False Gravitational Anomalies

Mitsuo Abe[1,*] and Noboru Nakanishi[2,**]

[1] *Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan*
[2] *12-20 Asahigaoka-cho, Hirakata 573-0026, Japan*

It is pointed out that the existence of gravitational anomalies cannot be established by the reasoning of Alvarez-Gaumé and Witten. Its serious drawback is the confusion of the T*-product quantities with the T-product ones. It is explicitly confirmed that the gravitational anomaly is non-existent in the 2-dimensional case.

§1. Introduction

In 1984, Alvarez-Gaumé and Witten[1)] claimed that there exist gravitational anomalies in the $(4k+2)$-dimensional spacetime $(k = 0, 1, 2, \ldots)$. That is, they calculated the 1-loop Feynman integrals of the $(2k+2)$-point functions involving the energy-momentum tensor, showing that the results were inconsistent with the conservation law of the energy-momentum tensor. The purpose of the present paper is to point out that their reasoning is not sufficient for establishing the existence of gravitational anomalies, and that the gravitational anomaly is indeed non-existent in the 2-dimensional case.[***)]

The serious drawback of their reasoning is caused by the fact that they were totally careless about the essential difference between the T-product and the T*-product. The former is converted into the latter when the Hamiltonian formalism is transcribed into the Lagrangian formalism. The expressions appearing in the covariant perturbation theory and in the path-integral formalism are written as the vacuum expectation values of the T*-product but not of the T-product. Nevertheless, Alvarez-Gaumé and Witten regarded the Feynman integrals as the quantities written in terms of the T-product.

The T-product is a sum over products of local operators multiplied by a product of θ-functions of time differences; therefore, in general, it becomes non-covariant when differentiated. On the contrary, the T*-product is defined in such a way that time differentiations always act after the vacuum expectation value of the T-product of the fundamental fields is taken. Hence the T*-product quantities are always covariant. The price to be paid for this bonus is that the T*-product is no longer a product in the mathematical sense; a T*-product quantity involving a factor 0 is not necessarily equal to 0. Accordingly, Feynman integrals are not necessarily consistent with the field equations and therefore with the Noether theorem. This violation of the Noether theorem should not be confused with anomaly; one can calculate it

*) E-mail: abe@kurims.kyoto-u.ac.jp
**) Professor Emeritus of Kyoto University. E-mail: nbr-nak@trio.plala.or.jp
***) A preliminary report was made half a decade ago.[2)]

exlicitly from the difference between the T*-product and the T-product.

As is well known, the Adler-Bell-Jackiw anomaly is established by calculating the Feynman integral corresponding to the triangle diagram. The reason why one correctly obtains the chiral anomaly in this case is that the expression for the chiral current involves no differentiation. On the contrary, the expression for the energy-momentum tensor necessarily contains time differentiations. In the T*-product quantity, those differentiations act from the outside of the vacuum expectation value, but not directly on the fields in the expression for the energy-momentum tensor. This fact yields a nontrivial difference between the T*-product and the T-product. But no analysis was made about it in the work of Alvarez-Gaumé and Witten. At least in the 2-dimensional case, what they called the gravitational anomaly can be shown to be nothing more than the contribution from the difference between the T*-product and the T-product. The non-existence of the gravitational anomaly is explicitly confirmed by constructing the exact solution in the Heisenberg picture.

The present paper is organized as follows. In §2, We discuss the apparent violation of the Noether theorem in the path-integral appraoch. In §3, we consider the 2-dimensional Weyl field, and show that what Alvarez-Gaumé and Witten regarded as the gravitational anomaly is nothing more than the apparent violation of the conservation law due to the use of the T*-product quantities. In §4, we discuss the BRS-formulated 2-dimensional quantum gravity coupled with the Weyl fields, and confirm the absence of the gravitational anomaly. In §5, some comments are made about the Virasoro anomaly. In the Appendix, some formulae for singular-function products are presented.

§2. T*-product and the Noether theorem

Previously, we discussed the pathological nature of the covariant perturbation theory and the path-integral formalism caused by the T*-product.[3)] We first reproduce the general formula for the field-equation-violating contribution due to the T*-product in the path integral.

