

# THE HEAT KERNEL TRANSFORM FOR THE HEISENBERG GROUP

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ABSTRACT. The heat kernel transform  $\mathcal{H}_t$  is studied for the Heisenberg group in detail. The main result shows that the image of  $\mathcal{H}_t$  is a direct sum of two weighted Bergman spaces, in contrast to the classical case of  $\mathbb{R}^n$  and compact symmetric spaces, and the weight functions are found to be (surprisingly) not non-negative.

## 1. INTRODUCTION

Over the last decade one could observe interesting developments on the *heat kernel transform* for various types of homogeneous Riemannian manifolds  $X$ . Complete results have been obtained for compact Lie groups (cf. [?, ?]) and, more generally, for compact symmetric spaces (cf. [?]). For non-compact spaces  $X$  the situation seems to be more complicated and little research has been undertaken in this direction: There is the well understood Euclidean case (e.g.  $X = \mathbb{R}^n$ , cf. [?]) and some partial results have been obtained for non-compact Riemannian symmetric spaces (cf. [?]). The objective of this paper is to give a complete and self-contained discussion for the Heisenberg group.

Our concern is with the  $(2n+1)$ -dimensional Heisenberg group  $\mathbb{H}$  and its universal complexification  $\mathbb{H}_{\mathbb{C}}$ . For  $t > 0$  we write  $k_t : \mathbb{H} \rightarrow \mathbb{R}^+$  for the heat kernel on  $\mathbb{H}$ . Contemplating on the spectral resolution of  $k_t$ , it is not hard to see that  $k_t$  admits an analytic continuation to a holomorphic function  $k_t^{\sim} : \mathbb{H}_{\mathbb{C}} \rightarrow \mathbb{C}$ . Consequently, for every  $f \in L^2(\mathbb{H})$  the convolution  $f * k_t$  continues holomorphically to  $H_{\mathbb{C}}$  and we obtain a map

$$\mathcal{H}_t : L^2(\mathbb{H}) \rightarrow \mathcal{O}(\mathbb{H}_{\mathbb{C}}), \quad f \mapsto (f * k_t)^{\sim}.$$

We refer to  $\mathcal{H}_t$  as the *heat kernel transform* on  $\mathbb{H}$  with parameter  $t > 0$ . The map  $\mathcal{H}_t$  is injective, left  $\mathbb{H}$ -equivariant and becomes continuous if  $\mathcal{O}(\mathbb{H}_{\mathbb{C}})$  is equipped with its natural Fréchet topology of compact convergence. It follows that  $\text{im } \mathcal{H}_t$  is a reproducing kernel Hilbert space. Standard abstract arguments readily yield an expression for the kernel function in terms of  $k_{2t}^{\sim}$  (see (3.1.2) below).

In all known cases (e.g.  $X$  a compact symmetric space or  $X = \mathbb{R}^n$ ) the image of the heat kernel transform has been a weighted Bergman space  $X_{\mathbb{C}}$  with regard to a positive weight function. It came to our surprise that the Heisenberg group deviates from this pattern. The main result of this paper asserts that

$$(1.1) \quad \text{im } \mathcal{H}_t = \mathcal{B}_t^+(\mathbb{H}_{\mathbb{C}}) \oplus \mathcal{B}_t^-(\mathbb{H}_{\mathbb{C}})$$

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is a direct sum of two weighted Bergman spaces on  $H_{\mathbb{C}}$ . Most interestingly, the weight functions  $W_t^{\pm}$  for  $\mathcal{B}_t^{\pm}(\mathbb{H}_{\mathbb{C}})$  have an oscillatory nature and attain positive and *negative* values. This fact forces the use of a certain exhaustion  $\mathbb{H}_{\mathbb{C}} = \bigcup_{R>0} K_R$  to define the inner product a suitable dense subspace  $\mathcal{V}_t^{\pm}(\mathbb{H}_{\mathbb{C}})$  of  $\mathcal{B}_t^{\pm}(\mathbb{H}_{\mathbb{C}})$  by

$$\langle f, g \rangle = \lim_{R \rightarrow \infty} \int_{K_R} f(z) \overline{g(z)} W_t^{\pm}(z) dz \quad (f, g \in \mathcal{V}_t^{\pm}(\mathbb{H}_{\mathbb{C}})),$$

quite reminiscent to the familiar notion of principal value.

Let us now describe the contents of this paper in more detail. In Section 2 we introduce our notation and recall some facts on the heat kernel  $k_t$  on  $\mathbb{H}$  and its analytic continuation to  $\mathbb{H}_{\mathbb{C}}$ . Subsequently in Section 3 we define the heat kernel transform and give a discussion of its general nature.

For the remainder it is useful to identify  $\mathbb{H}$  with  $\mathbb{R}^{2n} \times \mathbb{R}$ . In Section 4 we introduce for each spectral parameter  $\lambda \in \mathbb{R}^{\times}$  a partial heat kernel transform

$$H_t^{\lambda} : L^2(\mathbb{R}^{2n}) \rightarrow \mathcal{O}(\mathbb{C}^{2n})$$

and show that  $\text{im } H_t^{\lambda}$  is a weighted Bergman space  $\mathcal{B}_t^{\lambda}(\mathbb{C}^{2n})$  associated to an explicitly given positive weight function  $W_t^{\lambda} : \mathbb{C}^{2n} \rightarrow \mathbb{R}^{2n}$ . With these results we prove in Section 5 that there is a natural left  $\mathbb{H}$ -equivariant equivalence

$$(1.2) \quad L^2(\mathbb{H}) \simeq \int_{\mathbb{R}^{\times}}^{\oplus} \mathcal{B}_t^{\lambda}(\mathbb{C}^{2n}) e^{2t\lambda^2} d\lambda.$$

Moreover, within the identification (??) the heat kernel transform  $\mathcal{H}_t$  becomes the diagonal operator  $(H_t^{\lambda})_{\lambda}$ .

In Section 6 we combine all previously obtained results to establish our main result (??). It turns out that the global weight functions  $W_t^{\pm}$  admit an integral representation in terms of the partial weight functions  $W_t^{\lambda}$ . Finally, in the appendix we derive explicit expansions of  $W_t^{\pm}$  by Hermite polynomials and explain their oscillatory behavior.

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## 2. THE HEAT KERNEL ON THE HEISENBERG GROUP

**2.1. Notation.** Let  $\mathfrak{h}$  denote the  $(2n + 1)$ -dimensional Heisenberg algebra with generators, say,

$$X_1, \dots, X_n, U_1, \dots, U_n, Z$$

and relations  $[X_j, U_j] = Z$ . In the sequel we will often identify  $\mathfrak{h}$  with  $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ . For that let  $(\mathbf{x}, \mathbf{u}, \xi)$  with  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{u} = (u_1, \dots, u_n)$  denote the canonical coordinates on  $\mathbb{R}^{2n+1}$ . Then the map

$$\mathbb{R}^{2n+1} \rightarrow \mathfrak{h}, \quad (\mathbf{x}, \mathbf{u}, \xi) \mapsto \sum_{j=1}^n x_j X_j + \sum_{j=1}^n u_j U_j + \xi Z$$

is a linear isomorphism providing us with suitable coordinates for  $\mathfrak{h}$ .

Let  $\mathbb{H}$  denote a simply connected Lie group with Lie algebra  $\mathfrak{h}$ , the *Heisenberg group*. We will identify  $\mathbb{H}$  with  $\mathfrak{h}$  through the exponential function  $\exp = \text{id} : \mathfrak{h} \rightarrow \mathbb{H}$ . As  $\mathbb{H}$  is two step, the Baker-Campbell-Hausdorff formula provides the group law

$$(\mathbf{x}, \mathbf{u}, \xi)(\mathbf{x}', \mathbf{u}', \xi') = (\mathbf{x} + \mathbf{x}', \mathbf{u} + \mathbf{u}', \frac{1}{2}(\mathbf{x} \cdot \mathbf{u}' - \mathbf{u} \cdot \mathbf{x}') + \xi + \xi').$$

Here  $\mathbf{x} \cdot \mathbf{u} = \sum_{j=1}^n x_j u_j$ , as usual, denotes the standard pairing on  $\mathbb{R}^n$ . We notice in particular that

$$(2.1.1) \quad (\mathbf{x}, \mathbf{u}, \xi)^{-1} = (-\mathbf{x}, -\mathbf{u}, -\xi).$$

Write  $dh$  for a Haar measure on  $\mathbb{H}$ . We can and will normalize  $dh$  in such a way that it coincides with the product of Lebesgue measures, i.e.

$$\int_{\mathbb{H}} f(h) dh = \int_{\mathbb{R}^{2n+1}} f(\mathbf{x}, \mathbf{u}, \xi) d\mathbf{x} d\mathbf{u} d\xi$$

for all  $f \in C_c(\mathbb{H})$ .

Write  $\mathbb{H}_{\mathbb{C}}$  for the universal complexification of  $\mathbb{H}$ . Of course we can identify  $\mathbb{H}_{\mathbb{C}}$  with  $\mathbb{C}^{2n+1}$  and we will often do so. We will write  $(\mathbf{z}, \mathbf{w}, \zeta)$  for the coordinates on  $\mathbb{C}^{2n+1}$  where  $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ ,  $\mathbf{w} = \mathbf{u} + i\mathbf{v}$  and  $\zeta = \xi + i\eta$ .

For any simply connected nilpotent Lie group  $H$  the polar mapping

$$H \times \mathfrak{h} \rightarrow H_{\mathbb{C}}, \quad (h, X) \mapsto h \exp(iX)$$

is a homeomorphism. Furthermore the Haar measure on  $H_{\mathbb{C}}$  decomposes as

$$(2.1.2) \quad \int_{H_{\mathbb{C}}} f(g) dg = \int_H \int_{\mathfrak{h}} f(h \exp(iX)) dX dh$$

for all  $f \in C_c(H_{\mathbb{C}})$ .

For the Heisenberg group  $\mathbb{H}$ , the polar mapping is explicitly given by

$$((\mathbf{x}, \mathbf{u}, \xi), (\mathbf{x}', \mathbf{u}', \xi')) \mapsto (\mathbf{x} + i\mathbf{x}', \mathbf{u} + i\mathbf{u}', \frac{i}{2}(\mathbf{x} \cdot \mathbf{u}' - \mathbf{u} \cdot \mathbf{x}') + \xi + i\xi')$$

where  $h = (\mathbf{x}, \mathbf{u}, \xi)$  and  $X = (\mathbf{x}', \mathbf{u}', \xi')$ . In particular the Haar measure on  $\mathbb{H}_{\mathbb{C}}$  can be chosen as the product of Lebesgue measures  $d\mathbf{x} d\mathbf{y} d\mathbf{u} d\mathbf{v} d\xi d\eta$ .

For integrable functions  $f, g$  on  $\mathbb{H}$  we define their convolution by

$$(f * g)(x) = \int_{\mathbb{H}} f(h)g(h^{-1}x) dh \quad (x \in \mathbb{H}).$$

In coordinates this is explicitly given by

$$(f * g)(\mathbf{x}, \mathbf{u}, \xi) = \int_{\mathbb{R}^{2n+1}} f(\mathbf{x}', \mathbf{u}', \xi')g((- \mathbf{x}', -\mathbf{u}', -\xi')(\mathbf{x}, \mathbf{u}, \xi)) d\mathbf{x}' d\mathbf{u}' d\xi'.$$

**2.2. The heat kernel.** Write  $\mathcal{U}(\mathfrak{h})$  for the universal enveloping algebra of  $\mathfrak{h}$  and define the Laplace element in  $\mathcal{U}(\mathfrak{h})$  by

$$\mathcal{L} = \sum_{j=1}^n X_j^2 + \sum_{j=1}^n U_j^2 + Z^2.$$

For  $X \in \mathfrak{h}$  we write  $\tilde{X}$  for the left invariant vector field on  $\mathbb{H}$ , i.e.,

$$(\tilde{X}f)(h) = \left. \frac{d}{dt} \right|_{t=0} f(h \exp(tX))$$

for  $f$  a function on  $\mathbb{H}$  which is differentiable at  $h \in \mathbb{H}$ . Write  $\rho$  for the right regular representation of  $\mathbb{H}$  on  $L^2(\mathbb{H})$ , i.e.

$$(\rho(h)f)(x) = f(xh)$$

for  $h, x \in \mathbb{H}$  and  $f \in L^2(\mathbb{H})$ . With  $d\rho$  the derived representation we then have  $d\rho(X) = \tilde{X}$  for all  $X \in \mathfrak{h}$ . In particular if

$$\Delta = \sum_{j=1}^n \tilde{X}_j^2 + \sum_{j=1}^n \tilde{U}_j^2 + \tilde{Z}^2$$

denotes the Laplace operator on  $\mathbb{H}$ , then  $d\rho(\mathcal{L}) = \Delta$ .

Set  $\mathbb{R}^+ = (0, \infty)$ . Our concern will be with the heat equation on  $\mathbb{H} \times \mathbb{R}^+$

$$\partial_t u(h, t) = \Delta u(h, t)$$

for appropriate functions  $u(h, t)$  on  $\mathbb{H} \times \mathbb{R}^+$ . The fundamental solution is given by the heat kernel  $k_t(h)$  which can be computed as follows:

$$(2.2.1) \quad k_t(\mathbf{x}, \mathbf{u}, \xi) = c_n \int_{\mathbb{R}} e^{-i\lambda\xi} e^{-t\lambda^2} \left( \frac{\lambda}{\sinh \lambda t} \right)^n e^{-\frac{1}{4}\lambda(\coth t\lambda)(\mathbf{x}\cdot\mathbf{x} + \mathbf{u}\cdot\mathbf{u})} d\lambda$$

with  $c_n = (4\pi)^{-n}$  (this follows from a slight modification of [?, Theorem 2.8.1].) It satisfies the usual property of  $k_t * k_t = k_{2t}$  (see, for example, [?, (2.87) and Corollary 2.3.4]).

If  $f$  is an analytic function on  $\mathbb{H}$  which holomorphically extends to  $\mathbb{H}_{\mathbb{C}}$ , then we write  $f^{\sim}$  for this holomorphic extension. The explicit formula (2.2.1) now implies that  $k_t$  has a holomorphic continuation to  $\mathbb{H}_{\mathbb{C}}$  which is given by

$$(2.2.2) \quad k_t^{\sim}(\mathbf{z}, \mathbf{w}, \zeta) = c_n \int_{\mathbb{R}} e^{-i\lambda\zeta} e^{-t\lambda^2} \left( \frac{\lambda}{\sinh \lambda t} \right)^n e^{-\frac{1}{4}\lambda(\coth t\lambda)(\mathbf{z}\cdot\mathbf{z} + \mathbf{w}\cdot\mathbf{w})} d\lambda$$

for  $(\mathbf{z}, \mathbf{w}, \zeta) \in \mathbb{C}^{2n+1} = \mathbb{H}_{\mathbb{C}}$ . It follows from (2.1.1) and (2.2.2) that

$$(2.2.3) \quad k_t^{\sim}(z) = k_t^{\sim}(z^{-1}) \quad (z \in \mathbb{H}_{\mathbb{C}}).$$

Furthermore, as  $k_t \geq 0$  is real, we record

$$(2.2.4) \quad \overline{k_t^{\sim}(z)} = k_t^{\sim}(\bar{z}) \quad (z \in \mathbb{H}_{\mathbb{C}})$$

Here, as usual,  $z \mapsto \bar{z}$  denotes the complex conjugation of  $\mathbb{H}_{\mathbb{C}}$  with respect to the real form  $\mathbb{H}$ .

