Combinatorics and Geometry
of Higher Level Weyl Modules

B. Feigin, A.N. Kirillov, and S. Loktev

ABSTRACT. A higher level analog of Weyl modules over multi-variable currents is proposed. It is shown that the sum of their dual spaces form a commutative algebra. The structure of these modules and the geometry of the projective spectrum of this algebra is studied for the currents of dimension one and two. Along the way we prove some particular cases of the conjectures in [**FL1**] and propose a generalization of the notion of parking function representations.

Introduction

Let us start from some geometrical background. The Borel–Weyl theorem provides a construction of finite–dimensional representations of a simple Lie algebra in spaces of sections of line bundles on the corresponding flag variety. This theorem can be generalized for affine algebras, but the affine flag varieties are infinite–dimensional and it is not so convenient to work with them. One approach is to treat this variety as a limit of finite–dimensional Schubert subvarieties. And the space of sections of that line bundles on these subvarieties are known as *Demazure modules.*

The affine Demazure modules are well studied (see [**KMOTU**], [**FoL**], [**S**]), but there was a lack of construction for such modules, which are not bases on infinite–dimensional representation theory. Now it appears that for the level one case the Demazure modules are isomorphic to the classical Weyl modules introduced in [**CP1**] (see [**CL**] for sl_r case, still a conjecture otherwise). Then the higher level Demazure modules can be constructed in a standard way (See Section 1.1 and Section 2).

Another construction, based on the fusion product introduced in [**FL1**], is related to the tensor product structure as a module over constant currents (see [**CP1**] for Weyl modules and [**FoL**] for Demazure modules). Here we show that in

sl_r case the fusion product of irreducible representations produces the Demazure modules (see [**CL**] for the level one case).

In [**FL2**] an analog of Weyl modules for multi-dimensional currents was introduced. It can be considered as a candidate for a level one Demazure module over the multi–dimensional current algebra. Here we construct higher level Weyl modules that pretend to be Demazure modules of arbitrary level. In particular, their dual spaces form a commutative algebra, whose spectrum can be considered as a multi–dimensional analog of a corresponding Schubert variety.

In the case of double affine (toroidal) algebras things become exciting. Recall that the toroidal algebra is the universal central extension of $\mathfrak{g} \otimes \mathbb{C}[x, x^{-1}, y, y^{-1}]$. An analog of level one integrable representations (and their quantum version) was introduced in [**VV**], [**STU**].

Weyl modules are also studied for that case in [**FL2**], [**FL3**]. For $\mathfrak{g} = sl_r$ its structure was established for the weights, proportional to the weight of vector representation, and a conjecture for other weights was proposed. Since the construction in [**FL3**] is pretty similar to given in [**VV**], [**STU**], we are convinced that the Weyl modules over two-dimensional polynomial ring are isomorphic to $\mathfrak{g} \otimes \mathbb{C}[x, y]$–submodules of that toroidal modules. So the Weyl modules can be considered as an analog of level one Demazure modules also in toroidal settings. Moreover, in [**FL3**] an action of the universal central extension of $\mathfrak{g} \otimes \mathbb{C}[x, x^{-1}, y]$ on the limits of Weyl modules is proposed. We expect that these limits are just the restriction of the corresponding toroidal modules to $\mathfrak{g} \otimes \mathbb{C}[x, x^{-1}, y]$.

Recall that in [**FL3**] we construct $gl_r \otimes \mathbb{C}[x, y]$–modules from cyclic modules over $gl_r \otimes B_N$, where B_N is the associative algebra of upper–triangular $N \times N$ matrices. Note that $gl_r \otimes B_N$ is isomorphic to a parabolic subalgebra $\mathfrak{p} \subset gl_{Nr}$, so we can obtain such representations from the spaces of sections of $\mathfrak{p}$–equivariant bundles. Namely, suppose we have a line bundle on a closure of a $\mathfrak{p}$–orbit. Then the space, dual to its section, is a cyclic representation of $\mathfrak{p}$, so we can produce from it a representation of $gl_r \otimes \mathbb{C}[x, y]$. And our conjecture that the higher level Weyl modules can be obtained in this way from Schubert varieties.

We start Section 1 from some general notion and constructions. Namely, we introduce higher level cyclic modules and the structure of commutative algebra on their dual spaces. Then we discuss the geometry of the spectrum of this algebra and propose some useful examples. At the end of the section we recall the definition of Weyl modules.

In Section 2 we discuss in more detail higher level Weyl modules and fusion modules over one–dimensional currents. Here we relate them each to other as well as to Demazure modules. In sl_r case we construct each higher level Weyl module as a fusion module as well as a Demazure module.

In Section 3 we proceed to two–dimensional currents. Here we relate the higher level Weyl modules to the coordinate ring of the usual Schubert varieties in a Grassmann variety, using the deformation of Weyl modules proposed in [**FL3**]. Also we introduce a higher level generalization of the parking function notion.

1. Generalities

Recall that a module, generated by a single vector, is called a *cyclic* module and this vector is called a *cyclic* vector.

1.1. Higher level modules.

DEFINITION 1.1. Let W be a cyclic module over a Lie algebra $\mathfrak{a}$ generated by a fixed cyclic vector w. Introduce the module $W^{[k]}$ as the submodule of $W^{\otimes k}$ generated by $w^{\otimes k}$

PROPOSITION 1.2. *Suppose W_0 is a quotient of W, fix the image w_0 of w as the cyclic vector in W_0. Then $W_0^{[k]}$ is a quotient of $W^{[k]}$.*

PROOF. We have the natural map of modules $W^{\otimes k} \to W_0^{\otimes k}$ sending $w^{\otimes k}$ to $w_0^{\otimes k}$. Then the image of $W^{[k]}$ is the submodule of $W_0^{\otimes k}$, generated by $w_0^{\otimes k}$, that is $W_0^{[k]}$. $\qquad\square$

Now suppose that the action of $\mathfrak{a}$ on W is extended to an action of a corresponding connected Lie group $\mathcal{A}$.

PROPOSITION 1.3. *We have $W^{[k]}$ is the $\mathcal{A}$-submodule in $W^{\otimes k}$ generated by $w^{\otimes k}$.*

PROOF. Proceeding to the tangent spaces, we obtain that this $\mathcal{A}$-submodule contains $W^{[k]}$. On the other hand, since $\mathcal{A}$ is generated by the image of the exponent map, we have the inverse inclusion. $\qquad\square$

1.2. Multiplication. Note that we have the natural inclusion

$$m_{k_1,k_2} : W^{[k_1+k_2]} \hookrightarrow W^{[k_1]} \otimes W^{[k_2]}$$

of subspaces in $W^{\otimes(k_1+k_2)}$.

PROPOSITION 1.4. *Let $W^{[*]} = \bigoplus_{k \geq 0} \left(W^{[k]}\right)^*$, where $W^{[0]} = \mathbb{C}$. The map*

$$m^* = \bigoplus_{k_1,k_2} m_{k_1,k_2}^* : W^{[*]} \otimes W^{[*]} \to W^{[*]}$$

defines a structure of commutative algebra on $W^{[*]}$.

