ABSOLUTE ANABELIAN CUSPIDALIZATIONS OF PROPER HYPERBOLIC CURVES

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Abstract. In this paper, we develop the theory of *"cuspidalizations"* of the étale fundamental group of a proper hyperbolic curve over a finite or nonarchimedean local field. The ultimate goal of this theory is the *group-theoretic reconstruction* of the étale fundamental group of an arbitrary open subscheme of the curve from the étale fundamental group of the full proper curve. We then apply this theory to show that a certain *absolute p-adic version of the Grothendieck Conjecture* holds for *hyperbolic curves "of Belyi type"*. This includes, in particular, affine hyperbolic curves over a nonarchimedean local field which are defined over a number field and isogenous to a hyperbolic curve of genus zero. Also, we apply this theory to prove the *analogue for proper hyperbolic curves over finite fields* of the version of the Grothendieck Conjecture that was shown in [Tama].

Introduction

Let X be a *proper hyperbolic curve* over a field k which is either *finite* or *nonarchimedean local*; let $U \subseteq X$ be an *open subscheme* of X. Write Π_X for the *étale fundamental group* of X. In this paper, we study the extent to which the étale fundamental group of U may be *group-theoretically reconstructed* from Π_X.

In §1, we show that the *abelian portion* of the extension of Π_X determined by the étale fundamental group of U may be *group-theoretically reconstructed* from Π_X [cf. Theorem 1.16, (iii)], and, moreover, that this construction has certain *remarkable rigidity* properties [cf. Propositions 1.15, (i); 2.6, (i)].

In §2, we show that this abelian portion of the extension is sufficient to reconstruct [in essence] the multiplicative group of the *function field* of X [cf. Theorem 2.5, (ii)]. In the case of *nonarchimedean local fields*, this already implies various interesting consequences in the context of the *absolute anabelian geometry* studied in [Mzk5], [Mzk6], [Mzk8]. In particular, it implies that the *absolute p-adic version of the Grothendieck Conjecture* [i.e., an absolute version of [a certain portion of] the *relative* result that appears as the main result of [Mzk4]] holds for *hyperbolic curves "of Belyi type"* [cf. Definition 2.9; Corollary 2.12]. This includes, in particular, hyperbolic curves "of strictly Belyi type", i.e., affine hyperbolic curves over a nonarchimedean local field which are defined over a number field and isogenous to a hyperbolic curve of genus zero. In particular, we obtain a new countable class of *"absolute curves"* [in the terminology of [Mzk6]], whose absoluteness is, in certain respects, reminiscent of the absoluteness of the *canonical curves of p-adic Teichmüller theory* discussed in [Mzk6] [cf. Remark 2.13.1], but [in contrast to the class of canonical curves] appears [at least from the point of view of certain circumstantial evidence] unlikely to be Zariski dense in most moduli spaces [cf. Remark 2.13.2].

Finally, in §3, we apply the theory of the *weight filtration* [cf., e.g., [Kane], [Mtm]] to develop various *"higher order generalizations"* of the theory of §1, 2. In particular, we obtain various "higher order generalizations" of the *"remarkable rigidity"* referred to above [cf. Corollaries 3.8, 3.9, especially Corollary 3.9, (iii)], which we apply to show that, relative to the notation introduced above, the *geometrically pro-l portion* [where l is a prime number invertible in k] of the étale fundamental group of U may be *recovered from* Π_X, at least when U is obtained from X by removing a *single k-rational point* [cf. Theorem 3.10]. This, along with the theory of §2, allows one to verify the *analogue for proper hyperbolic curves over finite fields* of the version of the Grothendieck Conjecture that was shown in [Tama] [cf. Theorem 3.12].

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Section 0: Notations and Conventions

Numbers:

We shall denote by $\widehat{\mathbb{Z}}$ the *profinite completion* of the additive group of rational integers $\mathbb{Z}$. If p is a prime number, then $\mathbb{Z}_p$ denotes the ring of *p-adic integers*; $\mathbb{Q}_p$ denotes its quotient field. We shall refer to as a *p-adic local field* (respectively, *nonarchimedean local field*) any finite field extension of $\mathbb{Q}_p$ (respectively, a p-adic local field, for some p). A *number field* is defined to be a finite extension of the field of rational numbers. If Σ is a *set of prime numbers*, then we shall refer to a positive integer each of whose prime factors belongs to Σ as a *Σ-integer*. We shall refer to a *finite étale covering* of schemes whose degree is a Σ-integer as a *Σ-covering*. Also, we shall write $\mathfrak{Primes}$ for the *set of all prime numbers*.

Topological Groups:

Let G be a *Hausdorff topological group*, and $H \subseteq G$ a *closed subgroup*. Let us write

$$G^{\mathrm{ab}}$$

for the *abelianization* of G [i.e., the quotient of G by the topological subgroup of G generated by the commutators of G]. Let us write

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot h = h \cdot g, \ \forall\ h \in H\}$$

for the *centralizer* of H in G;

$$N_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$

for the *normalizer* of H in G; and

$$C_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid (g \cdot H \cdot g^{-1}) \bigcap H \text{ has finite index in } H,\ g \cdot H \cdot g^{-1}\}$$

for the *commensurator* of H in G. Note that: (i) $Z_G(H)$, $N_G(H)$ and $C_G(H)$ are *subgroups of* G; (ii) we have *inclusions*

$$H,\ Z_G(H) \subseteq N_G(H) \subseteq C_G(H)$$

and (iii) H is *normal* in $N_G(H)$. If $H = N_G(H)$ (respectively, $H = C_G(H)$), then we shall say that H is *normally terminal* (respectively, *commensurably terminal*) in G. Note that $Z_G(H)$, $N_G(H)$ are *always closed in* G, while $C_G(H)$ is *not necessarily closed in* G.

If G_1, G_2 are *Hausdorff topological groups*, then an *outer homomorphism* $G_1 \to G_2$ is defined to be an equivalence class of continuous homomorphisms $G_1 \to G_2$, where two such homomorphisms are considered equivalent if they differ by composition with an inner automorphism of G_2. The group of *outer automorphisms* of G [i.e., bijective bicontinuous outer homomorphisms $G \to G$] will be denoted $\mathrm{Out}(G)$. If G is *center-free*, then there is a *natural exact sequence*:

$$1 \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

[where the homomorphism $G \to \mathrm{Aut}(G)$ is defined by letting G act on G by *conjugation*].

If G is a *profinite group* such that, for every open subgroup $H \subseteq G$, we have $Z_G(H) = \{1\}$, then we shall say that G is *slim*. One verifies immediately that G is slim if and only if every open subgroup of G is center-free [cf. [Mzk5], Remark 0.1.3].

If G is a *profinite group* and Σ is *set of prime numbers*, then we shall say that G is a *pro-Σ group* if the order of every finite quotient group of G is a *Σ-integer*. If $\Sigma = \{l\}$ is of *cardinality one*, then we shall refer to a pro-Σ group as a *pro-l group*.

Curves:

Suppose that $g \geq 0$ is an *integer*. Then if S is a scheme, a *family of curves of genus g*

$$X \to S$$

is defined to be a smooth, proper, geometrically connected morphism of schemes $X \to S$ whose geometric fibers are curves of genus g.

Suppose that $g, r \geq 0$ are *integers* such that $2g - 2 + r > 0$. We shall denote the *moduli stack of r-pointed stable curves of genus g* (where we assume the points to be *unordered*) by $\overline{\mathcal{M}}_{g,r}$ [cf. [DM], [Knud] for an exposition of the theory of such curves; strictly speaking, [Knud] treats the finite étale covering of $\overline{\mathcal{M}}_{g,r}$ determined by *ordering* the marked points]. The open substack $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ of smooth curves will be referred to as the *moduli stack of smooth r-pointed stable curves of genus g* or, alternatively, as the *moduli stack of hyperbolic curves of type (g, r)*.

A *family of hyperbolic curves of type (g, r)*

$$X \to S$$

is defined to be a morphism which factors $X \hookrightarrow Y \to S$ as the composite of an open immersion $X \hookrightarrow Y$ onto the complement $Y \backslash D$ of a relative divisor $D \subseteq Y$ which is finite étale over S of relative degree r, and a family $Y \to S$ of curves of genus g. One checks easily that, if S is *normal*, then the pair (Y, D) is *unique up to canonical isomorphism*. (Indeed, when S is the spectrum of a field, this fact is well-known from the elementary theory of algebraic curves. Next, we consider an arbitrary *connected normal* S on which a prime l is *invertible* (which, by Zariski localization, we may assume without loss of generality). Denote by $S' \to S$ the finite étale covering parametrizing *orderings of the marked points* and *trivializations of the l-torsion points of the Jacobian of Y*. Note that $S' \to S$ is *independent* of the choice of (Y, D), since (by the normality of S), S' may be constructed as the *normalization* of S in the function field of S' (which is independent of the choice of (Y, D) since the restriction of (Y, D) to the generic point of S has already been shown to be unique). Thus, the uniqueness of (Y, D) follows by considering the classifying morphism (associated to (Y, D)) from S' to the finite étale covering of $(\mathcal{M}_{g,r})_{\mathbb{Z}[\frac{1}{l}]}$ parametrizing orderings of the marked points and trivializations of the l-torsion points of the Jacobian [since this covering is well-known to be a scheme, for l sufficiently large].) We shall refer to Y (respectively, D; D) as the *compactification* (respectively, *divisor of cusps*; *divisor of marked points*) of X. A *family of hyperbolic curves* $X \to S$ is defined to be a morphism $X \to S$ such that the restriction of this morphism to each connected component of S is a *family of hyperbolic curves of type* (g, r) for some integers (g, r) as above. A family of hyperbolic curves $X \to S$ of type $(0, 3)$ will be referred to as a *tripod*.

If X is a *hyperbolic curve* over a field K with *compactification* $X \subseteq \overline{X}$, then we shall write

$$X^{\mathrm{cl}}; \quad X^{\mathrm{cl}+}$$

for the *sets of closed points* of X and $\overline{X}$, respectively.

If X_K (respectively, Y_L) is a *hyperbolic curve over a field* K (respectively, L), then we shall say that X_K is *isogenous* to Y_L if there exists a hyperbolic curve Z_M over a field M together with *finite étale morphisms* $Z_M \to X_K$, $Z_M \to Y_L$.

If X is a *generically scheme-like algebraic stack* [i.e., an algebraic stack which admits a "scheme-theoretically" dense open that is isomorphic to a scheme] over a field K of *characteristic zero* that admits a [surjective] *finite étale* [or, equivalently, *finite étale Galois*] *covering* $Y \to X$, where Y is a hyperbolic curve over a finite extension of K, then we shall refer to X as a *hyperbolic orbicurve* over K. [Although this definition differs from the definition of a "hyperbolic orbicurve" given in [Mzk6], Definition 2.2, (ii), it follows immediately from a theorem of Bundgaard-Nielsen-Fox [cf., e.g., [Namba], Theorem 1.2.15, p. 29] that these two definitions are equivalent.] If $X \to Y$ is a dominant morphism of hyperbolic orbicurves, then we shall refer to $X \to Y$ as a *partial coarsification morphism* if the morphism induced by $X \to Y$ on *associated coarse spaces* [cf., e.g., [FC], Chapter I, §4.10] is an *isomorphism*.

Let X be a *hyperbolic orbicurve* over an algebraically closed field of characteristic zero; denote its *étale fundamental group* by Δ_X. We shall refer to the order of the [manifestly finite!] decomposition group of a closed point x of X as the *order of* x. We shall refer to the [manifestly finite!] *least common multiple* of the orders of the closed points of X as the *order of* X. Thus, it follows immediately from the definitions that X is a *hyperbolic curve* if and only if the order of X is equal to 1.

Section 1: Cuspidalizations

Let X be a *proper hyperbolic curve* over a *field* k which is either *finite* or *nonarchimedean local*. Write

$$\underline{d}_k$$

for the *cohomological dimension* of k. Thus, if k is finite (respectively, nonarchimedean local), then $\underline{d}_k = 1$ (respectively, $\underline{d}_k = 2$ [cf., e.g., [NSW], Chapter 7, Theorem 7.1.8, (i)]). If k is finite (respectively, nonarchimedean local), we shall denote the characteristic of k (respectively, of the residue field of k) by p and the number p (respectively, 1) by $p^\dagger$. Also, we shall write

$$\mathfrak{Primes}^\dagger \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \backslash (\mathfrak{Primes} \bigcap \{p^\dagger\})$$

[where $\mathfrak{Primes}$ is the set of all prime numbers [cf. §0]; the intersection is taken in the "ambient set" $\mathbb{Z}$].

Let Σ be a *set of prime numbers* that contains at least one prime number that is *invertible* in k. Write

$$\Sigma' \stackrel{\mathrm{def}}{=} \Sigma \backslash (\Sigma \bigcap \{p\}); \quad \Sigma^\dagger \stackrel{\mathrm{def}}{=} \Sigma \backslash (\Sigma \bigcap \{p^\dagger\})$$

[where the intersections are taken in the "ambient set" $\mathbb{Z}$]. Denote by $\widehat{\mathbb{Z}}'$ the *maximal pro-Σ' quotient* of $\widehat{\mathbb{Z}}$ and by $\widehat{\mathbb{Z}}^\dagger$ the *maximal pro-$\Sigma^\dagger$ quotient* of $\widehat{\mathbb{Z}}$.

If $\overline{k}$ is an *algebraic closure* of k, then we shall denote the result of base-changing objects over k to $\overline{k}$ by means of a subscript "$\overline{k}$". Any choice of a basepoint of X determines an algebraic closure $\overline{k}$ of k, and hence an *exact sequence*

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to G_k \to 1$$

where $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; $\pi_1(X)$, $\pi_1(X_{\overline{k}})$ are the *étale fundamental groups* of X, $X_{\overline{k}}$, respectively. Write Δ_X for the *maximal pro-Σ quotient* of $\pi_1(X_{\overline{k}})$ and $\Pi_X \stackrel{\mathrm{def}}{=} \pi_1(X)/\mathrm{Ker}(\pi_1(X_{\overline{k}}) \twoheadrightarrow \Delta_X)$. Thus, we have an *exact sequence*:

$$1 \to \Delta_X \to \Pi_X \to G_k \to 1$$

Similarly, if we write $X \times X \stackrel{\mathrm{def}}{=} X \times_k X$, then we obtain [by considering the *maximal pro-Σ quotient* of $\pi_1((X \times X)_{\overline{k}})$] an *exact sequence*

$$1 \to \Delta_{X \times X} \to \Pi_{X \times X} \to G_k \to 1$$

where $\Pi_{X \times X}$ (respectively, $\Delta_{X \times X}$) may be identified with $\Pi_X \times_{G_k} \Pi_X$ (respectively, $\Delta_X \times \Delta_X$). Let $\Pi_Z \subseteq \Pi_{X \times X}$ be an open subgroup that surjects onto G_k. Write $Z \to X \times X$ for the corresponding covering; $\Delta_Z \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_Z \twoheadrightarrow G_k)$.

Proposition 1.1. (Group-theoreticity of Étale Cohomology) *Let* $\widehat{\mathbb{Z}}^\dagger \twoheadrightarrow A$ *be a* **finite quotient**, *and* N *a finite* A*-module equipped with a continuous* Δ_X*- (respectively,* Π_X*-;* Δ_Z*-;* Π_Z*-) action. Then for* $i \in \mathbb{Z}$*, the natural homomorphism*

$$H^i(\Delta_X, N) \to H^i_{\text{ét}}(X_{\overline{k}}, N) \quad \textit{(respectively, } H^i(\Pi_X, N) \to H^i_{\text{ét}}(X, N);$$
$$H^i(\Delta_Z, N) \to H^i_{\text{ét}}(Z_{\overline{k}}, N); \quad H^i(\Pi_Z, N) \to H^i_{\text{ét}}(Z, N))$$

is an **isomorphism**.

Proof. In light of the Leray spectral sequence for the surjections $\Pi_X \twoheadrightarrow G_k$, $\Pi_Z \twoheadrightarrow \mathrm{Im}(\Pi_Z) \subseteq \Pi_X$ [i.e., where "$\mathrm{Im}(-)$" denotes the image via the natural homomorphism associated to one of the projections $Z \to X \times X \to X$], it suffices to verify the asserted isomorphism in the case of Δ_X. If $Y \to X_{\overline{k}}$ is a *connected finite étale Galois* Σ*-covering*, then the associated Leray spectral sequence has "E_2-term" given by the cohomology of $\mathrm{Gal}(Y/X)$ with coefficients in the étale cohomology of Y and abuts to the étale cohomology of $X_{\overline{k}}$. By allowing Y to vary, one then verifies immediately that it suffices to verify that every étale cohomology class of Y [with coefficients in N] vanishes upon pull-back to some [connected] finite étale Σ-covering $Y' \to Y$. Moreover, by passing to an appropriate Y, one reduces immediately to the case where $N = A$, equipped with the trivial Π_X-action. Then the vanishing assertion in question is a tautology for "H^1"; for "H^2", it suffices to take $Y' \to Y$ so that the degree of $Y' \to Y$ *annihilates* A [cf., e.g., the discussion at the bottom of [FK], p. 136]. ○

Set:

$$M_X \stackrel{\text{def}}{=} \mathrm{Hom}_{\widehat{\mathbb{Z}}^\dagger}(H^2(\Delta_X, \widehat{\mathbb{Z}}^\dagger), \widehat{\mathbb{Z}}^\dagger); \quad M_k \stackrel{\text{def}}{=} \mathrm{Hom}_{\widehat{\mathbb{Z}}^\dagger}(H^{\underline{d}_k}(G_k, M_X^{\otimes \underline{d}_k - 1}), M_X^{\otimes \underline{d}_k - 1})$$

Thus, M_k, M_X are *free* $\widehat{\mathbb{Z}}^\dagger$*-modules of rank one*; M_X is isomorphic as a G_k-module to $\widehat{\mathbb{Z}}^\dagger(1)$ [where the "(1)" denotes a "Tate twist" — i.e., G_k acts on $\widehat{\mathbb{Z}}^\dagger(1)$ via the cyclotomic character]; M_k is isomorphic as a G_k-module to $\widehat{\mathbb{Z}}^\dagger(\underline{d}_k - 1)$. [Indeed, this follows from Proposition 1.1; *Poincaré duality* [cf., e.g., [FK], Chapter II, Theorem 1.13]; the fact, in the finite field case, that $G_k \cong \widehat{\mathbb{Z}}$ [together with an easy computation of the group cohomology of $\widehat{\mathbb{Z}}$]; the well-known theory of the cohomology of nonarchimedean local fields [cf., e.g., [NSW], Chapter 7, Theorem 7.2.6].]

Remark 1.2.0. Note that for any open subgroup $\Pi_{X'} \subseteq \Pi_X$ [which we think of as corresponding to a finite étale covering $X' \to X$], we obtain a *natural isomorphism*

$$M_X \xrightarrow{\sim} M_{X'}$$

by applying the functor $\mathrm{Hom}_{\widehat{\mathbb{Z}}^\dagger}(-, \widehat{\mathbb{Z}}^\dagger)$ to the induced morphism on group cohomology $H^2(\Delta_X, \widehat{\mathbb{Z}}^\dagger) \to H^2(\Delta_{X'}, \widehat{\mathbb{Z}}^\dagger)$ [where $\Delta_{X'} \stackrel{\text{def}}{=} \mathrm{Ker}(\Pi_{X'} \to G_k)$] and *dividing* by $[\Delta_X : \Delta_{X'}]$. [One verifies easily that this does indeed yield an isomorphism — cf., e.g., the discussion at the bottom of [FK], p. 136.] Moreover, relative to the tautological isomorphisms $H^2(\Delta_X, M_X) \cong \widehat{\mathbb{Z}}^\dagger$, $H^2(\Delta_{X'}, M_{X'}) \cong \widehat{\mathbb{Z}}^\dagger$, the isomorphism $M_X \xrightarrow{\sim} M_{X'}$ just constructed induces [via the restriction morphism on group cohomology] the morphism $\widehat{\mathbb{Z}}^\dagger \to \widehat{\mathbb{Z}}^\dagger$ given by multiplication by $[\Delta_X : \Delta_{X'}]$. Similarly, if k' is the base field of X', then we obtain a *natural isomorphism*

$$M_k \xrightarrow{\sim} M_{k'}$$

by applying the natural isomorphism $M_X \xrightarrow{\sim} M_{X'}$ just constructed and the dual of the natural pull-back morphism on group cohomology and then *dividing* by $[k' : k]$ [cf., e.g., [NSW], Chapter 7, Corollary 7.1.4].

Proposition 1.2. **(Top Cohomology Modules)**

(i) There are natural isomorphisms:

$$H^{\underline{d}_k}(G_k, M_k) \cong \widehat{\mathbb{Z}}^\dagger; \quad H^2(\Delta_X, M_X) \cong \widehat{\mathbb{Z}}^\dagger; \quad H^{\underline{d}_k+2}(\Pi_X, M_X \otimes M_k) \cong \widehat{\mathbb{Z}}^\dagger$$
$$H^4(\Delta_Z, M_X^{\otimes 2}) \cong \widehat{\mathbb{Z}}^\dagger; \quad H^{\underline{d}_k+4}(\Pi_Z, M_X^{\otimes 2} \otimes M_k) \cong \widehat{\mathbb{Z}}^\dagger$$

(ii) There is a **unique** *isomorphism $M_X \xrightarrow{\sim} \widehat{\mathbb{Z}}^\dagger(1)$ such that the image of $1 \in \widehat{\mathbb{Z}}^\dagger$ maps via the composite of the isomorphism $\widehat{\mathbb{Z}}^\dagger \cong H^2(\Delta_X, M_X)$ of (i) with the isomorphism $H^2(\Delta_X, M_X) \xrightarrow{\sim} H^2(\Delta_X, \widehat{\mathbb{Z}}^\dagger(1))$ induced by the isomorphism $M_X \xrightarrow{\sim} \widehat{\mathbb{Z}}^\dagger(1)$ in question to the [first] Chern class of a line bundle of degree 1 on $X_{\overline{k}}$.*

Proof. Assertion (i) follows from the definitions; the Leray spectral sequence for the surjections $\Pi_X \twoheadrightarrow G_k$, $\Pi_Z \twoheadrightarrow \mathrm{Im}(\Pi_Z) \subseteq \Pi_X$ [i.e., where "Im(−)" denotes the image via the natural homomorphism associated to one of the projections $Z \to X \times X \to X$]. Assertion (ii) is immediate from the definitions. ○

Proposition 1.3. **(Duality)** *For $i \in \mathbb{Z}$, let $\widehat{\mathbb{Z}}^\dagger \twoheadrightarrow A$ be a* **finite quotient***, and N a finite A-module.*

(i) Suppose that N is equipped with a continuous G_k-action. Then the pairing

$$H^i(G_k, N) \times H^{\underline{d}_k - i}(G_k, \mathrm{Hom}_A(N, M_k \otimes A)) \to A$$

determined by the cup product in group cohomology and the natural isomorphisms of Proposition 1.2, (i), determines an isomorphism as follows:

$$H^i(G_k, N) \xrightarrow{\sim} \mathrm{Hom}_A(H^{\underline{d}_k - i}(G_k, \mathrm{Hom}_A(N, M_k \otimes A)), A)$$

(ii) Suppose that N is equipped with a continuous Π_X- (respectively, Δ_X-; Π_Z-; Δ_Z-) action. Then the pairing

$$H^i(\Pi_X, N) \times H^{\underline{d}_k+2-i}(\Pi_X, \mathrm{Hom}_A(N, M_X \otimes M_k \otimes A)) \to A$$

$$\textit{(respectively, } H^i(\Delta_X, N) \times H^{2-i}(\Delta_X, \mathrm{Hom}_A(N, M_X \otimes A)) \to A;$$

$$H^i(\Pi_Z, N) \times H^{\underline{d}_k+4-i}(\Pi_Z, \mathrm{Hom}_A(N, M_X^{\otimes 2} \otimes M_k \otimes A)) \to A;$$

$$H^i(\Delta_Z, N) \times H^{4-i}(\Delta_Z, \mathrm{Hom}_A(N, M_X^{\otimes 2} \otimes A)) \to A)$$

determined by the cup product in group cohomology and the natural isomorphisms of Proposition 1.2, (i), determines an isomorphism as follows:

$$H^i(\Pi_X, N) \xrightarrow{\sim} \mathrm{Hom}_A(H^{\underline{d}_k+2-i}(\Pi_X, \mathrm{Hom}_A(N, M_X \otimes M_k \otimes A)), A)$$

$$\textit{(respectively, } H^i(\Delta_X, N) \xrightarrow{\sim} \mathrm{Hom}_A(H^{2-i}(\Delta_X, \mathrm{Hom}_A(N, M_X \otimes A)), A);$$

$$H^i(\Pi_Z, N) \xrightarrow{\sim} \mathrm{Hom}_A(H^{\underline{d}_k+4-i}(\Pi_Z, \mathrm{Hom}_A(N, M_X^{\otimes 2} \otimes M_k \otimes A)), A);$$

$$H^i(\Delta_Z, N) \xrightarrow{\sim} \mathrm{Hom}_A(H^{4-i}(\Delta_Z, \mathrm{Hom}_A(N, M_X^{\otimes 2} \otimes A)), A))$$

Proof. Assertion (i) follows immediately from the fact that $G_k \cong \widehat{\mathbb{Z}}$ [together with an easy computation of the group cohomology of $\widehat{\mathbb{Z}}$] in the finite field case; [NSW], Chapter 7, Theorem 7.2.6, in the nonarchimedean local field case. Assertion (ii) then follows from assertion (i); the Leray spectral sequences associated to $\Pi_X \twoheadrightarrow G_k$, $\Pi_Z \twoheadrightarrow \mathrm{Im}(\Pi_Z) \subseteq \Pi_X$ [i.e., where "$\mathrm{Im}(-)$" denotes the image via the natural homomorphism associated to one of the projections $Z \to X \times X \to X$]; Proposition 1.1; *Poincaré duality* [cf., e.g., [FK], Chapter II, Theorem 1.13]. ◯

Proposition 1.4. (Automorphisms of Cyclotomic Extensions)

(i) We have: $H^0(G_k, H^1(\Delta_X, M_X)) = 0$.

(ii) There are natural isomorphisms

$$H^1(\Pi_X, M_X) \xrightarrow{\sim} H^1(G_k, M_X) \xrightarrow{\sim} (k^\times)^\wedge$$

$$H^1(\Pi_Z, M_X) \xrightarrow{\sim} H^1(G_k, M_X) \xrightarrow{\sim} (k^\times)^\wedge$$

— where the first isomorphisms in each line are induced by the surjections $\Pi_X \twoheadrightarrow G_k$, $\Pi_Z \twoheadrightarrow G_k$*; the second isomorphisms in each line are induced by the isomorphism of Proposition 1.2, (ii), and the Kummer exact sequence;* $(k^\times)^\wedge$ *is the* **maximal pro-$\Sigma^\dagger$-quotient** *of* $k^\times$*.*

Proof. Assertion (i) follows immediately from the "Riemann hypothesis for abelian varieties over finite fields" [cf., e.g., [Mumf], p. 206] in the finite field case; [Mzk8], Lemma 4.6, in the nonarchimedean local field case. The first isomorphisms of assertion (ii) follow immediately from assertion (i) and the Leray spectral sequences associated to $\Pi_X \twoheadrightarrow G_k$, $\Pi_Z \twoheadrightarrow G_k$; the second isomorphisms follow immediately from consideration of the Kummer exact sequence for $\mathrm{Spec}(k)$. $\bigcirc$

Definition 1.5.

(i) Let H be a profinite group equipped with a homomorphism $H \to \Pi_X$. Then we shall refer to the kernel I_H of $H \to \Pi_X$ as the *cuspidal subgroup* of H [relative to $H \to \Pi_X$]. We shall say that H is *cuspidally abelian* (respectively, *cuspidally pro-*Σ^* [where Σ^* is a set of prime numbers]) [relative to $H \twoheadrightarrow \Pi_X$] if I_H is abelian (respectively, a pro-Σ^* group). If H is cuspidally abelian, then observe that H/I_H acts naturally [by conjugation] on I_H; we shall say that H is *cuspidally central* [relative to $H \to \Pi_X$] if this action of H/I_H on I_H is trivial. Also, we shall use similar terminology to the terminology just introduced for $H \twoheadrightarrow \Pi_X$ when Π_X is replaced by Δ_X, $\Pi_{X\times X}$, $\Delta_{X\times X}$.

(ii) Let H be a profinite group; $H_1 \subseteq H$ a closed subgroup. Then we shall refer to as an H_1*-inner automorphism* of H an inner automorphism induced by conjugation by an element of H_1. If H' is also a profinite group, then we shall refer to as an H_1*-outer homomorphism* $H' \to H$ an equivalence class of homomorphisms $H' \to H$, where two such homomorphisms are considered equivalent if they differ by composition by an H_1-inner automorphism. If H is equipped with a homomorphism $H \to G_k$ [cf., e.g., the various groups introduced above], and $H_1 \stackrel{\mathrm{def}}{=} \mathrm{Ker}(H \to G_k)$, then we shall refer to an H_1-inner automorphism (respectively, H_1-outer homomorphism) as a *geometrically inner automorphism* (respectively, *geometrically outer homomorphism*). If H is equipped with a structure of extension of some other profinite group H_0 by a finite product H_1 of copies of M_X, or, more generally, a projective limit H_1 of such finite products, then we shall refer to an H_1-inner automorphism (respectively, H_1-outer homomorphism) as a *cyclotomically inner automorphism* (respectively, *cyclotomically outer homomorphism*). If H is equipped with a homomorphism to Π_X, Δ_X, $\Pi_{X\times X}$, or $\Delta_{X\times X}$ [cf. the situation of (i)], and H_1 is the kernel of this homomorphism, then we shall refer to an H_1-inner automorphism (respectively, H_1-outer homomorphism) as a *cuspidally inner automorphism* (respectively, *cuspidally outer homomorphism*).

Next, let

$$\Pi_{X'} \subseteq \Pi_X$$

be an *open normal subgroup*, corresponding to a finite étale Galois covering $X' \to X$. Set

$$\Pi_{Z'} \stackrel{\mathrm{def}}{=} \Pi_{X'\times X'} \cdot \Pi_X \subseteq \Pi_{X\times X}$$

[where we regard Π_X as a subgroup of $\Pi_{X\times X}$ via the diagonal map]; write $Z' \to X \times X$ for the covering determined by $\Pi_{Z'}$. Thus, it is a tautology that the diagonal morphism $\iota : X \hookrightarrow X \times X$ lifts to a morphism

$$\iota' : X \hookrightarrow Z'$$

which induces the inclusion $\Pi_X \hookrightarrow \Pi_{Z'}$ on fundamental groups. If $Z \to X \times X$ is a connected finite étale covering arising from an open subgroup of $\Pi_{X\times X}$, write:

$$U_{X\times X} \stackrel{\mathrm{def}}{=} (X \times X)\backslash\iota(X); \quad U_Z \stackrel{\mathrm{def}}{=} (U_{X\times X}) \times_{(X\times X)} Z$$

Denote by $\Delta_{U_{X\times X}}$ the *maximal cuspidally* [i.e., relative to the natural map to $\pi_1((X \times X)_{\overline{k}})$] *pro-*$\Sigma^\dagger$ *quotient* of the maximal pro-Σ quotient of the tame fundamental group of $(U_{X\times X})_{\overline{k}}$ [where "tame" is with respect to the divisor $\iota(X) \subseteq X \times X$] and by $\Pi_{U_{X\times X}}$ the quotient $\pi_1(U_{X\times X})/\mathrm{Ker}(\pi_1((U_{X\times X})_{\overline{k}}) \twoheadrightarrow \Delta_{U_{X\times X}})$; write $\Pi_{U_Z} \subseteq \Pi_{U_{X\times X}}$ for the open subgroup corresponding to the finite étale covering $U_Z \to U_{X\times X}$.

Proposition 1.6. (Characteristic Class of the Diagonal)

(i) The pull-back morphism arising from the natural inclusion

$$\Pi_X \hookrightarrow \Pi_{Z'} \;\; (\subseteq \Pi_{X\times X} = \Pi_X \times_{G_k} \Pi_X)$$

composed with the natural isomorphism of Proposition 1.2, (i), determines a homomorphism

$$H^{\underline{d}_k+2}(\Pi_{Z'}, M_X \otimes M_k) \to H^{\underline{d}_k+2}(\Pi_X, M_X \otimes M_k) \xrightarrow{\sim} \widehat{\mathbb{Z}}^\dagger$$

hence [by Proposition 1.3, (ii)] a class

$$\eta_{Z'}^{\mathrm{diag}} \in H^2(\Pi_{Z'}, M_X)$$

which is equal to the étale cohomology class associated to $\iota'(X) \subseteq Z'$, or, alternatively, the [first] Chern class of the line bundle $\mathcal{O}_{Z'}(\iota'(X))$.

(ii) Denote by

$$\mathbb{L}^\times_{\mathrm{diag}}[Z'] \to Z'$$

the complement of the zero section in the geometric line bundle [i.e., $\mathbb{G}_\mathrm{m}$-torsor] determined by $\mathcal{O}_{Z'}(\iota'(X))$, by $\Delta_{\mathbb{L}^\times_{\mathrm{diag}}[Z']}$ the maximal cuspidally pro-$\Sigma^\dagger$ quotient of the maximal pro-Σ quotient of the tame fundamental group of $(\mathbb{L}^\times_{\mathrm{diag}}[Z'])_{\overline{k}}$ [where "tame" is with respect to the divisor determined by the complement of the $\mathbb{G}_\mathrm{m}$-torsor $\mathbb{L}^\times_{\mathrm{diag}}[Z']$ in the naturally associated $\mathbb{P}^1$-bundle], and by $\Pi_{\mathbb{L}^\times_{\mathrm{diag}}[Z']}$ the quotient $\pi_1(\mathbb{L}^\times_{\mathrm{diag}}[Z'])/\mathrm{Ker}(\pi_1((\mathbb{L}^\times_{\mathrm{diag}}[Z'])_{\overline{k}}) \twoheadrightarrow \Delta_{\mathbb{L}^\times_{\mathrm{diag}}[Z']})$. Then [in light of the isomorphism of Proposition 1.2, (ii)] we have a natural exact sequence

$$1 \to M_X \to \Pi_{\mathbb{L}^\times_{\mathrm{diag}}[Z']} \to \Pi_{Z'} \to 1$$

whose associated extension class is equal to the class $\eta_{Z'}^{\mathrm{diag}}$.

