determines a *splitting* of this torsor by considering the $\widehat{\mathbb{Z}}(1)$-torsor over the formal completion $(\overline{X}_K)_x$ [i.e., of $\overline{X}_K$ at x] given by taking N-th roots [as N ranges over the positive integers] of any local coordinate $t \in \mathfrak{m}_{\overline{X}_K,x}$ such that $dt|_x = \theta$. In particular, if the *pointed stable curve* associated to X_K has *stable reduction* over $\mathcal{O}_K$, then the *cotangent module* to this stable reduction at the $\mathcal{O}_K$*-valued point* determined by x determines a *natural integral structure* on ω_x [i.e., a rank one free $\mathcal{O}_K$-submodule of the one-dimensional K-vector space ω_x]. In particular, this integral structure determines a *reduction of the structure group* of the *torsor of splittings* considered above from $(K^\times)^\wedge$ to $\mathcal{O}_K^\times$.

Definition 4.1.

(i) If $(K^\times)^\wedge \to A$ is a continuous homomorphism of topological groups, then the torsor obtained from the torsor of splittings considered above by changing the structure group via this homomorphism will be referred to as the *A-torsor at x*. If, moreover, $B \subseteq A$ is a closed subgroup, then any reduction of the structure group of the A-torsor at x from A to B will be referred to as a *B-torsor structure at x*.

(ii) A $\mathcal{O}_K^\times$- (respectively, $K^\times$-) torsor structure on the $(K^\times)^\wedge$-torsor at x will be referred to as a(n) *integral* (respectively, *discrete*) *structure* on the cuspidal decomposition group D_x. Let us think of $(K^\times)^\wedge \otimes \widehat{\mathbb{Z}}'$ as a quotient of $(K^\times)^\wedge$; write $(\mathcal{O}_K^\times)'$, $(K^\times)'$ for the images of $\mathcal{O}_K^\times$, $K^\times$, respectively, in $(K^\times)^\wedge \otimes \widehat{\mathbb{Z}}'$. Then a $(\mathcal{O}_K^\times)'$- (respectively, $(K^\times)'$-) torsor structure on the $(K^\times)^\wedge \otimes \widehat{\mathbb{Z}}'$-torsor at x will be referred to as a(n) *tame integral* (respectively, *tame discrete*) *structure* on the cuspidal decomposition group D_x.

(iii) If X_K has *stable reduction* over $\mathcal{O}_K$ (respectively, X_K is arbitrary), then the particular integral (respectively, discrete) structure on D_x arising [as discussed above] from a generator of the rank one free $\mathcal{O}_K$-submodule of ω_x determined by the stable reduction of X_K (respectively, any nonzero element of ω_x) will be referred to as the *canonical integral* (respectively, *discrete*) *structure* on the cuspidal decomposition group D_x. The canonical integral (respectively, discrete) structure on D_x induces a tame integral (respectively, tame discrete) structure on D_x which we shall also refer to as *canonical*.

(iv) An arbitrary closed point x' of $\overline{X}_K$ will be referred to as *absolute* if, for every Y_L, α as in Theorem 2.3, there exists a closed point y' of $\overline{Y}_L$ such that $\alpha(D_{x'}) = D_{y'}$. A nonconstant unit $U \in \Gamma(X_K, \mathcal{O}_{X_K}^\times)$ on X_K will be called *coabsolute* if $\overline{X}_K$ admits an absolute point at which U is invertible. The hyperbolic curve X_K will be called *coabsolute* if it admits a coabsolute unit. The hyperbolic curve X_K will be called *quasi-coabsolute* if it is isogenous to a coabsolute hyperbolic curve. If X_K has *stable reduction* over $\mathcal{O}_K$ (respectively, X_K is arbitrary), then the cusp x will be called *integrally absolute* (respectively, *discretely absolute*) if, for every Y_L, α as in Theorem 2.3, the isomorphism $D_x \xrightarrow{\sim} D_y$ [where y is a cusp of Y_L — cf. Theorem 1.3, (iii)] induced by α is compatible with the *canonical* integral (respectively, discrete) structures on D_x, D_y. Similarly, one has a notion of *tamely integrally absolute* and *tamely discretely absolute* cusps.