PROOF. Let $m_{k_1,k_2,k_3} : W^{[k_1+k_2+k_3]} \hookrightarrow W^{[k_1]} \otimes W^{[k_2]} \otimes W^{[k_3]}$ be the similar inclusion. Then we have

$$m_{k_1,k_2+k_3}^*(1 \otimes m_{k_2,k_3}^*) = m_{k_1,k_2,k_3}^* = m_{k_1+k_2,k_3}^*(m_{k_1,k_2}^* \otimes 1).$$

So $m^*(1 \otimes m^*) = m^*(m^* \otimes 1)$, that is associativity. $\qquad\square$

We can also view the algebra $W^{[*]}$ as the coordinate ring of the projective variety $\mathrm{Proj}\left(W^{[*]}\right)$.

Suppose that $\mathfrak{a}$ is the Lie algebra of a connected Lie group $\mathcal{A}$ acting on W. Let $\mathcal{O}(k)$ be the line bundles on $P(W)$ formed by homogeneous polynomials of degree k. Consider the $\mathcal{A}$–orbit of $\mathbb{C}w$ in the projective space $P(W)$. Note that the closure $\mathcal{N}_w$ of this orbit is algebraic.

PROPOSITION 1.5. *We have $\mathrm{Proj}\left(W^{[*]}\right) \cong \mathcal{N}_w$ and $W^{[k]} \cong \Gamma(\mathcal{N}_w, \mathcal{O}(k))^*$ as $\mathcal{A}$–modules.*

PROOF. First we have the natural isomorphism $ev : S^k(W) \to \Gamma(P(W), \mathcal{O}(k))^*$. Here $ev(u^{\otimes k})$ evaluate the sections at the point $\mathbb{C}u \in P(W)$, and $S^k(W)$ is spanned by such vectors.

The space $\Gamma(\mathcal{N}_w, \mathcal{O}(k))$ is the quotient of $\Gamma(P(W), \mathcal{O}(k))$ by the subspace of sections vanishing at $\mathcal{N}_w$. On the other hand, taking into account Proposition 1.3, we have $W^{[k]}$ is the subspace of $S^k(W)$ spanned by $u^{\otimes k}$ with $\mathbb{C}u \in \mathcal{N}_w$ (we can proceed to the closure because any linear subspace is closed). So they are dual each to other.

Concerning the multiplication, due to the action of $\mathfrak{a}$ the maps m_{k_1, k_2} are uniquely defined by the image of $w^{\otimes k}$. And for $\gamma_1 \in \mathcal{O}(k_1)$, $\gamma_2 \in \mathcal{O}(k_2)$ we have $ev(w^{\otimes k})(\gamma_1 \cdot \gamma_2) = ev(w^{\otimes k_1})(\gamma_1)\, ev(w^{\otimes k_2})(\gamma_2)$, so we have an isomorphism of algebras. $\square$

1.3. Examples.

EXAMPLE 1.6. Let $\mathfrak{a} = \mathfrak{g}$ be a simple Lie algebra, let $V(\lambda)$ be the irreducible finite-dimensional representation with highest weight λ. Fix the highest weight vector in $V(\lambda)$ as the cyclic one. Then $V(\lambda)^{[k]} \cong V(k\lambda)$ and $\mathrm{Proj}\left(V(\lambda)^{[*]}\right)$ is known as a *generalized flag variety* of the corresponding group G (that is the usual flag variety if λ is big enough).

EXAMPLE 1.7. Let $\mathfrak{a} = \mathfrak{b} \subset \mathfrak{g}$ be the Borel subalgebra (that stabilizes the highest weight vector). For an element $\mathbf{w}$ of the Weyl group introduce the *extremal vector* $v_{\mathbf{w}} \in V(\lambda)$ as the vector with the weight obtained from the highest weight by the action of $\mathbf{w}$. This vector is defined uniquely up to multiplication by a scalar.

Take $v_{\mathbf{w}}$ as the cyclic vector of $V_{v_{\mathbf{w}}}(\lambda) = U(\mathfrak{b})v_{\mathbf{w}} \subset V(\lambda)$. Then we have $V_{v_{\mathbf{w}}}(\lambda)^{[k]} = V_{v_{\mathbf{w}}}(k\lambda)$ and $\mathrm{Proj}\left(V_{v_{\mathbf{w}}}(\lambda)^{[*]}\right)$ is known as the *Schubert subvariety* of the generalized flag variety.

Note that these varieties are well–defined and the same situation holds for affine Lie algebras.

EXAMPLE 1.8. Let $\mathfrak{g} = \mathfrak{gl}_r$. Take $\lambda = \omega_n$, that is the n-th fundamental weight, so $V(\lambda) = \wedge^n V$, where V is the r–dimensional vector representation of $\mathfrak{gl}_r$. Then $\mathrm{Proj}\left(V(\lambda)^{[*]}\right)$ is the Grassmann variety $\mathrm{Gr}(n, r)$ formed by n–dimensional planes in $\mathbb{C}^r$. The restriction of $\mathcal{O}(1)$ to $\mathrm{Gr}(n, r)$ is dual to the determinant bundle with stalks $\wedge^n P$ over each plane P.

Choose a basis $v_1, \ldots, v_r$ in V, then fix the Borel subalgebra $\mathfrak{b}$ of upper–triangular matrices mapping each v_i to a linear combination of v_j with $j \leq i$. Then extremal vectors in $\wedge^n V$ are just monomials of the form $v_\eta = v_{\eta_1} \wedge \cdots \wedge v_{\eta_n}$, $\eta_1 < \cdots < \eta_n$. The corresponding Schubert subvariety $Sh_\eta = \mathrm{Proj}\left(V_{v_\eta}(\omega_n)^{[*]}\right)$ consists of planes, whose intersection with $\langle v_1, v_2, \ldots v_{\eta_i} \rangle$ has dimension at least i.

1.4. Cyclic adjoint module.
Suppose we have an increasing filtration on the Lie algebra $\mathfrak{a}$: $F^0\mathfrak{a} \subset F^1\mathfrak{a} \subset \ldots$. Then it can be extended to the filtration $F^0 U(\mathfrak{a}) \subset F^1 U(\mathfrak{a}) \subset \ldots$.

For a cyclic module W introduce the filtration and the adjoint graded space

$$F_C^i W = (F^i U(\mathfrak{a}))w, \qquad \mathrm{gr}_C W = \bigoplus_i F_C^i W / F_C^{i-1} W.$$

Then $\mathrm{gr}_C W$ is a module over $\mathrm{gr} A$.

For $u \in W$ by $\bar{u}$ denote the corresponding vector in $\mathrm{gr}_C W$. Fix $\overline{w}$ as a cyclic vector in $\mathrm{gr}_C W$.

PROPOSITION 1.9. *We have* $(\mathrm{gr}_C W)^{[k]}$ *is a quotient of* $\mathrm{gr}_C W^{[k]}$.

