HOROSPHERICAL MODEL FOR HOLOMORPHIC DISCRETE SERIES AND HOROSPHERICAL CAUCHY TRANSFORM

SIMON GINDIKIN, BERNHARD KRÖTZ, AND GESTUR ÓLAFSSON

Introduction

For some homogeneous spaces the method of horospheres delivers an effective way to decompose representations in irreducible ones. For Riemannian symmetric spaces $Y = G/K$ horospheres are orbits of maximal unipotent subgroups of G. They are parameterized by points of the horospherical homogeneous space $\Xi_{\mathbb{R}} = G/MN$ where N is a fixed maximal unipotent subgroup and $M = Z_K(A)$ as usual. The horospherical transform maps sufficiently regular functions on Y to the corresponding average along the horospheres. The crucial point is, that the abelian group A acts on Ξ and that this action commutes with the action of G. The decomposition of the natural representation of G in $L^2(\Xi)$ in irreducible ones reduces to the decomposition relative to A. In this way we obtain all unitary spherical representations on Y (with constant multiplicity), except the complementary series. The computation of the Plancherel measure on Y is equivalent to the inversion of the horospherical transform.

The method of horospheres works for several other types of homogeneous spaces, including complex semisimple Lie groups (considered as symmetric spaces) but it has very serious restrictions: discrete series representations lie in the kernel of the horospherical transform, as well as all representations induced from parabolic subgroups that are not minimal. In short, the kernel is the orthocomplement of the most continuous part of the spectrum. The simplest example when the horospherical transform can not be inverted is for the group $\mathrm{SL}(2, \mathbb{R})$. In [4, 5, 6] a modification of the method of horospheres was suggested: complex horospherical transform (the horospherical Cauchy-Radon transform). For a homogeneous space X we consider the complexification $X_{\mathbb{C}}$ and instead of real horospheres on X we consider complex horospheres on $X_{\mathbb{C}}$ without real points (they do not intersect X). The integration along a real horosphere is equivalent to the integration of a δ-function on X with support on this horosphere. In the complex version we replace this δ-function by a Cauchy type kernel with singularities on the complex horosphere without real point. In [4, 5] it is shown that such a complex horospherical transform has no kernel for $\mathrm{SL}(2; \mathbb{R})$ and that it reproduces the Plancherel formula; in [6] it is shown for all compact symmetric spaces.

The objective of this paper is to show that the complex horospherical transform has no kernel on the holomorphic discrete series. Holomorphic discrete series exist for affine symmetric spaces $X = G/H$ of Hermitian type; G is here a group of Hermitian type [7, 21] . The corresponding part of $L^2(X)$ can be realized as boundary values of Hardy space $\mathcal{H}^2(D_+)$ in a Stein tube $D_+ \subset X_{\mathbb{C}}$ with edge X [11]. Our aim is to define a complex horospherical transform which has no kernel on holomorphic H-spherical representations.

The first step is a construction of the space that is going to be the image of the complex horospherical transform. For that, we consider those complex horospheres in the Stein symmetric space $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$

1991 *Mathematics Subject Classification.* 22E46.

Key words and phrases. Semisimple Lie groups, symmetric spaces, horospheres, horospherical transform, Cauchy kernel, Hardy spaces.

The first author was supported by the Louisiana Board of Regents grant *Visiting Experts in Mathematics.* The last two authors were supported by the Research in Pairs program of the Mathematisches Forschungsinstitut, Oberwolfach. The last author was supported by NSF grant DMS-0139783 and DMS-0402068.

that are parameterized by points of the complex horospherical space $\Xi = G_{\mathbb{C}}/M_{\mathbb{C}}N_{\mathbb{C}}$. In Ξ we then consider an orbit Ξ_+ of $G \times \mathcal{T}_+$ where $\mathcal{T}_+$ is an abelian semigroup in the complex torus $T_{\mathbb{C}} = AT$ with the compact torus T as the edge. The space $\mathcal{O}(\Xi_+)$ of holomorphic functions on Ξ_+ is the Fréchet model of the holomorphic discrete series. More exactly, if we decompose this representation with respect to the compact torus T we obtain G-modules which are lowest weight modules (if they are irreducible); we obtain all such modules with multiplicity one. Using the abelian semigroup $\mathcal{T}_+$ we can define a Hardy type space $\mathcal{H}^2(\Xi_+)$ with spectrum "almost all" of the holomorphic discrete series.

The next step is a geometrical background for the construction of the horospherical transform. Firstly, we prove that the horospheres $E(\xi)$ parameterized by points $\xi \in \Xi_+$ do not intersect X. We construct a simple Cauchy type kernel which has no singularities on X and the edge of its singularities coincides with $E(\xi)$. Using this kernel we define the horospherical Cauchy transform from $L^1(X)$ to $\mathcal{O}(\Xi_+)$ which can be extended on $L^2(X)$. The horospherical transform decomposed under T yields the holomorphic spherical Fourier transform.

The last step is the inversion of the horospherical Cauchy transform. We give the Radon type inversion formula using results from [14] for the holomorphic discrete series. Let us remark that for $X = \mathrm{SL}(2,\mathbb{R})$ the inversion formula was obtained in [4, 5] with tools from integral geometry on quadrics. This method automatically extends on any symmetric spaces of Hermitian type of rank 1, i.e., the hyperboloids of signature $(2, n)$. Let us also pay attention to the complete similarity of formulas of this paper and formulas in [6] for compact symmetric spaces. It confirms the view that finite-dimensional spherical representations are similar to representations of holomorphic discrete series.

Special acknowledgement: The paper was written during the stay of the second named author at the RIMS, Kyoto. BK thanks his host Toshiyuki Kobayashi for giving him the opportunity to do his reserach in the stimulating atmosphere of the RIMS. In addition, it should be mentioned, that the material of the paper was lectured on in the RIMS Lie group seminar and enjoyed constructive critique of a knowledgeable audience: a quick look at the facial expressions of T. Kobayashi, K. Nishiyama or E. Opdam was usually enough.

1. Symmetric spaces of Hermitian type

The objective of this section is to set up a standard choice of terminology that will be used throughout the text.

Let us fix some conventions upfront. For a real Lie algebra $\mathfrak{g}$ let us denote by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Likewise, if not stated otherwise, for a connected Lie group G we write $G_{\mathbb{C}}$ for its universal complexification. If $\varphi : G \to H$ is a homomorphism of connected Lie groups, then we will also denote by φ

- the derived homomorphism $d\varphi(\mathbf{1}) : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$,
- the extension of φ to a holomorphic homomorphism $G_{\mathbb{C}} \to H_{\mathbb{C}}$.

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. We assume that $G \subset G_{\mathbb{C}}$ and that $G_{\mathbb{C}}$ is simply connected. Let $\tau : G \to G$ be a non-trivial involution and write H, resp. $H_{\mathbb{C}}$, for the τ-fixed points in G, resp. $G_{\mathbb{C}}$. The object of concern is the affine symmetric space $X = G/H$. We observe that X is contained in its complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ as a totally real submanifold. Write $x_0 = H_{\mathbb{C}}$ for the base point in $X_{\mathbb{C}}$.

Let $\mathfrak{h}$ be the Lie algebra of H and note that $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ with $\tau|_{\mathfrak{q}} = -\mathrm{id}_{\mathfrak{q}}$. The symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is called irreducible if $\mathfrak{g}$ does not contain any τ-invariant ideals except the trivial ones, $\{0\}$ and $\mathfrak{g}$. In that case, either $\mathfrak{g}$ is simple or $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_1$ with $\mathfrak{g}_1$ simple and $\tau(x, x') = (x', x)$. We say that X is irreducible, if $(\mathfrak{g}, \mathfrak{h})$ is irreducible. From now on we will assume, that X is irreducible.

Fix a Cartan involution $\theta : G \to G$ commuting with τ. Denote by $K < G$ the subgroup of θ-fixed points and write $Y = G/K$ for the associated Riemannian symmetric space. Write $\mathfrak{k}$ for the Lie algebra of K. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ with $\theta|_{\mathfrak{s}} = -\mathrm{id}_{\mathfrak{s}}$. Notice that the universal complexification $K_{\mathbb{C}}$ of K naturally identifies with the θ-fixed points in $G_{\mathbb{C}}$.

We will assume that G is a Lie group of Hermitian type, i.e. Y is Riemannian symmetric space of Hermitian type. The assumption can be phrased algebraically: $\mathfrak{z}(\mathfrak{k}) \neq \{0\}$ with $\mathfrak{z}(\mathfrak{k})$ the center of $\mathfrak{k}$.

We assume that τ induces an anti-holomorphic involution on Y and then call X an *affine symmetric space of Hermitian type.*

Remark 1.1. (a) Our assumptions on G and τ can be phrased algebraically, namely:

$$\mathfrak{z}(\mathfrak{k}) \cap \mathfrak{q} \neq \{0\}\ . \tag{A}$$

Let us mention that another way to formulate (A) is to say that $\mathfrak{q}$ admits an H-invariant regular elliptic cone, i.e. X is compactly causal [10].
(b) Symmetric spaces of Hermitian type resemble compact symmetric spaces on an analytical level. Combined they form the class of symmetric spaces which admit lowest weight modules in their L^2-spectrum (holomorphic discrete series).

Since X is irreducible, it follows that $\mathfrak{z}(\mathfrak{k}) \cap \mathfrak{q} = i\mathbb{R}Z_0$ is one dimensional. It is possible to normalize Z_0 in such a way that the spectrum of $\mathrm{ad}(Z_0)$ is $\{-1, 0, 1\}$. The zero-eigenspace is $\mathfrak{k}_{\mathbb{C}}$. We denote the $+1$-eigenspace in $\mathfrak{s}_{\mathbb{C}}$ by $\mathfrak{s}^+$, and the -1-eigenspace by $\mathfrak{s}^-$.

Let $\mathfrak{t}$ be a maximal abelian subspace in $\mathfrak{q}$ containing iZ_0. Then $\mathfrak{t}$ is contained in $\mathfrak{k} \cap \mathfrak{q}$. Set $\mathfrak{a} = i\mathfrak{t}$ and note that $\mathfrak{a}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}$.

Let Δ be the set of roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$,

$$\Delta_n = \{\alpha \in \Delta \mid \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{s}_{\mathbb{C}}\} = \{\alpha \in \Delta \mid \alpha(Z_0) \in \{-1, 1\}\}$$

and

$$\Delta_k = \{\alpha \in \Delta \mid \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{k}_{\mathbb{C}}\} = \{\alpha \in \Delta \mid \alpha(Z_0) = 0\}\,.$$

Then $\Delta = \Delta_k \dot{\cup} \Delta_n$. The elements of Δ_n are called *non-compact roots*, and the elements in Δ_k are called *compact roots*. We choose an ordering in $i\mathfrak{t}^*$ such that $\alpha(Z_0) > 0$ implies that $\alpha \in \Delta_n^+ \subseteq \Delta^+$. Let $\mathcal{W}$ be the Weyl group of Δ and $\mathcal{W}_k$ the subgroup generated by the reflections coming from the compact roots. As $s(Z_0) = Z_0$ for all $s \in \mathcal{W}_k$, it follows that Δ_n^+ is $\mathcal{W}_k$-invariant.

1.1. **Polyhedrons, cones and the minimal tubes.** Set $A = \exp(\mathfrak{a})$, $A_{\mathbb{C}} = \exp(\mathfrak{a}_{\mathbb{C}})$, $T = \exp(\mathfrak{t})$ and $T_{\mathbb{C}} = \exp(\mathfrak{t}_{\mathbb{C}})$. We note that

$$A_{\mathbb{C}} = T_{\mathbb{C}} = TA \simeq T \times A\,.$$

For $\alpha \in \Delta$ let $\check{\alpha} \in \mathfrak{a}$ be its coroot, i.e. $\check{\alpha} \in [\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{-\alpha}] \cap \mathfrak{a}$ and $\alpha(\check{\alpha}) = 2$. Then

$$\Omega = \sum_{\alpha \in \Delta_n^+} \mathbb{R}_{>0} \cdot \check{\alpha} \tag{1.1.1}$$

defines a $\mathcal{W}_k$-invariant open convex cone in $\mathfrak{a} = i\mathfrak{t}$ which contains Z_0. Often one refers to Ω as the *minimal cone* (it is denoted $c_{\min}$ in [10]). Let us remark that one can characterize Ω as the smallest $\mathcal{W}_k$-invariant open convex cone in $\mathfrak{a}$ which contains a long non-compact coroot, i.e.

$$\Omega = \mathrm{co}\left(\mathcal{W}_k(\mathbb{R}_{>0} \cdot \check{\alpha})\right) \qquad (\alpha \text{ long in } \Delta_n^+)\,. \tag{1.1.2}$$

Here $\mathrm{co}(\cdot)$ denotes the convex hull of $(\cdot)$.

We set $A_+ = \exp(\Omega)$ and note that $A_+ \subset A$ is an open semigroup. Moreover

$$\mathcal{T}_+ = T\exp(\Omega) = TA_+ \subset T_{\mathbb{C}}$$

defines a semigroup and complex polyhedron with edge T. We also use the notation $A_- = \exp(-\Omega)$ and $\mathcal{T}_- = TA_-$.

Define G-invariant subsets of $X_\mathbb{C}$ by

$$D_\pm = GA_\pm \cdot x_0 \subset X_\mathbb{C}\,.$$

According to [18] D_+ and D_- are Stein domains in $X_\mathbb{C}$ with $X = G\cdot x_0$ as Shilov boundary. Subsequently we will refer to D_+ and D_- as *minimal tube in $X_\mathbb{C}$ with edge X*.