(v) The cusp x will be called *subprincipal* if it is contained in the support of a cuspidal principal divisor on [i.e., principal divisor supported in the cusps of] $\overline{X}_K$. The hyperbolic curve X_K will be called *subprincipally ample* if every cusp of X_K is subprincipal. The hyperbolic curve X_K will be called *subprincipally quasi-ample* if it is isogenous to a subprincipally ample hyperbolic curve.

Remark 4.1.1. By Theorem 1.3, (iii), *cusps* are always *absolute*. By Corollaries 2.6, 3.2, once-punctured elliptic curves, as well as hyperbolic curves that are isogenous to a hyperbolic curve of genus zero which is defined over a number field, have *infinitely many absolute points*.

Next, let us write

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_{\overline{X}_K}(x)$$

for the *line bundle* determined by the cusp x;

$$\mathbb{L} \to \overline{X}_K$$

for the *geometric line bundle* determined by $\mathcal{L}$; and

$$(\mathbb{L} \supseteq)\ \mathbb{L}^\times \to \overline{X}_K$$

for the *complement of the zero section* in $\mathbb{L}$. Thus, the natural inclusion $\mathcal{O}_{\overline{X}_K} \hookrightarrow \mathcal{O}_{\overline{X}_K}(x)$ determines a section

$$\overline{X}_K \to \mathbb{L}$$

whose restriction to X_K determines a section $X_K \to \mathbb{L}^\times$, hence a morphism of fundmental groups:

$$\Pi_{X_K} \to \Pi_{\mathbb{L}^\times} \stackrel{\text{def}}{=} \pi_1(\mathbb{L}^\times)$$

Lemma 4.2. **(The Line Bundle Associated to a Cusp)** *Suppose that X_K is of type (g, r), where $g \geq 2$, $r = 1$. Then:*

(i) $\Pi_{\mathbb{L}^\times}$ fits into a short exact sequence:

$$1 \to \widehat{\mathbb{Z}}(1) \to \Pi_{\mathbb{L}^\times} \to \Pi_{\overline{X}_K} \to 1$$

Moreover, the resulting extension class $\in H^2(\Pi_{\overline{X}_K}, \widehat{\mathbb{Z}}(1))$ is the first Chern class of the line bundle $\mathcal{L}$.

(ii) The morphism of fundmental groups $\Pi_{X_K} \to \Pi_{\mathbb{L}^\times}$ induces an isomorphism $I_x \xrightarrow{\sim} \mathrm{Ker}(\Pi_{\mathbb{L}^\times} \to \Pi_{\overline{X}_K})$. In particular, the morphism $\Pi_{X_K} \to \Pi_{\mathbb{L}^\times}$ is surjective.

(iii) Write $\Delta_{X/\overline{X}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{X_K} \twoheadrightarrow \Pi_{\overline{X}_K})$. *Then the quotient of* $\Delta_{X/\overline{X}}$ *by*

$$\mathrm{Ker}(\Pi_{X_K} \to \Pi_{\mathbb{L}^\times}) \subseteq \Delta_{X/\overline{X}}$$

is the **maximal quotient** *of* $\Delta_{X/\overline{X}}$ *on which the conjugation action by* Δ_X *is* **trivial**.

Proof. Assertion (i) follows from [Mzk4], Lemmas 4.3, 4.4, 4.5. Assertion (ii) is immediate from the discussion preceding Definition 4.1 involving roots of local coordinates. As for assertion (iii), write $Q_1 \stackrel{\mathrm{def}}{=} \Delta_{X/\overline{X}}/\mathrm{Ker}(\Pi_{X_K} \to \Pi_{\mathbb{L}^\times})$; Q_2 for the *maximal quotient* of $\Delta_{X/\overline{X}}$ on which the conjugation action by Δ_X is *trivial*. Thus, we have a natural surjection $Q_2 \twoheadrightarrow Q_1$. Now assertion (iii) follows from assetion (ii) and the well-known fact that $\Delta_{X/\overline{X}}$ is *topologically generated* by the Δ_X-conjugates of I_x. ○