PROOF. First let us construct a map
$$\rho : \mathrm{gr}_C W^{[k]} \to (\mathrm{gr}_C W)^{\otimes k}.$$
We have
$$F_C^i W^{[k]} \subset \sum_{i_1 + \cdots + i_n = i} F_C^{i_1} W \otimes \cdots \otimes F_C^{i_n} W.$$
As
$$\left(F_C^{i_1} W \otimes \cdots \otimes F_C^{i_n} W \right) \cap \left(F_C^{i_1'} W \otimes \cdots \otimes F_C^{i_n'} W \right) = F_C^{\min(i_1, i_1')} W \otimes \cdots \otimes F_C^{\min(i_n, i_n')} W,$$
it gives a map
$$F_C^i W^{[k]} \to \bigoplus_{i_1 + \cdots + i_n = i} \mathrm{gr}_C^{i_1} W \otimes \cdots \otimes \mathrm{gr}_C^{i_n} W,$$
where $\mathrm{gr}_C^i W = F_C^i W / F_C^{i-1} W$. As the image of $F_C^{i-1} W^{[k]}$ under this map is zero, we obtain the map ρ.

Note that $\rho(\overline{w^{\otimes k}}) = \overline{w}^{\otimes k}$, so the image of ρ is $(\mathrm{gr}_C W)^{[k]}$ and we have the statement of the proposition. $\square$

1.5. Weyl modules. Let $\mathfrak{g}$ be a reductive Lie algebra. Choose a Cartan and a Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. In this paper we consider mainly $\mathfrak{a} = \mathfrak{g} \otimes \mathbb{C}[x^1, \ldots, x^d]$ and the following class of modules.

DEFINITION 1.10. For a weight $\lambda : \mathfrak{b} \to \mathfrak{h} \to \mathbb{C}$ let $W^d(\lambda)$ be the maximal finite-dimensional module over $\mathfrak{g} \otimes \mathbb{C}[x^1, \ldots, x^d]$ generated by w_λ such that

(1.1) $$(\mathfrak{g} \otimes P) w_\lambda = \lambda(g) P(0) w_\lambda \quad \text{for } g \in \mathfrak{b}.$$

By *maximal* we mean that any finite-dimensional module generated by w_λ is a quotient of $W^d(\lambda)$.

In [**FL2**] it is shown that $W^d(\lambda)$ exists and that it is non-trivial for a dominant λ. Also it is shown there that $W^d(\lambda)$ is graded as $\mathfrak{g} \otimes \mathbb{C}[x^1, \ldots, x^d]$–module, that is we have
$$W^d(\lambda) = \bigoplus_{i_1, \ldots, i_d \geq 0} W^d(\lambda)^{i_1, \ldots, i_d},$$
where $W^d(\lambda)^{i_1, \ldots, i_d}$ are $\mathfrak{g} \otimes 1$–modules and $\mathfrak{g} \otimes (x^1)^{j_1} \ldots (x^d)^{j_d}$ acts from $W^d(\lambda)^{i_1, \ldots, i_d}$ to $W^d(\lambda)^{i_1 + j_1, \ldots, i_d + j_d}$.

This grading can be extended in the usual way to the grading on the tensor product and therefore on the higher level Weyl modules.

Finally, let us introduce the notation for graded character of any graded module W by
$$\mathrm{ch}_d W = \sum_{i_1, \ldots, i_d \geq 0} t_1^{i_1} \ldots t_d^{i_d} \cdot \mathrm{ch} W^{i_1, \ldots, i_d},$$
where ch denotes the usual character of $\mathfrak{g}$–modules.

2. One-dimensional case

In this section $\mathfrak{g}$ denotes a semi–simple Lie algebra. Let R denotes the set of roots and Q denotes the root lattice of $\mathfrak{g}$. Let $\omega_1, \ldots \omega_r$ be the fundamental weights.

2.1. Demazure modules. Let $\widehat{\mathfrak{g}}$ be the central extension of $\mathfrak{g} \otimes \mathbb{C}[x^{-1}, x]$. Note that the restriction of the central extension to $\mathfrak{g} \otimes \mathbb{C}[x]$ is trivial, so we have $\mathfrak{g} \otimes \mathbb{C}[x] \subset \widehat{\mathfrak{g}}$.

Let L_k be the integrable level k vacuum representation of $\widehat{\mathfrak{g}}$, let $w_k \in L_k$ be the highest weight vector.

For a weight λ let $\iota_\lambda : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ be the automorphism mapping $g \otimes x^n$ to $g \otimes x^{n+\lambda(\alpha)}$ when g belongs to the root space $\mathfrak{g}_\alpha$, $\alpha \in R$. Note that it coincides with the action of the corresponding translation from the extended Weyl group. Let $\mathfrak{g}[x] = \mathfrak{g} \otimes \mathbb{C}[x]$ and let $\mathfrak{g}[x]_\lambda = \iota_\lambda(\mathfrak{g} \otimes \mathbb{C}[x])$. In other words, we have

$$\mathfrak{g}[x]_\lambda = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \otimes x^{\lambda(\alpha)} \mathbb{C}[x] \subset \widehat{\mathfrak{g}}.$$

DEFINITION 2.1. Introduce the *Demazure module* over $\mathfrak{g}[x]$ by

$$D(k, \lambda) = \iota_\lambda^* \left(U(\mathfrak{g}[x]_\lambda) w_k \right).$$

Namely, it is isomorphic to $U(\mathfrak{g}[x]_\lambda) w_k$ as the vector space, where $\mathfrak{g}[x]$ acts via identification ι_λ with $\mathfrak{g}[x]_\lambda$.

PROPOSITION 2.2. *We have* $D(1, \lambda)^{[k]} \cong D(k, \lambda)$.

PROOF. It follows from $L_1^{[k]} \cong L_k$. And this is because $L_1^{[k]}$ is integrable and generated by the highest weight vector with the corresponding highest weight. □

COROLLARY 2.3. *(see also* [**FF3**]*) Suppose* $\lambda \in Q$. *Then* $\mathrm{Proj}\left(D(1, \lambda)^{[*]}\right)$ *is a Schubert cell in the affine Grassmann variety for* $\widehat{\mathfrak{g}}$.

Concerning the dimension, one can use the following result.

THEOREM 2.4. [**FoL**] *We have*

$$\dim D(k, \lambda) = \prod_{i=1}^r \left(\dim D(k, \omega_i) \right)^{\lambda_i}.$$

PROPOSITION 2.5. *We have* $D(k, \lambda)$ *is a quotient of* $W^1(\lambda)^{[k]}$.

PROOF. The module $D(1, \lambda)$ is finite–dimensional and the image of w_1 in $D(1, \lambda)$ satisfies (1.1), so $D(1, \lambda)$ is a quotient of $W^1(\lambda)$.