(iii) The global section of $\mathcal{O}_{Z'}(\iota'(X))$ over Z' determined by the natural inclusion $\mathcal{O}_{Z'} \hookrightarrow \mathcal{O}_{Z'}(\iota'(X))$ defines a morphism

$$U_{Z'} \to \mathbb{L}^\times_{\mathrm{diag}}[Z']$$

over Z' which induces a **surjective** *homomorphism of groups over* $\Pi_{Z'}$*:*

$$\Pi_{U_{Z'}} \twoheadrightarrow \Pi_{\mathbb{L}^{\times}_{\text{diag}}[Z']}$$

Proof. Assertion (i) follows immediately from Propositions 1.1, 1.2, 1.3, together with well-known facts concerning Chern classes and associated cycles in étale cohomology [cf., e.g., [FK], Chapter II, Definition 1.2, Proposition 2.2]. Assertion (ii) follows from Proposition 1.1; [Mzk7], Definition 4.2, Lemmas 4.4, 4.5. Assertion (iii) follows from [Mzk8], Lemma 4.2, by considering fibers over one of the two natural projections $\Pi_{Z'} \to \Pi_{X\times X} \twoheadrightarrow \Pi_X$. [Here, we note that although in [Mzk7], §4; [Mzk8], the base field is assumed to be of characteristic zero, one verifies immediately that the same arguments as those applied in *loc. cit.* yield the corresponding results in the finite field case — so long as we restrict the coefficients of the cohomology modules in question to modules over $\widehat{\mathbb{Z}}^{\dagger}$.] ○

Definition 1.7.

(i) We shall refer to a covering $Z' \to X \times X$ as in the above discussion as the *diagonal covering associated to the covering* $X' \to X$. We shall refer to an extension of profinite groups

$$1 \to M_X \to \mathcal{D}' \to \Pi_{Z'} \to 1$$

whose associated extension class is the class $\eta^{\text{diag}}_{Z'}$ of Proposition 1.6, (i), as a *fundamental extension* [of $\Pi_{Z'}$]. In the following (ii) — (iv), we shall assume that $1 \to M_X \to \mathcal{D} \to \Pi_{X\times X} \to 1$ is a fundamental extension.

(ii) Let $x, y \in X(k)$; write $D_x, D_y \subseteq \Pi_X$ for the associated decomposition groups [which are well-defined up to conjugation by an element of Δ_X — cf. Remark 1.7.1 below]. Now set:

$$\mathcal{D}_x \stackrel{\text{def}}{=} \mathcal{D}|_{D_x \times_{G_k} \Pi_X}; \quad \mathcal{D}_{x,y} \stackrel{\text{def}}{=} \mathcal{D}|_{D_x \times_{G_k} D_y}$$

Thus, $\mathcal{D}_x$ (respectively, $\mathcal{D}_{x,y}$) is an extension of Π_X (respectively, G_k) by M_X. Similarly, if $D = \sum_i m_i \cdot x_i, E = \sum_j n_j \cdot y_j$ are divisors on X supported on points that are rational over k, then set:

$$\mathcal{D}_D \stackrel{\text{def}}{=} \sum_i m_i \cdot \mathcal{D}_{x_i}; \quad \mathcal{D}_{D,E} \stackrel{\text{def}}{=} \sum_{i,j} m_i \cdot n_j \cdot \mathcal{D}_{x_i,y_j}$$

[where the sums are to be understood as sums of extensions of Π_X or G_k by M_X — i.e., the sums are induced by the additive structure of M_X]. Also, we shall write $\mathcal{C} \stackrel{\text{def}}{=} -\mathcal{D}|_{\Pi_X}$ [where we regard Π_X as a subgroup of $\Pi_{X\times X}$ via the diagonal map]. [Thus, $\mathcal{C}$ is an extension of Π_X by M_X whose extension class is the Chern class of the *canonical bundle* of X.]

(iii) Let $S \subseteq X(k)$ be a finite subset. Then we shall write

$$\mathcal{D}_S \stackrel{\mathrm{def}}{=} \prod_{x \in S} \mathcal{D}_x$$

[where the product is to be understood as the fiber product over Π_X]. Thus, $\mathcal{D}_S$ is an extension of Π_X by a product of copies of M_X indexed by elements of S. We shall refer to $\mathcal{D}_S$ as an *S-cuspidalization [of Π_X at S]*. Observe that if $T \subseteq X(k)$ is a finite subset such that $S \subseteq T$, then we obtain a natural *projection morphism* $\mathcal{D}_T \to \mathcal{D}_S$.

(iv) We shall refer to a homomorphism

$$\Pi_{U_{X \times X}} \to \mathcal{D}$$

over $\Pi_{X \times X}$ as a *fundamental section* if, for some isomorphism of $\mathcal{D}$ with $\Pi_{\mathbb{L}^{\times}_{\mathrm{diag}}}$ that induces the identity on $\Pi_{X \times X}$, M_X, the resulting composite homomorphism $\Pi_{U_{X \times X}} \to \Pi_{\mathbb{L}^{\times}_{\mathrm{diag}}}$ is the homomorphism of Proposition 1.6, (iii).

Remark 1.7.1. Relative to the situation in Definition 1.7, (ii), conjugation by elements $\delta \in \Delta_X$ induces isomorphisms between the different possible choices of "D_x", all of which lie over the isomorphism between any of these choices and G_k induced by the projection $\Pi_X \twoheadrightarrow G_k$. Moreover, by lifting $(\delta, 1) \in \Delta_{X \times X} \subseteq \Pi_{X \times X}$ to an element $\delta_{\mathcal{D}} \in \mathcal{D}$, and conjugating by $\delta_{\mathcal{D}}$, we obtain natural isomorphisms between the various resulting "$\mathcal{D}_x$'s" which induce the *identity* on the quotient group $\mathcal{D}_x \twoheadrightarrow \Pi_X$, as well as on the subgroup $M_X \subseteq \mathcal{D}_x$. Note that this last property [i.e., of inducing the identity on Π_X, M_X] holds *precisely* because we are working with $\delta \in \Delta_X \subseteq \Pi_X$, as opposed to an arbitrary "$\delta \in \Pi_X$".

Remark 1.7.2. By Proposition 1.4, (ii), if $\mathcal{E}$ is any profinite group extension of Π_X (respectively, G_k; an open subgroup $\Pi_Z \subseteq \Pi_{X \times X}$ that surjects onto G_k) by M_X, then the *group of cyclotomically outer automorphisms of the extension* $\mathcal{E}$ [i.e., that induce the identity on Π_X (respectively, G_k; Π_Z) and M_X] may be naturally identified with $(k^{\times})^{\wedge}$. In particular, in the context of Definition 1.7, (iv), any two fundamental sections of $\mathcal{D}$ differ, up to composition with a cyclotomically inner automorphism of $\mathcal{D}$, by a "$(k^{\times})^{\wedge}$-multiple".

Proposition 1.8. **(Basic Properties of Cuspidalizations)** *Let*

$$1 \to M_X \to \mathcal{D} \to \Pi_{X \times X} \to 1$$

be a **fundamental extension***; $\phi : \Pi_{U_{X \times X}} \twoheadrightarrow \mathcal{D}$ a* **fundamental section***; $S \subseteq X(k)$ a finite subset. Then:*

(i) The profinite groups $\Delta_{X \times X}$, Δ_X, as well as any profinite group extension of $\Pi_{X \times X}$ or Π_X by a [possibly empty] finite product of copies of M_X is **slim** *[cf. §0]. In particular, the profinite group $\mathcal{D}_S$ is* **slim**.

(ii) For $x \in X(k)$, write $U_x \stackrel{\mathrm{def}}{=} X \backslash \{x\}$. Denote by Δ_{U_x} the maximal cuspidally [i.e., relative to the natural map to $\pi_1((U_x)_{\overline{k}})$] pro-$\Sigma^\dagger$ quotient of the maximal pro-Σ quotient of the tame fundamental group of $(U_x)_{\overline{k}}$ [where "tame" is with respect to the complement of U_x in X] and by Π_{U_x} the quotient $\pi_1(U_x)/\mathrm{Ker}(\pi_1((U_x)_{\overline{k}}) \twoheadrightarrow \Delta_{U_x})$. Then the inverse image via either of the natural projections $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi_X$ of the decomposition group $D_x \subseteq \Pi_X$ is naturally isomorphic to Π_{U_x}. In particular, $\Delta_{U_{X\times X}}$, $\Pi_{U_{X\times X}}$ are **slim**.

(iii) For $S \subseteq X(k)$ a finite subset, write:

$$U_S \stackrel{\mathrm{def}}{=} \prod_{x \in S} U_x$$

[where the product is to be understood as the fiber product over X]. Denote by Δ_{U_S} the maximal cuspidally [i.e., relative to the natural map to $\pi_1((U_S)_{\overline{k}})$] pro-$\Sigma^\dagger$ quotient of the maximal pro-Σ quotient of the tame fundamental group of $(U_S)_{\overline{k}}$ [where "tame" is with respect to the complement of U_S in X], and by Π_{U_S} the quotient $\pi_1(U_S)/\mathrm{Ker}(\pi_1((U_S)_{\overline{k}}) \twoheadrightarrow \Delta_{U_S})$. Then Δ_{U_S}, Π_{U_S} are **slim**. *Forming the product of the specializations of ϕ to the various $D_x \times_{G_k} \Pi_X \subseteq \Pi_{X\times X}$ yields homomorphisms*

$$\Pi_{U_S} \to \prod_{x \in S} \Pi_{U_x} \to \mathcal{D}_S$$

[where the product is to be understood as the fiber product over Π_X]. Moreover, the composite morphism $\Pi_{U_S} \to \mathcal{D}_S$ is **surjective***; the resulting quotient of $\Delta_{U_S} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{U_S} \twoheadrightarrow G_k)$ is the* **maximal cuspidally central quotient** *of Δ_{U_S} [relative to the surjection $\Delta_{U_S} \twoheadrightarrow \Delta_X$].*

(iv) The quotient of $\Delta_{U_{X\times X}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{U_{X\times X}} \twoheadrightarrow G_k)$ determined by $\phi : \Pi_{U_{X\times X}} \twoheadrightarrow \mathcal{D}$ is the **maximal cuspidally central quotient** *of $\Delta_{U_{X\times X}}$ [relative to the surjection $\Delta_{U_{X\times X}} \twoheadrightarrow \Delta_{X\times X}$].*

Proof. Assertion (i) follows immediately from the *slimness* of Π_X, Δ_X [cf., e.g., [Mzk5], Theorem 1.1.1, (ii); the proofs of [Mzk5], Lemmas 1.3.1, 1.3.10], together with the fact that G_k acts on M_X via the *cyclotomic character*. Next, we consider assertion (ii). The portion of assertion (ii) concerning Π_{U_x} follows immediately from the well-known *"base change theorem for smooth base change"* in étale cohomology [cf., e.g., [FK], Chapter I, Theorem 7.3, for the abelian version of this result]. The *slimness* assertion then follows from assertion (i) [applied to Π_X] and the slimness of Δ_{U_x} [cf. the proofs of [Mzk5], Lemmas 1.3.1, 1.3.10]. As for assertion (iii), the *slimness* of Δ_{U_S}, Π_{U_S} follows via the arguments given in the proofs of [Mzk5], Lemmas 1.3.1, 1.3.10. The existence of homomorphisms $\Pi_{U_S} \to \prod_{x\in S} \Pi_{U_x} \to \mathcal{D}_S$ as asserted is immediate from the definitions, assertion (ii). For $x \in S$, write

$$D_x[U_S] \subseteq \Pi_{U_S}$$

for the *decomposition group* of x; $I_x[U_S] \subseteq D_x[U_S]$ for the *inertia subgroup*. Now it is immediate from the definitions that $I_x[U_S]$ maps isomorphically onto the copy

M_X in $\mathcal{D}_S$ corresponding to the point x. This implies the desired *surjectivity*. Since, moreover, it is immediate from the definitions that the cuspidal subgroup of any cuspidally central quotient of Δ_{U_S} is generated by the image of the $I_x[U_S]$, as x ranges over the elements of S, the final assertion concerning the *maximal cuspidally central quotient* of Δ_{U_S} follows immediately. Assertion (iv) follows by a similar argument to the argument applied to the final portion of assertion (iii). ◯

Next, let $Z' \to X \times X$ (respectively, $Z'' \to X \times X$; $Z^* \to X \times X$) be the *diagonal covering* associated to a covering $X' \to X$ (respectively, $X'' \to X$; $X^* \to X$) arising from an open subgroup of Π_X; denote by $\iota' : X \hookrightarrow Z'$ (respectively, $\iota'' : X \hookrightarrow Z''$; $\iota^* : X \hookrightarrow Z^*$) the tautological lifting of the diagonal embedding $\iota : X \hookrightarrow X \times X$ and by k' (respectively, k''; k^*) the extension of k determined by X' (respectively, X''; X^*). Assume, moreover, that the covering $X'' \to X$ factors as follows:

$$X'' \to X' \to X^* \to X$$

Thus, we obtain a factorization $Z'' \to Z' \to Z^* \to X \times X$. Let

$$1 \to M_X \to \mathcal{D}'' \to \Pi_{Z''} \to 1$$

be a *fundamental extension* of $\Pi_{Z''}$.

Write

$$1 \to M_X \to \mathcal{D}''_{X''\times X''} \to \Pi_{X''\times X''} \to 1$$

for the *pull-back* of the extension $\mathcal{D}''$ via the inclusion $\Pi_{X''\times X''} \subseteq \Pi_{Z''}$. Now if we think of $\Pi_{X\times X}$ or $\Pi_{X''\times X''}$ as only being defined *up to* $\Delta_{X''} \times \{1\}$*-inner automorphisms*, then it makes sense, for $\delta \in \Delta_X/\Delta_{X''}$ to speak of the *pull-back of the extension* $\mathcal{D}''_{X''\times X''}$ *via* $\delta \times 1$:

$$1 \to M_X \to (\delta \times 1)^* \mathcal{D}''_{X''\times X''} \to \Pi_{X''\times X''} \to 1$$

In particular, we may form the *fiber product* over $\Pi_{X''\times X''}$:

$$\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X''\times X''} \stackrel{\text{def}}{=} \prod_{\delta \in \Delta_{X^*}/\Delta_{X''}} (\delta \times 1)^* \mathcal{D}''_{X''\times X''}$$

Thus, $\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X''\times X''}$ is an extension of $\Pi_{X''\times X''}$ by a product of copies of M_X indexed by $\Delta_{X^*}/\Delta_{X''}$; $\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X''\times X''}$ admits a *tautological* $\Delta_{X''} \times \{1\}$*-outer* [more precisely, a $(\Delta_{X''} \times \{1\}) \times_{\Pi_{X''\times X''}} \mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X''\times X''}$-outer] *action* by the finite group $\Delta_{X^*}/\Delta_{X''} \cong (\Delta_{X^*}/\Delta_{X''}) \times \{1\}$. Moreover, the natural outer action of $\mathrm{Gal}(X''/X) \cong \mathrm{Gal}((X'' \times X'')/Z'') \cong \Pi_X/\Pi_{X''}$ on $\Pi_{X''\times X''}$ [arising from the diagonal embedding $\Pi_X \hookrightarrow \Pi_{Z''}$] clearly lifts to an outer action of $\mathrm{Gal}(X''/X)$ on $\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X''\times X''}$, which is *compatible*, relative to the natural action of $\mathrm{Gal}(X''/X)$ on $\Delta_{X^*}/\Delta_{X''}$ by conjugation, with the $\Delta_{X''} \times \{1\}$-outer action of $\Delta_{X^*}/\Delta_{X''}$ on $\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X''\times X''}$. Thus, in summary, the natural isomorphism

$$\left\{(\Delta_{X^*}/\Delta_{X''}) \times \{1\}\right\} \rtimes \mathrm{Gal}(X''/X) \cong \mathrm{Gal}((X'' \times X'')/Z^*)$$

determines a homomorphism $\mathrm{Gal}((X'' \times X'')/Z^*) \to \mathrm{Out}(\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X'' \times X''})$ via which we may pull-back the extension "$1 \to (-) \to \mathrm{Aut}(-) \to \mathrm{Out}(-) \to 1$" [cf. §0; Proposition 1.8, (i)] for $\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X'' \times X''}$ to obtain an extension

$$1 \to \prod_{\Delta_{X^*}/\Delta_{X''}} M_X \to \mathbb{S}_{X''/X^*}(\mathcal{D}'') \to \Pi_{Z^*} \to 1$$

in which Π_{Z^*} is only determined up to $\Delta_{X''} \times \{1\}$-inner automorphisms. Note, moreover, that every cyclotomically outer automorphism of the extension $\mathcal{D}''$ — i.e., an element of $(k^\times)^\wedge$ [cf. Remark 1.7.2] — induces a *cyclotomically outer automorphism* of $\mathbb{S}_{X''/X^*}(\mathcal{D}'')$. In particular, we have a *natural cyclotomically outer action of* $(k^\times)^\wedge$ *on* $\mathbb{S}_{X''/X^*}(\mathcal{D}'')$.

Next, let us *push-forward* the extension $\mathbb{S}_{X''/X^*}(\mathcal{D}'')$ just constructed via the *natural surjection*

$$\prod_{\Delta_{X^*}/\Delta_{X''}} M_X \twoheadrightarrow \prod_{\Delta_{X^*}/\Delta_{X'}} M_X$$

[which induces the identity morphism $M_X \to M_X$ between the various factors of the domain and codomain], so as to obtain an extension $\mathrm{Tr}_{X''/X':X^*}(\mathcal{D}'')$ as follows:

$$1 \to \prod_{\Delta_{X^*}/\Delta_{X'}} M_X \to \mathrm{Tr}_{X''/X':X^*}(\mathcal{D}'') \to \Pi_{Z^*} \to 1$$

[in which Π_{Z^*} is only determined up to $\Delta_{X''} \times \{1\}$-inner automorphisms].

Proposition 1.9. **(Symmetrizations and Traces)** *In the notation of the discussion above:*

(i) The extension $\mathrm{Tr}_{X''/X':X'}(\mathcal{D}'')$ *of* $\Pi_{Z'}$ *by* M_X *is a* **fundamental extension** *of* $\Pi_{Z'}$.

(ii) There is a **natural commutative diagram***:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \prod_{\Delta_X/\Delta_{X''}} M_X & \longrightarrow & \mathbb{S}_{X''/X}(\mathcal{D}'') & \longrightarrow & \Pi_{X \times X} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \\
1 & \longrightarrow & \prod_{\Delta_X/\Delta_{X'}} M_X & \longrightarrow & \mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}'')) & \longrightarrow & \Pi_{X \times X} & \longrightarrow & 1
\end{array}$$

[which is well-defined up to $\Delta_{X'} \times \{1\}$*-inner automorphisms — cf. Remark 1.9.1 below].*

(iii) Relative to the commutative diagram of (ii), the natural cyclotomically outer action of $(k^\times)^\wedge$ *on* $\mathbb{S}_{X''/X}(\mathcal{D}'')$ *lies over the composite of the map* $(k^\times)^\wedge \to (k^\times)^\wedge$ *given by* **raising to the** $[\Delta_{X'} : \Delta_{X''}]$**-power** *with the natural cyclotomically outer action of* $(k^\times)^\wedge$ *on* $\mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}''))$*. In particular, if* N *is a positive integer that divides* $[\Delta_{X'} : \Delta_{X''}]$*, then the natural cyclotomically outer action of*

an element of $(k^\times)^\wedge$ *on* $\mathbb{S}_{X''/X}(\mathcal{D}'')$ *lies over the cyclotomically outer action of an element of* $\{(k^\times)^\wedge\}^N$ *on* $\mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}''))$.

Proof. To verify assertion (i), *observe* that it is immediate from the definitions that

$$\iota'(X) \times_{Z'} (X'' \times X'') \subseteq X'' \times X''$$

is equal to the $\Delta_{X'}/\Delta_{X''}$*-orbit* of $\iota''(X) \times_{Z''} (X'' \times X'') \subseteq X'' \times X''$. Now assertion (i) follows by translating this observation into the language of étale cohomology classes associated to subvarieties; assertions (ii), (iii) follow formally from assertion (i) and the definitions of the various objects involved. ○

Remark 1.9.1. Relative to the commutative diagram of Proposition 1.9, (ii), note that, although $\mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}''))$ is, by definition, only well-defined *up to* $\Delta_{X'} \times \{1\}$*-inner automorphisms*, the *push-forward* of $\mathbb{S}_{X''/X}(\mathcal{D}'')$ by

$$\prod_{\Delta_X/\Delta_{X''}} M_X \to \prod_{\Delta_X/\Delta_{X'}} M_X$$

is well-defined *up to* $\Delta_{X''} \times \{1\}$*-inner automorphisms.* That is to say, the push-forward extension implicit in this commutative diagram furnishes a *canonically more rigid* version of the extension $\mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}''))$.

Definition 1.10.

(i) We shall refer to the extension $\mathbb{S}_{X''/X^*}(\mathcal{D}'')$ [of Π_{Z^*}] constructed from the fundamental extension $\mathcal{D}''$ as the *[*X''/X^**-]symmetrization* of $\mathcal{D}''$, or, alternatively, as a *symmetrized fundamental extension.* We shall refer to the extension $\mathrm{Tr}_{X''/X':X^*}(\mathcal{D}'')$ [of Π_{Z^*}] constructed from the fundamental extension $\mathcal{D}''$ as the *[*$X''/X' : X^*$*-]trace* of $\mathcal{D}'$, or, alternatively, as a *trace-symmetrized fundamental extension.*

(ii) If $\mathcal{D}'$ is a fundamental extension of $\Pi_{Z'}$, then we shall refer to as a *morphism of trace type* any morphism

$$\mathbb{S}_{X''/X}(\mathcal{D}'') \to \mathbb{S}_{X'/X}(\mathcal{D}')$$

obtained by composing the morphism

$$\mathbb{S}_{X''/X}(\mathcal{D}'') \to \mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}''))$$

of Proposition 1.9, (ii), with a morphism

$$\mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}'')) \to \mathbb{S}_{X'/X}(\mathcal{D}')$$

arising [by the *functoriality* of the construction of "$\mathbb{S}_{X'/X}(-)$"] from an isomorphism of [fundamental] extensions $\mathrm{Tr}_{X''/X':X'}(\mathcal{D}'') \xrightarrow{\sim} \mathcal{D}'$ of $\Pi_{Z'}$ by M_X [which induces the identity on $\Pi_{Z'}$, M_X].

(iii) We shall refer to as a *pro-symmetrized fundamental extension* any compatible system [indexed by the natural numbers]

$$\ldots \twoheadrightarrow \mathcal{S}_i \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{S}_j \twoheadrightarrow \ldots \twoheadrightarrow \Pi_{X \times X}$$

of morphisms of trace type [up to inner automorphisms of the appropriate type] between symmetrized fundamental extensions, where $\mathcal{S}_i$ is the X_i/X-symmetrization of a fundamental extension of Π_{Z_i}; Z_i is the diagonal covering associated to an open normal subgroup $\Pi_{X_i} \subseteq \Pi_X$; the intersection of the Π_{X_i} is trivial. In this situation, we shall refer to the inverse limit profinite group

$$\mathcal{S}_\infty \stackrel{\mathrm{def}}{=} \varprojlim_{i} \ \mathcal{S}_i$$

as the *limit of the pro-symmetrized fundamental extension* $\{\mathcal{S}_i\}$; any profinite group $\mathcal{S}_\infty$ arising in this fashion will be referred to as a *pro-fundamental extension [of* $\Pi_{X \times X}$*]*.

(iv) Let $S \subseteq X(k)$ be a finite subset; $\mathcal{S}'$ an X'/X-symmetrization of a fundamental extension $\mathcal{D}'$ of $\Pi_{Z'}$. Then we shall write

$$\mathcal{S}'_S \stackrel{\mathrm{def}}{=} \prod_{x \in S} \mathcal{S}'_{D_x \times_{G_k} \Pi_X}$$

[where the product is to be understood as the fiber product over Π_X]. Thus, $\mathcal{S}'_S$ is an extension of Π_X by a product of copies of M_X. Similarly, given a projective system $\{\mathcal{S}_i\}$ as in (iii), we obtain a projective system $\{(\mathcal{S}_i)_S\}$, with inverse limit:

$$(\mathcal{S}_\infty)_S$$

We shall refer to $(\mathcal{S}_\infty)_S$ as an *S-pro-cuspidalization [of* Π_X *at S]*. Observe that if $T \subseteq X(k)$ is a finite subset such that $S \subseteq T$, then we obtain a natural *projection morphism* $(\mathcal{S}_\infty)_T \to (\mathcal{S}_\infty)_S$.

Remark 1.10.1. Let $\mathcal{D}$ be as in Definition 1.7, (iii); $\mathcal{S}'$, $\{\mathcal{S}_i\}$, $\mathcal{S}_\infty$ as in Definition 1.10, (iii), (iv). Then observe that it follows from Proposition 1.8, (i), that $\mathcal{D}$, $\mathcal{S}'$, $\mathcal{S}_i$, and $\mathcal{S}_\infty$ are *slim*. In particular, if $S \subseteq X^{\mathrm{cl}}$ is *any finite set of closed points* of X, then we may form the objects

$$\mathcal{D}_S; \quad \mathcal{S}'_S; \quad (\mathcal{S}_i)_S; \quad (\mathcal{S}_\infty)_S$$

by passing to a Galois covering $X_{k_S} \to X$ [i.e., the result of base-changing X to some finite Galois extension k_S of k] such that the closed points of X_{k_S} that lie over points of S are *rational* over k_S; forming the various objects in question over X_{k_S} [cf. Definition 1.7, (iii); Definition 1.10, (iv)]; and, finally, *"descending to X"* via the outer action of $\mathrm{Gal}(X_{k_S}/X) = \mathrm{Gal}(k_S/k)$ on the various objects in question [cf. the exact sequence "$1 \to (-) \to \mathrm{Aut}(-) \to \mathrm{Out}(-) \to 1$" of §0; the *slimness* mentioned above]. Thus, in the remainder of this paper, we shall often speak of the various objects defined in Definition 1.7, (iii); Definition 1.10, (iv), *even when the points of the finite set S are not necessarily rational over k.*

Before proceeding, we note the following:

Lemma 1.11. **(Conjugacy Estimate)** *Let $H \subseteq \Delta_X$ be a normal open subgroup; $a \in \Delta_X/H$ an element not equal to the identity; N a $\Sigma^\dagger$-integer [cf. §0]. Then there exists a normal open subgroup $H' \subseteq \Delta_X$ contained in H such that for any normal open subgroup $H'' \subseteq \Delta_X$ contained in H' and any $a'' \in \Delta_X/H''$ that lifts a, the cardinality of the H-conjugacy class $\mathrm{Conj}(a'', H'') \subseteq \Delta_X/H''$ of a'' in Δ_X/H'' is divisible by N.*

Proof. In the notation of the statement of Lemma 1.11, denote by $Z(a'', H'') \subseteq H$ the subgroup of elements $\delta \in H$ such that $\delta \cdot a'' \cdot \delta^{-1} = a''$ in Δ_X/H''. Then it is immediate that if a' is the image of a'' in Δ_X/H', then $Z(a'', H'') \subseteq Z(a', H')$, so the cardinality of $\mathrm{Conj}(a'', H'') \cong H/Z(a'', H'')$ is *divisible* by the cardinality of $\mathrm{Conj}(a', H') \cong H/Z(a', H')$. Thus, it suffices to find a normal open subgroup $H' \subseteq H$ such that for any $a' \in H'$ that lifts a, the cardinality of $\mathrm{Conj}(a', H')$ is divisible by N.

To this end, let us consider, for some prime number $l \in \Sigma^\dagger$, the *maximal pro-l quotient* $H[l]$ of the *abelianization* H^{ab} of H. Note that Δ_X/H acts by conjugation on H^{ab}, $H[l]$. Now I *claim* that there exists a [nonzero] $h_l \in H[l]$ such that $a(h_l) \notin \mathbb{Z}_l \cdot h_l$. Indeed, if this claim were false, then it would follow that a acts on $H[l]$ by multiplication by a single element $\lambda \in \mathbb{Z}_l$. Moreover, by considering the portion of $H[l]$ arising from the abelianization of Δ_X, it follows that $\lambda = 1$, i.e., that a acts trivially on $H[l]$. But since a induces a nontrivial automorphism of the covering of $X_{\overline{k}}$ determined by H, it follows that a induces a nontrivial automorphism of the l-power torsion points of the Jacobian of $X_{\overline{k}}$ [since these points are Zariski dense in this Jacobian] — a contradiction. This completes the proof of the *claim*.

Now let $j \in H$ be an element that lifts the various h_l obtained above for the [finite collection of] primes l that divide N; let $a_X \in \Delta_X$ be an element that lifts a. Then observe that for some integer power M of N that is *independent* of the choice of a_X, the image of $j^n \cdot a_X \cdot j^{-n} \cdot a_X^{-1}$ in $H^{\mathrm{ab}} \otimes (\mathbb{Z}/M\mathbb{Z})$ is *nonzero*, for $n \in \widehat{\mathbb{Z}}$ with nonzero image in $\widehat{\mathbb{Z}}/N \cdot \widehat{\mathbb{Z}}$. Thus, if we take H' equal to the inverse image of $M \cdot H^{\mathrm{ab}}$ in Δ_X, we obtain that the intersection of the subgroup $j^{\widehat{\mathbb{Z}}} \subseteq H$ with $Z(a', H')$ [where $a' \in \Delta_X/H$ lifts a] *does not contain* j^n, for $n \in \widehat{\mathbb{Z}}$ with nonzero image in $\widehat{\mathbb{Z}}/N \cdot \widehat{\mathbb{Z}}$. But this implies that the intersection $(j^{\widehat{\mathbb{Z}}}) \bigcap Z(a', H') \subseteq j^{N \cdot \widehat{\mathbb{Z}}}$, hence that $[H : Z(a', H')]$ is divisible by N, as desired. ○

Next, we consider the following *fundamental extensions* of $\Pi_{Z''}$, $\Pi_{Z'}$:

$$\underline{\mathcal{D}}'' \stackrel{\mathrm{def}}{=} \Pi_{\mathbb{L}^{\times}_{\mathrm{diag}}[Z'']}; \quad \underline{\mathcal{D}}' \stackrel{\mathrm{def}}{=} \mathrm{Tr}_{X''/X':X'}(\mathcal{D}'')$$

[cf. Proposition 1.6, (ii)]. Note that in this situation, it follows immediately from the definitions that we obtain a natural isomorphism $\underline{\mathcal{D}}' \xrightarrow{\sim} \Pi_{\mathbb{L}^{\times}_{\mathrm{diag}}[Z']}$, which we shall use in the following discussion to *identify* $\underline{\mathcal{D}}'$, $\Pi_{\mathbb{L}^{\times}_{\mathrm{diag}}[Z']}$. Thus, we have *fundamental sections*:

$$\Pi_{U_{Z''}} \twoheadrightarrow \underline{\mathcal{D}}''; \quad \Pi_{U_{Z'}} \twoheadrightarrow \underline{\mathcal{D}}'$$

[cf. Proposition 1.6, (iii)]. In particular, by pulling back from Z'' to $X'' \times X''$, we obtain a surjection:

$$\Pi_{U_{X''\times X''}} \twoheadrightarrow \underline{\mathcal{D}}''_{X''\times X''}$$

Now if we apply the natural *outer* $(\Delta_X/\Delta_{X''}) \times \{1\}$*-action* on $\Pi_{U_{X''\times X''}}$ to this surjection, it follows from the definition of "$\mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')$" that we obtain a natural homomorphism

$$\Pi_{U_{X''\times X''}} \twoheadrightarrow \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')_{X''\times X''}$$

which is easily verified [cf. Proposition 1.8, (ii), (iii)] to be *surjective*. Since, moreover, the construction of this surjective homomorphism is manifestly compatible with the outer actions of $\mathrm{Gal}(X''/X)$ on both sides, we thus obtain a *natural surjection*:

$$\Pi_{U_{X\times X}} \twoheadrightarrow \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')$$

Now let us denote by

$$D_X \subseteq \Pi_{U_{X\times X}}$$

the *decomposition group* of the subvariety $\iota(X) \subseteq X \times X$. [Thus, D_X is well-defined up to conjugation; here, we assume that we have chosen a conjugate that maps to the image of the diagonal embedding $\Pi_X \hookrightarrow \Pi_{X\times X}$ via the natural surjection $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi_{X\times X}$.] Observe that we have a *natural exact sequence*

$$1 \to I_X \to D_X \to \Pi_X \to 1$$

[where I_X — i.e., the *inertia subgroup* of D_X — is defined so as to make the sequence exact], together with a natural isomorphism $I_X \cong M_X$. Also, we shall write $D_{X'} \stackrel{\mathrm{def}}{=} D_X \bigcap \Pi_{U_{X'\times X'}}$; $D_{X''} \stackrel{\mathrm{def}}{=} D_X \bigcap \Pi_{U_{X''\times X''}}$. Since the construction just carried out for double primed objects may also be carried out for single primed objects, we thus obtain the following:

Proposition 1.12. **(Symmetrized Fundamental Sections)** *In the notation of the discussion above:*

(i) There is a natural commutative diagram:

$$\begin{array}{ccccc}
D_X & \subseteq & \Pi_{U_{X\times X}} & \twoheadrightarrow & \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'') \\
\downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \\
D_X & \subseteq & \Pi_{U_{X\times X}} & \twoheadrightarrow & \mathbb{S}_{X'/X}(\underline{\mathcal{D}}')
\end{array}$$

[where the vertical arrow on the right is the morphism in the diagram of Proposition 1.9, (ii)].

