A background story and some know-how of virtual turning points*

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Let $P(x, \eta^{-1}d/dx, \eta)$ be an m-th order linear ordinary differential operator with a large parameter η, and let $P_B(x, \partial_y^{-1}\partial_x, \partial_y)$ denote its Borel transform. Usually we consider an operator P of the form $\eta^m P(x, \eta^{-1}d/dx)$; then P_B is a linear partial differential operator, and the Borel transform ψ_B of the WKB solution of P satisfies the equation $P_B\psi_B = 0$. Now, microlocal analysis ([H], [SKK]) tells us that the most elementary carrier of the singularities of solutions of the equation $P_B u = 0$ is a bicharacteristic strip, which is, by definition, a curve $\{(x(t), y(t); \xi(t), \eta(t))\}$ in the cotangent bundle $T^*\mathbb{C}^2_{(x,y)}$

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that is determined by the following Hamilton-Jacobi equation:

$$(1) \qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial p}{\partial \xi} & \text{(1.a)} \\ \dfrac{dy}{dt} = \dfrac{\partial p}{\partial \eta} & \text{(1.b)} \\ \dfrac{d\xi}{dt} = -\dfrac{\partial p}{\partial x} & \text{(1.c)} \\ \dfrac{d\eta}{dt} = -\dfrac{\partial p}{\partial y} & \text{(1.d)} \\ p(x, \xi, \eta) = 0, & \text{(1.e)} \end{cases}$$

where $p(x, \xi, \eta)$ stands for the principal symbol of the partial differential operator P_B and we identify η with the principal symbol of the operator ∂_y. [In application, $p(x, \xi, \eta)$ normally has the form $\eta^m p(x, \eta^{-1}\xi)$.] Since the operator P_B is free from the multiplication operator $y\cdot, \partial p/\partial y$ vanishes identically. Hence η is a (non-zero) constant, which we usually normalize to be 1.

The point of the above result is, not only that the microlocal singularities of a solution u of the equation $P_B u = 0$ is confined to the characteristic variety $V = \{(x, y; \xi, \eta) \in T^*\mathbb{C}^2; p(x, \xi, \eta) = 0\}$, but also that they are consistent with the fiber space structure of V determined by the Hamiltonian flow (cf. Fig. 1); the microlocal singularities of a solution u of the equation $P_B u = 0$ are not arbitrarily scattered in V, but they are fabricated by fibers called "bicharacteristic strips".

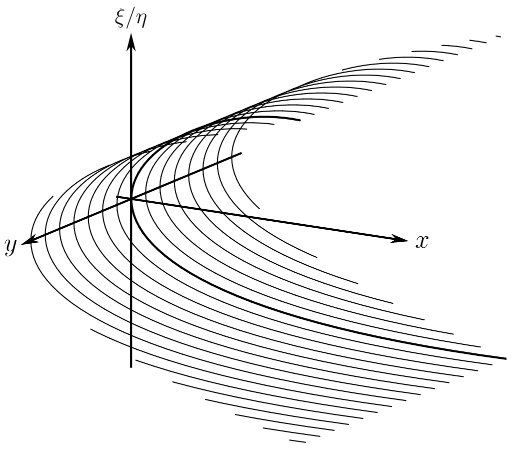

Figure 1 : Characteristic variety V and bicharacteristic strips on it.

Now, for a simple turning point a of a Schrödinger operator $P = \partial_x^2 - \eta^2 Q(x)$, the Borel transform of the WKB solution of P has two singularity loci $\{y = \pm \int_a^x \sqrt{Q} dx\}$, which form a cuspidal singularity at the point $O = \{(x, y) = (a, 0)\}$. Note, however, that this cuspidal singularity appears as a result of projection of a non-singular curve determined by (1); since a is supposed to be a simple turning point, $\partial p/\partial x (= -\eta^2 \partial Q/\partial x)$ does not vanish at $x = a$, and hence the curve determined by (1) is non-singular (cf. Fig. 2). This microlocal singularity structure of the Borel transformed WKB