As $D(k, \lambda) \cong D(1, \lambda)^{[k]}$, the proposition follows from Proposition 1.2. □

CONJECTURE 2.6. *(see* [**CP2**]*) For a dominant* λ *we have*

$$\dim D(1, \lambda) = \dim W^1(\lambda).$$

This conjecture is already proved in sl_r case ([**CL**]). Summarizing the statements above, we have the following description of the higher level Weyl modules.

PROPOSITION 2.7. *In the case when Conjecture 2.6 holds we have* $W^1(\lambda)^{[k]} \cong D(k, \lambda)$, *therefore for* $\lambda \in Q$ *we have* $\mathrm{Proj}\left(W^1(\lambda)^{[*]}\right)$ *is a Schubert cell in the affine Grassmann variety.*

2.2. Fusion. For $z \in \mathbb{C}^d$ let $\varphi(z)$ be the automorphism of $\mathfrak{g} \otimes \mathbb{C}[x]$ sending x to $x + z$. For a module W over $\mathfrak{g} \otimes \mathbb{C}[x]$ define the shifted module $W(z) = \varphi(z)^*W$, so $W(z) \cong W$ as a vector space and the action is combined with $\varphi(z)$.

Let $W_1, \ldots, W_n$ be cyclic modules over $\mathfrak{g} \otimes \mathbb{C}[x]$ such that for a certain N the subalgebra $\mathfrak{g} \otimes x^N \mathbb{C}[x]$ acts on each W_i by zero. By $w_1, \ldots, w_n$ denote their cyclic vectors.

PROPOSITION 2.8. [**FL1**] *If z_i are pairwise distinct then the vector $w_1 \otimes \cdots \otimes w_n$ is cyclic in $W_1(z_1) \otimes \cdots \otimes W_n(z_n)$.*

PROOF. Let

$$\mathfrak{g}[N(z_1, \ldots, z_n)] = \mathfrak{g} \otimes \left(\mathbb{C}[x]/(x - z_1)^N \ldots (x - z_n)^N \mathbb{C}[x] \right).$$

As the ideal $\mathfrak{g} \otimes (x - z_1)^N \ldots (x - z_n)^N \mathbb{C}[x]$ acts on the tensor produce by zero, $W_1(z_1) \otimes \cdots \otimes W_n(z_n)$ is a module over $\mathfrak{g}[N(z_1, \ldots, z_n)]$.

Next note that we have the natural projections $p_i : \mathfrak{g}[N(z_1, \ldots, z_n)] \to \mathfrak{g}[N(z_i)]$. Then it is known that the direct sum

$$\oplus p_i : \mathfrak{g}[N(z_1, \ldots, z_n)] \to \bigoplus_{i=1}^{n} \mathfrak{g}[N(z_i)]$$

is an isomorphism. Note that the preimage of each $\mathfrak{g}[N(z_i)]$ belongs to the ideal $\mathfrak{g} \otimes (x - z_1)^N \ldots (x - z_{i-1})^N (x - z_{i+1})^N \ldots (x - z_n)^N \mathbb{C}[x]$. So we have

(2.1) $U(\mathfrak{g}[N(z_1, \ldots, z_n)])(w_1 \otimes \cdots \otimes w_n) = (U(\mathfrak{g}[N(z_1)]) \, w_1 \otimes \cdots \otimes (U(\mathfrak{g}[N(z_n)]) \, w_n$

that is equal to $W_1(z_1) \otimes \cdots \otimes W_n(z_n)$. $\qquad\square$

Introduce the filtration on $\mathfrak{g} \otimes \mathbb{C}[x]$ such that $F^i(\mathfrak{g} \otimes \mathbb{C}[x])$ consists of $\mathfrak{g}$-valued polynomials whose degree does not exceed i. Then $\mathrm{gr}(\mathfrak{g} \otimes \mathbb{C}[x]) = \mathfrak{g} \otimes \mathbb{C}[x]$, so we can produce a graded module from any cyclic module.

DEFINITION 2.9. [**FL1**] For given $z_1, \ldots, z_n$ introduce the $\mathfrak{g} \otimes \mathbb{C}[x]$-module

$$W_1 * \cdots * W_n(z_1, \ldots, z_n) = \mathrm{gr}_C(W_1(z_1) \otimes \cdots \otimes W_n(z_n)).$$

We call it the *fusion module*.

PROPOSITION 2.10. *For any $\mathfrak{g} \otimes \mathbb{C}[x]$–modules $W_1, \ldots, W_n$ we have the module $(W_1 * \cdots * W_n(z_1, \ldots, z_n))^{[k]}$ is a quotient of $W_1^{[k]} * \cdots * W_n^{[k]}(z_1, \ldots, z_n)$.*

PROOF. Note that due to the isomorphism (2.1) we have

$$(W_1(z_1) \otimes \cdots \otimes W_n(z_n))^{[k]} \cong W_1^{[k]}(z_1) \otimes \cdots \otimes W_n^{[k]}(z_n).$$

Then the statement follows from Proposition 1.9. $\qquad\square$

For any $\mathfrak{g}$–module V introduce the evaluation $\mathfrak{g} \otimes \mathbb{C}[x]$–module $V[0]$, isomorphic to V as a vector space, where $\mathfrak{g} \otimes x\mathbb{C}[x]$ acts by zero and $\mathfrak{g} \otimes 1$ acts as $\mathfrak{g}$ on V. For a set of $\mathfrak{g}$–modules $V_1, \ldots, V_n$ let us set

$$V_1 * \cdots * V_n(z_1, \ldots, z_n) = V_1[0] * \cdots * V_n[0](z_1, \ldots, z_n).$$

CONJECTURE 2.11. *For any simple $\mathfrak{g}$ and any weights $\lambda^1, \ldots, \lambda^n$ we have*

$$\left(V(\lambda^1) * \cdots * V(\lambda^n)(z_1, \ldots, z_n) \right)^{[k]} \cong V(k\lambda^1) * \cdots * V(k\lambda^n)(z_1, \ldots, z_n).$$

Note that $V(k\lambda) \cong V(\lambda)^{[k]}$, so by Proposition 2.10 the left hand side in this Conjecture is a quotient of the right hand side.

2.3. Fusion of Weyl modules.

PROPOSITION 2.12. *For any $\lambda^1, \ldots, \lambda^n$ we have $W^1(\lambda^1) * \cdots * W^1(\lambda^n)(z_1, \ldots, z_n)$ is a quotient of $W^1(\lambda^1 + \cdots + \lambda^n)$.*

PROOF. The cyclic vector of the left hand side satisfies (1.1). □

Together with Proposition 1.9 it motivates the following conjecture.

CONJECTURE 2.13. *We have*
$$W^1(\lambda^1)^{[k]} * \cdots * W^1(\lambda^n)^{[k]}(z_1, \ldots, z_n) \cong W^1(\lambda^1 + \cdots + \lambda^n)^{[k]},$$
in particular, the left hand side is independent on $(z_1, \ldots, z_n)$.

THEOREM 2.14. *Conjecture 2.6 implies Conjecture 2.13.*

PROOF. In the case $k = 1$ Conjecture 2.6 together with Theorem 2.4 implies the equality of dimensions, so this case of Conjecture 2.13 follows from Proposition 2.12.