### 3. THE HEAT KERNEL TRANSFORM

**3.1. Definition and basic properties.** Let  $C \subseteq \mathbb{H}_{\mathbb{C}}$  be a compact subset. Then it follows from (2.2.2) that

$$(3.1.1) \quad \sup_{z \in C} \int_{\mathbb{H}} |k_t^{\sim}(h^{-1}z)|^2 dh < \infty.$$

Fix  $t > 0$ . Then (3.1.1) implies that  $f * k_t$  has an analytic continuation to  $\mathbb{H}_{\mathbb{C}}$  for all  $f \in L^2(\mathbb{H})$ . In particular we obtain a linear map

$$\mathcal{H}_t : L^2(\mathbb{H}) \rightarrow \mathcal{O}(\mathbb{H}_{\mathbb{C}}), \quad f \mapsto (f * k_t)^{\sim}; \quad \mathcal{H}_t(f)(z) = \int_{\mathbb{H}_{\mathbb{C}}} f(h) k_t^{\sim}(h^{-1}z) dh.$$

We will call  $\mathcal{H}_t$  the *heat kernel transform*.

In the sequel we wish to consider  $\mathcal{O}(\mathbb{H}_{\mathbb{C}})$  as a Fréchet space – the topology being the one of compact convergence. If  $h \in H$  and  $f$  is a function on  $\mathbb{H}$  or  $\mathbb{H}_{\mathbb{C}}$ , then we write  $\tau(h)f = f(h^{-1}\cdot)$ . The following properties of  $\mathcal{H}_t$  are immediate:

- $\mathcal{H}_t$  is continuous (because of (3.1.1))
- $\mathcal{H}_t$  is injective (note that  $\mathcal{H}_t(f) = e^{t\Delta}f$  and  $\Delta$  is a negative definite operator).
- $\mathcal{H}_t$  is  $\mathbb{H}$ -equivariant, i.e.  $\mathcal{H}_t \circ \tau(h) = \tau(h) \circ \mathcal{H}_t$  for all  $h \in \mathbb{H}$  (this is a general fact for the convolution on a locally compact group).

We will endow  $\text{im } \mathcal{H}_t$  with the Hilbert topology induced from  $L^2(\mathbb{H})$ . As  $\mathcal{H}_t$  is continuous we see that  $\text{im } \mathcal{H}_t$  is an  $\mathbb{H}$ -invariant Hilbert space of holomorphic functions on  $\mathbb{H}_{\mathbb{C}}$ . As such  $\text{im } \mathcal{H}_t$  has continuous point evaluations, i.e. for all  $z \in \mathbb{H}_{\mathbb{C}}$  the map

$$\text{im } \mathcal{H}_t \rightarrow \mathbb{C}, \quad f \mapsto f(z)$$

is continuous. Hence  $f(z) = \langle f, \mathcal{K}_z^t \rangle$  for a unique element  $\mathcal{K}_z^t \in \text{im } \mathcal{H}_t$ . We then obtain a positive definite kernel function

$$\mathcal{K}^t : \mathbb{H}_{\mathbb{C}} \times \mathbb{H}_{\mathbb{C}} \rightarrow \mathbb{C}; \quad \mathcal{K}^t(z, w) = \langle \mathcal{K}_w^t, \mathcal{K}_z^t \rangle = \mathcal{K}_w^t(z)$$

which is holomorphic in the first and anti-holomorphic in the second variable. Moreover, the  $\mathbb{H}$ -invariance of  $\text{im } \mathcal{H}_t$  translates into  $\mathcal{K}^t(hz, hw) = \mathcal{K}^t(z, w)$  for all  $h \in \mathbb{H}$  and  $z, w \in \mathbb{H}_{\mathbb{C}}$ .

Let us compute  $\mathcal{K}^t$ . Fix  $w \in \mathbb{H}_{\mathbb{C}}$ . Let  $g \in \text{im } \mathcal{H}_t$ . Then  $g = \mathcal{H}_t(f)$  for some  $f \in L^2(\mathbb{H})$  and

$$\langle g, \mathcal{K}_w^t \rangle = g(w) = \mathcal{H}_t(f)(w) = (f * k_t)^\sim(w) = \int_{\mathbb{H}} f(h) k_t^\sim(h^{-1}w) dh.$$

As this holds for all  $g \in \text{im } \mathcal{H}_t$ , we thus conclude that

$$\mathcal{H}_t^{-1}(\mathcal{K}_w^t)(h) = \overline{k_t^\sim(h^{-1}w)} = k_t^\sim(\overline{w^{-1}h}) \quad (h \in \mathbb{H})$$

where for the last equality we used the facts (2.2.3-4). From this we now get for all  $w, z \in \mathbb{H}_{\mathbb{C}}$  that

$$\begin{aligned} \mathcal{K}_w^t(z) &= \mathcal{H}_t(k_t^\sim(\overline{w^{-1}\cdot}))(z) = \int_{\mathbb{H}} k_t^\sim(\overline{w^{-1}h}) k_t^\sim(h^{-1}z) dh \\ &= \int_{\mathbb{H}} k_t(h) k_t^\sim(h^{-1}\overline{w^{-1}z}) dh \\ &= (k_t * k_t)^\sim(\overline{w^{-1}z}) \\ &= k_{2t}^\sim(\overline{w^{-1}z}). \end{aligned}$$

We have thus shown that the kernel function is given by

$$(3.1.2) \quad \mathcal{K}^t(z, w) = k_{2t}^\sim(\overline{w^{-1}z}) \quad (z, w \in \mathbb{H}_{\mathbb{C}}).$$

**3.2. General remarks on integral transforms and Bergman spaces.** The setup for this Section is as follows: We let  $N$  be a positive integer and  $G$  be a Lie group which acts on  $\mathbb{R}^N$  in a measure preserving manner. We assume that the action of  $G$  extends to an action on  $\mathbb{C}^N$  by measure preserving biholomorphisms. Our next data is a continuous (integral) transform

$$\Phi : L^2(\mathbb{R}^N) \hookrightarrow \mathcal{O}(\mathbb{C}^N)$$

which we assume to be  $G$ -equivariant. In this way  $\text{im } \Phi$  becomes a  $G$ -invariant Hilbert space of holomorphic functions on  $\mathbb{C}^N$ . We write  $\mathcal{K} : \mathbb{C}^N \times \mathbb{C}^N \rightarrow \mathbb{C}$  for the corresponding kernel function.

**Example 3.1.** (a) The heat kernel transform  $\mathcal{H}_t : L^2(\mathbb{H}) \rightarrow \mathcal{O}(\mathbb{H}_{\mathbb{C}})$  meets the general assumptions from above. In fact, for  $N = 2n + 1$  we may identify  $\mathbb{H}$  with  $\mathbb{R}^N$  and  $\mathbb{H}_{\mathbb{C}}$  with  $\mathbb{C}^N$ . Furthermore the group  $G = \mathbb{H}$  acts from the left on  $\mathbb{H} = \mathbb{R}^N$  and  $\mathbb{H}_{\mathbb{C}} = \mathbb{C}^N$  in a measure preserving manner.

(b) The partial heat kernel transforms  $H_t^\lambda : L^2(\mathbb{R}^{2n}) \hookrightarrow \mathcal{O}(\mathbb{C}^{2n})$  introduced in Section 4 below satisfy the general assumptions made above.  $\square$

For the remainder we will assume that  $\text{im } \Phi = \mathcal{B}(\mathbb{C}^N, W)$  is a weighted Bergman space for some measurable weight function  $W : \mathbb{C}^N \rightarrow \mathbb{R}$ , i.e.,

$$\mathcal{B}(\mathbb{C}^N, W) = \left\{ f \in \mathcal{O}(\mathbb{C}^N) : \int_{\mathbb{C}^N} |f(z)|^2 |W(z)| dz < \infty \right\}$$

Hilbert structure given by

$$(3.2.1) \quad \langle f, g \rangle = \int_{\mathbb{C}^n} f(z) \overline{g(z)} W(z) dz$$

As the action of  $G$  on  $\mathcal{B}(\mathbb{C}^N, W)$  is unitary, the weight function  $W$  should be left  $G$ -invariant, i.e.

$$(3.2.2) \quad W(g.z) = W(z) \quad (g \in G, z \in \mathbb{C}^N).$$

What we cannot expect however is that  $W$  is non-negative. It might then be a surprise that (3.2.1) still defines a Hilbert structure. As the following example shows, this is a phenomenon which already appears in one variable.

**Example 3.2.** We consider the unit disk  $D = \{z \in \mathbb{C} : |z| < 1\}$ . For a measurable subset  $A \subseteq D$  write  $\mathbf{1}_A$  for its characteristic function. Define a weight function  $W$  on  $D$  by

$$W = \mathbf{1}_{\{\frac{1}{2} \leq |z| < 1\}} - \mathbf{1}_{\{|z| < \frac{1}{2}\}}.$$

With  $W$  we form the weighted Bergman space

$$\mathcal{B}^2(D, W) := \left\{ f \in \mathcal{O}(D) : \int_D |f(z)|^2 |W(z)| dx dy < \infty \right\}$$

and endow it with the sesquilinear bracket

$$\langle f, g \rangle = \int_D f(z) \overline{g(z)} W(z) dx dy.$$

We will show that  $(\mathcal{B}^2(D, W), \langle \cdot, \cdot \rangle)$  is a Hilbert space. For that we first observe that  $\{z^n\}_{n \in \mathbb{N}_0}$  is an orthogonal system in  $\mathcal{B}^2(D, W)$ . This is because  $W$  is rotationally invariant. Next we compute

$$\begin{aligned} \langle z^n, z^n \rangle &= 2\pi \int_{\frac{1}{2}}^1 r^{2n+1} dr - 2\pi \int_0^{\frac{1}{2}} r^{2n+1} dr \\ &= \frac{\pi}{n+1} \left[ 1 - \left( \frac{1}{2} \right)^{2n+1} \right] > 0 \end{aligned}$$

for any  $n \in \mathbb{N}_0$ . Thus if  $f = \sum_n a_n z^n \in \mathcal{B}^2(D, W)$  is an arbitrary element, then

$$(3.2.1) \quad \langle f, f \rangle = \sum_n |a_n|^2 \frac{\pi}{n+1} \left[ 1 - \left( \frac{1}{2} \right)^{2n+1} \right] \geq 0$$

and  $\langle f, f \rangle = 0$  if and only if  $f = 0$ . This shows that  $\langle \cdot, \cdot \rangle$  defines a pre Hilbert structure on  $\mathcal{B}^2(D, W)$ . Next notice that

$$(3.2.2) \quad \int_D |f(z)|^2 |W(z)| dx dy = \sum_n |a_n|^2 \frac{\pi}{n+1}.$$

It follows from identities (??) and (??) that  $\langle \cdot, \cdot \rangle$  and the Hilbert bracket  $(f|g) = \int_D f(z) \overline{g(z)} |W(z)| dx dy$  induce equivalent norms. Hence  $(\mathcal{B}^2(D, W), \langle \cdot, \cdot \rangle)$  is a Hilbert space.

Finally we note that  $W$  is uniquely characterized by the Hilbert norm on  $\mathcal{B}^2(D, W)$ , i.e.  $\mathcal{B}^2(D, W) = \mathcal{B}^2(D, W')$  if and only if  $W = W'$  almost everywhere (use Stone-Weierstraß).  $\square$

We conclude this section with some general remarks on how to obtain the weight function  $W$ . Define a subspace of  $\text{im } \Phi$  by

$$(\text{im } \Phi)_0 = \text{span}\{\mathcal{K}_x : x \in \mathbb{R}^N\}.$$

Since a holomorphic function on  $\mathbb{C}^N$  which vanishes on  $\mathbb{R}^N$  is identically zero, we conclude that  $(\text{im } \Phi)_0$  is dense in  $\text{im } \Phi$ . Hence  $\text{im } \Phi = \mathcal{B}(\mathbb{C}^N, W)$  will hold precisely if

$$(3.2.3) \quad \mathcal{K}(x, x') = \langle \mathcal{K}_{x'}, \mathcal{K}_x \rangle = \int_{\mathbb{C}^N} \mathcal{K}_{x'}(z) \overline{\mathcal{K}_x(z)} W(z) dz$$

for all  $x, x' \in \mathbb{R}^N$ . The formula (3.2.3) is actually quite helpful and will be applied in Section 4 below.

#### 4. THE $\lambda$ -TWISTED HEAT-KERNEL TRANSFORM

For  $\lambda \in \mathbb{R}$ ,  $\lambda \neq 0$ , we will introduce a  $\lambda$ -twisted heat kernel transform  $H_t^\lambda : L^2(\mathbb{R}^{2n}) \rightarrow \mathcal{O}(\mathbb{C}^{2n})$ . We will show that the image of  $H_t^\lambda$  is a weighted Bergman space  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  on  $\mathbb{C}^{2n}$ . Further we provide an inversion formula for  $H_t^\lambda$ .

The results of this section are the building blocks for our general discussion of the heat kernel transform  $\mathcal{H}_t : L^2(\mathbb{H}) \rightarrow \mathcal{O}(\mathbb{H}_\mathbb{C})$  in the following sections.

**4.1. Notation.** Let  $\lambda \in \mathbb{R}$ ,  $\lambda \neq 0$ . For suitable functions  $F$  on  $\mathbb{H}$  we define a function  $F^\lambda$  on  $\mathbb{R}^{2n}$  by

$$F^\lambda(\mathbf{x}, \mathbf{u}) = \int_{\mathbb{R}} e^{i\lambda\xi} F(\mathbf{x}, \mathbf{u}, \xi) d\xi.$$

For  $f, g \in L^1(\mathbb{R}^{2n})$  the  $\lambda$ -twisted convolution is defined by

$$(f *_\lambda g)(\mathbf{x}, \mathbf{u}) = \int_{\mathbb{R}^{2n}} f(\mathbf{x}', \mathbf{u}') g(\mathbf{x} - \mathbf{x}', \mathbf{u} - \mathbf{u}') e^{-i\frac{\lambda}{2}(\mathbf{x}' \cdot \mathbf{u} - \mathbf{x} \cdot \mathbf{u}')} d\mathbf{x}' d\mathbf{u}'.$$

Notice that we have for Schwartz functions  $F, G \in \mathcal{S}(\mathbb{H}) = \mathcal{S}(\mathbb{R}^{2n+1})$  that

$$(4.1.1) \quad (F * G)^\lambda = F^\lambda *_\lambda G^\lambda.$$

Let  $\Delta_{\text{sub}} = d\rho \left( \sum_{j=1}^n (\tilde{X}_j^2 + \tilde{Y}_j^2) \right)$  denote the sublaplacian on  $\mathbb{H}$ . The heat kernel for  $\Delta_{\text{sub}}$  is denoted by  $p_t$  and its inverse Fourier transform in the central variable is explicitly given by

$$(4.1.2) \quad p_t^\lambda(\mathbf{x}, \mathbf{u}) = c_n \left( \frac{\lambda}{\sinh t\lambda} \right)^n e^{-\frac{\lambda}{4} \coth(\lambda t)(|\mathbf{x}|^2 + |\mathbf{u}|^2)}$$

with  $c_n = (4\pi)^{-n}$ .