The generating functional, $Z(J)$, of the Green functions is formally expressed as a path integral,

$$Z(J) = \int (\prod_A \mathcal{D}\varphi_A) \exp i \int d^N x (\mathcal{L} + \sum_A J_A \varphi_A). \tag{2·1}$$

Here $S = \int d^N x \mathcal{L}(x)$ is the action for the fields $\varphi_A(x)$ in the N-dimensional space-time; $\mathcal{D}\varphi_A$ is the path-integral measure normalized by $Z(0) = 1$, and $J_A(x)$ denotes the source function for $\varphi_A(x)$. Let $F(\varphi)$ be an arbitrary function of $\varphi_{A_1}(y_1), \ldots,$ $\varphi_{A_m}(y_m)$, and $\delta\varphi_A$ be a *field-independent* variation of φ_A. The path-integral measure should be invariant under the functional translation $\varphi_A \to \varphi_A + \delta\varphi_A$. Accordingly, by considering a variation of a particular field φ_A in $F(i^{-1}\partial/\partial J) Z|_{J=0}$, we obtain

$$i \langle \mathrm{T}^* F(\varphi) \frac{\delta}{\delta\varphi_A} S \rangle + \langle \mathrm{T}^* \frac{\delta}{\delta\varphi_A} F(\varphi) \rangle = 0. \tag{2·2}$$

This is the T*-product version of the field equation $(\delta/\delta\varphi_A) S = 0$. The second term of (2·2) is the field-equation-violating term due to the use of the T*-product. Thus it is not admissible to naively use the field equations, and therefore the Noether theorem, in the T*-product quantities. Hence we must treat the current conservation law very carefully in the covariant perturbation theory.

Now, we consider an infinitesimal symmetry transformation

$$\delta^\epsilon \varphi_A(x) \equiv \varphi'_A(x') - \varphi_A(x). \tag{2·3}$$

We set

$$\delta^\epsilon_* \varphi_A(x) \equiv \varphi'_A(x) - \varphi_A(x), \tag{2·4}$$

so that

$$\delta^\epsilon \varphi_A = \delta^\epsilon_* \varphi_A + \delta^\epsilon x^\mu \cdot \partial_\mu \varphi_A, \tag{2·5}$$

where $\delta^\epsilon x^\mu \equiv x'^\mu - x^\mu$.

The Noether current J^μ is defined by

$$\epsilon J^\mu \equiv \sum_A \delta^\epsilon_* \varphi_A \cdot \frac{\partial}{\partial(\partial_\mu \varphi_A)} \mathcal{L} + \delta^\epsilon x^\mu \cdot \mathcal{L}. \tag{2·6}$$

Then the Noether identity is

$$\epsilon \partial_\mu J^\mu = - \sum_A \delta^\epsilon_* \varphi_A \cdot \frac{\delta}{\delta \varphi_A} S + \delta^\epsilon \mathcal{L}, \tag{2·7}$$

where the last term of the right-hand side vanishes if the Lagrangian density is invariant under the symmetry transformation. However, the first term cannot be set equal to zero because the field equations are violated in the T*-product quantitites, as stated above. In particular, for the energy-momentum tensor $T^{\mu\nu}$, we have

$$\partial_\mu T^{\mu\nu} = - \sum_A \partial^\nu \varphi_A \frac{\delta}{\delta \varphi_A} S. \tag{2·8}$$

Hence, with the help of (2·2), we obtain

$$\langle \mathrm{T}^* \partial_\mu T^{\mu\nu}(x) \cdot F(y) \rangle = -i \langle \mathrm{T}^* \sum_A \frac{\delta}{\delta \varphi_A(x)} [\partial^\nu \varphi_A(x) \cdot F(y)] \rangle. \tag{2·9}$$

For illustration, we consider a free massive scalar field $\phi(x)$ in N dimensions. Its field equation is

$$(\Box + m^2)\phi = 0. \tag{2·10}$$

The energy-momentum tensor is given by

$$T^{\mu\nu} = \partial^\mu \phi \cdot \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} (\partial^\sigma \phi \cdot \partial_\sigma \phi - m^2 \phi^2), \tag{2·11}$$

which, of course, satisfies the conservation law:

$$\partial_\mu T^{\mu\nu} = (\Box + m^2)\phi \cdot \partial^\nu \phi = 0 \tag{2·12}$$

owing to the field equation. One should note, however, that the Feynman propagator,

$$\langle \mathrm{T}^*\phi(x)\phi(y)\rangle = \langle \mathrm{T}\phi(x)\phi(y)\rangle = \Delta_F(x-y), \tag{2·13}$$

does not satisfy the Klein-Gordon equation, but instead we have

$$(\Box + m^2)\Delta_F(x-y) = -i\delta^N(x-y). \tag{2·14}$$

Accordingly, by straightforward calculation, we find a *nonvanishing* result for the divergence of the 2-point function of the energy-momentum tensor:

$$\begin{aligned}\langle \mathrm{T}^*\partial_\mu T^{\mu\nu}(x)\cdot T^{\lambda\rho}(y)\rangle &\equiv \partial_\mu^x\langle \mathrm{T}^* T^{\mu\nu}(x)T^{\lambda\rho}(y)\rangle \\ &= -i(\eta^{\lambda\sigma}\partial^\rho + \eta^{\rho\sigma}\partial^\lambda - \eta^{\lambda\rho}\partial^\sigma)\partial^\nu\Delta_F(x-y)\cdot\partial_\sigma\delta^N(x-y) \\ &\quad -im^2\eta^{\lambda\rho}\partial^\nu\Delta_F(x-y)\cdot\delta^N(x-y). \end{aligned} \tag{2·15}$$

This is, of course, *not* a gravitational anomaly. Indeed, the right-hand side of (2·9) becomes

$$\begin{aligned}&-i\langle \mathrm{T}^*\frac{\delta}{\delta\phi(x)}[\partial^\nu\phi(x)\cdot T^{\lambda\rho}(y)]\rangle \\ &= -i\langle \mathrm{T}^*\partial^\nu\phi(x)\,[(-\partial_\sigma^x)\{(\eta^{\lambda\sigma}\partial^\rho + \eta^{\rho\sigma}\partial^\lambda)\phi(y)\cdot\delta^N(x-y)\} \\ &\quad -\eta^{\lambda\rho}(-\partial_\sigma^x)\{\partial^\sigma\phi(y)\cdot\delta^N(x-y)\} + m^2\eta^{\lambda\rho}\phi(y)\delta^N(x-y)]\,\rangle, \end{aligned} \tag{2·16}$$

which is exactly equal to (2·15).

In order to avoid the disturbance caused by the T*-product, it is convenient to define the anomaly in terms of Wightman functions.[4)] Let Q be a symmetry generator expressed as

$$Q = \int d^{N-1}x\, F(x). \tag{2·17}$$

Then the anomaly for this symmetry exists if

$$\partial_0^x \int d^{N-1}x\,\langle F(x)\varphi_1(y_1)\cdots\varphi_n(y_n)\rangle \neq 0 \tag{2·18}$$

for some fields $\varphi_1, \ldots, \varphi_n$. As for the gravitational anomaly of the above model, we have only to calculate

$$\partial_0^x \int d^{N-1}x\,\langle T^{0\nu}(x)\phi(y_1)\phi(y_2)\rangle. \tag{2·19}$$

It vanishes as is easily confirmed by using the Klein-Gordon equation for $\Delta^{(+)}(x-y_j)$ and by integrating by parts. Thus the gravitational anomaly does not exist.

§3. False 2-dimensional gravitational anomaly

We consider the 2-dimensional complex Weyl field $\psi(x)$. Its free-field action is given by

$$S = i\int d^2x\psi^\dagger\partial_-\psi. \tag{3·1}$$

We employ the light-cone coordinates $x^{\pm} = (x^0 \pm x^1)/\sqrt{2}$. The field equation and the 2-dimensional anticommutation relation are

$$\partial_- \psi = 0, \tag{3·2}$$
$$\{\psi(x), \psi^\dagger(y)\} = \delta(x^+ - y^+), \tag{3·3}$$

respectively. Accordingly, the 2-point Wightman function and the Feynman propagator are, respectively, as follows:

$$\langle \psi(x)\psi^\dagger(y)\rangle = \frac{1}{2\pi i}\cdot\frac{1}{x^+ - y^+ - i0}, \tag{3·4}$$

$$\begin{aligned}\langle \mathrm{T}^*\psi(x)\psi^\dagger(y)\rangle &= \frac{1}{2\pi i}\cdot\frac{1}{x^+ - y^+ - i0(x^- - y^-)} \\ &\equiv \frac{1}{2\pi i}\left[\frac{\theta(x^- - y^-)}{x^+ - y^+ - i0} + \frac{\theta(-x^- + y^-)}{x^+ - y^+ + i0}\right].\end{aligned} \tag{3·5}$$

It is very important to write *explicitly* the fact that the Feynman propagator depends on $x^- - y^-$.

