EULER SYSTEMS ON DRINFELD MODULAR VARIETIES AND ZETA VALUES

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(with Appendices by Seidai Yasuda)

Abstract. Special elements are constructed in the higher K-groups of Drinfeld modular varieties and are shown to form an Euler system. Then a regulator map is constructed from the K-groups to the space of automorphic forms using the local model, or the analytic uniformization, at infinity of the modular variety. It is shown that the image under the regulator map of the special elements is related to the special value of the L-function of Hecke eigenforms on Drinfeld modular varieties.

Our result in the case of Drinfeld modular curves is the function field analogue of Beilinson's result on elliptic modular forms. The proof is not analogous to that of Beilinson's; we use that the image under the regulator map is also an Euler system.

1. Introduction

Beilinson [Be] constructed special elements in the K-group of elliptic modular curves. He showed that the image under the regulator map is related to the special value of the L-function. One of our results is the analogue of Beilinson's result ([Be], see also [Ka] Theorem 2.6) in the context of Drinfeld modular varieties (Theorem 6.3). The special elements of Beilinson were shown to form an Euler system by Kato ([Ka] Propositions 2.3, 2.4). Another result of ours is the construction of Euler systems in the K-group of Drinfeld modular varieties (Theorem 3.7).

The naive positive characteristic analogue of the Beilinson conjectures, where one describes the special values of L-function up to $\mathbb{Q}^*$ in terms of the covolume of the regulator map, is not so interesting for the reason that the L-function of a variety over the function field of a curve over a finite field $\mathbb{F}_q$ is essentially a congruence zeta function, which is a rational function in q^{-s}. Over number fields, there is the conjecture of Bloch and Kato, which describes the relation between the special values of the L-function and arithmetic étale cohomology. The analogue of the prime-to-p part of the conjecture may be easily formulated but is of less importance, since the L-function over a function field is directly related to étale cohomology. A naive approach for the p-part fails due to the lack of a good theory of p-adic cohomology with integral structure for varieties of characteristic $p > 0$. Although not in print, we believe that there is a conjecture which describes the exact value of the L-function of a motive over a global field of positive characteristic in terms of the regulator maps from K-groups. Our formula (Theorem 6.3) is to be regarded as the first evidence in higher dimensions toward the conjecture.

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An Euler system is a series of elements in some cohomology theory, which satisfies certain properties under norm maps. The properties are described in terms of local L-factors. The application to Iwasawa theory is prominent, but not many examples are known. We take as our definition the L-factor of automorphic representation, whereas in the book of Rubin [Ru] the L-factor of Galois representation is used. This is natural in the sense that most of the existing Euler systems are constructed with a help from Shimura varieties. The bridge between the two definitions is given by Langlands' conjectural description of the cohomology of Shimura varieties. In this paper, we construct abstract Euler systems for the L-factors of GL_d (Theorem 2.9). This is applied to give Euler systems in the K-groups of Drinfeld modular varieties (Theorem 3.7).

The sections of this paper are organized as follows. See also the introduction at the beginning of each section for more technical details.