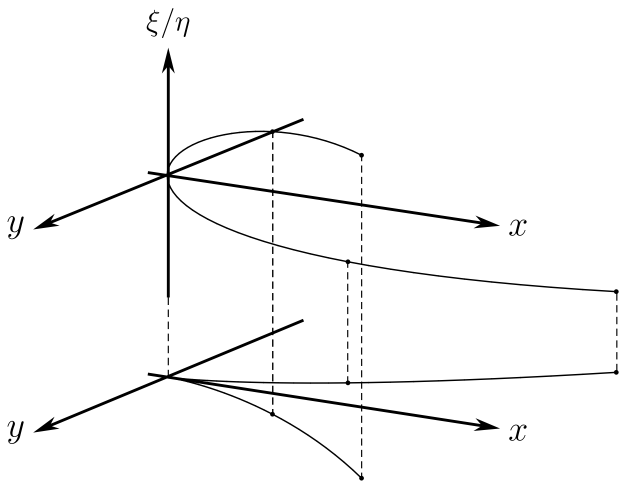

Figure 2 : A bicharacteristic strip (i.e., a non-singular curve determined by (1)) passing through a simple turning point and its projection.

solution near O indicates that the genuine notion of a "turning point" should be given as the x-component of a kink, i.e., a singular point, in the projection of a bicharacteristic strip, a thread of "microlocal singularity locus" of the Borel transform of a WKB solution of the operator in question, and a virtual turning point ([AKT]) is a typical example; it is the x-component of a self-intersection point of a bicharacteristic curve, that is, the projection of a bicharacteristic strip to the base space $\mathbb{C}^2_{(x,y)}$. The expectation that the notion of a virtual turning point should play an important role in WKB analysis of higher order differential operators was first validated by the following observation ([AKT]):

A new Stokes curve introduced by Berk et al. ([BNR]) *is an ordinary Stokes curve emanating from a virtual turning point.*

We will see this fact in a concrete example discussed below.

Since the notion of a virtual turning point does not find any similar — or even related — precedents in the traditional asymptotic analysis, we believe that showing some practical aspects of its analysis is worth doing for acquiring more users of this novel object. This we will do below.

[I] How to find a virtual turning point with the help of a computer.

To find the location of virtual turning points, we have to solve the Hamilton-Jacobi equation (1) globally; this is a scaring task. Fortunately, however, we can locate a "relevant virtual turning point" by evaluating some integrals of algebraic functions with the help of a computer (cf. [S], [SI]).

Let us consider the following situation: Suppose that simple turning points s_1 and s_2 of an operator P are given. Suppose further that the Stokes curve σ_1 (resp., σ_2) emanating from s_1 (resp., s_2) is of type (1,2) (resp., (2,3)) and that σ_1 and σ_2 cross at a point C. Here saying that σ_1(resp., σ_2) is of type (1,2) (resp., (2,3)) means that σ_1 and σ_2 are respectively given by

$$\mathrm{Im} \int_{s_1}^{x} (\xi_1 - \xi_2)dx = 0 \tag{2}$$

and

$$\mathrm{Im} \int_{s_2}^{x} (\xi_2 - \xi_3)dx = 0 \tag{3}$$

where $\xi_j (j = 1, 2, 3)$ denotes a root of the characteristic equation $p(x, \xi, \eta)|_{\eta=1} = 0$. In this situation the following procedure ([Step 1] $\sim$ [Step 4] below) enables us to locate a virtual turning point x_* such that the Stokes curve emanating from x_* passes through the crossing point C. As will be shown in [II] for the concrete example of BNR-equation ([BNR]), this Stokes curve is nothing but the curve that Berk et al. introduced into WKB analysis of higher order differential equations under the name of a "new Stokes' line".

We also note that, when the type (j, k) of σ_1 and the type (l, m) of σ_2 are disjoint in the sense that there is no common number among j, k, l and m, we do not need to seek for a virtual turning point relevant to C ([AKT]).

