Irreducible representations of crossed products of the Cuntz algebras and isotropy subgroups of the unitary group

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For an element x of the Cuntz algebra $\mathcal{O}_N$, define the isotropy subgroup $G_x \equiv \{g \in U(N) : \alpha_g(x) = x\}$ of the unitary group $U(N)$ with respect to the canonical action α of $U(N)$ on $\mathcal{O}_N$. We have irreducible representations of the crossed product $\mathcal{O}_N \rtimes G_x$ by extending irreducible generalized permutative representations of $\mathcal{O}_N$ and irreducible representation of G_x. ¿From this, the Peter-Weyl theorem for G_x is extended to the regular representation of $\mathcal{O}_N \rtimes G_x$.

1. Main theorem

By comparing the theory of Lie groups, representation theory of operator algebras are not well developed. The most different point among them is the uniqueness of irreducible decomposition of representation. In general, representations of C*-algebras do not have unique decomposition(up to unitary equivalence) into sums or integrals of irreducibles. However, the permutative representations of the Cuntz algebra $\mathcal{O}_N$ do ([**1**, **3**, **4**]). We generalized the permutative representations in [**5**, **6**, **7**, **8**] by keeping the uniqueness of decomposition. For a C*-dynamical system $(\mathcal{A}, G, \alpha)$, representations of the C*-crossed product $\mathcal{A}\rtimes G$ are written in [**9**]. However their irreducible representations are not well-known. In this paper, we show concrete irreducible representations of $\mathcal{O}_N \rtimes G$ by using generalized permutative(=GP) representations of $\mathcal{O}_N$ and unitary representations of a compact group G and classify them when G is a closed subgroup of the isotropy subgroup of the unitary group $U(N)$ with respect to the canonical action α of $U(N)$ on $\mathcal{O}_N$. In order to introduce representations of $\mathcal{O}_N \rtimes G$, we start to review GP representations of $\mathcal{O}_N$ with cycle.

Let $S(\mathbf{C}^N) \equiv \{w \in \mathbf{C}^N : \|w\| = 1\}$, $S(\mathbf{C}^N)^{\otimes k} \equiv \{w^{(1)} \otimes \cdots \otimes w^{(k)} : w^{(i)} \in S(\mathbf{C}^N),\, i = 1, \ldots, k\}$ for $k \geq 1$ and $S(\mathbf{C}^N)^{\otimes *} \equiv \bigcup_{k\geq 1} S(\mathbf{C}^N)^{\otimes k}$. Let $s_1, \ldots, s_N$ be canonical generators of $\mathcal{O}_N$. For $w = (w_i)_{i=1}^N \in \mathbf{C}^N$, define $s(w) \equiv w_1 s_1 + \cdots + w_N s_N$. For $w = w^{(1)} \otimes \cdots \otimes w^{(k)} \in S(\mathbf{C}^N)^{\otimes k}$, a

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representation $(\mathcal{H},\pi)$ of $\mathcal{O}_N$ is $GP(w)$ if there is a cyclic vector $\Omega\in\mathcal{H}$ such that $\pi(s(w))\Omega=\Omega$ where $s(w)\equiv s(w^{(1)})\cdots s(w^{(k)})$. We call Ω by the *GP vector* of $(\mathcal{H},\pi)$. We call $GP(w)$ by the *GP representation of* $\mathcal{O}_N$ *by* w. $w\in S(\mathbf{C}^N)^{\otimes k}$ is *non periodic* if there is no $y\in S(\mathbf{C}^N)^{\otimes l}$ such that w equals to the tensor power $y^{\otimes l}$ of y for some $l\geq 2$. If $w\in S(\mathbf{C}^N)^{\otimes k}$ is non periodic, then $GP(w)$ exists uniquely up to unitary equivalence. $GP(w)$ is irreducible if w is non periodic. If both $w\in S(\mathbf{C}^N)^{\otimes k}$ and $y\in S(\mathbf{C}^N)^{\otimes k'}$ are non periodic, then $GP(w)\sim GP(y)$ if and only if $k'=k$ and $w=y^{(p(1))}\otimes\cdots\otimes y^{(p(k))}$ for some $p\in\mathbf{Z}_k$ where $\sim$ means unitary equivalence. Any cyclic permutative representation with a cycle is a GP representation.

Let α be the canonical action of $U(N)$ on $\mathcal{O}_N$. For $w\in S(\mathbf{C}^N)^{\otimes *}$, let $U_w(N)\equiv\{g\in U(N):\alpha_g(s(w))=s(w)\}$. Then $U_w(N)$ is a closed subgroup of $U(N)$. For example, $U_w(N)\cong U(N-1)$ for any $w\in S(\mathbf{C}^N)$. For a closed subgroup G of $U_w(N)$, we define $\mathcal{O}_N\rtimes G$ by the C*-crossed product associated with a C*-dynamical system $(\mathcal{O}_N,G,\alpha)$.

Definition 1.1. *For* $w\in S(\mathbf{C}^N)^{\otimes *}$ *and a unitary representation* $(\mathcal{V},V)$ *of* G*, a representation* $(\mathcal{H},\pi)$ *of* $\mathcal{O}_N\rtimes G$ *is* $GP(w)\rtimes(\mathcal{V},V)$ *if there is a subspace* $\mathcal{V}'\subset\mathcal{H}$ *which is cyclic for* $(\mathcal{H},\pi)$ *such that* $(\mathcal{V}',\pi|_G)$ *is unitarily equivalent to* $(\mathcal{V},V)$ *and* $\pi(s(w))v=v$ *for each* $v\in\mathcal{V}'$*. We call* $\mathcal{V}'$ *by the GP subspace of* $(\mathcal{H},\pi)$*.*

Theorem 1.2. *For a non periodic element* $w\in S(\mathbf{C}^N)^{\otimes *}$ *and a closed subgroup* G *of* $U_w(N)$*, the following holds:*