Let d be a positive integer. The results in Section 2 is based on the basic observation that any set of d distributions gives rise to an Euler system of GL_d. By a distribution, we mean a system of elements which satisfy the distribution property. Two well-known examples are cyclotomic units and Siegel units on elliptic modular curves. Let $\mathbb{A}$ be the ring of finite adeles of some global field. A distribution is better understood as a $\mathrm{GL}_d(\mathbb{A})$-homomorphism $\mathcal{S}(\mathbb{A}^{\oplus d}) \to V$ where $\mathcal{S}(\mathbb{A}^{\oplus d})$ is the space of Schwartz-Bruhat functions (locally constant, compactly supported functions) on $\mathbb{A}^{\oplus d}$, and V is some representation of $\mathrm{GL}_d(\mathbb{A})$. Any set of d distributions $\mathcal{S}(\mathbb{A}^{\oplus d}) \to V_i$ $(i = 1, \dots, d)$ then gives rise to a $\mathrm{GL}_d(\mathbb{A})$-homomorphism $\mathcal{S}(\mathbb{A}^{\oplus d}) \to V_1 \otimes \cdots \otimes V_d$. We may call this homomorphism an Euler system. It is justified by the following observation. For a finite set S of places, let $\phi_S \in \mathcal{S}(\mathrm{Mat}_d(\mathbb{A}))$ denote the characteristic function of the set $\{X \in \mathrm{Mat}_d(\widehat{\mathcal{O}}) \mid X \bmod v \text{ is invertible for any } v \in S\}$, where $\widehat{\mathcal{O}}$ denotes the ring of integers of $\mathbb{A}$. If $S = S' \amalg \{v\}$, then the unramified local L-factor at v, which we consider as an element in the local Hecke algebra at v, appears in the description of the difference of the two functions ϕ_S and $\phi_{S'}$. This fact plays a key role in the construction of Euler systems of cyclotomic units ($d = 1$, $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}^{\infty}$ case) and of Kato's Euler system of Beilinson elements on modular curves ($d = 2$, $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}^{\infty}$ case); our construction is a generalization to arbitrary d.

For more general applications, for example in K-theory, we give a construction of Euler systems in an abstract, categorical setting. As a convenient tool in describing adelic calculations, we introduce the category $\mathcal{FC}^d$ and a certain topology on the category. We define special presheaves on this category which we call the Schwartz-Bruhat sheaves. A distribution is then a morphism of presheaves from the Schwartz-Bruhat sheaves, and an Euler system is the product of d distributions.

In the first half of Section 2, we list the basic definitions and properties of the sheaf theory on the new category $\mathcal{FC}^d$. The proofs are formal and elementary but long. We supply them in Appendix A for the reader's convenience.

In Section 3, we give the construction of special elements in the K-groups of Drinfeld modular varieties, and we prove that the system of those elements forms an Euler system. The global units called Siegel units are constructed, and the special elements are obtained as the image of those units under the symbol map. This is of course modeled on the construction by Beilinson [Be] (and more explicitly by Kato [Ka]) in the case of elliptic modular curves.

In Section 4, we prove a function field analogue of the Kronecker limit formula. This states that the logarithms of Siegel units are expressed as a limit of Eisenstein series. This is proved in the case of Drinfeld modules of rank one by Gross and Rosen ([Gr-Ro]), and in general by the first author ([Ko]). Here we give a shorter, more conceptual proof. The formula is applied to the computation of the regulator map in Proposition 6.2.

Studying the regulator map and the Kronecker limit formula, we arrive at an integral over $\mathrm{GL}_d(\mathbb{A})$, whose integrand is the product of a cusp form and the product of d Eisenstein series. In Section 5, we show that the integral is related to the L-function of the cusp form. The proof differs significantly from the case of elliptic modular curves. In [Be] (see also [Ka]), the Rankin-Selberg method was used to compute the integral. We prove that the product of Eisenstein series forms an Euler system, and use the norm property to unfold the integral.

In the first half of Section 6, we define a regulator map from the K-group of Drinfeld modular varieties to the group of harmonic cochains on the Bruhat-Tits building. Our regulator is motivated by the regulator map in the Beilinson conjectures. While the conjectures only handle the Archimedean place, there are a number of papers in which non-Archimedean places are considered. The very rough idea is that the regulator of a variety over a local field is simply the boundary map to the special fiber in the localization sequence. Our approach is very close to that of Consani ([Con]) and of Sreekantan ([Sr]), but differs from them in that we use rigid analytic geometry. We use the fact that Drinfeld modular varieties are analytically uniformized by Drinfeld symmetric spaces. In the second half of Section 6, we calculate the image under the regulator map of the members of the Euler system constructed in Section 3. Using the Kronecker limit formula and the result in Section 5, we derive our main theorem describing the cusp form part of the image in terms of the special values of L-functions of cuspidal automorphic representations. In this description a quantity $P(f)$ appears for each cusp form f. It is a certain integral concerning f on the normalizer of the diagonal torus of GL_d, and is called the period of f.