(ii) Denote by means of a subscript X'' the result of pulling back extensions of $\Pi_{X\times X}$, $\Pi_{Z''}$, $\Pi_{X''\times X''}$ to $\Pi_{X''}$ [via the diagonal inclusion]. Then the projection [cf. the fiber product defining $\mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')$] to the factor labeled "$\Delta_{X''}/\Delta_{X''}$" detemines a **natural surjection**

$$\zeta'' : \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')_{X''} \twoheadrightarrow \underline{\mathcal{D}}_{X''}$$

whose **restriction** *to $D_{X''}$ [i.e., relative to the arrows in the first line of the commutative diagram of (i)] defines an* **isomorphism** $D_{X''} \xrightarrow{\sim} \underline{\mathcal{D}}_{X''}$*. Moreover, the cuspidal subgroup of $D_{X''}$ maps isomorphically onto the factor of M_X in $\mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')$ labeled "$\Delta_{X''}/\Delta_{X''}$". In particular, if we denote by*

$$\mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')^{\neq}$$

the quotient of $\mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')$ by this factor of M_X, then ζ'' determines a surjection

$$\zeta''_{\neq} : \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')^{\neq}_{X''} \twoheadrightarrow \Pi_{X''}$$

whose **restriction** *to the quotient $D_{X''} \twoheadrightarrow \Pi_{X''}$ is equal to the identity $\Pi_{X''} \xrightarrow{\sim} \Pi_{X''}$ [up to geometric inner automorphisms]. Thus, we have a* **natural commutative diagram** *[well-defined up to geometric inner automorphisms]*

$$\begin{array}{ccccc} D_{X''} & \hookrightarrow & \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')_{X''} & \xrightarrow{\zeta''} & \underline{\mathcal{D}}''_{X''} \\ \downarrow & & \downarrow & & \downarrow \\ \Pi_{X''} & \hookrightarrow & \mathbb{S}_{X''/X}(\underline{\mathcal{D}}')^{\neq}_{X''} & \xrightarrow{\zeta''_{\neq}} & \Pi_{X''} \end{array}$$

in which the two horizontal composites are isomorphisms; the vertical arrows are surjections; the left-hand square is **cartesian**.

(iii) If we carry out the construction of (ii) for the single primed objects, then the commutative diagram of (i) induces a **natural commutative diagram** *[well-defined up to geometric inner automorphisms]:*

$$\begin{array}{ccccc} \Pi_{X''} & \hookrightarrow & \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')^{\neq}_{X''} & \xrightarrow{\zeta''_{\neq}} & \Pi_{X''} \\ \downarrow & & \downarrow & & \downarrow \\ \Pi_{X'} & \hookrightarrow & \mathbb{S}_{X'/X}(\underline{\mathcal{D}}')^{\neq}_{X'} & \xrightarrow{\zeta'_{\neq}} & \Pi_{X'} \end{array}$$

Moreover, there is a natural outer action of $\mathrm{Gal}(X''/X)$ (respectively, $\mathrm{Gal}(X'/X)$) on the first (respectively, second) line of this diagram; these outer actions are compatible with one another.

(iv) When considered up to cyclotomically inner automorphisms, the sections of $\zeta''_{\neq}$ form a **torsor** *over the following group:*

$$\prod_{(\Delta_X/\Delta_{X''})\backslash(\Delta_{X''}/\Delta_{X''})} ((k'')^{\times})^{\wedge}$$

[Here, the "$\backslash$" denotes the set-theoretic complement.] The $\mathrm{Gal}(X''/X)$**-equivariant sections** *of* $\zeta''_{\neq}$ *form a torsor over the* $\mathrm{Gal}(X''/X)$*-invariant subgroup of this group. Similar statements hold for the single primed objects.*

(v) The double and single primed torsors of equivariant sections of (iv) are related, via the right-hand square of the diagram of (iii), by a homomorphism

$$\Big\{ \prod_{(\Delta_X/\Delta_{X''})\backslash(\Delta_{X''}/\Delta_{X''})} ((k'')^\times)^\wedge \Big\}^{\mathrm{Gal}(X''/X)} \to \Big\{ \prod_{(\Delta_X/\Delta_{X'})\backslash(\Delta_{X'}/\Delta_{X'})} ((k')^\times)^\wedge \Big\}^{\mathrm{Gal}(X'/X)}$$

[where the superscripts denote the result of taking invariants with respect to the action of the superscripted group] with the following property:

> *An element* ξ'' *of the domain maps to an element of the codomain whose component in the factor labeled* $a' \in \Delta_X/\Delta_{X'}$ *is a product of elements of* $((k')^\times)^\wedge$ *of the form* $\mathcal{N}_{k'_{a''}/k'}(\lambda'')^{n''}$*.*

Here, $a'' \in (\Delta_X/\Delta_{X''})\backslash(\Delta_{X'}/\Delta_{X''})$ *maps to* a' *in* $\Delta_X/\Delta_{X'}$*;* $\lambda'' \in ((k'')^\times)^\wedge$ *is the component of* ξ'' *in the factor labeled* a''*;* $k'_{a''}$ *is an intermediate field extension between* k' *and* k'' *such that* $\lambda'' \in ((k'_{a''})^\times)^\wedge$*;* $\mathcal{N}_{k'_{a''}/k'} : ((k'_{a''})^\times)^\wedge \to ((k')^\times)^\wedge$ *is the norm map;* n'' *is the* **cardinality of the** $\Delta_{X'}$**-conjugacy class** *of* a'' *in* $(\Delta_X/\Delta_{X''})$*. In particular, by Lemma 1.11 [where we take "*H*" to be* $\Delta_{X'}$*, "*H''*" to be* $\Delta_{X''}$*], for a given* $\Delta_{X'}$*, if, for a given positive integer* N*,* $\Delta_{X''}$ *is "sufficiently small", then an* **arbitrary** $\mathrm{Gal}(X''/X)$**-equivariant section** *of* $\zeta''_{\neq}$ *lies over the* **canonical section** *of* $\zeta'_{\neq}$ *given in (iii), up to the cyclotomically outer action of some* N*-th power of an element of the single primed version of the group exhibited in the display of (iv).*

Proof. All of these assertions follow immediately from the definitions [and, in the case of assertion (iv), Proposition 1.4, (ii)]. ○

Definition 1.13. Let $\mathcal{D}'$ be a *fundamental extension* of $\Pi_{Z'}$; $\{\mathcal{S}_i\}$ a *pro-symmetrized fundamental extension*, with limit $\mathcal{S}_\infty$ [cf. Definition 1.10, (iii)].

(i) We shall refer to as a *symmetrized fundamental section* a homomorphism

$$\Pi_{U_{X\times X}} \twoheadrightarrow \mathbb{S}_{X'/X}(\mathcal{D}')$$

obtained by composing the surjection $\Pi_{U_{X\times X}} \twoheadrightarrow \mathbb{S}_{X'/X}(\underline{\mathcal{D}}')$ of Proposition 1.12, (i), with the isomorphism $\mathbb{S}_{X'/X}(\underline{\mathcal{D}}') \xrightarrow{\sim} \mathbb{S}_{X'/X}(\mathcal{D}')$ induced by an isomorphism $\underline{\mathcal{D}}' \xrightarrow{\sim} \mathcal{D}'$ of fundamental extensions of $\Pi_{Z'}$ by M_X [which induces the identity on $\Pi_{Z'}$, M_X]. We shall refer to an inclusion

$$D_X \hookrightarrow \mathbb{S}_{X'/X}(\mathcal{D}')$$

obtained by restricting a symmetrized fundamental section to $D_X \subseteq \Pi_{U_{X\times X}}$ [cf. Proposition 1.12, (i)] as a *fundamental inclusion.*

(ii) We shall refer to a compatible system of symmetrized fundamental sections $\Pi_{U_{X\times X}} \to \mathcal{S}_i$ as a *pro-symmetrized fundamental section* and to the resulting limit homomorphism $\Pi_{U_{X\times X}} \to \mathcal{S}_\infty$ as a *pro-fundamental section.* Similarly, we have a notion of *"pro-fundamental inclusions".*

Remark 1.13.1. Thus, by the above discussion, if we take the "$\mathcal{S}_i$" to be the symmetrizations of the $\Pi_{\mathbb{L}^\times_{\mathrm{diag}}[Z']}$ as in Proposition 1.6, (ii), then we obtain natural *pro-fundamental sections* and *pro-fundamental inclusions* [cf. Proposition 1.12, (i), (ii), (iii)].

Proposition 1.14. **(Maximal Cuspidally Abelian Quotients)** *Let $\{\mathcal{S}_i\}$ be a* **pro-symmetrized fundamental extension***, with* **limit** $\mathcal{S}_\infty$ *[cf. Definition 1.10, (iii)] and* **pro-fundamental section** $\Pi_{U_{X\times X}} \twoheadrightarrow \mathcal{S}_\infty$ *[cf. Definition 1.13, (ii)]; $S \subseteq X^{\mathrm{cl}}$ a finite set of closed points [cf. Remark 1.10.1]. Then:*

(i) The pro-fundamental section $\Pi_{U_{X\times X}} \twoheadrightarrow \mathcal{S}_\infty$ determines a **surjection**

$$\Pi_{U_S} \twoheadrightarrow (\mathcal{S}_\infty)_S$$

[cf. Proposition 1.8, (iii)]. The resulting quotient of Δ_{U_S} (respectively, Π_{U_S}) is the **maximal cuspidally abelian quotient** *of Δ_{U_S} (respectively, Π_{U_S}).*

(ii) The quotient of $\Delta_{U_{X\times X}}$ (respectively, $\Pi_{U_{X\times X}}$) induced by the pro-fundamental section $\Pi_{U_{X\times X}} \twoheadrightarrow \mathcal{S}_\infty$ is the **maximal cuspidally abelian quotient** *[which we shall denote by] $\Delta_{U_{X\times X}} \twoheadrightarrow \Delta^{\text{c-ab}}_{U_{X\times X}}$ (respectively, $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}}$) of $\Delta_{U_{X\times X}}$ (respectively, $\Pi_{U_{X\times X}}$).*

Proof. Indeed, this follows as in the proof of Proposition 1.8, (iii), (iv), by observing that the cuspidal subgroup of the maximal cuspidally abelian quotient of Δ_{U_S} (respectively, $\Delta_{U_{X\times X}}$) is naturally isomorphic to the inverse limit of the cuspidal subgroups of the maximal cuspidally *central* quotients of the $\Delta_{U_S} \times_{\Delta_X} \Delta_{X'}$ ($\subseteq \Delta_{U_S}$) (respectively, $\Delta_{U_{X'\times X'}}$) [as $\Delta_{X'} \subseteq \Delta_X$ ranges over the open normal subgroups of Δ_X]. ○

Proposition 1.15. **(Automorphisms and Commensurators)** *Let $\{\mathcal{S}_i\}$ be a* **pro-symmetrized fundamental extension***, with* **limit** $\mathcal{S}_\infty$ *[cf. Definition 1.10, (iii)] and* **pro-fundamental inclusion** $D_X \hookrightarrow \mathcal{S}_\infty$ *[cf. Definition 1.13, (ii)]. Then:*

(i) Any automorphism α of the profinite group $\Pi^{\text{c-ab}}_{U_{X\times X}}$ which

(a) is compatible with the natural surjection $\Pi^{\text{c-ab}}_{U_{X\times X}} \twoheadrightarrow \Pi_{X\times X}$ and induces the identity on $\Pi_{X\times X}$;

(b) preserves the image of $M_X \cong I_X \subseteq D_X$ *via the natural inclusion* $D_X \hookrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}}$

is **cuspidally inner**.

(ii) Π_X *(respectively,* Δ_X*) is* **commensurably terminal** *[cf. §0] in* $\Pi_{X\times X}$ *(respectively,* $\Delta_{X\times X}$*).*

(iii) D_X *is* **commensurably terminal** *in* $\mathcal{S}_i$, $\mathcal{S}_\infty \cong \Pi^{\text{c-ab}}_{U_{X\times X}}$.

Proof. First, we verify assertion (i). By Proposition 1.14, (ii), we have a natural isomorphism $\Pi^{\text{c-ab}}_{U_{X\times X}} \xrightarrow{\sim} \mathcal{S}_\infty$, so we may think of α as an automorphism of $\mathcal{S}_\infty$. In light of (a); Proposition 1.8, (iii), it follows that α is compatible with the natural surjections $\mathcal{S}_\infty \twoheadrightarrow \mathcal{S}_i$. Write α_i for the automorphism of $\mathcal{S}_i$ induced by α. By (a), (b), it follows that α_i is an automorphism of the *extension* $\mathcal{S}_i$ of $\Pi_{X\times X}$ by a product of copies of M_X which induces the *identity* on both $\Pi_{X\times X}$ and the product of copies of M_X [cf. the definition by a certain fiber product of the symmetrized fundamental extension $\mathcal{S}_i$]. [Here, we note that the fact that α_i induces the identity on each copy of M_X follows by considering the *non-torsion* [cf. Propositions 1.2, (ii); 1.6, (i), (ii)] *extension class* determined by that copy of M_X [which is preserved by α_i!], together with the fact that α_i induces the identity on the second cohomology groups of open subgroups of $\Delta_{X\times X}$ with coefficients in M_X.] Thus, up to cyclotomically inner automorphisms, α_i arises from a collection of elements of $(k_i^\times)^\wedge$, where k_i is some finite Galois extension of k [cf. Proposition 1.4, (ii)], one corresponding to each copy of M_X. Moreover, since these copies of M_X are *permuted* by the action of $\Pi_{X\times X}$ by conjugation, it follows that [up to cyclotomically inner automorphisms] α_i arises from a *single element* of $(k_i^\times)^\wedge$, which in fact belongs to $(k^\times)^\wedge$ $(\subseteq (k_i^\times)^\wedge)$ [as one sees by considering the conjugation action via the "G_k portion" of $\Pi_{X\times X}$]. On the other hand, since the α_i form a *compatible system* of automorphisms of the $\mathcal{S}_i$, it follows from Proposition 1.9, (iii), that this element of $(k^\times)^\wedge$ must be equal to 1, as desired.

Next, to verify assertion (ii), let us observe that it suffices to show that Δ_X is commensurably terminal in $\Delta_{X\times X}$. But this follows immediately from the fact that Δ_X is *slim* [cf. Proposition 1.8, (i)]. Finally, we consider assertion (iii). Clearly, it suffices to show that D_X is commensurably terminal in $\mathcal{S}_i$. By assertion (ii), to verify this commensurable terminality, it suffices to show that the [manifestly abelian] cuspidal subgroup $H_i \subseteq \mathcal{S}_i$ [i.e., relative to the natural surjection $\mathcal{S}_i \twoheadrightarrow \Pi_{X\times X}$] satisfies the following property: Every $h \in H_i$ such that $h^\delta - h \in D_X$, for all δ in some open subgroup J of D_X, satisfies $h \in D_X$. But this property follows immediately [cf. the definition by a certain fiber product of the symmetrized fundamental extension $\mathcal{S}_i$] from the fact that, for J sufficiently small, the J-module $H_i/(D_X \bigcap H_i)$ is isomorphic to a direct product of a finite number of copies of M_X. ○

The following result is the *main result* of the present §1:

Theorem 1.16. (Reconstruction of Maximal Cuspidally Abelian Quotients) *Let X, Y be* **hyperbolic curves** *over a* **finite** *or* **nonarchimedean local field***; denote the base fields of X, Y by k_X, k_Y, respectively. Let Σ_X (respectively, Σ_Y) be a* **set of prime numbers** *that contains at least one prime number that is invertible in k_X (respectively, k_Y); write Δ_X (respectively, Δ_Y) for the* **maximal cuspidally pro-$\Sigma^\dagger_X$** *(respectively,* **pro-$\Sigma^\dagger_Y$***)* **quotient of the maximal pro-Σ_X** *(respectively,* **pro-Σ_Y***)* **quotient** *of the* **tame fundamental group** *of $X_{\overline{k}_X}$ (respectively, $Y_{\overline{k}_Y}$) [where "tame" is with respect to the complement of $X_{\overline{k}_X}$ (respectively, $Y_{\overline{k}_Y}$) in its canonical compactification], and Π_X (respectively, Π_Y) for the corresponding quotient of the étale fundamental group of X (respectively, Y). Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be an **isomorphism** *of profinite groups. Then:*

(i) We have $\Sigma_X = \Sigma_Y$, $\Sigma^\dagger_X = \Sigma^\dagger_Y$; write $\Sigma \stackrel{\mathrm{def}}{=} \Sigma_X = \Sigma_Y$, $\Sigma^\dagger \stackrel{\mathrm{def}}{=} \Sigma^\dagger_X = \Sigma^\dagger_Y$. Moreover, k_X is a **finite field** *if and only if k_Y is; X is of* **type** *(g, r) [where $g, r \geq 0$ are integers such that $2g-2+r > 0$] if and only if Y is of type (g, r). Finally, if k_X, k_Y are* **nonarchimedean local***, then their* **residue characteristics coincide***.*

(ii) α is **compatible** *with the natural* **quotients** *$\Pi_X \twoheadrightarrow G_{k_X}$, $\Pi_Y \twoheadrightarrow G_{k_Y}$.*

(iii) Assume that X, Y are **proper***. Denote by $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}}$, $\Pi_{U_{Y\times Y}} \twoheadrightarrow \Pi^{\text{c-ab}}_{U_{Y\times Y}}$ the* **maximal cuspidally abelian quotients** *[cf. Proposition 1.14, (ii); the discussion preceding Proposition 1.6]. Then there is a commutative diagram, [well-defined up to* **cuspidally inner automorphisms***]:*

$$\begin{array}{ccc} \Pi^{\text{c-ab}}_{U_{X\times X}} & \xrightarrow{\alpha^{\text{c-ab}}} & \Pi^{\text{c-ab}}_{U_{Y\times Y}} \\ \downarrow & & \downarrow \\ \Pi_{X\times X} & \xrightarrow{\alpha\times\alpha} & \Pi_{Y\times Y} \end{array}$$

Here, the horizontal arrows are **isomorphisms** *which are* **compatible** *with the* **natural inclusions** *$D_X \hookrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}}$, $D_Y \hookrightarrow \Pi^{\text{c-ab}}_{U_{Y\times Y}}$ [cf. Proposition 1.12, (i)]; the vertical arrows are the natural surjections. Finally, the correspondence*

$$\alpha \mapsto \alpha^{\text{c-ab}}$$

is **functorial** *[up to cuspidally inner automorphisms] with respect to α.*

Proof. First, we consider assertions (i), (ii). Note that k_X is *finite* if and only if, for every open subgroup $H \subseteq \Pi_X$, the quotient of the abelianization H^{ab} by the closure of the torsion subgroup of H^{ab} is *topologically cyclic* [cf. [Tama], Proposition 3.3, (ii)]; a similar statement holds for k_Y, Π_Y. In the *finite field case*, assertion (ii) also follows from [Tama], Proposition 3.3, (ii); the portion of assertion (i) concerning Σ_X, Σ_Y follows from assertion (ii). The fact that $\Sigma^\dagger_X = \Sigma^\dagger_Y$ then follows from the following observation: *Either* the function

$$\Sigma \ni l \mapsto \dim_{\mathbb{Q}_l}((\Delta_X)^{\text{ab}} \otimes \mathbb{Q}_l)$$

is *constant* — in which case $\Sigma^\dagger_X = \Sigma$ — *or* this function attains a *minimum* at a unique element of Σ, which is equal to the residue characteristic of k_X [cf. [Tama], Proposition 3.1]; a similar statement holds for Y. Now by considering the respective outer actions of G_{k_X}, G_{k_Y} on the *maximal pro-l quotients* of Δ_X, Δ_Y, for some $l \in \Sigma^\dagger$, we obtain that X is of type (g, r) if and only if Y is of type (g, r), by [Mzk9], Corollary 2.7, (i). This completes the proof of assertions (i), (ii) in the *finite field case*.

Next, let us assume that k_X, k_Y are *nonarchimedean local*. Then the portion of assertion (i) concerning Σ_X, Σ_Y follows by considering the *cohomological dimension* of Π_X, Π_Y — cf., e.g., Proposition 1.3, (ii) [in the proper case]. As for assertion (ii), if the cardinality of Σ is ≥ 2, then assertion (ii) follows from the evident pro-Σ analogue of [Mzk5], Lemma 1.3.8; if the cardinality of Σ is 1, then assertion (ii) follows from Lemma 1.17 below. Now the portion of assertion (i) concerning the residue characteristics of k_X, k_Y follows from assertion (ii) and [Mzk5], Proposition 1.2.1, (i); the fact that X is of type (g, r) if and only if Y is of type (g, r) follows from [Mzk9], Corollary 2.7, (i). [Here, we note that the hypothesis of *"weak l-graphic fullness"* [which must be satisfied in order to apply [Mzk9], Corollary 2.7, (i)] follows immediately from the "Riemann hypothesis for abelian varieties over finite fields" — cf., e.g., [Mumf], p. 206 — if Σ contains a prime l that differs from the the residue characteristic p of k_X, k_Y, and follows from the well-known fact that the p-adic Galois modules $\Delta_X \otimes \mathbb{Q}_p$, $\Delta_Y \otimes \mathbb{Q}_p$ are [successive extensions of] *Hodge-Tate* modules, with weights contained in the set $\{0, 1\}$ — cf., e.g., [Tate] — if $\Sigma = \{p\}$.] This completes the proof of assertions (i), (ii) in the *nonarchimedean local field case*.

Finally, we consider assertion (iii). It follows from the definitions that α induces an isomorphism $M_X \xrightarrow{\sim} M_Y$. If, moreover, $Z'_X \to X$, $Z'_Y \to Y$ are *diagonal coverings* corresponding to [connected] finite étale coverings $X' \to X$, $Y' \to Y$ that arise from open subgroups of Π_X, Π_Y that correspond via α, then α induces an isomorphism of group cohomology modules

$$H^2(\Pi_{Z'_X}, M_X) \xrightarrow{\sim} H^2(\Pi_{Z'_Y}, M_Y)$$

that preserves the extension classes associated to *fundamental extensions* of $\Pi_{Z'_X}$, $\Pi_{Z'_Y}$ [cf. Proposition 1.6, (i)]. In particular, if $\mathcal{D}'$ (respectively, $\mathcal{E}'$) is a fundamental extension of $\Pi_{Z'_X}$ (respectively, $\Pi_{Z'_Y}$), then α induces an isomorphism

$$\mathcal{D}' \xrightarrow{\sim} \mathcal{E}'$$

which is compatible with the morphisms $M_X \xrightarrow{\sim} M_Y$, $\Pi_{Z'_X} \xrightarrow{\sim} \Pi_{Z'_Y}$ already induced by α, and, moreover, *uniquely determined*, up to cyclotomically inner automorphisms, and the action of $(k_X^\times)^\wedge$ (respectively, $(k_Y^\times)^\wedge$) [cf. Proposition 1.4, (ii)]. On the other hand, by allowing X', Y' to *vary*, taking *symmetrizations* of the fundamental extensions involved [which may be constructed entirely *group-theoretically*!], and making use of the vertical morphism in the center of the diagram of Proposition 1.9, (ii) [again an object which may be constructed entirely *group-theoretically*!], it follows from Proposition 1.9, (iii), that the indeterminacy of the isomorphism

$\mathcal{D}' \xrightarrow{\sim} \mathcal{E}'$ arising from the action of $(k_X^\times)^\wedge$, $(k_Y^\times)^\wedge$ *"converges to the identity indeterminacy"* [i.e., by taking $\mathcal{D}' \xrightarrow{\sim} \mathcal{E}'$ to arise as just described from an isomorphism of fundamental extensions $\mathcal{D}'' \xrightarrow{\sim} \mathcal{E}''$ associated to [connected] finite étale coverings $X'' \to X'$, $Y'' \to Y'$ [that arise from open subgroups of Π_X, Π_Y that correspond via α], where the open subgroups $\Pi_{X''} \subseteq \Pi_{X'}$, $\Pi_{Y''} \subseteq \Pi_{Y'}$ are *sufficiently small*]. Thus, in light of the manifest *functoriality* of the vertical morphism in the center of the diagram of Proposition 1.9, (ii) [the detailed explication of which, in terms of various commutative diagrams, is a routine task which we leave to the reader!], we obtain an isomorphism

$$\{\mathcal{S}_i\} \xrightarrow{\sim} \{\mathcal{T}_j\}$$

of *pro-symmetrized fundamental extensions* [cf. Definition 1.10, (iii)] of $\Pi_{X\times X}$, $\Pi_{Y\times Y}$, respectively, which arises from α and is *completely determined up to cyclotomically inner automorphisms*. Here, we pause to note that although in the construction of the symmetrization of a fundamental extension $\mathcal{D}'$ (respectively, $\mathcal{E}'$), one must, a priori, contend with a certain *indeterminacy* with respect to $\Delta_{X'} \times \{1\}$-(respectively, $\Delta_{Y'} \times \{1\}$-)inner automorphisms [cf., e.g., Proposition 1.9, (ii)], in fact, by allowing X', Y' to vary, this indeterminacy also *"converges to the identity indeterminacy"* [cf. Remark 1.9.1].

Thus, in summary, α induces an *isomorphism* [well-defined up to *cyclotomically* [or, alternatively, *cuspidally*] *inner automorphisms*]

$$\mathcal{S}_\infty \xrightarrow{\sim} \mathcal{T}_\infty$$

of *pro-fundamental extensions* of $\Pi_{X\times X}$, $\Pi_{Y\times Y}$, respectively. Moreover, by applying the fact that the left-hand square of the commutative diagram of Proposition 1.12, (ii), is *cartesian*, together with the fact that the *"canonical section"* of "$\zeta'_{\neq}$" that appears in Proposition 1.12, (iii), is *completely determined* [cf. Proposition 1.12, (v); Lemma 1.11] by the condition that it *lie under an arbitrary "equivariant section"* [cf. Proposition 1.12, (iv)] of the "$\zeta''_{\neq}$" associated to coverings "$X'' \to X'$" arising from arbitrarily small open subgroups $\Pi_{X''} \subseteq \Pi_X$, it follows that the isomorphism $\mathcal{S}_\infty \xrightarrow{\sim} \mathcal{T}_\infty$ just obtained is *compatible* with the *pro-fundamental inclusions* $D_X \hookrightarrow \mathcal{S}_\infty$, $D_Y \hookrightarrow \mathcal{T}_\infty$. In particular, by Proposition 1.14, (ii) [cf. also Proposition 1.12, (i)], we conclude that α induces an *isomorphism* [well-defined up to *cuspidally inner automorphisms*]

$$(\mathcal{S}_\infty \cong\ )\quad \Pi^{\text{c-ab}}_{U_{X\times X}} \xrightarrow{\sim} \Pi^{\text{c-ab}}_{U_{Y\times Y}} \quad (\ \cong \mathcal{T}_\infty)$$

which is compatible with *the natural inclusions* $D_X \hookrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}}$, $D_Y \hookrightarrow \Pi^{\text{c-ab}}_{U_{Y\times Y}}$. Finally, the *functoriality* of this isomorphism follows from the naturality of its construction. ◯

Remark 1.16.1. It follows immediately from the *naturality* of the constructions used in the proof of Theorem 1.16, (iii), that when "α" arises from an *isomorphism of schemes* $X \xrightarrow{\sim} Y$, the resulting $\alpha^{\text{c-ab}}$ of Theorem 1.16, (iii), coincides with the morphism induced on fundamental groups by the resulting isomorphism of schemes $U_{X\times X} \xrightarrow{\sim} U_{Y\times Y}$.

Lemma 1.17. **(Normal Subgroups of the Absolute Galois Group of a Nonarchimedean Local Field)** *Let k be a* **nonarchimedean local field** *of residue characteristic p; write G_k for the* **absolute Galois group** *of k. Also, let us write $I \subseteq G_k$ for the* **inertia subgroup** *of G_k and $W \subseteq I$ for the* **wild inertia subgroup**. *[Here, we recall that W is the unique Sylow pro-p subgroup of I.] Let $H \subseteq G_k$ be a* **closed subgroup** *that satisfies [at least] one of the following four conditions:*

(a) *H is a* **finite group**.

(b) *H* **commutes** *with W.*

(c) *H is a* **pro-prime-to-p group** *[i.e., the order of every finite quotient group of H is prime to p] that is* **normal** *in G_k.*

(d) *H is a* **topologically finitely generated pro-p group** *that is* **normal** *in G_k.*

Then $H = \{1\}$.

Proof. Indeed, suppose that H satisfies *condition (a)*. Then the fact that $H = \{1\}$ follows from [NSW], Corollary 12.1.3, Theorem 12.1.7. Now suppose that H satisfies *condition (b)*. Then by the well-known functorial isomorphism [arising from *local class field theory*] between the additive group underlying a finite field extension of k that corresponds to an open subgroup $J \subseteq G_k$ and the tensor product with $\mathbb{Q}_p$ of the image of $W \bigcap J$ in the abelianization J^{ab}, it follows immediately that the conjugation action of H on W is *nontrivial*, whenever H is *nontrivial*. Thus we conclude again that $H = \{1\}$. Next, suppose that H satisfies *condition (c)*. Then since H, W are both *normal* in G_k, it follows [by considering commutators of elements of H with elements of W] that arbitrary elements of H *commute* with arbitrary elements of W. In particular, H satisfies condition (b), so we conclude yet again that $H = \{1\}$.

Finally, we assume that H is *nontrivial* and satisfies *condition (d)*. Write $G_k \twoheadrightarrow G$ for the *maximal pro-p quotient* of G_k. By replacing k by a suitable finite extension of k, and applying the fact that H is *infinite* [since we have already shown that H does not satisfy condition (a)], we may assume without loss of generality that G is a *free pro-p group* [cf., e.g., [NSW], Theorem 7.5.8, (i)] such that the natural map

$$H^{\mathrm{ab}} \otimes \mathbb{F}_p \to G^{\mathrm{ab}} \otimes \mathbb{F}_p$$

is *injective*, but *not surjective*. Then it follows immediately from the well-known *theory of free pro-p groups* that there exists a set of *free topological generators* $\xi_1, \dots, \xi_n$ of G [where $n = \dim_{\mathbb{F}_p}(G^{\mathrm{ab}} \otimes \mathbb{F}_p)$] such that for some integer $1 \leq m < n$, $\xi_1, \dots, \xi_m$ lie in and topologically generate the image $\mathrm{Im}(H) \subseteq G$ of H in G. On the other hand, since $\mathrm{Im}(H)$ is *normal* in G, it follows from the well-known structure of free pro-p groups that we obtain a contradiction. This completes the proof of Lemma 1.17. $\bigcirc$

Remark 1.17.1. The author would like to thank *A. Tamagawa* for informing him of the content of Lemma 1.17.

Definition 1.18. In the situation of Theorem 1.16, (i), (ii):

(i) If, for every finite étale covering $X' \to X$ of X arising from an open subgroup $\Pi_{X'} \subseteq \Pi_X$, it holds that the map from $(X')^{\mathrm{cl}+}$ to conjugacy classes of closed subgroups of $\Pi_{X'}$ given by assigning to a closed point its associated decomposition group is *injective*, then we shall say that X is Σ*-separated*.

(ii) If the map induced by α on closed subgroups of Π_X, Π_Y induces a bijection between the decomposition groups of the points of $X^{\mathrm{cl}+}$, $Y^{\mathrm{cl}+}$, then we shall say that α is *quasi-point-theoretic*. If α is *quasi-point-theoretic*, and, moreover, X, Y are Σ*-separated* — in which case α induces *bijections*

$$X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}; \quad X^{\mathrm{cl}+} \xrightarrow{\sim} Y^{\mathrm{cl}+}$$

— then we shall say that α is *point-theoretic*.

(iii) Suppose further that we are in the *finite field case*. Then we shall say that α is *Frobenius-preserving* if the isomorphism $G_{k_X} \xrightarrow{\sim} G_{k_Y}$ induced by α [cf. Theorem 1.16, (ii)] maps the Frobenius element of G_{k_X} to the Frobenius element of G_{k_Y}.

Remark 1.18.1. In the *finite field case*, when $\Sigma^\dagger = \mathfrak{Primes}^\dagger$, the Frobenius element of G_{k_X} may be characterized as in [Tama], Proposition 3.4, (i), (ii); a similar statement holds for the Frobenius element of G_{k_Y}. [Moreover, in the *proper* case, the Frobenius element of G_{k_X} may be characterized as the element of G_{k_X} that acts on M_X via multiplication by the cardinality of k_X, i.e., the cardinality of $H^1(G_{k_X}, M_X)$ plus 1.] Thus, when $\Sigma^\dagger = \mathfrak{Primes}^\dagger$, any α as in Theorem 1.16, (i), (ii), is *automatically Frobenius-preserving*.

Remark 1.18.2. Note that in the *finite field case*, *any Frobenius-preserving* isomorphism α as in Theorem 1.16, (i), (ii), is *quasi-point-theoretic* [cf. the arguments of [Tama], Corollary 2.10, Proposition 3.8].

Remark 1.18.3. Note that in the *finite field case*, if α as in Theorem 1.16, (i), (ii), is *Frobenius-preserving*, then the *characteristics* of k_X, k_Y *coincide*. Indeed, this follows immediately from Theorem 1.16, (i), (ii); [Tama], Proposition 3.4, (i), (ii).

Now we return to the notation of the *discussion preceding Theorem 1.16*. Observe that the automorphism

$$\tau : X \times X \to X \times X$$

given by *switching the two factors* induces an outer automorphism of $\Pi_{U_{X\times X}}$. Moreover, by choosing the basepoints used to form the various fundamental groups involved in an appropriate fashion, it follows that there exists an automorphism

$$\Pi_\tau : \Pi_{U_{X\times X}} \to \Pi_{U_{X\times X}}$$

among those automorphisms induced by τ [i.e., all of which are related to one another by composition with an inner automorphism] which induces the automorphism on $\Pi_{X\times X} = \Pi_X \times_{G_k} \Pi_X$ given by *switching the two factors*; *preserves* the subgroup $D_X \subseteq \Pi_{U_{X\times X}}$; and *preserves and induces the identity automorphism* on the subgroup $I_X \subseteq D_X$ ($\subseteq \Pi_{U_{X\times X}}$). Note that by the *slimness* of Proposition 1.8, (i), together with the well-known *commensurable terminality* of $D_X \subseteq \Pi_{U_{X\times X}}$ in $\Pi_{U_{X\times X}}$ [cf., e.g., [the proof of] [Mzk5], Lemma 1.3.12], it follows that these three conditions [are more than sufficient to] *determine* Π_τ, up to composition with an inner automorphism arising from I_X.