Next, let us recall the notation of [Mzk2], §1.2: By *local class field theory*, we have a natural isomorphism

$$(K^\times)^\wedge \xrightarrow{\sim} G_K^{\mathrm{ab}}$$

which we may use to think of the group of roots of unity of $(K^\times)^\wedge$ as a subgroup:

$$\boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(K) \subseteq G_K^{\mathrm{ab}}$$

Also, we recall [cf. [Mzk2], Proposition 1.2.1, (iv)] that the subgroup $K^\times \subseteq (K^\times)^\wedge \xrightarrow{\sim} G_K$ may be recovered *group-theoretically* from the profinite group structure of G_K. Allowing "K" to vary among the various finite extensions of a given K inside an algebraic closure $\overline{K}$ of K, we obtain groups:

$$\boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{K}); \quad \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(\overline{K})); \quad \boldsymbol{\mu}_{\widehat{\mathbb{Z}}'}(\overline{K}) \stackrel{\mathrm{def}}{=} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}) \otimes \widehat{\mathbb{Z}}'$$

In particular, by considering roots of local coordinates as in the discussion preceding Definition 4.1, we obtain a *natural isomorphism* $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}) \xrightarrow{\sim} I_x$.

Theorem 4.3. (Rigidity of Cuspidal Geometric Decomposition Groups) *In the notation of Theorem 2.3, suppose that* α *induces isomorphisms*

$$I_x \xrightarrow{\sim} I_y; \quad \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{L})$$

where $x \in \overline{X}_K(K)$ *(respectively,* $y \in \overline{Y}_L(L)$*) is a cusp. Then these isomorphisms are* **compatible** *with the* **natural isomorphisms** $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}) \xrightarrow{\sim} I_x$; $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{L}) \xrightarrow{\sim} I_y$.

Proof. Indeed, by replacing X_K, Y_L by finite étale coverings, one reduces immediately to the case where both curves are of genus ≥ 2. By "filling in" [cf. Theorem 1.3, (iii)] all of the cusps other than those of interest [i.e., x, y], we may assume, moreover, that X_K, Y_L satisfy the hypotheses of Lemma 4.2. Thus, by Lemma 4.2, we conclude that the morphism

$$H^2(\Delta_{\overline{X}}, I_x) \xrightarrow{\sim} H^2(\Delta_{\overline{Y}}, I_y)$$

induced by α is *compatible* with the extension classes of Lemma 4.2. On the other hand, by [Mzk2], Lemma 2.5, (ii), the morphism

$$H^2(\Delta_{\overline{X}}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K})) \xrightarrow{\sim} H^2(\Delta_{\overline{Y}}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{L}))$$

induced by α is *compatible* with the elements determined by the Chern class of a point on either side. Since all of these "H^2's" are isomorphic to $\widehat{\mathbb{Z}}$, we thus obtain the compatibility asserted in the statement of Theorem 4.3. ○

Proposition 4.4. **(Tame Integral Absoluteness)** *Suppose that* X_K *has* **stable reduction** *over* $\mathcal{O}_K$. *Then:*

(i) Every cusp of X_K *is* **tamely integrally absolute**.

(ii) A cusp of X_K *is* **discretely absolute** *if and only if it is* **integrally absolute**.

Proof. Assertion (ii) follows formally from assertion (i) and the fact that the restriction of the projection $\widehat{\mathbb{Z}} \twoheadrightarrow \widehat{\mathbb{Z}}'$ to $\mathbb{Z} \subseteq \widehat{\mathbb{Z}}$ is *injective*. Now we consider assertion (i). First, let us observe that it is immediate from the definitions that it suffices to prove assertion (i) after replacing X_K by a finite étale covering of X_K that extends to an *admissible covering* of the stable model of X_K. In particular, we may assume without loss of generality that every irreducible component of the normalization of the geometric special fiber of this stable model has *genus* ≥ 1.