(i) $GP(w)\rtimes(\mathcal{V},V)$ *exists uniquely up to unitary equivalence.*
(ii) $GP(w)\rtimes(\mathcal{V}\oplus\mathcal{V}',V\oplus V')\sim(GP(w)\rtimes(\mathcal{V},V))\oplus(GP(w)\rtimes(\mathcal{V}',V'))$.
(iii) $GP(w)\rtimes(\mathcal{V},V)\sim GP(w)\rtimes(\mathcal{V}',V')$ *if and only if* $(\mathcal{V},V)\sim(\mathcal{V}',V')$.
(iv) $GP(w)\rtimes(\mathcal{V},V)$ *is irreducible if and only if* $(\mathcal{V},V)$ *is irreducible.*
(v) *Identify* $\mathcal{O}_N$ *with the subalgebra of* $\mathcal{O}_N\rtimes G$ *by the natural embedding of* $\mathcal{O}_N$ *into* $\mathcal{O}_N\rtimes G$*. Then the following branching law holds:*

$$(GP(w)\rtimes(\mathcal{V},V))|_{\mathcal{O}_N}\sim(GP(w))^{\oplus\dim\mathcal{V}}.$$

For a representation $(\mathcal{H},\pi)$ of a C*-algebra $\mathcal{A}$ and a C*-dynamical system $(\mathcal{A},G,\alpha)$ with a locally compact group G, the *regular representation* $(L_2(G,\mathcal{H}),\tilde{\pi}\rtimes\lambda)$ of $\mathcal{A}\rtimes G$ by $(\mathcal{H},\pi)$ is the representation which is induced by the following covariant representation $(L_2(G,\mathcal{H}),\tilde{\pi},\lambda)$ as follows(§ 7.7, [**9**]):

$$(\tilde{\pi}(a)\phi)(g)\equiv\pi(\alpha_{g-1}(a))\phi(g),\quad(\lambda_h\phi)(g)\equiv\phi(h^{-1}g)\quad(a\in\mathcal{A},\,g,h\in G).$$

For a non periodic element $w\in S(\mathbf{C}^N)^{\otimes *}$ and a closed subgroup G of $U_w(N)$, let $(\mathcal{H},\pi)$ be $GP(w)$ of $\mathcal{O}_N$. By the Peter-Weyl theorem for G and Theorem

1.2, the following irreducible decomposition holds:

$$(L_2(G,\mathcal{H}),\tilde{\pi}\rtimes\lambda)\sim\bigoplus_{\gamma\in\hat{G}}\{GP(w)\rtimes\gamma\}^{\oplus d_\gamma}$$

where $\hat{G}$ is the unitary dual of G and d_γ is the dimension of the representation of G associated with γ.

In §2, we prove the main theorem(Theorem 1.2). In §3, we show the state of $\mathcal{O}_N\rtimes G$ associated with $GP(w)\rtimes(\mathcal{V},V)$. In §4, we show example.

2. Proof of the main theorem

In this paper, any representation and embedding are unital and $*$-preserving. We assume that any group G is locally compact and any representation of G means a(possibly infinite dimensional)unitary representation in this section. For a representation $(\mathcal{K},\pi)$ of a C*-algebra $\mathcal{A}$, a subspace $\mathcal{V}\subset\mathcal{K}$ is *cyclic* for $(\mathcal{K},\pi)$ if $\overline{\pi(\mathcal{A})\mathcal{V}}=\mathcal{K}$. For a C*-dynamical system $(\mathcal{A},G,\alpha)$, let $\mathrm{Rep}(\mathcal{A}\rtimes G)$ and $\mathrm{Rep}_u(G)$ be the set of all representations of $\mathcal{A}\rtimes G$ and the set of all (possibly infinite dimensional)unitary representations of G, respectively. For a covariant representation $(\mathcal{H},\pi,U)$ of $(\mathcal{A},G,\alpha)$ and $(\mathcal{V},V)\in\mathrm{Rep}_u(G)$, define a new covariant representation $(\mathcal{H}\otimes\mathcal{V},\tilde{\pi},U\otimes V)$ of $(\mathcal{A},G,\alpha)$ as follows:

$$\tilde{\pi}(x)\equiv\pi(x)\otimes I,\quad (U\otimes V)_g\equiv U_g\otimes V_g\quad (x\in\mathcal{A},\ g\in G). \tag{2.1}$$

For $(\mathcal{H},\pi)\in\mathrm{Rep}(\mathcal{A}\rtimes G)$, we have a covariant representation $(\mathcal{H},\pi|_{\mathcal{A}},\pi|_G)$ of $(\mathcal{A},G,\alpha)$.

Definition 2.1. *For a representation $(\mathcal{H},\pi)$ of $\mathcal{A}\rtimes G$ and a unitary representation $(\mathcal{V},V)$ of G, a new representation $(\mathcal{H},\pi)\rtimes(\mathcal{V},V)$ of $\mathcal{A}\rtimes G$ is defined by the representation which is induced by a covariant representation $(\mathcal{H}\otimes\mathcal{V},\widetilde{\pi|_{\mathcal{A}}},(\pi|_G)\otimes V)$ of $(\mathcal{A},G,\alpha)$.*

This induces the following map:

$$\mathrm{Rep}(\mathcal{A}\rtimes G)\times\mathrm{Rep}_u(G)\ni((\mathcal{H},\pi),(\mathcal{V},V))\mapsto(\mathcal{H},\pi)\rtimes(\mathcal{V},V)\in\mathrm{Rep}(\mathcal{A}\rtimes G).$$

Define $\mathcal{R}(G)\equiv\mathrm{Rep}_u(G)/\!\sim$ and $\mathcal{R}(\mathcal{A}\rtimes G)\equiv\mathrm{Rep}(\mathcal{A}\rtimes G)/\!\sim$ where $\sim$ means unitary equivalence. Then we can verify that the following map is well-defined:

$$\mathcal{R}(\mathcal{A}\rtimes G)\times\mathcal{R}(G)\ni([(\mathcal{H},\pi)],[(\mathcal{V},V)])\mapsto[(\mathcal{H},\pi)\rtimes(\mathcal{V},V)]\in\mathcal{R}(\mathcal{A}\rtimes G).$$

We denote $[(\mathcal{H},\pi)]\rtimes[(\mathcal{V},V)]\equiv[(\mathcal{H},\pi)\rtimes(\mathcal{V},V)]$. Both $\mathcal{R}(G)$ and $\mathcal{R}(\mathcal{A}\rtimes G)$ have a sum by the direct sum of representations. The tensor product on $\mathcal{R}(G)$ is associative and distributive with respect to direct sum. For $x\in\mathcal{R}(G)$, define

$$R_x:\mathcal{R}(\mathcal{A}\rtimes G)\to\mathcal{R}(\mathcal{A}\rtimes G);\quad \xi R_x\equiv\xi\rtimes x. \tag{2.2}$$