Proposition 1.19. **(Switching the Two Factors)** *The automorphism*

$$\Pi_\tau^{\text{c-ab}} : \Pi_{U_{X\times X}}^{\text{c-ab}} \to \Pi_{U_{X\times X}}^{\text{c-ab}}$$

induced by Π_τ *is the* **unique** *automorphism of the profinite group* $\Pi_{U_{X\times X}}^{\text{c-ab}}$*, up to composition with a cyclotomically inner automorphism, that satisfies the following two conditions: (a) it preserves the quotient* $\Pi_{U_{X\times X}}^{\text{c-ab}} \twoheadrightarrow \Pi_{X\times X}$ *and induces on this quotient the automorphism on* $\Pi_{X\times X} = \Pi_X \times_{G_k} \Pi_X$ *given by switching the two factors; (b) it preserves the image of* $I_X \subseteq D_X \hookrightarrow \Pi_{U_{X\times X}}^{\text{c-ab}}$.

Proof. This follows immediately from Propositions 1.15, (i). ○

Section 2: Points and Functions

We maintain the notation of §1 [i.e., the discussion preceding Theorem 1.16]. If $x \in X^{\mathrm{cl}}$, then we shall denote by

$$D_x \subseteq \Pi_X$$

the *decomposition group* of x [well-defined up to conjugation by an element of Δ_X]. If $x \in X(k)$, then D_x determines a section $s_x : G_k \to \Pi_X$ [which is well-defined as a geometrically outer homomorphism].

Next, let $S \subseteq X^{\mathrm{cl}}$ be a *finite set.* If n is a $\Sigma^\dagger$-integer [cf. §0], then the *Kummer exact sequence*

$$1 \to \boldsymbol{\mu}_n \to \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}} \to 1$$

[where $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ is the n-th power map; $\boldsymbol{\mu}_n$ is defined so as to make the sequence exact] on the étale site of X determines a homomorphism $\mathrm{Pic}(X) \to H^2(\Delta_X, \boldsymbol{\mu}_n)$ [where $\mathrm{Pic}(X)$ is the Picard group of X]. Now there is a *unique isomorphism*

$$\boldsymbol{\mu}_n \xrightarrow{\sim} M_X/n \cdot M_X$$

such that the homomorphism $\mathrm{Pic}(X) \to H^2(\Delta_X, \boldsymbol{\mu}_n)$ sends line bundles of degree 1 to the element determined by $1 \in \mathbb{Z}/n\mathbb{Z}$ via the composite of the induced isomorphism $H^2(\Delta_X, \boldsymbol{\mu}_n) \xrightarrow{\sim} H^2(\Delta_X, M_X/n \cdot M_X)$ with the *tautological isomorphism* $H^2(\Delta_X, M_X/n \cdot M_X) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ [cf. Proposition 1.2, (i)]. In the following discussion, we shall identify $\boldsymbol{\mu}_n$ with $M_X/n \cdot M_X$ via this isomorphism.

If we consider the Kummer exact sequence on the étale site of $U_S \subseteq X$ [and pass to the inverse limit with respect to n], then we obtain a *natural homomorphism*

$$\Gamma(U_S, \mathcal{O}_{U_S}^\times) \to H^1(\Pi_{U_S}, M_X)$$

[where we note that here, it suffices to consider the group cohomology of Π_{U_S} [i.e., as opposed to the étale cohomology of U_S], since the extraction of n-th roots of an element of $\Gamma(U_S, \mathcal{O}_{U_S}^\times)$ yields finite étale coverings of U_S that correspond to open subgroups of Π_{U_S}] which is *injective* [since the abelian group $\Gamma(U_S, \mathcal{O}_{U_S}^\times)$ is clearly finitely generated and free of $p^\dagger$-torsion, hence *injects* into its prime-to-$p^\dagger$ completion] whenever $\Sigma^\dagger = \mathfrak{Primes}^\dagger$. In particular, by allowing S to *vary* we obtain a *natural homomorphism*

$$K_X^\times \to \varinjlim_{S} H^1(\Pi_{U_S}, M_X)$$

[where K_X is the *function field* of X; the direct limit is over all finite subsets S of X^{cl}] which is *injective* whenever $\Sigma^\dagger = \mathfrak{Primes}^\dagger$.

Proposition 2.1. **(Kummer Classes of Functions)** *If $S \subseteq X^{\mathrm{cl}}$ is a finite subset, write*

$$\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{\text{c-ab}} \twoheadrightarrow \Delta_{U_S}^{\text{c-cn}}$$

for the **maximal cuspidally abelian** *and* **maximal cuspidally central** *quotients, respectively, and*

$$\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\text{c-ab}} \twoheadrightarrow \Pi_{U_S}^{\text{c-cn}}$$

for the corresponding quotients of Π_{U_S}*. If* $x \in X^{\text{cl}}$*, then let us write*

$$D_x[U_S] \subseteq \Pi_{U_S}$$

for the **decomposition group** *of* x *in* Π_{U_S} *[which is well-defined up to conjugation by elements of* Δ_{U_S}*] and* $I_x[U_S] \subseteq D_x[U_S]$ *for the inertia subgroup. [Thus, when* $x \in S$*, we obtain [cf. Proposition 1.6, (ii), (iii)] a natural isomorphism of* M_X *with* $I_x[U_S] \stackrel{\text{def}}{=} D_x[U_S] \bigcap \Delta_{U_S}$*.]*

(i) The natural surjections induce **isomorphisms** *as follows:*

$$H^1(\Pi_{U_S}^{\text{c-cn}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}^{\text{c-ab}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}, M_X)$$

In particular, we obtain **natural homomorphisms** *as follows:*

$$\Gamma(U_S, \mathcal{O}_{U_S}^{\times}) \to H^1(\Pi_{U_S}^{\text{c-cn}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}^{\text{c-ab}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}, M_X)$$

$$K_X^{\times} \to \varinjlim_{S} H^1(\Pi_{U_S}^{\text{c-cn}}, M_X) \xrightarrow{\sim} \varinjlim_{S} H^1(\Pi_{U_S}^{\text{c-ab}}, M_X) \xrightarrow{\sim} \varinjlim_{S} H^1(\Pi_{U_S}, M_X)$$

These natural homomorphisms are **injective** *whenever* $\Sigma^\dagger = \mathfrak{Primes}^\dagger$*.*

(ii) Suppose that $S \subseteq X(k)$ *is a finite subset. Then restricting cohomology classes of* Π_{U_S} *to the various* $I_x[U_S]$*, for* $x \in S$*, yields a natural exact sequence*

$$1 \to (k^{\times})^{\wedge} \to H^1(\Pi_{U_S}, M_X) \to \Big(\bigoplus_{x \in S} \widehat{\mathbb{Z}}^\dagger\Big)$$

[where we identify $\text{Hom}_{\widehat{\mathbb{Z}}^\dagger}(I_x[U_S], M_X)$ *with* $\widehat{\mathbb{Z}}^\dagger$*]. Moreover, the image [via the natural homomorphism given in (i)] of* $\Gamma(U_S, \mathcal{O}_{U_S}^{\times})$ *in* $H^1(\Pi_{U_S}, M_X)$ *is equal to the inverse image in* $H^1(\Pi_{U_S}, M_X)$ *of the submodule of*

$$\Big(\bigoplus_{x \in S} \mathbb{Z}\Big) \subseteq \Big(\bigoplus_{x \in S} \widehat{\mathbb{Z}}^\dagger\Big)$$

determined by the **principal divisors** *[with support in* S*]. A similar statement holds when "*Π_{U_S}*" is replaced by "*$\Pi_{U_S}^{\text{c-ab}}$*" or "*$\Pi_{U_S}^{\text{c-cn}}$*".*

(iii) If $f \in \Gamma(U_S, \mathcal{O}_{U_S}^{\times})$*, write*

$$\kappa_f^{\text{c-cn}} \in H^1(\Pi_{U_S}^{\text{c-cn}}, M_X); \quad \kappa_f^{\text{c-ab}} \in H^1(\Pi_{U_S}^{\text{c-ab}}, M_X); \quad \kappa_f \in H^1(\Pi_{U_S}, M_X)$$

for the associated Kummer classes. If $x \in X^{\text{cl}} \backslash S$*, then* $D_x[U_S]$ *maps, via the natural surjection* $\Pi_{U_S} \twoheadrightarrow G_k$*, isomorphically onto the open subgroup* $G_{k(x)} \subseteq G_k$

[where $k(x)$ is the residue field of X at x]. Moreover, the images of the pulled back classes

$$\kappa_f^{\text{c-cn}}|_{D_x[U_S]} = \kappa_f^{\text{c-ab}}|_{D_x[U_S]} = \kappa_f|_{D_x[U_S]} \in H^1(D_x[U_S], M_X) \xrightarrow{\sim} H^1(G_{k(x)}, M_X) \xrightarrow{\sim} (k(x)^\times)^\wedge$$

in $(k(x)^\times)^\wedge$ are equal to the image in $(k(x)^\times)^\wedge$ of the **value** *of f at x.*

Proof. Assertion (i) follows immediately from the definitions. The exact sequence of assertion (ii) follows immediately from Proposition 1.4, (ii). The characterization of the image of $\Gamma(U_S, \mathcal{O}_{U_S}^\times)$ is immediate from the definitions and the exact sequence of assertion (ii). Assertion (iii) follows immediately from the definitions and the functoriality of the Kummer class. ○

Remark 2.1.1. If, in the situation of Proposition 2.1, (iii), we think of the extension of $\Pi_{U_S}^{\text{c-cn}}$ of Π_X as being given by the extension $\mathcal{D}_S$ [cf. Proposition 1.8, (iii)], where $\mathcal{D}$ is a *fundamental extension* of $\Pi_{X\times X}$ that appears as a *quotient* of $\Pi_{U_{X\times X}}$ [hence is *"rigid"* with respect to the action of $(k^\times)^\wedge$ — cf. Proposition 1.9, (iii); the proof of Theorem 1.16, (iii)], then it follows that the image of $D_x[U_S]$ in $\Pi_{U_S}^{\text{c-cn}}$ may be thought of as the image of $D_x[U_S]$ in $\mathcal{D}_S$. If, moreover, we assume, for simplicity, that $x \in X(k)$, $S \subseteq X(k)$, then this image of $D_x[U_S]$ in $\mathcal{D}_S$ amounts to a *section* of $\mathcal{D}_S \twoheadrightarrow \Pi_X \twoheadrightarrow G_k$ lying over the section s_x of $\Pi_X \twoheadrightarrow G_k$. Since $\mathcal{D}_S$ is defined as a certain *fiber product*, this section is equivalent to a *collection of sections* [regarded as cyclotomically outer homomorphisms]

$$\gamma_{y,x} : G_k \to \mathcal{D}_{y,x}$$

[where y ranges over the points of S]. [Here, we note that it is immediate from the definitions that, as the notation suggests, $\gamma_{y,x}$ depends only on x, y — i.e., that $\gamma_{y,x}$ is *independent* of the choice of S.] That is to say, from this point of view, Proposition 2.1, (iii), may be regarded as stating that:

> The image in $(k^\times)^\wedge = (k(x)^\times)^\wedge$ of the *value of a function* $\in \Gamma(U_S, \mathcal{O}_{U_S}^\times)$ at $x \in X(k)$ may be computed from its *Kummer class*, as soon as one knows the *sections* $\gamma_{y,x} : G_k \to \mathcal{D}_{y,x}$, for $y \in S$.

Also, before proceeding, we note that an arbitrary section of $\mathcal{D}_{y,x} \twoheadrightarrow G_k$ differs [as a cyclotomically outer homomorphism] from $\gamma_{y,x}$ by the action of an element of $H^1(G_k, M_X) \xrightarrow{\sim} (k^\times)^\wedge$. Thus, the datum of "$\gamma_{y,x}$" may be regarded as a *trivialization of a certain $(k^\times)^\wedge$-torsor.*

Remark 2.1.2. The finite field portion of Proposition 2.1 may be regarded as the evident finite field analogue of [a certain portion of] the theory of [Mzk8], §4. Also, we observe that the approach of "reconstructing the function field of the curve via *Kummer theory*, as opposed to *class field theory* [as was done in [Tama], [Uchi]]" has the advantage of being applicable to *nonarchimedean local fields*, as well as to finite fields.

Definition 2.2. For $x, y \in X(k)$, we shall refer to the section [regarded as a cyclotomically outer homomorphism]

$$\gamma_{y,x} : G_k \to \mathcal{D}_{y,x}$$

as the *Green's trivialization* of $\mathcal{D}$ at (y, x). If D is a divisor on X supported in the subset of k-rational points $X(k) \subseteq X^{\mathrm{cl}}$, then multiplication of the various Green's trivializations for the points in the support of D determines a section [regarded as a cyclotomically outer homomorphism]

$$\gamma_{D,x} : G_k \to \mathcal{D}_{D,x}$$

which we shall refer to as the *Green's trivialization* of $\mathcal{D}$ at (D, x). [Note that the definition of $\gamma_{D,x}$ generalizes immediately to the case where the divisor D, but not necessarily the points in its support, is rational over k — cf. Remark 1.10.1.]

Remark 2.2.1. The terminology of Definition 2.2, is intended to suggest the similarity between the $\gamma_{y,x}$ of the present discussion and the *"Green's functions"* that occur in the theory of *bipermissible metrics* — cf., e.g., [MB], §4.11.4.

Remark 2.2.2. Note that the Green's trivializations are *symmetric* with respect to the involution of $\mathcal{D}$ induced by the automorphism $\Pi_\tau^{\text{c-ab}}$ of Proposition 1.19. Indeed, relative to the natural projections

$$\Pi_{U_{X\times X}} \twoheadrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}} \twoheadrightarrow \mathcal{D}$$

the Green's trivialization at (y, x) is simply the section of $\mathcal{D} \twoheadrightarrow G_k$ arising [by composition] from the section of $\Pi_{U_{X\times X}} \twoheadrightarrow G_k$ determined by the *decomposition group* of the point $(x, y) \in U_{X\times X}(k)$. Thus, the asserted symmetry of the Green's trivializations follows from the fact that $\Pi_\tau^{\text{c-ab}}$ is compatible with Π_τ, together with the evident fact that [by "transport of structure"] Π_τ maps the decomposition group of $(x, y) \in U_{X\times X}(k)$ isomorphically onto the decomposition group of $(y, x) \in U_{X\times X}(k)$.

If $d \in \mathbb{Z}$, denote by J^d the subscheme of the *Picard scheme* of X that parametrizes line bundles of *degree* d; write $J \stackrel{\mathrm{def}}{=} J^0$. Thus, J^d is a *torsor over* J. Note that there is a natural morphism $X \to J^1$ [given by assigning to a point of X the line bundle of degree 1 determined by the point]. Thus, the basepoint of X [already chosen in §1] determines a basepoint of J^1. At the level of *"geometrically pro-*Σ*" étale fundamental groups*, this morphism induces a surjective homomorphism

$$\Pi_X \twoheadrightarrow \Pi_{J^1}$$

whose kernel is the kernel of the maximal abelian quotient $\Delta_X \twoheadrightarrow \Delta_X^{\mathrm{ab}}$. In particular, for $x \in X(k)$, the section s_x determines a section $t_x : G_k \to \Pi_{J^1}$. Note that applying the "change of structure group" given by the "multiplication by d map" on J to the J-torsor J^1 yields the J-torsor J^d. [Indeed, this follows by considering the group structure of the Picard scheme.] Thus, we obtain a morphism $J^1 \to J^d$ whose induced morphism on fundamental groups

$$\Pi_{J^1} \to \Pi_{J^d}$$

determines an *isomorphism* of Π_{J^d} with the *push-forward* of the extension Π_{J^1} [i.e., of G_k by $\Delta_{J^1} \cong \Delta_X^{\mathrm{ab}}$] via the homomorphism $\Delta_X^{\mathrm{ab}} \to \Delta_X^{\mathrm{ab}}$ given by *multiplication by d*. When $d \geq 1$, the group structure on the Picard scheme also determines a morphism

$$\prod \Pi_{J^1} \to \Pi_{J^d}$$

[where the product is a fiber product over G_k of d factors of Π_{J^1}] which determines an *isomorphism* of Π_{J^d} with the *push-forward* of the extension constituted by the fiber product via the homomorphism $\prod \Delta_X^{\mathrm{ab}} \to \Delta_X^{\mathrm{ab}}$ [i.e., from a product of d copies of Δ_X^{ab} to Δ_X^{ab} given by adding up the d components]. Moreover, one verifies immediately that when $d \geq 1$, these two constructions of "Π_{J^d}" from Π_{J^1} yield groups that are *naturally isomorphic*.

Thus, by applying the various homomorphisms induced on fundamental groups by the group structure of the Picard scheme, it follows that if D is any *divisor of degree* d on X whose support lies in the set of k-rational points $X(k) \subseteq X^{\mathrm{cl}}$, then D determines a section

$$t_D : G_k \to \Pi_{J^d}$$

which may be constructed *entirely group-theoretically*. In particular, if D is of *degree* 0, then the section $t_D : G_k \to \Pi_J$ may be compared with the *identity section* of Π_J to obtain a cohomology class:

$$\eta_D \in H^1(G_k, \Delta_X^{\mathrm{ab}})$$

Now we have the following well-known result:

Proposition 2.3. **(Points and Galois Sections)** *Suppose that $\Sigma = \mathfrak{Primes}$. Then, in the notation of the above discussion:*

(i) The divisor D is **principal** *if and only if $\eta_D = 0$.*

(ii) The map $x \mapsto D_x$ from X^{cl} to conjugacy classes of closed subgroups of Π_X is **injective**, *i.e., X is* **$\mathfrak{Primes}$-separated**.

Proof. First, we consider assertion (i). By well-known general nonsense [cf., e.g., [Naka], Claim (2.2); [NTs], Lemma (4.14); [Mzk4], the Remark preceding Definition 6.2], there is a *natural isomorphism*

$$H^1(k, \Delta_X^{\mathrm{ab}}) \xrightarrow{\sim} J(k)^\wedge \ (\supseteq J(k))$$

[where the "$\wedge$" denotes the profinite completion] which maps η_D to the element of $J(k)$ determined by D. [Here, we recall that this natural isomorphism arises by considering the long exact sequence obtained by applying the functors $H^*(G_k, -)$ to the short exact sequence of G_k-modules

$$1 \to J(\overline{k})[n] \to J(\overline{k}) \to J(\overline{k}) \to 1$$

— where n is a positive integer; the morphism $J(\overline{k}) \to J(\overline{k})$ is the "multiplication by n map"; $J(\overline{k})[n]$ is defined so as to make the sequence exact.] Thus, assertion (i) follows immediately.

To prove assertion (ii), it suffices [by possibly base-changing to a finite extension of k] to verify that two points $x_1, x_2 \in X(k)$ that induce Δ_X-conjugate sections s_{x_1}, s_{x_2} are necessarily equal [cf. also [Tama], Corollary 2.10]. But this follows formally from assertion (i), by considering the divisor $x_1 - x_2$ [and the well-known fact that the natural morphism $X \to J^1$ considered above is an *embedding*]. $\bigcirc$

Remark 2.3.1. From the point of view of Definition 1.7, (ii), the reader may feel tempted to expect that [still under the assumption that $\Sigma = \mathfrak{Primes}$] D is principal if and only if the extension $\mathcal{D}_D$ of Π_X [by M_X] is *trivial* [i.e., determines the zero class in $H^2(\Pi_X, M_X)$]. When k is *nonarchimedean local*, it is not difficult to verify, using Proposition 2.3, (i), that this is indeed the case. On the other hand, when k is *finite*, although this condition for principality is easily verified to be *necessary*, it is *not*, however, *sufficient*, since it only involves the *"prime-to-$p^\dagger$ portion"* of the point of $J(k)$ determined by D.

Definition 2.4. In the situation of Theorem 1.16, (iii), suppose that α is *point-theoretic*. Let $S \subseteq X^{\mathrm{cl}}$ be a [not necessarily finite] subset that corresponds via the bijection $X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$ induced by [the point-theoreticity of] α to a subset $T \subseteq Y^{\mathrm{cl}}$.

(i) Write $\mathcal{D}$ (respectively, $\mathcal{E}$) for the fundamental extension of $\Pi_{X \times X}$ (respectively, $\Pi_{Y \times Y}$) that arises as the quotient of $\Pi^{\text{c-ab}}_{U_{X \times X}}$ (respectively, $\Pi^{\text{c-ab}}_{U_{Y \times Y}}$) by the kernel of the *maximal cuspidally central quotient* $\Delta^{\text{c-ab}}_{U_{X \times X}} \twoheadrightarrow \Delta^{\text{c-cn}}_{U_{X \times X}}$ (respectively, $\Delta^{\text{c-ab}}_{U_{Y \times Y}} \twoheadrightarrow \Delta^{\text{c-cn}}_{U_{Y \times Y}}$) [cf. Proposition 1.8, (iv)]. Thus, $\alpha^{\text{c-ab}}$ induces an isomorphism:

$$\alpha^{\text{c-cn}} : \mathcal{D} \xrightarrow{\sim} \mathcal{E}$$

We shall say that α is *(S, T)-locally Green-compatible* if, for every pair of points $(x_1, x_2) \in X(k_X) \times X(k_X)$ corresponding via the bijection induced by α to a pair of points $(y_1, y_2) \in Y(k_Y) \times Y(k_Y)$, such that $x_2 \in S$, $y_2 \in T$, the isomorphism

$$\mathcal{D}_{x_1, x_2} \xrightarrow{\sim} \mathcal{E}_{y_1, y_2}$$

[obtained by restricting $\alpha^{\text{c-cn}}$] is compatible with the *Green's trivializations*. We shall say that α is *(S, T)-locally degree zero* (respectively, *(S, T)-locally principally*) *Green-compatible* if, for every $x \in X(k_X) \bigcap S$ and every divisor of degree zero (respectively, principal divisor) D supported in $X(k_X) \subseteq X^{\mathrm{cl}}$ corresponding via the bijection induced by α to a pair (y, E) of Y [so $y \in Y(k_Y) \bigcap T$], the isomorphism

$$\mathcal{D}_{D,x} \xrightarrow{\sim} \mathcal{E}_{E,y}$$

is compatible with the *Green's trivializations.*

(ii) We shall say that α is *totally* (S, T)*-locally Green-compatible* (respectively, *totally* (S, T)*-locally degree zero Green-compatible*; *totally* (S, T)*-locally principally Green-compatible*) if, for all pairs of connected finite étale coverings $X' \to X$, $Y' \to Y$ that arise from open subgroups of Π_X, Π_Y that correspond via α, the isomorphism

$$\Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$$

induced by α is (S', T')-locally Green-compatible (respectively, (S', T')-locally degree zero Green-compatible; (S', T')-locally principally Green-compatible), where $S' \subseteq (X')^{\mathrm{cl}}$, $T' \subseteq (Y')^{\mathrm{cl}}$ are the inverse images in X', Y' of S, T, respectively.

(iii) With respect to the terminology introduced in (i), (ii), when $S = X^{\mathrm{cl}}$, $T = Y^{\mathrm{cl}}$, then we shall replace the phrase "(S, T)-locally" by the phrase *"globally"*.

Remark 2.4.1. In the situation of Definition 2.4, if $X' \to X$, $Y' \to Y$ are connected finite étale coverings that arise from open subgroups of Π_X, Π_Y that correspond via α; $\mathcal{D} \xrightarrow{\sim} \mathcal{E}$ is the isomorphism of fundamental extensions of $\Pi_{X \times X}$, $\Pi_{Y \times Y}$ that arises from the isomorphism $\alpha^{\text{c-ab}}$ of Theorem 1.16, (iii); and the points x_1, x_2 (respectively, y_1, y_2) are Δ_X- (respectively, Δ_Y-) *conjugate*, then it follows immediately from the *compatibility* of $\alpha^{\text{c-ab}}$ with the natural inclusions $D_X \hookrightarrow \Pi^{\text{c-ab}}_{U_{X \times X}}$, $D_Y \hookrightarrow \Pi^{\text{c-ab}}_{U_{Y \times Y}}$ [cf. Theorem 1.16, (iii)] that the isomorphism $\mathcal{D}_{x_1,x_2} \xrightarrow{\sim} \mathcal{E}_{y_1,y_2}$ is *automatically* compatible with the *Green's trivializations.* [Indeed, this follows from the easily verified fact that the Green's trivializations in this case are, in essence, specializations of the *"canonical sections of* $\zeta'_{\neq}$*"* of Proposition 1.12.] Unfortunately, however, the author is unable, at the time of writing, to see how to generalize the argument applied in the proof of Theorem 1.16, (iii), involving Lemma 1.11; Proposition 1.12, (v), so as to cover the case where the points x_1, x_2 (respectively, y_1, y_2) *fail to be* Δ_X- (respectively, Δ_Y-) *conjugate.* Indeed, this sort of generalization appears to require the group-theoretic reconstructibility of some collection of *isomorphisms of extensions of* G_{k_X}, G_{k_Y} by M_X, M_Y, respectively,

$$\mathcal{D}_{x_1,x_2} \xrightarrow{\sim} \mathcal{D}_{x_1,x_3}; \quad \mathcal{E}_{y_1,y_2} \xrightarrow{\sim} \mathcal{E}_{y_1,y_3}$$

that are compatible both with α and with the respective Green's trivializations, for all collections of points $x_1, x_2, x_3 \in X(k)$ (respectively, $y_1, y_2, y_3 \in Y(k)$) such that x_2, x_3 (respectively, y_2, y_3) are Δ_X- (respectively, Δ_Y-) *conjugate.*

Remark 2.4.2. It is immediate that (S, T)-local Green-compatibility (respectively, (S, T)-local degree zero Green-compatibility) implies (S, T)-local degree zero Green-compatibility (respectively, (S, T)-local principal Green-compatibility), and that total (S, T)-local Green-compatibility (respectively, total (S, T)-local degree zero Green-compatibility) implies total (S, T)-local degree zero Green-compatibility (respectively, total (S, T)-local principal Green-compatibility).

Theorem 2.5. **(Reconstruction of Functions)** *In the situation of Theorem 1.16, (iii), suppose further that α is* **point-theoretic**. *Then:*

(i) Let $S \subseteq X^{\mathrm{cl}}$, $T \subseteq Y^{\mathrm{cl}}$ be finite subsets that correspond via the bijection $X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$ induced by α. Then α, $\alpha^{\text{c-ab}}$ induce **isomorphisms** *[well-defined up to* **cuspidally inner automorphisms***]*

$$\Pi_{U_S}^{\text{c-ab}} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab}}$$

[where $V_T \stackrel{\mathrm{def}}{=} Y \backslash T$] lying over α, which are **functorial** *with respect to α and S, T, as well as with respect to passing to* **connected finite étale coverings of X, Y** *[that do not necesarily arise from open subgroups of Π_X, Π_Y!].*

(ii) Suppose that $\Sigma = \mathfrak{Primes}$. Then the bijection $X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$ induced by α induces a bijection between the groups of **principal divisors** *on X, Y. This bijection, together with the isomorphisms of (i), induces a* **compatible isomorphism**

$$K_X^{\times} \cdot (k_X^{\times})^{\wedge} \xrightarrow{\sim} K_Y^{\times} \cdot (k_Y^{\times})^{\wedge}$$

between the push-forwards of the multiplicative groups associated to the **function fields** *of X, Y, relative to the homomorphisms $k_X^{\times} \hookrightarrow (k_X^{\times})^{\wedge}$, $k_Y^{\times} \hookrightarrow (k_Y^{\times})^{\wedge}$.*

Proof. Assertion (i) follows immediately from the definitions; Theorem 1.16, (iii). [Here, we note that the *functoriality* asserted in assertion (i), which is somewhat *stronger* than the functoriality asserted in Theorem 1.16, (iii), follows from the definitions, together with the *naturality* of the constructions applied in the proof of Theorem 1.16, (iii) — cf., e.g., the diagram of Proposition 1.9, (ii).] Assertion (ii) follows immediately from assertion (i); Proposition 2.3, (i); Proposition 2.1, (i), (ii). ◯

Remark 2.5.1. In fact, later in §3, we shall construct, in the *finite field case*, the *crucial isomorphism* $\Pi_{U_S}^{\text{c-ab}} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab}}$ of Theorem 2.5, (i), via a *different technique*, without applying Theorem 1.16, (iii). Thus, from this point of view, Theorem 1.16, (iii), is *not logically necessary* for the proof of the main results of the present paper in the finite field case. Nevertheless, we chose to include the proof of Theorem 1.16, (iii), via Propositions 1.9, 1.12 in the present paper for the following reasons: First of all, unlike the techniques of §3, the techniques of §1 apply to situations [e.g., the case of *nonarchimedean local fields*!] where the *weight filtration* [cf. §3] *does not admit a Galois-invariant splitting.* Indeed, the techniques of §1, essentially only require that the Galois cohomology of the base field admit a natural *duality pairing.* Secondly, even in the finite field case, in light of the importance of this isomorphism $\Pi_{U_S}^{\text{c-ab}} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab}}$ in the theory of the present paper, it is of interest to see that this isomorphism may be constructed via *two fundamentally different approaches.* Thirdly, although the approach of §3 is better suited to the reconstruction of the *Green's trivializations*, it has the drawback that it depends fundamentally on the *choice of a "basepoint"* $x_* \in X(k)$ [cf. the theory of §3, especially the proof of Theorem 3.10]. Thus, it is of interest to know that this isomorphism may be constructed [i.e., via the techniques of §1] *"cohomologically"* [cf. Proposition 1.6, (i)] without making such a choice.

Remark 2.5.2. In the case of *nonarchimedean local fields*, it is natural to ask, in the style of [Mzk8], §4, whether or not various *"canonical integral structures"* on the extensions $\mathcal{D}_{x,y}$ [where $x, y \in X(k)$] of G_k by M_X are preserved by arbitrary isomorphisms of arithmetic fundamental groups. When $x \neq y$, such a canonical integral structure is determined by the *Green's trivialization*; when $x = y$, such a canonical integral structure is determined by the integral structure [in the usual sense of scheme theory] on the canonical sheaf of the stable model of the curve [when the curve has stable reduction] — cf. [Mzk8], §4.

Before proceeding, we note the following *"analogue for* $\Pi^{\text{c-ab}}_{U_S}$*"* of Proposition 1.15, (i):

Proposition 2.6. **(Automorphisms and Commensurators)** *Let* $\Pi^{\text{c-ab}}_{U_S}$ *be as in Theorem 2.5, (i). For* $x \in S$*, write* $D_x[U_S] \hookrightarrow \Pi^{\text{c-ab}}_{U_S}$ *for the natural inclusion. Then:*

(i) Any automorphism α *of the profinite group* $\Pi^{\text{c-ab}}_{U_S}$ *which*

(a) is compatible with the natural surjection $\Pi^{\text{c-ab}}_{U_S} \twoheadrightarrow \Pi_X$ *and induces the identity on* Π_X*;*

(b) for each $x \in S$*, preserves the image of* $M_X \cong I_x[U_S] \subseteq D_x[U_S]$ *via the natural inclusion* $D_x[U_S] \hookrightarrow \Pi^{\text{c-ab}}_{U_S}$

is **cuspidally inner***.*

(ii) Suppose that X *is* Σ**-separated***. Then for* $x \in S$*,* D_x *is* **commensurably terminal** *in* Π_X*.*

(iii) Suppose that X *is* Σ**-separated***. Then the image of* $D_x[U_S] \hookrightarrow \Pi^{\text{c-ab}}_{U_S}$ *is* **commensurably terminal** *in* $\Pi^{\text{c-ab}}_{U_S}$*.*

Proof. First, we observe that assertion (ii) follows formally from the definition of a "decomposition group" and "Σ-separated". Thus, assertion (i) (respectively, (iii)) follows by an argument which is entirely similar to the argument that was used to prove assertion (i) (respectively, (iii)) of Proposition 1.15. ◯

Remark 2.6.1. In the situation of Definition 2.4, suppose that S, T are *finite*, and that α arises from an *isomorphism*

$$\Pi_{U_S} \xrightarrow{\sim} \Pi_{V_T}$$

which is *point-theoretic* [or, equivalently, *quasi-point-theoretic*] — a condition that is automatically satisfied in the *finite field case* whenever α is *Frobenius-preserving* [cf. Remark 1.18.2]. Then observe that, [in light of our *point-theoreticity* assumption] it follows from Proposition 2.6,(i), that the resulting induced isomorphism

$$\Pi_{U_S}^{\text{c-ab}} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab}}$$

coincides [up to cuspidally inner automorphisms] with the isomorphism of Theorem 2.5, (i). Thus, in light of Remark 2.2.2, it follows formally from the definitions that α is *totally* (S,T)*-locally Green-compatible.*

Corollary 2.7. (Point-theoretic Totally Locally Principally Green-compatible Isomorphisms) *In the situation of Theorem 1.16, (iii), assume further that* α *is* **point-theoretic** *and* **totally** (S,T)**-locally principally Green-compatible**, *for some* **nonempty** *subsets* $S \subseteq X^{\text{cl}}$, $T \subseteq Y^{\text{cl}}$ *which correspond via the bijection* $X^{\text{cl}} \xrightarrow{\sim} Y^{\text{cl}}$ *induced by* α, *and that* $\Sigma = \mathfrak{Primes}$. *Then* α *arises from a* **uniquely** *determined commutative diagram of* **schemes**

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\sim} & \widetilde{Y} \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\sim} & Y \end{array}$$

in which the horizontal arrows are **isomorphisms**; *the vertical arrows are the profinite étale coverings determined by the profinite groups* Π_X, Π_Y.