Next, let us recall the *"étale quotient"*

$$\Pi_{X_K} \twoheadrightarrow \Pi^{\mathrm{et}}_{X_K}$$

of [Mzk2], §2. Thus, the finite quotients of $\Pi^{\mathrm{et}}_{X_K}$ correspond to the coverings of X_K that arise from *finite étale coverings* of the stable model of X_K that are *tamely ramified* at the cusps. In particular, the quotient of G_K determined by $\Pi^{\mathrm{et}}_{X_K}$ is the natural quotient $G_K \twoheadrightarrow G_k$, where k is the *residue field* of K. If x is a *cusp* of X_K, then [in light of our assumption that every irreducible component of the normalization of the geometric special fiber of the stable model has *genus* ≥ 1] the quotient

$$D_x \twoheadrightarrow D'_x$$

determined by $\Pi^{\mathrm{et}}_{X_K}$ fits into an exact sequence:

$$1 \to I'_x \to D'_x \to G_k \to 1$$

[where $I'_x \stackrel{\mathrm{def}}{=} I_x \otimes \widehat{\mathbb{Z}}'$]. In particular, the splittings of this exact sequence form a torsor over $H^1(G_k, I'_x) \cong k^\times$. These splittings may be thought of as elements of

$H^1(D'_x, I'_x)$ whose restriction to I'_x is equal to the *identity element* of $H^1(I'_x, I'_x) = \mathrm{Hom}(I'_x, I'_x)$. Thus, unraveling the definitions, one verifies immediately that the pull-back to D_x of any such element of $H^1(D'_x, I'_x)$ forms an element of $H^1(D_x, I'_x)$ which determines the *canonical tame integral structure* on D_x. Since the *étale quotient* is compatible with isomorphisms α as in Theorem 2.3 [cf. [Mzk2], Lemma 2.2, (ii)], we thus conclude that x is *tamely integrally absolute*, as desired. ○

Proposition 4.5. **(Absoluteness and Coverings)** *Let* $Z \to X_K$ *be a* **finite étale covering**. *Let* z *be a closed point of the compactification* $\overline{Z}$ *of* Z *that maps to a closed point* x *of* $\overline{X}_K$. *Then:*

(i) z *is* **absolute** *(respectively, a* **discretely absolute cusp***) if and only if* x *is.*

(ii) Suppose that X_K, Z *have* **stable reduction** *[over the rings of integers of their respective fields of constants]. Then* z *is an* **integrally absolute cusp** *if and only if* x *is.*

Proof. Assertion (i) is immediate from the definitions; [the "$\mathrm{Loc}_K(-)$ portion" — already contained in [Mzk3], Theorem 2.4 — of] Theorem 2.3, (ii) [cf. the proof of Corollary 2.8]; Theorem 1.3, (ii); the fact that $\widehat{\mathbb{Z}}/\mathbb{Z}$ is *divisible*. Assertion (ii) is immediate from assertion (i) and Proposition 4.4, (ii). ○

Before proceeding, we recall the following well-known result:

Lemma 4.6. **(Vanishing of Galois Invariants of the Tate Module)** *We have:* $H^0(G_K, H^1(\Delta_{\overline{X}}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}))) = 0$.

Proof. Since $T_{\overline{X}} \stackrel{\mathrm{def}}{=} \Delta^{\mathrm{ab}}_{\overline{X}}$, i.e., the *Tate module* of the Jacobian J_X of $\overline{X}_K$, is isomorphic to its *Cartier dual*, it suffices to show that $H^0(G_K, T_{\overline{X}}) = 0$, i.e., that the torsion subgroup of $J_X(K)$ is *finite*. Since J_X is a proper group scheme over K, it follows that the p-adic topology on K determines a *p-adic topology* on $J_X(K)$ with respect to which $J_X(K)$ forms a *compact p-adic Lie group*. As is well-known [cf., e.g., [Serre], Chapter V, §7], the *exponential map* for this p-adic Lie group determines an isomorphism of a certain open neighborhood of the identity of $J_X(K)$ with a free $\mathbb{Z}_p$-module of finite rank. Thus, the desired *finiteness* follows formally from this isomorphism, together with the compactness of $J_X(K)$. ○

Remark 4.6.1. The author wishes to thank A. Tamagawa for informing him of the simple proof of Lemma 4.6 given above.