For all  $f \in L^2(\mathbb{R}^{2n})$  the twisted convolution  $f *_\lambda p_t^\lambda$  has an analytic continuation to  $\mathbb{C}^{2n}$ . In particular, there is a  $\lambda$ -twisted heat kernel transform

$$H_t^\lambda : L^2(\mathbb{R}^{2n}) \rightarrow \mathcal{O}(\mathbb{C}^{2n}), \quad f \mapsto (f *_\lambda p_t^\lambda)^\sim.$$

In coordinates we have

$$H_t^\lambda(f)(\mathbf{z}, \mathbf{w}) = \int_{\mathbb{R}^{2n}} f(\mathbf{x}', \mathbf{u}') p_t^\lambda(\mathbf{z} - \mathbf{x}', \mathbf{w} - \mathbf{u}') e^{-\frac{i}{2}\lambda(\mathbf{x}' \cdot \mathbf{w} - \mathbf{u}' \cdot \mathbf{z})} d\mathbf{x}' d\mathbf{u}'.$$

We define a unitary representation  $\tau^\lambda$  of  $\mathbb{R}^{2n}$  on  $L^2(\mathbb{R}^{2n})$  by

$$(\tau^\lambda(\mathbf{a}, \mathbf{b})f)(\mathbf{x}, \mathbf{u}) = e^{-\frac{i\lambda}{2}(\mathbf{a} \cdot \mathbf{u} - \mathbf{b} \cdot \mathbf{x})} f(\mathbf{x} - \mathbf{a}, \mathbf{u} - \mathbf{b})$$

for  $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$ ,  $f \in L^2(\mathbb{R}^{2n})$  and  $(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^{2n}$ . Likewise  $\tau^\lambda$  defines an action of  $\mathbb{R}^{2n}$  on  $\mathcal{O}(\mathbb{C}^{2n})$  via

$$(\tau^\lambda(\mathbf{a}, \mathbf{b})f)(\mathbf{z}, \mathbf{w}) = e^{-\frac{i\lambda}{2}(\mathbf{a} \cdot \mathbf{w} - \mathbf{b} \cdot \mathbf{z})} f(\mathbf{z} - \mathbf{a}, \mathbf{w} - \mathbf{b})$$

where  $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$ ,  $f \in \mathcal{O}(\mathbb{C}^{2n})$  and  $(\mathbf{z}, \mathbf{w}) \in \mathbb{C}^{2n}$ .

As for functions  $F, G \in L^1(\mathbb{H})$  we have  $\tau(h)F * G = \tau(h)(F * G)$  for all  $h \in \mathbb{H}$ , it is immediate from (??) that  $H_t^\lambda$  becomes  $\mathbb{R}^{2n}$ -equivariant, i.e.

$$(4.1.3) \quad H_t^\lambda(\tau^\lambda(\mathbf{a}, \mathbf{b})f) = \tau^\lambda(\mathbf{a}, \mathbf{b})(H_t^\lambda(f))$$

for all  $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$  and  $f \in L^2(\mathbb{R}^{2n})$ .

*Remark 4.1.* For the proofs in the sequel it is notationally convenient to prove the assertions for the ‘‘essential case’’  $\lambda = 1$  only. Whenever we do so we will use a simplified notation: we write  $f \times g$  instead of  $f *_{\mathbf{1}} g$  for the 1-twisted convolution; further we will drop all sub- and superscripts involving  $\lambda = 1$ , i.e.  $p_t^1$  becomes  $p_t$ ,  $H_t^1$  becomes  $H_t$  etc.

**4.2. Determination of the weight function.** Our objective is to find a non-negative weight function  $W_t^\lambda$  on  $\mathbb{C}^{2n}$  such that

$$(4.2.1) \quad \int_{\mathbb{C}^{2n}} |H_t^\lambda(f)(\mathbf{z}, \mathbf{w})|^2 W_t^\lambda(\mathbf{z}, \mathbf{w}) d\mathbf{z} d\mathbf{w} = \int_{\mathbb{R}^{2n}} |f(\mathbf{x}, \mathbf{u})|^2 d\mathbf{x} d\mathbf{u}$$

for all  $f \in L^2(\mathbb{R}^{2n})$ .

**Proposition 4.1.** *A weight function  $W_t^\lambda$  which satisfies (??) is given by*

$$(4.2.2) \quad W_t^\lambda(\mathbf{x} + i\mathbf{y}, \mathbf{u} + i\mathbf{v}) = 4^n e^{\lambda(\mathbf{u} \cdot \mathbf{y} - \mathbf{v} \cdot \mathbf{x})} p_{2t}^\lambda(2\mathbf{y}, 2\mathbf{v}).$$

*Remark 4.2.* The weight function  $W_t^\lambda$  is unique in the sense that is the unique measurable function  $W_t^\lambda : \mathbb{C}^{2n} \rightarrow \mathbb{R}_{\geq 0}$  which satisfies (??). This will be shown in Lemma ?? below.

*Proof.* We restrict our attention to the case  $\lambda = 1$ . As mentioned earlier we will write now  $p_t$  and  $W_t$  in place of  $p_t^\lambda$  and  $W_t^\lambda$ , respectively, and write  $f \times g$  for the 1-twisted convolution of  $f$  and  $g$ . Via  $H_t$  we can transfer the Hilbert space structure of  $L^2(\mathbb{R}^{2n})$  to  $\text{im } H_t$  and make it into Hilbert space of holomorphic functions. Write  $K^t(\mathbf{z}, \mathbf{w}; \mathbf{z}', \mathbf{w}')$  for the corresponding reproducing kernel. Arguing as in Subsection 3.2, the inner product  $\langle \cdot, \cdot \rangle_t$  on the image is uniquely determined by the equality

$$(4.2.3) \quad K^t(\mathbf{a}, \mathbf{b}; \mathbf{a}', \mathbf{b}') = \langle K_{(\mathbf{a}, \mathbf{b})}^t, K_{(\mathbf{a}', \mathbf{b}')}^t \rangle_t$$

for all real pairs  $(\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}') \in \mathbb{R}^n \times \mathbb{R}^n$ .

As the heat kernel transform  $f \rightarrow H_t(f) = (f \times p_t)^\sim$  commutes with the twisted translation (see equation (??)), we may assume  $(\mathbf{a}', \mathbf{b}') = 0$  in (??). As  $p_t \times p_t = p_{2t}$ , arguing as in Subsection 3.1 readily yields

$$K_{(\mathbf{a}, \mathbf{b})}^t(\mathbf{z}, \mathbf{w}) = p_{2t}(\mathbf{z} - \mathbf{a}, \mathbf{w} - \mathbf{b}) e^{-\frac{i}{2}(\mathbf{a} \cdot \mathbf{w} - \mathbf{b} \cdot \mathbf{z})}.$$

In particular,  $K_{(0,0)}^t = p_{2t}$  and  $K^t(\mathbf{a}, \mathbf{b}, 0, 0) = p_{2t}(\mathbf{a}, \mathbf{b})$ . Thus (??) translates into

$$p_{2t}(\mathbf{a}, \mathbf{b}) = \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} p_{2t}(\mathbf{z} - \mathbf{a}, \mathbf{w} - \mathbf{b}) e^{-\frac{i}{2}(\mathbf{a} \cdot \mathbf{w} - \mathbf{b} \cdot \mathbf{z})} \overline{p_{2t}(\mathbf{z}, \mathbf{w})} W_t(\mathbf{z}, \mathbf{w}) \, d\mathbf{z} \, d\mathbf{w}.$$

This is established in Lemma ?? below.  $\square$

**Lemma 4.2.** *For  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$  we have*

$$\int_{\mathbb{C}^n} \int_{\mathbb{C}^n} p_{2t}^\lambda(\mathbf{z} + \mathbf{a}, \mathbf{w} + \mathbf{b}) e^{\frac{i\lambda}{2}(\mathbf{a} \cdot \mathbf{w} - \mathbf{b} \cdot \mathbf{z})} \overline{p_{2t}^\lambda(\mathbf{z}, \mathbf{w})} W_t^\lambda(\mathbf{z}, \mathbf{w}) \, d\mathbf{z} \, d\mathbf{w} = p_{2t}^\lambda(\mathbf{a}, \mathbf{b}).$$

*Proof.* We will prove the assertion for  $\lambda = 1$ . Further, by the product nature of the functions involved, we may assume in addition that  $n = 1$ .

Expanding out and simplifying we have

$$\begin{aligned} p_{2t}(x + a + iy, u + b + iv) \overline{p_{2t}(x + iy, u + iv)} &= (4\pi)^{-2} (\sinh 2t)^{-2} e^{-\frac{1}{2}(\coth 2t)(x^2 + u^2)} \\ &\cdot e^{-\frac{1}{4}(\coth 2t)(a^2 + b^2)} e^{\frac{1}{2}(\coth 2t)(y^2 + v^2)} e^{-\frac{1}{2}(\coth 2t)(a(x + iy) + b(u + iv))}. \end{aligned}$$

We can combine the terms  $e^{-\frac{1}{2}(\coth 2t)(x^2 + u^2)}$  and

$$e^{(uy - vx)} = e^{(\coth 2t)(uy \tanh(2t) - xv \tanh(2t))}$$

to get

$$\begin{aligned} p_{2t}(x + a + iy, u + b + iv) \overline{p_{2t}(x + iy, u + iv)} e^{(uy - vx)} &= (4\pi)^{-2} (\sinh 2t)^{-2} e^{-\frac{1}{4}(\coth 2t)(a^2 + b^2)} e^{\frac{1}{2}(\coth 2t + \tanh 2t)(y^2 + v^2)} \\ &\cdot e^{-\frac{1}{2}(\coth 2t)((x + v \tanh(2t))^2 + (u - y \tanh(2t))^2)} e^{-\frac{1}{2}(\coth 2t)(a(x + iy) + b(u + iv))}. \end{aligned}$$

Using the identity  $\tanh 2t + \coth 2t = 2 \coth 4t$  and simplifying further we get

$$\begin{aligned} p_{2t}(z + a, w + b) e^{\frac{i}{2}(aw - bz)} \overline{p_{2t}(z, w)} W_t(z, w) &= 4^{-2} \pi^{-3} (\sinh 2t)^{-3} e^{-\frac{1}{8} \coth 2t (a^2 + b^2)} e^{(\coth 4t - \coth 2t)(y^2 + v^2)} \\ &\cdot e^{-\frac{1}{2}(\coth 2t)((x + \frac{a}{2} + \tanh(2t)v)^2 + (u + \frac{b}{2} - y \tanh(2t))^2)} e^{-\frac{i}{2}(\coth 2t)(ay + bv)} e^{\frac{i}{2}(au - bx)}, \end{aligned}$$

where  $z = x + iy$  and  $w = u + iv$ .

First consider the integral

$$\begin{aligned} & \int_{\mathbb{R}^2} e^{\frac{i}{2}(au-bx)} e^{-\frac{1}{2}(\coth 2t)\left((x+\frac{a}{2}+v \tanh(2t))^2+(u+\frac{b}{2}-y \tanh(2t))^2\right)} dx du \\ &= e^{\frac{i}{2}(\tanh 2t)(ay+bv)} \int_{\mathbb{R}^2} e^{\frac{i}{2}(au-bx)} e^{-\frac{1}{2}(\coth 2t)(x^2+u^2)} dx du \\ &= 2\pi(\tanh 2t)e^{\frac{i}{2}(\tanh 2t)(ay+bv)} e^{-\frac{1}{8}(\tanh 2t)(a^2+b^2)}. \end{aligned}$$

Up to an explicit factor the remaining integral is

$$\int_{\mathbb{R}^2} e^{-\frac{i}{2}(\coth 2t - \tanh 2t)(ay+bv)} e^{-(\coth 2t - \coth 4t)(y^2+v^2)} dy dv.$$

As  $\coth 2t - \tanh 2t = 2(\sinh 4t)^{-1}$  and  $\coth 2t - \coth 4t = (\sinh 4t)^{-1}$  the above integral reduces to

$$\int_{\mathbb{R}^2} e^{-i(\sinh 4t)^{-1}(ay+bv)} e^{-(\sinh 4t)^{-1}(y^2+v^2)} dy dv = \pi(\sinh 4t)e^{-\frac{1}{4}(\sinh 4t)^{-1}(a^2+b^2)}.$$

Combining results yields

$$\begin{aligned} & \int_{\mathbb{C}^2} p_{2t}(z+a, w+b) e^{\frac{i}{2}(aw-bz)} \overline{p_{2t}(z, w)} W_t(z, w) dz dw \\ &= 8^{-1} \pi^{-1} (\sinh 2t)^{-3} (\tanh 2t) (\sinh 4t) e^{-\frac{1}{8}(\coth 2t + \tanh 2t)(a^2+b^2)} e^{-\frac{1}{4}(\sinh 4t)^{-1}(a^2+b^2)}. \end{aligned}$$

Finally using the identities  $\coth 2t + \tanh 2t = 2 \coth 4t$  and  $\coth 4t + (\sinh 4t)^{-1} = \coth 2t$  and simplifying we get

$$\begin{aligned} & \int_{\mathbb{C}^2} p_{2t}(z+a, w+b) e^{\frac{i}{2}(aw-bz)} \overline{p_{2t}(z, w)} W_t(z, w) dz dw \\ &= \frac{1}{4\pi} (\sinh 2t)^{-1} e^{-\frac{1}{4} \coth 2t(a^2+b^2)} = p_{2t}(a, b). \end{aligned}$$

This proves the lemma.  $\square$

**4.3. The twisted Bergman space and surjectivity of  $H_t^\lambda$ .** For each  $\lambda \in \mathbb{R}$ ,  $\lambda \neq 0$ , we define the  $\lambda$ -twisted Bergman space by

$$\mathcal{B}_t^\lambda(\mathbb{C}^{2n}) = \{f \in \mathcal{O}(\mathbb{C}^{2n}) : \|f\|_\lambda^2 = \int_{\mathbb{C}^n \times \mathbb{C}^n} |f(\mathbf{z}, \mathbf{w})|^2 W_t^\lambda(\mathbf{z}, \mathbf{w}) dz d\mathbf{w} < \infty\}.$$

Clearly  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  is a Hilbert space of holomorphic functions on  $\mathbb{C}^{2n}$ . It follows from Proposition ?? that  $H_t^\lambda : L^2(\mathbb{R}^{2n}) \rightarrow \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  is an isometric embedding.

Our goal for this subsection is to show that  $H_t^\lambda$  is onto. We begin with a description of a useful orthonormal basis for  $\text{im } H_t^\lambda$  in terms of the special Hermite functions  $\Phi_{\alpha, \beta}^\lambda(\mathbf{x}, \mathbf{u})$  (see [?, Section 2.3]). For each  $\alpha, \beta \in \mathbb{N}_0^n$ , let us consider

$$\tilde{\Phi}_{\alpha, \beta}^\lambda(\mathbf{z}, \mathbf{w}) = (2\pi)^{-n} e^{-(2|\beta|+n)|\lambda|t} \Phi_{\alpha, \beta}^\lambda(\mathbf{z}, \mathbf{w})$$

where  $\Phi_{\alpha, \beta}^\lambda(\mathbf{z}, \mathbf{w})$  is the extension of  $\Phi_{\alpha, \beta}^\lambda(\mathbf{x}, \mathbf{u})$  to  $\mathbb{C}^n \times \mathbb{C}^n$ . The functions  $\Phi_{\alpha, \beta}^\lambda(\mathbf{x}, \mathbf{u})$  satisfy the orthogonal relation

$$(\Phi_{\alpha, \beta}^\lambda * \lambda \Phi_{\mu, \nu}^\lambda)(\mathbf{x}, \mathbf{u}) = \delta_{\beta, \mu} \Phi_{\alpha, \nu}^\lambda(\mathbf{x}, \mathbf{u}).$$

**Lemma 4.3.** *The set  $\{\tilde{\Phi}_{\alpha, \beta}^\lambda : \alpha, \beta \in \mathbb{N}_0^n\}$  is an orthonormal basis for  $\text{im } H_t^\lambda$ .*

*Proof.* It is enough to prove it for  $\lambda = 1$  and we drop the superscript when  $\lambda = 1$ . As the heat kernel  $p_t(\mathbf{x}, \mathbf{u})$  is given by

$$p_t(\mathbf{x}, \mathbf{u}) = (2\pi)^{-n} \sum_{\mu} e^{-(2|\mu|+n)t} \Phi_{\mu, \mu}(\mathbf{x}, \mathbf{u}),$$

we obtain the relation

$$(\Phi_{\alpha, \beta} \times p_t)(\mathbf{x}, \mathbf{u}) = (2\pi)^{-n} e^{-(2|\beta|+n)t} \tilde{\Phi}_{\alpha, \beta}(\mathbf{x}, \mathbf{u}).$$

Thus  $H_t(\Phi_{\alpha, \beta})(\mathbf{z}, \mathbf{w}) = \tilde{\Phi}_{\alpha, \beta}(\mathbf{z}, \mathbf{w})$  and, therefore, using Proposition ?? we obtain

$$\begin{aligned} & \int_{\mathbb{C}^{2n}} \tilde{\Phi}_{\alpha, \beta}(\mathbf{z}, \mathbf{w}) \overline{\tilde{\Phi}_{\mu, \nu}(\mathbf{z}, \mathbf{w})} W_t(\mathbf{z}, \mathbf{w}) \, d\mathbf{z} \, d\mathbf{w} \\ &= \int_{\mathbb{C}^{2n}} H_t(\Phi_{\alpha, \beta})(\mathbf{z}, \mathbf{w}) \overline{H_t(\Phi_{\mu, \nu})(\mathbf{z}, \mathbf{w})} W_t(\mathbf{z}, \mathbf{w}) \, d\mathbf{z} \, d\mathbf{w} \\ &= \int_{\mathbb{R}^{2n}} \Phi_{\alpha, \beta}(\mathbf{x}, \mathbf{u}) \overline{\Phi_{\mu, \nu}(\mathbf{x}, \mathbf{u})} \, d\mathbf{x} \, d\mathbf{u}. \end{aligned}$$

Hence  $\{\tilde{\Phi}_{\alpha, \beta} : \alpha, \beta \in \mathbb{N}_0^n\}$  is an orthonormal system in  $\text{im } H_t$ .