1.2. **Minimal $\overline{\theta}\tau$-stable parabolics.** Denote by $g \mapsto \overline{g}$ the complex conjugation in $G_\mathbb{C}$ with respect to the real form G. Let

$$\mathfrak{n}_\mathbb{C}^+ = \bigoplus_{\alpha\in\Delta_k^+} \mathfrak{k}_\mathbb{C}^\alpha \quad \text{and} \quad \mathfrak{n}_\mathbb{C}^- = \bigoplus_{\alpha\in\Delta_k^+} \mathfrak{k}_\mathbb{C}^{-\alpha}\,.$$

Set

$$\mathfrak{n}_\mathbb{C} = \mathfrak{n}_\mathbb{C}^+ \oplus \bigoplus_{\alpha\in\Delta_n^+} \mathfrak{g}_\mathbb{C}^\alpha = \mathfrak{n}_\mathbb{C}^+ \oplus \mathfrak{s}^+\,,$$

$$\mathfrak{m}_\mathbb{C} = \{U \in \mathfrak{h}_\mathbb{C} \mid (\forall V \in \mathfrak{t})\ [U,V] = 0\}\,,$$

and

$$\mathfrak{p}_\mathbb{C} = \mathfrak{m}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}_\mathbb{C}\,.$$

Notice, that $\mathfrak{m}_\mathbb{C}$ is contained in $\mathfrak{k}_\mathbb{C}$, as $Z_0 \in \mathfrak{t}_\mathbb{C}$. The Lie algebra $\mathfrak{p}_\mathbb{C}$ is a *minimal $\overline{\theta}\tau$-stable parabolic subalgebra of* $\mathfrak{g}_\mathbb{C}$. Define subgroups of $G_\mathbb{C}$ by $M_\mathbb{C} = Z_{H_\mathbb{C}}(\mathfrak{t}_\mathbb{C}) \subset K_\mathbb{C}$, and $N_\mathbb{C} = \exp(\mathfrak{n}_\mathbb{C})$.

Note that $T_\mathbb{C} = A_\mathbb{C}$. Then the prescription

$$P_\mathbb{C} = M_\mathbb{C}A_\mathbb{C}N_\mathbb{C} = M_\mathbb{C}T_\mathbb{C}N_\mathbb{C}$$

defines a *minimal $\overline{\theta}\tau$-stable parabolic subgroup of* $G_\mathbb{C}$ whose Lie algebra is $\mathfrak{p}_\mathbb{C}$. Write $\Gamma = M_\mathbb{C}\cap A_\mathbb{C} = M\cap T$ and observe that Γ is a finite 2-group. The isomorphic map

$$(M_\mathbb{C} \times_\Gamma A_\mathbb{C}) \times N_\mathbb{C} \to P_\mathbb{C}, \quad ([m,a],n) \mapsto man$$

yields the structural decomposition of $P_\mathbb{C}$.

We denote by $\mathfrak{t} \subseteq \mathfrak{c}$ a τ-stable Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. Then $\mathfrak{c} = \mathfrak{t}\oplus\mathfrak{c}_h$, where $\mathfrak{c}_h = \mathfrak{c}\cap\mathfrak{h}$. Denote by Σ the set of roots of $\mathfrak{c}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. Similarly we set Σ_n, the set of non-compact roots, Σ_k, the set of compact roots. We choose a positive system Σ^+ such that $\Sigma^+|_\mathfrak{t}\backslash\{0\} = \Delta^+$.

Define tori in G by $C = \exp\mathfrak{c}$ and $C_h = \exp\mathfrak{c}_h$. We note that $C = TC_h \simeq T\times_\Gamma C_h$.

2. Complex Horospheres I: Definition and basic properties

The objective of this section is to discuss (generic) horospheres on the complex symmetric space $X_\mathbb{C} = G_\mathbb{C}/H_\mathbb{C}$. We will show that the space of horospheres is $G_\mathbb{C}$-isomorphic to the homogeneous space $\Xi = G_\mathbb{C}/M_\mathbb{C}N_\mathbb{C}$. Further we will introduce a G-invariant subdomain $\Xi_+ \subset \Xi$ which will be a central object for the rest of this paper.

Set

$$\Xi = G_\mathbb{C}/M_\mathbb{C}N_\mathbb{C}$$

and write $\xi_0 = M_\mathbb{C}N_\mathbb{C}$ for the base point of Ξ. Usually we express elements $\xi\in\Xi$ as $\xi = g\cdot\xi_0$ for $g\in G_\mathbb{C}$.

Consider the double fibration

(2.2.1)
$$\begin{tikzcd} & G_\mathbb{C}/M_\mathbb{C} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Xi & & X_\mathbb{C}\,. \end{tikzcd}$$

By a *horosphere* in $X_\mathbb{C}$ we understand a subset of the form

(2.2.2) $$E(\xi) = \pi_2(\pi_1^{-1}(\xi)) \qquad (\xi\in\Xi)\,.$$

For $\xi = g \cdot \xi_0$ we record that

$$E(\xi) = gM_{\mathbb{C}}N_{\mathbb{C}} \cdot x_0 = gN_{\mathbb{C}} \cdot x_0 \subset X_{\mathbb{C}}$$

(use $M_{\mathbb{C}} \subset H_{\mathbb{C}}$).

Similarly, for $z \in X_{\mathbb{C}}$ we set

$$S(z) = \pi_1(\pi_2^{-1}(z)) \ . \tag{2.2.3}$$

If $z = g \cdot x_0$ for $g \in G_{\mathbb{C}}$, then notice $S(z) = gH_{\mathbb{C}} \cdot \xi_0$. Moreover, for $z \in X_{\mathbb{C}}$ and $\xi \in \Xi$ one has the incidence relations

$$z \in E(\xi) \iff \pi_1^{-1}(\xi) \cap \pi_2^{-1}(z) \neq \emptyset \iff \xi \in S(z) \, . \tag{2.2.4}$$

The space of horospheres on $X_{\mathbb{C}}$ shall be denoted by $\mathrm{Hor}(X_{\mathbb{C}})$, i.e.

$$\mathrm{Hor}(X_{\mathbb{C}}) = \{E(\xi) \mid \xi \in \Xi\} \, .$$

Our first objective is to show that Ξ parameterizes $\mathrm{Hor}(X_{\mathbb{C}})$:

Proposition 2.1. *The map*

$$E : \Xi \to \mathrm{Hor}(X_{\mathbb{C}}), \quad \xi \mapsto E(\xi)$$

is a $G_{\mathbb{C}}$-equivariant bijection.

Proof. Surjectivity and $G_{\mathbb{C}}$-equivariance are clear by definition. It remains to establish injectivity. For that write $G_{\mathbb{C}}^{E(\xi_0)}$ for the stabilizer of $E(\xi_0)$ in $G_{\mathbb{C}}$. By $G_{\mathbb{C}}$-equivariance it is enough to show that $G_{\mathbb{C}}^{E(\xi_0)} \subseteq M_{\mathbb{C}}N_{\mathbb{C}}$. Assume that $g \cdot E(\xi_0) = E(\xi_0)$. Then $gN_{\mathbb{C}} \subseteq N_{\mathbb{C}}H_{\mathbb{C}}$. In particular, $g = nh \in N_{\mathbb{C}}H_{\mathbb{C}}$. As $G_{\mathbb{C}}^{E(\xi_0)}$ is a group, and $n \in G_{\mathbb{C}}^{E(\xi_0)}$, it follows, that $h \in G_{\mathbb{C}}^{E(\xi_0)}$. By Lemma 2.2 from below it follows that $h \in M_{\mathbb{C}}$. Hence $g = h(h^{-1}nh) \in M_{\mathbb{C}}N_{\mathbb{C}}$, as $M_{\mathbb{C}}$ normalizes $N_{\mathbb{C}}$. ☐

Lemma 2.2. *Assume that $h \in H_{\mathbb{C}}$ is such that $h \cdot E(\xi_0) = E(\xi_0)$. Then $h \in M_{\mathbb{C}}$.*

Proof. Identify the tangent space $T_{x_0}(G_{\mathbb{C}}/H_{\mathbb{C}})$ with $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$. Then, as $(hN_{\mathbb{C}}h^{-1}) \cdot x_0 = N_{\mathbb{C}} \cdot x_0$, it follows that

$$\mathrm{Ad}(h)(\mathfrak{n}_{\mathbb{C}} \oplus \mathfrak{h}_{\mathbb{C}}) = \mathfrak{n}_{\mathbb{C}} \oplus \mathfrak{h}_{\mathbb{C}} \, .$$

Thus, if $U \in \mathfrak{n}_{\mathbb{C}}$, there exists $Z \in \mathfrak{n}_{\mathbb{C}}$ and $L \in \mathfrak{h}_{\mathbb{C}}$ such that $\mathrm{Ad}(h)U = Z + L$. Applying $(\mathbf{1} - \tau)$ this equality, we get $\mathrm{Ad}(h)(U - \tau(U)) = Z - \tau(Z)$. As $\mathfrak{q}_{\mathbb{C}} = (\mathbf{1} - \tau)(\mathfrak{n}_{\mathbb{C}}) \oplus \mathfrak{t}_{\mathbb{C}}$, and this sum is orthogonal with respect to Killing form, it follows that $\mathrm{Ad}(h)\mathfrak{t}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}$. In particular, $h \in N_{H_{\mathbb{C}}}(\mathfrak{t}_{\mathbb{C}})$.

We recall the Riemannian dual $X^r = G^r/K^r$ of $X = G/H$ which corresponds to the Lie algebras $\mathfrak{g}^r = \mathfrak{k}^r + \mathfrak{s}^r$ with $\mathfrak{k}^r = (\mathfrak{h} \cap \mathfrak{k}) + i(\mathfrak{h} \cap \mathfrak{s})$ and $\mathfrak{s}^r = i(\mathfrak{q} \cap \mathfrak{k}) + (\mathfrak{q} \cap \mathfrak{s})$. Notice that $\mathfrak{a}$ is maximal abelian in $\mathfrak{s}^r$.

To continue with the proof, we observe that $N_{H_{\mathbb{C}}}(\mathfrak{t}_{\mathbb{C}}) = N_{K^r}(\mathfrak{a})M_{\mathbb{C}}$. Thus we may assume that $h \in N_{K^r}(\mathfrak{a})$. Write σ_r for the complex conjugation in $G_{\mathbb{C}}$ with respect to the real form G^r. Then taking σ^r fixed points in $hN_{\mathbb{C}} \in N_{\mathbb{C}}H_{\mathbb{C}}$ yields $hN^r \in N^rK^r$ with $N^r = G^r \cap N_{\mathbb{C}}$. Thus the situation is reduced to the Riemannian case where it follows from [9], p.78. ☐

It is crucial to observe that there is a $T_{\mathbb{C}}$-action on Ξ which commutes with the left $G_{\mathbb{C}}$-action:

Proposition 2.3. *Let $\xi = g \cdot \xi_0 \in \Xi$, $g \in G_{\mathbb{C}}$. For $t \in T_{\mathbb{C}}$ the prescription*

$$\xi \cdot t = gt \cdot \xi_0 \tag{2.2.5}$$

defines an element of Ξ. In particular,

$$T_{\mathbb{C}} \times \Xi \to \Xi, \quad (t, \xi) \mapsto \xi \cdot t \tag{2.2.6}$$

defines an action of $T_{\mathbb{C}}$ on Ξ, which commutes with the natural action of G on Ξ.

Proof. As $T_{\mathbb{C}}$ normalizes $M_{\mathbb{C}}N_{\mathbb{C}}$ it follows that (2.2.5) is defined. Finally, (2.2.5) implies that (2.2.6). defines a left-action of $T_{\mathbb{C}}$. ☐

It is obvious that the map

$$(G_{\mathbb{C}} \times T_{\mathbb{C}}) \times \Xi \to \Xi, \quad ((g,t),\xi) \mapsto g \cdot \xi \cdot t \tag{2.2.7}$$

is a holomorphic action of the complex group $G_{\mathbb{C}} \times T_{\mathbb{C}}$ on the homogeneous space Ξ.

The remainder of this section will be devoted to the definition and basic discussion of an important $G \times T$-invariant subset Ξ_+ of Ξ.

We recall from Subsection 1.1 the polydisc $\mathcal{T}_+ = TA_+$ and define

$$\Xi_+ = G\mathcal{T}_+ \cdot \xi_0 = GA_+ \cdot \xi_0 \, .$$

We record that Ξ_+ is a $(G \times T)$-invariant subset of Ξ.

The set $GP_{\mathbb{C}}$ is open in $G_{\mathbb{C}}$ and $G \cap P_{\mathbb{C}} = MT$. Hence G/MT can be viewed as an open, complex submanifold of the flag manifold $F = G_{\mathbb{C}}/P_{\mathbb{C}}$. We write $F_+ = GP_{\mathbb{C}}/P_{\mathbb{C}}$ for the image of G/MT in F and call F_+ the *flag domain*. Although obvious we emphasize that F_+ is G-homogeneous.

Notice G/MT is the base space of the holomorphic fiber bundle $G/M \times_T \mathcal{T}_+ \to G/MT$ with fiber $\mathcal{T}_+/\Gamma$. There is a natural action of $G \times T$ on $G/M \times_T \mathcal{T}_+$ given by

$$(G \times T) \times (G/M \times_T \mathcal{T}_+) \to G/M \times_T \mathcal{T}_+, \quad ((g,t),[xM,a]) \mapsto [gxM, at] \, .$$

The next lemma gives us basic structural information on Ξ_+.

Lemma 2.4. *The set Ξ_+ is open in $\Xi = G_{\mathbb{C}}/M_{\mathbb{C}}N_{\mathbb{C}}$. Moreover, the mapping*

$$\Phi : G/M \times_T \mathcal{T}_+ \to \Xi, \quad [gM,t] \mapsto gt \cdot \xi_0$$

is a $G \times T$-equivariant biholomorphism onto Ξ_+.

Proof. Clearly, Φ is a defined $G \times T$ -equivariant map with $\operatorname{im}\Phi = \Xi_+$. By the definition of the complex structure of G/MT the holomorphicity of the map is clear, too. Let us show that Φ is injective. For that assume that $g_1t_1 \cdot \xi_0 = g_2t_2 \cdot \xi_0$, $g_j \in G$, $t_j \in \mathcal{T}_+$. By G-equivariance we may assume that $g_2 = \mathbf{1}$. Then $g_1 \in G \cap P_{\mathbb{C}} = MT$ and w.lo.g. we may assume that $g_1 \in M$. Consequently, as $T_{\mathbb{C}} \cap M_{\mathbb{C}}N_{\mathbb{C}} = \Gamma$, we obtain $t_1 \in t_2\Gamma$, i.e. $[M,t_1] = [M,t_2]$. Hence Φ is injective.

A standard computation yields that $d\Phi$ is an immersion and a simple dimension count shows that $\dim G/MT + \dim \mathcal{T}_+ = \dim \Xi$. In particular, Φ is a submersion and $\operatorname{im}\Phi = \Xi_+$ is open, concluding the proof of the lemma. □

2.1. **Fiberings.** To conclude this section we mention three natural fibrations in relation to Ξ_+ and F_+.

Write $S^+ = \exp(\mathfrak{s}^+)$ and recall that the map

$$Y = G/K \to G_{\mathbb{C}}/K_{\mathbb{C}}S^+, \quad gK \mapsto gK_{\mathbb{C}}S^+$$

is a G-equivariant open embedding. Henceforth Y will be understood as an open subset of the flag manifold $G_{\mathbb{C}}/K_{\mathbb{C}}S^+$.