Next, let us observe that for any integer $N \geq 1$, the *Kummer exact sequence*

$$1 \to \boldsymbol{\mu}_N \to \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}} \to 1$$

[where $\mathbb{G}_m \to \mathbb{G}_m$ denotes the N-th power map on $\mathbb{G}_m$; $\boldsymbol{\mu}_N$ is defined so as to make the sequence exact] on the étale site of X_K determines a long exact sequence in cohomology, hence, in particular, by letting N vary, an *injection*

$$H_X \stackrel{\text{def}}{=} \Gamma(X_K, \mathcal{O}^\times_{X_K}) \hookrightarrow H^\wedge_X \hookrightarrow H^1(\Pi_{X_K}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}))$$

[where we use the easily verified fact that H_X is *residually finite*]. On the other hand, the Leray spectral sequence for the quotient $\Pi_{X_K} \to G_K$ yields an exact sequence:

$$0 \to (K^\times)^\wedge \to H^1(\Pi_{X_K}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K})) \to D_X \stackrel{\text{def}}{=} H^0(G_K, H^1(\Delta_X, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K})))$$

Moreover, since, by Lemma 4.6, $H^0(G_K, H^1(\Delta_{\overline{X}}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}))) = 0$, it follows that, if we assume, for simplicity, that X_K is *cuspidally split*, then restriction to the various "I_x" in Δ_X determines [by applying the *natural isomorphisms* $I_x \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K})$] an injection

$$D_X \hookrightarrow P_X \stackrel{\text{def}}{=} \prod_x \widehat{\mathbb{Z}}$$

[where the product ranges over the *cusps* x of X_K]. In particular, we obtain *exact sequences*:

$$0 \to K^\times \to H_X \to P_X; \quad 0 \to (K^\times)^\wedge \to H^\wedge_X \to P_X$$

Write $E_X \stackrel{\text{def}}{=} \mathrm{Im}(H_X) \subseteq P_X$ for the image of H_X in P_X [so we obtain an induced injection $E^\wedge_X \hookrightarrow P_X$]. Thus, the maps

$$H_X \to P_X; \quad H^\wedge_X \to P_X$$

are the maps obtained by associating to a function in H_X its *divisor of zeroes and poles*. Put another way, $E_X \subseteq P_X$ may be characterized as the *submodule of cuspidal principal divisors*.

Proposition 4.7. (Principal Cuspidal Divisors) *In the notation of Theorem 2.3, assume that X_K, Y_L are* **cuspidally split**. *Then the isomorphism*

$$P_X \xrightarrow{\sim} P_Y$$

induced [cf. Theorem 1.3, (iii)] by α maps E_X onto E_Y.

Remark 4.7.1. In the statement of Proposition 4.7, as well as in the discussion to follow, we shall use similar notation for the objects associated to Y_L to the notation used for the various objects just defined for X_K.

Proof. Write J_X (respectively, A_X) for the *Jacobian* (respectively, *Albanese variety*) of X_K. Thus, the natural map $\overline{X}_K \to A_X$ determines a surjection on fundamental groups $\Pi_{\overline{X}_K} \twoheadrightarrow \Pi_{A_X} \stackrel{\text{def}}{=} \pi_1(A_X)$ whose kernel is the kernel of $\Delta_{\overline{X}} \twoheadrightarrow \Delta^{\text{ab}}_{\overline{X}}$.

In particular, any pair of sections of $\Pi_{\overline{X}_K} \to G_K$ determines a pair of sections of $\Pi_{A_X} \to G_K$ whose difference determines an element of $H^1(G_K, \Delta_{\overline{X}}^{\mathrm{ab}})$. Moreover, if these sections arise from points $\in \overline{X}_K(K)$, then the resulting element of $H^1(G_K, \Delta_{\overline{X}}^{\mathrm{ab}})$ *completely determines* the point of $J_X(K)$ given by forming the difference of these two points [cf., e.g., [Mzk1], the discussion preceding Definition 6.2; [BK], Example 3.11]. More generally, given any divisor of cusps on $\overline{X}_K$ with $\mathbb{Z}$-coefficients of degree 0, the divisor is *principal* if and only if the resulting element of $H^1(G_K, \Delta_{\overline{X}}^{\mathrm{ab}})$ *vanishes*. Since the sections of $\Pi_{\overline{X}_K} \to G_K$ arising from cusps are *preserved* by α [cf. Theorem 1.3, (iii)], we thus conclude that the isomorphism $P_X \to P_Y$ induced by α maps E_X onto E_Y, as desired. ◯