The energy-momentum tensor $T^\mu{}_\nu$ is given by

$$T^-{}_+ = T_{++} = \frac{i}{2}(\psi^\dagger\partial_+\psi - \partial_+\psi^\dagger\cdot\psi), \tag{3·6}$$
$$T^+{}_+ = T_{-+} = \frac{i}{2}(-\psi^\dagger\partial_-\psi + \partial_-\psi^\dagger\cdot\psi). \tag{3·7}$$

Although T_{-+} vanishes owing to the field equation, it *cannot* be neglected in the calculation of the T*-product quantities.

A straightforward calculation yields

$$\begin{aligned}\langle \mathrm{T}^* T_{++}(x) T_{++}(y)\rangle &= \frac{1}{8\pi^2}\cdot\frac{1}{(x^+ - y^+ - i0(x^- - y^-))^4} \\ &= \frac{1}{8\pi^2}\left[\frac{\theta(x^- - y^-)}{(x^+ - y^+ - i0)^4} + \frac{\theta(-x^- + y^-)}{(x^+ - y^+ + i0)^4}\right].\end{aligned} \tag{3·8}$$

Hence

$$\partial_-^x \langle \mathrm{T}^* T_{++}(x) T_{++}(y)\rangle = \frac{1}{24\pi i}\delta'''(x^+ - y^+)\delta(x^- - y^-). \tag{3·9}$$

We may rewrite the above results into those in the momentum space. With the help of the formula

$$\int d^2 z\, \frac{\theta(\pm z^-)}{z^+ \mp i0}\, e^{ipz} = -2\pi\cdot\frac{\theta(\mp p_+)}{p_- \mp i0}, \tag{3·10}$$

we see that the Fourier transform of (3·8) is given by

$$\frac{i}{24\pi}\cdot\frac{{p_+}^3}{p_- + i0p_+}. \tag{3·11}$$

It is important to write explicitly the infinitesimal imaginary part of the denominator. To differentiate (3·8) with respect to x^- is equivalent to multiplying (3·11) by $-ip_-$, so that we obtain $(1/24\pi){p_+}^3$, which is, of course, the Fourier transform of (3·9). This quantity is nothing but what Alvarez-Gaumé and Witten regarded as the gravitational anomaly.[1)] We can show, however, that it is nothing more than the contribution from the difference between the T*-product quantity and the T-product one.

It is convenient to work in the spacetime representation. Using the formula $\epsilon(z)\delta(z) = 0$, we obtain

$$\begin{aligned}&\langle \mathrm{T}^* T_{-+}(x) T_{++}(y)\rangle \\ &= \frac{1}{4\pi i}\left[\mathrm{Pf}\frac{1}{(x^+-y^+)^2}\cdot\delta(x^+-y^+) + \mathrm{Pf}\frac{1}{x^+-y^+}\cdot\delta'(x^+-y^+)\right]\delta(x^- - y^-),\end{aligned} \tag{3·12}$$

where Pf denotes “finite part”, i.e., $\mathrm{Pf}(1/z^n) \equiv \Re[1/(z-i0)^n]$. Hence

$$\begin{aligned}&\partial_+^x\langle \mathrm{T}^* T_{-+}(x) T_{++}(y)\rangle \\ &= \frac{1}{4\pi i}\left[-2\mathrm{Pf}\frac{1}{(x^+-y^+)^3}\cdot\delta(x^+-y^+) + \mathrm{Pf}\frac{1}{x^+-y^+}\cdot\delta''(x^+-y^+)\right]\delta(x^- - y^-).\end{aligned} \tag{3·13}$$

From (3.9) and (3.13), we obtain

$$\begin{aligned}&\langle \mathrm{T}^*[\partial_- T_{++}(x) + \partial_+ T_{-+}(x)] T_{++}(y)\rangle \\ &= \frac{1}{4\pi i}\left[-\frac{1}{6}\delta'''(x^+-y^+) - 4\mathrm{Pf}\frac{1}{(x^+-y^+)^3}\cdot\delta(x^+-y^+)\right]\delta(x^- - y^-),\end{aligned} \tag{3·14}$$

where we have made use of an identity (A·6) presented in the Appendix.

On the other hand, the quantity of the right-hand side of (2·9) becomes

$$\begin{aligned}&i\langle \mathrm{T}^*\frac{\delta}{\delta\psi(x)}[\partial_+\psi(x)\cdot T_{++}(y)]\rangle + i\langle \mathrm{T}^*\frac{\delta}{\delta\psi^\dagger(x)}[\partial_+\psi^\dagger(x)\cdot T_{++}(y)]\rangle \\ &= \frac{1}{2\pi i}\left[-\mathrm{Pf}\frac{1}{(x^+-y^+)^2}\cdot\delta'(x^+-y^+) - 2\mathrm{Pf}\frac{1}{(x^+-y^+)^3}\cdot\delta(x^+-y^+)\right]\delta(x^- - y^-).\end{aligned} \tag{3·15}$$

With the help of an identity (A·4), we see that *(3·15) exactly coincides with (3·14).* Thus, we have established that what Alvarez-Gaumé and Witten called the gravitational anomaly is nothing but the contribution from the T*-product.