We see that $\xi R_{\mathbf{1}} = \xi$, $(\xi\oplus\eta)R_x = \xi R_x \oplus \eta R_x$, $(\xi R_x)R_y = \xi R_{xy}$, $\xi R_{x\oplus y} = \xi R_x \oplus \xi R_y$ for $x, y \in \mathcal{R}(G)$ and $\xi, \eta \in \mathcal{R}(\mathcal{A}\rtimes G)$ where $\mathbf{1}$ is the trivial representation of G. ¿From this, the following holds:

Proposition 2.2. *For a C*-dynamical system $(\mathcal{A}, G, \alpha)$, $\mathcal{R}(\mathcal{A}\rtimes G)$ is a right $\mathcal{R}(G)$-module by R in (2.2).*

For $N \geq 2$, let $\mathcal{O}_N$ be the Cuntz algebra([**2**]), that is, it is a C*-algebra which is universally generated by generators $s_1, \ldots, s_N$ satisfying $s_i^* s_j = \delta_{ij} I$ for $i, j = 1, \ldots, N$ and $s_1 s_1^* + \cdots + s_N s_N^* = I$. For a non periodic element $w = w^{(1)} \otimes \cdots \otimes w^{(k)} \in S(\mathbf{C}^N)^{\otimes k}$, define $w[l] \equiv w^{(l)} \otimes \cdots \otimes w^{(k)}$ for $1 \leq l \leq k$ and choose an orthonormal set $\{y^{(l,m)}\}_{m=1}^N \subset \mathbf{C}^N$ such that $y^{(l,1)} \equiv w^{(l)}$ for each $l = 1, \ldots, k$. Let $\{1, \ldots, N\}^* \equiv \coprod_{k\geq 0}\{1, \ldots, N\}^k$, $\{1, \ldots, N\}_1^* \equiv \coprod_{k\geq 1}\{1, \ldots, N\}^k$, $\{1, \ldots, N\}^0 \equiv \emptyset$, $\{1, \ldots, N\}^k \equiv \{(j_1, \ldots, j_k) : j_1, \ldots, j_k \in \{1, \ldots, N\}\}$. Define a subset $\Lambda(w)$ of $S(\mathbf{C}^N)^{\otimes *}$ by

$$\Lambda(w) \equiv \Lambda_1(w) \sqcup \Lambda_2(w) \sqcup \Lambda_3(w), \tag{2.3}$$

$\Lambda_1(w) \equiv \{w[l] : l = 1, \ldots, k\}$, $\Lambda_2(w) \equiv \coprod_{l=1}^k \Lambda_{2,l}(w)$, $\Lambda_{2,1}(w) \equiv \{y^{(k,m)} \otimes w : m = 2, \ldots, N\}$, $\Lambda_{2,l}(w) \equiv \{y^{(l-1,m)} \otimes w[l] : m = 2, \ldots, N\}$ for $2 \leq l \leq k$, $\Lambda_3(w) \equiv \{\varepsilon_J \otimes x : x \in \Lambda_2(w),\ J \in \{1, \ldots, N\}_1^*\}$ where $\varepsilon_J = \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k}$ for $J = (j_1, \ldots, j_k)$ and $\{\varepsilon_j\}_{j=1}^N$ is the standard basis of $\mathbf{C}^N$. If $(\mathcal{H}, \pi)$ is $GP(w)$ of $\mathcal{O}_N$ with the GP vector Ω, then $\{\pi(s(x))\Omega : x \in \Lambda(w)\}$ is a complete orthonormal basis of $\mathcal{H}$.

Lemma 2.3. *For a non periodic element $w \in S(\mathbf{C}^N)^{\otimes k}$ and a closed subgroup G of $U_w(N)$, let $(\mathcal{K}, \pi)$ be $GP(w)$ with the GP vector Ω. Let $\Lambda(w)$ be in (2.3). For $g \in G$, define an operator Y_g on $\mathcal{K}$ by*

$$Y_g \pi(s(x))\Omega \equiv \pi(\alpha_g(s(x)))\Omega \quad (x \in \Lambda(w)).$$

Then Y is a unitary action of G on $\mathcal{K}$ and $(\mathcal{K}, \pi, Y)$ is a covariant representation of a C-dynamical system $(\mathcal{O}_N, G, \alpha)$.*

Proof. For each $g \in G$, Y_g is well-defined because $G \subset U_w(N)$. We show that $K_{x,y} \equiv < Y_g\pi(s(x))\Omega | Y_g\pi(s(y))\Omega >= \delta_{x,y}$ for each $x, y \in \Lambda(w)$. Because $\pi(s(w))\Omega = \Omega$ and $\alpha_g(s(w)) = s(w)$,

$$\pi_g(s(w^{\otimes L}))\Omega = \Omega \quad (\forall L \geq 0) \tag{2.4}$$

where $\pi_g \equiv \pi \circ \alpha_g$. By (2.4), $K_{x,y} =< \Omega | \pi_g(\{s(x)\}^* s(y \otimes w^{\otimes L}))\Omega >$ for each $L \geq 0$. Assume that $x = x^{(1)} \otimes \cdots \otimes x^{(a)}$ and $y = y^{(1)} \otimes \cdots \otimes y^{(b)}$ for $a, b \geq 0$. <u>If $a = b + Lk + j$,</u> $L \geq 0$ and $0 < j < k$, then $K_{x,y} = c\cdot < \Omega | \pi_g(s(w[j+1]))\Omega >$ where $c \equiv < x | y \otimes w^{\otimes L} \otimes w^{(1)} \otimes \cdots \otimes w^{(j)} >$. By (2.4), $< \Omega | \pi_g(s(w[j+1]))\Omega >= d\cdot < \Omega | \pi_g(s(w[j+1]))\Omega >$ where $d \equiv < w | w[j] \otimes w^{(1)} \otimes \cdots \otimes w^{(j)} >$. Because w is non periodic, $|d| < 1$. This implies that $K_{x,y} = 0$. <u>Assume that $a = b + Lk$.</u> Because $< \pi(s(x))\Omega | \pi(s(y))\Omega >=$