[Step 1] We introduce a cut γ_1 (resp., γ_2) starting from s_1 (resp., s_2) so that $\xi_j(x)$ $(j = 1, 2, 3)$ are well-defined on this cut plane. All the computation below should be done on this cut plane.

[Step 2] This step is intended to give a bridge over some psychological difficulties in relating the theoretical definition of a virtual turning point and its practical computation with the help of a computer.

Let us suppose that $p(x,\xi,\eta)$ has the form $\prod_{j=1}^{m}(\xi-\xi_j(x)\eta)$, where $\xi_j(x)(j=1,\ldots,m)$ are mutually distinct. Then, except at turning points, we may assume that the characteristic variety is locally defined by $\{\xi=\xi_j(x)\eta\}$ for some j. Hence we find there

$$\frac{\partial p}{\partial \xi}\Big|_V=\Big(\prod_{l\neq j}(\xi_j-\xi_l)\Big)\eta^{m-1} \tag{4}$$

and

$$\frac{\partial p}{\partial \eta}\Big|_V=-\xi_j\Big(\prod_{l\neq j}(\xi_j-\xi_l)\Big)\eta^{m-1}. \tag{5}$$

Therefore (1) entails

$$\frac{dy}{dx}=-\xi_j. \tag{6}$$

In particular, the bicharacteristic curves passing through $(x,y)=(s_1,0)$ are given either by

$$y=-\int_{s_1}^{x}\xi_1 dx \tag{7}$$

or by

$$y=-\int_{s_1}^{x}\xi_2 dx. \tag{8}$$

Then, starting from a point (x_*,y_*) on the curve given by (7) [that is, $y_*=-\int_{s_1}^{x_*}\xi_1 dx$], we reach $(s_1,0)$, then continue our journey following the relation (8) to reach $(x,y)=(s_2,-\int_{s_1}^{s_2}\xi_2 dx)$ and then further continue our journey following the relation

$$y=-\int_{s_2}^{x}\xi_3 dx-\int_{s_1}^{s_2}\xi_2 dx. \tag{9}$$

If we return to our starting point (x_*, y_*) after this journey, that is, if we find a point x_* which satisfies

$$\int_{s_1}^{x_*} \xi_1 dx = \int_{s_1}^{s_2} \xi_2 dx + \int_{s_2}^{x_*} \xi_3 dx, \tag{10}$$

then it is clear from the definition of a virtual turning point that x_* is a virtual turning point.

[Step 3] We next show

$$\mathrm{Im} \int_{x_*}^{C} (\xi_1 - \xi_3) dx = 0. \tag{11}$$

As the real variety defined by the relation

$$\mathrm{Im} \int_{x_*}^{x} (\xi_1 - \xi_3) dx = 0 \tag{12}$$

may have several connected components, (11) does not necessarily entail that the point C is connected with the virtual turning point x_* by a Stokes curve of type $(1,3)$. However, computers are good at discriminating different components; with the help of a computer, one may regard (11) as an almost guarantee that C lies on the Stokes curve σ of type $(1,3)$ that emanates from x_*. To prove (11) we first consider the integral I:

$$I = \int_{s_1}^{x_*} (\xi_1 - \xi_2) dx. \tag{13}$$

It then follows from the definition of x_* that the following relation holds:

$$I = \int_{s_1}^{s_2} \xi_2 dx + \int_{s_2}^{x_*} \xi_3 dx - \int_{s_1}^{x_*} \xi_2 dx = - \int_{s_2}^{x_*} \xi_2 dx + \int_{s_2}^{x_*} \xi_3 dx. \tag{14}$$