Finally, we make a remark on the Majorana Weyl field ψ, used by Green, Schwarz and Witten.[5)] It has only one field degree of freedom. Hence, without introducing an extra field, *it cannot be quantized* because there is no canonical conjugate independent of ψ. Nevertheless, setting up the 2-dimensional anticommutation relation

$$\{\psi(x), \psi(y)\} = \delta(x^+ - y^+) \tag{3·16}$$

by hand and replacing (3·6) and (3·7) by the expressions without dagger, we can repeat the above consideration. We obtain the same results as above except for an overall multiplicative factor 2 *on the left-hand side only.* Thus, (2·9) does not hold in this model. The reason for this is the non-existence of the path integral.

§4. 2-dimensional gravitational theory

In this section, in order to demonstrate positively the non-existence of the gravitational anomaly, we consider the BRS-formulated conformal-gauge 2-dimensional quantum gravity coupled with D Weyl fields. If coupled with D scalar fields instead, the model can be interpreted as a string theory in the D-dimensional spacetime. Previously, we fully investigated this model and found the complete solution in terms of Wightman functions.[6)–9)] Extension to the case of the Weyl fields is straightforward.

In the conformal gauge, the gravitational field $g^{\mu\nu}$ is parametrized as $g^{\pm\mp} = \exp(-\theta)$ and $g^{\pm\pm} = \exp(-\theta)h_\pm$. Hence, to first order, the zweibein $\tilde{h}_{\mu a}$ is given by $\tilde{h}_{\pm\mp} = \exp(\frac{1}{2}\theta)$ and $\tilde{h}_{\pm\pm} = -\frac{1}{2}\exp(\frac{1}{2}\theta)h_\mp$ in the symmetric gauge $\tilde{h}_{\mu a} = \tilde{h}_{a\mu}$.

The action for D Weyl fields coupled with the zweibein is given by

$$S_W = \frac{i}{2}\int d^2x\,[\tilde{\psi}_M^\dagger(\tilde{h}_{+-}\partial_-\tilde{\psi}_M - \tilde{h}_{--}\partial_+\tilde{\psi}_M) - (\tilde{h}_{+-}\partial_-\tilde{\psi}_M^\dagger - \tilde{h}_{--}\partial_+\tilde{\psi}_M^\dagger)\cdot\tilde{\psi}_M], \quad (4\cdot1)$$

where the sum over $M = 1, \ldots, D$ should be understood. Setting $\tilde{\psi}_M = \exp(-\frac{1}{4}\theta)\psi_M$, we have

$$S_W = \frac{i}{2}\int d^2x\,[\psi_M^\dagger(\partial_-\psi_M + \frac{1}{2}h_+\partial_+\psi_M) - (\partial_-\psi_M^\dagger + \frac{1}{2}h_+\partial_+\psi_M^\dagger)\cdot\psi_M] \quad (4\cdot2)$$

to first order.

The action proper to the zweibein is the same as in our previous work.[6)] Let $\tilde{b}^\mu$, c_μ and $\bar{c}^\mu$ be the B field, the FP ghost and the FP anti-ghost, respectively. Then, from (4·2), we see that the total action is given by

$$S = \int d^2x\,(\mathcal{L}_0 + \mathcal{L}_I) \quad (4\cdot3)$$

with

$$\mathcal{L}_0 = -\frac{1}{2}\tilde{b}^+h_+ - i\bar{c}^+\partial_-c^+ + (+\leftrightarrow-) + \frac{i}{2}\left[\psi_M^\dagger\partial_-\psi_M - \partial_-\psi_M^\dagger\cdot\psi_M\right], \quad (4\cdot4)$$

$$\mathcal{L}_I = \frac{1}{2}h_+[-2i\bar{c}^+\partial_+c^+ - i(\partial_+\bar{c}^+\cdot c^+ + \partial_-\bar{c}^+\cdot c^-)] + (+\leftrightarrow-) + \frac{1}{2}h_+T_{++} + O(h^2), \quad (4\cdot5)$$

where

$$T_{++} \equiv \frac{i}{2}(\psi_M^\dagger\partial_+\psi_M - \partial_+\psi_M^\dagger\cdot\psi_M) \quad (4\cdot6)$$

and $(+\leftrightarrow-)$ means to interchange *scripts only* in the preceding expression.

