

THE IMAGE OF THE HEAT KERNEL TRANSFORM ON RIEMANNIAN SYMMETRIC SPACES OF THE NONCOMPACT TYPE

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ABSTRACT. The heat kernel transform on G/K , a Riemannian symmetric space of noncompact type, maps an L^2 -function on G/K to a holomorphic function on the complex crown. In this article we determine the image of this transform on L^2 .

1. INTRODUCTION

The heat equation, and the associated heat kernel transform (also known as the Bargmann-Segal transform), is a natural counterpart of a Riemannian metric. For the homogeneous spaces \mathbb{R}^n or a compact symmetric space endowed with the usual metric, many properties of the heat kernel transform are known [2, 9, 18]. The relevant property for this note is the fundamental observation made by Bargmann and Segal that the image of the heat kernel transform for \mathbb{R}^n consists of functions with holomorphic extension to \mathbb{C}^n . Let X denote one of the aforementioned Riemannian manifolds (\mathbb{R}^n or a compact symmetric space) and k_t the heat kernel. The isometry group of X acts transitively, consequently X is diffeomorphic and isometric to G/K , K the isotropy group of a basepoint. One knows that the Riemannian manifold X has a natural complexification, denoted here by Ξ , which is G -diffeomorphic to the tangent bundle TX , and on which k_t has a holomorphic extension, say k_t^\sim . For a function f on X , $k_t * f$, the solution of the heat equation with initial data f , has holomorphic extension to Ξ given by $k_t^\sim * f$. For f in the Hilbert space $L^2(X)$ one obtains thus a G -equivariant linear map, the *heat kernel transform*, $H_t : L^2(X) \rightarrow \mathcal{O}(\Xi)$. For these spaces X it is known that the image of H_t is a weighted Bergman space on Ξ , and that H_t is a G -equivariant unitary map between these Hilbert spaces.

That this might not be a general phenomenon was observed first in [15] for the Heisenberg group with a left invariant Laplace operator. For this Riemannian space the image of the heat kernel transform is no longer one weighted Bergman space but a sum of two, each of which corresponds to a nonpositive weight function with strong oscillatory behavior. Surprisingly, neither the Heisenberg group nor the other examples suggest the form of the result for symmetric spaces of noncompact type.

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One problem to be resolved had been to determine the *natural* complexification of $X = G/K$. Introduced in [1] for this purpose, Ξ , or as it was later called the *complex crown* of G/K , is a G -invariant domain in $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ containing X , and biholomorphic to an open complex subdomain in TX endowed with the adapted complex structure. The G -orbit structure of Ξ is nicely given by $\Xi = G \exp(i\Omega) \cdot x_o$, where the basepoint x_o is $eK_{\mathbb{C}} \in X_{\mathbb{C}}$, X is identified with the G -orbit through x_o , and Ω is a specific polyhedral subset of a maximal flat subspace $\mathfrak{a} \subset T_{x_o}X$.

In [13, 14] it was shown that Ξ is a maximal domain of holomorphy in $X_{\mathbb{C}}$ for eigenfunctions of $\mathbb{D}(X)$, the algebra of invariant differential operators on X . Consequently, Ξ provides a canonical domain on which to investigate the image of the heat kernel transform on X . Such a study was initiated in [14] where a G -equivariant holomorphic heat kernel transform for X was introduced. There it was shown that the heat kernel k_t extends to a holomorphic function k_t^{\sim} on Ξ , and that, similarly, one has a G -equivariant heat kernel transform $H_t : L^2(X) \rightarrow \mathcal{O}(\Xi)$

$$H_t(f)(z) = \int_X k_t^{\sim}(g^{-1}z) f(gK) d\mu_X(gK).$$

The purpose of this note is to give an explicit determination of the image of the heat kernel transform for the noncompact Riemannian symmetric spaces. The description of the image is unlike the previously known cases. That a different type of description is necessary we note in Remark 3.1 (and prove in §4) for symmetric spaces with G complex as we show that the image can not be a weighted Bergman space, i.e., there is no measure of the form $\mu_t = w_t(z) dz$ on Ξ with w_t G -invariant, and with the norm on $\text{im } H_t$ given by $\int_{\Xi} |F(z)|^2 d\mu_t(z)$.

Some of the techniques we use were developed in [13] and [14]. The main new tool, however, is the *G-orbital integral*, first appearing in [5] and further developed in [8],

$$\mathcal{O}_h(iY) = \int_G h(g \exp(\frac{i}{2}Y) \cdot x_o) dg,$$

h a function on Ξ suitably decreasing at the boundary and $Y \in 2\Omega$. In Theorem 3.3 we show that there exists a pseudo-differential shift operator D such that for $f \in L^2(X)$ the function $D\mathcal{O}_{|H_t f|^2}$, initially defined on 2Ω , has a natural holomorphic extension to $\mathfrak{a}_{\mathbb{C}}$, and that there is an equality of norms

$$\|f\|^2 = \int_{\mathfrak{a}} (D\mathcal{O}_{|H_t f|^2})(iY) w_t(Y) dY,$$

the weight function w_t being given by

$$w_t(Y) = \frac{1}{|\mathcal{W}|} \cdot \frac{e^{2t|\rho|^2}}{(2\pi t)^{n/2}} \cdot e^{-\frac{|Y|^2}{2t}} \quad (Y \in \mathfrak{a}),$$

\mathcal{W} the usual Weyl group and $n = \text{rank } X$. To describe $\text{im } H_t$, in §2 we introduce the space $\mathcal{G}(\Xi)$ consisting of holomorphic functions F on Ξ satisfying the following two properties:

- $f := F|_X \in L^2(X)$
- $\int_X |\widehat{f}(b, \lambda)|^2 \psi_\lambda(iY) d\mu(b, \lambda) < \infty$ for all $Y \in \mathfrak{a}$ and $\psi_\lambda(iY) = \sum_{w \in \mathcal{W}} e^{\lambda(iwY)}$ the Weyl-symmetrized exponential.

Then we show that $\text{im } H_t$ is the space of those functions $F \in \mathcal{G}(\Xi)$ with $\int_{\mathfrak{a}} (D\mathcal{O}_{|F|^2})(iY) w_t(Y) dY < \infty$.

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2. RIEMANNIAN SYMMETRIC SPACES

We recall the necessary background on the geometry and analysis on Ξ , the *complex crown* of X .

2.1. The complex crown. Let G be a connected semisimple Lie group. We will assume that G is contained in its universal complexification $G_{\mathbb{C}}$ and that $G_{\mathbb{C}}$ is simply connected. Let K be a maximal compact subgroup of G and form the homogeneous space $X = G/K$. The complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ contains X as a totally real submanifold.

Write \mathfrak{g} (resp. \mathfrak{k}) for the Lie algebras of G (resp. K). We denote the Cartan-Killing form on \mathfrak{g} by $\langle \cdot, \cdot \rangle$. Then $\mathfrak{p} := \mathfrak{k}^{\perp}$ is a complementary subspace on which $\langle \cdot, \cdot \rangle$ defines an inner product and hence a Riemannian symmetric structure on $X = G/K$. Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and set

$$\Omega = \left\{ Y \in \mathfrak{a} : \text{Spec}(\text{ad}Y) \subset \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\right\}.$$

Ω is open, convex, and invariant under the Weyl group $\mathcal{W} = N_K(\mathfrak{a})/M$, $M = Z_K(\mathfrak{a})$.