Our main interest lies in the case where the local component of f at the specified place is an Iwahori spherical vector of the Steinberg representations. Then $L(f,s)$ is conjecturally related to the L-functions of the Galois representation provided by the $(d-1)$-st l-adic cohomology groups of Drinfeld modular varieties. If we further assume $d=2$, the classical theory of Hecke and Jacquet-Langlands gives us an expression for $P(f)$ in terms of the special value of $L(f,s)$ at $s=\frac{1}{2}$. In Section 7, under certain conditions on f, we express $P(f)$ as the product of $L(f,0)$ and $L(f,1)$ when $d=3$ and when the global field which we are considering is the rational function field.

In Appendix A, we provide the detailed proofs of the results stated in the first half of Section 2.

In Appendix B, we express the L-factor at a bad prime as the sum of Hecke operators. This is included since we were not able to find an appropriate reference which suits our application. Perhaps it follows from [Ja-Pi-Sh], and it is known to experts. The approach taken there is again that of Section 2.

In Appendix C, we give an explicit formula (Proposition C.3) of the Iwahori-spherical Whittaker functions of the Steinberg representation of GL_d over a non-Archimedean local field. This formula is a direct consequence of Li's generalization [Li] of the Casselman-Shalika formula, which expresses the values of the Whittaker function on the diagonal torus. This is included since we need it for the calculation of $P(f)$ in Section 7.

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2. Abstract construction of Euler systems for GL_d from distributions

We introduce the categories $\mathcal{C}^d$ and $\mathcal{FC}^d$ and develop sheaf theory on them following Verdier [SGA4]. Those categories are not closed under fiber products, hence do not convey topology in the sense of Verdier, but we have a cofinality lemma (Lemma A.3) to circumvent the difficulty. The notion of Galois coverings is peculiar to our setting; it facilitates our exposition.

We consider three kinds of presheaves: sheaves, semi-sheaves, and presheaves with transfers. An example of a sheaf is the space of automorphic forms as discussed in Section 5. The presheaf constructed from K-theory (Section 3) is not a sheaf but it is equipped with transfers. The theorems are stated in the more general case of presheaves with semi-transfers. We work in this general setting, since there is a semi-sheaf of interest which is not a sheaf. An example is the integral structure of the l-adic sheaf of elliptic modular forms of weight greater than 2.

The connection with the adelic language is given by the functor ω (Section 2.1.4). We have an equivalence of categories between the category of abelian sheaves on $\mathcal{FC}^d$ and the category of smooth $\mathrm{GL}_d(\mathbb{A}_X)$-modules (Section A.1.10).

We could have refrained from using these new categories if we were interested only in Theorem 6.3, since its proof involves sheaves only. The translation of the notion of presheaves with transfers in the adelic setting seems difficult.

The functor ω may be interpreted as the fiber functor in Galois theory [SGA1]. From this point of view, our sheaf theory is the Galois theory of the locally profinite group $\mathrm{GL}_d(\mathbb{A}_X)$. While classically only profinite groups appear as the Galois group, this generalization suggests the extension of Galois theory to other algebraic groups.

For the motivation of the Schwartz-Bruhat sheaves, the reader is referred to Section 1.

In Section 2.1, we list the basic definitions and properties. The proofs are elementary but long, hence we give them in Appendix A. Our use of the term "fibration" should not be interpreted as alluding to homotopical algebra, and the term "presheaf with transfers" to Voevodsky's work.

The reader is advised to work out the case when $d = 1$. It may be applied to proving the norm compatibility of cyclotomic units. The case when $d = 2$ is closely related to the Euler system of Kato. We warn that the normalization is different from [Ka]; we choose $(m_{*,*})_*$ (see text for the notation) for our norm maps, while Kato chooses $(r_{*,*})_*$ as his norm maps (see Remark 2.10 for more details).