$\delta_{x,y}$, we see that $< x|y \otimes w^{\otimes L} >= \delta_{x,y}$. On the other hand, $K_{x,y} =< x|y \otimes w^{\otimes L} >$. In consequence, $K_{x,y} = \delta_{x,y}$ for each $x, y \in \Lambda(w)$. Therefore Y_g is an isometry for each $g \in G$. Because $Y_g^{-1} = Y_{g^{-1}}$ and $Y_g Y_h = Y_{gh}$ for each $g, h \in G$, Y is a unitary action of G on $\mathcal{K}$. By definition of Y_g, we see that $Y_g \pi(s_J)\Omega = \pi(\alpha_g(s_J))\Omega$ for each $J \in \{1, \ldots, N\}^*$. ¿From this, we can verify that $\mathrm{Ad}Y_g \circ \pi = \pi \circ \alpha_g$ for each $g \in G$. Hence the statement holds. □

Let φ be the natural embedding of $\mathcal{O}_N$ into $\mathcal{O}_N \rtimes G$. When $(\mathcal{K}, \pi)$ is $GP(w)$ of $\mathcal{O}_N$, there is a representation $(\mathcal{K}, \tilde{\pi})$ such that $\tilde{\pi} \circ \varphi = \pi$ by Lemma 2.3. If $(\mathcal{K}^{'}, \pi^{'})$ is $GP(w)$ of $\mathcal{O}_N$, then $(\mathcal{K}^{'}, \tilde{\pi}^{'}) \sim (\mathcal{K}, \tilde{\pi})$ by construction. We denote $(\mathcal{K}, \tilde{\pi})$ by $GP(w) \rtimes \mathbf{1}$. By Definition 2.1, we have $(GP(w) \rtimes \mathbf{1}) \rtimes (\mathcal{V}, V)$ for $(\mathcal{V}, V) \in \mathrm{Rep}_u(G)$.

Lemma 2.4. *For a non periodic element $w \in S(\mathbf{C}^N)^{\otimes *}$, let G be a closed subgroup of $U_w(N)$ and $(\mathcal{K}, \pi)$ be $GP(w)$ with the GP vector Ω. Then $GP(w) \rtimes (\mathcal{V}, V) \sim (GP(w) \rtimes \mathbf{1}) \rtimes (\mathcal{V}, V)$ for each $(\mathcal{V}, V) \in \mathrm{Rep}_u(G)$.*

Proof. Let $(\mathcal{H}, \Pi)$ be $GP(w) \rtimes (\mathcal{V}, V)$ and $\mathcal{V}^{'} \subset \mathcal{H}$ such that $(\mathcal{V}^{'}, \Pi|_G) \sim (\mathcal{V}, V)$. We construct a unitary from $\mathcal{H}$ to $\mathcal{K} \otimes \mathcal{V}$. By definition, there is a unitary u from $\mathcal{V}^{'}$ to $\mathcal{V}$ such that $\mathrm{Ad}u(\pi(g)) = V_g$ for each $g \in G$. Choose an orthonormal basis $\{e_n\}_{n \in \Xi}$ of $\mathcal{V}^{'}$. Define an operator T on $\mathcal{H}$ by $T\Pi(s(x))e_n \equiv \pi(s(x))\Omega^{'} \otimes ue_n$ for $x \in \Lambda(w)$ and $n \in \Xi$. Then we see that $< T\Pi(s(x))e_n | T\Pi(s(y))e_m >= \delta_{x,y}\delta_{n,m}$ for $x, y \in \Lambda(w)$ and $n, m \in \Xi$. Hence T is an isometry. Because $\{\pi(s(x))\Omega^{'} \otimes ue_n : x \in \Lambda(x),\, n \in \Xi\}$ is a complete orthonormal basis of $\mathcal{K} \otimes \mathcal{V}$, T is a unitary. Further we see that $T\Pi(s_i) = (\pi(s_i) \otimes I)T$ and $T\Pi(g) = (\pi(g) \otimes V_g)T$ for each $i = 1, \ldots, N$ and $g \in G$. Therefore T gives the unitary equivalence between $GP(w) \rtimes (\mathcal{V}, V)$ and $(GP(w) \rtimes \mathbf{1}) \rtimes (\mathcal{V}, V)$. □

For a non periodic element $w \in S(\mathbf{C}^N)^{\otimes k}$, let $(\mathcal{K}, \pi)$ be $GP(w)$ of $\mathcal{O}_N$ with the GP vector Ω. If $v \in \mathcal{K}$ satisfies $< v|\Omega >= 0$, then we can verify that $\lim_{n\to\infty}(\pi(s(w))^*)^n v = 0$.

Proof of Theorem 1.2. (i) By Lemma 2.4, the statement holds.
(ii) and (iii) hold by (i) and Proposition 2.2.
(iv) Assume that $(\mathcal{V}, V)$ is irreducible. Let $(\mathcal{H}, \pi)$ be $GP(w) \rtimes (\mathcal{V}, V)$ with the GP subspace $\mathcal{V}^{'}$. For $v \in \mathcal{H}$, $v \neq 0$, it is sufficient to show that $\mathcal{V}^{'} \subset \overline{\pi(\mathcal{O}_N \rtimes G)v}$. By the proof of Lemma 2.4, v is written by $\sum_{(x,n)\in\Lambda\times\Xi} a_{x,n}\pi(s(x))e_n$ for $a_{x,n} \in \mathbf{C}$. When $a_{x,n} \neq 0$, put $v^{'} \equiv a_{x,n}^{-1} \cdot (\pi(s(x)))^* v$. Then we can denote $v^{'} = e_n + y$ for $y \in \mathcal{H}$, $< y|e_n >= 0$. Then $v^{'} \equiv \lim_{n\to\infty}(\pi(s(w))^*)^n v = \sum_{l=1}^{M} c_l e_n \in \mathcal{V}^{'}$ and $c_n = 1$. Because $(\mathcal{V}, V)$ is irreducible, $\mathrm{Lin} < \pi(G)v^{'} >=$

$\mathcal{V}'$. Therefore $\mathcal{V}' \subset \text{Lin} < \pi(G) \cdot \overline{\pi(\mathcal{O}_N)v} > \subset \overline{\pi(\mathcal{O}_N \rtimes G)v}$. Hence $GP(w) \rtimes (\mathcal{V}, V)$ is irreducible. If $(\mathcal{V}, V)$ is not irreducible, then $GP(w) \rtimes (\mathcal{V}, V)$ is not irreducible by (ii). Therefore the statement holds. $\square$

For $w, w' \in S(\mathbf{C}^N)^{\otimes *}$, $w \sim w'$ if there is $p \in \mathbf{Z}_k$ such that $w' = w^{(p(1))} \otimes \cdots \otimes w^{(p(k))}$ where $w = w^{(1)} \otimes \cdots \otimes w^{(k)}$. When $w, w' \in S(\mathbf{C}^N)^{\otimes *}$ are non periodic, $GP(w) \sim GP(w')$ if and only if $w \sim w'$.