To show that it is an orthonormal basis for  $\text{im } H_t$ , we only need to show that

$$\int_{\mathbb{C}^{2n}} H_t(f)(\mathbf{z}, \mathbf{w}) \overline{\tilde{\Phi}_{\alpha, \beta}(\mathbf{z}, \mathbf{w})} W_t(\mathbf{z}, \mathbf{w}) \, d\mathbf{z} \, d\mathbf{w} = 0$$

for all  $\alpha, \beta$  implies  $f \equiv 0$ . But the above simply means, by Proposition ??, that

$$\int_{\mathbb{R}^{2n}} f(\mathbf{x}, \mathbf{u}) \overline{\Phi_{\alpha, \beta}(\mathbf{x}, \mathbf{u})} \, d\mathbf{x} \, d\mathbf{u} = 0$$

for all  $\alpha, \beta$  and we know that  $\{\Phi_{\alpha, \beta} : \alpha, \beta \in \mathbb{N}_0^n\}$  is an orthonormal basis for  $L^2(\mathbb{R}^{2n})$ . Hence  $f \equiv 0$  and the proof is complete.  $\square$

We will show that  $\{\tilde{\Phi}_{\alpha, \beta}^\lambda : \alpha, \beta \in \mathbb{N}_0^n\}$  is also an orthonormal basis for  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ . Clearly this implies that  $H_t^\lambda : L^2(\mathbb{R}^{2n}) \rightarrow \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  is onto.

Note that  $\tilde{\Phi}_{\alpha, \beta}^\lambda \in \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  for any  $t > 0$  and  $\{\tilde{\Phi}_{\alpha, \beta}^\lambda : \alpha, \beta \in \mathbb{N}_0^n\}$  will be an orthonormal basis for any  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ .

As  $\mathbf{z} = \mathbf{x} + i\mathbf{y}$  and  $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ , we note that  $\mathbf{u} \cdot \mathbf{y} - \mathbf{v} \cdot \mathbf{x} = \Im(\mathbf{z} \cdot \overline{\mathbf{w}})$  is the symplectic form on  $\mathbb{R}^{2n}$ . Thus  $\Im(\sigma\mathbf{z} \cdot \overline{\sigma\mathbf{w}}) = \Im(\mathbf{z} \cdot \overline{\mathbf{w}})$  for  $\sigma \in U(n)$ .

We introduce the *twisted Fock space*  $\mathcal{F}_t^\lambda(\mathbb{C}^{2n})$  by

$$\begin{aligned} \mathcal{F}_t^\lambda(\mathbb{C}^{2n}) &= \{G \in \mathcal{O}(\mathbb{C}^{2n}) : \\ &\|G\|^2 = \int_{\mathbb{C}^n \times \mathbb{C}^n} |G(\mathbf{z}, \mathbf{w})|^2 e^{\lambda \Im(\mathbf{z} \cdot \overline{\mathbf{w}})} e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2 + |\mathbf{w}|^2)} \, d\mathbf{z} \, d\mathbf{w} < \infty\}. \end{aligned}$$

Clearly, the prescription

$$U(n) \times \mathcal{F}_t^\lambda(\mathbb{C}^{2n}) \rightarrow \mathcal{F}_t^\lambda(\mathbb{C}^{2n}), \quad (\sigma, G) \mapsto G^\sigma; \quad G^\sigma(\mathbf{z}, \mathbf{w}) = G(\sigma\mathbf{z}, \sigma\mathbf{w})$$

defines a unitary representation of  $U(n)$  on  $\mathcal{F}_t^\lambda(\mathbb{C}^{2n})$ .

The Hilbert spaces  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  and  $\mathcal{F}_t^\lambda(\mathbb{C}^{2n})$  are related through (4.3.1)

$$F(\mathbf{z}, \mathbf{w}) \in \mathcal{B}_t^\lambda(\mathbb{C}^{2n}) \quad \text{if and only if} \quad F(\mathbf{z}, \mathbf{w}) e^{\frac{\lambda}{4}(\coth 2t\lambda)(\mathbf{z} \cdot \mathbf{z} + \mathbf{w} \cdot \mathbf{w})} \in \mathcal{F}_t^\lambda(\mathbb{C}^{2n}).$$

Let  $T \simeq (\mathbb{S}^1)^n$  be the diagonal subgroup of  $U(n)$ . We write the elements of  $T$  as  $\sigma = (e^{i\varphi_1}, \dots, e^{i\varphi_n})$ . For each  $n$ -tuple of integers  $\mathbf{m} = (m_1, m_2, \dots, m_n)$  let  $\chi_{\mathbf{m}}(\sigma)$

be the character of  $T$  defined by  $\chi_{\mathbf{m}}(\sigma) = e^{i \sum_{j=1}^n m_j \varphi_j}$ . For each  $G \in \mathcal{F}_t^\lambda(\mathbb{C}^{2n})$  define

$$G_{\mathbf{m}}(\mathbf{z}, \mathbf{w}) = \int_T G(\sigma \mathbf{z}, \sigma \mathbf{w}) \overline{\chi_{\mathbf{m}}(\sigma)} d\sigma.$$

As  $G$  is holomorphic it is clear that  $G_{\mathbf{m}} = 0$  unless  $\mathbf{m}$  is a multi-index in  $\mathbb{N}_0^n$ . By the Fourier expansion

$$G(\sigma \mathbf{z}, \sigma \mathbf{w}) = \sum_{\mathbf{m} \in \mathbb{N}_0^n} G_{\mathbf{m}}(\mathbf{z}, \mathbf{w}) \chi_{\mathbf{m}}(\sigma)$$

and by the Plancherel theorem we have

$$(4.3.2) \quad \int_T |G(\sigma \mathbf{z}, \sigma \mathbf{w})|^2 d\sigma = \sum_{\mathbf{m} \in \mathbb{N}_0^n} |G_{\mathbf{m}}(\mathbf{z}, \mathbf{w})|^2.$$

Note that the functions  $G_{\mathbf{m}}$  satisfy the homogeneity condition

$$G_{\mathbf{m}}(\sigma \mathbf{z}, \sigma \mathbf{w}) = \chi_{\mathbf{m}}(\sigma) G_{\mathbf{m}}(\mathbf{z}, \mathbf{w}).$$

For any  $G \in \mathcal{F}_t^\lambda(\mathbb{C}^{2n})$  we observe that, as  $\Im(\mathbf{z} \cdot \overline{\mathbf{w}}) = \Im(\sigma \mathbf{z} \cdot \sigma \overline{\mathbf{w}})$ ,

$$\begin{aligned} & \int_{\mathbb{C}^{2n}} G(\mathbf{z}, \mathbf{w}) e^{\lambda \Im(\mathbf{z} \cdot \overline{\mathbf{w}})} e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2 + |\mathbf{w}|^2)} d\mathbf{z} d\mathbf{w} \\ &= \int_T \int_{\mathbb{C}^{2n}} G(\sigma \mathbf{z}, \sigma \mathbf{w}) e^{\lambda \Im(\mathbf{z} \cdot \overline{\mathbf{w}})} e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2 + |\mathbf{w}|^2)} d\mathbf{z} d\mathbf{w} d\sigma. \end{aligned}$$

In view of this and the homogeneity condition we arrive at the orthogonality relations

$$\int_{\mathbb{C}^{2n}} G_{\mathbf{m}}(\mathbf{z}, \mathbf{w}) \overline{G_{\mathbf{m}'}(\mathbf{z}, \mathbf{w})} e^{\lambda \Im(\mathbf{z} \cdot \overline{\mathbf{w}})} e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2 + |\mathbf{w}|^2)} d\mathbf{z} d\mathbf{w} = 0,$$

whenever  $\mathbf{m}$  and  $\mathbf{m}'$  are different. We also note that each  $G_{\mathbf{m}}$  has an expansion of the form

$$G_{\mathbf{m}}(\mathbf{z}, \mathbf{w}) = \sum_{\alpha + \beta = \mathbf{m}} c_{\alpha, \beta} \mathbf{z}^\alpha \mathbf{w}^\beta.$$

Hence each  $G_{\mathbf{m}}$  is a polynomial.

**Lemma 4.4.** *The linear span of  $P_{\alpha, \beta}(\mathbf{z}, \mathbf{w}) = \mathbf{z}^\alpha \mathbf{w}^\beta$ ,  $\alpha, \beta \in \mathbb{N}_0^n$ , is dense in  $\mathcal{F}_t^\lambda(\mathbb{C}^{2n})$ .*

*Proof.* If  $G \in \mathcal{F}_t^\lambda(\mathbb{C}^{2n})$  is orthogonal to all  $P_{\alpha, \beta}$  then

$$\int_{\mathbb{C}^{2n}} G(\mathbf{z}, \mathbf{w}) \overline{G_{\mathbf{m}}(\mathbf{z}, \mathbf{w})} e^{\lambda \Im(\mathbf{z} \cdot \overline{\mathbf{w}})} e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2 + |\mathbf{w}|^2)} d\mathbf{z} d\mathbf{w} = 0$$

for any  $\mathbf{m} \in \mathbb{N}_0^n$ . In view of the homogeneity property of  $G_{\mathbf{m}}$  this means that

$$\int_{\mathbb{C}^{2n}} |G_{\mathbf{m}}(\mathbf{z}, \mathbf{w})|^2 e^{\lambda \Im(\mathbf{z} \cdot \overline{\mathbf{w}})} e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2 + |\mathbf{w}|^2)} d\mathbf{z} d\mathbf{w} = 0.$$

Hence  $G_{\mathbf{m}}(\mathbf{z}, \mathbf{w}) = 0$  for every  $\mathbf{m}$  and so  $G = 0$  in view of (??).  $\square$

It follows from Lemma ?? and (??) that every  $F \in \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  has the orthonormal expansion

$$(4.3.3) \quad F(\mathbf{z}, \mathbf{w}) = \sum_{\mathbf{m}} \sum_{\alpha + \beta = \mathbf{m}} c_{\alpha, \beta} P_{\alpha, \beta}(\mathbf{z}, \mathbf{w}) e^{-\frac{\lambda}{4}(\coth 2t\lambda)(\mathbf{z} \cdot \mathbf{z} + \mathbf{w} \cdot \mathbf{w})}.$$

The functions

$$\Psi_{\mathbf{m}}(\mathbf{z}, \mathbf{w}) = \sum_{\alpha+\beta=\mathbf{m}} c_{\alpha,\beta} P_{\alpha,\beta}(\mathbf{z}, \mathbf{w}) e^{-\frac{\lambda}{4}(\coth 2t\lambda)(\mathbf{z}\cdot\mathbf{z}+\mathbf{w}\cdot\mathbf{w})}$$

are orthogonal in  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  but not orthogonal in any other  $\mathcal{B}_s^\lambda(\mathbb{C}^{2n})$  when  $s \neq t$ . Another crucial property of these functions is proved in the next lemma.

**Lemma 4.5.** *All the functions  $\Psi_{\alpha,\beta}^{\mathbf{m}}(\mathbf{z}, \mathbf{w}) = P_{\alpha,\beta}(\mathbf{z}, \mathbf{w}) e^{-\frac{\lambda}{4}(\coth 2t\lambda)(\mathbf{z}\cdot\mathbf{z}+\mathbf{w}\cdot\mathbf{w})}$  belong to the image  $\text{im } H_t^\lambda$  of the heat kernel transform.*

*Proof.* We may restrict ourselves to the case of  $\lambda = 1$ . It will suffice to show that for each pair  $\alpha, \beta \in \mathbb{N}_0^n$  there exists a function  $f_{\alpha,\beta} \in L^2(\mathbb{R}^{2n})$  such that

$$H_t(f_{\alpha,\beta})(\mathbf{z}, \mathbf{w}) = (f_{\alpha,\beta} \times p_t)^\sim(\mathbf{z}, \mathbf{w}) = \mathbf{z}^\alpha \mathbf{w}^\beta e^{-\frac{1}{4}(\coth 2t)(\mathbf{z}^2 + \mathbf{w}^2)}.$$

As both sides are holomorphic it is enough to prove this for  $\mathbf{z} = \mathbf{x}$  and  $\mathbf{w} = \mathbf{u}$  where  $\mathbf{x}, \mathbf{u} \in \mathbb{R}^n$ . Thus we need to solve the equation

$$(4.3.4) \quad (f_{\alpha,\beta} \times p_t)(\mathbf{x}, \mathbf{u}) = \mathbf{x}^\alpha \mathbf{u}^\beta p_{2t}(\mathbf{x}, \mathbf{u}).$$

In the sequel it will be convenient to identify  $\mathbb{R}^{2n}$  with  $\mathbb{C}^n$  via  $z = \mathbf{x} + i\mathbf{u}$ . Then  $\mathbf{x}^\alpha \mathbf{u}^\beta = 2^{-|\alpha|} (2i)^{-|\beta|} (z + \bar{z})^\alpha (z - \bar{z})^\beta$ . It is then sufficient to solve the equation

$$(f_{\alpha,\beta} \times p_t)(z) = z^\alpha \bar{z}^\beta p_{2t}(z)$$

where  $p_t(z) = p_t(\mathbf{x}, \mathbf{u})$ . We solve this equation using properties of the Weyl transform.