In the operator formalism, higher-order terms $O(h^2)$ have no contributions because of the field equation $h_\pm = 0$, and moreover the left-moving mode and the right-moving mode decouple completely because of the field equations $\partial_\mp c^\pm = 0$, $\partial_\mp\bar{c}^\pm = 0$, etc. One should note that such simplicity is *never* realized in the path-integral formalism because of the field-equation-violating nature of the T*-product. The terms of first order in $h_\pm$ yield the B-field equations :

$$\mathcal{T}^+ \equiv -\tilde{b}^+ - 2i\bar{c}^+\partial_+c^+ - i\partial_+\bar{c}^+\cdot c^+ + T_{++} = 0, \quad (4\cdot7)$$

$$\mathcal{T}^- \equiv -\tilde{b}^- - 2i\bar{c}^-\partial_-c^- - i\partial_-\bar{c}^-\cdot c^- = 0. \quad (4\cdot8)$$

Everything goes in the same way as in our previous work except for the part in which the Weyl fields are relevant. Hence we skip the description of our reasoning and directly goes to the Wightman functions involving the Weyl fields. The nonvanishing n-point truncated Wightman functions involving the Weyl fields are only those which consist of one ψ_M, one $\psi_M^\dagger$ and $(n-2)$ $\tilde{b}^+$ fields ($n \geqq 2$). The explicit expressions are

$$\begin{aligned}
&\langle \psi_M(x_1)\tilde{b}^+(x_2)\cdots\tilde{b}^+(x_{n-1})\psi_N^\dagger(x_n)\rangle_{\mathrm{T}} \\
&=\delta_{MN}\left(\frac{i}{2}\right)^{n-2}\frac{1}{(2\pi i)^{n-1}}\sum_{P(j_2,\ldots,j_{n-1})}^{(n-2)!}(\partial_{j_2}^R-\partial_{j_2}^L)(\partial_{j_3}^R-\partial_{j_3}^L)\cdots(\partial_{j_{n-1}}^R-\partial_{j_{n-1}}^L) \\
&\quad\cdot\left(\frac{1}{x_1^+-x_{j_2}^+-i0}\cdot\frac{1}{x_{j_2}^+-x_{j_3}^+\mp i0}\cdots\frac{1}{x_{j_{n-2}}^+-x_{j_{n-1}}^+\mp i0}\cdot\frac{1}{x_{j_{n-1}}^+-x_n^+-i0}\right)
\end{aligned} \tag{4·9}$$

and similar expressions for other orderings of field operators. Here $P(j_2,\ldots,j_{n-1})$ denotes a permutation of $(2,\ldots,n-1)$; $x_j^+-x_k^+\pm i0 = x_j^+-x_k^+-i0$ for $j<k$ and $= x_j^+-x_k^++i0$ for $j>k$; $\partial_j^{R/L}$ acts only on the x_j^+ involved in the right/left factor.*)

All truncated Wightman functions are consistent with translational invariance. The non-existence of the gravitational anomaly is easily seen as follows. The Noether currents for translational invariance are

$$J^-{}_+ = -i\bar{c}^+\partial_+c^+ + T_{++}, \tag{4·10}$$
$$J^+{}_- = -i\bar{c}^-\partial_-c^- \tag{4·11}$$

with $J^\pm{}_\pm = 0$. Because any nonvanishing truncated vacuum expectation value of a product of $J^-{}_\nu(x)$ and fundamental fields $\varphi_1(y_1),\ldots,\varphi_n(y_n)$ depends only on $x^+, {y_1}^+,\ldots,{y_n}^+$ but not on x^- at all, we trivially have

$$\partial_-^x\int dx^+\langle J^-{}_\nu(x)\varphi_1(y_1)\cdots\varphi_n(y_n)\rangle = 0. \tag{4·12}$$

Likewise for $J^+{}_\nu$. Thus there is no gravitational anomaly (see §2).