Proposition 2.5. *For non periodic elements $w, w' \in S(\mathbf{C}^N)^{\otimes *}$, assume that G is a closed subgroup of $U_w(N) \cap U_{w'}(N)$. If $w \not\sim w'$, then $GP(w) \rtimes (\mathcal{V}, V) \not\sim GP(w') \rtimes (\mathcal{V}', V')$ for any representations $(\mathcal{V}, V)$ and $(\mathcal{V}', V')$ of G.*

Proof. We see the following branching laws: $(GP(w) \rtimes (\mathcal{V}, V))|_{\mathcal{O}_N} = (GP(w))^{\oplus \dim \mathcal{V}}$, $(GP(w') \rtimes (\mathcal{V}', V'))|_{\mathcal{O}_N} = (GP(w'))^{\oplus \dim \mathcal{V}'}$. Because $w \not\sim w'$, $GP(w) \rtimes (\mathcal{V}, V) \not\sim GP(w') \rtimes (\mathcal{V}', V')$. $\square$

3. State associated with $GP(w) \rtimes (\mathcal{V}, V)$

Operator algebraists prefer *state* than representation. We realize $GP(w) \rtimes (\mathcal{V}, V)$ as the GNS representation of a state of $\mathcal{O}_N \rtimes G$. We denote generators of $\mathcal{O}_N \rtimes G$ by s_i, λ_g for $i = 1, \ldots, N$ and $g \in G$.

Proposition 3.1. *Assume that $w = w^{(1)} \otimes \cdots \otimes w^{(k)} \in S(\mathbf{C}^N)^{\otimes k}$ with $k \geq 1$ is non periodic and G is a closed subgroup of $U_w(N)$. For a finite dimensional unitary representation $(\mathcal{V}, V)$ of G with $\dim \mathcal{V} = M$ and an orthonormal basis $\{e_n\}_{n=1}^M$ of $\mathcal{V}$, let $V_g = ((V_g)_{ij})_{i,j=1}^M$ be the matrix representation of V_g for $g \in G$ with respect to $\{e_n\}_{n=1}^M$. define a state ρ of $\mathcal{O}_N \rtimes G$ by*

$$\rho(s_J s_K^* \lambda_g) = \begin{cases} (V_g)_{11} \cdot \overline{w_J} \cdot w_K & (|J| \equiv |K| \ mod\ k), \\ \\ 0 & (otherwise) \end{cases}$$

for $g \in G$ and $J, K \in \{1, \ldots, N\}^*$ where $w_J \equiv \prod_{l=1}^k w_{j_l}^{(\sigma^{l-1}(1))}$ for $J = (j_1, \ldots, j_k)$. Then the following holds:

(i) *ρ is pure if and only if $(\mathcal{V}, V)$ is irreducible.*
(ii) *If $(\mathcal{V}, V)$ is irreducible, then the GNS representation of $\mathcal{O}_N \rtimes G$ by ρ is equivalent to $GP(w) \rtimes (\mathcal{V}, V)$.*

Proof. Let $(\mathcal{H}, \pi)$ be $GP(w) \rtimes (\mathcal{V}, V)$ with the GP subspace $\mathcal{V}' \subset \mathcal{H}$. Then there is a unitary u from $\mathcal{V}$ to $\mathcal{V}'$ such that $\text{Ad}u \circ (\pi|_G) = V$. Define $\Omega \equiv ue_1$ and a state ρ' of $\mathcal{O}_N \rtimes G$ by $\rho'(s_J s_K^* \lambda_g) \equiv < \Omega | \pi(s_J s_K^*) \lambda_g \Omega >$ for $g \in G$ and $J, K \in \{1, \ldots, N\}^*$. Then we can verify that $\rho' = \rho$. Therefore the statements hold by Theorem 1.2. $\square$

4. Example

Let $\{\varepsilon_j\}_{j=1}^N$ be the standard basis of $\mathbf{C}^N$ and d_γ be the dimension of the representation associated with $\gamma \in \hat{G}$ for a group G in this section.

4.1. Examples of Theorem 1.2.

Example 4.1. Let $(l_2(\mathbf{N}), \pi)$ be a representation of $\mathcal{O}_N$ defined by $\pi(s_i)e_n \equiv e_{N(n-1)+i}$ for $n \in \mathbf{N}$ and $i = 1, \ldots, N$. $(l_2(\mathbf{N}), \pi)$ is an irreducible permutative representation. Define an action β of $U(N-1)$ on $\mathcal{O}_N$ by $\beta_g(s_1) \equiv s_1$ and $\beta_g(t_i) \equiv \sum_{j=1}^{N-1} g_{ji} t_j$ for $g = (g_{ij}) \in U(N-1)$ where $t_i \equiv s_{i+1}$ for $i = 1, \ldots, N-1$. Then $(l_2(\mathbf{N}), \pi)$ is $GP(\varepsilon_1)$ of $\mathcal{O}_N$ and $G \equiv U_{\varepsilon_1}(N) = \{1\} \times U(N-1) \cong U(N-1)$. For the regular representation $(L_2(U(N-1), l_2(\mathbf{N})), \Pi)$ of $\mathcal{O}_N \rtimes_\beta U(N-1)$ induced by $(l_2(\mathbf{N}), \pi)$ of $\mathcal{O}_N$, the following irreducible decomposition holds:

$$(L_2(U(N-1), l_2(\mathbf{N})), \Pi) = \bigoplus_{\gamma \in \widehat{U(N-1)}} (GP(\varepsilon_1) \rtimes \gamma)^{\oplus d_\gamma}.$$

When $N = 2$, the action β of $U(1)$ on $\mathcal{O}_2$ is given by $\beta_c(s_1) = s_1$, $\beta(s_2) = cs_2$ for $c \in U(1)$. For $\mathcal{O}_2 \rtimes_\beta U(1)$, the following irreducible decomposition holds:

$$(L_2(U(1), l_2(\mathbf{N})), \Pi) = \bigoplus_{n \in \mathbf{Z}} GP(\varepsilon_1) \rtimes \chi_n$$

where $\chi_n(c) \equiv c^n$ for $c \in U(1)$. Especially this decomposition is multiplicity free.