Recall that the Weyl transform  $\mathbb{W}(f)$  of a function  $f \in L^1(\mathbb{C}^n)$ , is defined to be the bounded operator on  $L^2(\mathbb{R}^n)$  given by

$$\mathbb{W}(f)\varphi(\xi) = \int_{\mathbb{C}^n} f(z)\pi(z)\varphi(\xi) dz \quad (\xi \in \mathbb{R}^n)$$

where  $\pi(z) = \pi_1(z, 0)$  and  $\pi_1$  is the Schrödinger representation of the Heisenberg group  $\mathbb{H}$  with parameter  $\lambda = 1$  (see [?, Section 2.2]). Then for  $f \in L^1 \cap L^2(\mathbb{C}^n)$ ,  $\mathbb{W}(f)$  is a Hilbert-Schmidt operator and  $\mathbb{W}$  extends to  $L^2(\mathbb{C}^n)$  as an isometry onto the space of Hilbert-Schmidt operators. Moreover  $\mathbb{W}(f \times g) = \mathbb{W}(f)\mathbb{W}(g)$  and  $\mathbb{W}(p_t) = e^{-tH}$ . Here  $H$  denotes the Hermite operator

$$H = (-\Delta + |\xi|^2) = \frac{1}{2} \sum_{j=1}^n (A_j A_j^* + A_j^* A_j),$$

in which  $A_j = -\frac{\partial}{\partial \xi_j} + \xi_j$  and  $A_j^* = \frac{\partial}{\partial \xi_j} + \xi_j$  are the creation and annihilation operators. The eigenfunctions of  $H$  are the Hermite functions  $\Phi_\alpha$ . They satisfy

$$A_j \Phi_\alpha = (2\alpha_j + 2)^{\frac{1}{2}} \Phi_{\alpha+e_j}, \quad A_j^* \Phi_\alpha = (2\alpha_j)^{\frac{1}{2}} \Phi_{\alpha-e_j}$$

where  $e_j$  are the coordinate vectors. Given a bounded linear operator  $T$  on  $L^2(\mathbb{R}^n)$ , define the derivations

$$\delta_j T = [A_j^*, T] = A_j^* T - T A_j^*, \quad \bar{\delta}_j T = [T, A_j] = T A_j - A_j T.$$

Then it can be shown that (see [?])

$$\mathbb{W}(z_j f) = \delta_j \mathbb{W}(f), \quad \text{and} \quad \mathbb{W}(\bar{z}_j f) = \bar{\delta}_j \mathbb{W}(f).$$

By iteration we obtain

$$\mathbb{W}(z^\alpha \bar{z}^\beta f) = \delta^\alpha \bar{\delta}^\beta \mathbb{W}(f)$$

where  $\delta^\alpha \bar{\delta}^\beta$  are defined in an obvious way.

Returning to our equation (??), we take the Weyl transform on both sides and obtain that

$$\mathbb{W}(f_{\alpha,\beta})e^{-tH} = \delta^\alpha \bar{\delta}^\beta e^{-2tH}.$$

Testing against the Hermite basis it is easy to see that the densely defined operator

$$T = (\delta^\alpha \bar{\delta}^\beta e^{-2tH})e^{tH}$$

extends to the whole  $L^2(\mathbb{R}^n)$  as a Hilbert-Schmidt operator. Hence,  $T = \mathbb{W}(f_{\alpha,\beta})$  for some  $f_{\alpha,\beta} \in L^2(\mathbb{C}^n)$ . This completes the proof of the lemma.  $\square$

**Theorem 4.6.** *Let  $t > 0$  and  $\lambda \in \mathbb{R}$ ,  $\lambda \neq 0$ . Then the  $\lambda$ -twisted heat kernel transform  $H_t^\lambda : L^2(\mathbb{R}^{2n}) \rightarrow \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  is an isometric isomorphism. Moreover,  $\{\tilde{\Phi}_{\alpha,\beta}^\lambda : \alpha, \beta \in \mathbb{N}_0^n\}$  is an orthonormal basis for  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ .*

*Proof.* As usual we restrict our attention to the case  $\lambda = 1$ . All what is left to show is that  $H_t$  is onto. Suppose that  $F \in \mathcal{B}_t(\mathbb{C}^{2n})$  is orthogonal to all  $\tilde{\Phi}_{\alpha,\beta}$ . We have to verify that  $F \equiv 0$ . The function

$$G(\mathbf{z}, \mathbf{w}) = F(\mathbf{z}, \mathbf{w})e^{\frac{1}{4}(\coth 2t)(\mathbf{z} \cdot \mathbf{z} + \mathbf{w} \cdot \mathbf{w})}$$

is orthogonal in  $\mathcal{F}_t$  to all functions of the form

$$f \times p_t(\mathbf{z}, \mathbf{w})e^{-\frac{1}{4}(\coth 2t)(\mathbf{z} \cdot \mathbf{z} + \mathbf{w} \cdot \mathbf{w})}.$$

In view of Lemma ??,  $G$  is orthogonal to all  $P_{\alpha,\beta}$ . Hence by Lemma ?? we get  $G = 0$  and so  $F = 0$  as desired.  $\square$

We conclude this subsection with a proof of the uniqueness of the weight function  $W_t^\lambda$ .

**Lemma 4.7.**  *$W_t^\lambda$  is the unique non-negative measurable weight function for the  $\lambda$ -twisted Bergman space  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ .*

*Proof.* In view of (??), the statement is equivalent to the assertion that

$$(4.3.5) \quad \mathcal{W}_t^\lambda(\mathbf{w}, \mathbf{z}) = e^{\lambda \Im(\mathbf{z} \cdot \bar{\mathbf{w}})} e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2 + |\mathbf{w}|^2)}$$

is the unique weight function for the twisted Fock space  $\mathcal{F}_t^\lambda(\mathbb{C}^{2n})$ . This will be verified in the sequel.

We may restrict ourselves to the notationally convenient case  $n = 1$ ,  $\lambda = 1$  and drop all sub and superscripts involving  $\lambda$ . Let  $\mathcal{U}_t : \mathbb{C}^2 \rightarrow \mathbb{R}_{\geq 0}$  be a measurable function such that

$$(4.3.6) \quad \int_{\mathbb{C}^2} f(z, w) \overline{g(z, w)} \mathcal{W}_t(z, w) dz dw = \int_{\mathbb{C}^2} f(z, w) \overline{g(z, w)} \mathcal{U}_t(z, w) dz dw$$

holds for all  $f, g \in \mathcal{F}_t(\mathbb{C}^2)$ . We have to show that  $\mathcal{W}_t = \mathcal{U}_t$  almost everywhere. Recall from Lemma ?? that all polynomials  $z^m w^n$  lie in  $\mathcal{F}_t(\mathbb{C}^2)$ . In particular the constant function belongs to  $\mathcal{F}_t(\mathbb{C}^2)$  and (??) implies that  $\mathcal{U}_t$  is integrable.

Let us introduce polar coordinates on  $\mathbb{C}^2$  by  $(z, w) = (re^{i\phi}, se^{i\theta})$ . Consider the Fourier expansions of  $\mathcal{W}_t$  and  $\mathcal{U}_t$  given by

$$\mathcal{W}_t(re^{i\phi}, se^{i\theta}) = \sum_{m,n \in \mathbb{Z}} a_{m,n}(r, s) e^{im\phi} e^{in\theta}$$

and

$$\mathcal{U}_t(re^{i\phi}, se^{i\theta}) = \sum_{m,n \in \mathbb{Z}} b_{m,n}(r,s) e^{im\phi} e^{in\theta}.$$

Identity (??) applied to  $f = g = z^k w^l$  yields the estimates

$$(4.3.7) \quad \int_0^\infty \int_0^\infty r^{2k+1} s^{2l+1} |a_{m,n}(r,s)| dr ds \leq \|z^k w^l\|_{\mathcal{F}_t(\mathbb{C}^2)}^2$$

$$(4.3.8) \quad \int_0^\infty \int_0^\infty r^{2k+1} s^{2l+1} |b_{m,n}(r,s)| dr ds \leq \|z^k w^l\|_{\mathcal{F}_t(\mathbb{C}^2)}^2$$

for all  $m, n \in \mathbb{Z}$ .

We finish the proof and show  $c_{m,n} = a_{m,n} - b_{m,n} = 0$  for all  $m, n \in \mathbb{Z}$ . In fact for  $f = z^{m_1} w^{n_1}$  and  $g = z^{m_2} w^{n_2}$  for  $m_1, m_2, n_1, n_2 \in \mathbb{N}_0$  we obtain from (??) that

$$(4.3.9) \quad \int_0^\infty \int_0^\infty r^{m_1+m_2+1} s^{n_1+n_2+1} c_{m_2-m_1, n_2-n_1}(r,s) dr ds = 0.$$

Note that the integral on the left is absolutely convergent by (??)-(??). Fix now  $m, n \in \mathbb{Z}$ . Reformulating (??) reads

$$(4.3.10) \quad \int_0^\infty \int_0^\infty r^{|m|+2k+1} s^{|n|+2l+1} c_{m,n}(r,s) dr ds = 0$$

for all  $k, l \in \mathbb{N}_0$ . In view of (??)-(??), we have the estimate

$$(4.3.11) \quad \int_0^\infty \int_0^\infty r^{|m|+2k+1} s^{|n|+2l+1} |c_{m,n}(r,s)| dr ds \leq 2 \|z^{|m|+k} w^{|n|+l}\|_{\mathcal{F}_t(\mathbb{C}^2)}^2 + C$$

with  $C = \int_{|z|<1, |w|<1} (\mathcal{W}_t(z,w) + \mathcal{U}_t(z,w)) dz dw > 0$  a constant independent of  $m, n$ .

Denote by  $\mathcal{R}_+ = \{\zeta \in \mathbb{C} : \Re \zeta > 0\}$  the right halfplane. Let us recall the elementary fact that a bounded holomorphic function  $f : \mathcal{R}_+ \rightarrow \mathbb{C}$  which vanishes on  $\alpha + \beta \mathbb{N}_0$  for some  $\alpha \geq 0, \beta > 0$  is identically zero (see [?], Lemma A.1 for a proof).

The explicite formula for  $\mathcal{W}_t$  in (??) yields a crude but sufficient estimate for the norm of monomials: there exists constants  $c, \gamma > 0$  such that for all  $k, l \in \mathbb{N}_0$  one has

$$(4.3.12) \quad \|z^k w^l\|^2 \leq c \cdot e^{\gamma(k+l)}.$$

Now define the function

$$F_{m,n} : \mathcal{R}_+ \times \mathcal{R}_+ \rightarrow \mathbb{C},$$

$$(\zeta_1, \zeta_2) \mapsto e^{-3\gamma(\zeta_1+\zeta_2)} \int_0^\infty \int_0^\infty r^{|m|+2\zeta_1+1} s^{|n|+2\zeta_2+1} c_{m,n}(r,s) dr ds.$$

It is a consequence of (??) and (??) that  $F_{m,n}$  is bounded and holomorphic on  $\mathcal{R}_+ \times \mathcal{R}_+$ . As  $F_{m,n}|_{\mathbb{N} \times \mathbb{N}} = 0$  by (??), we conclude that  $F_{m,n} = 0$ . But then  $c_{m,n} = 0$  by the properties of the Mellin transform.  $\square$

4.4. **The inversion formula for  $H_t^\lambda$ .** We conclude this section by proving a formula for the inverse map of the  $\lambda$ -twisted heat kernel transform  $H_t^\lambda : L^2(\mathbb{R}^{2n}) \rightarrow \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ . It is in the nature of the problem that  $(H_t^\lambda)^{-1}$  can only be defined nicely on a dense subspace of  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ . The precise statement is as follows:

**Theorem 4.8.** *The inverse of  $H_t^\lambda : L^2(\mathbb{R}^{2n}) \rightarrow \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  is given by*

$$(H_t^\lambda)^{-1}(F) = \lim_{s \rightarrow 0^+} F_s \quad (F \in \mathcal{B}_t^\lambda(\mathbb{C}^{2n})),$$

where

$$F_s(\mathbf{a}, \mathbf{b}) = \int_{\mathbb{C}^{2n}} F(\mathbf{z} + \mathbf{a}, \mathbf{w} + \mathbf{b}) e^{\frac{i\lambda}{2}(\mathbf{a} \cdot \mathbf{w} - \mathbf{b} \cdot \mathbf{z})} \overline{p_{t+s}^\lambda(\mathbf{z}, \mathbf{w})} W_t^\lambda(\mathbf{z}, \mathbf{w}) \, d\mathbf{z} \, d\mathbf{w}.$$

*Proof.* As before we only need to handle the case of  $\lambda = 1$ .

Let  $F \in \mathcal{B}_t(\mathbb{C}^{2n})$ . Since the space  $\mathcal{B}_t(\mathbb{C}^{2n})$  is twisted-translation invariant, it is clear that the function

$$(\tau(-\mathbf{a}, -\mathbf{b})F)(\mathbf{z}, \mathbf{w}) = F(\mathbf{z} + \mathbf{a}, \mathbf{w} + \mathbf{b}) e^{\frac{i}{2}(\mathbf{a} \cdot \mathbf{w} - \mathbf{b} \cdot \mathbf{z})}$$

belongs to  $\mathcal{B}_t(\mathbb{C}^{2n})$ . Hence, by Cauchy-Schwarz inequality, the integral defining  $F_s$  converges. According to Theorem ?? we have  $F = H_t(f) = (f \times p_t)^\sim$  for some  $f \in L^2(\mathbb{R}^{2n})$ . It is easy to see that  $F_s \in L^2(\mathbb{R}^{2n})$  and that  $F_s$  converges to  $f$ . In fact, we have

$$\begin{aligned} F_s(\mathbf{a}, \mathbf{b}) &= \int_{\mathbb{C}^{2n}} (\tau(-\mathbf{a}, -\mathbf{b})H_t(f))(\mathbf{z}, \mathbf{w}) \overline{H_t(p_s)}(\mathbf{z}, \mathbf{w}) W_t(\mathbf{z}, \mathbf{w}) \, d\mathbf{z} \, d\mathbf{w} \\ &= \int_{\mathbb{C}^{2n}} H_t(\tau(-\mathbf{a}, -\mathbf{b})f)(\mathbf{z}, \mathbf{w}) \overline{H_t(p_s)}(\mathbf{z}, \mathbf{w}) W_t(\mathbf{z}, \mathbf{w}) \, d\mathbf{z} \, d\mathbf{w} \\ &= \int_{\mathbb{R}^{2n}} (\tau(-\mathbf{a}, -\mathbf{b})f)(\mathbf{x}, \mathbf{u}) p_s(\mathbf{x}, \mathbf{u}) \, d\mathbf{x} \, d\mathbf{u}. \end{aligned}$$

As  $(p_s)_{s>0}$  is a Dirac sequence, it therefore follows that

$$F_s(\mathbf{a}, \mathbf{b}) \rightarrow (\tau(-\mathbf{a}, -\mathbf{b})f)(0, 0) = f(\mathbf{a}, \mathbf{b})$$

for  $s \rightarrow 0^+$ . This proves the theorem.  $\square$

## 5. THE IMAGE OF $\mathcal{H}_t$ AS A DIRECT INTEGRAL

The goal of this section is to give a natural  $\mathbb{H}$ -equivariant identification of the image of the heat kernel transform  $\mathcal{H}_t : L^2(\mathbb{H}) \rightarrow \mathcal{O}(\mathbb{H}_\mathbb{C})$  with a direct integral of twisted Bergman-spaces.

We set  $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ . For each  $\lambda \in \mathbb{R}^\times$  we write  $\langle \cdot, \cdot \rangle_\lambda$  for the inner product on  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ . Recall the orthonormal basis  $\{\tilde{\Phi}_{\alpha, \beta}^\lambda : \alpha, \beta \in \mathbb{N}_0^n\}$  of  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  from Theorem 4.6.

We now introduce a measurable structure on  $\coprod_{\lambda \in \mathbb{R}^\times} \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ . By a *section*  $s$  of  $\coprod_{\lambda \in \mathbb{R}^\times} \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  we understand an assignment

$$s : \mathbb{R}^\times \rightarrow \coprod_{\lambda \in \mathbb{R}^\times} \mathcal{B}_t^\lambda(\mathbb{C}^{2n}), \quad \lambda \mapsto s_\lambda \in \mathcal{B}_t^\lambda(\mathbb{C}^{2n}).$$

We declare a section  $s = (s_\lambda)$  to be *measurable* if for all  $\alpha, \beta \in \mathbb{N}_0^n$  the map

$$\mathbb{R}^\times \rightarrow \mathbb{C}, \quad \lambda \mapsto \langle s_\lambda, \tilde{\Phi}_{\alpha, \beta}^\lambda \rangle_\lambda$$

is measurable. With that we can define a direct integral of Hilbert spaces by

$$\int_{\mathbb{R}^\times}^{\oplus} \mathcal{B}_t^\lambda(\mathbb{C}^{2n}) e^{2t\lambda^2} d\lambda = \{s : \mathbb{R}^\times \rightarrow \prod_{\lambda \in \mathbb{R}^\times} \mathcal{B}_t^\lambda(\mathbb{C}^{2n}) : s \text{ measurable,} \\ \|s\|^2 = \int_{\mathbb{R}^\times} \|s_\lambda\|_\lambda^2 e^{2t\lambda^2} d\lambda < \infty\}.$$

Recall the unitary representation  $\tau^\lambda$  of  $\mathbb{H}$  on  $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$  from Subsection 4.1. We then obtain a unitary representation  $\int_{\mathbb{R}^\times} \tau^\lambda d\lambda$  on  $\int_{\mathbb{R}^\times}^{\oplus} \mathcal{B}_t^\lambda(\mathbb{C}^{2n}) e^{2t\lambda^2} d\lambda$  by

$$\left( \int_{\mathbb{R}^\times} \tau^\lambda d\lambda \right) (h)(s) = (\tau^\lambda(h)s_\lambda)_\lambda$$

for  $h \in \mathbb{H}$  and  $s = (s_\lambda)$  a square integrable section.