Definition 4.8. We shall say that X_K is *unitwise absolute* if, in the notation of Theorem 2.3, the isomorphism

$$H^1(\Pi_{X_K}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K})) \xrightarrow{\sim} H^1(\Pi_{Y_L}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{L}))$$

induced by α maps the image of $\Gamma(X_K, \mathcal{O}_{X_K}^\times)$ via the Kummer map onto the image of $\Gamma(Y_L, \mathcal{O}_{Y_L}^\times)$ via the Kummer map.

Corollary 4.9. **(Divisor-Theoretic Properties)** *In the notation of Theorem 2.3, let x, y (respectively, $A \in E_X$, $B \in E_Y$) be* **cusps** *(respectively,* **cuspidal principal divisors***) of X_K, Y_L, respectively, that correspond via α [cf. Proposition 4.7]. Then:*

(i) x is **subprincipal** *if and only if y is.*

(ii) A is the divisor of a **coabsolute unit** *if and only if B is.*

(iii) X_K is **coabsolute** *(respectively,* **quasi-coabsolute***) if and only if Y_L is.*

(iv) X_K is **subprincipally ample** *(respectively,* **subprincipally quasi-ample***) if and only if Y_L is.*

Proof. In light of Proposition 4.7, all of these statements follow formally from the definitions. Also, we note that for the various "quasi-" properties, one must apply [the "$\mathrm{Loc}_K(-)$ portion" — already contained in [Mzk3], Theorem 2.4 — of] Theorem 2.3, (ii), as in the proof of Corollary 2.8. ◯

Theorem 4.10. **(Units and Canonical Integral Structures)** *Let X_K be a* **hyperbolic curve** *over a* **local field** *K. Then:*

(i) If X_K is **quasi-coabsolute***, then it admits a* **discretely absolute cusp***.*

(ii) If X_K admits a **discretely absolute cusp** *or an* **absolute noncusp** *[i.e., an absolute point which is not a cusp], then X_K is* **unitwise absolute***.*

(iii) If X_K is **unitwise absolute** *and* **subprincipally ample**, *then every cusp of X_K is* **discretely absolute**.

(iv) Suppose that X_K has **stable reduction** *over $\mathcal{O}_K$. Then if X_K is* **quasi-coabsolute** *and* **subprincipally quasi-ample**, *then every cusp of X_K is* **integrally absolute**.

Proof. First, we consider assertion (i). In light of Proposition 4.5, (i), we may assume that X_K is *coabsolute*. Let $U \in H_X$ be a *coabsolute unit* of X_K; let x be a cusp of X_K at which U *fails* to be invertible. If U has a *zero* of order $(\mathbb{Z} \ni)\ n \neq 0$ at x, then the *restriction* of the class

$$\eta_U \in H^1(\Pi_{X_K}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}))$$

determined by U to D_x determines a *splitting* of the torsor obtained by applying a change of structure group to the $(K^\times)^\wedge$-torsor at x via the map $(K^\times)^\wedge \to (K^\times)^\wedge$ given by *multiplication by n*. Since $n \neq 0$, and $\widehat{\mathbb{Z}}/\mathbb{Z}$ is *divisible*, it thus follows that this splitting is sufficient to determine the *canonical discrete structure* on D_x. Let us write $\epsilon_U \in E_X$ for the image of η_U in E_X. Then ϵ_U determines the set $(K^\times)^\wedge \cdot \eta_U$. On the other hand, since U is *coabsolute*, it follows that $\overline{X}_K$ admits an *absolute point* x' at which U is *invertible*. Thus, the subset

$$K^\times \cdot \eta_U \subseteq (K^\times)^\wedge \cdot \eta_U$$

may be characterized as the set of elements of $(K^\times)^\wedge \cdot \eta_U$ whose restriction to $D_{x'}$ — which [by the invertibility of U at x'] necessarily lies in

$$(K^\times)^\wedge \cong H^1(G_K, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K})) \subseteq H^1(D_{x'}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}))$$