Example 4.2. Let $\mathfrak{S}_N \hookrightarrow U(N)$ be the natural embedding of the symmetric group $\mathfrak{S}_N$ by the permutation of $\{\varepsilon_j\}_{j=1}^N$ and $w \equiv c \cdot (\varepsilon_1 + \cdots + \varepsilon_N)/\sqrt{N} \in S(\mathbf{C}^N)$ for $c \in U(1)$. Since $p(w) = w$ for each $p \in \mathfrak{S}_N$, $\mathfrak{S}_N \subset U_w(N)$. Let $(\mathcal{H}, \pi)$ be $GP(w)$. Then we have the irreducible decomposition of the regular representation of $\mathcal{O}_N \rtimes \mathfrak{S}_N$ as follows:

$$L_2(\mathfrak{S}_N, \mathcal{H})(\cong \mathcal{H}^{\oplus N!}) = \bigoplus_{\gamma \in \hat{\mathfrak{S}}_N} (GP(w) \rtimes \gamma)^{\oplus d_\gamma}.$$

4.2. Other cases.

The following are not examples of Theorem 1.2.

Example 4.3. For $\mathbf{N} \equiv \{1, 2, 3, \ldots\}$ and $N \geq 2$, let $\sigma(i) \equiv i+1$ for $i = 1, \ldots, N-1$ and $\sigma(N) \equiv 1$. Define operators T and P on $l_2(\mathbf{N})$ by

$$Te_n \equiv e_{Nn}, \quad Pe_{N(n-1)+i} \equiv e_{N(n-1)+\sigma(i)} \quad (n \in \mathbf{N},\, i = 1, \ldots, N).$$

Define a representation $(l_2(\mathbf{N}), \pi)$ of $\mathcal{O}_N$ by

$$\pi(s_i) \equiv P^i T P^{-i} \quad (i = 1, \ldots, N). \tag{4.1}$$

Then $(l_2(\mathbf{N}), \pi)$ is $GP(\varepsilon_1 \otimes \cdots \otimes \varepsilon_N)$ of $\mathcal{O}_N$. Define $\beta \in \mathrm{Aut}\mathcal{O}_N$ by $\beta(s_i) \equiv s_{\sigma(i)}$ for $i = 1, \ldots, N$. Then $\mathrm{Ad}P \circ \pi = \pi \circ \beta$. ¿From this, $(l_2(\mathbf{N}), \pi, P)$

is a covariant representation of a C*-dynamical system $(\mathcal{O}_N, \mathbf{Z}_N, \beta)$. This induces a representation $(l_2(\mathbf{N}), \tilde{\pi})$ of $\mathcal{O}_N \rtimes \mathbf{Z}_N$ naturally. Because $(l_2(\mathbf{N}), \pi)$ is irreducible, $(l_2(\mathbf{N}), \tilde{\pi})$ is irreducible.

Example 4.4. Let $\mathcal{O}_N \rtimes U(1)$ be the C*-crossed product by the gauge action on $\mathcal{O}_N$ and $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N \rtimes U(1)$. If there are $w \in S(\mathbf{C}^N)^{\otimes *}$ and $\Omega \in \mathcal{H}$ such that $\pi(s(w))\Omega = \Omega$, then we can show that $\Omega = 0$. ¿From this, there is no permutative representation of cycle type for $\mathcal{O}_N \rtimes U(1)$.

4.3. Permutative representation of $\mathcal{O}_N \rtimes (\mathbf{T}^{N-1} \rtimes \mathbf{Z}_N)$. For $N \geq 2$, let transformations τ on $\mathbf{T}^{N-1} \equiv \{(z_1, \ldots, z_{N-1}) : \forall i,\ z_i \in \mathbf{C}, |z_i| = 1\}$ and κ on $\mathbf{Z}^{N-1}$ as follows: when $N = 2$, $\tau(z) \equiv \bar{z}$, $\kappa(n) \equiv 1 - n$ and when $N \geq 3$,

$$(4.2)\quad \begin{cases} \tau(z_1, \ldots, z_{N-1}) \equiv & (z_2, \ldots, z_{N-1}, \overline{z_1 \cdots z_{N-1}}), \\ \kappa(n_1, \ldots, n_{N-1}) \equiv & (n_2 - n_1 + 1, n_3 - n_1, \ldots, n_{N-1} - n_1, -n_1). \end{cases}$$

τ and κ induce actions of the cyclic group $\mathbf{Z}_N \equiv \{\sigma^{i-1} : i = 1, \ldots, N\}$ where σ is a cyclic permutation on $\{1, \ldots, N\}$ defined by $\sigma(i) \equiv i + 1$ for $i = 1, \ldots, N-1$ and $\sigma(N) \equiv 1$. For $n = (n_1, \ldots, n_{N-1}) \in \mathbf{Z}^{N-1}$ and $z = (z_1, \ldots, z_{N-1}) \in \mathbf{T}^{N-1}$, define $z^n \equiv z_1^{n_1} \cdots z_{N-1}^{n_{N-1}}$. The action κ is free. Define $[n] \equiv \{\kappa^{i-1}(n) : i = 1, \ldots, N\}$ and $G \equiv \mathbf{T}^{N-1} \rtimes \mathbf{Z}_N$ by τ. We denote an element in G by (z, σ^j). We identify both $\mathbf{T}^{N-1}$ and $\mathbf{Z}_N$ as subgroups of G. G is realized as a subgroup $G' \equiv < \{\mathrm{diag}(z_1, \ldots, z_{N-1}, \overline{z_1 \cdots z_{N-1}}), P\} >$ of $U(N)$ where $P \in U(N)$ is defined by $P\varepsilon_i \equiv \varepsilon_{\sigma(i)}$ for $i = 1, \ldots, N$. We identify G and G'. We denote the action of G on $\mathcal{O}_N$ by α. Define the crossed product $\mathcal{O}_N \rtimes G$ for the C*-dynamical system $(\mathcal{O}_N, G, \alpha)$.