Lemma 2.5. *The following assertions hold:*

(i) *The natural map*
$$\Xi_+ \to F_+, \quad zM_{\mathbb{C}}N_{\mathbb{C}} \mapsto zP_{\mathbb{C}}$$
is a holomorphic fibration with fiber $\mathcal{T}_+/\Gamma$.

(ii) *The natural map*
$$F_+ \to Y, \quad gMT \mapsto gK$$
is a holomorphic fibration with fiber the flag variety K/MT.

(iii) *The natural map*
$$\Xi_+ \to Y, \quad gt \cdot \xi_0 \mapsto gK$$
is a holomorphic fibration with fiber $K/M \times_T \mathcal{T}_+$.

Proof. (i) follows from $G \cap P_{\mathbb{C}} = MT$ and (ii) is obvious. Finally (iii) is a consequence (i) and (ii). □

3. The $G \times T$-Fréchet module $\mathcal{O}(\Xi_+)$

The natural action of $G\times T$ on Ξ_+ gives rise to a representation of $G\times T$ on the Fréchet space $\mathcal{O}(\Xi_+)$ of holomorphic functions on Ξ_+. We will decompose $\mathcal{O}(\Xi_+)$ with respect to this action. By the compactness of T, it is clear that $\mathcal{O}(\Xi_+)$ decomposes discretely under T. It turns out that each T-isotypical component is the section module of a holomorphic line bundle over the flag domain F_+ and that all such section modules arise in this manner.

In the second part of this section we turn our attention to $G\times T$-invariant Hilbert spaces of holomorphic functions on Ξ_+. By definition these are unitary $G \times T$-modules $\mathcal{H}$ with continuous $G \times T$-equivariant embeddings into $\mathcal{O}(\Xi_+)$. There are many interesting examples such as weighted Bergman and weighted Hardy spaces. We will discuss the Hardy space $\mathcal{H}^2(\Xi_+)$ on Ξ_+ with constant weight and show that $\mathcal{H}^2(\Xi_+)$ constitutes a natural model for the the H-spherical holomorphic discrete series of G.

3.1. **The decomposition of $\mathcal{O}(\Xi_+)$.**

In Section 2 we exhibited a natural action of $G\times T$ on Ξ_+, namely

$$(G \times T) \times \Xi_+ \to \Xi_+, \quad ((g,t),\xi) \mapsto g\cdot\xi\cdot t\ . \tag{3.3.1}$$

We recall that $\mathcal{O}(\Xi_+)$ becomes a Fréchet space when endowed with the topology of compact convergence.

Remark 3.1. Finite dimensional representation theory of $G_\mathbb{C}$ shows that Ξ (and hence Ξ_+) is holomorphically separable. In particular $\mathcal{O}(\Xi_+) \neq \{0\}$.

Denote by $\mathrm{GL}(\mathcal{O}(\Xi_+))$ the group of bounded invertible operators on $\mathcal{O}(\Xi_+)$.

The action (3.3.1) induces a continuous representation of $G \times T$ on $\mathcal{O}(\Xi_+)$:

$$L\otimes R : G\times T \to \mathrm{GL}(\mathcal{O}(\Xi_+)), \quad \left((L\otimes R)(g,t)f\right)(\xi) = f(g^{-1}\cdot\xi\cdot t^{-1})\,,$$

$(g,t) \in G\times T$, $f \in \mathcal{O}(\Xi_+)$, and $\xi \in \Xi_+$.

We first decompose $\mathcal{O}(\Xi_+)$ under the action of the compact torus T. Denote by $\widehat{T/\Gamma}$ the character group of T/Γ, i.e. $\widehat{T/\Gamma} = \mathrm{Hom}_{\mathrm{cont}}(T/\Gamma, \mathbb{S}^1)$. In the sequel we identify $\widehat{T/\Gamma}$ with the lattice

$$\Lambda = \{\lambda \in \mathfrak{a}^* \mid \forall U \in (\exp|_\mathfrak{t})^{-1}(\Gamma)\ \lambda(U) \in 2\pi i\mathbb{Z}\}\,.$$

Explicitly, to $\lambda \in \Lambda$ one associates the character $\chi_\lambda(t\Gamma) = e^{\lambda(\log t)}$. Often we will write t^λ for $\chi_\lambda(t\Gamma)$.

The assumption that $G_\mathbb{C}$ is simply connected allows an uncomplicated description of the lattice Λ.

Lemma 3.2. $\Lambda = \left\{\lambda \in \mathfrak{a}^* \mid (\forall\alpha\in\Delta)\ \frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}\right\}$.

Proof. "$\subseteq$": Let $\lambda \in \Lambda$. We first show that $\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha \in \Delta_k$. For that observe that the compact symmetric space $K/H\cap K$ embeds into G/H via the natural map

$$K/H\cap K \to G/H,\ k(H\cap K) \mapsto kH\,.$$

Thus [8], Ch. V, Th. 4.1, yields that $\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha \in \Delta_k$. To complete the proof of "$\subseteq$" we still have to verify $\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha\in\Delta_n$. Fix $\alpha \in \Delta_n$. Standard structure theory implies that there is an embedding of symmetric Lie algebras $(\mathfrak{su}(1,1), \mathfrak{so}(1,1)) \to (\mathfrak{g},\mathfrak{h})$ such that $\begin{bmatrix} i & 0 \\ 0 & -i\end{bmatrix} \in \mathfrak{su}(1,1)$ is mapped to $i\breve{\alpha} \in \mathfrak{t}$. As $G_\mathbb{C}$ is simply connected, we thus obtain an immersive map $\mathrm{SU}(1,1)/\mathrm{SO}(1,1) \to G/H$. In particular, $\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ must hold.

"$\supseteq$": Suppose that $\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ holds for all α. Recall the extension $\mathfrak{t} \subseteq \mathfrak{c}$ of $\mathfrak{t}$ to a compact Cartan subalgebra of $\mathfrak{g}$. In the sequel we consider λ as an element of $\mathfrak{c}^*$ which is trivial on $\mathfrak{c}\cap\mathfrak{h}$. On p. 537 in [8], it is shown that λ is analytically integral for $C = \exp\mathfrak{c}$ (again this needs that $G_\mathbb{C}$ is simply connected).

In particular λ defines an element $\chi_\lambda \in \hat{T}$. It remains to show that $\chi_\lambda|_\Gamma = \mathbf{1}$. As $M = Z_{H\cap K}(\mathfrak{a})$ and $\Gamma = M \cap T$, this reduces to an assertion on the compact symmetric space $K/H \cap K$, where it follows from [8], Ch. V, Th. 4.1. $\square$

For each $\lambda \in \Lambda$ define the λ-isotypical component of $\mathcal{O}(\Xi_+)$ by

$$\mathcal{O}(\Xi_+)_\lambda = \{f \in \mathcal{O}(\Xi_+) \mid (\forall t \in T)\ R(t)f = t^\lambda f\}\ . \tag{3.3.2}$$

As $(R, \mathcal{O}(\Xi_+))$ is a continuous representation of the compact torus T on a Fréchet space, the Peter-Weyl theorem yields

$$\mathcal{O}(\Xi_+) = \bigoplus_{\lambda\in\Lambda} \mathcal{O}(\Xi_+)_\lambda \tag{3.3.3}$$

Each $\mathcal{O}(\Xi_+)_\lambda$ is a G-module for the representation L. In order to describe them explicitly we recall some facts on holomorphic line bundles.

For $\lambda \in \Lambda$ we write $\mathbb{C}_\lambda$ for $\mathbb{C}$ when considered as a MT-module with trivial M-action and T acting by χ_λ. Recall that G/MT inherits a complex manifold structure through its identification with the flag domain F_+. In particular, to each $\lambda \in \Lambda$ one associates the holomorphic line bundle

$$\mathcal{L}_\lambda = G \times_{MT} \mathbb{C}_{-\lambda}\,. \tag{3.3.4}$$

Write $\mathcal{O}(\mathcal{L}_\lambda)$ for its G-module of holomorphic sections, i.e. $\mathcal{O}(\mathcal{L}_\lambda)$ consists of smooth functions $f : G \to \mathbb{C}$ such that

- $f(gmt) = t^{-\lambda} f(g)$ for $g \in G$, $t \in T$ and $m \in m$.
- $G/MT \to \mathcal{L}_\lambda,\ \ gMT \mapsto [gMT, f(g)]$ is holomorphic.

The restriction of $\mathcal{L}_\lambda$ to the flag variety K/MT yields the holomorphic line bundle

$$\mathcal{K}_\lambda = K \times_{MT} \mathbb{C}_{-\lambda}$$

over K/MT. Write Λ_0 for the Δ_k^--dominant elements of Λ, i.e.

$$\Lambda_0 = \left\{\lambda \in \Lambda \mid (\forall \alpha \in \Delta_k^+)\ \langle\lambda, \alpha\rangle \leq 0\right\}\ . \tag{3.3.5}$$

According to Bott [1], $V_\lambda = \mathcal{O}(\mathcal{K}_\lambda)$ is of finite dimension, and non-trivial if and only if $\lambda \in \Lambda_0$. By $\mathcal{L}_\lambda = G \times_{MT} \mathbb{C}_{-\lambda} \simeq G \times_K (K \times_{MT} \mathbb{C}_{-\lambda})$ we retrieve the standard isomorphism

$$\mathcal{O}(\mathcal{L}_\lambda) \simeq \mathcal{O}(G \times_K V_\lambda)\,.$$

In particular,

$$\mathcal{O}(\mathcal{L}_\lambda) \neq \{0\} \quad\iff\quad \lambda \in \Lambda_0\,. \tag{3.3.6}$$

We remind the reader that the T-weight spectrum of π_λ is contained in $\lambda + \mathbb{Z}_{\geq 0}[\Delta^+]$. In particular, $\mathcal{O}(\mathcal{L}_\lambda)$, if irreducible, is a lowest weight module for G with respect to the positive system Δ^+ and lowest weight λ.

Finally we establish the connection between $\mathcal{O}(\Xi_+)_\lambda$ and $\mathcal{O}(\mathcal{L}_\lambda)$. For that let us denote by Ξ_0 the pre-image of F_+ in Ξ, i.e.

$$\Xi_0 = GT_\mathbb{C} \cdot \xi_0\,.$$

Notice that $\Xi_+ \subset \Xi_0$. Holomorphicity and T-equivariance yield $\mathcal{O}(\Xi_+)_\lambda = \mathcal{O}(\Xi_0)_\lambda$. Likewise holds for $\mathcal{O}(\mathcal{L}_\lambda)$. Thus holomorphic extension and restriction gives a natural G-isomorphism $\mathcal{O}(\Xi_+)_\lambda \simeq \mathcal{O}(\mathcal{L}_\lambda)$.

We summarize our discussion.

Proposition 3.3. *The $G \times T$-Fréchet module $\mathcal{O}(\Xi_+)$ decomposes as*

$$\mathcal{O}(\Xi_+) = \bigoplus_{\lambda \in \Lambda_0} \mathcal{O}(\Xi_+)_\lambda \ .$$

Moreover, holomorphic extension and restriction canonically identifies $\mathcal{O}(\Xi_+)_\lambda$ with the section module $\mathcal{O}(\mathcal{L}_\lambda)$.

We conclude this subsection with some comments on unitarization of the section modules $\mathcal{O}(\mathcal{L}_\lambda)$.

Remark 3.4. Let $\lambda \in \Lambda_0$ and let us denote by $\mathcal{O}(\mathcal{L}_\lambda)^{K-\mathrm{fin}}$ the $(\mathfrak{g}, K)$-module of K-finite sections of $\mathcal{O}(\mathcal{L}_\lambda)$. Let us assume that $\mathcal{O}(\mathcal{L}_\lambda)^{K-\mathrm{fin}}$ is irreducible. Then $\mathcal{O}(\mathcal{L}_\lambda)^{K-\mathrm{fin}}$ identifies with the generalized Verma module $N(\lambda) = \mathcal{U}(\mathfrak{g}_\mathbb{C}) \otimes_{\mathcal{U}(\mathfrak{k}_\mathbb{C}+\mathfrak{s}^-)} V_\lambda$ and the Shapovalov form on $N(\lambda)$ gives rise to the (up to scalar unique) contravariant Hermitian form on $\mathcal{O}(\mathcal{L}_\lambda)^{K-\mathrm{fin}}$. We say that $\mathcal{O}(\mathcal{L}_\lambda)^{K-\mathrm{fin}}$ is *unitarizable* if the Shapovalov form is positive definite. Another way to formulate it is that there exists a unitary lowest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ such that the $(\mathfrak{g}, K)$-module of K-finite vectors $\mathcal{H}_\lambda^{K-\mathrm{fin}}$ is $(\mathfrak{g}, K)$-isomorphic to $\mathcal{O}(\mathcal{L}_\lambda)^{K-\mathrm{fin}}$. In this situation $\mathcal{O}(\mathcal{L}_\lambda)$ is then naturally G-isomorphic to the hyperfunction vectors $\mathcal{H}_\lambda^{-\omega}$ of π_λ.

We want to emphasize that not all $\lambda \in \Lambda_0$ correspond to unitarizable modules $\mathcal{O}(\mathcal{L}_\lambda)^{K-\mathrm{fin}}$ (a necessary condition is $\lambda|_\Omega \geq 0$ and we refer to [3] for more precise information). However, we want to stress that $\mathcal{O}(\mathcal{L}_\lambda)^{K-\mathrm{fin}}$ is automatically unitarizable if $\lambda|_\Omega$ is sufficiently positive (for example if condition (3.3.12) below is satisfied).

3.2. **The Hardy space on Ξ_+.** The objective of this section is to introduce the Hardy space on Ξ_+ and to prove some of its basic properties.

We begin with some measure theoretic preliminaries. The groups $G_\mathbb{C}$ and $M_\mathbb{C} N_\mathbb{C}$ are unimodular, and hence $\Xi = G_\mathbb{C}/M_\mathbb{C} N_\mathbb{C}$ carries a $G_\mathbb{C}$-invariant measure μ.

Recall that M is a compact subgroup of G and denote by dm a normalized Haar measure on M. Further we let dg and $d(gM)$ denote left G-invariant measures on G, resp. G/M, normalized subject to the condition

$$\int_G f(g)\ dg = \int_{G/M} \int_M f(gm)\ dm\ d(gM)$$

for all $f \in L^1(G)$.