Proof. Corollary 2.7 follows immediately from the definitions; Theorem 2.5, (ii); Proposition 2.1, (iii); Remark 2.1.1; and [Tama], Lemma 4.7. Here, we note that, in the present situation, the isomorphism

$$K_X^\times \cdot (k_X^\times)^\wedge \xrightarrow{\sim} K_Y^\times \cdot (k_Y^\times)^\wedge$$

of Theorem 2.5, (ii), necessarily induces an isomorphism $K_X^\times \xrightarrow{\sim} K_Y^\times$ [cf. the assumption that $\Sigma^\dagger = \mathfrak{Primes}^\dagger$]. Indeed, this is immediate in the *finite field case*. In the *nonarchimedean local field case*, it follows via the arguments applied in the proof of [Mzk8], Theorem 4.10: That is to say, we assume for simplicity that $S \subseteq X(k_X)$; then if $f \in K_X^\times$, and $x \in S$ is a point that does not lie in the divisor of zeroes and poles of f, then let us *observe* that the subset

$$f \cdot k_X^\times \subseteq f \cdot (k_X^\times)^\wedge$$

may be characterized as the subset of elements whose *values* [cf. Proposition 2.1, (iii)] at x lie in $k_X^\times \subseteq (k_X^\times)^\wedge$. Note that since, for a given $x_1 \in S$, there clearly exist $f \in K_X^\times$ [at least after possibly passing to an appropriate connected finite étale covering of X] that have a zero or pole at x_1 but not at some other $x \in S$, this *observation* allows us to recover the *canonical discrete structure* [cf. [Mzk8], Definition 4.1, (iii); the proof of [Mzk8], Theorem 4.10] on the decomposition groups in $\Pi^{\text{c-ab}}_{U_{S_1}}$ [where $S_1 \subseteq X^{\text{cl}}$ is an arbitrary finite subset containing S, which corresponds, say, to a subset $T_1 \subseteq Y^{\text{cl}}$ that contains T] at arbitrary points [i.e., arbitrary "x_1"] of S. Thus, by applying this canonical discrete structure [as in the proof of [Mzk8], Theorem 4.10], we may recover the subset

$$f \cdot k_X^\times \subseteq f \cdot (k_X^\times)^\wedge$$

for arbitrary $f \in K_X^\times$ [i.e., even f that have a zero or pole at every point of S] as the subset of elements for which the restriction to each point x of S *either* lies in $k_X^\times \subseteq (k_X^\times)^\wedge$ *or* [when the element in question has a zero or pole at x] is compatible with the canonical discrete structure at x. Since this characterization of the subset $f \cdot k_X^\times \subseteq f \cdot (k_X^\times)^\wedge$ is *manifestly compatible* [in light of the *Green-compatibility* assumption on α] with the isomorphisms $\Pi^{\text{c-ab}}_{U_{S_1}} \xrightarrow{\sim} \Pi^{\text{c-ab}}_{V_{T_1}}$ induced by α, we thus conclude that the isomorphism

$$K_X^\times \cdot (k_X^\times)^\wedge \xrightarrow{\sim} K_Y^\times \cdot (k_Y^\times)^\wedge$$

of Theorem 2.5, (ii), maps the subset $K_X^\times \subseteq K_X^\times \cdot (k_X^\times)^\wedge$ onto the subset $K_Y^\times \subseteq K_Y^\times \cdot (k_Y^\times)^\wedge$, as desired. ◯

Remark 2.7.1. Suppose, in the situation of Corollary 2.7, that $S = X^{\text{cl}}$, $T = Y^{\text{cl}}$. Then unlike the situation discussed in [Tama], one has the freedom to evaluate functions at arbitrary points of the *entire sets* X^{cl}, Y^{cl}, as opposed to just certain restricted subsets $S \subseteq X^{\text{cl}}$, $T \subseteq Y^{\text{cl}}$. Thus, instead of applying [Tama], Lemma 4.7, one may instead apply the somewhat *easier* argument *implicit* in [Uchi], §3, Lemmas 8-11 [which is used to treat the function field case].

Thus, in light of Remark 2.6.1, Corollary 2.7 implies the following result, in the *affine* case:

Corollary 2.8. **(Point-theoretic Isomorphisms in the Affine Case)** *Let U, V be* **affine hyperbolic curves** *over a* **finite** *or* **nonarchimedean local field***. Suppose that $\Sigma = \mathfrak{Primes}$. Then any* **point-theoretic isomorphism**

$$\beta : \Pi_U \xrightarrow{\sim} \Pi_V$$

arises from a **uniquely** *determined commutative diagram of* **schemes**

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\sim} & \widetilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are **isomorphisms***; the vertical arrows are the profinite étale coverings determined by the profinite groups* Π_U, Π_V.

Remark 2.8.1. In light of the results of [Tama] [cf. Remarks 1.18.1, 1.18.2], Corollary 2.8 is only truly of interest in the case of *nonarchimedean local fields*.

Definition 2.9. Suppose that k is a *nonarchimedean local field*.

(i) A [necessarily affine] hyperbolic curve U over k will be said to be of *strictly Belyi type* if it is defined over a number field and isogenous [cf. §0] to a hyperbolic curve of genus zero.

(ii) A [necessarily affine] hyperbolic curve U over k will be said to be of *Belyi type* if it is defined over a number field, and, moreover, for some positive integer m, there exists a finite sequence

$$U = U_1 \rightsquigarrow U_2 \rightsquigarrow \ldots \rightsquigarrow U_{m-1} \rightsquigarrow U_m$$

of *hyperbolic orbicurves* [cf. §0] U_j such that U_m is a *tripod* [cf. §0], and, moreover, for each $j = 1, \ldots, m-1$, U_{j+1} is related to U_j in one of the following ways:

(a) there exists a *finite étale morphism* $U_{j+1} \to U_j$ [i.e., "U_{j+1} is a *finite étale covering* of U_j"];

(b) there exists a *finite étale morphism* $U_j \to U_{j+1}$ [i.e., "U_{j+1} is a *finite étale quotient* of U_j"];

(c) there exists an *open immersion* $U_j \hookrightarrow U_{j+1}$ [i.e., in the terminology of [Mzk8], "U_{j+1} is a *[hyperbolic] partial compactification* of U_j"];

(d) there exists a *partial coarsification morphism* [cf. §0] $U_j \to U_{j+1}$ [i.e., "U_{j+1} is a *partial coarsification* of U_j"].

(iii) A [necessarily affine] hyperbolic curve U over k will be said to be of *quasi-Belyi type* if it is defined over a number field and admits a connected finite étale covering $V \to U$ such that V admits a [not necessarily finite or étale!] dominant morphism $V \to W$ to a tripod W.

Remark 2.9.1. It is immediate that every hyperbolic curve of strictly Belyi type is also of Belyi type [as the terminology suggests]. Moreover, one verifies easily by "induction on m" [where "m" is as in Definition 2.9, (ii)] that every hyperbolic curve of Belyi type is also of quasi-Belyi type [as the terminology suggests]. It is not difficult to see that there exist [multiply] punctured elliptic curves that are of Belyi type, but not of strictly Belyi type [cf. Remark 2.13.2 below]. On the other hand, it is not clear to the author at the time of writing whether or not there exist hyperbolic curves of quasi-Belyi type that are not of Belyi type.

Remark 2.9.2. Hyperbolic curves of strictly Belyi type are precisely the sort of curves considered in [Mzk8], Corollaries 2.8, 3.2.

Remark 2.9.3. The author would like to thank *A. Tamagawa* for useful discussions concerning Definition 2.9, (ii), especially Definition 2.9, (ii), (d).

Proposition 2.10. (Decomposition Groups of Curves of Quasi-Belyi Type) *Let U (respectively, V) be a* **hyperbolic curve** *over a nonarchimedean local field. Denote the base field of U (respectively, V) by k_U (respectively, k_V), the étale fundamental group of U (respectively, V) by Π_U (respectively, Π_V) [i.e., "we take $\Sigma = \mathfrak{Primes}$"]. Let*

$$\beta : \Pi_U \xrightarrow{\sim} \Pi_V$$

be an isomorphism of profinite groups. Then:

(i) If U is of **quasi-Belyi type***, then the closed points of "*DLoc*-type" [in the sense of [Mzk8], Definition 2.4] are* **p_U-adically dense** *[where p_U is the residue characteristic of k_U] in $U(k_U)$.*

(ii) If U is of **quasi-Belyi type***, then β maps every decomposition group of a closed point of U isomorphically onto a decomposition group of a closed point of V.*

(iii) If **both** *U, V are of* **quasi-Belyi type***, then β is* **point-theoretic***.*

(iv) If U is of **Belyi type***, then so is V.*

Proof. The proof of assertion (i) is similar to the proof of [Mzk8], Corollary 2.8: That is to say, in the terminology of *loc. cit.*, it follows formally from the fact that U is of *quasi-Belyi type* that the *"algebraic" closed points* [i.e., closed points defined over a number field] of U are of "DLoc-type" [cf. the proof of [Mzk8], Corollary 2.8]: Indeed, it suffices to consider the following commutative diagram of hyperbolic curves, whose existence follows from the assumption that U is of *quasi-Belyi type*:

$$\begin{array}{ccccccc} & & V' & \longrightarrow & W' & \hookrightarrow & U' & \longrightarrow & U \\ & & \downarrow & & \downarrow & & & & \\ U & \longleftarrow & V & \longrightarrow & W & & & & \end{array}$$

Here, the "hooked arrow $\hookrightarrow$" is an *open immersion*; all of the "non-hooked arrows" *except for $V \to W$, $V' \to W'$* are *finite étale morphisms*; $V \to W$, $V' \to W'$ are *dominant*; the finite étale morphism $U' \to U$ is obtained by a *base-change* to a finite extension of the base field k_U; and W is a *tripod* [so $W' \to W$ is a *"Belyi map"*]. Note that the composite arrow $V' \to W' \hookrightarrow U' \to U$ may be thought of as an arrow in the category $\mathrm{DLoc}_{k_U}(U)$ of [Mzk8], §2. Observe, moreover, that the arrow $W' \hookrightarrow U'$ may be chosen to have *arbitrarily designated algebraic closed points* in the complement of its image. Thus, we conclude that this diagram exhibits the [arbitrarily designated] algebraic closed points in the complement of the image of

$W' \hookrightarrow U' \to U$ as *points of* DLoc-*type*, as desired. This completes the proof of assertion (i).

In light of assertion (i) [applied to the various connected finite étale coverings of U], the proof of assertion (ii) is entirely similar to the proof of [Mzk8], Corollary 3.2: That is to say, by [Mzk8], Corollary 2.5, it follows that *β maps decomposition groups of* DLoc-*type of U to decomposition groups of* DLoc-*type of V*. Thus, assertion (ii) follows by applying [Mzk8], Lemma 3.1 [where the density statement of assertion (i) concerning points of DLoc-type allows one to replace the "algebraicity" condition of [Mzk8], Lemma 3.1, (iii), by the condition that the points in question be of DLoc-type]. Finally, assertion (iii) follows formally from assertion (ii) [and Proposition 2.3, (ii)].

Finally, we consider assertion (iv). First, I *claim* that by applying the isomorphism β [and thinking of hyperbolic orbicurves as being represented by their associated étale fundamental groups], one may *transform* the sequence

$$U = U_1 \rightsquigarrow U_2 \rightsquigarrow \ldots \rightsquigarrow U_{m-1} \rightsquigarrow U_m$$

of Definition 2.9, (ii), into a sequence

$$V = V_1 \rightsquigarrow V_2 \rightsquigarrow \ldots \rightsquigarrow V_{m-1} \rightsquigarrow V_m$$

that also satisfies the conditions of Definition 2.9, (ii), in such a way that we also obtain *compatible isomorphisms* $\beta_j : \Pi_{U_j} \xrightarrow{\sim} \Pi_{V_j}$ [where $j = 1, \ldots, m$; $\beta_1 = \beta$]. Indeed, we reason by induction on m. If [for $j = 1, \ldots, m-1$] U_{j+1} is related to U_j as in (a) [of Definition 2.9, (ii)], then it is immediate [by thinking in terms of open subgroups of Π_{U_j}, Π_{V_j}] that one may construct [from V_j] a V_{j+1} related to V_j as in (a). If U_{j+1} is related to U_j as in (b) (respectively, (c)), then it follows from [Mzk6], Theorem 2.4 (respectively, [Mzk8], Theorem 1.3, (iii) [cf. also [Mzk8], Theorem 2.3]), that one may construct [from V_j] a V_{j+1} related to V_j as in (b) (respectively, (c)). If U_{j+1} is related to U_j as in (d), then $\Pi_{U_{j+1}}$ is obtained from Π_{U_j} by forming the quotient of Π_{U_j} by the closed normal subgroup of Π_{U_j} generated by some finite collection of elements of Δ_{U_j} that belong to the *decomposition groups* of points of U_j in Δ_{U_j}. Thus, by Lemma 2.11, (v), below, we conclude that the quotient $\Pi_{U_j} \twoheadrightarrow \Pi_{U_{j+1}}$ determines a quotient $\Pi_{V_j} \twoheadrightarrow \Pi_{V_{j+1}}$ that corresponds to a *partial coarsification* $V_j \to V_{j+1}$, as desired. Finally, if U_m is a tripod, the existence of the isomorphism $\Pi_{U_m} \xrightarrow{\sim} \Pi_{V_m}$ implies that V_m is also a *tripod* [cf. [Mzk5], Lemma 1.3.9]. This completes the proof of the *claim*.

Thus, to complete the proof of assertion (iv), it suffices to verify that V is *defined over a number field*. But observe that since U is defined over a number field, there exists a *diagram of hyperbolic curves* [i.e., in essence, a "Belyi map"]

$$U_m \quad \longleftarrow \quad U'_m \quad \hookrightarrow \quad U' \quad \longrightarrow \quad U$$

where the "hooked arrow $\hookrightarrow$" is an *open immersion*; the "non-hooked arrows" are *finite étale morphisms*; and the finite étale morphism $U' \to U$ is obtained by a *base-change* to a finite extension of the base field k_U. Now the isomorphisms

$\Pi_{U_m} \xrightarrow{\sim} \Pi_{V_m}$, $\Pi_U \xrightarrow{\sim} \Pi_V$ allow us to *transform* [cf. [Mzk8], Theorem 2.3 and its proof] this diagram into a similar diagram

$$V_m \quad \longleftarrow \quad V'_m \quad \hookrightarrow \quad V' \quad \longrightarrow \quad V$$

whose existence [since V_m is also a *tripod*!] shows that V is also *defined over a number field*, as desired. This completes the proof of assertion (iv). ○

Remark 2.10.1. Note that the essential reason that the author is unable to prove the stronger statement of Proposition 2.10, (iv), in the *quasi-Belyi* case is that, in the notation of the proof of Proposition 2.10, (i), it is *unclear how to construct* [at the level of arithmetic fundamental groups] *the dominant morphism* $V \to W$ *from* V. That is to say, unlike the situation involving the operations of Definition 2.9, (ii), (a), (b), (c), (d), it is by no means clear how to construct, via *purely group-theoretic operations*, the quotient of an arithmetic fundamental group arising from an arbitrary dominant morphism.

Lemma 2.11. (Finite Subgroups of Fundamental Groups of Hyperbolic Orbicurves) *Let* W *be a* **hyperbolic orbicurve** *over an algebraically closed field of characteristic zero;* Σ_W *a nonempty set of prime numbers. Denote the maximal pro-*Σ_W *quotient of the* **étale fundamental group** *of* W *by* Δ_W. *Let* $A \subseteq \Delta_W$ *(respectively,* $B \subseteq \Delta_W$*) be the* **decomposition group** *[well-defined up to conjugation in* Δ_W*] of a closed point* w_A *(respectively,* w_B*) of* W*; suppose that* $w_A \neq w_B$. *Then:*

(i) A, B *are* **cyclic**.

(ii) $A \bigcap B = \{1\}$. *In particular, if* $A \neq \{1\}$, *then* A *is* **normally terminal** *in* Δ_W.

(iii) The order of **every finite cyclic subgroup** $C \subseteq \Delta_W$ *divides the* **order of** W *[cf. §0].*

(iv) Every **finite subgroup** $C \subseteq \Delta_W$ *is contained in a* **unique decomposition group of a closed point** *of* W.

(v) The decomposition groups of closed points of W *may be characterized as the* **maximal finite subgroups** *of* Δ_W.

Proof. Assertion (i) follows immediately from the well-known [and easily verified] fact that the absolute Galois group of a complete discrete valuation field with algebraically closed residue field of characteristic zero is *cyclic*.

Next, we consider assertion (ii). Let $C \subseteq A \bigcap B$ be a subgroup of *prime order* $l \in \Sigma_W$. Now consider a normal open subgroup $H \subseteq \Delta_W$ such that the covering $W_H \to W$ determined by H is a *hyperbolic curve*. Note that this implies that $A \bigcap H = B \bigcap H = C \bigcap H = \{1\}$. Write $W_H \to W_C \to W$ for the covering determined by the open subgroup $C \cdot H \subseteq \Delta_W$. Observe that there exist closed points w'_A, w'_B of W_C that lift w_A, w_B, respectively, and whose decomposition groups [well-defined up to conjugation in $C \cdot H$] are equal to C. Note that since W_H is a *hyperbolic curve*, and C is of prime order l, it follows that the order of every closed point of W_H is equal to either 1 or l. Now if W_C is *affine*, then let v be a *cusp* of W_C. If W_C is *proper* and admits ≥ 3 *points of order* l, then let v be a point of W_C of order l such that $v \neq w'_A, w'_B$. Note that if W_C is *proper* and admits ≤ 2 *points of order* l, then it follows from the *hyperbolicity* assumption that the *coarsification* of W_C is a proper smooth curve of genus ≥ 1; thus, by *replacing* H by an appropriate open subgroup of H, one verifies immediately that one may assume without loss of generality that *either* W_C is affine *or* W_C admits ≥ 3 points of order l. Now observe that W_C admits a finite étale cyclic covering $W'_C \to W_C$ of degree l which is *étale* over the compactification of the coarsification of W_C, *except* over the points in the compactification of the coarsification of W_C corresponding to v, w'_B, over which W'_C is *totally ramified*. In particular, it follows that any point of W'_C lying over w'_A (respectively, w'_B) is of order l (respectively, 1), thus contradicting the observation that the decomposition groups [well-defined up to conjugation in $C \cdot H$] of w'_A, w'_B are equal to C. This completes the proof that $A \bigcap B = \{1\}$. By applying this fact to arbitrary finite étale coverings of W, it follows formally [cf. Proposition 2.6, (ii)] that A is *normally terminal* in Δ_W, whenever $A \neq \{1\}$.

Next, we consider assertion (iii). Denote the *order of* W by n. Now if $C \subseteq \Delta_W$ is a *nontrivial finite cyclic subgroup*, then C maps *injectively* to the inverse limit of the abelianizations H^{ab} of the open subgroups $H \subseteq \Delta_W$ of Δ_W that contain C. Thus, there exists such an H such that the natural map $C \to H^{\mathrm{ab}}$ is *injective*. On the other hand, if we denote by $W_H \to W$ the covering determined by H, then it is clear that the order of W_H divides n, hence that H^{ab} is the extension of a *torsion-free* profinite abelian group by a *finite abelian group annihilated by* n. Thus, we conclude from the injection $C \hookrightarrow H^{\mathrm{ab}}$ that the *order of* C *divides* n, as desired. This completes the proof of assertion (iii).

Next, we consider assertion (iv). First, let us observe that *uniqueness* follows formally from assertion (ii). Next, let us verify assertion (iv) under the further assumption that C is *solvable*. By induction on the order of C, we may assume that [at least] *one* of the following conditions is satisfied: (a) C is an extension of a group of prime order by a nontrivial subgroup $C_1 \subseteq C$ which is contained in the decomposition group A; (b) C is of prime order $l \in \Sigma_W$. If (a) is satisfied, then by replacing W by a finite étale covering of W determined by a suitable open subgroup containing C, we may assume that ($C_1 \subseteq$) $A \subseteq C$. Thus, if $A \neq C$, then $A = C_1$ is normal in C. But this implies, by the *normal terminality* portion of assertion (ii), that $A = C$, a contradiction. If (b) is satisfied, then we argue as follows: Observe that by assertion (iii), every open subgroup $H \subseteq \Delta_W$ that contains C determines a finite étale covering $W_H \to W$ such that the *order of* W_H *is divisible by* l. Write

$$\mathrm{Stack}_l(W_H)$$

for the set of closed points of W_H whose order is *divisible by* l. Now observe that since the order of W_H is divisible by the *prime number* l, it follows that $\mathrm{Stack}_l(W_H)$ is *nonempty*. Since the set $\mathrm{Stack}_l(W_H)$ is *finite* and *nonempty*, we thus conclude that, if we allow H to *vary* [among open subgroups $H \subseteq \Delta_W$ that contain C], then the inverse limit

$$\varprojlim_{H} \text{Stack}_l(W_H)$$

is *nonempty*. But, unraveling the definitions, this means precisely that C contains the decomposition group D associated to some compatible system of points of the sets $\text{Stack}_l(W_H)$. Since D is of order divisible by l, we thus conclude that $D = C$, as desired. This completes the proof of assertion (iv) for C *solvable*. On the other hand, a well-known theorem from the theory of finite groups asserts that a finite group in which every Sylow subgroup is cyclic is *solvable* [cf. [Scott], p. 356]. Thus, in light of assertion (i), we conclude that assertion (iv) for C solvable implies assertion (iv) for C *arbitrary*.

Finally, we observe that assertion (v) follows formally from assertions (ii), (iv). ○

Remark 2.11.1. The author would like to thank *A. Tamagawa* for informing him of Lemma 2.11 and, in particular, of the theorem on finite groups that was applied in the proof of Lemma 2.11, (iv).

We are now ready to state the following *"absolute p-adic version of the Grothendieck Conjecture"* for hyperbolic curves of Belyi or quasi-Belyi type:

Corollary 2.12. **(Curves of Belyi or Quasi-Belyi Type)** *Let U (respectively, V) be a* **hyperbolic curve** *over a nonarchimedean local field. Denote the base field of U (respectively, V) by k_U (respectively, k_V), the étale fundamental group of U (respectively, V) by Π_U (respectively, Π_V) [i.e., "we take $\Sigma = \mathfrak{Primes}$"]. Suppose further that* **at least one** *of the following conditions holds:*

(a) **both** *U and V are of* **quasi-Belyi type**;

(b) **either** *U* **or** *V [but not necessarily both!] is of* **Belyi type**.

Then any isomorphism of profinite groups

$$\beta : \Pi_U \xrightarrow{\sim} \Pi_V$$

arises from a **uniquely** *determined commutative diagram of* **schemes**

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\sim} & \widetilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are **isomorphisms**; *the vertical arrows are the profinite étale coverings determined by the profinite groups Π_U, Π_V.*

Proof. In light of Proposition 2.10, (iii), (iv) [cf. also Remark 2.9.1], Corollary 2.12 follows formally from Corollary 2.8. ○

Remark 2.12.1. Note that in the proof of Proposition 2.10, Corollary 2.12, it is necessary, in the *quasi-Belyi* case, to apply the full "Hom version" of [Mzk4], Theorem A. This differs from the situation of [Mzk8], Corollaries 2.8, 3.2 — i.e., where one only treats hyperbolic curves of *strictly* Belyi type — or, indeed, of the portion of Proposition 2.10, Corollary 2.12, that concerns curves of *Belyi* type, in which the *"isomorphism version"* of [Mzk4], Theorem A, suffices [cf. [Mzk8], Remark 2.8.1].

Thus, in the terminology of [Mzk6], Definition 3.7, the portion of Corollary 2.12 concerning hyperbolic curves of Belyi type admits the following formal consequence:

Corollary 2.13. **(Absoluteness of Curves of Belyi Type)** *Every hyperbolic curve of* **Belyi type** *over a nonarchimedean local field is* **absolute***.*

Remark 2.13.1. It is interesting to note that the essential property that underlies the absoluteness of Corollary 2.13 is the existence of a *Belyi map* [since the curve is defined over a number field], which, in the context of the theory of [Mzk8], §2, may be regarded as a sort of *endomorphism* of the curve. From this point of view, Corollary 2.13 is *reminiscent* of [Mzk6], Corollary 3.8, which states that the *"canonical curves" of p-adic Teichmüller theory* are absolute. Indeed, from the point of view of the theory of [Mzk2], this canonicality may be regarded as the existence of a sort of *"Frobenius endomorphism"* of the curve. It is also interesting to note that both of these results assert that every member of some *countable* collection of nonarchimedean hyperbolic curves is absolute. This suggests that the property of being absolute is a somewhat unusual property, i.e., a property not satisfied by "most" nonarchimedean hyperbolic curves [cf., e.g., the Introduction to [Mzk5]].

Remark 2.13.2. In the context of Remark 2.13.1, it is interesting to note that, unlike the canonical curves discussed in [Mzk6], §3, the set of points determined by the hyperbolic curves of *strictly* Belyi type *fails*, for all pairs (g, r) such that $2g - 2 + r \geq 3$, $g \geq 1$, to be *Zariski dense* in the moduli stack of hyperbolic curves of type (g, r). Indeed, this follows immediately from [Mzk1], Theorem B. On the other hand, it is not clear to the author at the time of writing whether or not the set of points determined by the hyperbolic curves of Belyi (respectively, quasi-Belyi) type is Zariski dense in the moduli stack of hyperbolic curves of type (g, r) [when, say, $2g - 2 + r \geq 3$, $g \geq 2$]. Note, however, that when $g = 0, 1$, [one verifies easily that] every hyperbolic curve of type (g, r) that is defined over a number field is *automatically of Belyi type.*

Remark 2.13.3. Recall [in the context of Remark 2.13.1] that in [Mzk6], Remark 3.6.3, the point of view is advanced that the absoluteness of canonical curves should be regarded as the analogue for hyperbolic curves of the fact that Serre-Tate canonical liftings of abelian varieties are *defined over number fields*. Thus, from this point of view, curves of Belyi type and canonical curves have in common not only the property that they admit some sort of nontrivial *endomorphism*, but also that this endomorphism appears to be related [indeed, in the Belyi case, is *literally related*] to some sort of "generalized version" of the property of *being defined over a number field*. Put another way, this state of affairs suggests that:

> Perhaps the property of *absoluteness* is related to some sort of natural p-adic generalization of the property of being *"defined over a number field"*.

Indeed, this state of affairs further suggests that:

> Perhaps the property of *absoluteness* is *equivalent* to some sort of natural p-adic generalization for hyperbolic curves of the condition on an abelian variety that the abelian variety be *"defined over a number field"* and *admit sufficiently many complex multiplications*.

In particular, this state of affairs suggests that perhaps a natural way to look for *more examples* of absolute p-adic hyperbolic curves is to look for other situations in which p-adic hyperbolic curves admit, in some sort of generalized sense, endomorphisms that are reminiscent of endomorphisms of abelian varieties that allow one to show that the object on which the endomorphism acts is *defined over a number field*.

Section 3: Characterization of Green's Trivializations over Finite Fields

In this §, we apply the theory of the *weight filtration* [cf. [Kane], [Mtm]] to show, in the *finite field* case, that, under quite general conditions [cf. Corollary 3.11 below], an isomorphism "α" as in Theorem 1.16, (iii), is always *totally globally Green-compatible.*

In the following discussion, we maintain the notation of §2, and assume further throughout the present §3 that we are in the *finite field* case.

Definition 3.1. Let l be a prime number; G, H, A *topologically finitely generated pro-l groups*; $\phi : H \to A$ a [continuous] homomorphism. Suppose further that A is *abelian*, and that G is an *l-adic Lie group.*

(i) We shall refer to as the *ϕ-central filtration* on H the filtration defined as follows:

$$H(1) \stackrel{\text{def}}{=} H$$

$$H(2) \stackrel{\text{def}}{=} \mathrm{Ker}(\phi)$$

$$H(m) \stackrel{\text{def}}{=} \Big(\text{the subgroup topologically generated by the commutators}$$
$$[H(a), H(b)], \text{ where } a+b=m\Big), \ \forall\ m \geq 3$$

Thus, in words, this filtration on H is the *"fastest decreasing central filtration among those central filtrations whose top quotient factors through ϕ".* We shall say that H is *ϕ-nilpotent* if $H(m) = \{1\}$ for sufficiently large ϕ. If H is ϕ-nilpotent when ϕ is taken to be the natural surjection $H \twoheadrightarrow H^{\mathrm{ab}}$ to its abelianization H^{ab}, then we shall say that H is *nilpotent.* In the following, for $a, b, n \in \mathbb{Z}$ such that $1 \leq a \leq b$, $n \geq 1$, we shall write

$$H(a/b) \stackrel{\text{def}}{=} H(a)/H(b)$$

and

$$\mathrm{Gr}(H)(n) \stackrel{\text{def}}{=} \bigoplus_{m \geq n} H(m/m+1) \subseteq \mathrm{Gr}(H) \stackrel{\text{def}}{=} \mathrm{Gr}(H)(1)$$

$$\mathrm{Gr}(H)(a/b) \stackrel{\text{def}}{=} \mathrm{Gr}(H)(a)/\mathrm{Gr}(H)(b)$$

and append a *subscript* $\mathbb{Q}_l$ to these objects to denote the result of tensoring over $\mathbb{Z}_l$ with $\mathbb{Q}_l$. Thus, $\mathrm{Gr}(H)$, $\mathrm{Gr}_{\mathbb{Q}_l}(H)$ are *graded Lie algebras* over $\mathbb{Z}_l$, $\mathbb{Q}_l$, respectively; $\mathrm{Gr}(H)(n) \subseteq \mathrm{Gr}(H)$ is a [Lie algebra-theoretic] *ideal.* Also, if $\mathbb{Z} \ni a \geq 1$, then we shall write:

$$H(a/\infty) \stackrel{\text{def}}{=} \varprojlim_{b} H(a/b)$$

[where b ranges over the integers $\geq a+1$].

(ii) We shall denote by $\mathrm{Lie}(G)$ the *Lie algebra over* $\mathbb{Q}_l$ determined by G. If G is *nilpotent*, then $\mathrm{Lie}(G)$ is a nilpotent Lie algebra over $\mathbb{Q}_l$, hence determines a connected, unipotent linear algebraic group $\mathrm{Lin}(G)$, which we shall refer to as the *linear algebraic group associated to* G. In this situation, there is a *natural [continuous] homomorphism* [with *open image*]

$$G \to \mathrm{Lin}(G)(\mathbb{Q}_l)$$

[from G to the l-adic Lie group determined by the $\mathbb{Q}_l$-valued points of $\mathrm{Lin}(G)$] which is determined by the condition that it induce the identity morphism on the associated Lie algebras. In the situation of (i), if $\mathbb{Z} \ni a \geq 1$, then we shall write:

$$\mathrm{Lie}(H(a/\infty)) \stackrel{\mathrm{def}}{=} \varprojlim_b \mathrm{Lie}(H(a/b)); \quad \mathrm{Lin}(H(a/\infty)) \stackrel{\mathrm{def}}{=} \varprojlim_b \mathrm{Lin}(H(a/b))$$

[where b ranges over the integers $\geq a+1$; we recall that it is well-known [or easily verified] that each $H(a/b)$ is an l-adic Lie group].

Now let us *fix* a prime number $l \in \Sigma^\dagger$. For $S \subseteq X(k)$ a finite subset, let us denote by

$$\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{(l)}; \quad \Delta_X \twoheadrightarrow \Delta_X^{(l)}$$

the *maximal pro-l quotients* and by

$$\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{(l)}; \quad \Pi_X \twoheadrightarrow \Pi_X^{(l)}$$

the quotients of Π_{U_S}, Π_X by the kernels of $\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{(l)}$, $\Delta_X \twoheadrightarrow \Delta_X^{(l)}$. Also, for $x \in X^{\mathrm{cl}}$, let us write

$$D_x^{(l)}[U_S] \subseteq \Pi_{U_S}^{(l)}; \quad I_x^{(l)}[U_S] \subseteq \Delta_{U_S}^{(l)}$$

for the images of $D_x[U_S]$, $I_x[U_S]$, respectively, in $\Pi_{U_S}^{(l)}$.

Note that we have a natural surjection:

$$\Delta_{U_S}^{(l)} \twoheadrightarrow \Delta_X^{(l)} \twoheadrightarrow (\Delta_X^{(l)})^{\mathrm{ab}}$$

The cup product on the group cohomology of $\Delta_X^{(l)}$ determines an *isomorphism* [cf. Proposition 1.3, (ii)]

$$\mathrm{Hom}((\Delta_X^{(l)})^{\mathrm{ab}}, M_X^{(l)}) \stackrel{\sim}{\to} (\Delta_X^{(l)})^{\mathrm{ab}}$$

[where we write $M_X^{(l)} \stackrel{\mathrm{def}}{=} M_X \otimes \mathbb{Z}_l$], hence a *natural G_k-equivariant injection*

$$M_X^{(l)} \hookrightarrow \wedge^2 \, (\Delta_X^{(l)})^{\mathrm{ab}}$$

whose image we denote by $I_{\mathrm{cup}}^{(l)}$.

Definition 3.2. We shall refer to the central filtration

$$\{\Delta_{U_S}^{(l)}(m)\}$$

on $\Delta^{(l)}_{U_S}$ with respect to the natural surjection $\Delta^{(l)}_{U_S} \twoheadrightarrow (\Delta^{(l)}_X)^{\mathrm{ab}}$ as the *weight filtration* on $\Delta^{(l)}_{U_S}$ [cf., e.g., [Mtm], §3, p. 200].

Proposition 3.3. **(Free Lie Algebras)** *Let R be a* **commutative ring with unity***; V a* **finitely generated free** *R-module. Write $\mathfrak{Lie}_R(V)$ for the* **free Lie algebra** *over R associated to V; for $\mathbb{Z} \ni b \geq 1$, denote by $\mathfrak{Lie}^b_R(V) \subseteq \mathfrak{Lie}_R(V)$ the R-submodule generated by the "alternants of degree b" [cf. [Bour], Chapter II, §2.6]. Also, we shall denote by $\mathcal{U}_R(V)$ the* **enveloping algebra** *of $\mathfrak{Lie}_R(V)$. [Thus, we have a natural inclusion $\mathfrak{Lie}_R(V) \hookrightarrow \mathcal{U}_R(V)$.] Then:*

(i) Each $\mathfrak{Lie}^b_R(V)$ is a **finitely generated free** *R-module. Moreover, there is a natural isomorphism $V \xrightarrow{\sim} \mathfrak{Lie}^1_R(V)$.*

(ii) Let $v \in V$ be a nonzero element such that the quotient module $V/R \cdot v$ is **free***. Then the* **centralizer** *of v in $\mathcal{U}_R(V)$ is equal to the R-submodule of $\mathcal{U}_R(V)$ generated by the nonnegative powers of v. In particular, if R is a field of characteristic zero, then the* **centralizer** *of v in $\mathfrak{Lie}_R(V)$ is equal to $R \cdot v$.*

(iii) Suppose that the rank of V over R is ≥ 2. Then the Lie algebra $\mathfrak{Lie}_R(V)$ is **center-free***. In particular, the* **adjoint representation** *of $\mathfrak{Lie}_R(V)$ is* **faithful**.