Let $x_o = eK_{\mathbb{C}}$ be the base point in $X_{\mathbb{C}}$. The *complex crown* of X is defined by

$$\Xi = G \exp(i\Omega) \cdot x_o.$$

From several sources (for example [3] or [14] and references therein) one knows that Ξ is a G -invariant Stein domain in $X_{\mathbb{C}}$ which contains X in its “middle” as a totally real submanifold. The definition of Ξ is independent of the choice of \mathfrak{a} (all maximal abelian subspaces in \mathfrak{p} are conjugate under K), hence Ξ is canonically defined by X .

Denote by $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots. Then

$$\Omega = \left\{ Y \in \mathfrak{a} : (\forall \alpha \in \Sigma) |\alpha(Y)| < \frac{\pi}{2} \right\}.$$

In particular $\overline{\Omega}$ is a compact polyhedron.

To each $\alpha \in \Sigma$ we associate the root space $\mathfrak{g}^\alpha = \{Y \in \mathfrak{g} : (\forall H \in \mathfrak{a}) [H, Y] = \alpha(H)Y\}$. If \mathfrak{m} denotes the Lie algebra of M , one has the root space decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha.$$

Fix a positive system $\Sigma^+ \subset \Sigma$ and set $\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$. If $A := \exp(\mathfrak{a})$ and $N := \exp(\mathfrak{n})$ are the (closed) analytic subgroups corresponding to \mathfrak{a} and \mathfrak{n} , and $A_{\mathbb{C}} := \exp(\mathfrak{a}_{\mathbb{C}})$ and $N_{\mathbb{C}} := \exp(\mathfrak{n}_{\mathbb{C}})$ are the corresponding complex groups, then one has the inclusion

$$\Xi \subset N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_o$$

(cf. [13], [10], [16]). As Ξ is contractible one can define an N -invariant holomorphic map

$$a : \Xi \rightarrow A_{\mathbb{C}}$$

with $a(x_o) = \mathbf{1}$ and having $z \in N_{\mathbb{C}} a(z) \cdot x_o$ for all $z \in \Xi$. The image of $\text{Im log} \circ a$ is described by the complex convexity theorem (cf. [7], [12]):

$$(2.1) \quad \text{Im log } a(G \exp(iY) \cdot x_o) = \text{conv}(\mathcal{W} \cdot Y).$$

2.2. Spherical functions. Spherical functions on X have a holomorphic extension to the G -domain Ξ (cf. [13]). Moreover, Ξ is a maximal G -domain in $X_{\mathbb{C}}$ with this property (see Theorem 5.1 below). Thus, it seems reasonable to define spherical functions ab initio as holomorphic functions on Ξ . Detailed proofs of the material below can be found in [13].

Set $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} (\dim \mathfrak{g}^\alpha) \alpha \in \mathfrak{a}^*$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we set $a(\cdot)^\lambda := e^{\lambda \log a(\cdot)}$. Notice that a^λ is a holomorphic function on Ξ .

The *spherical function with parameter* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is the holomorphic function on Ξ defined by

$$\varphi_\lambda(z) = \int_K a(kz)^{\rho+\lambda} dk \quad (z \in \Xi),$$

dk , as usual, a normalized Haar measure on the compact group K .

Later we shall restrict our attention to spherical functions with imaginary parameters, i.e. $\lambda \in i\mathfrak{a}^*$, as only these appear in the Plancherel decomposition of $L^2(X)$.

While Ξ is canonical as a G -domain in $X_{\mathbb{C}}$ for spherical functions, the situation is slightly different if one considers instead the action of $K_{\mathbb{C}}$ on $X_{\mathbb{C}}$. Write $\widehat{X}_{\mathbb{C}, 2\Omega}$ for the smallest $K_{\mathbb{C}}$ -invariant domain in $X_{\mathbb{C}}$ which contains $A \exp(i2\Omega) \cdot x_o$ (for existence see [11]). Then a spherical function φ_λ can be considered equally as a $K_{\mathbb{C}}$ -invariant holomorphic function on $\widehat{X}_{\mathbb{C}, 2\Omega}$. To avoid unnecessary notation, we write φ_λ also when viewed as a function on $\widehat{X}_{\mathbb{C}, 2\Omega}$. For $u \in A$ and $Y \in \Omega$ we have then

$$(2.2) \quad |\varphi_\lambda(u \exp(i2Y) \cdot x_o)| \leq \varphi_\lambda(\exp(i2Y) \cdot x_o) = \int_K |a(k \exp(iY) \cdot x_o)^{\rho+\lambda}|^2 dk.$$

For $\lambda \in i\mathfrak{a}^*$ we set

$$(2.3) \quad \psi_\lambda(Z) = \sum_{w \in \mathcal{W}} e^{\lambda(wZ)} \quad (Z \in \mathfrak{a}_{\mathbb{C}}).$$

Notice that ψ_λ is a \mathcal{W} -invariant function that is positive on $i\mathfrak{a}$ and of exponential growth there.

Lemma 2.1. *For each $Y \in \Omega$ there exists a constant $C_Y > 0$ such that*

$$(2.4) \quad \varphi_\lambda(\exp(i2Y) \cdot x_o) \leq C_Y \cdot \psi_\lambda(i2Y)$$

for all $\lambda \in i\mathfrak{a}^*$.

Proof. This follows from equations (2.1) and (2.2). □

Remark 2.2. C_Y is locally bounded on Ω .

2.3. Harmonic analysis on Ξ . We conclude this section with some aspects of harmonic analysis on Ξ . We begin with the Fourier transform and Plancherel theorem for X following Helgason and Harish-Chandra.

Let $B = M \backslash K$ and db a normalized Haar measure on B . Set $\mathcal{X} = B \times i\mathfrak{a}^*$ and define a measure μ on \mathcal{X} by $d\mu(b, \lambda) = db \otimes \frac{d\lambda}{|\mathfrak{c}(\lambda)|^2}$ where $d\lambda$ is the measure on $i\mathfrak{a}^*$ as normalized by Harish-Chandra, $\mathfrak{c}(\lambda)$ is the Harish-Chandra \mathfrak{c} -function. For an integrable function f on X we define its Fourier transform to be the function \widehat{f} on \mathcal{X} defined by

$$\widehat{f}(b, \lambda) = \int_X f(x) a(bx)^{\rho-\lambda} dx, \quad (b, \lambda) \in \mathcal{X},$$

where dx is a G -invariant measure on X normalized so that $f \mapsto \widehat{f}$ extends to an isometry $L^2(X) \rightarrow L^2(\mathcal{X}, \mu)$. The Fourier inversion theorem in this formulation becomes

$$f(x) = \int_{\mathcal{X}} \widehat{f}(b, \lambda) a(bx)^{\rho+\lambda} d\mu(b, \lambda).$$

Let $\mathcal{G}(\Xi)$ be the space of holomorphic functions F on Ξ such that the following two properties hold:

- $f := F|_X \in L^2(X)$
- $\int_{\mathcal{X}} |\widehat{f}(b, \lambda)|^2 \psi_\lambda(iY) d\mu(b, \lambda) < \infty$ for all $Y \in \mathfrak{a}$.