Notice that the stabilizer in G of any point $\xi \in \mathcal{T}_+ \cdot \xi_0 \subset \Xi_+$ is the compact subgroup M. In particular one has

$$\int_G f(g \cdot \xi)\, dg = \int_{G/M} f(g \cdot \xi)\, d(gM) \tag{3.3.7}$$

for all $\xi \in \mathcal{T}_+ \cdot \xi_0$ and integrable functions f on Ξ_+.

Write $\|\cdot\|_2$ for the L^2-norm on $L^2(G)$. Let us remark that the representation $(R, \mathcal{O}(\Xi_+))$ of T naturally extends to a representation of the semigroup $t \in \mathcal{T}_- \cup T$, also denoted by R. Furthermore if $f \in \mathcal{O}(\Xi_+)$ and $t \in \mathcal{T}_-$ then we can define the restriction of $R(t)f$ to G by $R(t)f|_G : G \to \mathbb{C}$ by $R(t)f|_G(g) = f(gt^{-1} \cdot \xi_0)$. The Hardy norm of $f \in \mathcal{O}(\Xi_+)$ is defined by

$$\|f\|^2 = \sup_{t \in \mathcal{T}_+} \int_G |f(gt \cdot \xi_0)|^2\, dg = \sup_{t \in \mathcal{T}_-} \|R(t)f|_G\|_2^2\,. \tag{3.3.8}$$

Let

$$\mathcal{H}^2(\Xi_+) = \{f \in \mathcal{O}(\Xi_+) \mid \|f\| < \infty\}\,. \tag{3.3.9}$$

Obviously

$$\|R(t)f\| \leq \|f\| \qquad \text{for all } t \in \mathcal{T}_- \tag{3.3.10}$$

and hence $\mathcal{T}_-$ acts on $\mathcal{H}^2(\Xi_+)$ by contractions. Note, that $R(t)f|_G$ is right M-invariant, and, by the definition of the Hardy space norm $R(t)f|_G \in L^2(G/M) \subseteq L^2(G)$.

Lemma 3.5. *The space $\mathcal{H}^2(\Xi_+)$ is a Hilbert space. Furthermore, the following holds:*

(i) *For $\xi \in \Xi_+$ the point evaluation map $\mathrm{ev}_\xi : \mathcal{H}^2(\Xi_+) \ni f \mapsto f(\xi) \in \mathbb{C}$ is continuous.*

(ii) *The boundary value map $\beta : \mathcal{H}^2(\Xi_+) \to L^2(G/M) \subseteq L^2(G)$*

$$\beta(f) = \lim_{\mathcal{T}_- \ni t \to e} R(t)f|_G$$

is an isometry into $L^2(G/M)$.

Proof. The proof follows a standard procedure and will be more a sketch. We refer to [11], in particular the proof of Theorem 2.2, for a detailed discussion of the underlying methods.

Let $\xi \in \Xi_+$. Then there exist relatively compact open sets $U_G \subseteq G$ and $U_T \subseteq \mathcal{T}_+$ such that $\xi \in U_G U_T \cdot \xi_0$. Thus, there is a constant $c > 0$ such that the Bergman-type estimate

$$\int_{U_G U_T \cdot \xi_0} |f(\xi)|^2 \, d\mu(\xi) \leq c \cdot \|f\|^2$$

holds for all $f \in \mathcal{H}^2(\Xi_+)$. This implies that $\mathcal{H}^2(\Xi_+)$ is complete, and that point evaluations are continuous.

Write $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ for the upper half plane and fix $Z \in i\Omega$. We notice that the map $\mathcal{T}_- \ni t \mapsto R(t)f|_G \in L^2(G)$ is well defined and holomorphic. Hence

$$L_f : \mathbb{C}_+ \to L^2(G/M);\ L_f(z) = R(\exp(zZ))f|_G \in L^2(G/M)$$

defines a holomorphic function on $\mathbb{C}_+$.

By Lemma 2.3 in [11] it follows, that $\lim_{z\to 0} L_f(z)$ exists, and is monotonically increasing as $s \searrow 0$ along each line segment $\exp(siZ)$, or, because of the right invariance of dg, on each $t\exp(siZ)$, $t \in T$. As in [11], one shows, that this limit is independent of Z. Thus, we get a boundary value map $\beta : \mathcal{H}^2(\Xi_+) \to L^2(G/M)$, defined by

$$\beta(f) = \lim_{t\to e} R(t)f|_G\,.$$

By the definition of the Hardy space norm, we obviously have

$$\|\beta(f)\|_2 \leq \|f\|\,.$$

But, as the norm $\|R(\exp(sZ))f\|_2$ is monotonically increasing for $s \searrow 0$, it follows that

$$\|R(\exp(sZ))f\|_2 \leq \|\beta(f)\|$$

for all $s \in \mathbb{R}^+$. Thus

$$\|R(t)f|_G\| \leq \|\beta(f)\|\,.$$

It follows, that $\beta : \mathcal{H}^2(\Xi_+) \to L^2(G)$ is an isometry, and hence $\mathcal{H}^2(\Xi_+)$ is a Hilbert space. □

Clearly $L \otimes R$ defines a unitary representation of $G \times T$ on $\mathcal{H}^2(\Xi_+)$. We are going to decompose $\mathcal{H}^2(\Xi_+)$ with respect to this action. As before we begin with the decomposition under T. For $\lambda \in \Lambda$ the λ-isotypical component of $\mathcal{H}^2(\Xi_+)$ is given by $\mathcal{H}^2(\Xi_+)_\lambda = \mathcal{H}^2(\Xi_+) \cap \mathcal{O}(\Xi_+)_\lambda$. The Peter-Weyl theorem yields the orthogonal decomposition

$$\mathcal{H}^2(\Xi_+) = \bigoplus_{\lambda \in \Lambda_0} \mathcal{H}^2(\Xi_+)_\lambda \tag{3.3.11}$$

of $\mathcal{H}^2(\Xi_+)$ in G-modules.

We draw our attention to the unitary G-modules $\mathcal{H}^2(\Xi_+)_\lambda$ inside of $\mathcal{O}(\Xi_+)_\lambda$.

Suppose that $\mathcal{H}^2(\Xi_+)_\lambda \neq \{0\}$. Then $\mathcal{O}(\mathcal{L}_\lambda) \neq \{0\}$ and the restriction mapping

$$\mathcal{H}^2(\Xi_+)_\lambda \to \mathcal{O}(\mathcal{L}_\lambda)$$

gives a G-equivariant embedding. Moreover $\beta(\mathcal{H}^2(\Xi_+)_\lambda) \subset L^2(G)$. Thus $\mathcal{H}^2(\Xi_+)_\lambda$ is a module of the holomorphic discrete series of G. In terms of λ this means that λ satisfies the Harish-Chandra condition [7]

$$\langle \lambda - \rho(\mathfrak{c}), \alpha \rangle > 0 \qquad (\forall \alpha \in \Sigma_n^+)\,, \tag{3.3.12}$$

where $\rho(\mathfrak{c}) = \frac{1}{2}\sum_{\alpha\in\Sigma^+} \alpha$.

Write $\Lambda_{\rm sd}$ for the set of all $\lambda \in \Lambda_0$ which satisfy (3.3.12).

Conversely, let $\lambda \in \Lambda_{\rm sd}$ and write $\mathcal{H}_\lambda$ for a corresponding unitary lowest weight module with lowest weight λ. Denote by $v_\lambda \in \mathcal{H}_\lambda$ a normalized lowest weight vector and write $d(\lambda)$ for the formal dimension (see [7] or (5.5.13) below). It is then straightforward that

$$\mathcal{H}_\lambda \to \mathcal{H}^2(\Xi_+), \quad v \mapsto \Big(gt\cdot\xi_0 \mapsto \sqrt{d(\lambda)}\cdot t^{-\lambda}\langle \pi_\lambda(g^{-1})v, v_\lambda\rangle\Big)$$

defines a G-equivariant isometric embedding. Hence $\mathcal{H}^2(\Xi_+)_\lambda \simeq \mathcal{H}_\lambda \neq \{0\}$.

Summarizing our discussion we obtain the Plancherel decomposition for $\mathcal{H}^2(\Xi_+)$.

Proposition 3.6. *As a G-module the Hardy space decomposes as*

$$\mathcal{H}^2(\Xi_+) \simeq \bigoplus_{\lambda\in\Lambda_{\rm sd}} \mathcal{H}_\lambda\,.$$

Remark 3.7. (a) The set $\Lambda_{\rm sd}$ describes the set of all H-spherical unitary lowest weight representations (up to equivalence) whose matrix coefficients are square integrable on G, i.e. $\Lambda_{\rm sd}$ is the spectrum of the H-spherical holomorphic discrete series of G.
(b) Later we well mainly deal with the spectrum Λ_2 of the holomorphic discrete series on X. One has

$$\Lambda_2 \subseteq \Lambda_{\rm sd}$$

with equality precisely for the equal rank cases [20, 21].

4. Complex Horospheres II: Horospheres with no real points

We continue our discussion of complex horospheres from Section 2. We will introduce the notion of horosphere without real points and investigate Ξ_+ with respect to this property. In addition we will prove some dual statements for the minimal tubes $D_\pm$.

Definition 4.1. We say that the complex horosphere $E(\xi) \subset X_\mathbb{C}$ has *no real points* if $E(\xi)\cap X = \emptyset$. We denote by $\Xi_{nr} \subset \Xi$ the subset of those ξ which correspond to horospheres with no real points.

Lemma 4.2. *The set Ξ_{nr} is a G-invariant subset of Ξ.*

Proof. Let $\xi \in \Xi_{nr}$ and $g \in G$. Assume, that $x \in E(g\cdot\xi)\cap X$. Then $g^{-1}x \in E(\xi)\cap X$, contradicting the assumption that $E(\xi)$ has no real points. □

Recall the open G-invariant subset $\Xi_+ = GA_+\cdot\xi_0 \subset \Xi$. In the sequel it will be useful to consider with Ξ_+ its pre-image $\widetilde{\Xi}_+$ in $G_\mathbb{C}$, i.e.

$$\widetilde{\Xi}_+ = GA_+M_\mathbb{C}N_\mathbb{C}\ .$$

It is clear that $\widetilde{\Xi}_+$ is a left G and right $M_\mathbb{C}N_\mathbb{C}$ invariant open subset of $G_\mathbb{C}$.

Next we draw our attention to the Zariski open subset $N_\mathbb{C}A_\mathbb{C}H_\mathbb{C}$ of $G_\mathbb{C}$. Our objective is to study $\widetilde{\Xi}_+$ in relation to $N_\mathbb{C}A_\mathbb{C}H_\mathbb{C}$.

Remark 4.3. Notice that $\widetilde{\Xi}_+^{-1} \subset N_{\mathbb{C}} A_{\mathbb{C}} H_{\mathbb{C}}$ is equivalent to $G \subset N_{\mathbb{C}} A_{\mathbb{C}} H_{\mathbb{C}}$. However the latter is true only for $\operatorname{rank} X = 1$, i.e. $\dim \mathfrak{t} = 1$. In general, $G \cap N_{\mathbb{C}} A_{\mathbb{C}} H_{\mathbb{C}}$ is an open and dense subset of G (cf. Theorem 4.5 below).

There is a right $H_{\mathbb{C}}$ and left $N_{\mathbb{C}}$-invariant holomorphic middle-projection

$$a_H : N_{\mathbb{C}} A_{\mathbb{C}} H_{\mathbb{C}} \to A_{\mathbb{C}}/\Gamma, \quad z \mapsto a_H(x)$$

In particular, for each $\lambda \in \Lambda$ we obtain natural $(N_{\mathbb{C}}, H_{\mathbb{C}})$-invariant holomorphic maps

$$N_{\mathbb{C}} A_{\mathbb{C}} H_{\mathbb{C}} \to \mathbb{C}, \quad x \mapsto a_H(x)^\lambda \ .$$

The holomorphic function $N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0 \to A_{\mathbb{C}}/\Gamma$ induced by a_H shall also be denoted by a_H.

The function a_H enables us to give a useful geometric description of horospheres.

Lemma 4.4. *Let $\xi = g \cdot \xi_0 \in \Xi$ for $g \in G_{\mathbb{C}}$. Then*

$$\begin{aligned} E(\xi) &= \{z \in X_{\mathbb{C}} \mid g^{-1} z \in N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0,\ a_H(g^{-1}z) = \Gamma\} \\ &= \{z \in X_{\mathbb{C}} \mid g^{-1} z \in N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0,\ a_H(g^{-1}z)^\lambda = 1 \quad \textit{for all } \lambda \in \Lambda\} \end{aligned}$$

Proof. $'' \subseteq''$: If $z \in E(\xi)$, then $z = gn \cdot x_o$ for some $n \in N_{\mathbb{C}}$. Thus $g^{-1} z = n \cdot \xi_0 \in N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0$ and $a_H(g^{-1}z) = a_H(n \cdot x_0) = \Gamma$.

$'' \supseteq''$: Conversely, let $z \in X_{\mathbb{C}}$ be such that $g^{-1}z \in N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0$ and $a_H(g^{-1}z) = \Gamma$. ¿From the first condition follows that $g^{-1}z = na \cdot x_0$ for some $n \in N_{\mathbb{C}}$ and $a \in A_{\mathbb{C}}$; the second condition implies $a \in \Gamma$. Thus $z \in g \cdot \xi_0$, as was to be shown. □

We define a subset of Λ_0 by

$$\Lambda_{\geq 0} = \{\lambda \in \Lambda_0 \mid \lambda|_\Omega \geq 0\} \tag{4.4.1}$$

$$= \left\{\lambda \in \Lambda \mid \lambda|_\Omega \geq 0,\ (\forall \alpha \in \Delta_k^+) \quad \langle \lambda, \alpha \rangle \leq 0 \right\} . \tag{4.4.2}$$

The following theorem is the main geometric result of the paper.