2.1. Let $d \geq 1$ be a positive integer. Let X be a regular noetherian scheme of Krull dimension one such that the residue field at each closed point is finite.

2.1.1. We define the category $\mathcal{C}^d = \mathcal{C}^d_X$ as follows. An object in $\mathcal{C}^d$ is a coherent $\mathcal{O}_X$-module of finite length which admits a surjection from $\mathcal{O}_X^{\oplus d}$. For two objects N and N' in $\mathcal{C}^d$, the set $\mathrm{Hom}_{\mathcal{C}^d}(N, N')$ of morphisms from N to N' is the set of isomorphism classes of diagrams

$$N' \twoheadleftarrow N'' \hookrightarrow N$$

in the category of coherent $\mathcal{O}_X$-modules where the left arrow is surjective and the right arrow is injective. This definition of morphisms is due to Quillen ([Qu]) except that here we take morphisms in the opposite direction.

We often consider the following two types of morphisms in $\mathcal{C}^d$. Let N be an object in $\mathcal{C}^d$. For a sub $\mathcal{O}_X$-module N' of N, the morphism $N' = N' \hookrightarrow N$ in $\mathcal{C}^d$ is denoted by $r_{N,N'} : N \to N'$. For a quotient $\mathcal{O}_X$-module N'' of N, the morphism $N'' \twoheadleftarrow N = N$ in $\mathcal{C}^d$ is denoted by $m_{N,N''} : N \to N'$.

2.1.2. Let $\mathcal{FC}^d$ denote the category of finite families of objects in $\mathcal{C}^d$. An object in $\mathcal{FC}^d$ is a pair $(J, (N_j)_{j\in J})$ where J is a finite set and $(N_j)_{j\in J}$ is a family of objects in $\mathcal{C}^d$ indexed by J. We denote the object $(J, (N_j)_{j\in J})$ by $\coprod_{j\in J} N_j$. We regard $\mathcal{C}^d$ as a full subcategory of $\mathcal{FC}^d$. We define $\pi_0(\coprod_{j\in J} N_j)$ to be the set J.

Definition 2.1. A *presheaf* on $\mathcal{FC}^d$ is a contravariant functor from $\mathcal{FC}^d$ to the category of sets. A presheaf F on $\mathcal{FC}^d$ is a *sheaf* if it satisfies the following conditions (1), (2) and (3):

(1) The image of the empty set $F(\emptyset)$ is the set of one element.
(2) For two objects N and N' in $\mathcal{FC}^d$, the canonical map $F(N \amalg N') \to F(N) \times F(N')$ is an isomorphism.
(3) Let $N \to N'$ be a covering in $\mathcal{FC}^d$. If the fiber product $N \times_{N'} N$ exists in $\mathcal{FC}^d$, then $F(N')$ is canonically isomorphic to the difference kernel of $F(N) \rightrightarrows F(N \times_{N'} N)$ where the maps are induced by the first and the second projections.

Let $f : N' \to N$ be a morphism in $\mathcal{FC}^d$, and let G be a subgroup of $\mathrm{Aut}_N(N')$. We say that f is a *Galois covering* of Galois group G if the fiber product $N' \times_N N'$ exists and if the morphism $\coprod_{g\in G}(g, \mathrm{id}) : \coprod_{g\in G} N' \to N' \times_N N'$ is an isomorphism. The inclusion of the category of sheaves on $\mathcal{FC}^d$ into the category of presheaves on $\mathcal{FC}^d$ has a right adjoint, which provides us with the notion of the sheaf associated to a presheaf.