Note that by Proposition 2.10 the module $W^1(\lambda^1)^{[k]} * \cdots * W^1(\lambda^s)^{[k]}(z_1, \ldots, z_s)$ has the quotient
$$W^1(\lambda^1) * \cdots * W^1(\lambda^s)(z_1, \ldots, z_s)^{[k]} \cong D(1, \lambda^1 + \cdots + \lambda^s)^{[k]}$$
which is isomorphic to $D(k, \lambda^1 + \cdots + \lambda^s)$ as well as to $W^1(\lambda^1 + \cdots + \lambda^s)^{[k]}$.

By Theorem 2.4 we have
$$\dim D(k, \lambda^1 + \cdots + \lambda^s) = \dim D(k, \lambda^1) \cdots \dim D(k, \lambda^s),$$
that by assumption is equal to $\dim W^1(\lambda^1)^{[k]} * \cdots * W^1(\lambda^s)^{[k]}(z_1, \ldots, z_s)$. So this quotient is the whole space and we have the isomorphism proposed in Conjecture 2.13. □

2.4. gl_r case. Now suppose $\mathfrak{g} = \mathfrak{gl}_r$.

THEOREM 2.15. [**CL**] *For $\mathfrak{g} = \mathfrak{gl}_r$ we have $W^1(\lambda) \cong D(1, \lambda)$.*

So Conjecture 2.6 and therefore Conjecture 2.13 is already proved in this case.

Note that for $\mathfrak{g} = \mathfrak{gl}_r$ we have $W(\omega_i) \cong V(\omega_i)[0]$, where the right hand side is the evaluation representation defined above.

COROLLARY 2.16. *We have*
$$V(k\omega_1)^{*\lambda_1} * \cdots * V(k\omega_{r-1})^{*\lambda_{r-1}}(z_1, \ldots, z_{|\lambda|}) \cong D(k, \lambda),$$
in particular, the left hand side does not depend on $z_1, \ldots, z_n$.

So we proved a substantial case of the Conjecture 1.8 of [**FL1**]. And we can deduce the similar case of Conjecture 2.11.

COROLLARY 2.17. *For $1 \leq i_1, \ldots, i_n \leq r - 1$ we have*
$$(V(l\omega_{i_1}) * \cdots * V(l\omega_{i_n})(z_1, \ldots, z_n))^{[k]} \cong V(kl\omega_{i_1}) * \cdots * V(kl\omega_{i_n})(z_1, \ldots, z_n).$$

REMARK 2.18. It seems that Conjecture 2.11 can be proved for the set of weights $l_1\omega_{i_1}, \ldots, l_n\omega_{i_n}$ using the methods of [**FF2**], where fusion product embedded into a direct sum of integrable modules. Then the corresponding variety $\text{Proj}\left(V(l_1\omega_{i_1}) * \cdots * V(l_n\omega_{i_n})(z_1, \ldots, z_n)^{[*]}\right)$ coincides with the *generalized Schubert varieties* introduced and described for $\mathfrak{g} = sl_2$ in [**FF3**], [**F**].

REMARK 2.19. Note that due to the result of [**S**] on Demazure modules we also have a formula for the graded character $\text{ch}_1 V(k\omega_1)^{*\lambda_1} * \cdots * V(k\omega_{r-1})^{*\lambda_{r-1}}$ in terms of parabolic Kostka polynomials as expected in [**FL1**].

3. Two-dimensional case

Here let us consider the case $d = 2$ and $\mathfrak{g} = \mathfrak{gl}_r$. Here we use the partition notation for weights of $\mathfrak{gl}_r$.

3.1. Deformation of Weyl modules.

Let us recall a construction from [**FL3**].

The Lie algebra $\mathfrak{gl}_r \otimes \mathbb{C}[x^1, x^2]$ can be deformed into the Lie algebra $\mathfrak{gl}_r \otimes \mathbb{C}\langle X, Y \rangle$, where $\mathbb{C}\langle X, Y \rangle$ is the associative algebra, generated by X and Y under the relation $YX - XY = X$. The algebra $\mathfrak{gl}_r \otimes \mathbb{C}\langle X, Y \rangle$ has the natural representation in $\mathbb{C}[t, t^{-1}]$, where X acts by multiplication on t and Y acts as $t\partial/\partial t$.

By V denote the r–dimensional vector representation of $\mathfrak{gl}_r$. Let $v_1, \ldots, v_r$ be the standard basis vectors in V. For a partition ξ introduce the $\mathfrak{gl}_r \otimes \mathbb{C}[x^1, x^2]$–module V_ξ as the submodule of

$$\bigwedge^{|\xi|} \left(V \otimes \mathbb{C}[t, t^{-1}]/\mathbb{C}[t] \right), \quad \text{generated by} \quad v_\xi = \bigwedge_{i=1}^{r} \bigwedge_{j=1}^{\xi_i} v_i \otimes t^{-j}.$$

THEOREM 3.1. [**FL3**] *There is a filtration $F^i \mathfrak{gl}_r \otimes \mathbb{C}\langle X, Y \rangle$ such that $\mathrm{gr}_C V_\xi$ is a quotient of $W^2(\xi)$. Moreover, if $\xi = (n)$ then $\mathrm{gr}_C V_\xi \cong W^2(\xi)$.*

CONJECTURE 3.2. *We have $W^2(\xi)^{[k]} \cong \mathrm{gr}_C \left(V_\xi^{[k]} \right)$.*

Note that for $\xi = (n)$ we know by Proposition 1.9 that the left hand side is a quotient of the right hand side.

3.2. Relation to Schubert cells.

Let us enumerate the basis vectors of $V \otimes \mathbb{C}[t^{-1}, t]$ as follows. Denote $v_i \otimes t^{-j}$ by u_{rj-i+1}.

For a partition $\xi = \xi_1 \geq \cdots \geq \xi_r \geq 0$ introduce $\eta(\xi)$ as the ordered set of numbers $\eta_1 < \eta_2 < \cdots < \eta_n$ equal to $lr - s$ with $0 \leq s < \xi_l^t$, $l = 1, 2, \ldots$, where ξ^t is the transposed partition. So

$$v_\xi = \bigwedge_{i \in \eta(\xi)} u_i.$$

Then our modules are related to Example 1.8 in the following way.

PROPOSITION 3.3. *We have $V_\xi^{[k]} \cong \Gamma(\mathrm{Sh}_{\eta(\xi)}, \mathcal{O}(k))^*$ and $\mathrm{Proj}\left(V_\xi^{[*]} \right) \cong \mathrm{Sh}_{\eta(\xi)}$.*

PROOF. Note that V_ξ is indeed a submodule of $\bigwedge^{|\xi|} U$, where

$$U = V \otimes \left(t^{-N} \mathbb{C}[t]/\mathbb{C}[t] \right), \qquad N = \xi_1,$$

or, in other words, $U = \langle u_i \rangle_{i=1\ldots Nr}$.