Theorem 4.5. *The following assertions hold:*

(i) *$G \cap N_{\mathbb{C}} A_{\mathbb{C}} H_{\mathbb{C}}$ is open and dense in G.*

(ii) *Let $\lambda \in \Lambda_{\geq 0}$. Then the function $a_H^\lambda|_{G \cap N_{\mathbb{C}} A_{\mathbb{C}} H_{\mathbb{C}}}$ extends to a continuous function on G and*

$$|a_H(g)^\lambda| \leq 1 \qquad (g \in G).$$

Proof. The approach to prove this theorem lies in the use of the structural decomposition

$$G = K A_q H \tag{4.4.3}$$

where $A_q = \exp(\mathfrak{a}_q)$ with $\mathfrak{a}_q \subseteq \mathfrak{s} \cap \mathfrak{q}$ a maximal abelian subspace. There is a natural way to construct a flat $\mathfrak{a}_q$ out of the weight space decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} + \mathfrak{m}_{\mathbb{C}} + \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C}}^\alpha$. It will be briefly reviewed. Let $\gamma_1, \ldots, \gamma_r \in \Delta_n^+$ be a maximal set of long strongly orthogonal roots. Then one can find $Z_j \in \mathfrak{g}_{\mathbb{C}}^{\gamma_j}$, $j = 1, \ldots, r$, such that

$$\mathfrak{a}_q = \bigoplus_{j=1}^r \mathbb{R}(Z_j - \tau(Z_j)) \tag{4.4.4}$$

is a maximal abelian subspace of $\mathfrak{s} \cap \mathfrak{q}$; further

$$A_q \subset S^+ \overline{A_-} H_{\mathbb{C}} \tag{4.4.5}$$

(see [10], p. 210-211 for all that).

(i) As $S^+ \subseteq N_{\mathbb{C}}$, we obtain from (4.4.3) and (4.4.5) that

$$G \subset N_{\mathbb{C}} K \overline{A_-} H_{\mathbb{C}} \ .$$

Hence it is sufficient to show that

$$K\overline{A_-} \cap N_{\mathbb{C}}^+ A_{\mathbb{C}}(H_{\mathbb{C}} \cap K_{\mathbb{C}}) \quad \text{is open and dense in } K\overline{A_-} \tag{4.4.6}$$

To continue, we first have to recall some facts related to the Iwasawa decomposition of $K_{\mathbb{C}}$. Write $\widetilde{N}_{\mathbb{C}}^+$ for a maximal C-stable unipotent subgroup of $K_{\mathbb{C}}$ containing $N_{\mathbb{C}}^+$ and set $\widetilde{A} = \exp(i\mathfrak{c})$. Then $K_{\mathbb{C}} = \widetilde{N}_{\mathbb{C}}^+ \widetilde{A} K$ is an Iwasawa decomposition of $K_{\mathbb{C}}$. We recall that Ω and hence $\overline{A_-}$ is $\mathcal{W}_k$-invariant. Thus Kostant's non-linear convexity theorem (cf. [8], Ch. IV, Th. 10.5) implies that $K\overline{A_-} \subset \tilde{N}_{\mathbb{C}}^+ \overline{A_-} K$. As $\widetilde{N}_{\mathbb{C}}^+ \subset N_{\mathbb{C}}^+ M_{\mathbb{C}}$ and $A \subseteq \widetilde{A} \subseteq AM_{\mathbb{C}}$, we thus get $K\overline{A_-} \subset N_{\mathbb{C}}^+ \overline{A_-} M_{\mathbb{C}} K$. In particular, in order to establish (4.4.6) it is enough to verify that $K \cap N_{\mathbb{C}}^+ A_{\mathbb{C}}(H_{\mathbb{C}} \cap K_{\mathbb{C}})$ is dense in K. But this is known (for example it follows from Lemme 2.1 in [2]).

(ii) In the proof of (i) we have seen that $G \subset N_{\mathbb{C}} M_{\mathbb{C}} \overline{A_-} K H_{\mathbb{C}}$. Thus we only have to show that a_H^λ can be defined as a holomorphic function on $K_{\mathbb{C}}$ with $|a_H(ak)^\lambda| \leq 1$ for all $k \in K$ and $a \in \overline{A_-}$. For that let $(\tau_\lambda, V_\lambda)$ denote the holomorphic $(H_{\mathbb{C}} \cap K_{\mathbb{C}})$-spherical representation of $K_{\mathbb{C}}$ with lowest weight λ. Write $(\cdot, \cdot)$ for a K-invariant inner product on V_λ. Let v_λ be a normalized lowest weight vector and v_H be the spherical vector with $(v_H, v_\lambda) = 1$. Then for all $x \in N_{\mathbb{C}}^+ A_{\mathbb{C}}(H_{\mathbb{C}} \cap K_{\mathbb{C}}) \subset K_{\mathbb{C}}$ we have

$$(\pi_\lambda(x) v_H, v_\lambda) = a_H(x)^\lambda \ .$$

As the left hand side has a holomorphic extension to $K_{\mathbb{C}}$, the same holds for a_H^λ. Finally, for $a \in \overline{A_-}$ and $k \in K$ we have

$$a_H(ak)^\lambda = a^\lambda a_H(k)^\lambda \ .$$

Observe that $a^\lambda \leq 1$ as $\lambda \in \Lambda_{\geq 0}$ and that $|a_H(k)^\lambda| \leq 1$ for all $k \in K$ by Lemma 2.3 in [2]. This completes the proof of (ii). $\square$

Theorem 4.5 features interesting and important corollaries.

Corollary 4.6. *Let $\lambda \in \Lambda_{\geq 0}$ be such that $\lambda|_\Omega > 0$. Then $a_H^\lambda|_{\widetilde{\Xi}_+^{-1} \cap N_{\mathbb{C}} A_{\mathbb{C}} H_{\mathbb{C}}}$ extends to a holomorphic function on $\widetilde{\Xi}_+^{-1}$ with*

$$|a_H(x)^\lambda| < 1 \qquad (x \in \widetilde{\Xi}_+^{-1}) \, .$$

Corollary 4.7. $\Xi_+ \subseteq \Xi_{nr}$, *i.e.* $E(\xi) \cap X = \emptyset$ *for all* $\xi \in \Xi_+$.

Proof. Suppose that there exists $\xi \in \Xi_+$ such that $E(\xi) \cap X \neq \emptyset$. In other words $\widetilde{\Xi}_+ \cap H_{\mathbb{C}} \neq \emptyset \iff \widetilde{\Xi}_+^{-1} \cap H_{\mathbb{C}} \neq \emptyset$; a contradiction to the previous corollary. $\square$

Remark 4.8. (Monotonicity/Convexity) Theorem 4.5 (ii) has a natural interpretation in terms of convexity/monotonicity. Write $\mathrm{pr}_{\mathfrak{a}} = \Im \log a_H$ and note that $\mathrm{pr}_{\mathfrak{a}} : N_{\mathbb{C}} A_{\mathbb{C}} H_{\mathbb{C}} \to \mathfrak{a}$ is a well defined continuous map. Theorem 4.5 (ii) is then equivalent to the inclusion

$$\mathrm{pr}_{\mathfrak{a}}(G \cap N_{\mathbb{C}} A_{\mathbb{C}} H_{\mathbb{C}}) \subseteq \bigoplus_{\alpha \in \Delta_n^- \cup \Delta_k^+} \mathbb{R}_{\geq 0} \cdot \check{\alpha} \, . \tag{4.4.7}$$

4.1. **Dual statements for the minimal tubes.** Recall from Subsection 1.1 the minimal tubes $D_\pm = GA_\pm \cdot x_0$ in $X_{\mathbb{C}}$ with edge X.

It follows from Neeb's non-linear convexity theorem [17] that

$$GA_- \subseteq N_{\mathbb{C}} M_{\mathbb{C}} A_- G \, . \tag{4.4.8}$$

This fact combined with Theorem 4.5 yields

$$GA_- H_{\mathbb{C}} \cap N_{\mathbb{C}} A_{\mathbb{C}} H_{\mathbb{C}} \subseteq N_{\mathbb{C}} T A_- \exp\left(\bigoplus_{\alpha \in \Delta_k^+} \mathbb{R}_{\geq 0} \cdot \check{\alpha} \right) H_{\mathbb{C}} \ . \tag{4.4.9}$$

We have shown:

Corollary 4.9. *Let $\lambda \in \Lambda_{\geq 0}$ be such that $\lambda|_\Omega > 0$. Then, $a_H^\lambda|_{D_- \cap N_\mathbb{C} A_\mathbb{C} \cdot x_0}$ extends to a holomorphic function on D_- such that*

$$|a_H(x)^\lambda| < 1 \qquad (x \in D_-)\,.$$

We recall the definition of the orbits $S(z) \subset \Xi$ for $z \in X_\mathbb{C}$ (cf. equation 2.2.3). The convexity inclusion (4.4.9) delivers the dual statement to Corollary 4.7:

Corollary 4.10. *$S(z) \cap G/M = \emptyset$ for all $z \in D_-$.*

Proof. Let $z = ga \cdot x_0$ for $g \in G$ and $a \in A_-$. Suppose that $S(z) \cap G/M \neq \emptyset$. As $S(z) = gaH_\mathbb{C} \cdot \xi_0$, this is equivalent to $aH_\mathbb{C} N_\mathbb{C} \cap G \neq \emptyset$. In other words $Ga \cap N_\mathbb{C} H_\mathbb{C} \neq \emptyset$; a contradiction to (4.4.9). □

Remark 4.11. Note that (4.4.8) is equivalent to $A_+ G \subseteq GA_+ M_\mathbb{C} N_\mathbb{C}$. This inclusion exhibits interesting additional structure of Ξ_+; it implies

$$\Xi_+ = GA_+ G \cdot \xi_0\,. \tag{4.4.10}$$

Remark 4.12. (Generalization to other cones) Let $\tilde{\Omega}$ be a $\mathcal{W}_k$-invariant convex open sharp cone in $\mathfrak{a}$ containing Ω. A particular interesting example is the maximal cone (denoted by $c_{\max}$ in [10]). In this context we would like to mention that the results in this section remain true for Ω replaced by $\tilde{\Omega}$, the obvious adjustment of $\Lambda_{\geq 0}$ understood.

5. The horospherical Cauchy transform

Our geometric results from Section 4 enable us to define a natural horospherical Cauchy kernel on Ξ_+. The kernel gives rise to the horospherical Cauchy transform $L^1(X) \to \mathcal{O}(\Xi_+)$. The main result is a geometric inversion formula for the horospherical Cauchy transform for functions in the holomorphic discrete series on X.

5.1. **The horospherical Cauchy kernel.** In this subsection we define the horospherical Cauchy kernel and the corresponding horospherical Cauchy transform. We will introduce the holomorphic spherical Fourier transform and relate it the horospherical Cauchy transform.

To begin with we have to recall some features of the root system Δ. Let us denote by

$$\Pi = \{\alpha_1, \ldots, \alpha_m\}$$

a basis of Δ corresponding to the positive system $\Delta_n^+ \cup \Delta_k^-$. As $\operatorname{Spec} \operatorname{ad}(Z_0) = \{-1, 0, 1\}$, it follows that exactly one member of Π is non-compact, say α_m. Define weights $\omega_1, \ldots, \omega_m \in \mathfrak{a}^*$ by

$$\frac{\langle \omega_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij} \qquad (1 \leq i, j \leq n)\,.$$

Set

$$\Lambda_{>0} = \mathbb{Z}_{\geq 0} \cdot \omega_1 + \ldots + \mathbb{Z}_{\geq 0} \cdot \omega_{m-1} + \mathbb{Z}_{>0} \cdot \omega_m\,.$$

Recall the definition of $\Lambda_{\geq 0}$ from (4.4.1).

Lemma 5.1. *The following assertions hold:*

(i) *$\omega_i|_\Omega > 0$ for all $1 \leq i \leq n$. In particular, $\lambda|_\Omega > 0$ for all $\lambda \in \Lambda_{>0}$.*
(ii) *$\Lambda_{\geq 0} = \mathbb{Z}_{\geq 0} \cdot \omega_1 + \ldots + \mathbb{Z}_{\geq 0} \cdot \omega_m$. In particular, $\Lambda_{>0} \subset \Lambda_{\geq 0}$.*

Proof. (i) Fix $x \in \Omega$. Then $x = \sum_{\alpha \in \Delta_n^+} k_\alpha \check{\alpha}$ with $k_\alpha > 0$. Now each $\alpha \in \Delta_n^+$ can be uniquely expressed as $\alpha = \alpha_m + \gamma$ with $\gamma \in \mathbb{Z}_{\geq 0}[\Delta_k^-]$. Moreover if $\alpha = \beta$ is the highest root, then $\gamma \in \mathbb{Z}_{>0}[\alpha_1, \ldots, \alpha_{n-1}]$. As $k_\beta > 0$, the assertion follows.

(ii) Set $\Lambda_{\geq 0}' = \mathbb{Z}_{\geq 0}\cdot\omega_1 + \ldots + \mathbb{Z}_{\geq 0}\cdot\omega_m$. We first show that $\Lambda_{\geq 0}' \subseteq \Lambda_{\geq 0}$. For that let $\lambda\in\Lambda_{\geq 0}'$, say $\lambda = \sum_{i=1}^m k_i\omega_i$ with $k_i\in\mathbb{Z}_{\geq 0}$. As $\alpha_1,\ldots,\alpha_{n-1}$ constitutes a basis of Δ_k^-, it follows that $\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}_{\leq 0}$ for all $\alpha\in\Delta_k^+$. Furthermore, $\lambda|_\Omega \geq 0$ by (i). Hence $\Lambda_{\geq 0}'\subseteq\Lambda_{\geq 0}$.

Finally we establish $\Lambda_{\geq 0}\subseteq\Lambda_{\geq 0}'$. For that fix $\lambda\in\Lambda_{\geq 0}$. Then $\lambda=\sum_{i=1}^m k_i\omega_i$ with some real numbers k_i. We have to show that $k_i\in\mathbb{Z}_{\geq 0}$. Now $\lambda\in\Lambda_{\geq 0}$ means in particular that $\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}_{\leq 0}$ for all $\alpha\in\Delta_k^+$. Hence $\frac{\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\in\mathbb{Z}_{\geq 0}$ for all $1\leq i\leq n-1$. It remains to show that $\frac{\langle\lambda,\alpha_m\rangle}{\langle\alpha_m,\alpha_m\rangle}\in\mathbb{Z}_{\geq 0}$. Integrality is clear. Also since $\mathbb{R}_{\geq 0}\cdot\check{\alpha_m}$ constitutes a boundary ray of the cone Ω, non-negativity follows. □

Define the *horospherical Cauchy kernel* on Ξ_+ as the function

$$\mathcal{K}(\xi) = \frac{1}{a_H(\xi^{-1})^{-\omega_m}-1}\cdot\prod_{j=1}^{m-1}\frac{1}{1-a_H(\xi^{-1})^{\omega_j}}\qquad(\xi\in\Xi_+)\,.$$

In view of Corollary 4.6 and Lemma 5.1(i), the function $\mathcal{K}$ is holomorphic, left H-invariant and bounded on subsets of the form $GU\cdot\xi_0$ for $U\subset A_+$ compact.