*(iv) Let R' be an R-***algebra** *which is* **finitely generated** *and* **free** *as an R-module. Let $\phi : R' \twoheadrightarrow R$ be a surjection of R-algebras; suppose that $V = V' \otimes_{R',\phi} R$, for some finitely generated free R'-module V' [so we obtain a natural surjection $V' \twoheadrightarrow V$ compatible with ϕ]. Then the natural surjection $V' \twoheadrightarrow V$ induces a surjection of R-modules $\mathfrak{Lie}^b_R(V') \twoheadrightarrow \mathfrak{Lie}^b_R(V)$ that factors as a composite of natural surjections as follows:*

$$\mathfrak{Lie}^b_R(V') \twoheadrightarrow \mathfrak{Lie}^b_{R'}(V') \twoheadrightarrow \mathfrak{Lie}^b_R(V)$$

Here, the first arrow of this factorization is the arrow naturally induced by observing that every Lie algebra over R' naturally determines a Lie algebra over R; the second arrow of this factorization is the arrow functorially induced by the natural ϕ-compatible surjection $V' \twoheadrightarrow V$. Finally, this second arrow induces an isomorphism $\mathfrak{Lie}^b_{R'}(V') \otimes_{R',\phi} R \xrightarrow{\sim} \mathfrak{Lie}^b_R(V)$.

Proof. Assertion (i) follows immediately from [Bour], Chapter II, §2.11, Theorem 1, Corollary. Assertion (ii) follows from the well-known structure of the *enveloping algebra* $\mathcal{U}_R(V)$ [i.e., the natural isomorphism of $\mathcal{U}_R(V)$ with the *free associative algebra* determined by V over R; the fact that when R is a field of characteristic zero, the image of $\mathfrak{Lie}_R(V)$ in $\mathcal{U}_R(V)$ may be identified with the set of *primitive elements* — cf. [Bour], Chapter II, §3, Theorem 1, Corollaries 1,2], by considering the *effect on "words" of forming the commutator with v* — cf. the argument of [Mtm], Proposition 3.1 [which is given only in the case where R is a field of characteristic zero, but does not, in fact, make use of this assumption on R in an essential way]. Assertion (iii) follows immediately from assertion (ii) [by allowing the element "v"

of assertion (ii) to range over the elements of an R-basis of V]. Assertion (iv) follows formally from the *universal property of a free Lie algebra*, together with the well-known *functoriality* of a free Lie algebra with respect to *tensor products* [cf. [Bour], Chapter II, §2.5, Proposition 3]. ◯

Proposition 3.4. **(Freeness and Centralizers of Inertia)** *Let $x \in S$. Write $S_x \stackrel{\mathrm{def}}{=} S \backslash \{x\}$; r for the cardinality of S, g for the genus of X. For $x' \in S$, let $\zeta_{x'}$ be a generator of $I_{x'}^{(l)}[U_S]$. By abuse of notation, we shall also denote by $\zeta_{x'}$ the image of $\zeta_{x'}$ in $\Delta_{U_S}^{(l)}(2/3)$. Then:*

(i) $\mathrm{Gr}(\Delta_{U_S}^{(l)})$ *is a* **free Lie algebra** *over $\mathbb{Z}_l$ [hence, in particular, is torsion-free as a $\mathbb{Z}_l$-module] which is freely generated by $2g$ elements*

$$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \in \Delta_{U_S}^{(l)}(1/2)$$

together with the $\zeta_{x'} \in \Delta_{U_S}^{(l)}(2/3)$, for $x' \in S_x$. Alternatively, for an appropriate choice of the elements $\zeta_{x'}$, $\mathrm{Gr}(\Delta_{U_S}^{(l)})$ *is the quotient of the free Lie algebra generated by $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, together with the $\zeta_{x'} \in \Delta_{U_S}^{(l)}(2/3)$, for $x' \in S$, by the* **single relation***:*

$$\sum_{x' \in S} \zeta_{x'} + \sum_{n=1}^{g} [\alpha_n, \beta_n] = 0$$

At a more intrinsic level, this relation is a generator of the image of the **natural G_k-equivariant morphism**

$$M_X^{(l)} \hookrightarrow \Big(\bigoplus_{x' \in S} I_{x'}^{(l)}[U_S] \Big) \oplus I_{\mathrm{cup}}^{(l)}$$

[determined by the various natural isomorphisms $M_X^{(l)} \xrightarrow{\sim} I_{x'}^{(l)}[U_S]$, $M_X^{(l)} \xrightarrow{\sim} I_{\mathrm{cup}}^{(l)}$], whose codomain maps to $\mathrm{Gr}(\Delta_{U_S}^{(l)})$ *via the* **natural G_k-equivariant morphism**

$$\Big(\bigoplus_{x' \in S} I_{x'}^{(l)}[U_S] \Big) \oplus I_{\mathrm{cup}}^{(l)} \to \Delta_{U_S}^{(l)}(2/3)$$

[determined by the natural inclusions $I_{x'}^{(l)}[U_S] \hookrightarrow \Delta_{U_S}^{(l)}(2/3)$ and the bracket operation $\wedge^2 \ (\Delta_X^{(l)})^{\mathrm{ab}} \to \Delta_{U_S}^{(l)}(2/3)$].

(ii) Let ξ be any of the elements $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$; $\zeta_{x'}$, where $x' \in S_x$, of (i). Then the **centralizer** *in $\mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{U_S}^{(l)})$ of [the image of] ξ [in $\mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{U_S}^{(l)})$] is equal to $\mathbb{Q}_l \cdot \xi$. In particular, the Lie algebra $\mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{U_S}^{(l)})$ is* **center-free***.*

(iii) Let ξ be as in (ii). Then for $m \geq 1$, the **centralizer** *in $\Delta_{U_S}^{(l)}(1/m+2)$ of [the image of] ξ [in $\Delta_{U_S}^{(l)}(1/m+2)$] is contained in the subgroup of $\Delta_{U_S}^{(l)}(1/m+2)$ generated by [the image of] ξ and $\Delta_{U_S}^{(l)}(m/m+2)$.*

(iv) Let $S_* \subseteq S$ be a subset of S. Write

$$\mathrm{New}^{(l)}_{S_*} \subseteq \mathrm{Gr}(\Delta^{(l)}_{U_S})$$

for the sub-Lie algebra over $\mathbb{Z}_l$ generated by the image of the restriction

$$\Big(\bigoplus_{x' \in S_*} I^{(l)}_{x'}[U_S]\Big) \subseteq \Big(\bigoplus_{x' \in S} I^{(l)}_{x'}[U_S]\Big) \to \Delta^{(l)}_{U_S}(2/3)$$

to the direct summands indexed by elements of S_* of the morphism of (i), and $\mathrm{New}^{(l)}_{S_*}(a) \stackrel{\mathrm{def}}{=} \mathrm{Gr}(\Delta^{(l)}_{U_S})(a) \bigcap \mathrm{New}^{(l)}_{S_*}$; $\mathrm{New}^{(l)}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{New}^{(l)}_{S_*}(a)/\mathrm{New}^{(l)}_{S_*}(b)$ for $a, b \in \mathbb{Z}$ such that $1 \leq a \leq b$. Then, in the notation of (i), $\mathrm{New}^{(l)}_{S_*}$ is a **free Lie algebra** *over $\mathbb{Z}_l$ generated by the elements $\zeta_{x'}$, for $x' \in S_*$. Moreover, the [***"new"*** and* ***"co-new"****] $\mathbb{Z}_l$-modules*

$$\mathrm{New}^{(l)}_{S_*}(a/b); \quad \mathrm{Cnw}^{(l)}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Gr}(\Delta^{(l)}_{U_S})(a/b)/\mathrm{New}^{(l)}_{S_*}(a/b)$$

are **free***. In the following discussion, we shall write $\mathrm{New}^{\mathrm{tor},(l)}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{New}^{(l)}_{S_*}(a/b) \otimes \mathbb{Q}/\mathbb{Z}$.*

Proof. Assertion (i) (respectively, (ii)) is, in essence, the content of [Kane], Proposition 1 (respectively, Proposition 3.3, (ii), (iii)). Assertion (iii) follows formally from assertion (ii). Finally, we consider assertion (iv). By Proposition 3.3, (iii), it follows that any free Lie algebra over $\mathbb{F}_l$ with ≥ 2 generators is *center-free*. Thus, let M be the module determined by any *faithful* representation [e.g., when the cardinality of S_* is ≥ 2, the *adjoint representation*] of the free Lie algebra $\mathcal{F}$ over $\mathbb{F}_l$ in the formal generators $\zeta_{x'}$, where $x' \in S_*$. Now observe that we obtain an action of $\mathrm{Gr}_{\mathbb{F}_l}(\Delta^{(l)}_{U_S})$ on $M' \stackrel{\mathrm{def}}{=} M \oplus M$ as follows: We let $\alpha_2, \dots \alpha_g$; $\beta_2, \dots \beta_g$; $\zeta_{x'}$, where $x' \in S_0 \stackrel{\mathrm{def}}{=} S \backslash S_*$, act by multiplication by 0 on M'. We let α_1, β_1 act on $M' = M \oplus M$ via the matrices

$$\begin{pmatrix} 0 & \sum_{x' \in S_*} \zeta_{x'} \\ 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

respectively. Finally, we let $\zeta_{x'}$, where $x' \in S_*$, act on M' via the following matrix:

$$\begin{pmatrix} \zeta_{x'} & 0 \\ 0 & -\zeta_{x'} \end{pmatrix}$$

Thus, [by assertion (i)] M' determines a representation of $\mathrm{Gr}_{\mathbb{F}_l}(\Delta^{(l)}_{U_S})$ whose restriction to the image of $\mathrm{New}^{(l)}_{S_*} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$ in $\mathrm{Gr}_{\mathbb{F}_l}(\Delta^{(l)}_{U_S})$ determines [via the natural surjection $\mathcal{F} \twoheadrightarrow \mathrm{New}^{(l)}_{S_*} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$] a *faithful* representation of $\mathcal{F}$. Thus, we conclude that the natural surjection $\mathcal{F} \twoheadrightarrow \mathrm{New}^{(l)}_{S_*} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$ is an *isomorphism*, and that $\mathrm{New}^{(l)}_{S_*} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$ *injects* into $\mathrm{Gr}_{\mathbb{F}_l}(\Delta^{(l)}_{U_S})$. Assertion (iv) now follows formally. $\bigcirc$

Remark 3.4.1. The author wishes to thank A. Tamagawa for pointing out to him the content of Proposition 3.4, (i).

Next, let us *fix* an $x_* \in S$, as well as a *choice of decomposition group*

$$D_{x_*}[U_S] \subseteq \Pi_{U_S}$$

[i.e., among the various Π_{U_S}-conjugates of this subgroup] associated to x_*. [Thus, $D_{x_*}[U_S]$ determines a *specific* subgroup [i.e., not just a conjugacy class of subgroups] $D^{(l)}_{x_*}[U_S] \subseteq \Pi^{(l)}_{U_S}$.] Recall that the natural exact sequence

$$1 \to I^{(l)}_{x_*}[U_S] \to D^{(l)}_{x_*}[U_S] \to G_k \to 1$$

splits. [Indeed, extracting l-power roots of any local uniformizer of X at x_* determines such a splitting — cf., e.g., the discussion at the beginning of [Mzk8], §4.] In the following discussion, we shall *fix a splitting*

$$G_k \to D^{(l)}_{x_*}[U_S]$$

of this exact sequence. Thus, this splitting determines a natural action of G_k [by conjugation] on $\Delta^{(l)}_{U_S}$, hence also on

$$\mathrm{Lin}^{(l)}_{U_S}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Lin}(\Delta^{(l)}_{U_S}(a/b))(\mathbb{Q}_l); \quad \mathrm{Lie}^{(l)}_{U_S}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Lie}(\Delta^{(l)}_{U_S}(a/b))$$
$$\mathrm{Gr}_{\mathbb{Q}_l}(\Delta^{(l)}_{U_S})(a/b)$$

[where $a, b \in \mathbb{Z}$; $1 \leq a \leq b$]. Write

$$F_k \in G_k$$

for the *Frobenius element* of G_k. In the following, we shall denote the *cardinality of k* by q_k.

Proposition 3.5. (Galois Invariant Splitting) *Let $a, b \in \mathbb{Z}$, $1 \leq a \leq b$.*

(i) The **eigenvalues** *of the action of F_k on $\mathrm{Lie}^{(l)}_{U_S}(a/a+1)$ are algebraic numbers all of whose complex absolute values are equal to $q_k^{a/2}$.*

(ii) There is a **unique G_k-equivariant isomorphism** *of Lie algebras*

$$\mathrm{Lie}^{(l)}_{U_S}(a/b) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^{(l)}_{U_S})(a/b)$$

which induces the identity isomorphism $\mathrm{Lie}^{(l)}_{U_S}(c/c+1) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^{(l)}_{U_S})(c/c+1)$, for all $c \in \mathbb{Z}$ such that $a \leq c \leq b-1$.

(iii) The isomorphism of (ii) together with the natural inclusions $I_x^{(l)}[U_S] \hookrightarrow \Delta_{U_S}^{(l)}$ for $x \in S$ [which are well-defined up to $\Delta_{U_S}^{(l)}$-conjugation] determine a G_k- **equivariant morphism**

$$\left(\bigoplus_{x \in S} I_x^{(l)}[U_S] \otimes \mathbb{Q}_l\right) \oplus \mathrm{Lie}_{U_S}^{(l)}(1/2) \to \mathrm{Lie}_{U_S}^{(l)}(1/\infty)$$

which exhibits, in a G_k-equivariant fashion, $\mathrm{Lie}_{U_S}^{(l)}(1/\infty)$ as the quotient of the completion [with respect to the filtration topology] of the **free Lie algebra** *generated by the finite dimensional $\mathbb{Q}_l$-vector space*

$$\left(\bigoplus_{x \in S} I_x^{(l)}[U_S] \otimes \mathbb{Q}_l\right) \oplus \mathrm{Lie}_{U_S}^{(l)}(1/2)$$

[equipped with a **natural grading**, *hence also a* **filtration**, *by taking the $I_x^{(l)}[U_S] \otimes \mathbb{Q}_l$ to be of weight 2, $\mathrm{Lie}_{U_S}^{(l)}(1/2)$ to be of weight 1], by the* **single relation** *determined by the image of the morphism*

$$M_X^{(l)} \otimes \mathbb{Q}_l \hookrightarrow \left(\bigoplus_{x \in S} I_x^{(l)}[U_S] \otimes \mathbb{Q}_l\right) \oplus (I_{\mathrm{cup}}^{(l)} \otimes \mathbb{Q}_l)$$

of Proposition 3.4, (i), tensored with $\mathbb{Q}_l$.

(iv) For each $g \in \mathrm{Lin}_{U_S}^{(l)}(1/\infty)$, there exists a **unique** *$h \in \mathrm{Lin}_{U_S}^{(l)}(1/\infty)$ such that*

$$F_k \circ \mathrm{Inn}_g = \mathrm{Inn}_h \circ F_k \circ \mathrm{Inn}_{h^{-1}}$$

[where "Inn" denotes the inner automorphism of $\mathrm{Lin}_{U_S}^{(l)}(1/\infty)$ defined by conjugation by the subscripted element]. Moreover, when g lies in the image of $I_{x_*}^{(l)} \otimes \mathbb{Q}_l$ [which is stabilized by the action of F_k], h also lies in the image of $I_{x_*}^{(l)} \otimes \mathbb{Q}_l$.

Proof. Assertion (i) follows immediately from the "Riemann hypothesis for abelian varieties over finite fields" — cf., e.g., [Mumf], p. 206. Assertion (ii) (respectively, (iii); (iv)) follows formally from assertion (i) (respectively, and Proposition 3.4, (i); and successive approximation of h with respect to the natural filtration $\mathrm{Lin}_{U_S}^{(l)}(a/\infty) \subseteq \mathrm{Lin}_{U_S}^{(l)}(1/\infty)$). ○

Next, let

$$S_* \subseteq S$$

be a subset such that $x_* \in S_*$; $S_0 \stackrel{\mathrm{def}}{=} S \backslash S_*$. In the following, we shall regard $\mathrm{Lin}_{U_S}^{(l)}(a/b)$ as being equipped with its natural *l-adic topology*. Thus, G_k acts continuously on $\mathrm{Lin}_{U_S}^{(l)}(a/b)$, $\mathrm{Lie}_{U_S}^{(l)}(a/b)$, and we have *natural G_k-equivariant surjections*:

$$\mathrm{Lin}_{U_S}^{(l)}(a/b) \twoheadrightarrow \mathrm{Lin}_{U_{S_0}}^{(l)}(a/b); \quad \mathrm{Lie}_{U_S}^{(l)}(a/b) \twoheadrightarrow \mathrm{Lie}_{U_{S_0}}^{(l)}(a/b)$$

Let us write

$$\mathrm{Lin}^{(l)}_{U_S/U_{S_0}}(a/b); \quad \mathrm{Lie}^{(l)}_{U_S/U_{S_0}}(a/b)$$

for the *kernels* of these surjections. In the following, to *simplify the notation*, we shall often *omit the superscript* (l) from the objects "$\mathrm{Lin}^{(l)}$", "$\mathrm{Lie}^{(l)}$", "$\mathrm{New}^{(l)}$", "$\mathrm{New}^{\mathrm{tor},(l)}$" introduced above and write:

$$\mathrm{Lin}_{U_S}(a/b); \quad \mathrm{Lie}_{U_S}(a/b); \quad \mathrm{Lin}_{U_{S_0}}(a/b); \quad \mathrm{Lie}_{U_{S_0}}(a/b)$$
$$\mathrm{Lin}_{U_S/U_{S_0}}(a/b); \quad \mathrm{Lie}_{U_S/U_{S_0}}(a/b); \quad \mathrm{New}_{S_*}(a/b); \quad \mathrm{New}^{\mathrm{tor}}_{S_*}(a/b)$$

Also, we shall write:

$$\mathrm{New}^{\mathbb{Q}}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{New}_{S_*}(a/b) \otimes \mathbb{Q}; \quad \Delta^{\mathrm{Lie}}_{U_S} \stackrel{\mathrm{def}}{=} \mathrm{Lin}_{U_S}(1/\infty) \times_{\mathrm{Lin}_{U_{S_0}}(1/\infty)} \Delta_{U_{S_0}}$$

Note that, for $\mathbb{Z} \ni b \geq 1$, we have a *natural* G_k*-equivariant inclusion*

$$\mathrm{Lin}_{U_S/U_{S_0}}(b+1/\infty) \xrightarrow{\sim} \mathrm{Lin}_{U_S/U_{S_0}}(b+1/\infty) \times_{\{1\}} \{1\} \hookrightarrow \mathrm{Lin}_{U_S}(1/\infty) \times_{\mathrm{Lin}_{U_{S_0}}(1/\infty)} \Delta_{U_{S_0}} = \Delta^{\mathrm{Lie}}_{U_S}$$

whose image forms a normal subgroup of $\Delta^{\mathrm{Lie}}_{U_S}$; write

$$\Delta^{\mathrm{Lie}}_{U_S} \twoheadrightarrow \Delta^{\mathrm{Lie}\leq b}_{U_S}$$

for the quotient of $\Delta^{\mathrm{Lie}}_{U_S}$ by this normal subgroup. Also, we have a *natural* G_k*-equivariant [composite] inclusion*

$$\mathrm{New}^{\mathbb{Q}}_{S_*}(b+1/b+2) \hookrightarrow \mathrm{Lie}_{U_S/U_{S_0}}(b+1/b+2) \xrightarrow{\sim} \mathrm{Lin}_{U_S/U_{S_0}}(b+1/b+2) \hookrightarrow \Delta^{\mathrm{Lie}\leq b+1}_{U_S}$$

whose image forms a normal subgroup of $\Delta^{\mathrm{Lie}\leq b+1}_{U_S}$; write

$$\Delta^{\mathrm{Lie}\leq b+1}_{U_S} \twoheadrightarrow \Delta^{\mathrm{Lie}\leq b+}_{U_S}$$

for the quotient of $\Delta^{\mathrm{Lie}\leq b+1}_{U_S}$ by this normal subgroup. Thus, we have *natural* G_k*-equivariant homomorphisms of topological groups*:

$$\Delta_{U_S} \to \Delta^{\mathrm{Lie}}_{U_S} \twoheadrightarrow \Delta^{\mathrm{Lie}\leq b+}_{U_S} \twoheadrightarrow \Delta^{\mathrm{Lie}\leq b}_{U_S} \twoheadrightarrow \Delta_{U_{S_0}}$$

[the last three of which are easily verified to be *surjective*]. Moreover, forming the *semi-direct product* with G_k [via the natural actions of G_k] yields topological groups and homomorphisms as follows:

$$\Pi_{U_S} \to \Pi^{\mathrm{Lie}}_{U_S} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq b+}_{U_S} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq b}_{U_S} \twoheadrightarrow \Pi_{U_{S_0}}$$

Also, we note that we have *natural exact sequences*:

$$1 \to \mathrm{Lin}_{U_S/U_{S_0}}(1/\infty) \to \Delta^{\mathrm{Lie}}_{U_S} \to \Delta_{U_{S_0}} \to 1$$
$$1 \to \mathrm{Lin}_{U_S/U_{S_0}}(1/\infty) \to \Pi^{\mathrm{Lie}}_{U_S} \to \Pi_{U_{S_0}} \to 1$$

Definition 3.6.

(i) We shall refer to $\Delta_{U_S}^{\mathrm{Lie}}$ (respectively, $\Pi_{U_S}^{\mathrm{Lie}}$; $\Delta_{U_S}^{\mathrm{Lie}\leq b}$; $\Pi_{U_S}^{\mathrm{Lie}\leq b}$; $\Delta_{U_S}^{\mathrm{Lie}\leq b+}$; $\Pi_{U_S}^{\mathrm{Lie}\leq b}$) as the [$l$-adic] *Lie-ification* (respectively, *Lie-ification*; *Lie-ification, truncated to order* b; *Lie-ification, truncated to order* b; *Lie-ification, truncated to order* $b+$; *Lie-ification, truncated to order* $b+$) of Δ_{U_S} (respectively, Π_{U_S}; Δ_{U_S}; Π_{U_S}; Δ_{U_S}; Π_{U_S}) [over $\Delta_{U_{S_0}}$ (respectively, $\Pi_{U_{S_0}}$; $\Delta_{U_{S_0}}$; $\Pi_{U_{S_0}}$; $\Delta_{U_{S_0}}$; $\Pi_{U_{S_0}}$)].

(ii) Observe that it follows immediately from the definitions that, for $\mathbb{Z} \ni b \geq 1$, we have *natural exact sequences*

$$1 \to \mathrm{New}_{S_*}^{\mathbb{Q}}(b+1/b+2) \to \Delta_{U_S}^{\mathrm{Lie}\leq b+1} \to \Delta_{U_S}^{\mathrm{Lie}\leq b+} \to 1$$
$$1 \to \mathrm{New}_{S_*}^{\mathbb{Q}}(b+1/b+2) \to \Pi_{U_S}^{\mathrm{Lie}\leq b+1} \to \Pi_{U_S}^{\mathrm{Lie}\leq b+} \to 1$$

on which $\Pi_{U_S}^{\mathrm{Lie}\leq b+1}$ acts naturally by conjugation. [Here, we note in passing that it is immediate from the definitions that the submodule

$$\mathrm{New}_{S_*}(b+1/b+2) \subseteq \mathrm{New}_{S_*}^{\mathbb{Q}}(b+1/b+2)$$

is contained in the image of Δ_{U_S}.] In particular, we obtain a *natural inclusion*:

$$\mathrm{New}_{S_*}(b+1/b+2) \hookrightarrow \Delta_{U_S}^{\mathrm{Lie}\leq b+1} \ (\subseteq \Pi_{U_S}^{\mathrm{Lie}\leq b+1})$$

We shall refer to the quotients of $\Delta_{U_S}^{\mathrm{Lie}\leq b+1}$, $\Pi_{U_S}^{\mathrm{Lie}\leq b+1}$ by the image of this natural inclusion as the *toral Lie-ifications* $\Delta_{U_S}^{\mathrm{tor}\leq b+1}$, $\Pi_{U_S}^{\mathrm{tor}\leq b+1}$ of Δ_{U_S}, Π_{U_S} [over $\Delta_{U_{S_0}}$, $\Pi_{U_{S_0}}$]. Thus, we have *natural exact sequences*

$$1 \to \mathrm{New}_{S_*}^{\mathrm{tor}}(b+1/b+2) \to \Delta_{U_S}^{\mathrm{tor}\leq b+1} \to \Delta_{U_S}^{\mathrm{Lie}\leq b+} \to 1$$
$$1 \to \mathrm{New}_{S_*}^{\mathrm{tor}}(b+1/b+2) \to \Pi_{U_S}^{\mathrm{tor}\leq b+1} \to \Pi_{U_S}^{\mathrm{Lie}\leq b+} \to 1$$

on which $\Pi_{U_S}^{\mathrm{Lie}\leq b+1}$ acts naturally by conjugation.

(iii) Suppose that $U'_{S'_0} \to U_{S_0}$ is a *connected finite étale covering* that arises from an open subgroup $\Pi_{U'_{S'_0}} \subseteq \Pi_{U_{S_0}}$; write $X' \to X$ for the *normalization* of X in $U'_{S'_0}$. Then we shall say that the [ramified] covering $X' \to X$ is (S, S_0, Σ)-*admissible* if every closed point of X' that lies over a point of S is rational over the base field k' of X', and, moreover, $\Pi_{U'_{S'_0}}$ is a *characteristic* subgroup of $\Pi_{U_{S_0}}$.

Remark 3.6.1. Note that it follows immediately from the definition of $\Pi_{U_S}^{\mathrm{Lie}}$ [cf. also Proposition 3.5, (iii)] that we obtain a *natural subgroup*

$$D_{x_*}^{\mathrm{Lie}} \stackrel{\mathrm{def}}{=} \left(I_{x_*}^{(l)}[U_S] \otimes \mathbb{Q}\right) \rtimes G_k \subseteq \Pi_{U_S}^{\mathrm{Lie}}$$

which contains the image of the decomposition group $D_{x_*}[U_S] \subseteq \Pi_{U_S}$ via the natural homomorphism $\Pi_{U_S} \to \Pi_{U_S}^{\mathrm{Lie}}$. Let us write, for $\mathbb{Z} \ni b \geq 1$, $D_{x_*}^{\mathrm{Lie}\leq b} \subseteq \Pi_{U_S}^{\mathrm{Lie}\leq b}$

for the image of $D^{\mathrm{Lie}}_{x_*}$ in $\Pi^{\mathrm{Lie}\leq b}_{U_S}$; $I^{\mathrm{Lie}}_{x_*} \stackrel{\mathrm{def}}{=} D^{\mathrm{Lie}}_{x_*} \bigcap \Delta^{\mathrm{Lie}}_{U_S}$; $I^{\mathrm{Lie}\leq b}_{x_*} \stackrel{\mathrm{def}}{=} D^{\mathrm{Lie}\leq b}_{x_*} \bigcap \Delta^{\mathrm{Lie}\leq b}_{U_S}$. [Also, we shall use similar notation when "b" is replaced by "$b+$".]

Proposition 3.7. **(Center-freeness of Lie-ification)** $\Delta^{\mathrm{Lie}}_{U_S}$ *is center-free.*

Proof. Since $\Delta_{U_{S_0}}$ is *center-free* [cf. Proposition 1.8, (iii)], and the natural morphism $\Delta^{\mathrm{Lie}}_{U_S} \to \Delta_{U_{S_0}}$ is *surjective*, it suffices to verify that the centralizer in $\mathrm{Lin}_{U_S}(1/\infty)$ of the image of $\Delta^{\mathrm{Lie}}_{U_S}$ is trivial. But the image of $\Delta^{\mathrm{Lie}}_{U_S}$ in $\mathrm{Lin}_{U_S}(1/\infty)$ contains the image of Δ_{U_S} in $\mathrm{Lin}_{U_S}(1/\infty)$. In particular, it follows that the centralizer in question lies in the *center* of $\mathrm{Lin}_{U_S}(1/\infty)$. Thus, Proposition 3.7 follows from Propositions 3.4, (ii) [or, alternatively, (iii)]. ◯

Remark 3.7.1. Observe that changing the *choice of splitting*

$$G_k \to D^{(l)}_{x_*}[U_S]$$

affects the image of the element $F_k \in G_k$ via the composite of the inclusion $G_k \hookrightarrow \Pi_{U_S}$ with the morphisms

$$\Pi_{U_S} \to \Pi^{\mathrm{Lie}}_{U_S}; \quad \Pi_{U_S} \to \Pi^{\mathrm{Lie}\leq b}_{U_S}; \quad \Pi_{U_S} \to \Pi^{\mathrm{Lie}\leq b+}_{U_S}$$

by *conjugation by an element* $h \in I^{\mathrm{Lie}}_{x_*}$, which, up to a denominator dividing $q_k - 1$, lies in the image of $I_{x_*}[U_S] \subseteq \Delta_{U_S}$ — cf. Proposition 3.5, (iv); Proposition 3.7. In particular, it follows that changing the choice of splittings $G_k \to D^{(l)}_{x_*}[U_S]$ affects the *Galois invariant splittings* of Proposition 3.5, (ii), by *conjugation by* h. Put another way, if we *identify* the "$\mathrm{Lin}_{U_S}(1/\infty)$", "$\mathrm{Lin}_{U_{S_0}}(1/\infty)$" portions of $\Delta^{\mathrm{Lie}}_{U_S}$ [cf. the definition of $\Delta^{\mathrm{Lie}}_{U_S}$] with the [completions of the] corresponding graded objects "$\mathrm{Gr}_{\mathbb{Q}_l}(-)(1/\infty)$" via the Galois invariant splittings of Proposition 3.5, (ii), then it follows that: *Changing the choice of splitting* $G_k \to D^{(l)}_{x_*}[U_S]$ *affects the images of the morphisms*

$$\Pi_{U_S} \to \Pi^{\mathrm{Lie}}_{U_S}; \quad \Pi_{U_S} \to \Pi^{\mathrm{Lie}\leq b}_{U_S}; \quad \Pi_{U_S} \to \Pi^{\mathrm{Lie}\leq b+}_{U_S}$$

[where $\mathbb{Z} \ni b \geq 1$*] by conjugation by* h.

In light of Proposition 3.7, we may apply the exact sequence "$1 \to (-) \to \mathrm{Aut}(-) \to \mathrm{Out}(-) \to 1$" [cf. §0] to construct the following *topological group*:

$$\Delta^{\mathrm{LIE}}_{U_S} \stackrel{\mathrm{def}}{=} \varprojlim_{X'} \mathrm{Aut}(\Delta^{\mathrm{Lie}}_{U'_{S'}}) \times_{\mathrm{Out}(\Delta^{\mathrm{Lie}}_{U'_{S'}})} \mathrm{Gal}(X'_{\overline{k}}/X_{\overline{k}})$$

[where $X' \to X$ ranges over the (S, S_0, Σ)-*admissible coverings* of X; $U'_{S'} \subseteq X'$ is the open subscheme determined by the complement of the set S' of closed points of X' that lie over points of S]. Note that G_k *acts naturally* on $\Delta^{\mathrm{LIE}}_{U_S}$; thus, we may form the *semi-direct product* of $\Delta^{\mathrm{LIE}}_{U_S}$ with G_k to obtain a *topological group* $\Pi^{\mathrm{LIE}}_{U_S}$.

Next, let us observe that, for $\mathbb{Z} \ni b \geq 1$, the various quotients $\Delta^{\mathrm{Lie}}_{U'_{S'}} \twoheadrightarrow \Delta^{\mathrm{tor}\leq b+1}_{U'_{S'}} \twoheadrightarrow \Delta^{\mathrm{Lie}\leq b+}_{U'_{S'}} \twoheadrightarrow \Delta^{\mathrm{Lie}\leq b}_{U'_{S'}}$ determine *quotients of topological groups* $\Delta^{\mathrm{LIE}}_{U_S} \twoheadrightarrow \Delta^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Delta^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Delta^{\mathrm{LIE}\leq b}_{U_S}$, $\Pi^{\mathrm{LIE}}_{U_S} \twoheadrightarrow \Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b}_{U_S}$. Thus, we obtain *natural homomorphisms of topological groups*:

$$\Delta_{U_S} \to \Delta^{\mathrm{LIE}}_{U_S} \twoheadrightarrow \Delta^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Delta^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Delta^{\mathrm{LIE}\leq b}_{U_S} \twoheadrightarrow \Delta_{U_{S_0}}$$
$$\Pi_{U_S} \to \Pi^{\mathrm{LIE}}_{U_S} \twoheadrightarrow \Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b}_{U_S} \twoheadrightarrow \Pi_{U_{S_0}}$$

We shall denote by

$$\Delta^{\leq b+}_{U_S} \subseteq \Delta^{\mathrm{LIE}\leq b+}_{U_S}; \quad \Pi^{\leq b+}_{U_S} \subseteq \Pi^{\mathrm{LIE}\leq b+}_{U_S}; \quad \Delta^{\leq b}_{U_S} \subseteq \Delta^{\mathrm{LIE}\leq b}_{U_S}; \quad \Pi^{\leq b}_{U_S} \subseteq \Pi^{\mathrm{LIE}\leq b}_{U_S}$$

the respective images of Δ_{U_S}, Π_{U_S} via these natural homomorphisms. Thus, one may think of $\Delta^{\leq b}_{U_S}$, $\Pi^{\leq b}_{U_S}$ as being a sort of *"canonical integral structure"* on the "inverse limit truncated Lie-ifications" $\Delta^{\mathrm{LIE}\leq b}_{U_S}$, $\Pi^{\mathrm{LIE}\leq b}_{U_S}$.