But this model is not totally free of anomaly. As in the scalar-field case,[6] the B-field equations exhibit the "field-equation anomaly", namely, a slight violation of field equations at the representation level. We find that

$$\langle\tilde{b}^+(x_1)\mathcal{T}^+(x_2)\rangle = -\langle\mathcal{T}^+(x_1)\mathcal{T}^+(x_2)\rangle = (D-26)\varPhi^{++}(x_1^+-x_2^+), \tag{4·13}$$
$$\langle\tilde{b}^-(x_1)\mathcal{T}^-(x_2)\rangle = -\langle\mathcal{T}^-(x_1)\mathcal{T}^-(x_2)\rangle = -26\,\varPhi^{++}(x_1^+-x_2^+), \tag{4·14}$$

where

$$\varPhi^{++}(z^+) \equiv \frac{1}{8\pi^2}\cdot\frac{1}{(z^+-i0)^4}, \tag{4·15}$$

just as in the scalar-field case. The perturbation-theoretical counterparts of (4·13) and (4·14) are the unexpected 1-loop contributions to $\langle \mathrm{T}^*\tilde{b}^\pm(x_1)\tilde{b}^\pm(x_2)\rangle$, which are

*) For example, $\partial_j^R[f(x_j^+)g(x_j^+)] \equiv f(x_j^+)\partial_j g(x_j^+)$.

easily calculated by using (4.5) according to the Feynman rules; the reason for their presence is that the Feynman propagator $\langle \mathrm{T}^* \tilde{b}^{\pm}(x) h_{\pm}(y)\rangle$ is *nonvanishing* owing the field-equation-violating nature of the T*-product.

The Noether currents for translational invariance contain an anomalous contribution from the above field-equation anomaly at the representation level, but we can define the anomaly-free currents by

$$\tilde{J}^{\mp}{}_{\pm} \equiv J^{\mp}{}_{\pm} - \mathcal{T}^{\pm} - i\partial_{\pm}(\bar{c}^{\pm} c^{\pm}) = \tilde{b}^{\pm}, \tag{4·16}$$

so that the anomaly-free translational generators are given by

$$P_{\pm} \equiv \int dx^{\pm} \tilde{b}^{\pm}. \tag{4·17}$$

We can likewise define the anomaly-free BRS generator.[6]

§5. Discussions

In the present paper, we have clarified that what were calculated by Alvarez-Gaumé and Witten[1] are *not* the quantities which show the existence of gravitational anomalies. In order to claim the existence of the gravitational anomaly, it is necessary to calculate the T-product quantities, but *not* the T*-product quantities directly calculable by Feynman integrals. We have exlicitly confirmed that there is no gravitational anomaly at least in the 2-dimensional case.

As was pointed out previously,[2] a similar confusion concerning T*-product is found in the description on the Virasoro anomaly in a book of Green, Schwarz and Witten.[5] We here reproduce their description in essence. In their notation, spacetime coordinates are denoted by (τ, σ) and $\sigma^{\pm} = \tau \pm \sigma$ (without a factor $1/\sqrt{2}$).

"We consider a 2-dimensional massless scalar field ϕ, which satisfies

$$\partial_+ \partial_- \phi = 0. \tag{5·1}$$

The ++ component of its energy-momentum tensor is

$$T_{++} = \partial_+ \phi \cdot \partial_+ \phi, \tag{5·2}$$

which satisfies the conservation law

$$\partial_- T_{++} = 0. \tag{5·3}$$

But, because the T-product is noncommutative with the time differentiation, we obtain

$$\partial_- \langle \mathrm{T}\, T_{++}(\sigma,\tau) T_{++}(\sigma',\tau')\rangle = \frac{1}{2}\delta(\tau-\tau')\langle [T_{++}(\sigma,\tau),\, T_{++}(\sigma',\tau')]\rangle. \tag{5·4}$$

Perturbative calculation yields

$$\langle \mathrm{T}\, T_{++}(\sigma,\tau) T_{++}(\sigma',\tau')\rangle = \frac{1}{8} \cdot \frac{1}{(\sigma^+ - \sigma'^+)^4}. \tag{5·5}$$

Substituting (5·5) into (5·4) and making use of the formula

$$\partial_- \frac{1}{\sigma^+} = i\pi\delta(\sigma)\delta(\tau), \tag{5·6}$$

we find

$$\langle [T_{++}(\sigma,\tau),\, T_{++}(\sigma',\tau)]\rangle = -\frac{i\pi}{24}\delta'''(\sigma-\sigma'), \tag{5·7}$$

which is the equal-time *anomalous* commutator."