This allows us to define for a function $f\in L^1(X)$ its *horospherical Cauchy transform* by

$$\widehat{f}(\xi) = \int_X f(x)\cdot\mathcal{K}(x^{-1}\xi)\,dx\qquad(\xi\in\Xi_+)\,.$$

We notice that the horospherical Cauchy transform is a G-equivariant continuous map

$$L^1(X)\to\mathcal{O}(\Xi_+),\quad f\mapsto\widehat{f}\,.$$

Remark 5.2. (a) The horospherical Cauchy kernel $\mathcal{K}$ is tied to the geometry of the minimal cone Ω: there is no larger $\mathcal{W}_K$-invariant open convex cone $\tilde{\Omega}$ such that $\mathcal{K}$ would be holomorphic on $G\exp(\tilde{\Omega})\cdot\xi_0$ (this follows from Lemma 5.1 and (1.1.2)). In this context we wish to point the difference to the results of Section 4 which are valid for a wider class of convex cones (cf. Remark 4.12).
(b) For each $\lambda\in\Lambda_0$ and $\xi\in\Xi_+$ consider the complex hypersurface

$$L(\lambda,\xi) = \{z\in X_\mathbb{C}\mid a_H(\xi^{-1}z)^\lambda - 1 = 0\}$$

in $X_\mathbb{C}$. Their intersection is $E(\xi)$ and they do not intersect X. The singular set of the horospherical Cauchy kernel is the union of the m hypersurfaces $L(\omega_i,\lambda)$ and the edge of this set is just $E(\xi)$. It means that if f is boundary value of a holomorphic function on D_+ then $\widehat{f}$ is a residue on $E(\xi)$.

The horospherical Cauchy transform can be decomposed in its constituents associated to the elements $\lambda\in\Lambda_{>0}$. More precisely, for $\lambda\in\Lambda_{>0}$ and $f\in L^1(X)$ let us define $\widehat{f}_\lambda\in\mathcal{O}(\Xi_+)$ by

$$\widehat{f}_\lambda(\xi) = \int_X f(x)\cdot a_H(\xi^{-1}x)^\lambda\,dx\,.$$

We will call the map $\lambda\mapsto\widehat{f}_\lambda\in\mathcal{O}(\Xi_+)$ the *spherical holomorphic Fourier transform of f*.

Lemma 5.3. *The following assertions hold:*

(i) *Let $U\subset A_+$ be a compact subset. The series*

$$\sum_{\lambda\in\Lambda_{>0}} a_H(\xi^{-1})^\lambda\qquad(\xi\in\Xi_+)$$

converges uniformly on $GU\cdot\xi_0\subset\Xi_+$.

(ii) *For all $\xi\in\Xi_+$ one has*

$$\sum_{\lambda\in\Lambda_{>0}} a_H(\xi^{-1})^\lambda = \mathcal{K}(\xi)\,.$$

Proof. Uniform convergence on $GU\cdot\xi_0$ is immediate from Corollary 4.6 and Lemma 5.1. Summing up the geometric series one obtains

$$\begin{aligned}\sum_{\lambda\in\Lambda_{>0}} a_H(\xi^{-1})^\lambda &= \sum_{k_1=\ldots=k_{m-1}=0}^{\infty}\sum_{k_m=1}^{\infty} a_H(\xi^{-1})^{k_1\omega_1+\ldots+k_m\omega_m}\\ &= \left(\frac{1}{1-a_H(\xi^{-1})^{\omega_m}}-1\right)\cdot\prod_{j=1}^{m-1}\frac{1}{1-a_H(\xi^{-1})^{\omega_j}}\\ &= \frac{1}{a_H(\xi^{-1})^{-\omega_m}-1}\cdot\prod_{j=1}^{m-1}\frac{1}{1-a_H(\xi^{-1})^{\omega_j}}\\ &= \mathcal{K}(\xi)\end{aligned}$$

□

We conclude from Lemma 5.3 that the horospherical Cauchy transform of a function $f\in L^1(X)$ can be decomposed as

$$\widehat{f}=\sum_{\lambda\in\Lambda_{>0}}\widehat{f}_\lambda$$

with the right hand side converging uniformly on compacta.

Remark 5.4. We wish to point out that the horospherical Cauchy kernel is a product of geometrical and not functional analytic reasoning. We emphasize that in general not all parameters $\lambda\in\Lambda_{>0}$ in the decomposition of the horospherical Cauchy kernel correspond to unitarizable lowest weight modules (see Remark 5.6 below for a more detailed discussion).

5.2. **Holomorphic Fourier transform on lowest weight representations.** The objective of this subsection is to give a more detailed discussion of the holomorphic Fourier transform for functions $f\in L^2(X)$ which are contained in lowest weight module.

To begin with we collect some material on spherical unitary lowest weight representations. A reasonable source might be the overview article [16].

Let $(\pi_\lambda,\mathcal{H}_\lambda)$ be a non-trivial H-spherical unitary lowest weight representation of G. As before we denote by v_λ a normalized lowest weight vector. Write v_H for the unique H-fixed distribution vector which satisfies $\langle v_\lambda,v_H\rangle=1$.

We record the fundamental identity

$$a_H(x)^\lambda=\langle\pi_\lambda(x)v_H,v_\lambda\rangle\qquad(x\in X)\,, \tag{5.5.1}$$

which allows us to link our geometric discussion in Section 4 with representation theory.

Remark 5.5. It follows from Corollary 4.9 that a_H^λ admits a holomorphic extension to the minimal tube D_-. Traditionally this fact was explained via (5.5.1) in the context of holomorphic extension of unitary lowest weight modules (see [18]). We wish to point out that Corollary 4.9 asserts more, namely that $a_H^\lambda|_{D_-}$ is bounded by 1. In addition Corollary 4.9 is more geometric, i.e. not restricted to unitary parameters λ.

Pairing the G-module of smooth vectors $\mathcal{H}_\lambda^\infty$ with v_H yields the G-equivariant embedding

$$\iota:\mathcal{H}_\lambda^\infty\to C^\infty(X),\quad v\mapsto\big(x\mapsto\langle\pi_\lambda(x^{-1})v,v_H\rangle\big)\ . \tag{5.5.2}$$

We say that π_λ is X*-square integrable* if there exists a constant $d_s(\lambda)>0$, the *spherical formal dimension* (cf. [14]), such that $\sqrt{d_s(\lambda)}\cdot\iota$ extends to an isometric map $\mathcal{H}_\lambda\to L^2(X)$.

X-square integrable parameters λ are characterized by the condition [21]

$$\langle \lambda - \rho, \alpha\rangle > 0 \qquad \text{for all } \alpha \in \Delta_n^+ \ . \tag{5.5.3}$$

Here $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} m_\alpha \alpha$ with $m_\alpha = \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^\alpha$.

Likewise we say π_λ is X*-integrable* if $\iota(\mathcal{H}_\lambda^{\mathrm{K-fin}}) \subset L^1(X)$. Integrability is described by the inequality

$$\langle \lambda - 2\rho, \alpha\rangle > 0 \qquad \text{for all } \alpha \in \Delta_n^+ \ . \tag{5.5.4}$$

The set of parameters $\lambda \in \Lambda_{>0}$ which satisfy condition (5.5.4), resp. (5.5.3), shall be denoted by Λ_1, resp. Λ_2. Note that $\Lambda_1 \subset \Lambda_2$.

Remark 5.6. We will discuss the lattice $\Lambda_{>0}$ with regard to Λ_1 and Λ_2. One recognizes a strong dependence on the multiplicities m_α which we will exemplify for three basic cases below. Recall that elements $\lambda \in \Lambda_{>0}$ are described by $\lambda = \sum_{i=1}^n \lambda_i \omega_i$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$ and $\lambda_m > 0$. In addition let us keep in mind that conditions (5.5.3) and (5.5.4) are equivalent to $\langle \lambda - \rho, \alpha_m\rangle > 0$, resp. $\langle \lambda - 2\rho, \alpha_m\rangle > 0$.

The equal rank case: In this situation one has $\mathfrak{t} = \mathfrak{c}$ and $m_\alpha = 1$ for all α. Thus $\rho = \frac{1}{2}\sum_{i=1}^m \omega_i$ and therefore $\lambda - \rho = \sum_{i=1}^m (\lambda_i - \frac{1}{2})\omega_i$. In particular $\langle \lambda - \rho, \alpha_m\rangle = \langle \alpha_m, \alpha_m\rangle(\lambda_m - \frac{1}{2})$ and thus $\Lambda_{>0} \subset \Lambda_2$ as $\lambda_m \geq 1$ for elements $\lambda \in \Lambda_{>0}$.

The group case: In this situation one has $m_\alpha = 2$ for all α and so $\rho = \sum_{i=1}^n \omega_i$. Accordingly we obtain $\langle \lambda - \rho, \alpha_m\rangle = \langle \alpha_m, \alpha_m\rangle(\lambda_m - 1)$. It follows that $\Lambda_{>0}$ parameterizes the holomorphic discrete series and their limits; in particular $\Lambda_2 \subset \Lambda_{>0}$.

The rank one case: Here one has $\Lambda_{>0} = \mathbb{Z}_{>0} \cdot \omega$ and $\rho = \frac{m_\alpha}{2}\alpha$. Thus $\Lambda_2 = (\mathbb{Z}_{>0} + \left[\frac{m_\alpha}{2}\right])\omega$ and $\Lambda_2 \subset \Lambda_{>0}$ with equality precisely for $m_\alpha = 1$, i.e. $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$.

For $\lambda \in \Lambda_2$ we set $L^2(X)_\lambda = \iota(\mathcal{H}_\lambda)$.

Lemma 5.7. *Let $\lambda, \mu \in \Lambda_2$. Fix $v \in \mathcal{H}_\lambda$ and define $f(x) = \langle \pi_\lambda(x^{-1})v, v_H\rangle \in L^2(X)_\lambda$. Then for all $\xi \in \Xi_+$, the function*

$$X \to \mathbb{C}, \quad x \mapsto f(x) a_H(\xi^{-1}x)^\mu$$

is integrable and

$$\int_X f(x) a_H(\xi^{-1}x)^\mu \, dx = \frac{\delta_{\lambda\mu}}{d_s(\lambda)} \langle v, \pi_\lambda(\overline{\xi}) v_\lambda\rangle \ . \tag{5.5.5}$$

Here

$$\pi_\lambda(\overline{\xi}) v_\lambda = a^{-\lambda} \pi_\lambda(g) v_\lambda \in \mathcal{H}_\lambda \qquad \textit{for } \xi = ga\cdot\xi_0, \ g \in G \textit{ and } a \in A_+ \ .$$

Proof. Fix $\xi \in \Xi_+$. Holomorphic extension of (5.5.1) yields

$$a_H(\xi^{-1}g)^\mu = \langle \pi_\lambda(g) v_H, \pi_\lambda(\overline{\xi}) v_\lambda\rangle$$

for all $g \in G$ and $\xi \in \Xi_+$. It follows that $x \mapsto a_H(\xi^{-1}x)^\mu$ is square integrable on X. Thus $x \mapsto f(x) a_H(\xi^{-1}x)^\mu$ is integrable. Finally we apply Schur-orthogonality (cf. [14], Prop. 3.2) and obtain

$$\begin{aligned}
\int_X f(x) a_H(\xi^{-1}x)^\mu \, dx &= \int_X \langle \pi_\lambda(x^{-1})v, v_H^\lambda\rangle \langle \pi_\mu(x) v_H^\mu, \pi_\mu(\overline{\xi}) v_\mu\rangle \, dx \\
&= \int_X \langle \pi_\lambda(x^{-1})v, v_H^\lambda\rangle \overline{\langle \pi_\mu(x^{-1}) \pi_\mu(\overline{\xi}) v_\mu, v_H^\mu\rangle} \, dx \\
&= \frac{\delta_{\mu\lambda}}{d_s(\lambda)} \langle v, \pi_\lambda(\overline{\xi}) v_\lambda\rangle \, .
\end{aligned}$$

□

For $\lambda \in \Lambda_1$ let us write $L^1(X)_\lambda$ for the closure of $\iota(\mathcal{H}_\lambda^{\mathrm{K-fin}})$ in $L^1(X)$. The next lemma can be understood as an L^1-version of Schur-orthogonality for the Cauchy-transform.