— in fact lies inside $K^\times \subseteq (K^\times)^\wedge$. Thus, Theorem 4.3, Proposition 4.7, together with the *absoluteness* of x', imply that x is *discretely absolute*, as desired.

Next, we consider assertion (ii). Let x be a *discretely absolute cusp* or an *absolute noncusp* of X_K. Then, as in the argument applied in the proof of assertion (i), the image of H_X in $H^1(\Pi_{X_K}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}))$ may be characterized as the set of elements lying over elements of E_X whose restriction to D_x determines a class in $H^1(D_x, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}))$ that lies in the submodule of this cohomology module generated by the elements that define splittings "compatible with the *canonical discrete structure* on D_x" [where in the noncuspidal case, we take this compatibility to mean that the restriction to D_x lies in $K^\times \subseteq (K^\times)^\wedge$, as in the proof of assertion (i)]. Thus, assertion (ii) follows from Theorem 4.3, Proposition 4.7, together with the *discrete absoluteness* [in the cuspidal case] or *absoluteness* [in the noncuspidal case] of x.

Next, we observe that assertion (iii) follows via the argument applied in the proof of assertion (i), since the hypothesis that X_K is *unitwise absolute* and *subprincipally ample* implies that for every cusp x of X_K, there exists a unit $U \in H_X$ that is not invertible at x and whose class in $H^1(\Pi_{X_K}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{K}))$ is mapped [in the notation of Theorem 2.3] to a class in $H^1(\Pi_{Y_L}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\overline{L}))$ that lies in the image of H_Y.

Finally, we observe that assertion (iv) is a formal consequence of Proposition 4.4, (ii); Proposition 4.5, (i); assertions (i), (ii), (iii). ◯

Corollary 4.11. **(The Case of Genus Zero)** *Let X_K be a* **hyperbolic curve** *over a* **local field** *K which is* **isogenous** *to a hyperbolic curve of* **genus zero**. *Then X_K is* **quasi-coabsolute**, **subprincipally quasi-ample**, *and* **unit-wise absolute**. *In particular, if X_K has* **stable reduction** *over $\mathcal{O}_K$, then every cusp of X_K is* **integrally absolute**.

Proof. In light of Theorem 4.10, it suffices to show that if X_K is *of genus zero* and *cuspidally split*, then X_K is *coabsolute* and *subprincipally ample*. But since cusps are always *absolute* [cf. Theorem 1.3, (iii)], these properties follow formally from the following two elementary facts: (a) every divisor of degree 0 on $\overline{X}_K$ is principal; (b) X_K has at least 3 cusps. ◯

In certain situations, the $\mathcal{O}_K^\times$-torsor determined by the canonical integral structure on the cuspidal decomposition group D_x admits an even *"finer reduction of structure group"*, as follows:

Corollary 4.12. **(The Case of Once-punctured Elliptic Curves)** *Let X_K be a* **once-punctured elliptic curve** *over a* **local field** *K of residue characteristic $\neq 2$. Suppose that X_K has* **stable reduction** *over $\mathcal{O}_K$. Also, if $n \geq 1$ is an integer, we shall write $\boldsymbol{\mu}_n(K) \subseteq \mathcal{O}_K^\times$ for the subgroup of n-th roots of unity. Then there exists a* **$\boldsymbol{\mu}_{12}(K)$-torsor structure** *at the unique cusp x of X_K which is* **compatible** *with the canonical integral structure arising from the stable model $\mathcal{X}^{\log}$ and, moreover, is* **preserved** *by arbitrary automorphisms of Π_{X_K}.*