2.1.3. Let N be an object in $\mathcal{FC}^d$ and H be a subgroup of $\mathrm{Aut}_{\mathcal{FC}^d}(N)$. We denote by N/H the sheaf associated to the presheaf $\mathrm{Hom}_{\mathcal{FC}^d}(-, N)/H$. Let $\widetilde{\mathcal{FC}}^d$ denote the full subcategory of the category of sheaves on $\mathcal{FC}^d$ whose objects are sheaves of the form N/H with N in $\mathcal{FC}^d$ and H a subgroup of $\mathrm{Aut}(N)$.

The notions of π_0 and covering are canonically extended to the category $\widetilde{\mathcal{FC}}^d$. We define the notions of sheaves on $\widetilde{\mathcal{FC}}^d$ and Galois coverings in $\widetilde{\mathcal{FC}}^d$ in a similar manner. There is an equivalence of categories between the category of sheaves on $\mathcal{FC}^d$ and the category of sheaves on $\widetilde{\mathcal{FC}}^d$.

2.1.4. We define the functor $\omega : \widetilde{\mathcal{FC}}^d \to (\mathrm{Sets})$ as follows. We consider $\mathbb{A}_X^{\oplus d}$ as the space of row vectors. Given a presheaf $F \in \mathrm{Presh}(\mathcal{FC}^d)$, we define $\omega(F)$ to be

$$\omega(F) = \varinjlim_{L_1 \subset L_2 \subset \mathbb{A}_X^d} F(L_2/L_1)$$

where the inductive limit is taken over the filtered ordered set of the pairs of two $\widehat{\mathcal{O}}_X$-lattices (L_1, L_2) in $\mathbb{A}_X^d$ with $L_1 \subset L_2$. The order is defined as follows: for two such pairs (L_1, L_2) and (L_1', L_2'), $(L_1, L_2) \geq (L_1', L_2')$ if and only if $L_1' \subset L_1 \subset L_2 \subset L_2'$.

We define the category $(\mathrm{GL}_d(\mathbb{A}_X)\text{-sets}^*)$. An object S in $(\mathrm{GL}_d(\mathbb{A}_X)\text{-sets}^*)$ is a set with left $\mathrm{GL}_d(\mathbb{A}_X)$-action such that S has finitely many $\mathrm{GL}_d(\mathbb{A}_X)$-orbits, and for any $s \in S$, the stabilizer at s is a compact open subgroup of $\mathrm{GL}_d(\mathbb{A}_X)$.

Lemma 2.2. *The functor* $\omega : \widetilde{\mathcal{FC}}^d \to (\mathrm{Sets})$ *gives an equivalence between the category* $\widetilde{\mathcal{FC}}^d$ *and the category* $(\mathrm{GL}_d(\mathbb{A}_X)\text{-sets}^*)$.

2.1.5. We use the following notations in Section 3.1.5. Let $\overline{N}$ be an object in $\widetilde{\mathcal{FC}}^d$ and let H be a finite group acting on $\overline{N}$. Let us consider the presheaf $N \mapsto \overline{N}(N)/H$. Then its associated sheaf defines an object in $\widetilde{\mathcal{FC}}^d$. We denote this sheaf by $\overline{N}/H$. It is easily checked that the set $\omega(\overline{N}/H)$ is canonically isomorphic to $\omega(\overline{N})/H$. We say that the action of H on $\overline{N}$ is free if the action of H on the set $\omega(\overline{N})$ is free. In other words, the action of H on $\overline{N}$ is free if the canonical morphism $\overline{N} \to \overline{N}/H$ is Galois and its Galois group is equal to H.

2.1.6. *Variant.* A morphism in $\mathcal{C}^d$ is called a *fibration* if it is isomorphic to a morphism of the form $m_{N,N'}$. A morphism $f : M \to M'$ in the category $\mathcal{FC}^d$ is said to be a fibration if it is a fibration in $\mathcal{C}^d$ on each component of M. We note that in the category $\mathcal{FC}^d$, the fiber product of two fibrations always exists.