Lemma 2.3 (Gutzmer's identity d'après Faraut). *Let $F \in \mathcal{G}(\Xi)$. Then*

$$(2.5) \quad \int_G |F(g \exp(iY) \cdot x_o)|^2 dg = \int_{\mathcal{X}} |\widehat{f}(b, \lambda)|^2 \varphi_\lambda(\exp(i2Y) \cdot x_o) d\mu(b, \lambda)$$

for all $Y \in \Omega$.

Proof. This is the Gutzmer identity of [5], Th. 1. □

For a sufficiently decreasing function h on Ξ we consider its G -orbital integral

$$\mathcal{O}_h(iY) = \int_G h(g \exp(\frac{i}{2}Y) \cdot x_o) dg \quad (Y \in 2\Omega).$$

In particular, if $F \in \mathcal{G}(\Xi)$ one has from (2.5) that $\mathcal{O}_{|F|^2}(iY)$ is finite for all $Y \in 2\Omega$. We consider in $\mathfrak{a}_{\mathbb{C}}$ the abelian tube domain $\mathcal{T}(2\Omega) = \mathfrak{a} + i2\Omega$. Then for $F \in \mathcal{G}(\Xi)$ it follows from Lemma 2.3 and the holomorphic extension of spherical functions [13] that $\mathcal{O}_{|F|^2}$ admits a natural extension to a holomorphic function on $\mathcal{T}(2\Omega)$, namely

$$(2.6) \quad \mathcal{O}_{|F|^2}(Z) = \int_{\mathcal{X}} |\widehat{f}(b, \lambda)|^2 \varphi_{\lambda}(\exp(Z) \cdot x_o) d\mu(b, \lambda) \quad (Z \in \mathcal{T}(2\Omega)).$$

3. THE HEAT KERNEL TRANSFORM

In this section we shall determine the image of the heat kernel transform on $L^2(X)$ and its holomorphic extension to Ξ . We start with a brief review of the results from [14].

3.1. Definition and basic properties. In the following, t denotes a positive number. According to Gangoli [6] the heat kernel k_t , suitably normalized, on X has the spectral resolution

$$(3.1) \quad k_t(x) = \int_{i\mathfrak{a}^*} e^{-t(|\lambda|^2 + |\rho|^2)} \varphi_{\lambda}(x) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \quad (x \in X).$$

If f is an analytic function on X which admits a holomorphic extension to Ξ , then we write f^{\sim} for the holomorphically extended function. It is proved in [14] (but also immediate from (3.1) and the results collected in Subsection 2.2) that k_t admits an analytic continuation k_t^{\sim} given by

$$k_t^{\sim}(z) = \int_{i\mathfrak{a}^*} e^{-t(|\lambda|^2 + |\rho|^2)} \varphi_{\lambda}(z) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \quad (z \in \Xi).$$

Our concern here is the heat kernel transform, viz. the operator given by convolution of k_t on $L^2(X)$:

$$(k_t * f)(x) = \int_X k_t(g^{-1}x) f(gK) d(gK) \quad (x \in X).$$

It was proved in [14] that $k_t * f$ admits a holomorphic extension given by

$$(k_t * f)^{\sim}(z) = \int_X k_t^{\sim}(g^{-1}z) f(gK) d(gK) \quad (z \in \Xi).$$

Thus one gets a linear map

$$H_t : L^2(X) \rightarrow \mathcal{O}(\Xi), \quad f \mapsto (k_t * f)^{\sim}$$

which is referred to as the *heat kernel transform with parameter $t > 0$* . In what follows we consider $\mathcal{O}(\Xi)$ as a topological vector space endowed with the usual Fréchet topology of compact convergence.

In [14] the following properties of the heat kernel transform are proved:

- H_t is continuous;
- H_t is G -equivariant;
- H_t is injective.

Thus $\text{im } H_t$, when endowed with the Hilbert topology from $L^2(X)$, becomes a G -invariant Hilbert space of holomorphic functions on Ξ . As such it admits a reproducing kernel which was spectrally characterized in ([14]) although the image of H_t was not described in [14].

Remark 3.1. Contrary to expectations, we suspect that $\text{im } H_t$ is not a weighted Bergman space, i.e. there does *not* exist a measure μ_t on Ξ , absolutely continuous with respect to the standard Lebesgue measure, such that the norm on $\text{im } H_t$ is given by

$$\int_{\Xi} |F(z)|^2 d\mu_t(z) \quad (F \in \text{im } H_t).$$

In Section 4 we will present an argument that shows for complex groups that there does not exist a G -invariant μ_t .

3.2. The image of the heat kernel transform. Let $f \in L^2(X)$ and set $F = H_t(f)$. In [14] it was shown that $F \in \mathcal{O}(\Xi)$. As ψ_λ is only of exponential growth it is easily seen that $F \in \mathcal{G}(\Xi)$. In particular, we can form the orbital integral $\mathcal{O}_{|F|^2}$ (cf. (2.6)) and for all $Z \in \mathcal{T}(2\Omega)$ one has from Lemma 2.3

$$(3.2) \quad \mathcal{O}_{|F|^2}(Z) = \int_{\mathcal{X}} |\widehat{F}(b, \lambda)|^2 \varphi_\lambda(\exp(Z) \cdot x_o) d\mu(b, \lambda)$$

$$(3.3) \quad = \int_{\mathcal{X}} |\widehat{f}(b, \lambda)|^2 e^{-2t(|\lambda|^2 + |\rho|^2)} \varphi_\lambda(\exp(Z) \cdot x_o) d\mu(b, \lambda)$$

$$(3.4) \quad = \int_{ia^*} g(\lambda) \varphi_\lambda(\exp(Z) \cdot x_o) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2},$$

where the function g is given by

$$(3.5) \quad g(\lambda) = e^{-2t(|\lambda|^2 + |\rho|^2)} \int_B |\widehat{f}(b, \lambda)|^2 db.$$

Next we define $\mathcal{F}(\mathcal{T}(\Omega))$ as the space of holomorphic functions h on $\mathcal{T}(2\Omega)$ such that

$$h(Z) = \int_{ia^*} g(\lambda) \varphi_\lambda(\exp(Z) \cdot x_o) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \quad (Z \in \mathcal{T}(2\Omega))$$

for an integrable \mathcal{W} -invariant function g on $i\mathfrak{a}^*$ with $\int_{i\mathfrak{a}^*} |g(\lambda)| \psi_\lambda(iY) \frac{d\lambda}{|\mathfrak{c}(\lambda)|^2} < \infty$ for all $Y \in \mathfrak{a}$. Notice that g is uniquely determined by h . Thus we can define a linear operator

$$D : \mathcal{F}(\mathcal{T}(\Omega)) \rightarrow \mathcal{O}(\mathfrak{a}_{\mathbb{C}}), \quad (Dh)(Z) = \int_{i\mathfrak{a}^*} g(\lambda) \psi_\lambda(Z) \frac{d\lambda}{|\mathfrak{c}(\lambda)|^2},$$

where ψ_λ is defined in 2.3. Observe that for $f \in L^2(X)$ and $F = H_t(f)$ we have $\mathcal{O}_{|F|^2} \in \mathcal{F}(\mathcal{T}(2\Omega))$ by (3.2) and (3.5), and so $D\mathcal{O}_{|F|^2} \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}})$.