The action of $\mathfrak{gl}_r \otimes \mathbb{C}\langle X, Y \rangle$ on U defines a map from $\mathfrak{gl}_r \otimes \mathbb{C}\langle X, Y \rangle$ to $\mathrm{End}(U) = \mathfrak{gl}_{Nr}$. It is shown in [**FL3**] that the image of this map is the "block upper-triangular" Lie subalgebra $\mathfrak{p}$ mapping each u_i to a linear combination of u_j with the integer part of $(j-1)/r$ not exceeding the integer part of $(i-1)/r$. Then $V_\xi^{[k]}$ is the $\mathfrak{p}$–submodule of $\bigwedge^{|\xi|} U$, generated by $v_\xi^{\otimes k}$.

Let $\mathfrak{b} \subset \mathrm{End}(U)$ be the Lie algebra of upper-triangular matrices, that is endomorphisms mapping each u_i to a linear combination of u_j with $j \leq i$. Consider the projection $p^+ : \mathfrak{p} \to \mathfrak{b}$ along the strictly lower–triangular matrices. Since for any $g \in \mathfrak{p}$ we have $g v_\xi^{\otimes k} = p^+(g) v_\xi^{\otimes k}$, the subspace $V_\xi^{[k]}$ is indeed the $\mathfrak{b}$–submodule generated by $v_\xi^{\otimes k}$.

So we are in the situation of Example 1.8.   □

For a matrix $A = (a_{ij})$, $i = 1 \ldots k$, $j = 1 \ldots n$, of positive integers introduce the functional u_A^* on $V_\xi^{[k]}$ by

$$u_A^* = \bigotimes_{i=1}^{k} \bigwedge_{j=1}^{n} u_{a_{ij}}^*.$$

COROLLARY 3.4. [**HP**] *Let*

$$\mathcal{M}_\eta^{[k]} = \{A = (a_{ij}) \mid a_{i1} < \cdots < a_{in}, \qquad 1 \le a_{1j} \le \cdots \le a_{kj} \le \eta_j\}.$$

Then elements u_A^* with $A \in \mathcal{M}_{\eta(\xi)}^{[k]}$ form a basis in $\left(V_\xi^{[k]}\right)^*$.

REMARK 3.5. Another way to describe the set $\mathcal{M}_\eta^{[k]}$ is the notion of *plane partitions*. Recall that a plane partition of shape λ is a filling of the diagram of partition λ by non-negative integers weakly increasing along rows and columns. Let $PP^\lambda(k)$ denotes the set of such plane partitions, filled by integers not exceeding k, and let $pp^\lambda(k)$ denotes their number.

Then we have the following bijection between $PP^\lambda(k)$ and $\mathcal{M}_\eta^{[k]}$, where $\lambda = (\eta_n - n \ge \cdots \ge \eta_1 - 1)$. For any plane partition from $PP^\lambda(k)$ let $a_{ij} - j$ be the number of integers in $n - j + 1$-th row less than i. Then the set (a_{ij}) belongs to $\mathcal{M}_\eta^{[k]}$ and one can see that it gives us a bijection.

REMARK 3.6. Note that Conjecture 3.2 implies that $\mathrm{Proj}\left(W^2(\xi)^{[*]}\right)$ is a degeneration of the Schubert variety $Sh_{\eta(\xi)}$.

3.3. Module structure. First let us calculate the character of $V_\xi^{[k]}$ as $\mathfrak{gl}_r{}^-$ module.

PROPOSITION 3.7. *We have*

$$\mathrm{ch} V_\xi^{[k]} = \sum_{A \in \mathcal{M}_{\eta(\xi)}^{[k]}} \prod_{i,j} x_{a_{ij} \bmod r},$$

where $a \bmod r$ takes values from 1 to r.

Let $\mathcal{F}_n^r$ be the map from the Grothendiek ring of representations of the symmetric group Σ_n to the Grothendiek ring of $\mathfrak{gl}_r$–modules defined by

$$\mathcal{F}_n^r(\pi) = \left(V^{\otimes n} \otimes \pi\right)^{\Sigma_n}.$$

In other word, it maps the representation of Σ_n corresponding to a partition ξ to the $\mathfrak{gl}_r$–module corresponding to the same partition if $\xi_{r+1} = 0$ and to zero otherwise.

Recall the notion of skew Schur function:

$$s_{\lambda \backslash \mu} = \sum_{\substack{k_{ij} \ge 1, \, (i,j) \in \lambda \backslash \mu \\ k_{ij} \le k_{i+1,j}; \, k_{ij} < k_{i,j+1}}} \prod_{(i,j) \in \lambda \backslash \mu} x_{k_{ij}}.$$

and the representation of $\Sigma_{|\lambda| - |\mu|}$

$$\pi_{\lambda \backslash \mu} = \sum C_{\lambda\mu}^\nu \pi_\nu,$$

where $C_{\lambda\mu}^\nu$ are Littlewood–Richardson coefficients. The representation $\pi_{\lambda \backslash \mu}$ corresponds to the symmetric function $s_{\lambda \backslash \mu}$ as follows.

PROPOSITION 3.8. [**M**] *We have*

$$\mathrm{ch}\mathcal{F}^r_{|\lambda|-|\mu|}(\pi_{\lambda\backslash\mu}) = s_{\lambda\backslash\mu}(x_1,\dots,x_r,0,0,\dots).$$

At last for representations π_1 of Σ_{n_1} and π_2 of Σ_{n_2} introduce the outer product

$$\pi_1 \odot \pi_2 = \mathrm{Ind}^{\Sigma_{n_1+n_2}}_{\Sigma_{n_1}\times\Sigma_{n_2}} \pi_1 \boxtimes \pi_2.$$

Note that we have $\mathcal{F}^r_n(\pi_1 \odot \pi_2) = \mathcal{F}^r_n(\pi_1) \otimes \mathcal{F}^r_n(\pi_2)$.

For our purpose introduce the following representation.

DEFINITION 3.9. The *Higher Parking Functions* representation of Σ_{kn} is given by

$$\mathbb{CPF}^{[\mathrm{k}]}_{\mathrm{Sign}}(\xi) = \bigoplus_{\substack{\emptyset=\lambda^0\subset\lambda^1\subset\cdots\subset\lambda^n=k^n \\ \lambda^s\supset k^{\xi^t_1+\cdots+\xi^t_s}}} \overset{n}{\underset{s=1}{\odot}} \pi_{\lambda^s\backslash\lambda^{s-1}},$$

where ξ^t is the transposed partition and $k^m = (k \geq \cdots \geq k)$, where m is the number of entrees.

REMARK 3.10. For $k=1$ this representation is the tensor product of the sign representation and the representation in ρ–parking functions introduced in [**PSt**] for $\rho = (1^{\xi^t_1}2^{\xi^t_2}\dots)$, that is the partition where each j appears ξ^t_j times, and ξ^t is the transposed partition.