Proof. We may assume without loss of generality that all of the 2-*torsion points* of the underlying elliptic curve of X_K are defined over K. Write $Y_K \to X_K$ for the Galois covering of degree 4 determined by the "multiplication by 2" map on the underlying elliptic curve of X_K [so Y_K is hyperbolic of type $(1,4)$] and $\mathcal{Y}^{\log}$ for the stable model over $\mathrm{Spec}(\mathcal{O}_K)^{\log}$ [where the log structure on $\mathrm{Spec}(\mathcal{O}_K)$ is that determined by the closed point] of the smooth log curve $\overline{Y}_K^{\log}$ determined by Y_K. Also, let us write e_1, e_2, e_3, e_4 for the four cusps of Y_K.

Let $\alpha : \Pi_{X_K} \xrightarrow{\sim} \Pi_{X_K}$ be an automorphism of Π_{X_K}. Note that, by Theorem 1.3, (ii), *any* $\boldsymbol{\mu}_{12}(K)$-torsor structure at x is preserved by arbitrary *inner automorphisms* of Π_{X_K}. Thus, we may assume [by composing with a suitable inner automorphism that induces a suitable element of $\mathrm{Gal}(Y_K/X_K)$] that the natural action of α on the cusps of Y_K [cf. Theorem 1.3, (iii)] *preserves* e_1.

Next, let us observe that [by the well-known definition of the group law on an elliptic curve; the definition of $Y_K \to X_K$] the divisor $D \stackrel{\mathrm{def}}{=} 2[e_1] - 2[e_2]$ on Y_K is *principal*. Thus, there exists a unique rational function f on Y_K whose divisor of zeroes and poles is D and whose value at e_3 is 1. Since D has multiplicity 2 at

e_1, it follows that f determines a $\boldsymbol{\mu}_2(K)$*-torsor structure* at e_1, hence also at x. Write η for the *Kummer class* [i.e., the image under the Kummer map] of f. In the following, we shall write Kummer classes *additively*.

Now, observe that, by Proposition 4.7; Theorem 1.3, (iii), if α *fixes all four cusps* of Y_K, then it follows that α preserves the class η, hence also the $\boldsymbol{\mu}_2(K)$*-torsor structure* at x determined by η.

Next, let us write Σ for the *group of permutations* of the three cusps e_2, e_3, e_4 that arise from automorphisms $\beta \in \mathrm{Aut}(\Pi_{X_K})$ that preserve e_1. Thus, the order s of Σ divides 6. Let $\beta_1, \dots, \beta_s \in \mathrm{Aut}(\Pi_{X_K})$ be a collection of automorphisms that give rise to the elements of Σ. Set:

$$\eta' \stackrel{\mathrm{def}}{=} (6/s) \cdot \sum_{j=1}^{s} \eta^{\beta_j}$$

Since Y_K is *unitwise absolute* by Corollary 4.11, it follows that η' arises from a rational function f' on Y_K which has a pole of order 12 at e_1. In particular, η' determines a $\boldsymbol{\mu}_{12}(K)$*-torsor structure* at e_1, hence also at x. Moreover, it follows formally from the preceding observation concerning automorphisms α that fix all four cusps of Y_K that *arbitrary* α [i.e., that are only assumed to fix e_1] *preserve* the $\boldsymbol{\mu}_{12}(K)$-torsor structure determined by η'. Finally, the fact that this $\boldsymbol{\mu}_{12}(K)$-torsor structure is *compatible* with the canonical integral structure follows from the easily verified fact that the rational function f is *generically invertible* [in light of our assumption that the residue charactertistic of K is $\neq 2$] on the special fiber of $\mathcal{Y}$. This completes the proof. ○

Remark 4.12.1. The number "12" appearing in Corollary 4.12 is interesting in light of the well-known fact that the line bundle on the moduli stack of elliptic curves determined by the cotangent bundle at the origin of the tautological family of elliptic curves has *order* 12 in the Picard group of this moduli stack.

Remark 4.12.2. It seems natural to expect that a(n) [perhaps somewhat more complicated] analogue of Corollary 4.12 should hold for more general hyperbolic curves X_K. This topic, however, lies beyond the scope of this paper.

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