Remark 3.2. In general, the operator D is a pseudo-differential shift operator, but if the multiplicities of all the restricted roots are even, it is a differential shift operator. This is most easily seen by relating the operator D to the Abel transform (Harish-Chandra's map $f \rightarrow F_f$). So let $h \in \mathcal{F}(\mathcal{T}(2\Omega))$. Then $h \circ \log|_A$ is a \mathcal{W} -invariant smooth function on A . By the C^∞ Chevalley theorem (cf. [4]) it has a natural extension to a K -invariant smooth function on X , say $H \in C^\infty(X)^K$. From the growth restriction on g it follows that H belongs to the Harish-Chandra Schwartz space on X . Hence the Abel transform

$$(3.6) \quad (\mathcal{A}H)(u) = u^\rho \int_N H(un) \, dn \quad (u \in A)$$

of H is well defined, and $\mathcal{A}H$ is a \mathcal{W} -invariant smooth function on A . (We remark that (3.6) cannot be extended to all of $u \in \exp(\mathcal{T}(2\Omega))$.)

Using the well known identity of the spherical transform of H and the Fourier transform of $\mathcal{A}H$ (for example cf. [17]) one obtains

$$(\mathcal{A}H)(u) = \int_{i\mathfrak{a}^*} g(\lambda) \psi_\lambda(\log u) \, d\lambda.$$

This shows that D is given by \mathcal{A} composed with the Fourier multiplier operator corresponding to $\frac{1}{|\mathfrak{c}(\lambda)|^2}$. The remark follows from the well known properties of \mathcal{A} and the structure of $\frac{1}{|\mathfrak{c}(\lambda)|^2}$.

To formulate the main result of the paper, with $n = \dim \mathfrak{a}$, we define a weight function on \mathfrak{a} by

$$w_t(Y) = \frac{1}{|\mathcal{W}|} \cdot \frac{e^{2t|\rho|^2}}{(2\pi t)^{\frac{n}{2}}} \cdot e^{-\frac{|Y|^2}{2t}} \quad (Y \in \mathfrak{a}).$$

Theorem 3.3. *For $f \in L^2(X)$ and $F = H_t(f)$ the following norm identity holds*

$$(3.7) \quad \|f\|^2 = \int_{\mathfrak{a}} (D\mathcal{O}_{|F|^2})(iY) w_t(Y) \, dY.$$

Moreover, one has

$$(3.8) \quad \text{im } H_t = \{F \in \mathcal{G}(\Xi) : \int_{\mathfrak{a}} (D\mathcal{O}_{|F|^2})(iY) w_t(Y) \, dY < \infty\}.$$

Proof. As observed earlier, $F = H_t(f)$ is in $\mathcal{G}(\Xi)$. So consider the norm identity (3.7). By the Plancherel theorem the left hand side of (3.7) is given by

$$(3.9) \quad \|f\|^2 = \int_{\mathcal{X}} |\widehat{f}(b, \lambda)|^2 d\mu(b, \lambda)$$

while, according to (3.2), the right hand side of (3.7) is given by

$$(3.10) \quad \int_{\mathfrak{a}} (D\mathcal{O}_{|F|^2})(iY) w_t(Y) dY = \int_{\mathfrak{a}} \int_{\mathcal{X}} |\widehat{f}(b, \lambda)|^2 e^{-2t(|\lambda|^2 + |\rho|^2)} \psi_{\lambda}(i2Y) w_t(Y) d\mu(b, \lambda) dY.$$

Comparing (3.9) and (3.10), we see that (3.7) will be established provided we can show

$$\int_{\mathfrak{a}} \psi_{\lambda}(i2Y) w_t(Y) dY = e^{2t(|\lambda|^2 + |\rho|^2)}$$

for $\psi_{\lambda}(i2Y)$ the Weyl symmetrized exponential function, and for each $\lambda \in i\mathfrak{a}^*$. But, using the definition of w_t , we have

$$\begin{aligned} \int_{\mathfrak{a}} \psi_{\lambda}(i2Y) w_t(Y) dY &= \frac{1}{|\mathcal{W}|} \cdot \frac{e^{2t|\rho|^2}}{(2\pi t)^{\frac{n}{2}}} \int_{\mathfrak{a}} \psi_{\lambda}(i2Y) e^{-\frac{|Y|^2}{2t}} dY \\ &= \frac{e^{2t|\rho|^2}}{(2\pi t)^{\frac{n}{2}}} \int_{\mathfrak{a}} e^{\lambda(i2Y)} e^{-\frac{|Y|^2}{2t}} dY \\ &= \frac{e^{2t|\rho|^2}}{(2\pi)^{\frac{n}{2}}} \int_{\mathfrak{a}} e^{\lambda(i2\sqrt{t}Y)} e^{-\frac{|Y|^2}{2}} dY \\ &= e^{2t|\rho|^2} e^{\frac{|2\sqrt{t}\lambda|^2}{2}} = e^{2t(|\lambda|^2 + |\rho|^2)}, \end{aligned}$$

establishing (3.7) and also proving that H_t is an injective isometry into the space described in (3.8).

For surjectivity, take an $F \in \mathcal{G}(\Xi)$. Then $f = F|_X \in L^2(X)$. We shall use the Paley-Wiener space from [14]. The main properties of $PW(G/K)$ needed here are that it is dense in $L^2(X)$ and that it consists of functions with compactly supported Fourier transforms. We will assume that f is in $PW(G/K)$. Multiply its Fourier transform by $e^{t(|\lambda|^2 + |\rho|^2)}$, then invert the result. We obtain a function g still in $PW(G/K)$. To g we apply the heat transform H_t . Then $H_t(g)$ is in $\mathcal{G}(\Xi)$ and its restriction to X has the same Fourier transform as f . Since $H_t(g)$ and F are in $\mathcal{G}(\Xi)$ and their restrictions to X have the same Fourier transform, we have $H_t(g) = F$. To conclude the argument we use that $PW(G/K)$ is dense and the identity of norms in (3.7). Finally, the characterization (3.8) follows from the observation that only equalities have been used in the above proof. \square

4. $\text{im } H_t$ NEED NOT BE WEIGHTED

In this appendix we will prove the assertion mentioned in Remark 3.1. The amount of detail presented is at the insistence of the referee. The material is extracted from unpublished notes of the first and last named authors.

In many ways, Riemannian symmetric spaces of the non-compact type $X = G/K$ resemble the real numbers \mathbb{R} , e.g. both X and \mathbb{R} are analytic complete Riemannian manifolds with Laplace operator having purely continuous L^2 -spectrum. There is however a basic difference between their natural complexifications, $\mathbb{C} = T\mathbb{R}$ and Ξ , in that for the fibration $\mathbb{C} = T\mathbb{R} \rightarrow \mathbb{R}$ the fibers are complete and $T\mathbb{R} = \mathbb{C}$ is a complete manifold; whereas, $\Xi = G \times_K \text{Ad}(K)\Omega \subset TX = G \times_K \mathfrak{p}$ resembles is a disk bundle over X with neither the fibers nor the total space Ξ complete with respect to the natural (adapted) metric ([14], Sect. 4). Thus in the abelian setting a better analog of the crown is a strip domain $S = \{z \in \mathbb{C} : |\text{Im } z| < \gamma\}$, $\gamma > 0$. For this simple situation it is not difficult to see that the image of the heat kernel transform $H_t : L^2(\mathbb{R}) \rightarrow \mathcal{O}(\mathbb{C})$, when considered in $\mathcal{O}(S)$, is not a weighted Bergman space. We include a detailed discussion of the flat case, then move to the curved situation Ξ .