PROPOSITION 3.11. *We have* $V^{[k]}_\xi \cong \mathcal{F}^r_{kn}\left(\mathbb{CPF}^{[\mathrm{k}]}_{\mathrm{Sign}}(\xi)\right)$ *as* $\mathfrak{gl}_r$*-modules.*

PROOF. It is enough to compare the characters.

For $\Lambda = (\emptyset = \lambda^0 \subset \lambda^1 \subset \lambda^2 \subset \cdots \subset \lambda^n = k^n)$ introduce the set $\mathcal{M}(\Lambda)$ consisting of matrices (a_{ij}) such that

$$a_{i1} < \cdots < a_{in}, \qquad a_{1j} \leq \cdots \leq a_{kj}, \qquad r(s-1) < a_{ij} \leq rs \text{ for } (i,j) \in \lambda^s\backslash\lambda^{s-1}.$$

Then $\mathcal{M}^{[k]}_{\eta(\xi)}$ is union of $\mathcal{M}(\Lambda)$ for all Λ satisfying $\lambda^s \supset k^{\xi^t_1+\cdots+\xi^t_s}$.

Next note that

$$\sum_{A\in\mathcal{M}(\Lambda)} \prod_{i,j} x_{a_{ij}\,\mathrm{mod}\,r} = \prod_{s=1}^n \sum_{\substack{r\geq k_{ij}\geq 1,\, (i,j)\in\lambda^s\backslash\lambda^{s-1} \\ k_{ij}\leq k_{i+1,j};\, k_{ij}<k_{i,j+1}}} \prod_{(i,j)\in\lambda^s\backslash\lambda^{s-1}} x_{k_{ij}}$$

by setting $k_{ij} = a_{ij}\,\mathrm{mod}\,r$. And the right hand side is equal to

$$\prod_{s=1}^n s_{\lambda^s\backslash\lambda^{s-1}}(x_1,\dots,x_r,0,0,\dots) = \mathrm{ch}\mathcal{F}^r_{rn}\left(\overset{n}{\underset{s=1}{\odot}} \pi_{\lambda^s\backslash\lambda^{s-1}}\right)$$

□

REMARK 3.12. The module $W^2(\xi)^{[k]}$ is bi-graded and one of these grading remains in $V^{[k]}_\xi$ as the grading by degree of t. This grading can be also viewed in $\mathbb{CPF}^{[\mathrm{k}]}_{\mathrm{Sign}}(\xi)$ by fixing $|\lambda^1| + \cdots + |\lambda^n|$ in the direct sum.

3.4. Dimension formula. The dimension of $\Gamma(\mathrm{Sh}_\eta, \mathcal{O}(k))$ is given by the following *Hodge postulation formula.*

THEOREM 3.13. [**HP**][**St1**] *For $\eta = (\eta_1 < \cdots < \eta_n)$ we have*

$$\dim \Gamma(\mathrm{Sh}_\eta, \mathcal{O}(k)) = \mathrm{Det}\left(\binom{\eta_j + k - j}{i + k - j}\right)_{1 \le i,j \le n}.$$

For the case $\xi = (n)$, that is $\eta = (r, 2r, \ldots, nr)$ let us deduce an explicit formula. Note that for $k = 1$ it gives the higher Catalan number $C_n^{(r)}$ (see e.g. [**FL3**])

$$\dim V_{(n)} = C_n^{(r)} := \frac{1}{n+1}\binom{r(n+1)}{n}.$$

THEOREM 3.14. *We have*

$$(3.1) \qquad \dim V_{(n)}^{[k]} = \prod_{j=1}^n \frac{(jr + k - j)!}{(jr - 1)!} \cdot \frac{(kr + jr - 1)!}{(kr + jr - j)!} \cdot \frac{(j-1)!}{(k+j-1)!}.$$

PROOF. Let $d_{i,j}(k,r) = \binom{rj+k-j}{i+k-j}$. We have $\dim V_{(n)}^{[k]} = \mathrm{Det}\,(d_{i,j})_{1 \le i,j \le n}$. First let us make the entrees of this matrix polynomials in k and r. To do it note that

$$d_{i,j}(k,r) = \frac{(jr + k - j)!}{(k - j + n)!(jr - 1)!} d'_{i,j}(k,r)$$

for $d'_{i,j}(k,r) = (k-j+i+1)_{n-i}(rj-i+1)_{i-1}$, where $(m)_i = m(m+1)\ldots(m+i-1)$. So

$$\dim V_{(n)}^{[k]} = \Delta(k,r)\prod_{j=1}^n \frac{(jr + k - j)!}{(k - j + n)!(jr - 1)!}, \qquad \Delta(k,r) = \mathrm{Det}\,\bigl(d'_{i,j}(k,r)\bigr)_{1 \le i,j \le n}.$$

Note that $\Delta(k,r)$ is a polynomial in k and r of degree $n(n-1)/2$ in k and of the same degree in r.

Let us show that $\Delta(k,r) = 0$ for $r = b/(a+k)$, $1 \le b < a \le n$. In this case for any j we have

$$\sum_{i=1}^n d'_{i,j}\left(k\binom{a-b-1}{i-b-1} + i\binom{a-b}{i-b}\right) = 0,$$

so the rows of this matrix are linearly dependent.

As the polynomials $kr + ar - b$, $1 \le b < a \le n$ are irreducible and have no common divisors, $\Delta(k,r)$ is divisible by $\prod_{1 \le b < a \le n}(kr + ar - b)$ and therefore proportional to this polynomial.

Then note that the maximal degree term in $\Delta(k,r)$ is equal to

$$\begin{aligned}
\mathrm{Det}\left(k^{n-i}(rj)^{i-1}\right)_{1 \le i,j \le n} &= (kr)^{n(n-1)/2}\,\mathrm{Det}\left(j^{i-1}\right)_{1 \le i,j \le n} = \\
&= (kr)^{n(n-1)/2}\prod_{1 \le b < a \le n}(a-b).
\end{aligned}$$

Summarizing, we obtain

$$\dim V_{(n)}^{[k]} = \prod_{1 \le b < a \le n}(a-b)(kr + ar - b)\prod_{j=1}^n \frac{(jr + k - j)!}{(k - j + n)!(jr - 1)!}.$$

At last writing this in the uniform way using

$$\prod_{1 \le b < a \le n} (a - b) = \prod_{j=1}^{n} (j - 1)!, \qquad \prod_{1 \le b < a \le n} (kr + ar - b) = \prod_{j=1}^{n} \frac{(kr + jr - 1)!}{(kr + jr - j)!},$$

and $\prod_{j=1}^{n} (k-j+n)! = \prod_{j=1}^{n} (k+j-1)!$, we obtain the statement of the theorem.   □

COROLLARY 3.15. *We have*

$$\dim W^2(n\omega_1)^{[k]} \ge \prod_{j=1}^{n} \frac{(jr + k - j)!}{(jr - 1)!} \cdot \frac{(kr + jr - 1)!}{(kr + jr - j)!} \cdot \frac{(j - 1)!}{(k + j - 1)!}.$$

REMARK 3.16. To the best of our knowledge the product formula for the number of plane partitions $pp^\lambda(k)$ in the case $\lambda = (nr + p, (n - 1)r + p, \cdots, r + p, p)$, has been obtained for the first time by R.Proctor (unpublished manuscript dated January 1984, but see [**P**], Corollary 4.1, for the case $r = 1$). As far as we aware the first published proof of the product formula for $pp^\lambda(k)$ in question, is due to C. Krattenthaler [**Kr**]. We refer the reader to [**St2**], p.550, for an elegant product formula for the number $pp^\lambda(k)$, due to R.Proctor, as well as for additional historical comments. We include a (new) proof of Theorem 3.14 since it directly furnishes the product formula (3.1) which is more suitable for our purposes.