In our next step we will identify  $\text{im } \mathcal{H}_t$  with our direct integral from above. For that let  $f \in S(\mathbb{H})$  be a Schwartz function. Then  $\mathcal{H}_t(f) = (k_t * f)^\sim$  and from  $(f * k_t)^\lambda = e^{-t\lambda^2} f^\lambda *_\lambda p_t^\lambda$  it hence follows that

$$(5.1) \quad (\mathcal{H}_t(f))^\lambda = e^{-t\lambda^2} H_t^\lambda(f^\lambda).$$

**Theorem 5.1.** *Let  $t > 0$ . The map*

$$\mathcal{J}_t : S(\mathbb{H}) \rightarrow \int_{\mathbb{R}^\times}^{\oplus} \mathcal{B}_t^\lambda(\mathbb{C}^{2n}) e^{2t\lambda^2} d\lambda, \quad f \mapsto ((\mathcal{H}_t(f))^\lambda)_\lambda$$

*extends to an  $\mathbb{H}$ -equivariant unitary equivalence*

$$(\tau, L^2(\mathbb{H})) \simeq \left( \int_{\mathbb{R}^\times} \tau^\lambda d\lambda, \int_{\mathbb{R}^\times}^{\oplus} \mathcal{B}_t^\lambda(\mathbb{C}^{2n}) e^{2t\lambda^2} d\lambda \right).$$

*Proof.* Let  $f \in S(\mathbb{H})$ . Then

$$\|f\|^2 = \int_{\mathbb{R}^{2n+1}} |f(\mathbf{x}, \mathbf{u}, \xi)|^2 d\mathbf{x} d\mathbf{u} d\xi = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}} |f^\lambda(\mathbf{x}, \mathbf{u})|^2 d\mathbf{x} d\mathbf{u} d\lambda.$$

By Theorem ?? we have for each  $\lambda$  that

$$\int_{\mathbb{R}^{2n}} |f^\lambda(\mathbf{x}, \mathbf{u})|^2 d\mathbf{x} d\mathbf{u} = \|H_t^\lambda(f^\lambda)\|^2.$$

Thus it follows from (??) that  $\mathcal{J}_t$  extends to an isometric embedding

$$\mathcal{J}_t : L^2(\mathbb{H}) \rightarrow \int_{\mathbb{R}^\times}^{\oplus} \mathcal{B}_t^\lambda(\mathbb{C}^{2n}) e^{2t\lambda^2} d\lambda,$$

denoted by the same symbol. The discussion leading up to (??) shows that  $\mathcal{J}_t$  is  $\mathbb{H}$ -equivariant.

It remains to show that  $\mathcal{J}_t$  is onto. For that observe if

$$f(\mathbf{x}, \mathbf{u}, \xi) = F(\mathbf{x}, \mathbf{u})\varphi(\xi)$$

for Schwartz functions  $F \in S(\mathbb{R}^{2n})$ ,  $\varphi \in \mathcal{S}(\mathbb{R})$ , then

$$(\mathcal{H}_t(f))^\lambda(\mathbf{x}, \mathbf{u}) = \hat{\varphi}(\lambda) e^{-t\lambda^2} H_t^\lambda(F)(\mathbf{x}, \mathbf{u}).$$

From that the surjectivity of  $\mathcal{J}_t$  easily follows.  $\square$

## 6. THE IMAGE OF $\mathcal{H}_t$ AS A SUM OF WEIGHTED BERGMAN SPACES

In this section we prove the main result of this paper:  $\text{im } \mathcal{H}_t = \mathcal{B}_t^+(\mathbb{H}_{\mathbb{C}}) \oplus \mathcal{B}_t^-(\mathbb{H}_{\mathbb{C}})$  is a direct sum of two weighted Bergman spaces. Very surprisingly, the corresponding weight functions  $W_t^+$  and  $W_t^-$  attain also negative values (see the phenomenon explained in Example 3.1).

We will begin our discussion by showing that  $\text{im } \mathcal{H}_t$  is not a weighted Bergman space corresponding to a non-negative weight function. This will lead naturally to the definition of the partial weight functions  $W_t^+$  and  $W_t^-$  and to a proof of the main theorem.

**6.1. Non-existence of a non-negative weight function.** The goal of this subsection is to discuss the non-existence of a non-negative weight function  $W_t$  on  $\mathbb{H}_{\mathbb{C}}$  such that

$$(6.1.1) \quad \|f\|^2 = \int_{\mathbb{H}_{\mathbb{C}}} |\mathcal{H}_t(f)(z)|^2 W_t(z) dz$$

holds for all  $f \in L^2(\mathbb{H})$ . In other words,  $\text{im } \mathcal{H}_t$  is not a weighted Bergman space corresponding to a non-negative weight function  $W_t$ . Subject to the natural assumption that  $W_t$  is  $\mathbb{H}$ -invariant, this will be established in Theorem ?? below.

Recall that we identify  $\mathbb{H}_{\mathbb{C}}$  with  $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$ . If  $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ ,  $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ ,  $\zeta = \xi + i\eta$  then  $(\mathbf{z}, \mathbf{w}, \zeta) = he^{iX}$  with  $X = (\mathbf{y}, \mathbf{v}, \eta + \frac{1}{2}(\mathbf{x} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{y}))$  and  $h = (\mathbf{x}, \mathbf{u}, \xi)$ .

Suppose that (??) holds. As  $\mathcal{H}_t$  is  $\mathbb{H}$ -equivariant, it is natural to assume that  $W_t(he^{iX}) = W_t(e^{iX})$  for all  $h \in \mathbb{H}$ . In coordinates  $(\mathbf{z}, \mathbf{w}, \zeta)$  this means that

$$(6.1.2) \quad W_t(\mathbf{x} + i\mathbf{y}, \mathbf{u} + i\mathbf{v}, \xi + i\eta) = W_t\left(i\mathbf{y}, i\mathbf{v}, i\eta + \frac{i}{2}(\mathbf{x} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{y})\right).$$

Thus the weight function is uniquely determined by its restriction to  $(i\mathbf{y}, i\mathbf{v}, i\eta)$ . Furthermore  $W_t$  is independent of the  $\xi$  variable. Hence (??) reads as

$$(6.1.3) \quad \|f\|^2 = \int_{\mathbb{H}_{\mathbb{C}}} |\mathcal{H}_t(f)(\mathbf{z}, \mathbf{w}, \zeta)|^2 W_t(\mathbf{z}, \mathbf{w}, i\eta) dz d\mathbf{w} d\zeta$$

**Proposition 6.1.** *Let  $W_t(\mathbf{z}, \mathbf{w}, i\eta)$  be a non-negative measurable function on  $\mathbb{H}_{\mathbb{C}}$ . If (??) holds for all  $f \in L^2(\mathbb{H})$ , then it is necessary that  $W_t$  satisfies*

$$(6.1.4) \quad W_t^\lambda(\mathbf{z}, \mathbf{w}) = e^{-2t\lambda^2} \int_{\mathbb{R}} e^{2\lambda\eta} W_t(\mathbf{z}, \mathbf{w}, i\eta) d\eta$$

for all  $\lambda \in \mathbb{R}^\times$  and  $W_t^\lambda$  the function given in (??).

*Proof.* Write

$$\mathcal{W}_t^\lambda(\mathbf{z}, \mathbf{w}) = e^{-2t\lambda^2} \int_{\mathbb{R}} e^{2\lambda\eta} W_t(\mathbf{z}, \mathbf{w}, i\eta) d\eta.$$

We have to show that  $W_t^\lambda = \mathcal{W}_t^\lambda$ .

It follows from (2.2.2) that

$$\int_{\mathbb{R}} k_t^\sim(\mathbf{z}, \mathbf{w}, \xi + i\eta) e^{i\lambda\xi} d\xi = e^{\lambda\eta} e^{-t\lambda^2} p_t^\lambda(\mathbf{z}, \mathbf{w}).$$

An easy calculation shows that

$$(6.1.5) \quad \begin{aligned} \int_{\mathbb{R}} \mathcal{H}_t(f)(\mathbf{z}, \mathbf{w}, \xi + i\eta) e^{i\lambda\xi} d\xi &= e^{\lambda\eta} e^{-t\lambda^2} (f^\lambda *_\lambda p_t^\lambda)(\mathbf{z}, \mathbf{w}) \\ &= e^{\lambda\eta} e^{-t\lambda^2} H_t^\lambda(f^\lambda)(\mathbf{z}, \mathbf{w}). \end{aligned}$$

Therefore, upon applying Plancherel theorem in the  $\xi$ -variable, the equation (??) becomes

$$\|f\|^2 = \int_{\mathbb{R}} \int_{\mathbb{C}^{2n}} \int_{\mathbb{R}} |H_t^\lambda(f^\lambda)(\mathbf{z}, \mathbf{w})|^2 e^{-2t\lambda^2} e^{2\lambda\eta} W_t(\mathbf{z}, \mathbf{w}, i\eta) d\eta d\mathbf{x} d\mathbf{u} d\mathbf{y} d\mathbf{v} d\lambda .$$

Here we applied Fubini's theorem which is justified as  $W_t$  is by assumption non-negative. Employing the definition of  $\mathcal{W}_t$  we therefore get

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{2n}} |f^\lambda(\mathbf{x}, \mathbf{u})|^2 d\mathbf{x} d\mathbf{u} d\lambda = \int_{\mathbb{R}} \int_{\mathbb{C}^{2n}} |H_t^\lambda(f^\lambda)(\mathbf{z}, \mathbf{w})|^2 \mathcal{W}_t^\lambda(\mathbf{z}, \mathbf{w}) d\mathbf{x} d\mathbf{u} d\mathbf{y} d\mathbf{v} d\lambda .$$

Let now  $\varphi$  be a Schwartz class function on  $\mathbb{R}$  with unit  $L^2$ -norm and define  $f$  by  $f(\mathbf{x}, \mathbf{u}, \xi) = \widehat{\varphi}(\xi)F(\mathbf{x}, \mathbf{u})$  with  $F \in L^2(\mathbb{R}^{2n})$ . Then  $f^\lambda(\mathbf{x}, \mathbf{u}) = \varphi(\lambda)F(\mathbf{x}, \mathbf{u})$  and  $H_t^\lambda(f^\lambda) = \varphi(\lambda)H_t^\lambda(F)$ . For such  $f$  the above displayed equation becomes

$$(6.1.6) \quad \int_{\mathbb{R}^{2n}} |F(\mathbf{x}, \mathbf{u})|^2 d\mathbf{x} d\mathbf{u} = \int_{\mathbb{R}} \int_{\mathbb{C}^{2n}} |\varphi(\lambda)|^2 |H_t^\lambda(F)(\mathbf{z}, \mathbf{w})|^2 \mathcal{W}_t^\lambda(\mathbf{z}, \mathbf{w}) d\mathbf{x} d\mathbf{u} d\mathbf{y} d\mathbf{v} d\lambda .$$

From (??) it is easy to see that for every  $\lambda \neq 0$  and all  $F \in L^2(\mathbb{R}^{2n})$

$$\int_{\mathbb{R}^{2n}} |F(\mathbf{x}, \mathbf{u})|^2 d\mathbf{x} d\mathbf{u} = \int_{\mathbb{C}^{2n}} |H_t^\lambda(F)(\mathbf{z}, \mathbf{w})|^2 \mathcal{W}_t^\lambda(\mathbf{z}, \mathbf{w}) d\mathbf{x} d\mathbf{u} d\mathbf{y} d\mathbf{v} .$$

By Lemma ??, the weight function  $\mathcal{W}_t^\lambda$  is given by (??). □

**Theorem 6.2.** *There is no non-negative left  $\mathbb{H}$ -invariant weight function  $W_t$  for which (??) holds for all  $f \in L^2(\mathbb{H})$ , i.e.  $\text{im } \mathcal{H}_t$  is not a weighted Bergman spaces corresponding to a left  $\mathbb{H}$ -invariant non-negative weight function.*

*Proof.* By (??),  $W_t$  is uniquely determined by its restriction to  $(i\mathbf{y}, i\mathbf{v}, i\eta)$ . By (??) and (??),

$$\int_{\mathbb{R}} e^{2\lambda\eta} W_t(i\mathbf{y}, i\mathbf{v}, i\eta) d\eta = e^{2t\lambda^2} p_{2t}^\lambda(2\mathbf{y}, 2\mathbf{v}) \quad (\lambda \in \mathbb{R}^\times) .$$

If  $W_t$  were non-negative, then for fixed  $\mathbf{y}, \mathbf{v}$  and  $\lambda$  the function  $\eta \mapsto e^{2\lambda\eta} W_t(i\mathbf{y}, i\mathbf{v}, i\eta)$  would belong to  $L^1(\mathbb{R})$ . Consequently, we would have

$$(6.1.7) \quad \int_{\mathbb{R}} e^{2(\lambda+is)\eta} W_t(i\mathbf{y}, i\mathbf{v}, i\eta) d\eta = e^{2t(\lambda+is)^2} p_{2t}^{\lambda+is}(2\mathbf{y}, 2\mathbf{v}) .$$

The left hand side of (??) would be holomorphic in  $\lambda + is$  since for every  $n \in \mathbb{N}_0$  there exists an  $\varepsilon > 0$  such that  $|\eta|^n e^{2\lambda\eta} W_t(i\mathbf{y}, i\mathbf{v}, i\eta) \leq e^{2\lambda\eta + \varepsilon|\eta|} W_t(i\mathbf{y}, i\mathbf{v}, i\eta)$ . However, the right side of (??) is holomorphic only for  $\lambda \neq 0$ . If  $\lambda = 0$ , it becomes

$$p_{2t}^{is}(2\mathbf{y}, 2\mathbf{v}) = c_n \left( \frac{s}{\sin(2st)} \right)^n e^{-s(\cot 2st)(\mathbf{y}^2 + \mathbf{v}^2)},$$

which has an essential singularity at the points  $s \in \mathbb{Z}^\times(\pi/t)$ . Therefore there is no non-negative  $W_t$  that will satisfy (??) or (??). □

**6.2. The partial weight functions  $W_t^+$  and  $W_t^-$ .** Recall the twisted weight function  $W_t^\lambda$  from (??).

Let  $\lambda > 0$  and define a function  $W_t^+$  on  $\mathbb{H}_\mathbb{C}$  by

$$(6.2.1) \quad W_t^+(\mathbf{z}, \mathbf{w}, \zeta) = \int_{\mathbb{R}} e^{2t(\lambda + \frac{i}{2}s)^2} e^{-2\eta(\lambda + \frac{i}{2}s)} W_t^{\lambda + \frac{i}{2}s}(\mathbf{z}, \mathbf{w}) ds.$$

It is easy to see that  $W_t^+$  is well-defined. Notice that  $W_t^+$  does not depend on  $\xi$ . In Proposition ?? below we will show that  $W_t^+$  is independent of the choice of  $\lambda > 0$ .