Definition 4.5. *For $n \in \mathbf{Z}^{N-1}$, a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N \rtimes G$ is $P(1 \cdots N|n)$ if there is a cyclic vector $\Omega \in \mathcal{H}$ such that*

$$\pi(s_1 \cdots s_N)\Omega = \Omega, \quad T_z\Omega = z^n\Omega \quad (\forall z \in \mathbf{T}^{N-1})$$

where $T_z \equiv \pi((z, id))$ for $z \in \mathbf{T}^{N-1}$. Ω is called the GP vector of $(\mathcal{H}, \pi)$.

A representation $(\mathcal{K}, \pi)$ of $\mathcal{O}_N$ is $P(1 \cdots N)$ if there is a cyclic unit vector Ω such that $\pi(s_1 \cdots s_N)\Omega = \Omega$. $(l_2(\mathbf{N}), \pi)$ in Example 4.3 is $P(1 \cdots N)$.

Theorem 4.6. (i) *For each $n \in \mathbf{Z}^{N-1}$, $P(1 \cdots N|n)$ is unique up to unitary equivalence.*

(ii) *For each $n \in \mathbf{Z}^{N-1}$, $P(1 \cdots N|n)$ is irreducible.*

(iii) *For $n, m \in \mathbf{Z}^{N-1}$, $P(1 \cdots N|n) \sim P(1 \cdots N|m)$ if and only if $[n] = [m]$.*

(iv) *Let $(\mathcal{K}, \pi)$ be $P(1 \cdots N)$ of $\mathcal{O}_N$ and $(L_2(G, \mathcal{K}), \Pi)$ be the regular representation of $\mathcal{O}_N \rtimes G$ induced by $(\mathcal{K}, \pi)$. Then there is an orthonormal family $\{v_n \in L_2(G, \mathcal{K}) : n \in \mathbf{Z}^{N-1}\}$ such that v_n is the GP vector of $P(1 \cdots N|n)$ of $\mathcal{O}_N \rtimes G$ and the following irreducible decomposition* *with respect to the action of $\mathcal{O}_N \rtimes G$ holds:*

$$L_2(G,\mathcal{K}) = \bigoplus_{n\in\mathbf{Z}^{N-1}} \mathcal{W}_n, \quad \mathcal{W}_n \equiv \overline{\Pi(\mathcal{O}_N \rtimes G)v_n}. \tag{4.3}$$

The multiplicity of each component in the decomposition in (4.3) is N. (4.3) implies that the existence of $P(1\cdots N|n)$ for each $n \in \mathbf{Z}^{N-1}$.

We show the case $N = 2$ in Theorem 4.6. Let τ be an action of $\mathbf{Z}_2$ on $U(1) \equiv \{z \in \mathbf{C} : |z| = 1\}$ by $z \mapsto \bar{z}$. $\kappa(n) = 1 - n$. $G = U(1) \rtimes \mathbf{Z}_2$ is realized as a subgroup $\left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} : z \in U(1) \right\}$ of $U(2)$. Let $(\mathcal{K},\pi)$ be $P(12)$ of $\mathcal{O}_2$ with the GP vector Ω and $(L_2(G,\mathcal{K}),\Pi)$ be the regular representation of $\mathcal{O}_2 \rtimes G$ induced by $(\mathcal{K},\pi)$. Define

$$\Omega_{n,+}(z,\sigma^j) \equiv \delta_{0,j}\cdot z^{-n}\Omega, \quad \Omega_{n,-}(z,\sigma^j) \equiv \delta_{1,j}\cdot z^{-n}\Omega$$

for $z \in U(1)$ and $j = 0, 1$. Then $V_{n,\pm} \equiv \overline{\Pi(\mathcal{O}_2)\Omega_{n,\pm}}$ is $P(12)$ of $\mathcal{O}_2$, $L_2(G,\mathcal{K}) = \bigoplus_{n\in\mathbf{Z}}(V_{n,+} \oplus V_{n,-})$. Therefore $\mathcal{W}_n = V_{n,+} \oplus V_{1-n,-}$.

$$L_2(G,\mathcal{K}) = \bigoplus_{n\in\mathbf{Z}}(V_{n,+} \oplus V_{n,-}) = \bigoplus_{n\in\mathbf{Z}} \mathcal{W}_n \sim \bigoplus_{n\in\mathbf{N}} (P(12|n))^{\oplus 2}$$

since $\mathcal{W}_n \sim \mathcal{W}_{1-n}$ and $\mathcal{W}_n$ is $P(12|n)$.

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Appendix A. Proof of Theorem 4.6

Let $(\mathcal{H},\pi)$ be a representation of $\mathcal{O}_N$. If $\Omega, \Omega' \in \mathcal{H}$ satisfy $<\Omega|\Omega'>= 0$, $\pi(s_1\cdots s_N)\Omega = \Omega$ and $\pi(s_1\cdots s_N)\Omega' = \Omega'$, then $\overline{\pi(\mathcal{O}_N)\Omega}$ and $\overline{\pi(\mathcal{O}_N)\Omega'}$ are orthogonal.

Lemma A.1. *Let τ, κ and G be in (4.2) and $(\mathcal{H},\pi)$ be $P(1\cdots N|n)$ for $n \in \mathbf{Z}^{N-1}$ with the GP vector Ω. Define $P \equiv \pi(\mathbf{1},\sigma)$, $R \equiv \pi(s_1)P$ and $\Omega_i \equiv R^{i-1}\Omega$ for $i = 1,\ldots,N$ where $\mathbf{1} \equiv (1,\ldots,1) \in \mathbf{T}^{N-1}$. Then the following holds:*

(i) $T_z\Omega_i = z^{\kappa^{i-1}(n)}\Omega_i$ *and* $\pi(s_1\cdots s_N)\Omega_i = \Omega_i$ *for* $i = 1,\ldots,N$.
(ii) $\mathcal{H} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_N$ *where* $\mathcal{V}_i \equiv \overline{\pi(\mathcal{O}_N)\Omega_i}$ *for* $i = 1,\ldots,N$.
(iii) $\{\pi(s(x))R^{i-1}\Omega : x \in \Lambda,\ i = 1,\ldots,N\}$ *is a complete orthonormal basis of $\mathcal{H}$ where $\Lambda \equiv \Lambda(\varepsilon_1 \otimes \cdots \otimes \varepsilon_N)$ in (2.3).*