REMARK 3.17. Using the same method we can prove more generally that for $\lambda = (n(r - 1) + p, (n - 1)(r - 1) + p, \cdots, r - 1 + p)$ we have

$$pp^\lambda(k) = \mathrm{Det} \left(\binom{rj + p + k - j}{i + k - j} \right)_{1 \le i,j \le n} =$$

$$\prod_{j=1}^{n} \frac{(jr + p + k - j)!}{(jr + p - 1)!} \cdot \frac{(kr + jr + p - 1)!}{(kr + jr + p - j)!} \cdot \frac{(j - 1)!}{(k + j - 1)!}.$$

The observation that the higher Catalan number $C_n^{(r)}$ is equal to the number of plane partitions $pp^{(n(r-1),(n-1)(r-1),\cdots,r-1)}(1)$ can be generalized to the natural bijection between the set of *trapezoidal paths* [**L**] of type (n, p, r) and the set of plane partitions $PP^{((n-1)(r-1)+p,\cdots,r-1+p)}(1)$.

References

[CL] V. Chari, S. Loktev, *Weyl, Demazure and Fusion modules for the current algebra of sl_{r+1}*, preprint math.QA/0502165.

[CP1] V. Chari, A. Pressley, *Weyl Modules for Classical and Quantum Affine Algebras*, Represent. Theory 5 (2001), 191–223, math.QA/0004174.

[CP2] V. Chari, A. Pressley, *Integrable and Weyl modules for quantum affine sl_2*, Proceedings of the LMS symposium on Quantum Groups, math.QA/0007123.

[F] E. Feigin, *Schubert varieties and the fusion products. The general case*, Int. Math. Res. Not. 2004, no. **59**, 3153–3175, math.QA/0311278.

[FF1] B. L. Feigin and E. Feigin, *q-characters of the tensor products in sl_2-case*, Mosc. Math. J. **2** (2002), no. 3, 567–588, math.QA/0201111.

[FF2] B. L. Feigin and E. Feigin, *Integrable $\widehat{sl_2}$-modules as infinite tensor products*, Fundamental mathematics today (Russian), 304–334, Nezavis. Mosk. Univ., Moscow, 2003, math.QA/0205281.

[FF3] B. L. Feigin and E. Feigin, *Schubert varieties and the fusion products*, Publ. Res. Inst. Math. Sci. **40** (2004), no. 3, 625–668, math.QA/0305437.

[FL1] B. Feigin, S. Loktev, *On Generalized Kostka Polynomials and the Quantum Verlinde Rule*, Differential topology, infinite–dimensional Lie algebras, and applications, Amer. Math. Soc. Transl. Ser. 2, Vol. 194 (1999), p. 61–79, math.QA/9812093.

[FL2] B. Feigin, S. Loktev, *Multi-dimensional Weyl modules and Symmetric Functions*, Comm. Math. Phys., **251** (2004), no. 3, 427–445, math.QA/0212001.

[FL3] B. Feigin, S. Loktev, *Deformation of Weyl Modules and Generalized Parking Functions*, IMRN 2004, no. 51, 2719-2750, math.QA/0312158.

[FoL] G. Fourier and P. Littelmann, *Tensor product structure of affine Demazure modules and limit constructions*, math.RT/0412432.

[HHKOTY] Goro Hatayama, Anatol N. Kirillov, Atsuo Kuniba, Masato Okado, Taichiro Takagi, Yasuhiko Yamada, *Character Formulae of $\hat{sl}_n$-Modules and Inhomogeneous Paths* Nuclear Physics B 536 [PM] (1999), 575-616, math.QA/9802085.

[HP] W.V.D. Hodge, D. Pedoe, *Methods of Algebraic Geometry*, Vol. II, Cambridge University Press, 1952.

[K] V.Kac, *Infinite dimensional Lie algebras*, Cambridge Univ. Press, 1985.

[Ke] R. Kedem, *Fusion products, cohomology of GL(N) flag manifolds and Kostka polynomials*, Int. Math. Res. Not. 2004, no. 25, 1273–1298, math.RT/0312478.

[Kr] C. Krattenthaler, *A determinant evaluation and some enumeration results for plane partitions*, Lect. Notes in Math., **1452** (1990), 121–131, Springer–Verlag.

[KMOTU] A. Kuniba, K. C. Misra, M. Okado, T. Takagi, J. Uchiyama, *Paths, Demazure Crystals and Symmetric Functions*, Talk presented at the Nankai-CRM meeting "Extended and Quantum Algebras and their Applications to Physics", q-alg/9612018.

[L] N. Loehr, *Trapezoidal lattice paths and multivariate analogues*, Adv. Applied Math. **31** (2003), 597-629.

[M] Ian Macdonald, *Symmetric functions and Hall polynomials*, Oxford mathematical monographs.

[P] R. Proctor, *Odd symplectic groups*, Invent. Math. **92** (1988), 307-332.

[PSt] J. Pitman, R. Stanley, *A polytope related to empirical distributions, plane trees, parking functions, and the associahedron*, Discrete and Computational Geometry **27** (2002), 603–634.

[S] M. Shimozono, *Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties*, J. Algebraic Combin. **15** (2002), no. 2, 151–187, math.QA/9804039.

[St1] R. Stanley, *Some combinatorial aspects of the Schubert Calculus*, Lect. Notes in Math. **579**, Springer–Verlag, 1977, 217–251.

[St2] R. Stanley, *Enumerative Combinatorics*, Cambridge Univ. press, 1999.

[STU] Y. Saito, K. Takemura, D. Uglov, *Toroidal actions on level 1 modules of $U_q(\hat{sl}_n)$*, Transform. Groups **3** (1998), no. 1, 75–102, q-alg/9702024.

[VV] M. Varagnolo, E. Vasserot, *Double-loop algebras and the Fock space*, Invent. Math. **133** (1998), no. 1, 133–159, q-alg/9612035.

LANDAU INSTITUTE FOR THEORETICAL PHYSICS, CHERNOGOLOVKA 142432, RUSSIA
E-mail address: `feigin@mccme.ru`

RIMS, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN
E-mail address: `kirillov@kurims.kyoto-u.ac.jp`

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN
E-mail address: `loktev@math.kyoto-u.ac.jp`