**Proposition 6.3.** *The function  $W_t^+$  satisfies the following properties:*

(i)  $W_t^+$  is independent of the choice of  $\lambda > 0$ . In particular,

$$W_t^+(\mathbf{z}, \mathbf{w}, \zeta) = \lim_{\lambda \rightarrow 0^+} \int_{\mathbb{R}} e^{2t(\lambda + \frac{i}{2}s)^2} e^{-2\eta(\lambda + \frac{i}{2}s)} W_t^{\lambda + \frac{i}{2}s}(\mathbf{z}, \mathbf{w}) ds.$$

(ii) Let  $a > 0$  and  $Q \subseteq \mathbb{C}^{2n}$  be a compact set. Then there exists a constant  $C = C(Q, a) > 0$  such that for all  $\varepsilon \in [a^{-1}, a]$  and  $\xi \in \mathbb{R}$

$$\sup_{(\mathbf{z}, \mathbf{w}) \in Q} \int_{\mathbb{R}} |e^{2\varepsilon\eta} W_t^+(\mathbf{z}, \mathbf{w}, \xi + i\eta)| d\eta \leq C.$$

(iii)  $W_t^+$  satisfies (??) with  $\lambda > 0$ , i.e.

$$(6.2.2) \quad W_t^\lambda(\mathbf{z}, \mathbf{w}) = e^{-2t\lambda^2} \int_{\mathbb{R}} e^{2\eta\lambda} W_t^+(\mathbf{z}, \mathbf{w}, i\eta) d\eta$$

for  $\lambda > 0$ .

(iv)  $W_t^+$  is real valued and left  $\mathbb{H}$ -invariant.

*Proof.* (i) Let  $\lambda > 0$ . We have to show that

$$W_t^+(\mathbf{z}, \mathbf{w}, \zeta) = \int_{\mathbb{R}} e^{2t(\lambda + \frac{i}{2}s)^2} e^{-2\eta(\lambda + \frac{i}{2}s)} W_t^{\lambda + \frac{i}{2}s}(\mathbf{z}, \mathbf{w}) ds$$

is independent of the choice of  $\lambda > 0$ . This will be a consequence of Cauchy's theorem. Indeed, let us denote the right hand side by  $I(\lambda)$ . For  $R > 0$  and  $\lambda_2 > \lambda_1 > 0$ , let  $\Gamma_R$  be the contour consisting four lines,  $\Gamma_R(\lambda_1) := \{\lambda_1 + is/2 : -2R < s < 2R\}$ ,  $\gamma_{-R} = \{\lambda - iR : \lambda_1 \leq \lambda \leq \lambda_2\}$ ,  $\Gamma_R(\lambda_2) = \{\lambda_2 + is/2 : -2R < s < 2R\}$  and  $\gamma_R = \{\lambda + iR : \lambda_1 \leq \lambda \leq \lambda_2\}$ , going counterclockwise. As  $R \rightarrow \infty$ , the integral on  $\Gamma_R(\lambda)$  becomes  $I(\lambda)$ . Cauchy's theorem shows that

$$\int_{\Gamma_R} e^{-2\eta z} e^{2tz^2} W_t^z(\mathbf{z}, \mathbf{w}) dz = 0.$$

It is easy to see that  $|\sinh(\lambda + iR)t| \geq \sinh(\lambda t)$  and  $|\cosh(\lambda + iR)t| \leq \cosh(\lambda t)$ . Thus,

$$|p_{2t}^{\lambda \pm iR}(2\mathbf{y}, 2\mathbf{v})| \leq \left( \frac{\lambda + R}{\sinh \lambda t} \right)^n e^{(\lambda + R) \coth(\lambda t)(|\mathbf{y}|^2 + |\mathbf{v}|^2)}.$$

Together with  $|e^{2t(\lambda \pm iR)^2}| = e^{2t\lambda^2} e^{-2tR^2}$ , this shows that the integrals on  $\gamma_{-R}$  and on  $\gamma_R$  go to zero as  $R \rightarrow +\infty$ . Thus, taking  $R \rightarrow \infty$  shows that  $I(\lambda_1) = I(\lambda_2)$ . This completes the proof of (i).

(ii) It follows from (i) that  $W_t^+$  satisfies the bound

$$|W_t^+(\mathbf{z}, \mathbf{w}, \xi + i\eta)| \leq e^{-2\eta\lambda} e^{2t\lambda^2} \int_{\mathbb{R}} e^{-\frac{1}{2}ts^2} |W_t^{\lambda + \frac{i}{2}s}(\mathbf{z}, \mathbf{w})| ds.$$

for any  $\lambda > 0$ . Notice that the integral on the right is independent of  $\eta$ . Thus if we let  $\lambda > \epsilon$  if  $\eta > 0$  and  $\lambda < \epsilon$  if  $\eta < 0$ , we see that  $\eta \mapsto e^{2\epsilon\eta} W_t^+(\mathbf{z}, \mathbf{w}, \xi + i\eta)$  is integrable. This implies (ii).

(iii) This is immediate from the definition (??) and Fourier inversion (which is justified by (ii)). In fact, we have

$$W_t^+(\mathbf{z}, \mathbf{w}, \zeta) = e^{-2\eta\lambda} \int_{\mathbb{R}} e^{-i\eta s} e^{2t(\lambda + \frac{i}{2}s)^2} W_t^{\lambda + \frac{i}{2}s}(\mathbf{z}, \mathbf{w}) ds$$

and so

$$\int_{\mathbb{R}} e^{2\lambda\eta} W_t^+(\mathbf{z}, \mathbf{w}, \xi + i\eta) e^{i\eta s} d\eta = e^{2t(\lambda + \frac{i}{2}s)^2} W_t^{\lambda + \frac{i}{2}s}(\mathbf{z}, \mathbf{w}) .$$

Setting  $s = 0$  gives the the stated result.

(iv) We first show that  $W_t^+$  is real valued. In fact, taking the conjugate of the integral (??) and then changing variable  $s \rightarrow -s$  shows that the weight function  $W_t^+$  is real. Finally, the fact that  $W_t^\lambda$  is twisted-translation invariant forces that  $W_t^+$  is left  $\mathbb{H}$ -invariant.  $\square$

The function  $W_t^+$  has a natural counterpart  $W_t^-$ . For  $\lambda < 0$  we define  $W_t^-$  by

$$(6.2.3) \quad W_t^-(\mathbf{z}, \mathbf{w}, \zeta) = \int_{\mathbb{R}} e^{2t(\lambda + \frac{i}{2}s)^2} e^{-2\eta(\lambda + \frac{i}{2}s)} W_t^{\lambda + \frac{i}{2}s}(\mathbf{z}, \mathbf{w}) ds.$$

It is more or less obvious that  $W_t^-$  satisfies the same properties as  $W_t^+$  listed in Proposition (??), i.e.  $W_t^-$  is independent of the choice of  $\lambda < 0$  etc. In fact, a simple change of variable in the integral and the fact that  $p_t^\lambda(2\mathbf{y}, 2\mathbf{v})$  is even in  $\lambda$  leads to the relation

$$W_t^+(\mathbf{z}, \mathbf{w}, i\eta) = W_t^-(\mathbf{z}, \mathbf{w}, -i\eta).$$

We refer to  $W_t^+$  and  $W_t^-$  as the *partial weight functions*. Their importance will become clear in the next subsection.

*Remark 6.1.* We will show in the appendix that both  $W_t^+$  and  $W_t^-$  attain positive and negative values. In addition we shall discuss their oscillatory behaviour. A more heuristic explanation of these phenomena might be the following: Both  $W_t^+(i\mathbf{y}, i\mathbf{v}, i\eta)$  and  $W_t^-(i\mathbf{y}, i\mathbf{v}, i\eta)$  satisfy the differential equation

$$(6.2.4) \quad 2 \frac{\partial}{\partial t} U = \left( \Delta + (1 - |\mathbf{y}|^2 - |\mathbf{v}|^2) \frac{\partial^2}{\partial \eta^2} \right) U.$$

Indeed, this follows from a straightforward computation starting from

$$\frac{\partial}{\partial t} p_t^\lambda(\mathbf{y}, \mathbf{v}) = \left( \Delta - \frac{\lambda^2}{4} (|\mathbf{y}|^2 + |\mathbf{v}|^2) \right) p_t^\lambda(\mathbf{y}, \mathbf{v})$$

for all  $\lambda \neq 0$  (see [?]). We note that the differential equation (??) is parabolic only for  $|\mathbf{y}|^2 + |\mathbf{v}|^2 < 1$ . If  $|\mathbf{y}|^2 + |\mathbf{v}|^2 > 1$ , then the right hand side of (??) resembles a wave equation which in turn might explain the oscillatory behaviour of  $W_t^+$  and  $W_t^-$  on the large scale.

**6.3. The image of the heat kernel transform.** The objective of this section is to prove our main theorem:  $\text{im } \mathcal{H}_t = \mathcal{B}_t^+(\mathbb{H}_{\mathbb{C}}) \oplus \mathcal{B}_t^-(\mathbb{H}_{\mathbb{C}})$  is a sum of two weighted Bergman spaces.

To exhibit the Bergman structure of the spaces  $\mathcal{B}_t^+(\mathbb{H}_{\mathbb{C}})$  and  $\mathcal{B}_t^-(\mathbb{H}_{\mathbb{C}})$  needs some preparation.

First we define subspaces of  $L^2(\mathbb{H})$  by

$$L_+^2(\mathbb{H}) = \{f \in L^2(\mathbb{H}) : f^\lambda = 0, \quad \lambda \leq 0\}$$

and

$$L_-^2(\mathbb{H}) = \{f \in L^2(\mathbb{H}) : f^\lambda = 0, \quad \lambda \geq 0\}.$$

Notice that both subspaces are  $\mathbb{H}$ -invariant and

$$L^2(\mathbb{H}) = L_+^2(\mathbb{H}) \oplus L_-^2(\mathbb{H}).$$

Next we recall some facts on the heat kernel transform on the real line. The heat kernel on  $\mathbb{R}$  is given by

$$q_t(x) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \quad (x \in \mathbb{R}).$$

Define a weighted Bergman space on  $\mathbb{C}$  by

$$\mathcal{B}_t(\mathbb{C}) = \{g \in \mathcal{O}(\mathbb{C}) : \|g\|^2 = \int_{\mathbb{C}} |g(x+iy)|^2 e^{-\frac{x^2}{2t}} dx dy < \infty\}$$

and recall that the mapping

$$h_t : L^2(\mathbb{R}) \rightarrow \mathcal{B}_t(\mathbb{C}), \quad g \mapsto (f * q_t)^\sim$$

is (up to scale) an  $\mathbb{R}$ -equivariant isometric isomorphism.

Set  $\mathbb{R}^+ = (0, \infty)$  and  $\mathbb{R}^- = (-\infty, 0)$ . With  $L_+^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \text{supp } \hat{f} \subseteq \mathbb{R}^+\}$  and  $L_-^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \text{supp } \hat{f} \subseteq \mathbb{R}^-\}$  we have  $L^2(\mathbb{R}) = L_+^2(\mathbb{R}) \oplus L_-^2(\mathbb{R})$ . Finally, let us write  $\mathcal{B}_t^\pm(\mathbb{C}) = h_t(L_\pm^2(\mathbb{R}))$ . Clearly we have  $\mathcal{B}_t(\mathbb{C}) = \mathcal{B}_t^+(\mathbb{C}) \oplus \mathcal{B}_t^-(\mathbb{C})$ .

Let  $R > 0$ . Denote by  $B_R$  the open ball centered at 0 with radius  $R$  in  $\mathbb{C}^n$ . Further define  $K_R = B_R \times B_R \times \mathbb{C} \subseteq \mathbb{H}_{\mathbb{C}}$  and note that  $\bigcup_{R>0} K_R = \mathbb{H}_{\mathbb{C}}$ .

We define  $\mathcal{V}_t^+(\mathbb{H}_{\mathbb{C}})$  as the vector space consisting of all holomorphic functions  $F$  on  $\mathbb{H}_{\mathbb{C}}$  such that

- $F|_{K_R} \in L^2(K_R, |W_t^+| dz)$  for all  $R > 0$ ,
- $\lim_{R \rightarrow \infty} \int_{K_R} |F(z)|^2 |W_t^+(z)| dz < \infty$ ,
- $F(\mathbf{z}, \mathbf{w}, \cdot) \in \mathcal{B}_t^+(\mathbb{C})$  for all  $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$ .

We endow  $\mathcal{V}_t^+(\mathbb{H}_{\mathbb{C}})$  with a sesquilinear bracket

$$(6.3.1) \quad \langle F, G \rangle_+ = \lim_{R \rightarrow \infty} \int_{K_R} F(z) \overline{G(z)} |W_t^+(z)| dz,$$

for  $F, G \in \mathcal{V}_t^+(\mathbb{H}_{\mathbb{C}})$ . Similarly one defines  $\mathcal{V}_t^-(\mathbb{H}_{\mathbb{C}})$  and  $\langle \cdot, \cdot \rangle_-$ .

*Remark 6.2.* One might ask if one cannot define  $\mathcal{V}_t^\pm(\mathbb{H}_{\mathbb{C}})$  in a simpler manner: avoid the exhaustion  $\bigcup_{R>0} K_R = \mathbb{H}_{\mathbb{C}}$  and just require  $|F|^2 |W_t^\pm|$  to be absolutely integrable on  $\mathbb{H}_{\mathbb{C}}$ . However, this will not work, and the reason for this is the bad oscillatory behaviour of  $W_t^\pm$  (see the appendix).

A priori it is not clear that  $\langle F, F \rangle_\pm \geq 0$ . This will be shown next.

**Lemma 6.4.** *The bracket  $\langle \cdot, \cdot \rangle_\pm$  induces on  $\mathcal{V}_t^\pm(\mathbb{H}_{\mathbb{C}})$  a pre Hilbert space structure.*

*Proof.* It is sufficient to treat the case “+” only. All what is left to show is that  $\langle F, F \rangle_+ \geq 0$  and  $\langle F, F \rangle_+ = 0$  if and only if  $F = 0$ .

Fix  $F \in \mathcal{V}_t^+(\mathbb{H}_{\mathbb{C}})$ . Then  $F(\mathbf{z}, \mathbf{w}, \cdot) \in \mathcal{B}_t^+(\mathbb{C})$  implies the existence of a function  $g(\mathbf{z}, \mathbf{w}, \cdot) \in L_+^2(\mathbb{R})$  such that

$$F(\mathbf{z}, \mathbf{w}, \zeta) = h_t(g(\mathbf{z}, \mathbf{w}, \cdot))(\zeta) = \int_{\mathbb{R}} g(\mathbf{z}, \mathbf{w}, s) q_t(\zeta - s) ds.$$

Therefore, up to an irrelevant constant only depending on  $t$ , the following equality holds:

$$\int_{\mathbb{R}} F(\mathbf{z}, \mathbf{w}, \xi + i\eta) e^{i\lambda\xi} d\xi = e^{\lambda\eta} e^{-t\lambda^2} g^\lambda(\mathbf{z}, \mathbf{w}) .$$

Consequently, as  $W_t^+$  is independent of  $\xi$ ,

$$\int_{K_R} |F(z)|^2 W_t^+(z) dz = \int_{B_R^2} \int_0^\infty \int_{\mathbb{R}} |g^\lambda(\mathbf{z}, \mathbf{w})|^2 e^{2\lambda\eta} e^{-2t\lambda^2} W_t^+(\mathbf{z}, \mathbf{w}, i\eta) d\eta d\lambda d\mathbf{z} d\mathbf{w} .$$

In view of (??) we thus get

$$\int_{K_R} |F(z)|^2 W_t^+(z) dz = \int_{B_R} \int_{B_R} \int_0^\infty |g^\lambda(\mathbf{z}, \mathbf{w})|^2 W_t^\lambda(\mathbf{z}, \mathbf{w}) d\lambda d\mathbf{z} d\mathbf{w} .$$

But  $W_t^\lambda \geq 0$  and so

$$\langle F, F \rangle_+ = \lim_{R \rightarrow \infty} \int_{B_R} \int_{B_R} \int_0^\infty |g^\lambda(\mathbf{z}, \mathbf{w})|^2 W_t^\lambda(\mathbf{z}, \mathbf{w}) d\lambda d\mathbf{z} d\mathbf{w} \geq 0$$

and  $\langle F, F \rangle_+ = 0$  if and only if  $g^\lambda = 0$  for all  $\lambda$ , i.e.  $F = 0$ . This completes the proof of the lemma.  $\square$

Let us write  $\mathcal{H}_t^\pm$  for the heat kernel transform when restricted to  $L_\pm^2(\mathbb{H})$ . Define Hilbert spaces of holomorphic functions by  $\mathcal{B}_t^\pm(\mathbb{H}_\mathbb{C}) = \text{im } \mathcal{H}_t^\pm$  and note that

$$\text{im } \mathcal{H}_t = \mathcal{B}_t^+(\mathbb{H}_\mathbb{C}) \oplus \mathcal{B}_t^-(\mathbb{H}_\mathbb{C}) .$$

Let us remark that this decomposition can be also achieved using the Hilbert transform in the last variable.