4.1. Some features of the heat kernel transform on \mathbb{R} . Write $\Delta = \frac{d^2}{dx^2}$ for the Laplace operator on \mathbb{R} and consider the heat equation

$$(\Delta - \partial_t)u(x, t) = 0 \quad ((x, t) \in \mathbb{R} \times \mathbb{R}_{>0}).$$

The fundamental solution is given by the heat kernel

$$(4.1) \quad k_t(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} e^{-ty^2} dy$$

$$(4.2) \quad = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

where $x \in \mathbb{R}, t > 0$.

If f is an analytic function on \mathbb{R} which extends holomorphically to \mathbb{C} , then we denote by f^\sim this holomorphic extension. From the explicit formula (4.2) one sees that k_t extends holomorphically to \mathbb{C} , and

$$(4.3) \quad k_t^\sim(z) = \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{z^2}{4t}} \quad (z \in \mathbb{C}).$$

Fix $f \in L^2(\mathbb{R})$. The convolution

$$\begin{aligned} (f * k_t)(x) &= \int_{\mathbb{R}} f(y) k_t(x - y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) e^{-\frac{(x-y)^2}{4t}} dy \end{aligned}$$

admits a holomorphic continuation to \mathbb{C} . Thus we obtain a map

$$H_t : L^2(\mathbb{R}) \rightarrow \mathcal{O}(\mathbb{C}), \quad f \mapsto (f * k_t)^\sim,$$

the *heat kernel transform*. We will consider $\mathcal{O}(\mathbb{C})$ as a Fréchet topological vector space (topology of compact convergence).

It is an elementary task to verify the following properties of H_t :

- H_t is injective.
- H_t is continuous.
- H_t is equivariant with respect to the \mathbb{R} -action, i.e. $H_t(f(\cdot + x)) = H_t(f)(\cdot + x)$ for all $x \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$.

From these properties one deduces that $\mathcal{F}_t(\mathbb{C}) := \text{im } H_t$, when endowed with the Hilbert inner product from $L^2(\mathbb{R})$, becomes a Hilbert space of holomorphic functions. As such it admits a reproducing kernel

$$\mathcal{K}^t : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}, \quad (z, w) \mapsto \mathcal{K}^t(z, w).$$

This is a continuous function which is holomorphic in the first variable and anti-holomorphic in the second variable. For $w \in \mathbb{C}$ we define a function $\mathcal{K}_w^t \in \mathcal{F}_t(\mathbb{C})$ by

$$\mathcal{K}_w^t(z) = \mathcal{K}^t(z, w).$$

If $\langle \cdot, \cdot \rangle$ denotes the Hilbert inner product on $\mathcal{F}_t(\mathbb{C})$, then \mathcal{K}_w^t describes the point-evaluation at w , i.e.

$$(\forall f \in \mathcal{F}_t(\mathbb{C})) \quad f(w) = \langle f, \mathcal{K}_w^t \rangle.$$

Notice that $H_t^{-1}(\mathcal{K}_z^t) = \overline{k_t^\sim(z - \cdot)}$. Thus

$$\begin{aligned} (4.4) \quad \mathcal{K}^t(z, w) &= \langle \mathcal{K}_w^t, \mathcal{K}_z^t \rangle = \langle H_t^{-1}(\mathcal{K}_w^t), H_t^{-1}(\mathcal{K}_z^t) \rangle_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} k_t^\sim(z - x) \overline{k_t^\sim(w - x)} \, dx \\ &= \int_{\mathbb{R}} k_t^\sim(z - x) k_t^\sim(\bar{w} - x) \, dx \\ &= \int_{\mathbb{R}} k_t^\sim(-x) k_t^\sim(\bar{w} - z - x) \, dx \\ &= \int_{\mathbb{R}} k_t^\sim(x) k_t^\sim(\bar{w} - z - x) \, dx \\ &= (k_t * k_t)^\sim(\bar{w} - z) \\ &= k_{2t}^\sim(z - \bar{w}) = \frac{1}{\sqrt{8\pi t}} \cdot e^{-\frac{(z - \bar{w})^2}{8t}}. \end{aligned}$$

The natural question as to whether $\mathcal{F}_t(\mathbb{C})$ admits a description as a weighted Bergman space was answered affirmatively by Bargmann and Segal. With

$$(4.5) \quad w_t(y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \quad (y \in \mathbb{R})$$

one has

$$\mathcal{F}_t(\mathbb{C}) = \left\{ f \in \mathcal{O}(\mathbb{C}) : \|f\|^2 = \int_{\mathbb{C}} |f(x+iy)|^2 w_t(y) \, dx dy < \infty \right\}.$$

This can be easily verified either by direct computation or, perhaps preferably, by abstract Hilbert space techniques (use that the point evaluations $\{\mathcal{K}_z^t : z \in \mathbb{C}\}$ form a dense subspace of $\mathcal{F}_t(\mathbb{C})$).

Let us pass now to a strip domain $S = \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \gamma\}$ in the complex plane. We denote by $\operatorname{Res} : \mathcal{O}(\mathbb{C}) \rightarrow \mathcal{O}(S)$ the restriction map and consider the restriction of the heat kernel transform $\operatorname{Res} \circ H_t : L^2(\mathbb{R}) \rightarrow \mathcal{O}(S)$. We notice, as Res is continuous and S is translation invariant, that $\operatorname{Res} \circ H_t$ satisfies the bulleted items from before as well. In conclusion, $\operatorname{im} \operatorname{Res} \circ H_t = \mathcal{F}_t(S)$ is an \mathbb{R} -invariant Hilbert space of holomorphic functions on S . However, the nature of $\mathcal{F}_t(S)$ is different from $\mathcal{F}_t(\mathbb{C})$ because

Proposition 4.1. *There does not exist a positive \mathbb{R} -translation invariant measurable weight function $w_t : S \rightarrow \mathbb{R}_{>0}$ such that*

$$\mathcal{F}_t(S) = \{f \in \mathcal{O}(S) : \|f\|^2 = \int_S |f(z)|^2 w_t(z) \, dx dy < \infty\};$$

in other words, $\mathcal{F}_t(S)$ is not a weighted Bergman space with respect to a translation invariant weight.

Proof. The reproducing kernel for $\mathcal{F}_t(S)$ is simply the restriction of the kernel of $\mathcal{F}_t(\mathbb{C})$ to $S \times S$ – a quick inspection of our derivation of the identity $\mathcal{K}^t(z, w) = k_{2t}^\sim(z - \bar{w})$ shows that. To simplify notation, we write H_t instead of $\operatorname{Res} \circ H_t$.