Here, we note in passing, relative to the theory of §1, 2, that [it is immediate from the definitions that] when $S = S_*$ [so $U_{S_0} = X$], the quotient $\Pi_{U_S} \twoheadrightarrow \Pi^{\leq 2}_{U_S}$ is the *maximal cuspidally abelian quotient* of Π_{U_S} [cf. Proposition 1.14, (i)].

Next, let us observe that in the inverse limit used to define $\Delta^{\mathrm{LIE}}_{U_S}$, $\Pi^{\mathrm{LIE}}_{U_S}$, the various "$I^{\mathrm{Lie}}_{x_*}$", "$D^{\mathrm{Lie}}_{x_*}$" [cf. Remark 3.6.1] form a *compatible system*, hence give rise to subgroups

$$I^{LIE}_{x_*} \subseteq D^{\mathrm{LIE}}_{x_*} \subseteq \Pi^{\mathrm{LIE}}_{U_S}; \quad I^{\mathrm{LIE}\leq b}_{x_*} \subseteq D^{\mathrm{LIE}\leq b}_{x_*} \subseteq \Pi^{\mathrm{LIE}\leq b}_{U_S}$$

together with natural exact sequences and isomorphisms [when $b \geq 2$]

$$1 \to I^{LIE}_{x_*} \to D^{\mathrm{LIE}}_{x_*} \to G_k \to 1$$
$$1 \to I^{\mathrm{LIE}\leq b}_{x_*} \to D^{\mathrm{LIE}\leq b}_{x_*} \to G_k \to 1$$
$$I^{LIE}_{x_*} \cong I^{\mathrm{LIE}\leq b}_{x_*} \cong I^{(l)}_{x_*}[U_S] \otimes \mathbb{Q}$$

[and similarly when "b" is replaced by "$b+$"]. Also, the images of the subgroups $I_{x_*}[U_S]$, $D_{x_*}[U_S]$ of Π_{U_S} determine subgroups

$$I^{\leq b}_{x_*} \subseteq D^{\leq b}_{x_*} \subseteq \Pi^{\leq b}_{U_S}$$

[and similarly when "b" is replaced by "$b+$"].

In the following, let us write [cf. Proposition 3.4, (iv)]

$$\mathrm{Cnw}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Cnw}^{(l)}_{S_*}(a/b); \quad \mathrm{Cnw}^{\mathbb{Q}}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Cnw}^{(l)}_{S_*}(a/b) \otimes \mathbb{Q}$$

[where $a, b \in \mathbb{Z}$, $1 \leq a \leq b$].

Before proceeding, let us observe that [it is immediate from the definitions that] the natural surjections

$$\Delta_{U_S}^{\mathrm{LIE}\leq 1+} \twoheadrightarrow \Delta_{U_S}^{\mathrm{LIE}\leq 1} \twoheadrightarrow \Delta_{U_{S_0}}; \quad \Pi_{U_S}^{\mathrm{LIE}\leq 1+} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq 1} \twoheadrightarrow \Pi_{U_{S_0}}$$

are *isomorphisms*. On the other hand, for $b \geq 2$, we have the following result:

Proposition 3.8. (Plus Liftings of Canonical Integral Structures) *For* $\mathbb{Z} \ni b \geq 2$*:*

(i) The natural surjections $\Delta_{U_S}^{\leq b+} \twoheadrightarrow \Delta_{U_S}^{\leq b}$, $\Pi_{U_S}^{\leq b+} \twoheadrightarrow \Pi_{U_S}^{\leq b}$ *are* **isomorphisms***.*

(ii) Any two liftings of the natural inclusion $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b}$ *to inclusions* $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}$ *differ by* **conjugation** *in* $\Pi_{U_S}^{\mathrm{LIE}\leq b+}$ *by a* **unique** *element of the kernel of* $\Pi_{U_S}^{\mathrm{LIE}\leq b+} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b}$*.*

(iii) Any two liftings of the natural inclusion $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b}$ *to inclusions* $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}$ *whose images* **contain** $D_{x_*}^{\leq b+}$ *in fact* **coincide***.*

Proof. First, we consider assertion (i). It follows immediately from the definitions that the kernel in question

$$\mathrm{Ker}(\Delta_{U_S}^{\leq b+} \twoheadrightarrow \Delta_{U_S}^{\leq b}) = \mathrm{Ker}(\Pi_{U_S}^{\leq b+} \twoheadrightarrow \Pi_{U_S}^{\leq b})$$

is given by the inverse limit

$$\varprojlim_{X'} \ \mathrm{Cnw}_{S'_*}(b+1/b+2)$$

[where $X' \to X$ ranges over the (S, S_0, Σ)-*admissible coverings* of X; S'_* (respectively, S') is the set of closed points of X' that lie over points of S_* (respectively, S)]. On the other hand, it follows from the definition of "$\mathrm{Cnw}_{S'_*}(b+1/b+2)$" that $\mathrm{Cnw}_{S'_*}(b+1/b+2)$ is generated by certain successive brackets of the various generators of the Lie algebra $\mathrm{Gr}(\Delta_{U'_{S'}}^{(l)})$ [cf. Proposition 3.4, (i)] with the property that at least one of the generators appearing in the successive bracket is [in the notation of Proposition 3.4, (i)] either one of the [analogue for X' of the] "$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$" or one of the "$\zeta_{x'}$", where $x' \in S'_0 \stackrel{\mathrm{def}}{=} S' \backslash S'_*$. Moreover, since, by taking $\Pi_{U''_{S''_0}} \subseteq \Pi_{U'_{S'_0}}$ to be *sufficiently small*, one may arrange that the image of $\Delta^{\mathrm{ab}}_{U''_{S''_0}}$ in $\Delta^{\mathrm{ab}}_{U'_{S'_0}}$ be contained in an *arbitrarily small* open subgroup of $\Delta^{\mathrm{ab}}_{U'_{S'_0}}$, it thus follows that the above inverse limit *vanishes*. This completes the proof of assertion (i).

Next, let us observe that to prove assertion (ii), it suffices — in light of the *natural isomorphism*

$$\mathrm{Ker}(\Pi_{U_S}^{\mathrm{LIE}\leq b+} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b}) \xrightarrow{\sim} \varprojlim_{X'} \ \mathrm{Cnw}^{\mathbb{Q}}_{S'_*}(b+1/b+2)$$

[where X', S'_* are as above] — to show that

$$H^i(\Pi_{U_S}^{\leq b}, \mathrm{Cnw}_{S'_*}^{\mathbb{Q}}(b+1/b+2)) = 0$$

for $i = 0, 1$, each S'_* as above. Since the action of $\Delta_{U_S}^{\leq b}$ on $\mathrm{Cnw}_{S'_*}^{\mathbb{Q}}(b+1/b+2)$ clearly factors through a *finite quotient* of $\Delta_{U_S}^{\leq b} \twoheadrightarrow \Delta_{U_{S_0}}$, it thus suffices to observe [by considering the Leray spectral sequence associated to the surjection $\Pi_{U_S}^{\leq b} \twoheadrightarrow G_k$] that the action of F_k on $\mathrm{Cnw}_{S'_*}^{\mathbb{Q}}(b+1/b+2)$ is *"of weight* $b+1 \geq 3$*"*, while the action of F_k on $(\Delta_{U'_{S'}}^{(l)})^{\mathrm{ab}}$ is *"of weight* ≤ 2*"* [cf. Proposition 3.5, (i)]. This completes the proof of assertion (ii).

Finally, we consider assertion (iii). First, let us observe that any two liftings of the natural inclusion $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b}$ to inclusions $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}$ whose images contain $D_{x_*}^{\leq b+} \xrightarrow{\sim} D_{x_*}^{\leq b}$ [since $b \geq 2$] in fact *coincide* on $D_{x_*}^{\leq b} \subseteq \Pi_{U_S}^{\leq b}$. Thus, by assertion (ii), it suffices to verify that the submodule of F_k*-invariants* of

$$\mathrm{Ker}(\Pi_{U_S}^{\mathrm{LIE}\leq b+} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b})$$

is zero. But in light of the natural isomorphism

$$\mathrm{Ker}(\Pi_{U_S}^{\mathrm{LIE}\leq b+} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b}) \xrightarrow{\sim} \varprojlim_{X'} \mathrm{Cnw}_{S'_*}^{\mathbb{Q}}(b+1/b+2)$$

[where S'_* is as above], this follows from Proposition 3.5, (i). This completes the proof of assertion (iii). ○

Next, for $\mathbb{Z} \ni b \geq 1$, let us denote by

$$\Delta_{U_S}^{\leq b++} \subseteq \Delta_{U_S}^{\mathrm{TOR}\leq b+1}; \quad \Pi_{U_S}^{\leq b++} \subseteq \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$$

the respective images of Δ_{U_S}, Π_{U_S} via the natural homomorphisms considered above and by

$$I_{x_*}^{\leq b++} \subseteq D_{x_*}^{\leq b++} \subseteq \Pi_{U_S}^{\leq b++}$$

the images of the subgroups $I_{x_*}[U_S]$, $D_{x_*}[U_S]$ of Π_{U_S}. Observe that it follows from the definition of $\Delta_{U_S}^{\mathrm{TOR}\leq b+1}$, $\Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ that the natural surjections $\Delta_{U_S}^{\leq b++} \twoheadrightarrow \Delta_{U_S}^{\leq b+}$, $\Pi_{U_S}^{\leq b++} \twoheadrightarrow \Pi_{U_S}^{\leq b+}$ are, in fact, *isomorphisms*. Thus, by Proposition 3.8, (i), we obtain a *commutative diagram of natural homomorphisms*

$$\begin{array}{ccccccc}
\Pi_{U_S}^{\leq b+1} & \twoheadrightarrow & \Pi_{U_S}^{\leq b++} & \xrightarrow{\sim} & \Pi_{U_S}^{\leq b+} & \xrightarrow{\sim} & \Pi_{U_S}^{\leq b} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Pi_{U_S}^{\mathrm{LIE}\leq b+1} & \twoheadrightarrow & \Pi_{U_S}^{\mathrm{TOR}\leq b+1} & \twoheadrightarrow & \Pi_{U_S}^{\mathrm{LIE}\leq b+} & \twoheadrightarrow & \Pi_{U_S}^{\mathrm{LIE}\leq b}
\end{array}$$

[where the vertical arrows are the *natural inclusions*; all of the horizontal arrows are surjections; the second two upper horizontal arrows are isomorphisms]. Moreover, it follows immediately from the definitions that the *first square* in this commutative diagram is *cartesian*. That is to say, the subgroup $\Pi_{U_S}^{\leq b+1} \subseteq \Pi_{U_S}^{\mathrm{LIE}\leq b+1}$ may be thought of as the *inverse image* via the natural surjection $\Pi_{U_S}^{\mathrm{LIE}\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ of the image of a certain *lifting* of the natural inclusion $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}$ [cf. Proposition 3.8, (i)] to an inclusion $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$.

Let us write:

$$\Pi_{U_S}^{\leq b}[\mathrm{new}] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{U_S}^{\leq b} \twoheadrightarrow \Pi_{U_{S_0}})$$
$$\Pi_{U_S}^{\leq b++}[\mathrm{new}] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{U_S}^{\leq b++} \twoheadrightarrow \Pi_{U_{S_0}})$$

for the *"new-cuspidal" subgroup* of $\Pi_{U_S}^{\leq b}$.

Proposition 3.9. (Extensions of Canonical Integral Structures)

(i) Let $b \stackrel{\mathrm{def}}{=} 1$. Then any two liftings of the natural inclusion $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}$ to inclusions $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ whose restrictions to the cuspidal subgroup $\Pi_{U_S}^{\leq b}[\mathrm{csp}] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{U_S}^{\leq b} \twoheadrightarrow \Pi_X)$ of $\Pi_{U_S}^{\leq b}$ **coincide** *differ by* **conjugation** *in $\Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ by an element of the kernel of $\Pi_{U_S}^{\mathrm{TOR}\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}$.*

(ii) Let $\mathbb{Z} \ni b \geq 2$; suppose that S_* is of **cardinality one**. *Then any two liftings of the natural inclusion $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}$ to inclusions $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ whose images* **contain** *$I_{x_*}^{\leq b++}$ differ by* **conjugation** *in $\Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ by an element of the kernel of $\Pi_{U_S}^{\mathrm{TOR}\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}$.*

(iii) Suppose that S_* is of **cardinality one**. *Let β be an automorphism of the profinite group $\Pi_{U_S}^{\leq b+1}$ that satisfies the following two conditions: (a) β preserves and induces the identity on the quotient $\Pi_{U_S}^{\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\leq b}$; (b) β preserves the subgroup $I_{x_*}^{\leq b+1} \subseteq \Pi_{U_S}^{\leq b+1}$. If $b = 1$, then we also assume that β induces the identity on the cuspidal subgroup $\Pi_{U_S}^{\leq b+1}[\mathrm{csp}] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{U_S}^{\leq b+1} \twoheadrightarrow \Pi_X)$ of $\Pi_{U_S}^{\leq b+1}$. Then β is a* $\mathrm{Ker}(\Pi_{U_S}^{\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\leq b})$**-inner automorphism**.

(iv) Suppose that S_* is of **cardinality one**. *Let β be an inner automorphism of the group $\Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ arising from an element of $\mathrm{Ker}(\Pi_{U_S}^{\mathrm{TOR}\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+})$. Suppose that for each $\gamma \in \Pi_{U_S}^{\leq b++} \subseteq \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$, β preserves the $\Pi_{U_S}^{\leq b++}[\mathrm{new}]$-conjugacy class of subgroups of $\Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ defined by $\gamma \cdot D_{x_*}^{\leq b++} \cdot \gamma^{-1}$. Then β is the* **identity** *automorphism.*

(v) Write

$$\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\leq\infty} \stackrel{\mathrm{def}}{=} \varprojlim_{b} \Pi_{U_S}^{\leq b}$$

for the the quotient of Π_{U_S} defined by the inverse limit of the $\Pi_{U_S}^{\leq b}$. Then $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\leq\infty}$ (respectively, the resulting quotient $\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{\leq\infty}$) is the **maximal "new-cuspidally" pro-*l* quotient** *of Π_{U_S} (respectively, Δ_{U_S}).*

Proof. First, we consider assertions (i), (ii). Observe that, for $\mathbb{Z} \ni b \geq 1$, the *difference* of any two liftings of the natural inclusion $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}$ to inclusions $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ determines a *compatible collection of cohomology classes*

$$\eta_{S'} \in H^1(\Pi_{U_S}^{\leq b}, \mathrm{New}_{S'_*}^{\mathrm{tor}}(b+1/b+2))$$

[where $X' \to X$ ranges over the (S, S_0, Σ)-*admissible coverings* of X; S'_* (respectively, S') is the set of closed points of X' that lie over points of S_* (respectively, S)].

Next, let us observe that by Proposition 3.5, (i), the zeroth cohomology module

$$H^0(\Pi_{U_S}^{\leq b}, \mathrm{New}_{S'_*}^{\mathrm{tor}}(b+1/b+2))$$

is *finite*. This finiteness implies that any [not necessarily compatible!] system of sections of a compatible system of torsors over $H^0(\Pi_{U_S}^{\leq b}, \mathrm{New}_{S'_*}^{\mathrm{tor}}(b+1/b+2))$ always admits a *compatible cofinal subsystem*. In light of the natural isomorphism

$$\mathrm{Ker}(\Pi_{U_S}^{\mathrm{TOR}\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}) \xrightarrow{\sim} \varprojlim_{X'} \mathrm{New}_{S'_*}^{\mathrm{tor}}(b+1/b+2))$$

[where X', S'_* are as described above], we thus conclude that in order to show that the two inclusions $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ differ by *conjugation* by an element of $\mathrm{Ker}(\Pi_{U_S}^{\mathrm{TOR}\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+})$, it suffices to show that the $\eta_{S'} = 0$.

Note that $\Pi_{U_S}^{\leq b}[\mathrm{new}]$ acts *trivially* on $\mathrm{New}_{S'_*}^{\mathrm{tor}}(b+1/b+2))$. Now I *claim* that:

> *If $b \geq 2$ (respectively, $b = 1$), then each $\eta_{S'}$ arises from a unique class [which, by abuse of notation, we shall also denote by $\eta_{S'}$] in*
>
> $$H^1(\Pi_{U_{S_0}}, \mathrm{New}_{S'_*}^{\mathrm{tor}}(b+1/b+2)) \textit{ (respectively, } H^1(\Pi_X, (\mathrm{New}_{S'_*}^{\mathrm{tor}}(b+1/b+2))^{\Pi_{U_{S_0}}[\mathrm{csp}]}))$$
>
> *[where $\Pi_{U_{S_0}}[\mathrm{csp}] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{U_{S_0}} \twoheadrightarrow \Pi_X)$].*

Indeed, if $b = 1$, this claim is immediate [cf. the statement of assertion (i)], so assume that $b \geq 2$, and that we are in the situation of assertion (ii). Now observe that, in light of our assumption that S_* is of *cardinality one*, it follows that $\Pi_{U_S}^{\leq b}[\mathrm{new}]$ (respectively, $\Pi_{U_S}^{\leq b++}[\mathrm{new}]$) is *topologically generated* by the $\Pi_{U_S}^{\leq b}$- (respectively, $\Pi_{U_S}^{\leq b++}$-) *conjugates* of $I_{x_*}^{\leq b}$ (respectively, $I_{x_*}^{\leq b++}$). Note, moreover, that it is immediate from the definitions that every element of $\mathrm{Ker}(\Pi_{U_S}^{\mathrm{TOR}\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+})$ *commutes* with $I_{x_*}^{\leq b++}$). In particular, it follows that the images of $\Pi_{U_S}^{\leq b}[\mathrm{new}]$ via the two inclusions $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ under consideration necessarily *coincide*. But this implies that each $\eta_{S'}$ arises from a unique class in $H^1(\Pi_{U_{S_0}}, \mathrm{New}_{S'_*}^{\mathrm{tor}}(b+1/b+2))$, thus completing the proof of the *claim*.

Next, [returning to the general situation involving *both* assertions (i) and (ii)] let

$$X'' \to X'$$

be a morphism of (S, S_0, Σ)-*admissible coverings* of X. Write $U''_{S''} \subseteq X'$ for the open subscheme determined by the complement of the set S'' of closed points of X'' that lie over points of S. Note that since the cohomology group $H^1(\Pi_{U_{S_0}}, \mathrm{New}^{\mathrm{tor}}_{S'_*}(b + 1/b+2))$ is unaffected by replacing X' by the result of base-changing X' to some finite extension of the base field of X', we may assume without loss of generality [from the point of viewing of showing that $\eta_{S'} = 0$] that $X'' \to X'$ induces an *isomorphism* between the *base fields* of X'', X'. Also, let us assume that the open subgroup $\Delta_{U''_{S''_0}} \subseteq \Delta_{U'_{S'_0}}$ *arises from some open subgroup* $H' \subseteq \Delta^{\mathrm{ab}}_{U'_{S'_0}}$ that is *preserved by the action of* $\Pi_{U_{S_0}}$. Thus, it follows that the covering $X'' \to X'$ is *abelian.* Set:

$$R' \stackrel{\mathrm{def}}{=} \mathbb{Z}_l; \quad R'' \stackrel{\mathrm{def}}{=} \mathbb{Z}_l[\mathrm{Gal}(X''/X')]$$

Thus, R'' is a commutative ring with unity whose underlying R'-module is *finite and free*; moreover, R'' admits a natural $\Pi_{U_{S_0}}$-action [induced by the conjugation action of $\mathrm{Gal}(X/X')$ on $\mathrm{Gal}(X''/X')$].

Next, let us observe that S'_*, S''_* admit *natural* $\Pi_{U_{S_0}}$-*actions* with respect to which we have *natural isomorphisms of* $\Pi_{U_{S_0}}$-*modules* [cf. Proposition 3.4, (i), (iv)]

$$\mathrm{New}_{S'_*}(1/2) \xrightarrow{\sim} R'[S'_*] \otimes M^{(l)}_X; \quad \mathrm{New}_{S''_*}(1/2) \xrightarrow{\sim} R'[S''_*] \otimes M^{(l)}_X$$

which determine *natural isomorphisms of* $\Pi_{U_{S_0}}$-*modules* as follows:

$$\mathrm{New}_{S'_*}(b+1/b+2) \xrightarrow{\sim} \mathfrak{Lie}^{b+1}_{R'}(R'[S'_*] \otimes M^{(l)}_X)$$
$$\mathrm{New}_{S''_*}(b+1/b+2) \xrightarrow{\sim} \mathfrak{Lie}^{b+1}_{R'}(R'[S''_*] \otimes M^{(l)}_X)$$

In the following, we shall *identify* the domains and codomains of these isomorphisms via these isomorphisms.

Next, let us observe that the R'-module $R'[S''_*]$ admits a natural R''-*module structure* that is compatible with the $\Pi_{U_{S_0}}$-action on R'', $R'[S''_*]$. Note, moreover, that $R'[S''_*]$ is a *free* R''-*module*, and that the natural augmentation $R'' \twoheadrightarrow R'$ [given by mapping all of the elements of $\mathrm{Gal}(X''/X')$ to 1] induces a natural isomorphism $R'[S''_*] \otimes_{R''} R' \xrightarrow{\sim} R'[S'_*]$. Also, we observe that any *choice of representatives* in S''_* of the $\Delta_{U'_{S'_0}}/\Delta_{U''_{S''_0}} = \mathrm{Gal}(X''/X')$-orbits of S''_* [where we note that the set of such orbits may be naturally identified with S'_*] determines an R''-*basis* of $R'[S''_*]$, hence [by considering *"Hall bases"* — cf., e.g., [Bour], Chapter II, §2.11] an R''-*basis* of $\mathfrak{Lie}^{b+1}_{R''}(R'[S''_*])$. In particular, it follows that the $\mathrm{Gal}(X''/X')$-module $\mathfrak{Lie}^{b+1}_{R''}(R'[S''_*])$ is an *"induced"* $\mathrm{Gal}(X''/X')$-module [in the terminology of the cohomology theory of finite groups]. Consideration of such bases also shows that we obtain *natural,* $\Pi_{U_{S_0}}$-*equivariant isomorphisms* [which are independent of the choices of representatives/bases!]

$$R'[S'_*] \xrightarrow{\sim} R'[S''_*]^{\mathrm{Gal}(X''/X')}; \quad \mathfrak{Lie}^{b+1}_{R'}(R'[S'_*]) \xrightarrow{\sim} \mathfrak{Lie}^{b+1}_{R''}(R'[S''_*])^{\mathrm{Gal}(X''/X')}$$

[where the superscript "$\mathrm{Gal}(X''/X')$" denotes the submodule of $\mathrm{Gal}(X''/X')$-*invariants*]. Relative to these natural isomorphisms, the restrictions of the natural surjections $R'[S''_*] \twoheadrightarrow R'[S'_*]$, $\mathfrak{Lie}^{b+1}_{R''}(R'[S''_*]) \twoheadrightarrow \mathfrak{Lie}^{b+1}_{R'}(R'[S'_*])$ to the respective submodules of $\mathrm{Gal}(X''/X')$-invariants thus induce the endomorphisms of $R'[S'_*]$, $\mathfrak{Lie}^{b+1}_{R'}(R'[S'_*])$ given by *multiplication by the order of* $\mathrm{Gal}(X''/X')$.

In light of the above observations [together with Propositions 3.3, (iv); 3.4, (iv)], we conclude the following:

(A) The natural surjection of $\Pi_{U_{S_0}}$-modules

$$\mathrm{New}^{\mathrm{tor}}_{S''_*}(b+1/b+2) \twoheadrightarrow \mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2)$$

admits a *factorization*

$$\mathrm{New}^{\mathrm{tor}}_{S''_*}(b+1/b+2) \twoheadrightarrow \mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2) \otimes_{R'} R'' \twoheadrightarrow \mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2)$$

which is compatible with the natural action of $\Pi_{U_{S_0}}$ on $\mathrm{New}^{\mathrm{tor}}_{S''_*}(b+1/b+2)$, $\mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2)$ and with a certain *"unnatural action"* of $\Pi_{U_{S_0}}$ on $\mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2) \otimes_{R'} R''$ whose restriction to $\Delta_{U'_{S'_0}}$ is equal to the tensor product of the trivial action of $\Delta_{U'_{S'_0}}$ on $\mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2)$ with the action of $\Delta_{U'_{S'_0}}$ on R'' given by multiplication, relative to the ring structure of R'', via the natural map $\Delta_{U'_{S'_0}} \twoheadrightarrow \mathrm{Gal}(X''/X') \hookrightarrow R''$. Nevertheless, this "unnatural action" of $\Pi_{U_{S_0}}$ on $\mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2) \otimes_{R'} R''$ is *compatible* with the natural R''-module structure of $\mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2) \otimes_{R'} R''$, relative to the natural action of $\Pi_{U_{S_0}}$ on R''.

(B) The induced morphism on $\Delta_{U'_{S'_0}}$*-invariants*

$$\mathrm{New}^{\mathrm{tor}}_{S''_*}(b+1/b+2)^{\Delta_{U'_{S'_0}}} \to \mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2)^{\Delta_{U'_{S'_0}}} = \mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2)$$

of the natural surjection of (A) *factors*, in a $\Pi_{U_{S_0}}$*-equivariant* fashion, through the morphism

$$\mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2) \to \mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2)$$

given by *multiplication by the order of* $\mathrm{Gal}(X''/X')$.

Now let us take $H' \stackrel{\mathrm{def}}{=} l^n \cdot \Delta^{\mathrm{ab}}_{U'_{S'_0}} \subseteq \Delta^{\mathrm{ab}}_{U'_{S'_0}}$, where n is some *"sufficiently large"* positive integer, to be chosen below. Write:

$$\mathcal{H}_{X'} \stackrel{\mathrm{def}}{=} H^1(\Delta_{U'_{S'_0}}, M^{(l)}_X); \quad \mathcal{H}_{X''} \stackrel{\mathrm{def}}{=} H^1(\Delta_{U''_{S''_0}}, M^{(l)}_X) \xrightarrow{\sim} H^1(\Delta_{U'_{S'_0}}, M^{(l)}_X[\mathrm{Gal}(X''/X')])$$

Now if we compute the cohomology of $\Pi_{U_{S_0}}$ via the *Leray spectral sequence* associated to the surjection $\Pi_{U_{S_0}} \twoheadrightarrow \Pi_{U_{S_0}}/\Delta_{U'_{S'_0}}$, then (A) implies that the natural morphism

$$H^1(\Delta_{U'_{S'_0}}, \mathrm{New}^{\mathrm{tor}}_{S''_*}(b+1/b+2)) \to H^1(\Delta_{U'_{S'_0}}, \mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2))$$

[which maps the image of $\eta_{S''}$ to the image of $\eta_{S'}$!] *factors* through a direct sum of copies of the [result of tensoring with $\mathbb{Q}/\mathbb{Z}$] the *"trace map"*

$$\mathrm{Tr}_{\mathcal{H}} : \mathcal{H}_{X''} \to \mathcal{H}_{X'}$$

— i.e., the map induced by the morphism of coefficients $M_X^{(l)}[\mathrm{Gal}(X''/X')] \twoheadrightarrow M_X^{(l)}$ that maps each element of $\mathrm{Gal}(X''/X')$ to 1.

Now I *claim* that the image of $\mathrm{Tr}_{\mathcal{H}}$ lies in $l^n \cdot \mathcal{H}_{X'}$. Indeed, if $S_0 = \emptyset$ [so $U_{S_0} = X$, $U'_{S'_0} = X'$], then this trace map $\mathrm{Tr}_{\mathcal{H}}$ is well-known to be *dual* [via Poincaré duality — cf., e.g., [FK], pp. 135-136] to the pull-back morphism; we thus conclude that, relative to the natural isomorphisms $\mathcal{H}_{X''} \xrightarrow{\sim} \Delta_{X''}^{\mathrm{ab}} \otimes \mathbb{Z}_l$, $\mathcal{H}_{X'} \xrightarrow{\sim} \Delta_{X'}^{\mathrm{ab}} \otimes \mathbb{Z}_l$ [arising from Poincaré duality — cf., e.g., Proposition 1.3, (ii)], the trace map corresponds to the natural morphism

$$\mathcal{H}_{X''} = \Delta_{X''}^{\mathrm{ab}} \to \Delta_{X'}^{\mathrm{ab}} = \mathcal{H}_{X'}$$

induced by the inclusion $\Delta_{X''} \subseteq \Delta_{X'}$ — hence *factors* through the endomorphism of $\mathcal{H}_{X'}$ given by *multiplication by* l^n, as claimed. If, on the other hand, S_0 is *not empty*, then observe that [since the order of $\mathrm{Gal}(X''/X')$ is a power of l] the construction of the morphism $\mathrm{Tr}_{\mathcal{H}}$ only involves the *maximal pro-l quotient* $\Delta_{U'_{S'_0}}^{(l)}$ of $\Delta_{U'_{S'_0}}$, which is a *free pro-l group on finitely many generators* $\xi_1, \ldots, \xi_m$. For $j = 1, \ldots, m$, write $(\mathbb{Z}_l \cong)\ \Xi_j \subseteq \Delta_{U'_{S'_0}}^{(l)}$ for the subgroup topologically generated by ξ_j. Since restriction to the cohomology of the Ξ_j determines an *isomorphism* of $\mathcal{H}_{X'}$ with the product of the $H^1(\Xi_j, M_X^{(l)})$, and the composite of $\mathrm{Tr}_{\mathcal{H}}$ with the restriction morphism to Ξ_j clearly *factors* through the *"trace map"*

$$\mathrm{Tr}_j : H^1(\Xi_j, M_X^{(l)}[\mathrm{Gal}(X''/X')]) \to H^1(\Xi_j, M_X^{(l)})$$

[i.e., the map induced by the morphism of coefficients $M_X^{(l)}[\mathrm{Gal}(X''/X')] \twoheadrightarrow M_X^{(l)}$ that maps each element of $\mathrm{Gal}(X''/X')$ to 1], it follows that to complete the proof of the *claim*, it suffices to verify that the image of Tr_j lies in $l^n \cdot H^1(\Xi_j, M_X^{(l)})$. But in light of the simple structure of $\Xi_j \cong \mathbb{Z}_l$, this is an easy computation. This completes the proof of the *claim*.

In light of the claim just verified, we thus conclude that $\mathrm{Tr}_{\mathcal{H}}$ *factors* through the endomorphism of $\mathcal{H}_{X'}$ given by *multiplication by* l^n. In particular, in the situation of assertion (ii), since the submodule of $\Pi_{U_{S_0}}$-invariants of $H^1(\Delta_{U'_{S'_0}}, \mathrm{New}_{S'_*}^{\mathrm{tor}}(b + 1/b + 2))$ is *finite* [cf. our assumption that $b \geq 2$; Proposition 3.5, (i)], we conclude that the image of $\eta_{S''}$ in $H^1(\Delta_{U'_{S'_0}}, \mathrm{New}_{S''_*}^{\mathrm{tor}}(b + 1/b + 2))$ [which is $\Pi_{U_{S_0}}$-invariant] maps to an l^n*-multiple of an* $\Pi_{U_{S_0}}$*-invariant* in $H^1(\Delta_{U'_{S'_0}}, \mathrm{New}_{S'_*}^{\mathrm{tor}}(b + 1/b + 2))$, which will be *zero* if we take n to be *"sufficiently large"*, hence that the image of $\eta_{S'}$ in $H^1(\Delta_{U'_{S'_0}}, \mathrm{New}_{S'_*}^{\mathrm{tor}}(b + 1/b + 2))$ is *zero*.

On the other hand, in the situation of assertion (i) [so $b = 1$], by replacing "$H^1(\Delta_{U'_{S'_0}}, -)$" by "$H^1(\Delta_{X'}, (-)^{\Delta_{U'_{S'_0}}[\mathrm{csp}]})$" [where $\Delta_{U'_{S'_0}}[\mathrm{csp}] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Delta_{U'_{S'_0}} \stackrel{\mathrm{def}}{=} \Delta_{X'})$] and taking $H' \stackrel{\mathrm{def}}{=} l^n \cdot \Delta^{\mathrm{ab}}_{U'_{S'_0}} + \mathrm{Im}(\Delta_{U'_{S'_0}}[\mathrm{csp}]) \subseteq \Delta^{\mathrm{ab}}_{U'_{S'_0}}$, a similar argument shows that image of $\eta_{S'}$ in $H^1(\Delta_{U'_{S'_0}}, \mathrm{New}^{\mathrm{tor}}_{S'_*}(b + 1/b + 2))$ is *zero* in the situation of assertion (i), as well.

Now I *claim* that the image of $\eta_{S''}$ in

$$H^1(\Delta_{U'_{S'_0}}, \mathrm{New}^{\mathrm{tor}}_{S'_*}(b + 1/b + 2) \otimes_{R'} R'')$$

[obtained by applying the surjection

$$\mathrm{New}^{\mathrm{tor}}_{S''_*}(b + 1/b + 2)) \twoheadrightarrow \mathrm{New}^{\mathrm{tor}}_{S'_*}(b + 1/b + 2) \otimes_{R'} R''$$

of (A)] is *zero*. Indeed, by applying the conclusion of the above discussion *concerning* X' to X'', we obtain first of all that the image of $\eta_{S''}$ in $H^1(\Delta_{U''_{S''_0}}, \mathrm{New}^{\mathrm{tor}}_{S''_*}(b + 1/b + 2))$ is *zero*, hence that the image in question in the claim arises from a class in the following cohomology module:

$$\begin{aligned} H^1(\mathrm{Gal}(X''/X'), &(\mathrm{New}^{\mathrm{tor}}_{S'_*}(b + 1/b + 2) \otimes_{R'} R'')^{\Delta_{U''_{S''_0}}}) \\ &= H^1(\mathrm{Gal}(X''/X'), \mathrm{New}^{\mathrm{tor}}_{S'_*}(b + 1/b + 2) \otimes_{R'} R'') = 0 \end{aligned}$$

[where the last cohomology module vanishes since $\mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2) \otimes_{R'} R''$ is an *induced* $\mathrm{Gal}(X''/X')$-module]. This completes the proof of the *claim*.