A serious mistake is (5·5): It is a perturbative result, and therefore it is a T*-product quantity. Because differentiation commutes with the T*-product, it cannot be substituted into the formula (5·4), which is a formula for the T-product. The curious formula (5·6) should be correctly written

$$\partial_- \frac{1}{\sigma^+ - i0\sigma^-} = i\pi\delta(\sigma)\delta(\tau). \tag{5·8}$$

Correspondingly, the correct version of (5.5) is

$$\langle \mathrm{T}^* T_{++}(\sigma,\tau)T_{++}(\sigma',\tau')\rangle = \frac{1}{8}\cdot\frac{1}{[\sigma^+ - \sigma'^+ - i0(\sigma^- - \sigma'^-)]^4}. \tag{5·9}$$

The correct calculation is easily carried out if one employs the Wightman functions. Because

$$\langle T_{++}(\sigma,\tau)T_{++}(\sigma',\tau')\rangle = \frac{1}{8}\cdot\frac{1}{(\sigma^+ - \sigma'^+ - i0)^4}, \tag{5·10}$$

$$\langle T_{++}(\sigma',\tau')T_{++}(\sigma,\tau)\rangle = \frac{1}{8}\cdot\frac{1}{(\sigma^+ - \sigma'^+ + i0)^4}, \tag{5·11}$$

we have

$$\begin{aligned}\langle [T_{++}(\sigma,\tau),\, T_{++}(\sigma',\tau')]\rangle &= \frac{1}{8}\left[\frac{1}{(\sigma^+ - \sigma'^+ - i0)^4} - \frac{1}{(\sigma^+ - \sigma'^+ + i0)^4}\right] \\ &= -\frac{i\pi}{24}\delta'''(\sigma^+ - \sigma'^+). \end{aligned} \tag{5·12}$$

In particular, when $\tau = \tau'$, (5·12) reduces to (5·7), but (5·12) is *not* anomalous. Note that δ''' can be expressed in terms of δ', as is seen in (A·4).

We emphasize that it is, in general, quite dangerous to discuss the existence of anomaly in the framework of the covariant perturbation theory or the path-integral formalism. The anomaly problem can be investigated more safely in the operator formalism.

Appendix A
——— *Singular function identities* ———

We present some identities for products of singular functions.

Taking the imaginary part of a self-evident identity

$$\left[\frac{1}{(z-i0)^{n+1}}\right]^2 = \frac{1}{(z-i0)^{2n+2}}, \tag{A·1}$$

we obtain

$$\mathrm{Pf}\frac{1}{z^{n+1}} \cdot \delta^{(n)}(z) = \frac{(-1)^{n+1}n!}{2 \cdot (2n+1)!}\delta^{(2n+1)}(z). \tag{A·2}$$

In particular, for $n = 0$ and for $n = 1$, it becomes

$$\mathrm{Pf}\frac{1}{z} \cdot \delta(z) = -\frac{1}{2}\delta'(z), \tag{A·3}$$

and

$$\mathrm{Pf}\frac{1}{z^2} \cdot \delta'(z) = \frac{1}{12}\delta'''(z), \tag{A·4}$$

respectively.

Differentiating (A·3) twice, we obtain

$$2\mathrm{Pf}\frac{1}{z^3} \cdot \delta(z) - 2\mathrm{Pf}\frac{1}{z^2} \cdot \delta'(z) + \mathrm{Pf}\frac{1}{z} \cdot \delta''(z) = -\frac{1}{2}\delta'''(z). \tag{A·5}$$

Adding (A·4) twice to (A·5), we have

$$2\mathrm{Pf}\frac{1}{z^3} \cdot \delta(z) + \mathrm{Pf}\frac{1}{z} \cdot \delta''(z) = -\frac{1}{3}\delta'''(z). \tag{A·6}$$

References

1) L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B **234** (1984), 269.
2) N. Nakanishi, Soryusiron Kenkyu (mimeographed circular in Japanese) **100** (1999), 167.
3) M. Abe and N. Nakanishi, Prog. Theor. Phys. **102** (1999), 1187.
4) M. Abe and N. Nakanishi, Int. J. Mod. Phys. **A11** (1996), 2623.
5) M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory* Vol. I (Cambridge University Press, 1987), Chap. 3.
6) M. Abe and N. Nakanishi, Int. J. Mod. Phys. **A14** (1999), 521.
7) M. Abe and N. Nakanishi, Int. J. Mod. Phys. **A14** (1999), 1357.
8) N. Nakanishi, in *Proceedings of the 30th International Conference on High Energy Physics, Osaka, 2000* Vol. II, edited by C. S. Lim and T. Yamanaka (World Scientific, 2001), 1402.
9) N. Nakanishi, Prog. Theor. Phys. **111** (2004), 301.