For each $z \in S$ the function $w \mapsto \mathcal{K}_z^t(w) = \mathcal{K}^t(w, z)$ is in the Hilbert space $\operatorname{im} H_t$. As $k_t * k_t = k_{2t}$, we have $\mathcal{K}_x^t = H_t(k_t(\cdot - x))$ for all $x \in \mathbb{R}$. Notice that the collection of \mathcal{K}_x^t , $x \in \mathbb{R}$, spans a dense subspace of $\mathcal{F}_t(S)$, hence the existence of a translation invariant weight for $\mathcal{F}_t(S)$ is equivalent to

$$\begin{aligned} (4.6) \quad \langle k_t(\cdot - x), k_t(\cdot - y) \rangle_{L^2(\mathbb{R})} &= \int_S \mathcal{K}_x^t(z) \overline{\mathcal{K}_y^t(z)} w_t(z) dz \\ &= \int_S \mathcal{K}_x^t(u+iv) \overline{\mathcal{K}_y^t(u+iv)} w_t(v) \, dudv \quad (x, y \in \mathbb{R}). \end{aligned}$$

Using translation invariance we may assume $y = 0$. Straightforward computation transforms the identity (4.6) into

$$(4.7) \quad k_{2t}(x) = \int_{-\gamma}^{\gamma} k_{4t}^\sim(x - 2iv) w_t(v) dv \quad (x \in \mathbb{R}).$$

As $k_{4t}^\sim(x - 2iv) = \operatorname{const} \cdot k_{4t}(x) e^{ixv/4t} e^{v^2/4t}$, moving $k_{4t}(x)$ to the left side of (4.7) gives

$$\begin{aligned}
e^{-x^2/16t} &= \text{const} \cdot \int_{-\gamma}^{\gamma} e^{ixv/4t} e^{v^2/4t} w_t(v) dv \\
e^{-ty^2} &= \text{const} \cdot \int_{-\gamma}^{\gamma} e^{iyv} e^{v^2/4t} w_t(v) dv \quad (y \in \mathbb{R}),
\end{aligned}$$

and we obtain a contradiction to the Fourier transform of the function e^{-ty^2} . \square

4.2. From flat to curved. Now moving from \mathbb{R} to X we will show that $\mathcal{F}_t(\Xi) = \text{im } H_t$ is NOT a weighted Bergman space.

At first we make no restriction on G , such as G is complex. We recall from [14], Th. 6.4 the formula for the reproducing kernel of $\mathcal{F}_t(\Xi)$,

$$\begin{aligned}
\mathcal{K}^t(z, w) &= \int_X k_t^\sim(g^{-1}z) \overline{k_t^\sim(g^{-1}w)} d(gK) \\
&= \int_X k_t^\sim(g^{-1}z) k_t^\sim(g^{-1}\bar{w}) d(gK) \quad (z, w \in \Xi).
\end{aligned}$$

When $\bar{w}^{-1}z$ is in $K_{\mathbb{C}}\Xi$ we can use integration by parts and $k_t * k_s = k_{t+s}$ to arrive at

$$(4.8) \quad \mathcal{K}^t(z, w) = k_{2t}^\sim(\bar{w}^{-1}z) \quad (\forall z, w \in \Xi \text{ with } \bar{w}^{-1}z \in K_{\mathbb{C}}\Xi).$$

Then (4.8) is the curved analogue of the formula (4.4).

Suppose for a moment that $\mathcal{F}_t(\Xi)$ were a weighted Bergman space with respect to an absolutely continuous G -invariant measure μ_t , i.e.

$$\mathcal{F}_t(\Xi) = \{f \in \mathcal{O}(\Xi) : \|f\|^2 = \int_{\Xi} |f(z)|^2 d\mu_t(z) dz < \infty\}.$$

As $X \subseteq \Xi$ is a totally real submanifold, it follows that the real point evaluation K_x^t , $x \in X$, form a dense subspace of $\mathcal{F}_t(\Xi)$. Thus the measure μ_t would be uniquely determined by the values

$$(4.9) \quad \langle \mathcal{K}_x^t, \mathcal{K}_y^t \rangle = \int_{\Xi} \mathcal{K}_x^t(z) \overline{\mathcal{K}_y^t(z)} d\mu_t(z)$$

for $x, y \in X$. Using G -invariance we may actually assume that $y = x_o$ and $x = a \cdot x_o$ for some $a \in A$. Apply (4.8) and transform the left hand side of (4.9) to

$$(4.10) \quad \langle \mathcal{K}_x^t, \mathcal{K}_y^t \rangle = \mathcal{K}^t(a \cdot x_o, x_o) = k_{2t}(a^{-1}) = k_{2t}(a) \quad (a \in A).$$

We will write Ω^+ for the intersection of Ω with a Weyl chamber and recall from [14], Cor. 4.2, the fact that the map

$$G/M \times \Omega^+ \rightarrow \Xi, \quad (gM, Y) \mapsto g \exp(iY) \cdot x_o$$

is a G -equivariant diffeomorphism with open dense image. It follows that the G -equivariant measure μ_t can be expressed as $w_t(Y)dYd(gM)$ with $w_t : \Omega^+ \rightarrow \mathbb{R}_{>0}$ a measurable weight function.

Now we note that that the right hand side of (4.9) can be computed to be

$$\begin{aligned}
\int_{\Xi} \mathcal{K}_x^t(z) \overline{\mathcal{K}_y^t(z)} d\mu_t(z) &= \int_{\Xi} k_{2t}^{\sim}(a^{-1}z) \overline{k_{2t}^{\sim}(z)} d\mu_t(z) \\
&= \int_G \int_{\Omega^+} k_{2t}^{\sim}(a^{-1}g \exp(iY) \cdot x_o) \overline{k_{2t}^{\sim}(g \exp(iY) \cdot x_o)} w_t(Y) dY dg \\
(4.11) \quad &= \int_G \int_{\Omega^+} k_{2t}^{\sim}(a^{-1}g \exp(iY) \cdot x_o) k_{2t}^{\sim}(g \exp(-iY) \cdot x_o) w_t(Y) dY dg.
\end{aligned}$$

We perform an intermediate step.

Lemma 4.2. *For all $a \in A$ and $Y \in \Omega$ the following identity holds:*

$$\begin{aligned}
(4.12) \quad &\int_G k_{2t}^{\sim}(a^{-1}g \exp(iY) \cdot x_o) k_{2t}^{\sim}(g \exp(-iY) \cdot x_o) dg \\
&= \int_{ia^*} e^{-4t(|\lambda|^2 + |\rho|^2)} \varphi_{\lambda}(a) \varphi_{\lambda}(\exp(2iY) \cdot x_o) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}.
\end{aligned}$$