Thus, in summary, we conclude that the image of $\eta_{S''}$ in $H^1(\Pi_{U_{S_0}}, \mathrm{New}^{\mathrm{tor}}_{S'_*}(b + 1/b + 2) \otimes_{R'} R'')$ [obtained by applying the surjection of (A)] arises from a unique class in

$$\begin{aligned} H^1(\Pi_{U_{S_0}}/\Delta_{U'_{S'_0}}, &(\mathrm{New}^{\mathrm{tor}}_{S'_*}(b + 1/b + 2) \otimes_{R'} R'')^{\Delta_{U'_{S'_0}}}) \\ &\stackrel{\sim}{\to} H^1(\Pi_{U_{S_0}}/\Delta_{U'_{S'_0}}, \mathrm{New}^{\mathrm{tor}}_{S'_*}(b + 1/b + 2)) \end{aligned}$$

which maps to the unique class in

$$H^1(\Pi_{U_{S_0}}/\Delta_{U'_{S'_0}}, \mathrm{New}^{\mathrm{tor}}_{S'_*}(b + 1/b + 2))$$

that gives rise to $\eta_{S'}$ via *multiplication by the order of* $\mathrm{Gal}(X''/X')$ [cf. (B)]. In particular, by taking n to be *"sufficiently large"* [cf. Proposition 3.5, (i); the fact that $b+1 \geq 2 > 0$; the finiteness of $\Delta_{U_{S_0}}/\Delta_{U'_{S'_0}}$], we may conclude that $\eta_{S'} = 0$, as desired. That is to say:

This completes the proof that the two inclusions $\Pi^{\leq b}_{U_S} \hookrightarrow \Pi^{\mathrm{TOR}\leq b+1}_{U_S}$ *differ by conjugation by an element of* $\mathrm{Ker}(\Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S})$.

In particular, the proof of assertions (i), (ii) is complete.

Next, we consider assertion (iii). First, let us observe that when $b = 1$, assertion (iii) follows immediately from [the argument of] Proposition 2.6, (i); Proposition 3.5, (i) [cf. Remark 3.9.1 below]. Thus, in the remainder of the proof of assertion (iii), we assume that $b \geq 2$. Note that since the elements of $\mathrm{Ker}(\Pi_{U_S}^{\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\leq b})$ manifestly commute with the elements of $I_{x_*}^{\leq b+1}$, it follows from conditions (a), (b), the fact that $b \geq 2$, and the assumption that S_* is of *cardinality one* that β induces the *identity* on $\Pi_{U_S}^{\leq b+1}[\mathrm{new}]$ [cf. the proof of assertion (ii) above]. Thus, to complete the proof of assertion (iii), it suffices to show that the *compatible system* of classes

$$\lambda_{S'} \in H^1(\Pi_{U_{S_0}}, \mathrm{New}_{S'_*}(b+1/b+2))$$

determined by β [cf. Proposition 3.8, (i)] *vanishes*. Note that since $(\Delta_{U_{S_0}}^{(l)})^{\mathrm{ab}}$ is of "weight ≤ 2", and $\mathrm{New}_{S'_*}(b+1/b+2)$ is of "weight $b+1 \geq 3$" [cf. Proposition 3.5, (i)], it follows immediately from the Leray spectral sequence for $\Pi_{U_{S_0}} \twoheadrightarrow G_k$ that we have a *natural injection*

$$H^1(\Pi_{U_{S_0}}, \mathrm{New}_{S'_*}(b+1/b+2)) \hookrightarrow H^1(G_k, (\mathrm{New}_{S'_*}(b+1/b+2))^{\Delta_{U_{S_0}}})$$

[where the superscript "$\Delta_{U_{S_0}}$" denotes the $\Delta_{U_{S_0}}$-invariants] and that the module $H^1(G_k, (\mathrm{New}_{S'_*}(b+1/b+2))^{\Delta_{U_{S_0}}})$ is *finite*. Thus, to show that the $\lambda_{S'} = 0$, it suffices to show that the inverse limit

$$\varprojlim_{X'} (\mathrm{New}_{S'_*}(b+1/b+2))^{\Delta_{U_{S_0}}}$$

[where X', S'_* are as described in the proof of assertions (i), (ii)] is *zero*. But this follows from *observation (B)* of the proof of assertions (i), (ii). This completes the proof of assertion (iii).

Next, we consider assertion (iv). First, I *claim* that β *preserves* and induces the *identity* on each subgroup $\gamma \cdot D_{x_*}^{\leq b++} \cdot \gamma^{-1}$ [where $\gamma \in \Pi_{U_S}^{\leq b++} \subseteq \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$]. Indeed, this follows immediately by projecting via $\Pi_{U_S}^{\mathrm{TOR}\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}$ [which induces an isomorphism $\Pi_{U_S}^{\leq b++} \xrightarrow{\sim} \Pi_{U_S}^{\leq b+}$]. Thus, the element of $\mathrm{Ker}(\Pi_{U_S}^{\mathrm{TOR}\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+})$ that gives rise to β *centralizes* each subgroup $\gamma \cdot D_{x_*}^{\leq b++} \cdot \gamma^{-1}$. Put another way, we may think of the element of $\mathrm{Ker}(\Pi_{U_S}^{\mathrm{TOR}\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+})$ that gives rise to β as a *compatible system of elements*

$$\kappa_{S'} \in \mathrm{New}_{S'_*}^{\mathrm{tor}}(b+1/b+2)$$

such that each $\kappa_{S'}$ is *fixed* by every $\Pi_{U_{S_0}}$-conjugate of the Frobenius element determined by F_k [which, by abuse of notation, we shall also denote by F_k] in $\Pi_{U_{S_0}}$. Thus, to complete the proof of assertion (iv), it suffices to show that such a compatible system of elements $\{\kappa_{S'}\}$ is necessarily *zero*.

Note that instead of thinking of $\kappa_{S'}$ as being held fixed by every $\Pi_{U_{S_0}}$-conjugate of F_k, we may [equivalently] think of $\kappa_{S'}$ as being held fixed by F_k and by all

"Frobenius commutators" in $\Delta_{U_{S_0}}$ [i.e., elements of $\Delta_{U_{S_0}}$ that may be written as the commutator of F_k with an element of $\Delta_{U_{S_0}}$]. Note, moreover, that by Proposition 3.5, (i), it follows that the *Frobenius commutators topologically generate an open subgroup of* $(\Delta^{(l)}_{U_{S_0}})^{\mathrm{ab}}$. Let us refer to an open subgroup

$$H' \subseteq (\Delta^{(l)}_{U'_{S'_0}})^{\mathrm{ab}}$$

that is stabilized by $\Pi_{U_{S_0}}$ as *Frobenius-admissible* if it arises as the inverse image via the natural morphism $(\Delta^{(l)}_{U'_{S'_0}})^{\mathrm{ab}} \to (\Delta^{(l)}_{U_{S_0}})^{\mathrm{ab}}$ of the image $H \subseteq (\Delta^{(l)}_{U_{S_0}})^{\mathrm{ab}}$ of H' in $(\Delta^{(l)}_{U_{S_0}})^{\mathrm{ab}}$ and, moreover, satisfies the condition that the image of $(\Delta^{(l)}_{U'_{S'_0}})^{\mathrm{ab}}$ in $(\Delta^{(l)}_{U_{S_0}})^{\mathrm{ab}}$ lies in the submodule of $(\Delta^{(l)}_{U_{S_0}})^{\mathrm{ab}}$ generated by H and the Frobenius commutators. Note that by taking the open subgroups $\Pi_{U'_{S'_0}} \subseteq \Pi_{U_{S_0}}$ [i.e., that index the system $\{\kappa_{S'}\}$] to be *sufficiently small*, we may assume that each $\Pi_{U'_{S'_0}}$ satisfies the condition that $(\Delta^{(l)}_{U'_{S'_0}})^{\mathrm{ab}}$ contains *Frobenius-admissible* open subgroups $H' \subseteq (\Delta^{(l)}_{U'_{S'_0}})^{\mathrm{ab}}$ which are of *arbitrarily large index* in $(\Delta^{(l)}_{U'_{S'_0}})^{\mathrm{ab}}$.

Now let us apply the *observation (B)* made in the proof of assertions (i), (ii), to a covering $X'' \to X'$ that arises from a *Frobenius-admissible* $H' \subseteq (\Delta^{(l)}_{U'_{S'_0}})^{\mathrm{ab}}$. Note that it follows from the definitions of "Frobenius-admissible" that $\Delta_{U'_{S'_0}}$ is *contained* in the subgroup of $\Delta_{U_{S_0}}$ generated by $\Delta_{U''_{S''_0}}$ and the Frobenius commutators. On the other hand,

$$\kappa_{S''} \in \mathrm{New}^{\mathrm{tor}}_{S''_*}(b + 1/b + 2)$$

is *manifestly* fixed by $\Delta_{U''_{S''_0}}$. Since, as observed above, $\kappa_{S''}$ is also fixed by the Frobenius commutators of $\Delta_{U_{S_0}}$, we thus conclude that $\kappa_{S''}$ is $\Delta_{U'_{S'_0}}$*-invariant*. Since, on the other hand, $\kappa_{S''}$ is also fixed by F_k, and the submodule of $\mathrm{New}^{\mathrm{tor}}_{S'_*}(b+1/b+2)$ consisting of elements fixed by F_k is *clearly finite* [cf. Proposition 3.5, (i)], we thus conclude from *observation (B)* that, for $H' \subseteq (\Delta^{(l)}_{U'_{S'_0}})^{\mathrm{ab}}$ of *sufficiently large index* in $(\Delta^{(l)}_{U'_{S'_0}})^{\mathrm{ab}}$, $\kappa_{S''}$ *maps to zero* in $\mathrm{New}^{\mathrm{tor}}_{S'_*}(b + 1/b + 2)$. But this implies that $\kappa_{S'} = 0$, thus completing the proof of assertion (iv).

Finally, we consider assertion (v). It is immediate from the definitions that the quotients $\Pi_{U_S} \twoheadrightarrow \Pi^{\leq\infty}_{U_S}$, $\Delta_{U_S} \twoheadrightarrow \Delta^{\leq\infty}_{U_S}$ are *new-cuspidally pro-l*. That these quotients are the *maximal new-cuspidally pro-l quotients* follows from the construction of $\Pi^{\leq\infty}_{U_S}$ and the easily verified fact that each $\Delta^{(l)}_{U'_{S'}}$ *injects* into $\mathrm{Lin}(\Delta^{(l)}_{U'_{S'}}(1/\infty))(\mathbb{Q}_l)$. ○

Remark 3.9.1. Proposition 3.9, (iii), may be regarded as a *"higher order, pro-l, possibly affine analogue"* of Proposition 2.6, (i).

We are now ready to prove the *main technical result* of the present §3:

Theorem 3.10. (Reconstruction of Once-Punctured Maximal New-cuspidally Pro-l Extensions) *Let X, Y be* **proper hyperbolic curves** *over a* **finite field***; denote the base fields of X, Y by k_X, k_Y, respectively. Suppose further that we have been given* **finite subsets**

$$S \subseteq X(k_X); \quad T \subseteq Y(k_Y)$$

as well as subsets $S_* = \{x_*\} \subseteq S$, $T_* = \{y_*\} \subseteq T$ of **cardinality one***; write $S_0 \stackrel{\mathrm{def}}{=} S \backslash S_*$, $U_S \stackrel{\mathrm{def}}{=} X \backslash S$, $U_{S_0} \stackrel{\mathrm{def}}{=} X \backslash S_0$, $T_0 \stackrel{\mathrm{def}}{=} T \backslash T_*$, $V_T \stackrel{\mathrm{def}}{=} Y \backslash T$, $V_{T_0} \stackrel{\mathrm{def}}{=} Y \backslash T_0$. Let Σ be a* **set of prime numbers** *that contains at least one prime number that is invertible in k_X, k_Y; thus, Σ determines various* **quotients** *Π_{U_S}, $\Pi_{U_{S_0}}$, Π_{V_T}, $\Pi_{V_{T_0}}$ [cf. Proposition 1.8, (iii)] of the* **étale fundamental groups** *of U_S, V_T, respectively. Let*

$$\alpha : \Pi_{U_{S_0}} \xrightarrow{\sim} \Pi_{V_{T_0}}$$

be a **Frobenius-preserving** *[hence also* **quasi-point-theoretic** *— cf. Remark 1.18.2]* **isomorphism** *of profinite groups that maps the decomposition group of x_* in $\Pi_{U_{S_0}}$ [which is well-defined up to conjugation] to the decomposition group of y_* in $\Pi_{V_{T_0}}$ [which is well-defined up to conjugation]. Then for each prime $l \in \Sigma$ such that $l \neq p$, there exists a* **commutative diagram**

$$\begin{array}{ccc} \Pi_{U_S}^{\leq\infty} & \xrightarrow{\alpha_\infty} & \Pi_{V_T}^{\leq\infty} \\ \downarrow & & \downarrow \\ \Pi_{U_{S_0}} & \xrightarrow{\alpha} & \Pi_{V_{T_0}} \end{array}$$

— in which $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\leq\infty}$, $\Pi_{V_T} \twoheadrightarrow \Pi_{V_T}^{\leq\infty}$ are the **maximal new-cuspidally pro-l quotients** *[cf. Proposition 3.9, (v)]; the vertical arrows are the natural morphisms; α_∞ is an* **isomorphism***, well-defined up to composition with a new-cuspidally inner automorphism, that is* **compatible***, relative to the natural surjections*

$$\Pi_{U_S}^{\leq\infty} \twoheadrightarrow \Pi_{U_S}^{\leq 2} \twoheadrightarrow \Pi_{U_S}^{\text{c-ab},l}; \quad \Pi_{V_T}^{\leq\infty} \twoheadrightarrow \Pi_{V_T}^{\leq 2} \twoheadrightarrow \Pi_{V_T}^{\text{c-ab},l}$$

— where we write $\Pi_{U_S}^{\text{c-ab}} \twoheadrightarrow \Pi_{U_S}^{\text{c-ab},l}$, $\Pi_{V_T}^{\text{c-ab}} \twoheadrightarrow \Pi_{V_T}^{\text{c-ab},l}$ for the respective **maximal cuspidally pro-l quotients** *— with the isomorphism $\Pi_{U_S}^{\text{c-ab}} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab}}$ of Theorem 2.5, (i).*

Proof. In the following argument, let us *identify* the "$\mathrm{Lin}_{U_S}(1/\infty)$", "$\mathrm{Lin}_X(1/\infty)$" portions of $\Delta_{U_S}^{\mathrm{Lie}}$ with the [completions, relative to the natural filtration topology, of the] corresponding graded objects "$\mathrm{Gr}_{\mathbb{Q}_l}(-)(1/\infty)$" via the Galois invariant splittings of Proposition 3.5, (ii), and similarly for V_T. Then, in light of our assumption that α is *Frobenius-preserving* [hence also *quasi-point-theoretic* — cf. Remark 1.18.2], it follows immediately from the *naturality* of our constructions [cf., especially, Proposition 3.5, (iii)] that α induces, for each $\mathbb{Z} \ni b \geq 1$, an *isomorphism*

$$\alpha^{\mathrm{LIE}\leq b} : \Pi_{U_S}^{\mathrm{LIE}\leq b} \xrightarrow{\sim} \Pi_{V_T}^{\mathrm{LIE}\leq b}$$

that is compatible, with respect to the natural projections $\Pi_{U_S}^{\mathrm{LIE}\leq b} \twoheadrightarrow \Pi_X$, $\Pi_{V_T}^{\mathrm{LIE}\leq b} \twoheadrightarrow \Pi_Y$, with α. Moreover, it follows from the *construction* of "$\Pi_{(-)}^{\mathrm{LIE}\leq b}$" that this isomorphism *maps* $D_{x_*}^{\mathrm{LIE}\leq b} \subseteq \Pi_{U_S}^{\mathrm{LIE}\leq b}$ *bijectively onto* $D_{y_*}^{\mathrm{LIE}\leq b} \subseteq \Pi_{V_T}^{\mathrm{LIE}\leq b}$, and that the resulting isomorphism $D_{x_*}^{\mathrm{LIE}\leq b} \xrightarrow{\sim} D_{y_*}^{\mathrm{LIE}\leq b}$ induces an *isomorphism*

$$D_{x_*}^{\leq b} \xrightarrow{\sim} D_{y_*}^{\leq b}$$

which is compatible [again by construction!] with the respective *Frobenius elements* "F_k" on either side.

Now I *claim* that the isomorphism $\alpha^{\mathrm{LIE}\leq b}$ maps $\Pi_{U_S}^{\leq b}$ bijectively onto $\Pi_{V_T}^{\leq b}$, thus inducing a *compatible inverse system* [parametrized by b] of *isomorphisms*

$$\alpha^{\leq b} : \Pi_{U_S}^{\leq b} \xrightarrow{\sim} \Pi_{V_T}^{\leq b}$$

that are compatible, with respect to the natural projections $\Pi_{U_S}^{\leq b} \twoheadrightarrow \Pi_X$, $\Pi_{V_T}^{\leq b} \twoheadrightarrow \Pi_Y$, with α. Note that since these isomorphisms were constructed via the explicit presentation of Proposition 3.5, (iii), it follows from Proposition 3.9, (iii), that, when $b \geq 2$, these isomorphisms will be *compatible*, relative to the natural surjections $\Pi_{U_S}^{\leq b} \twoheadrightarrow \Pi_{U_S}^{\leq 2} \twoheadrightarrow \Pi_{U_S}^{\text{c-ab},l}$, $\Pi_{V_T}^{\leq b} \twoheadrightarrow \Pi_{V_T}^{\leq 2} \twoheadrightarrow \Pi_{V_T}^{\text{c-ab},l}$, with the isomorphism $\Pi_{U_S}^{\text{c-ab}} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab}}$ of Theorem 2.5, (i). Thus, to complete the proof of Theorem 3.10, it suffices to verify the above *claim*.

To verify this *claim*, we apply induction on b. The case $b = 1$ is vacuous. Thus, we assume that $b \geq 1$, and that the claim has been verified for "b" that are $\leq$ the b under consideration. Now observe that by Propositions 3.8, (iii); 3.9, (i), (ii), it follows that the isomorphism

$$\Pi_{U_S}^{\mathrm{LIE}\leq b+1} \xrightarrow{\sim} \Pi_{V_T}^{\mathrm{LIE}\leq b+1}$$

maps $\Pi_{U_S}^{\leq b+1}$ bijectively onto a $\mathrm{Ker}(\Pi_{V_T}^{\mathrm{LIE}\leq b+1} \twoheadrightarrow \Pi_{V_T}^{\mathrm{LIE}\leq b+})$-conjugate of $\Pi_{V_T}^{\leq b+1}$. [Here, we note that when $b = 1$, the fact that the hypotheses of Proposition 3.9, (i), are satisfied follows immediately from the fact that the isomorphism $\alpha^{\mathrm{LIE}\leq 2}$ was constructed via the explicit presentation of Proposition 3.5, (iii).] In particular, by *conjugating* by an appropriate element $\gamma \in \mathrm{Ker}(\Pi_{V_T}^{\mathrm{LIE}\leq b+1} \twoheadrightarrow \Pi_{V_T}^{\mathrm{LIE}\leq b+})$, we obtain an *isomorphism*

$$\beta_{b+1} : \Pi_{U_S}^{\leq b+1} \xrightarrow{\sim} \Pi_{V_T}^{\leq b+1}$$

that is *compatible* with $\alpha^{\leq b}$ and, moreover, [since γ *commutes* with $I_{y_*}^{\leq b+1}$] *maps* $I_{x_*}^{\leq b+1}$ *bijectively onto* $I_{y_*}^{\leq b+1}$. Also, we observe in passing that by Propositions 3.5, (i); 3.9, (iii), it follows that the choice of γ is *unique*, modulo $\mathrm{Ker}(\Pi_{V_T}^{\leq b+1} \twoheadrightarrow \Pi_{V_T}^{\leq b+})$. Note that conjugation by γ on $\Pi_{V_T}^{\mathrm{TOR}\leq b+1}$ also determines an *isomorphism*

$$\beta_{b+1}^{\mathrm{TOR}} : \Pi_{U_S}^{\mathrm{TOR}\leq b+1} \xrightarrow{\sim} \Pi_{V_T}^{\mathrm{TOR}\leq b+1}$$

which may be *constructed directly* from β_{b+1} via appropriate "Lie-ification" and "push-forward" operations whose detailed explication is a routine task which we leave to the reader. Also, we observe that since β_{b+1} is compatible with $\alpha^{\leq b}$, and the natural projections determine isomorphisms $\Pi_{U_S}^{\leq b++} \xrightarrow{\sim} \Pi_{U_S}^{\leq b}$, $\Pi_{V_T}^{\leq b++} \xrightarrow{\sim} \Pi_{V_T}^{\leq b}$ [cf. Proposition 3.8, (i); the discussion preceding Proposition 3.9], it follows that $\beta_{b+1}^{\mathrm{TOR}}$ *maps* the subgroup $D_{\overline{x}_*}^{\leq b++} \subseteq \Pi_{U_S}^{\leq b++} \subseteq \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ *bijectively onto* the subgroup $D_{\overline{y}_*}^{\leq b++} \subseteq \Pi_{V_T}^{\leq b++} \subseteq \Pi_{V_T}^{\mathrm{TOR}\leq b+1}$.

Next, let us observe that all of the constructions executed so far *depend* on the choice of subgroups $D_{x_*}[U_S] \subseteq \Pi_{U_S}$, $D_{y_*}[V_T] \subseteq \Pi_{V_T}$ among the various conjugates of these subgroups. Now we would like to consider what happens when we make a *different* choice for these subgroups, i.e., by conjugating $D_{x_*}[U_S]$ (respectively, $D_{y_*}[V_T]$) by an element $\zeta_{x_*} \in \Pi_{U_S}$ (respectively, $\zeta_{y_*} \in \Pi_{V_T}$), where we assume that the images of ζ_{x_*}, ζ_{y_*} in $\Pi_{U_S}^{\leq b}$, $\Pi_{V_T}^{\leq b}$ are *compatible* with respect to $\alpha^{\leq b}$. By *transport of structure*, the various objects obtained for these *alternative choices* may be computed by *conjugating* by ζ_{x_*}, ζ_{y_*}, respectively. Note that conjugating by ζ_{x_*} on $\Pi_{U_S}^{\mathrm{LIE}\leq b+1}$ *differs* [relative to the isomorphism $\alpha^{\mathrm{LIE}\leq b+1} : \Pi_{U_S}^{\mathrm{LIE}\leq b+1} \xrightarrow{\sim} \Pi_{V_T}^{\mathrm{LIE}\leq b+1}$ obtained above] from conjugating by ζ_{y_*} on $\Pi_{V_T}^{\mathrm{LIE}\leq b+1}$ by conjugation by some element $\zeta_\delta \in \mathrm{Ker}(\Pi_{V_T}^{\mathrm{LIE}\leq b+1} \twoheadrightarrow \Pi_{V_T}^{\mathrm{LIE}\leq b})$. Nevertheless, *computing with the data for the alternative choices* yields an isomorphism

$$\beta_{b+1}^{\mathrm{alt}} : \Pi_{U_S}^{\leq b+1} \xrightarrow{\sim} \Pi_{V_T}^{\leq b+1}$$

that is *still compatible* with $\alpha^{\leq b}$ [by our assumption that the images of ζ_{x_*}, ζ_{y_*} in $\Pi_{U_S}^{\leq b}$, $\Pi_{V_T}^{\leq b}$ are *compatible* with respect to $\alpha^{\leq b}$]. Since, moreover, ζ_δ *commutes* with $\zeta_{y_*} \cdot I_{\overline{y}_*}^{\leq b+1} \cdot \zeta_{y_*}^{-1}$, it follows that $\beta_{b+1}^{\mathrm{alt}}$ *maps* $\zeta_{x_*} \cdot I_{\overline{x}_*}^{\leq b+1} \cdot \zeta_{x_*}^{-1}$ *bijectively onto* $\zeta_{y_*} \cdot I_{\overline{y}_*}^{\leq b+1} \cdot \zeta_{y_*}^{-1}$, hence that $\beta_{b+1}^{\mathrm{alt}}$ *maps* $I_{\overline{x}_*}^{\leq b+1}$ *bijectively onto* $I_{\overline{y}_*}^{\leq b+1}$. Thus, by Proposition 3.9, (iii), it follows that $\beta_{b+1}^{\mathrm{alt}}$ *differs* from β_{b+1} by *composition with a* $\mathrm{Ker}(\Pi_{V_T}^{\leq b+1} \twoheadrightarrow \Pi_{V_T}^{\leq b})$*-inner automorphism.* [Here, we note that when $b = 1$, the fact that the hypotheses of Proposition 3.9, (iii), are satisfied follows immediately from the fact that the isomorphism $\alpha^{\mathrm{LIE}\leq 2}$ was constructed via the explicit presentation of Proposition 3.5, (iii).]

On the other hand, by *construction* [relative to the *alternative choices*!] it follows that $\beta_{b+1}^{\mathrm{alt}}$ induces [via appropriate "Lie-ification" and "push-forward" operations] an *isomorphism*

$$\beta_{b+1}^{\mathrm{alt,TOR}} : \Pi_{U_S}^{\mathrm{TOR}\leq b+1} \xrightarrow{\sim} \Pi_{V_T}^{\mathrm{TOR}\leq b+1}$$

which *maps* the subgroup $\zeta_{x_*} \cdot D_{\overline{x}_*}^{\leq b++} \cdot \zeta_{x_*}^{-1} \subseteq \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ *bijectively onto* the subgroup $\zeta_{y_*} \cdot D_{\overline{y}_*}^{\leq b++} \cdot \zeta_{y_*}^{-1} \subseteq \Pi_{V_T}^{\mathrm{TOR}\leq b+1}$. Thus, since $\beta_{b+1}^{\mathrm{TOR}}$, $\beta_{b+1}^{\mathrm{alt,TOR}}$ differ, as observed above, by conjugation by an element of $\mathrm{Ker}(\Pi_{V_T}^{\leq b+1} \twoheadrightarrow \Pi_{V_T}^{\leq b})$, we conclude that, for each $\eta_{x_*} \in \Pi_{U_S}^{\leq b++} \subseteq \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$, $\eta_{y_*} \in \Pi_{V_T}^{\leq b++} \subseteq \Pi_{V_T}^{\mathrm{TOR}\leq b+1}$ that correspond via $\alpha^{\leq b}$, $\beta_{b+1}^{\mathrm{TOR}}$ maps the subgroup of $\Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ defined by $\eta_{x_*} \cdot D_{\overline{x}_*}^{\leq b++} \cdot \eta_{x_*}^{-1}$ to the subgroup of $\Pi_{V_T}^{\mathrm{TOR}\leq b+1}$ defined by $\eta_{y_*} \cdot D_{\overline{y}_*}^{\leq b++} \cdot \eta_{y_*}^{-1}$. But by Proposition 3.9, (iv) [and the *uniqueness* of γ, modulo $\mathrm{Ker}(\Pi_{V_T}^{\leq b+1} \twoheadrightarrow \Pi_{V_T}^{\leq b+})$, that was observed above], this implies that the element $\gamma \in \mathrm{Ker}(\Pi_{V_T}^{\mathrm{LIE}\leq b+1} \twoheadrightarrow \Pi_{V_T}^{\mathrm{LIE}\leq b+})$ in fact *belongs to*

$\mathrm{Ker}(\Pi_{\overline{V}_T}^{\leq b+1} \twoheadrightarrow \Pi_{\overline{V}_T}^{\leq b+})$. That is to say, we may conclude that the *original isomorphism*

$$\alpha^{\mathrm{LIE}\leq b+1} : \Pi_{U_S}^{\mathrm{LIE}\leq b+1} \xrightarrow{\sim} \Pi_{V_T}^{\mathrm{LIE}\leq b+1}$$

maps $\Pi_{\overline{U}_S}^{\leq b+1}$ *bijectively onto* $\Pi_{\overline{V}_T}^{\leq b+1}$. This completes the proof of the *claim*.

Finally, we note that the *indeterminacy*, referred to in the statement of Theorem 3.10, of the isomorphism α_∞ up to composition with a *new-cuspidally inner automorphism* arises precisely from the *indeterminacy of the choice of the subgroups* $D_{x_*}[U_S] \subseteq \Pi_{U_S}$, $D_{y_*}[V_T] \subseteq \Pi_{V_T}$ with respect to *new-cuspidally inner automorphisms* of Π_{U_S}, Π_{V_T}, respectively. ◯

Remark 3.10.1. The argument of the proof of Theorem 3.10 involving Proposition 3.9, (i), (ii), (iii), (iv), may be regarded as a sort of *"higher order analogue"* of the argument applied in the proof of Theorem 1.16, (iii), involving Lemma 1.11; Proposition 1.12, (v).

Remark 3.10.2. It seems reasonable to expect that, when, say, $\Sigma = \{l\}$, the techniques applied in the proof of Theorem 3.10, together with the theory of [Mtm], should allow one to reconstruct the [geometrically pro-Σ] étale fundamental groups of the various *configuration spaces* [i.e., finite products of copies of X over k_X, with the various diagonals removed] "group-theoretically" from Π_X [under, say, an appropriate hypothesis of *"Frobenius-preservation"* as in Theorem 3.10]. This topic, however, lies beyond the scope of the present paper.

Remark 3.10.3. When $\Sigma = \{l\}$, it is *tempting* to try to generalize Theorem 3.10 to the case where $S_* \subseteq S$, $T_* \subseteq T$ are subsets of *arbitrary finite cardinality*, by applying Theorem 3.10 [as stated above] *recursively*. Although such a recursive argument is formally possible, it appears, however, to lead to *complications* [whose resolution or, indeed, detailed explication is beyond the scope of the present paper!] when X *fails to be* Σ*-separated*. Nevertheless, this approach appears, to the author, to be an interesting direction for further research.

Remark 3.10.4. One essential portion of the proof of Theorem 3.10 is the *Galois invariant splitting* of Proposition 3.5, (ii). Although it does not appear likely that such a splitting exists in the case of a *nonarchimedean local* base field [cf., e.g., the theory of [Mzk4]], it would be interesting to investigate the extent to which a result such as Theorem 3.10 may be generalized to the nonarchimedean local case, perhaps by making use of some sort of splitting such as the *Hodge-Tate decomposition*, or a splitting that arises via *crystalline* methods. In the context of *absolute anabelian* geometry over nonarchimedean local fields, however, such p-adic Hodge-theoretic splittings might not be available, since the isomorphism class of the Galois module "$\mathbb{C}_p$" is not preserved by arbitrary automorphisms of the absolute Galois group of a nonarchimedean local field [cf. the theory of [Mzk3]].

The development of the theory underlying Theorem 3.10 was motivated by the following *important consequence*:

Corollary 3.11. (Total Global Green-compatibility) *In the situation of Theorem 1.16, (iii) [in the finite field case], suppose further that* $\Sigma^\dagger = \mathfrak{Primes}^\dagger$*, and that* X*,* Y *are* **Σ-separated** *[which implies that* α *is* **Frobenius-preserving** *and* **point-theoretic** *— cf. Remarks 1.18.1, 1.18.2]. Then the isomorphism* α *is* **totally globally Green-compatible***.*

Proof. Indeed, we may apply Theorem 3.10 to the isomorphism α of Theorem 1.16, (iii), and arbitrary choices of sets of cardinality one $S = \{x_*\}$, $T = \{y_*\}$ [so $U_{S_0} = X$, $V_{T_0} = Y$] that correspond via α. Then let us *observe* that the quotient $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\leq\infty}$ satisfies the following property:

> If $\Pi_{U_S} \twoheadrightarrow Q$ is a finite quotient of Π_{U_S} such that for some quotient $Q \twoheadrightarrow Q'$ whose kernel has order a power of l, $\Pi_{U_S} \twoheadrightarrow Q'$ factors through $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\leq\infty}$, then $\Pi_{U_S} \twoheadrightarrow Q$ also factors through $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\leq\infty}$.

A similar statement holds for the quotient $\Pi_{V_T} \twoheadrightarrow \Pi_{V_T}^{\leq\infty}$. In light of this *observation*, together with our assumption that $\Sigma^\dagger = \mathfrak{Primes}^\dagger$ [which implies that α is *Frobenius-preserving*], it follows that the reasoning of [Tama], Corollary 2.10, Proposition 3.8, may be applied to the isomorphism

$$\alpha_\infty : \Pi_{U_S}^{\leq\infty} \xrightarrow{\sim} \Pi_{V_T}^{\leq\infty}$$

of Theorem 3.10 to conclude that the isomorphism α_∞ maps the set of decomposition subgroups of the domain *bijectively onto* the set of decomposition subgroups of the codomain.

On the other hand, sorting through the definitions, the datum of the lifting of a decomposition group of Π_X, Π_Y corresponding to a point that does not belong to S, T to a [noncuspidal] decomposition group of the domain or codomain of α_∞ *determines*, by projection to $\Pi_{U_S}^{\text{c-ab},l}$, $\Pi_{V_T}^{\text{c-ab},l}$, the *$l$-adic portion of the Green's trivialization* associated to this point and the unique point of S or T. Since l is an *arbitrary* element of $\Sigma^\dagger = \mathfrak{Primes}^\dagger$, and the points x_*, y_* are *arbitrary* points that correspond via α, this shows that α is *globally Green-compatible*. That α is *totally globally Green-compatible* follows by applying this argument to the isomorphism induced by α between open subgroups of Π_X, Π_Y. ◯

Theorem 3.12. (The Grothendieck Conjecture for Proper Hyperbolic Curves over Finite Fields) *Let* X*,* Y *be* **proper hyperbolic curves** *over a* **finite field***; denote the base fields of* X*,* Y *by* k_X*,* k_Y*, respectively. Write* Π_X*,* Π_Y *for the* **étale fundamental groups of** X**,** Y*, respectively. Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be an **isomorphism** *of profinite groups. Then* α *arises from a* **uniquely** *determined commutative diagram of* **schemes**

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\sim} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \end{array}$$

in which the horizontal arrows are **isomorphisms***; the vertical arrows are the profinite étale universal coverings determined by the profinite groups* Π_X, Π_Y.

Proof. Theorem 3.12 follows formally from Corollaries 2.7, 3.11; Remarks 1.18.1, 1.18.2. ◯

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