Hence we find

$$\int_{s_1}^{C} (\xi_1 - \xi_2) dx + \int_{C}^{x_*} (\xi_1 - \xi_2) dx = \int_{s_2}^{C} (\xi_3 - \xi_2) dx + \int_{C}^{x_*} (\xi_3 - \xi_2) dx, \tag{15}$$

that is,

$$\int_{x_*}^{C} (\xi_1 - \xi_3) dx = \int_{s_1}^{C} (\xi_1 - \xi_2) dx + \int_{s_2}^{C} (\xi_2 - \xi_3) dx. \tag{16}$$

On the other hand, C is a crossing point of σ_1 and σ_2. Hence

$$\text{Im}\int_{s_1}^{C}(\xi_1-\xi_2)dx = \text{Im}\int_{s_2}^{C}(\xi_2-\xi_3)dx = 0. \tag{17}$$

Therefore we obtain from (16) and (17) the following:

$$\text{Im}\int_{x_*}^{C}(\xi_1-\xi_3)dx = 0. \tag{18}$$

This completes the proof of (11).

[Step 4] Let us now look on this situation in [Step 3] upside down: If we evaluate

$$\varrho(x) = \text{Re}\int_{C}^{x}(\xi_1-\xi_3)dx \tag{19}$$

along the curve

$$\text{Im}\int_{C}^{x}(\xi_1-\xi_3)dx = 0, \tag{20}$$

$\varrho(x)$ is, as the real part of a holomorphic function, monotonically increasing or decreasing. Hence we normally find that the relation

$$\varrho(x_0) = \int_{s_1}^{C}(\xi_2-\xi_1)dx + \int_{s_2}^{C}(\xi_3-\xi_2)dx \tag{21}$$

holds at a point x_0 in the curve given by (20). (Note that the right-hand side of (21) is a real number thanks to (17).) But, then we find

$$\int_{x_0}^{C}(\xi_1-\xi_3)dx = \int_{s_1}^{C}(\xi_1-\xi_2)dx + \int_{s_2}^{C}(\xi_2-\xi_3)dx. \tag{22}$$

Therefore, by setting $x_* = x_0$, we find (16), and hence (14); that is, x_0 satisfies

$$\int_{s_1}^{x_0}(\xi_1-\xi_2)dx = \int_{s_2}^{x_0}(\xi_3-\xi_2)dx. \tag{23}$$

Reversing the computation in obtaining (14) from (10), we obtain the following relation (24) from (23):

$$\int_{s_1}^{x_0} \xi_1 dx = \int_{s_1}^{x_0} \xi_2 dx - \int_{s_2}^{x_0} \xi_2 dx + \int_{s_2}^{x_0} \xi_3 dx = \int_{s_1}^{s_2} \xi_2 dx + \int_{s_2}^{x_0} \xi_3 dx. \tag{24}$$

Thus, on the condition that all the computations are legitimately done on the cut plane given by [Step 1], x_0 is a virtual turning point in the sense that it satisfies the equation (10), the same equation as (24).

Summing up, the first trial in locating a virtual turning point in our context is to find a solution x_0 of the equation (21) in the curve given by (20).

Remark. One practical importance in locating a virtual turning point lies in the fact that the portion of the Stokes curve containing a virtual turning point is inert in the connection problem; no Stokes phenomena occur near a virtual turning point. This fact can be confirmed by the same reasoning as in [V, p.244]. To emphasize this fact we often use a dotted line to designate the portion of a Stokes curve along which no Stokes phenomenon is observed.

[II] Evidences which persuade skeptics to believe in the reality of a virtual turning point — virtual is real, real is virtual.

Judging from our experience, we imagine that the most serious obstacle in manipulating virtual turning points is a psychological barrier: We cannot see them with the naked eye; do they exist in the real world? The best way to get rid of such skepticism is to handle virtual turning points in actual problems. Then one will be filled with wonder by observing the subtle harmony that is provided by virtual turning points. We ourselves often shouted with joyful wonder "Subtle is the Lord!" Here we present some of them. See [AKSST], [S] and [SI] for more detailed expositions.