Proof. (i) We see that $T_zR = z_1RT_{\tau^{-1}(z)}$ for $z = (z_1,\ldots,z_{N-1}) \in \mathbf{T}^{N-1}$. By this and the induction with respect to $i = 1,\ldots,N$, the statement holds.
(ii) By (i), $\{\Omega_i\}_{i=1}^N$ is an orthogonal family of vectors in $\mathcal{H}$. ¿From this and the eigenequation of Ω_i, $\{\mathcal{V}_i\}_{i=1}^N$ is an orthogonal family of subspaces

of $\mathcal{H}$. Because $T_z\mathcal{V}_i\subset\mathcal{V}_i$ and $PV_i\subset\pi(s_1^*)\mathcal{V}_{\sigma(i)}\subset\mathcal{V}_{\sigma(i)}$ for $i=1,\ldots,N$, $\mathcal{H}=\overline{\pi(\mathcal{A})\Omega}\subset\mathcal{V}_1\oplus\cdots\oplus\mathcal{V}_N\subset\mathcal{H}$. Hence the statement holds.
(iii) By construction, $(\mathcal{V}_i,\pi|_{\mathcal{V}_i})$ is $P(1\cdots N)$ of $\mathcal{O}_N$ with the GP vector Ω_i. Hence $\{\pi(s(x))\Omega_i : x\in\Lambda\}$ is a complete orthonormal basis of $\mathcal{V}_i$. By (ii), the statement holds. □

Proof of Theorem 4.6. Let $\mathcal{A}\equiv\mathcal{O}_N\rtimes G$ and R be in Lemma A.1.
(i) By Lemma A.1 (iii), the existence of the canonical basis implies the uniqueness.
(ii) Let $v\in\mathcal{H}$, $v\neq 0$. Because Ω is a cyclic vector for $(\mathcal{H},\pi)$, it is sufficient to show that $\Omega\in\overline{\pi(\mathcal{A})v}$. Define $e_{J,i}\equiv\pi(s_J)R^{i-1}\Omega$ for $J\in\Lambda$ and $i=1,\ldots,N$. Then $v=\sum_{J,i}a_{J,i}e_{J,i}$. Choose J,i such that $a_{J,i}\neq 0$. Then $<\Omega|(\pi(s_J)R^{i-1})^*v>\neq 0$. We can assume that $v=\Omega+y$ for $y\in\mathcal{H}$, $<\Omega|y>=0$. Then $v'\equiv\lim_{n\to\infty}(\pi(s_1\cdots s_N)^*)^n v=\Omega+\sum_{j=2}^N c_j\Omega_j\in\overline{\pi(\mathcal{O}_N)v}$ for some $c_j\in\mathbf{C}$ for $j=2,\ldots,N$. ¿From this, $\Omega=\int_{\mathbf{T}^{N-1}}\bar{z}^nT_zv'\,d\mu(z)\in\overline{\pi(\mathcal{A})v}$. Hence the statement holds.
(iii) Assume that $P(1\cdots N|n)\sim P(1\cdots N|m)$. By Lemma A.1, $P(1\cdots N|n)$ and $P(1\cdots N|m)$ have vectors $\Omega_1,\ldots,\Omega_N$ and $\Omega_1',\ldots,\Omega_N'$ which satisfy the statement, respectively. By the description above Lemma A.1, the vector in the statement is unique up to scalar multiple. By checking the eigenvalues by T_z, the statement holds.
(iv) Because $L_2(G,\mathcal{K})=\bigoplus_{i=1}^N(\mathcal{K}\otimes L_2(\mathbf{T}^{N-1}\cdot\sigma^{i-1}))$, $\{\pi(s_J)\Omega\otimes f_{n,i} : J\in\Lambda,\, i=1,\ldots,N,\, n\in\mathbf{Z}^{N-1}\}$ is a complete orthonormal basis of $L_2(G,\mathcal{K})$ where $f_{n,i}(z,\sigma^j)\equiv z^{-n}\cdot\delta_{j,1-i}$. Define $\{\phi_{n,i} : n\in\mathbf{Z}^{N-1},\, i=1,\ldots,N\}\subset L_2(G,\mathcal{K})$ by $\phi_{n,i}(z,\sigma^j)\equiv\delta_{j,1-i}\cdot z^{-n}\pi(s_i\cdots s_N)\Omega$. Then $R\phi_{n,i}=\phi_{\kappa(n),\sigma^{-1}(i)}$ and $\Pi(s_1\cdots s_N)\phi_{n,i}=\phi_{n,i}$ where $R\equiv\Pi(s_1)P$. ¿From this, $L_2(G,\mathcal{K})=\bigoplus_{n\in\mathbf{Z}^{N-1}}\bigoplus_{i=1}^N V_{n,i}$ where $V_{n,i}\equiv\overline{\Pi(\mathcal{O}_N)\phi_{n,i}}$. Define $\mathcal{W}_n\equiv\overline{\Pi(\mathcal{A})\phi_{n,1}}$. We see that $\bigoplus_{i=1}^N V_{\kappa^{i-1}(n),\sigma^{1-i}(1)}\subset\mathcal{W}_n$. Because $T_zV_{n,i}\subset V_{n,i}$ and $PV_{n,i}\subset\Pi(s_1)^*V_{\kappa(n),\sigma^{-1}(i)}\subset V_{\kappa(n),\sigma^{-1}(i)}$, $\mathcal{W}_n\subset\bigoplus_{i=1}^N V_{\kappa^{i-1}(n),\sigma^{1-i}(1)}$. In consequence, $\mathcal{W}_n=\bigoplus_{i=1}^N V_{\kappa^{i-1}(n),\sigma^{1-i}(1)}$. By these, the decomposition holds. v_n is obtained by normalizing $\phi_{n,1}$. By definition, $(\mathcal{W}_n,\Pi|_{\mathcal{W}_n})$ is $P(1\cdots N|n)$ of $\mathcal{A}$. Hence each component is irreducible. □

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