Lemma 5.8. *Let $\lambda \in \Lambda_1$. Then*

$$\widehat{f} = \widehat{f}_\lambda \qquad \text{for all } f \in L^1(X)_\lambda \,.$$

Proof. Fix $f \in L^1(X)_\lambda$. We have to show that $\widehat{f}_\mu = 0$ for all $\mu \in \Lambda_{>0} \backslash \{\lambda\}$. For $\mu \in \Lambda_2$ this is a consequence of Lemma 5.7. Therefore, we may assume that $\mu \in \Lambda_{>0} \backslash \Lambda_2$. It means that condition (5.5.3) is violated which we will express as

$$\langle \mu - \rho, \alpha_m \rangle \leq 0 \,. \tag{5.5.6}$$

We now show that

$$\widehat{f}_\mu(\xi) = \int_X f(x) a_H(\xi^{-1}x)^\mu \, dx = 0 \qquad \text{for all } \xi \in \Xi_+ \,.$$

Equation above has boundary values on $G/M \subset \partial \Xi_+$ and it will be sufficient to prove that

$$\widehat{f}_\mu(gM) = \int_X f(x) a_H(g^{-1}x)^\mu \, dx = 0 \qquad \text{for all } g \in G \,.$$

We compute

$$\begin{aligned} \widehat{f}_\mu(gM) &= \int_X f(x) a_H(g^{-1}x)^\mu \, dx \\ &= \int_X f(gx) a_H(x)^\mu \, dx \\ &= \int_T \int_X f(tgx) a_H(tx)^\mu \, dx \, dt \\ &= \int_X \left(\int_T t^\mu f(tgx) \, dt \right) a_H(x)^\mu \, dx \,. \end{aligned}$$

To arrive at a contradiction, suppose that $\int_T t^\mu f(tgx)\, dt \neq 0$. This can only happen if μ belongs to the T-weight spectrum of π_λ. Now the T-weights of π_λ are contained in $\lambda + \mathbb{Z}_{\geq 0}[\Delta^+]$. Thus $\mu = \lambda + \gamma$ for some $\gamma \in \Delta^+$. But then

$$\langle \mu - \rho, \alpha_m \rangle = \langle \lambda - \rho, \alpha_m \rangle + \langle \gamma, \alpha_m \rangle \,.$$

Observe that both summands on the right hand side are positive, the desired contradiction to (5.5.6) $\square$

Remark 5.9. (Analytic continuation) Let $\lambda \in \Lambda_1$ and $f \in L^1(X)_\lambda \cap L^2(X)_\lambda$. Write $f(x) = \langle \pi_\lambda(x^{-1})v, v_H \rangle$ for some $v \in \mathcal{H}_\lambda$. Then Lemma 5.7 and Lemma 5.8 imply

$$\widehat{f}(\xi) = \frac{1}{d_s(\lambda)} \langle v, \pi_\lambda(\overline{\xi}) v_\lambda \rangle \,. \tag{5.5.7}$$

Clearly, the right hand side makes sense for all $v \in \mathcal{H}_\lambda$ and all X-square integrable parameters $\lambda \in \Lambda_2$. We now explain how passing to parameters $\lambda \in \Lambda_2$ in (5.5.7) has a natural explanation in terms of analytic continuation. For that let $\tilde{G}$ denote the universal cover of G. Write $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$ for the sets of $\tilde{G}$-integral parameters which satisfy (5.5.4), resp. (5.5.3). Clearly $\Lambda_{1,2} \subset \tilde{\Lambda}_{1,2}$. The effect of passing to the universal cover is that the parameter spaces involved become continuous in the central variable, i.e. there exists constants $0 < c_2 < c_1$ such that

$$\tilde{\Lambda}_1|_{\mathbb{R}Z_0} =]c_1, \infty[\cdot (\alpha_m|_{\mathbb{R}Z_0}) \quad \text{and} \quad \tilde{\Lambda}_2|_{\mathbb{R}Z_0} =]c_2, \infty[\cdot (\alpha_m|_{\mathbb{R}Z_0}) \,.$$

By the concrete formula for $d_s(\lambda)$ from [14], Th. 4.15, we know that $\lambda \mapsto d_s(\lambda)$ is a meromorphic function on $\mathfrak{a}_\mathbb{C}^*$ which is positive on $\tilde{\Lambda}_2$. Now familiar techniques show that that the assignment $\tilde{\Lambda}_2 \ni \lambda \mapsto \frac{1}{d_s(\lambda)} \langle v, \pi_\lambda(\overline{\xi}) v_\lambda \rangle \in \mathbb{C}$ becomes analytic in the central variable (the Shapovalov form is polynomial in λ and in [13] it is explained how to make consistent analytic choices for v and v_λ in dependence of the central coordinate of λ).

Motivated by Remark 5.9 we define the horospherical Cauchy transform for functions $f \in L^2(X)_\lambda$, $\lambda \in \Lambda_2$ by

$$\widehat{f} = \widehat{f}_\lambda \, .$$

5.3. **Hyperfunctions and generalized matrix coefficients.** In order to discuss the horospherical Cauchy transform and its inverse in a more comprehensive way we need some results on the analytic continuation of generalized matrix coefficients of lowest weight representations. Proofs of the facts cited below can be found in [15].

Let $(\pi, \mathcal{H})$ be a unitary lowest weight representation of G. Write $\mathcal{H}^\omega$ and $\mathcal{H}^{-\omega}$ for the associated G-modules of analytic, resp. hyperfunction vectors. The nature of the T-spectrum of π shows that $\pi|_T$ extends holomorphically to $\mathcal{T}_-$. Moreover the so obtained self adjoint operators $\pi(a)$, $a \in A_-$, are of trace class and strongly mollifying, i.e

$$\pi(a)\mathcal{H}^{-\omega} \subset \mathcal{H}^\omega \qquad (a \in A_-) \, . \tag{5.5.8}$$

Assume that π is H-spherical and denote by v_H the (up to scalar) unique H-fixed distribution vector. Let $v \in \mathcal{H}^{-\omega}$ be a hyperfunction vector. We wish to interpret the generalized matrix coefficient $f(x) = \langle \pi(x^{-1})v, v_H \rangle$ as a generalized function on $X = G/H$. It follows essentially from (5.5.8) that the prescription

$$\tilde{f}(ga \cdot x_0) = \langle \pi(g^{-1})\pi(a^{-1})v, v_H \rangle \qquad \text{for } g \in G \text{ and } a \in A_+ \, . \tag{5.5.9}$$

defines a holomorphic function on $D_+ = GA_+ \cdot x_0$. The minimal tube D_+ has X as edge and this allows us to interpret f as the boundary value of $\tilde{f}$. Henceforth we will identify f with the holomorphic function $\tilde{f}$.

Suppose that $\mathcal{H} \subset L^2(X)$, i.e. $\pi = \pi_\lambda$ with $\lambda \in \Lambda_2$ and $\mathcal{H} = L^2(X)_\lambda$. We now show how the horospherical Cauchy transform restricted to $L^2(X)_\lambda$ can be extended to $L^2(X)_\lambda^{-\omega} \subset \mathcal{O}(D_+)$. In other words for $\xi \in \Xi_+$ we wish to give meaning to

$$\begin{aligned} \widehat{f}(\xi) &= \int_X f(x) a_H(\xi^{-1}x)^\lambda \; dx \\ &= \int_X \langle \pi(x^{-1})v, v_H \rangle a_H(\xi^{-1}x)^\lambda \; dx \end{aligned}$$

as a holomorphic function on Ξ_+. Express ξ as $\xi = ga \cdot \xi_0$ with $g \in G$ and $a \in A_+$. By the usual holomorphic change of variables one obtains that

$$\begin{aligned} \widehat{f}(\xi) &= \int_X f(x) a_H(a^{-1}g^{-1}x)^\lambda \; dx \\ &= \int_X \langle \pi(x^{-1})\pi(g^{-1})\pi(a^{-1})v, v_H \rangle a_H(x)^\lambda \; d(x) \, . \end{aligned}$$

Now the last expression is well defined by (5.5.8). Of course one has

$$\widehat{f}(\xi) = \frac{1}{d_s(\lambda)} \langle \pi(a^{-1})\pi(g^{-1})v, v_\lambda \rangle \tag{5.5.10}$$

by the same argument as in Lemma 5.7. Thus we have shown that the horospherical Cauchy transform on $L^2(X)_\lambda$ extends to a G-equivariant continuous map

$$L^2(X)_\lambda^{-\omega} \to \mathcal{O}(\Xi_+)_\lambda \, .$$

We conclude this section with a conjecture related to the holomorphic intertwining of $\mathcal{O}(D_+)$ and $\mathcal{O}(\Xi_+)$. It can be seen as a holomorphic analogue of Helgason's conjecture (actually a theorem ny [12]).

In order to state the conjecture some new terminology is needed. Let us call a holomorphic function f on Ξ_+ *bounded away from the boundary* if its restriction to $g\mathcal{T}_+ a$ is bounded for all choices of $g \in G$

and $a \in A_+$. We denote by $\mathcal{O}_{\text{b.a.b.}}(\Xi_+)$ the space of all holomorphic function on Ξ_+ which are bounded away from the boundary. Note that $\mathcal{O}_{\text{b.a.b.}}(\Xi_+)$ is a closed G-subspace of the Fréchet space $\mathcal{O}(\Xi_+)$.

Conjecture 1. *Let $\mathbb{D}(X)$ be the algebra of G-invariant differential operators on X. Naturally we can view $\mathbb{D}(X)$ as holomorphic differential operators on $X_{\mathbb{C}}$. Write $\mathcal{O}(D_+)_\lambda$ for the common holomorphic $\mathbb{D}(X)$-eigenfunctions on D_+ with infinitesimal character $\lambda - \rho$. Let $\lambda \in \Lambda_2$. We conjecture*

$$\mathcal{O}_{\text{b.a.b.}}(D_+)_\lambda = L^2(X)^{-\omega}_\lambda \ . \tag{5.5.11}$$

Notice that the inclusion $'' \supset''$ is clear by (5.5.8).

We have already remarked that $\mathcal{O}(\Xi_+)_\lambda \simeq \mathcal{H}_\lambda^{-\omega}$ [15]. *Hence our conjectured equality means that the horospherical Cauchy transform induces an intertwining isomorphism $\mathcal{O}_{\text{b.a.b.}}(D_+)_\lambda \to \mathcal{O}(\Xi_+)_\lambda$.*

It is also interesting problem to formulate (and prove) the conjecture for other parameters.

Remark 5.10. We illustrate Conjecture 1 for the one-sheeted hyperboloid $X = \mathrm{Sl}(2,\mathbb{R})/\mathrm{SO}(1,1)$. Fix $\lambda \in 2\mathbb{Z}_{>0} = \Lambda_2$ and denote by $\mathcal{H}_\lambda$, resp. $\mathcal{H}_{-\lambda}$, the lowest (resp. highest) weight module of $G = \mathrm{Sl}(2,\mathbb{R})$ with lowest (resp. highest) weight λ, resp. $-\lambda$. Denote by $V_{\lambda-2}$ the finite dimensional G-module of highest weight $\lambda - 2$. Write $C^\infty(X)_\lambda$ for the $\mathbb{D}(X)$-eigenspace with eigenvalue $\lambda - 1$. Then

$$C^\infty(X)_\lambda \simeq \mathcal{H}_\lambda^\infty \oplus \mathcal{H}_{-\lambda}^\infty \oplus V_{\lambda-2}$$

Now, the functions of $\mathcal{H}_{\pm\lambda}^\infty$ extend holomorphically to $D_\pm$ but not beyond, while the functions of $V_{\lambda-2}$ extend holomorphically to all of $X_{\mathbb{C}}$. One deduces that $\mathcal{O}(D_+)_\lambda = \mathcal{H}_\lambda^{-\omega} \oplus V_{\lambda-2}$. Finally, the holomorphic functions in $V_{\lambda-2}$ grow exponentially at infinity and hence are not bounded away from the boundary. Thus $\mathcal{O}_{\text{b.a.b.}}(D_+)_\lambda = L^2(X)^{-\omega}_\lambda$ as conjectured.

5.4. **Inversion of the horospherical Cauchy transform.** To begin with we first have to explain certain facts on incidence geometry between the Shilov boundary X of D_+ and the boundary piece G/M of Ξ_+.

We keep in mind that we realized G/M in the boundary of Ξ_+ by $G/M \simeq G \cdot \xi_0 \subset \partial\Xi_+$.

Recall the orbits $S(z) \subset \Xi$ from 2.2.3. For a point $x \in X$ we define the real form of $S(x)$ by

$$S_{\mathbb{R}}(x) = S(x) \cap G/M \ .$$

In view of the incidence relation (2.2.4), one has

$$S_{\mathbb{R}}(x) = \{\xi \in G/M \mid \xi \in S(x)\} = \{\xi \in G/M \mid x \in E(\xi)\}\,.$$

Lemma 5.11. *Let $x = g \cdot x_0 \in X$, $g \in G$. Then*

$$S_{\mathbb{R}}(x) = gH \cdot \xi_0 \simeq H/M \ .$$

Proof. First notice that for $x = g \cdot x_0$ with $g \in G$ one has $S_{\mathbb{R}}(x) = g \cdot S_{\mathbb{R}}(x_0)$. Hence it suffices to show that

$$S_{\mathbb{R}}(x_0) = H \cdot \xi_0 \simeq H/M \ .$$

Let $\xi \in S_{\mathbb{R}}(x_0)$ and write $\xi = y \cdot \xi_0$ for some $y \in G$. We have to show that $y \in H$ and that y is uniquely determined modulo M. First observe that $\xi \in S(x_0)$ means $yN_{\mathbb{C}} \subset H_{\mathbb{C}}N_{\mathbb{C}}$ and so $y \in H_{\mathbb{C}}N_{\mathbb{C}} \cap G$. Now $H_{\mathbb{C}}N_{\mathbb{C}} \cap G = H$ implies $y \in H$. Finally, uniqueness modulo M is immediate from Lemma 2.2. □

It is possible to view the boundary orbits $S_{\mathbb{R}}(x)$ as certain limits. For $z = ga \cdot x_0 \in D_+$ we define

$$S_{\mathbb{R}}(z) = gaH \cdot \xi_0 \simeq H/M\,.$$

We note that $S_{\mathbb{R}}(z) \subset \Xi_+$ by (4.4.10). Furthermore there is the obvious limit relation

$$\lim_{\substack{a \to \mathbf{1} \\ a \in A_+}} S_{\mathbb{R}}(ga \cdot \xi) = S_{\mathbb{R}}(g \cdot x_0)\,.$$

Write $d_z(\xi)$ for the measure on $S_{\mathbb{R}}(z)$ which is induced from a Haar measure $d(hM)$ on H/M via the identification $S_{\mathbb{R}}(z) \simeq H/M$. Define the space of *fiber integrable* holomorphic functions on Ξ_+ by

$$\mathcal{O}_{\text{f.i.}}(\Xi_+) = \{\phi \in \mathcal{O}(\Xi_+) \mid D_+ \ni z \to \int_{S_{\mathbb{R}}(z)} |\phi(\xi)|\, d_z(\xi) \quad \text{is locally bounded}\} \,.$$

For a function $\phi \in \mathcal{O}_{\text{f.i.}}(\Xi_+)$ we define its *inverse horospherical transform* $\phi^\vee \in \mathcal{O}(D_+)$ by

$$\phi^\vee(z) = \int_{S_{\mathbb{R}}(z)} \phi(\xi)\, d_z(\xi) \qquad (z \in D_+)\,.$$

We note that

$$\mathcal{O}_{\text{f.i.}}(\Xi_+) \to \mathcal{O}(D_+), \quad \phi \mapsto \phi^\vee$$

is a G-equivariant continuous map.

Finally, we define a subset $\Lambda_c \subset \Lambda_2$ of large parameters by

$$\Lambda_c = \{\lambda \in \Lambda_2 \mid (\forall \alpha \in \Delta_n^+)\ (\lambda - \rho)(\check{\alpha}) > 2 - m_\alpha\}\,.$$

The inversion formula for the horospherical Cauchy transform is based on the following key result.

Lemma 5.12. *Let $\lambda \in \Lambda_c$. Let $f \in L^2(X)_\lambda^{-\omega} \subset \mathcal{O}(D_+)$. Then $\widehat{f} \in \mathcal{O}_{\text{f.i}}(\Xi_+)$ and*

$$f(z) = d(\lambda) \cdot \int_{S_{\mathbb{R}}(z)} \widehat{f}(\xi)\ d_z(\xi) \qquad (z \in D_+)\,. \tag{5.5.12}$$

In other words, $f = d(\lambda) \cdot (\widehat{f})^\vee$.