**Theorem 6.5.** *Let  $t > 0$ . Then  $\mathcal{B}_t^\pm(\mathbb{H}_\mathbb{C})$  is the Hilbert completion of  $(\mathcal{V}_t^\pm(\mathbb{H}_\mathbb{C}), \langle \cdot, \cdot \rangle_\pm)$  with  $\langle \cdot, \cdot \rangle_\pm$  given by (??).*

*Proof.* We restrict ourselves to the “+”-case. Define a dense subspace of  $L_+^2(\mathbb{H})^0$  of  $L_+^2(\mathbb{H})$  by

$$L_+^2(\mathbb{H})^0 = \{f \in L_+^2(\mathbb{H}) : \lambda \mapsto f^\lambda \text{ compactly supported in } (0, \infty)\}$$

We claim that  $\mathcal{H}_t(L_+^2(\mathbb{H})^0) \subset \mathcal{V}_t^+(\mathbb{H})$ . Let  $f \in L_+^2(\mathbb{H})^0$  and set  $F = \mathcal{H}_t^+(f)$ . Choose  $a > 0$  such that  $f^\lambda = 0$  for  $\lambda$  outside of  $(a^{-1}, a)$ . Proceeding as in Lemma 6.4 and using the estimate Proposition 6.3 (ii) we see that  $\mathcal{H}_t^+(f)$  satisfies the first condition in the definition of  $\mathcal{V}_t^+(\mathbb{H}_\mathbb{C})$ . Furthermore (6.1.5) implies that

$$\int_{K_R} |F(z)|^2 W_t^+(z) dz = \int_{B_R} \int_{B_R} \int_0^\infty |H_t^\lambda(f^\lambda)(\mathbf{z}, \mathbf{w})|^2 W_t^\lambda(\mathbf{z}, \mathbf{w}) d\lambda d\mathbf{z} d\mathbf{w} .$$

As  $W_t^\lambda \geq 0$ , it hence follows that  $\int_{K_R} |F(z)|^2 W_t^+(z) dz$  is increasing in  $R$ . Similar reasoning as in (6.1.6) now shows that

$$\lim_{R \rightarrow \infty} \int_{K_R} |F(z)|^2 W_t^+(z) dz = \|f\|^2 < \infty .$$

Furthermore, for fixed  $(\mathbf{z}, \mathbf{w})$  we have  $F(\mathbf{z}, \mathbf{w}, \cdot) \in \mathcal{B}_t(\mathbb{C})$  as a quick inspection of (6.1.5) shows. This proves our claim.

As a byproduct of our reasoning above we have shown that  $\mathcal{H}_t^+ : L_+^2(\mathbb{H})^0 \rightarrow \mathcal{V}_t^+(\mathbb{H})$  is an isometric map. It remains to verify that each function  $F \in \mathcal{V}_t^+(\mathbb{H}_\mathbb{C})$  can be written as  $\mathcal{H}_t^+(f)$  for some  $f \in L_+^2(\mathbb{H})$ . Let  $g^\lambda(\mathbf{z}, \mathbf{w})$  be the function associated to

$F$  as in the proof of Lemma 6.4. Then for almost all  $\lambda$  there exists an  $f^\lambda \in L^2(\mathbb{R}^{2n})$  such that  $g^\lambda = H_t^\lambda(f^\lambda)$ . It is easy to check that the prescription

$$f(\mathbf{x}, \mathbf{u}, \xi) = \int_{\mathbb{R}} e^{-i\lambda\xi} f^\lambda(\mathbf{x}, \mathbf{u}) d\lambda$$

defines a function in  $L_+^2(\mathbb{H})$  such that  $\mathcal{H}_t^+(f) = F$ . This completes the proof of the theorem.  $\square$

## 7. APPENDIX: THE OSCILLATORY BEHAVIOUR OF THE PARTIAL WEIGHT FUNCTIONS

This appendix is devoted to a closer study of the partial weight functions  $W_t^\pm$ . In particular we will detect “good” and “bad” directions for  $W_t^\pm$ , meaning rays in  $H_C$  on which  $W_t^\pm$  stays positive resp. starts to oscillate. It is no loss of generality to treat the case of  $W_t^+$  only.

We start with an explicit formula for the function  $W_t^+$ . Recall that the kernel  $p_t^\lambda$  admits an expansion of the type [?, p. 85]

$$p_t^\lambda(\mathbf{y}, \mathbf{v}) = (2\pi)^{-n} \lambda^n \sum_{k=0}^{\infty} e^{-(2k+n)|\lambda|t} L_k^{n-1}\left(\frac{|\lambda|}{2}(|\mathbf{y}|^2 + |\mathbf{v}|^2)\right) e^{-\frac{|\lambda|}{4}(|\mathbf{y}|^2 + |\mathbf{v}|^2)},$$

where  $L_k^{n-1}$  is the Laguerre polynomial of degree  $k$  with parameter  $n-1$ , which can be extended analytically to  $\lambda + is$  for  $\lambda \neq 0$ . Let  $H_k(x)$  denote the Hermite polynomial, which can be defined by Rodrigue’s formula  $H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$ .

**Proposition 7.1.** *For  $n = 1$  and  $\beta := (y^2 + v^2)$ ,*

$$W_{t/2}^+(iy, iv, i\eta) = c \sqrt{\frac{\pi}{t}} \sum_{k=0}^{\infty} e^{-\frac{1}{4}\mu_k^2} \times \left[ \mu_k \sum_{j=0}^k \frac{1}{j!} \left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\mu_k \sqrt{t}) \binom{k}{j} + \frac{\beta}{t} \sum_{j=0}^{k-1} \frac{1}{j!} \left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\mu_k \sqrt{t}) \binom{k}{j+1} \right]$$

where  $\mu_k = (2k + 1 + (2\eta + \beta)/t)/2$ .

*Proof.* The integral formula of  $W_t^+$  shows that, for a fixed  $\lambda > 0$ ,

$$\begin{aligned} W_{t/2}^+(iy, iv, i\eta) &= c \int_{\mathbb{R}} e^{t(\lambda+is)^2} e^{-2\eta(\lambda+is)} p_t^{\lambda+is}(2y, 2v) ds \\ &= c e^{-(t+2\eta)\lambda+t\lambda^2-\lambda\beta} \sum_{k=0}^{\infty} e^{-2kt\lambda} \int_{\mathbb{R}} (\lambda+is) e^{-ts^2} L_k^{n-1}(2\lambda\beta+2is\beta) \\ &\quad \times e^{-ist(2\lambda-2k-\beta-2\eta/t)} e^{-is\beta} ds \\ &= c e^{-(t+2\eta)\lambda+t\lambda^2-\lambda\beta} \sum_{k=0}^{\infty} e^{-2kt\lambda} (\lambda+\partial_\alpha) L_k^{n-1}(2\beta(\lambda+\partial_\alpha)) \int_{\mathbb{R}} e^{i\alpha s} e^{-ts^2} ds \\ &= c \sqrt{\frac{\pi}{t}} e^{-(t+2\eta)\lambda+t\lambda^2-\lambda\beta} \sum_{k=0}^{\infty} e^{-2kt\lambda} (\lambda+\partial_\alpha) L_k^{n-1}(2\beta(\lambda+\partial_\alpha)) e^{-\frac{1}{4t}\alpha^2} \end{aligned}$$

where  $\alpha = t(2\lambda - 2k - 1 - (2\eta + \beta)/t)$  and  $\partial_\alpha = \partial/\partial\alpha$ .

Using the Rodrigue's formula of the Hermite polynomials and the explicit formula of  $L_k^\gamma$ , we conclude that

$$\begin{aligned} L_k^{n-1}(2\beta(\lambda + \partial_\alpha))e^{-\frac{1}{4t}\alpha^2} &= \sum_{l=0}^k \frac{(-k)_l}{l!l!} (2\beta)^l (\lambda + \partial_\alpha)^l e^{-\frac{1}{4t}\alpha^2} \\ &= \sum_{l=0}^k \frac{(-k)_l}{l!l!} (2\beta)^l \sum_{j=0}^l \binom{l}{j} \frac{1}{(2\sqrt{t})^j} (-1)^j e^{-\frac{1}{4t}\alpha^2} H_j\left(\frac{\alpha}{2\sqrt{t}}\right) \\ &= e^{-\frac{1}{4t}\alpha^2} \sum_{j=0}^k \frac{1}{j!} \frac{\beta}{(\sqrt{t})^j} H_j\left(\frac{\alpha}{2\sqrt{t}}\right) L_{k-j}^j(2\beta\lambda), \end{aligned}$$

upon changing summations, simplifying and using the explicit formula of  $L_k^j$ . Let  $\alpha$  be fixed. It turns out that the generating function of the above quantity is given by

$$e^{-\frac{1}{4t}\alpha^2} \sum_{k=0}^{\infty} \left( \sum_{j=0}^k \frac{1}{j!} \frac{\beta}{(\sqrt{t})^j} H_j\left(\frac{\alpha}{2\sqrt{t}}\right) L_{k-j}^j(2\beta\lambda) \right) s^k = \exp \left[ -\frac{2\beta s \mu}{1-s} - \frac{\beta s}{(1-s)\sqrt{t}} \right]$$

where  $\alpha = 2t(\lambda - \mu)$ . Since the generating function is independent of  $\lambda$ , this shows that the inner sum is in fact independent of  $\lambda$ . We can, in particular, set  $\lambda = 0$  in the inner sum and set  $\mu = (2k + 1 + (2\eta + \beta)/t)/2$ . Recall that  $L_{k-j}^j(0) = \binom{k}{j}$ . The change of variable from  $\alpha$  to  $\mu$  also leads to  $\partial_\alpha = -\frac{1}{2t}\partial_\mu$ . A simple computation then leads to

$$\begin{aligned} (\lambda + \partial_\alpha)L_k^{n-1}(2\beta(\lambda + \partial_\alpha))e^{-\frac{1}{4t}\alpha^2} &= e^{-t(\lambda - \mu)^2} \\ &\times \left[ \mu \sum_{j=0}^k \frac{1}{j!} \left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\sqrt{t}\mu) \binom{k}{j} + \frac{\beta}{t} \sum_{j=0}^{k-1} \frac{1}{j!} \left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\sqrt{t}\mu) \binom{k}{j+1} \right] \end{aligned}$$

from which the stated formula follows readily.  $\square$

We note that the formula proved above shows explicitly that  $W_t^+$  is independent of  $\lambda$  without using the contour integral and Cauchy's theorem.

**Proposition 7.2.** *The function  $W_t^+$  is positive in a neighborhood of  $(0, 0, 0)$ . Furthermore,  $W_t^+(0, 0, i\eta)$  is non-negative for all  $\eta$ .*

*Proof.* Setting  $\beta = 0$  in the explicit formula of  $W_t^+$  gives

$$W_{t/2}^+(0, 0, i\eta) = c \sqrt{\frac{\pi}{t}} \sum_{k=0}^{\infty} e^{-\frac{1}{4}(2k+1+2\eta/t)^2} \left( 2k + 1 + \frac{2\eta}{t} \right),$$

which is clearly positive if  $\eta \geq 0$ . Furthermore, if  $\eta/t = -m$  for  $m \in \mathbb{N}$  then the sum can be written as

$$\sum_{k=0}^{\infty} e^{-\frac{1}{4}(2k+1-2m)^2} (2k + 1 - 2m) = \sum_{k=0}^{\infty} e^{-\frac{1}{4}(2k+1)^2} (2k + 1) - \sum_{k=1}^m e^{-\frac{1}{4}(2k-1)^2} (2k - 1)$$

which is strictly positive. Similarly, the sum is strictly positive if  $\eta/t = -m - 1/2$ . Hence, we are left with the case of  $2\eta/t = -2m - 1 + r$ , where  $0 < r < 1$ . In this

case, the sum becomes

$$S_m := 2 \sum_{k=0}^{\infty} e^{-(k-m+r/2)^2 t} (k-m+r/2) = 2 \sum_{k=0}^{\infty} e^{-(k+s)^2 t} (k+s) - \sum_{k=1}^m e^{-(k-s)^2 t} (k-s)$$

where  $0 < s = r/2 < 1/2$ . Set  $g_k(s) = e^{-(k+s)^2 t} (k+s) - e^{-(k+1-s)^2 t} (k+1-s)$ . It is easy to see that  $g'_k(s) > 0$  for  $0 < s < 1$ . Hence  $g_k$  is increasing. It follows that

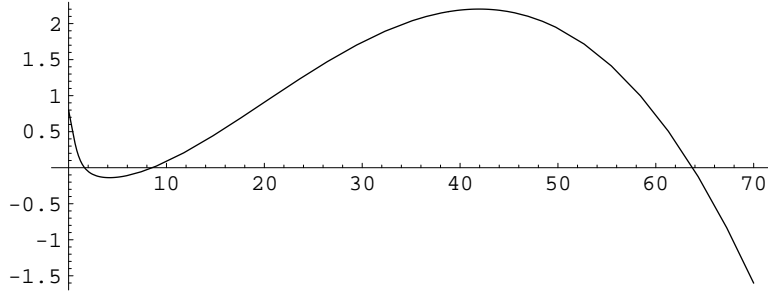
$$\sum_{k=0}^{\infty} e^{-\frac{t}{4}(2k+1)^2} (2k+1) - \sum_{k=1}^{\infty} e^{-\frac{t}{4}(2k-1)^2} (2k-1) = \sum_{k=0}^{\infty} g_k(s) \geq \sum_{k=0}^{\infty} g_k(0) = 0,$$

from which the stated result follows.  $\square$

However, the weight function  $W_t^+(iy, iv, i\eta)$  is not non-negative for all  $(y, v, \eta)$ . In fact, if  $2\eta = -(y^2 + v^2)$ , then  $2\eta + \beta = 0$  and

$$W_{t/2}^+(iy, iv, i\eta) = c \sqrt{\frac{\pi}{t}} \sum_{k=0}^{\infty} e^{-(k+\frac{1}{2})^2 t} \left[ \left(k + \frac{1}{2}\right) \sum_{j=0}^k \frac{1}{j!} \left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\sqrt{t}(k + \frac{1}{2})) \binom{k}{j} + \frac{\beta}{t} \sum_{j=0}^{k-1} \frac{1}{j!} \left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\sqrt{t}(k + \frac{1}{2})) \binom{k}{j+1} \right].$$

For each fixed  $t$ , this is a function of  $\beta$  and it appears to be oscillatory. The graph for  $t = 1$  is shown below.



The function oscillates in growing intervals and increasing amplitudes. To demonstrate the oscillatory nature of the function, what we have shown above is the function  $W(iy, iv, -i\beta/2)/\log(2 + \beta^2)$  without the factor  $c\sqrt{\pi}$ . It is a function of  $\beta$ , where  $\beta = y^2 + v^2$ .

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