To illustrate our discussion, let us consider the following third order differential operator P, which we call BNR-operator after [BNR]:

$$P = \frac{d^3}{dx^3} + 3\eta^2 \frac{d}{dx} + 2ix\eta^3. \tag{25}$$

One can easily solve the equation (1) in this case to find that there exists one and only one virtual turning point of (25): $x = 0$. We also note that $x = \pm 1$ are ordinary turning points. The resulting Stokes geometry is given in Fig. 3 when η is real and positive. Now, let us see what happens when we change

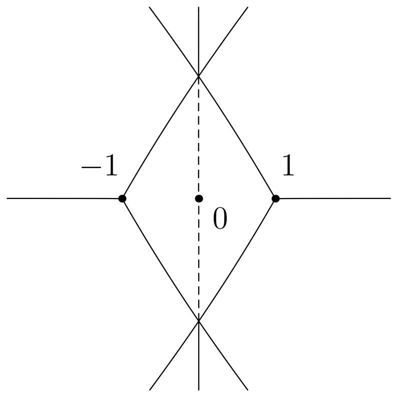

Figure 3 : Stokes geometry of (25) for $\arg \eta = 0$.

$\arg \eta$. Note that Stokes curves do depend on $\arg \eta$ by their definition but that virtual turning points remain intact just like ordinary turning points remain intact. Then we find the Stokes geometry for (a) $\arg \eta = \left(\frac{1}{2} - \frac{1}{12}\right)\pi$, (b) $\arg \eta = \frac{1}{2}\pi$ and (c) $\arg \eta = \left(\frac{1}{2} + \frac{1}{12}\right)\pi$ respectively in Fig. 4 (a), (b) and (c). One then observes in Fig. 4 (a) and (c) the interchange of the relative location of a Stokes curve emanating from an ordinary turning point and that emanating from a virtual turning point. One also finds that the bifurcation of Stokes curves in Fig. 4 (b) should look awkward if the Stokes curves emanating from virtual turning points were not included in Fig. 4 (a), (c). The bifurcation of Stokes curves observed in Fig. 4 (b) is a consequence of the (square-root type) singularity of the roots of the characteristic equation at ordinary turning points, and it is commonly observed in the case of higher order differential equations when a Stokes curve of type (j, k) hits a simple turning point such that the Stokes curves emanating from it are of type (k, l) with $l \neq j$. The relevance of a virtual turning point and the clearly visible phenomenon of bifurcation of Stoke curves enable one to realize the naturality of the notion of a virtual turning point. Furthermore the above mentioned interchange of the relative location of a Stokes curve emanating from an ordinary turning point and a Stokes curve emanating from a virtual turning