Proof. Let $f(ga \cdot \xi_0) = \langle \pi_\lambda(a^{-1})\pi_\lambda(g^{-1})v, v_H\rangle$ for some $v \in \mathcal{H}_\lambda^{-\omega}$. Then by (5.5.10)

$$\widehat{f}(\xi) = \frac{1}{d_s(\lambda)} \langle v, \pi_\lambda(\overline{\xi})v_\lambda\rangle\,.$$

As $\lambda \in \Lambda_c$, [14], Th. 2.16 and Th. 3.6, imply that

$$\int_{H/M} \pi_\lambda(h)v_\lambda\, d(hM) = \frac{d_s(\lambda)}{d(\lambda)} \cdot v_H$$

with the left hand side understood as convergent $\mathcal{H}_\lambda^{-\omega}$-valued integral.

Thus with $z = ga \cdot x_0$ one obtains that

$$\begin{aligned} \int_{S_{\mathbb{R}}(z)} \widehat{f}(\xi)\, d_z(\xi) &= \int_{H/M} \frac{1}{d_s(\lambda)} \langle \pi_\lambda(a^{-1})\pi_\lambda(g^{-1})v, \pi_\lambda(h)v_\lambda\rangle\, d(hM) \\ &= \frac{1}{d(\lambda)} \cdot f(z)\,, \end{aligned}$$

completing the proof of the lemma. □

Remark 5.13. If $\lambda \in \Lambda_2 \backslash \Lambda_c$ and $0 \neq f \in L^2(X)_\lambda^{-\omega}$, then the integral $\int_{S_{\mathbb{R}}(z)} \widehat{f}(\xi)\, d_z(\xi)$ does not converge. However, using the results from [14] it can be shown that the identity (5.5.12) can be analytically continued (cf. Remark 5.9) to all $\lambda \in \Lambda_2$. Henceforth we understand (5.5.12) as an identity valid for all $\lambda \in \Lambda_2$.

The formal dimension $d(\lambda)$ is a polynomial in λ, explicitly given by [7]

$$d(\lambda) = c \cdot \prod_{\alpha \in \Sigma^+} \langle \lambda - \rho(\mathfrak{c}), \alpha\rangle \tag{5.5.13}$$

with $c \in \mathbb{R}$ a constant depending on the normalization of measures.

The right action of T on Ξ_+ induces an identification of $\mathcal{U}(\mathfrak{t}_\mathbb{C})$ with G-invariant differential operators on Ξ_+. As usual we identify $\mathcal{U}(\mathfrak{t}_\mathbb{C})$ with polynomial functions on $\mathfrak{t}_\mathbb{C}$. In this way $d(\lambda)$ corresponds to a G-invariant differential operator $\mathcal{L}$ on Ξ_+ which acts along the fibers of $\Xi_+ \to F_+$ and has constant coefficients in logarithmic coordinates. In particular,

$$\mathcal{L}\phi = d(\lambda) \cdot \phi \qquad (\phi \in \mathcal{O}(\Xi_+)_\lambda) \ .$$

Combining this fact with equation (5.5.12) we obtain the main result of this paper.

Theorem 5.14. *Let* $f \in \sum_{\lambda \in \Lambda_2} L^2(X)_\lambda^{-\omega} \subset \mathcal{O}(D_+)$. *Then*

$$f = (\mathcal{L}\widehat{f})^\vee \ .$$

6. The example of the hyperboloid of one sheet

This section is devoted to the discussion of the case $G = \mathrm{Sl}(2,\mathbb{R})$ and $H = \mathrm{SO}(1,1)$. Notice that $G/H \simeq \mathrm{SO}_e(2,1)/\mathrm{SO}_e(1,1)$. For what follows it is inconsequential to assume that $G = \mathrm{SO}_e(2,1)$ and $H = \mathrm{SO}_e(1,1)$ although the universal complexification of $G = \mathrm{SO}_e(2,1)$ is not simply connected.

The map

$$G/H \to \mathbb{R}^3, \quad gH \mapsto g \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

identifies $X = G/H$ with the one sheeted hyperboloid

$$X = \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = 1\} \ .$$

The base point x_0 becomes $(1,0,0)^T$. Let us define a complex bilinear pairing on $\mathbb{C}^3$ by

$$\langle z, w\rangle = z_1w_1 + z_2w_2 - z_3w_3 \qquad \text{for} \quad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{C}^3 \ .$$

If we set $\Delta(z) = \langle z, z\rangle$ for $z \in \mathbb{C}^3$, then $X = \{x \in \mathbb{R}^3 \mid \Delta(x) = 1\}$. Further one has $G_\mathbb{C} = \mathrm{SO}(2,1;\mathbb{C}) \simeq \mathrm{SO}(3,\mathbb{C})$ and $H_\mathbb{C} = \mathrm{SO}(1,1;\mathbb{C}) \simeq \mathrm{SO}(2,\mathbb{C})$. Clearly

$$X_\mathbb{C} = G_\mathbb{C}/H_\mathbb{C} = \{z \in \mathbb{C}^3 \mid \Delta(z) = 1\} \ .$$

Our choice of T will be

$$T = K = \left\{ \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid \theta \in \mathbb{R} \right\} \ .$$

In particular $\mathfrak{a} = \mathbb{R}U_0$ where

$$U_0 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\Delta = \Delta_n = \{\alpha, -\alpha\}$ with $\alpha(U_0) = 1$. If we demand α to be the positive root, then

$$N_\mathbb{C} = \left\{ \begin{bmatrix} 1 - \frac{z^2}{2} & i\frac{z^2}{2} & iz \\ i\frac{z^2}{2} & 1 + \frac{z^2}{2} & z \\ iz & z & 1 \end{bmatrix} \mid z \in \mathbb{C} \right\} \ .$$

The homogeneous space $G_\mathbb{C}/M_\mathbb{C}N_\mathbb{C}$ naturally identifies with the isotropic vectors $\Xi = \{\zeta \in \mathbb{C}^3 \backslash \{0\} \mid \Delta(\zeta) = 0\}$ via the $G_\mathbb{C}$-equivariant map

$$G_\mathbb{C}/M_\mathbb{C}N_\mathbb{C} \to \Xi, \quad gM_\mathbb{C}N_\mathbb{C} \mapsto g \cdot \zeta_0 \qquad \text{where} \quad \zeta_0 = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}.$$

The correspondence between elements of $\zeta \in \Xi$ and horospheres on $X_{\mathbb{C}}$ is explicitly given by

$$\zeta \leftrightarrow E(\zeta) = \{z \in X_{\mathbb{C}} \mid \langle z, \zeta\rangle = 1\} \ .$$

Elements $\zeta \in \Xi$ can be expressed as $\zeta = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}^3 \backslash \{0\}$ subject to

$$\Delta(\xi) = \Delta(\eta) \qquad \text{and} \qquad \langle \xi, \eta\rangle = 0 \,.$$

A simple computation yields

$$\Xi_+ = \{\zeta = \xi + i\eta \in \Xi \mid \Delta(\xi) = \Delta(\eta) > 1\}$$

and

$$D_+ = \{z = x + iy \in X_{\mathbb{C}} \mid \Delta(x) > 1\} \,.$$

Next we compute the kernel function.

Lemma 6.1. *For all $z \in X_{\mathbb{C}}$ and $\zeta \in \Xi$ one has*

$$a_H(\zeta^{-1} z)^{-\alpha} = \langle z, \zeta\rangle \ .$$

Proof. We first show that

$$a_H(g)^{-\alpha} = \langle g \cdot x_0, \zeta_0\rangle \qquad (g \in G_{\mathbb{C}}) \ . \tag{6.6.1}$$

Observe that both sides are holomorphic functions on $G_{\mathbb{C}}$ which are left $N_{\mathbb{C}}$ and right $H_{\mathbb{C}}$-invariant. Thus it is enough to test with elements $a \in A_{\mathbb{C}}$. Then $a_H(a)^{-\alpha} = a^{-\alpha}$. On the other hand for $a = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $\theta \in \mathbb{C}$ we specifically obtain

$$\langle a \cdot x_0, \zeta_0\rangle = \langle \begin{bmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \rangle = \cos\theta + i \sin\theta = a^{-\alpha} \ .$$

This proves (6.6.1).

It is now easy to prove the asserted statement of the lemma. For that write $\zeta = g \cdot \zeta_0$ and $z = y \cdot x_0$ for $g, y \in G_{\mathbb{C}}$. Then (6.6.1) implies

$$a_H(\zeta^{-1} z)^{-\alpha} = a_H(g^{-1} y)^{-\alpha} = \langle g^{-1} y \cdot x_0, \zeta_0\rangle = \langle y \cdot x_0, g \cdot \zeta_0\rangle = \langle z, \zeta\rangle \,.$$

□

We observe that $\Lambda_{>0} = \Lambda_2 = \mathbb{Z}_{>0} \cdot \alpha$. Hence Lemma 6.1 implies that the horospherical Cauchy kernel is

$$\mathcal{K}(\zeta) = \frac{1}{a_H(\zeta^{-1})^{-\alpha} - 1} = \frac{1}{\langle \zeta, x_0\rangle - 1} \qquad (\zeta \in \Xi_+) \ .$$

The horospherical Cauchy transform for $f \in L^1(X)$ is given by

$$\widehat{f}(\zeta) = \int_X \frac{f(x)}{\langle \zeta, x\rangle - 1} \, dx \qquad (\zeta \in \Xi_+)$$

with dx the invariant measure on the hyperboloid X. Finally, we will discuss inversion. Let the inner product on $\mathfrak{a}$ be normalized such that $\langle \alpha, \alpha\rangle = 1$ and identify $\mathbb{R}$ with $\mathfrak{a}^*$ by means of the bijection $\mathbb{R} \ni \lambda \mapsto \lambda\alpha \in \mathfrak{a}^*$. Then $\Lambda_{>0} = \mathbb{Z}_{>0}$ and $d(\lambda) = \lambda - \frac{1}{2}$. An easy calculation gives

$$\mathcal{L} = \sum_{j=1}^{3} \zeta_j \frac{\partial}{\partial \zeta_j} \ - \ \frac{1}{2} \ .$$

For $f \in \sum_{\lambda>0} L^2(X)_\lambda^{-\omega} \subset \mathcal{O}(D_+)$ the inversion formula reads

$$f(z) = \int_{-\infty}^{\infty} (\mathcal{L}f) \begin{pmatrix} z_1 - i\frac{z_2}{r}\cosh t - i\frac{z_1 z_3}{r}\sinh t \\ z_2 + i\frac{z_1}{r}\cosh t - i\frac{z_2 z_3}{r}\sinh t \\ z_3 - ir\sinh t \end{pmatrix} \, dt,$$

where $r = \sqrt{z_1^2 + z_2^2}$.

References

[1] R. Bott: Homogeneous vector bundles. *Ann. of Math.* **(2) 66** (1957), 203–248.

[2] J.-L. Clerc: Fonctions sphériques des espaces symétriques compacts. *Trans. Amer. Math. Soc.* **306** (1988), 421–431.

[3] Enright, T. J., R. Howe, and N. Wallach: A classification of unitary highest weight modules. In Proc. "Representation theory of reductive groups" (Park City, UT, 1982), pp. 97-149; Progr. Math. **40** (1983), 97–143.

[4] S. Gindikin: Integral Geometry on $\mathrm{SL}(2;\mathbb{R})$. *Math. Res. Lett.* **7**(2000), no. **4**, 417–432.

[5] ———: An analytic separation of series of representations for $\mathrm{SL}(2;\mathbb{R})$. *Mosc. Math. J.* **2** (2002), no. **4**, 635–645, 804–805.

[6] ———: Horospherical Cauchy-Radon Transform on Compact Symmetric Spaces. Preprint 2004.

[7] Harish-Chandra: Representations of semisimple Lie groups. VI. Integrable and square-integrable representations. *Amer. J. Math.* **78** (1956), 564–62.

[8] S. Helgason: Groups and Geometric Analysis. Academic Press, 1984.

[9] ———: Geometric Analysis on Symmetric Spaces, Math. Surveys and Monographs **39**, Amer. Math. Soc., 1994.

[10] J. Hilgert, G. Ólafsson: Causal Symmetric Spaces, Geometry and Harmonic Analsysis. Acad. Press. 1996.

[11] J. Hilgert, G. Ólafsson, and B. Ørsted: Hardy spaces on affine symmetric spaces. *J. Reine Angew. Math.* **415** (1991), 189–218.

[12] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, and M. Tanaka: Eigenfunctions of invariant differential operators on a symmetric space. *Ann. of Math.* **(2) 107** (1978), no. **1**, 1–39.

[13] B. Krötz: Norm estimates for unitarizable highest weight modules. *Ann. Inst. Fourier* (Grenoble) **49** (1999), no. **4**, 1241–1264.

[14] B. Krötz: Formal Dimension For Semisimple Symmetric Spaces. *Compositio Math.* **125** (2001), 155-191.

[15] B. Krötz, K.-H. Neeb, and G. Ólafsson: Spherical representations and mixed symmetric spaces. *Represent. Theory* **1** (1997), 424–461.

[16] B. Krötz, and G. Ólafsson: The c-function for non-compactly causal symmetric spaces and its relations to harmonic analysis and representation theory. Lie groups and symmetric spaces, 171–194, Amer. Math. Soc. Transl. Ser. 2, **210**, Amer. Math. Soc., 2003.

[17] K.-H. Neeb: A convexity theorem for semisimple symmetric spaces. *Pacific J. Math.* **162** (1994), no. **2**, 305–349.

[18] ———: On the complex geometry of invariant domains in complexified symmetric spaces. *Ann. Inst. Fourier* (Grenoble) **49** (1999), no. **1**, 177–225.

[19] ———: Holomorphy and convexity in Lie theory. de Gruyter Expositions in Mathematics, **28**. Walter de Gruyter & Co., Berlin.

[20] G. Ólafsson and B. Ørsted: The holomorphic discrete series for affine symmetric spaces. *J. Funct. Anal.* **81** (1988), 126–159.

[21] ———: The holomorphic discrete series of an affine symmetric space and representations with reproducing kernels. *Trans. Amer. Math. Soc.* **326** (1991), 385–405.

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA
E-mail address: gindikin@math.rutgers.edu

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
E-mail address: kroetz@kurims.kyoto-u.ac.jp

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
E-mail address: olafsson@math.lsu.edu