Proof. Both sides of (4.12) depend analytically on Y and hence it is sufficient to prove the identity for $Y \in \frac{1}{2}\Omega$. Change of variables transforms the left hand side of (4.12) into

$$(4.13) \quad \int_G k_{2t}^{\sim}(a^{-1}g \exp(2iY) \cdot x_o) k_{2t}(g \cdot x_o) dg$$

and we start manipulating (4.13):

$$\begin{aligned}
&\int_G k_{2t}^{\sim}(a^{-1}g \exp(2iY) \cdot x_o) k_{2t}(g \cdot x_o) dg \\
&= \int_{ia^*} \int_G e^{-2t(|\lambda|^2 + |\rho|^2)} \varphi_{\lambda}(a^{-1}g \exp(2iY) \cdot x_o) k_{2t}(g) dg \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \\
&= \int_{ia^*} \int_G \int_K e^{-2t(|\lambda|^2 + |\rho|^2)} \varphi_{\lambda}(a^{-1}kg \exp(2iY) \cdot x_o) k_{2t}(g) dk dg \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \\
&= \int_{ia^*} \int_G e^{-2t(|\lambda|^2 + |\rho|^2)} \varphi_{\lambda}(a^{-1}) \varphi_{\lambda}(g \exp(2iY) \cdot x_o) k_{2t}(g) dg \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \\
&= \int_{ia^*} e^{-4t(|\lambda|^2 + |\rho|^2)} \varphi_{-\lambda}(a) \varphi_{-\lambda}(\exp(2iY) \cdot x_o) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \\
&= \int_{ia^*} e^{-4t(|\lambda|^2 + |\rho|^2)} \varphi_{\lambda}(a) \varphi_{\lambda}(\exp(2iY) \cdot x_o) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}.
\end{aligned}$$

□

We combine Lemma 4.2, (4.11) and (4.10) to arrive at the curved analogue of the abelian identity (4.7).

Lemma 4.3. *The existence of a G -invariant weight function for $\mathcal{F}_t(\Xi)$ implies*

$$(4.14) \quad k_{2t}(a) = \int_{\Omega^+} \int_{i\mathfrak{a}^*} e^{-4t(|\lambda|^2 + |\rho|^2)} \varphi_\lambda(a) \varphi_\lambda(\exp(2iY) \cdot x_o) w_t(Y) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} dY \quad (a \in A).$$

Finally, we derive a contradiction from (4.14). We replace k_{2t} on the left hand side of (4.14) by its spectral resolution (3.1), use uniqueness of the Fourier transform and conclude that (4.14) is equivalent to

$$(4.15) \quad e^{-2t(|\lambda|^2 + |\rho|^2)} = \int_{\Omega^+} e^{-4t(|\lambda|^2 + |\rho|^2)} \varphi_\lambda(\exp(2iY) \cdot x_o) w_t(Y) dY \quad (\lambda \in i\mathfrak{a}^*),$$

or, in other words,

$$(4.16) \quad e^{2t(|\lambda|^2 + |\rho|^2)} = \int_{\Omega^+} \varphi_\lambda(\exp(2iY) \cdot x_o) w_t(Y) dY \quad (\lambda \in i\mathfrak{a}^*).$$

At this point estimates for $\varphi_\lambda(\exp(2iY))$ slightly better than (2.4) would allow us to obtain a contradiction. For complex groups, however, there is an alternative approach using Harish-Chandra's formula

$$(4.17) \quad \varphi_\lambda(a) = \frac{\mathbf{c}(\lambda)}{\delta(a)} \sum_{w \in W} \epsilon(w) e^{\lambda(w \log a)} \quad (a \in A)$$

with the functions $\delta(a) = (-1)^{|\Sigma^+|/2} \prod_{\alpha \in \Sigma^+} [e^{\alpha(\log a)} - e^{-\alpha(\log a)}] = (-1)^{|\Sigma^+|/2} \sum_{w \in W} \epsilon(w) e^{\rho(w \log a)}$ and $\mathbf{c}(\lambda) = \frac{\prod_{\alpha \in \Sigma^+} \langle \rho, \alpha \rangle}{\prod_{\alpha \in \Sigma^+} \langle \lambda, \alpha \rangle}$. Hence for G complex, and setting $W_t(Y) := \delta(\exp(2iY))^{-1} w_t(Y)$ we derive from (4.16) the identity

$$(4.18) \quad e^{2t(|\lambda|^2 + |\rho|^2)} = \int_{\Omega^+} \mathbf{c}(\lambda) \sum_{w \in W} \epsilon(w) e^{\lambda(i2wY)} W_t(Y) dY \quad (\lambda \in i\mathfrak{a}^*).$$

Since $\varphi_\lambda(w \cdot a) = \varphi_\lambda(a)$, we have $W_t(w \cdot Y) = \epsilon(w) W_t(Y)$. Then using the standard unfolding argument we get

$$(4.19) \quad e^{2t(|\lambda|^2 + |\rho|^2)} = \int_{\Omega} \mathbf{c}(\lambda) e^{\lambda(i2Y)} W_t(Y) dY \quad (\lambda \in i\mathfrak{a}^*).$$

From the definition of Ω it is clear that $\delta(a)$ is non-zero at the boundary of Ω , so introduces no singularity to the integrand there. Whereas the smoothness of the spherical function shows there is no singularity of the integrand caused by the vanishing on $\delta(a)$ on root planes. Using the discussion on p. 329 of [19], we see that behaviour near the root

planes contributes at most an additional factor of polynomial growth in λ ; on the other hand, the integral away from the root planes can be estimated by exponential growth in λ as Ω is compact. This contradicts that $e^{2t(|\lambda|^2+|\rho|^2)}$ has exponential growth in $|\lambda|^2$ and completes the argument for complex groups. The general case should follow similarly using Gangolli's expansion for $\delta(a)\varphi_\lambda(a)$ and his estimates.

5. THE CROWN AS A G -DOMAIN OF HOLOMORPHY FOR A SPHERICAL FUNCTION

This section too is included to resolve a disagreement between the authors and the referee. The material is extracted from unpublished notes of the first and last named authors.

Theorem 5.1. *The crown is the maximal G -invariant domain in $X_{\mathbb{C}}$ to which a spherical function φ_λ with $\lambda \in i\mathfrak{a}^*$ extends holomorphically.*

Remark 5.2. We stress the importance of G -invariance in the statement of the theorem. Without this assumption the statement becomes false, see [13], Th. 4.2. In order to obtain a feeling for this situation consider the function $f(z) = \sqrt{1-z}$ on the complex plane \mathbb{C} . Now the maximal \mathbb{S}^1 -invariant domain of definition for f is the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and there the function f is bounded. Similarly, all non-trivial positive definite spherical functions are bounded on Ξ by [11] and the theorem asserts that they do not extend to a bigger G -invariant domain.

The theorem can be reduced to the basic case of $G = \mathrm{Sl}(2, \mathbb{R})$ which will be presented first.

5.1. The basic case. For this section we shall assume that $X = G/K = \mathrm{Sl}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$. We follow the custom and make the standard choices

$$A = \left\{ \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} : t > 0 \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

Then

$$\Omega = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \in \mathfrak{a} : |x| < \frac{\pi}{4} \right\} = (-\pi/4, \pi/4)$$

Define a $K_{\mathbb{C}}$ -invariant holomorphic function on $X_{\mathbb{C}} = \mathrm{Sl}(2, \mathbb{C})/\mathrm{SO}(2, \mathbb{C})$ by

$$P : X_{\mathbb{C}} \rightarrow \mathbb{C}, \quad z \cdot x_o \mapsto \mathrm{tr}(zz^t) = a^2 + b^2 + c^2 + d^2 \quad \left(z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

Of course, P is nothing other than the elementary spherical function.