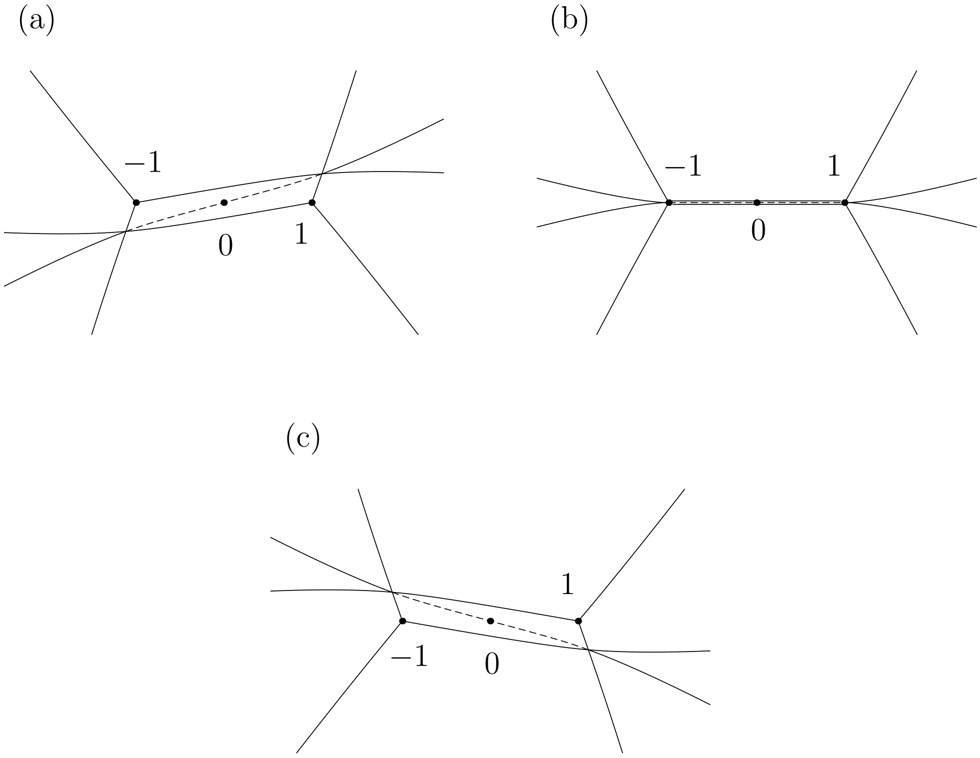

Figure 4 : Stokes geometry of (25) for (a) $\arg \eta = \left(\frac{1}{2} - \frac{1}{12}\right)\pi$, (b) $\arg \eta = \frac{1}{2}\pi$ and (c) $\arg \eta = \left(\frac{1}{2} + \frac{1}{12}\right)\pi$.

point plays an important role in resolving the following paradox (P) which was found by Sasaki [S] in his study of Noumi-Yamada hierarchy ([NY]):

(P) *In some* 3×3 *system of linear differential equations depending on a parameter* t*, no degeneracy of Stokes geometry is observed at a point* $t = t_*$*, where the results in* [KT] *predict the occurrence of some degeneracy, if only ordinary turning points are taken into account. Here "degeneracy" means the existence of a Stokes curve that connects two turning points.*

As is expounded in [AKSST], this paradox (P) is neatly resolved by taking into account virtual turning points. The reasoning is as follows: at a point t_{**} where we observe the "normal" degeneracy in the sense that a simple turning point s_1 and a double turning point d are connected by a Stokes curve, a pair of virtual turning points v_1 and v_2 are also connected by a Stokes curve. As the parameter t moves from t_{**} to t_*, another simple turning point s_2 crosses the Stokes curve that connects s_1 and d (and at the same time the Stokes curve that connects v_1 and v_2). Then the switching of the relative location of Stokes curves forces s_1 to be connected with v_2, not d, and it also forces d to be connected with v_1, not s_1. Thus the partner has become the object in the virtual world — to live happily in such circumstances we are forced to accept the following idea:

A virtual turning point is also an object in the real world.

This is the conclusion of [AKSST].

References

[AKSST] T. Aoki, T. Kawai, S. Sasaki, A. Shudo and Y. Takei, Virtual turning points and bifurcation of Stokes curves for higher order ordinary defferential equations, RIMS preprint 1478 (2004).

[AKT] T. Aoki, T. Kawai and Y. Takei, in Analyse algébrique des perturbations singulières, I. (ed. by L. Boutet de Monvel), Hermann, pp.69-84 (1994). A virtual turning point is called a new turning point in this article.

[BNR] H. L. Berk, W. M. Nevins and K. V. Roberts, J. Math. Phys., **23**, 988 (1982).

[H] L. Hörmander, Acta Math., **27**, 79 (1971).

[KT] T. Kawai and Y. Takei, Adv. in Math., **118**, 1 (1996).

[NY] M. Noumi and Y. Yamada, Funkcialaj Ekvacioj, **41**, 483 (1998).

[S] S. Sasaki, to be published.

[SKK] M. Sato, T. Kawai and M. Kashiwara, in Lect. Notes in Math., No.287, Springer, pp.265-529 (1973).

[SI] A. Shudo and K. S. Ikeda, to be published.

[V] A. Voros, Ann. Inst. Henri Poincaré, **39**, 211 (1983).