For a convex \mathcal{W} -invariant open set $\omega \subseteq 2\Omega$ one defines in [11] the smallest $K_{\mathbb{C}}$ -domain $\hat{X}_{\mathbb{C},\omega}$ in $X_{\mathbb{C}}$ which contains $A \exp(i\omega) \cdot x_o$. Specifically, we draw attention to the $K_{\mathbb{C}}$ -domains

$$(5.1) \quad \hat{X}_{\mathbb{C},\Omega} = \{z \in X_{\mathbb{C}} : \mathrm{Re} P(z) > 0\}$$

$$(5.2) \quad \hat{X}_{\mathbb{C},2\Omega} = \{z \in X_{\mathbb{C}} : P(z) \in \mathbb{C} \setminus]-\infty, -2]\}.$$

We recall the significance of $\hat{X}_{\mathbb{C},2\Omega}$ as the largest $K_{\mathbb{C}}$ -domain to which spherical functions extend, [14] Th. 2.4. Next, as $\Xi = G \exp(i\Omega) \cdot x_o$, it is immediate that

$$(5.3) \quad \Xi \subset \hat{X}_{\mathbb{C},\Omega}$$

and we note that (5.3) holds in full generality [11].

The open interval $\Omega = (-\pi/4, \pi/4)$ is best possible for even the weaker inclusion

$$(5.4) \quad \Xi \subset \hat{X}_{\mathbb{C},2\Omega}$$

to hold. The precise statement is as follows.

Lemma 5.3. *Let $G = \mathrm{Sl}(2, \mathbb{R})$. Then for $Y \in 2\Omega \setminus \bar{\Omega}$,*

$$(5.5) \quad G \exp(iY) \cdot x_o \not\subset \hat{X}_{\mathbb{C},2\Omega}.$$

More precisely, there exists a curve $\gamma(s)$, $s \in [0, 1]$, in G such that the assignment

$$s \mapsto \sigma(s) = P(\gamma(s) \exp(iY) \cdot x_o)$$

is strictly decreasing with values in $[-2, 2]$ such that $\sigma(0) = P(x_o) = 2$ and $\sigma(1) = -2$.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \in \exp(2i\Omega) \setminus \exp(i\bar{\Omega})$. This means $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ and $\frac{\pi}{4} < |\phi| < \frac{\pi}{2}$ for $\phi \in \mathbb{R}$. Thus

$$\begin{aligned} P(gz \cdot x_o) &= P \begin{pmatrix} ae^{i\phi} & be^{-i\phi} \\ ce^{i\phi} & de^{-i\phi} \end{pmatrix} = a^2 e^{2i\phi} + b^2 e^{-2i\phi} + c^2 e^{2i\phi} + d^2 e^{-2i\phi} \\ &= \cos(2\phi)(a^2 + b^2 + c^2 + d^2) + i \sin 2\phi(a^2 - b^2 + c^2 - d^2) \end{aligned}$$

Using that $G = KAN$ and that P is left K -invariant, we may actually assume that $g \in AN$, i.e.

$$g = \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}$$

for some $a > 0$ and $b \in \mathbb{R}$. Then

$$P(gz \cdot x_o) = \cos(2\phi)(a^2 + \frac{1}{a^2} + b^2) + i \sin 2\phi(a^2 - \frac{1}{a^2} - b^2).$$

We now show that $P(gz \cdot x_o) = -2$ has a solution for fixed $\frac{\pi}{4} < |\phi| < \frac{\pi}{2}$. This is because $P(gz \cdot x_o) = -2$ forces $\mathrm{Im} P(gz \cdot x_o) = 0$ and so $b^2 = a^2 - \frac{1}{a^2}$. Thus

$$P(gz \cdot x_o) = 2a^2 \cos(2\phi) = -2.$$

Thus if we choose $a = \frac{1}{\sqrt{-\cos 2\phi}}$ we obtain a solution. The desired curve $\gamma(s)$ is now given by

$$\gamma(s) = \begin{pmatrix} a(s) & b(s) \\ 0 & \frac{1}{a(s)} \end{pmatrix}$$

with $a(s) = \frac{1}{\sqrt{-\cos 2\phi}}(\sqrt{-\cos 2\phi} + s(1 - \sqrt{-\cos 2\phi}))$ and $b(s) = \sqrt{a(s)^2 - \frac{1}{a(s)^2}}$. \square

Theorem 5.4. *Let $G = \mathrm{Sl}(2, \mathbb{R})$. Then the crown Ξ is a maximal G -domain to which a spherical function φ_λ with $\lambda \in i\mathfrak{a}^*$ extends holomorphically.*

Proof. Fix $\lambda \in i\mathfrak{a}^*$. We consider the spherical function φ_λ on its maximal $K_{\mathbb{C}}$ -domain $\hat{X}_{\mathbb{C}, 2\Omega}$ of definition. Thus for each φ_λ there exists a holomorphic function Φ_λ on $\mathbb{C} \setminus (-\infty, 2] = P(\hat{X}_{\mathbb{C}, 2\Omega})$ such that

$$(5.6) \quad \varphi_\lambda(z) = \Phi_\lambda(P(z)) \quad (z \in \hat{X}_{\mathbb{C}, 2\Omega}).$$

Let $Y \in 2\Omega \setminus \overline{\Omega}$. Let $\gamma \subset G$ and $\sigma \subset [-2, 2]$ be curves as in the previous lemma.

Note that $\gamma(s) \exp(iY) \cdot x_o \subset G$ for all $s \in [0, 1)$. Hence (5.6) gives

$$\varphi_\lambda(\gamma(s) \exp(iY) \cdot x_o) = \Phi_\lambda(\sigma(s)) \quad (s \in [0, 1)).$$

Now recall that $s \mapsto \Phi_\lambda(\sigma(s))$ is positive (cf. [13], Th. 4.2) and tends to infinity for $s \nearrow 1$ (cf. [14], Th. 2.4). Thus the assertion of the theorem will be proved if we can show that a G -invariant domain $\Xi' \subset X_{\mathbb{C}}$ properly containing Ξ contains points in $\exp(i2\Omega \setminus \overline{\Omega})$. But this follows from the fact that each G -orbit in the boundary $\partial\Xi$ contains the point $\begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \cdot x_o \in \exp(i\partial\Omega) \cdot x_o$ in its closure (cf. [14], Lemma 2.3(ii)). \square

5.2. Proof of Theorem 5.1. Now that we understand $G = \mathrm{Sl}(2, \mathbb{R})$, the general case will follow easily. We recall some facts on the boundary of Ξ . First, each boundary G -orbit contains a point of $\exp(i\partial\Omega) \cdot x_o$ in its closure ([14], Lemma 2.3 (ii)). Next, to each $Y \in \partial\Omega$ we can associate an $\mathrm{Sl}(2, \mathbb{R})$ -crown Ξ_{Sl_2} (cf. [14], Th. 2.4 with formula (2.2) in its proof) which embeds into Ξ in such a way that

$$\partial\Xi_{\mathrm{Sl}_2} \ni \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \cdot x_o \mapsto \exp(iY) \cdot x_o \in \partial\Xi.$$

We restrict the spherical function ϕ_λ to Ξ_{Sl_2} , see [13], Prop. 4.5, and apply Theorem 5.4.

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