A presheaf F on $\mathcal{FC}^d$ is a *semi-sheaf* if it satisfies the conditions (1), (2) in Definition 2.1 and the following condition $(3)'$:

- $(3)'$ If $N \to N'$ is a covering in $\mathcal{FC}^d$ which is a fibration, then $F(N')$ is canonically isomorphic to the difference kernel of $F(N) \rightrightarrows F(N \times_{N'} N)$ where the maps are induced by the first and the second projections.

The inclusion of the category of semi-sheaves on $\mathcal{FC}^d$ into the category of presheaves on $\mathcal{FC}^d$ has a right adjoint, which provides us with the notion of the semi-sheaf associated to a presheaf.

Let N be an object in $\mathcal{FC}^d$ and H be a subgroup of $\mathrm{Aut}_{\mathcal{FC}^d}(N)$. We denote by $(N/H)^\dagger$ the semi-sheaf associated to the presheaf $\mathrm{Hom}_{\mathcal{FC}^d}(\ ,N)/H$. Let $\widetilde{\mathcal{FC}}^{\dagger,d}$ denote the full subcategory of the category of sheaves on $\mathcal{FC}^d$ whose objects are semi-sheaves of the form $(N/H)^\dagger$. The notions of π_0 and covering are canonically extended to the category $\widetilde{\mathcal{FC}}^{\dagger,d}$.

We say that a morphism f in $\widetilde{\mathcal{FC}}^{\dagger,d}$ is a fibration if there exist two coverings g_1, g_2 in $\widetilde{\mathcal{FC}}^{\dagger,d}$ such that $g_2 \circ f \circ g_1$ is a fibration in $\mathcal{FC}^d$. We define semi-sheaves on $\widetilde{\mathcal{FC}}^{\dagger,d}$ in a similar way. The fiber product of two fibrations in $\widetilde{\mathcal{FC}}^{\dagger,d}$ always exists.

There is an equivalence of categories between the category of semi-sheaves on $\mathcal{FC}^d$ and the category of semi-sheaves on $\widetilde{\mathcal{FC}}^{\dagger,d}$.

The following lemma is used in the proof of Lemma 2.11. See Lemma A.16 for the proof.

Lemma 2.3. *Let $f : M \to N$ be a fibration in $\mathcal{FC}^d$. Suppose a finite group H acts equivariantly on M and N. Then the induced morphism $m : (M/H)^\dagger \to (N/H)^\dagger$ is a fibration in $\widetilde{\mathcal{FC}}^{\dagger,d}$.*

2.1.7. We define the functor $\omega^\dagger : \mathrm{Presh}(\mathcal{FC}^d) \to (\mathrm{Sets})$ as follows. Given a presheaf $F \in \mathrm{Presh}(\mathcal{FC}^d)$, we define $\omega^\dagger(F)$ to be

$$\omega^\dagger(F) = \varinjlim_{L \subset \widehat{\mathcal{O}}_X^{\oplus d}} F(\widehat{\mathcal{O}}_X^{\oplus d}/L)$$

where the inductive limit is taken over the $\widehat{\mathcal{O}}_X$-lattices in $\widehat{\mathcal{O}}_X^{\oplus d}$ ordered by inclusion.

Let $F = (\coprod_{j \in J} N_j)/H$ be an object in $\widetilde{\mathcal{FC}}^{\dagger,d}$. We have a map

$$\omega^\dagger(F) = (\coprod_{j \in J} \omega^\dagger(\mathrm{Hom}(-, N_j)))/H \to J/H = \pi_0(F)$$

induced by the map which sends the elements in $\omega^\dagger(\mathrm{Hom}(-, N_j))$ to j.

Let $f : F_0 \to F$ be a fibration in $\widetilde{\mathcal{FC}}^{\dagger,d}$. The map $\omega^\dagger(F) \to \mathbb{Z}_{\geq 0}$, which sends $x \in \omega^\dagger(F)$ to $\#\omega^\dagger(f)^{-1}(x)$, factors through $\pi_0(F)$. We call the induced map $\deg f : \pi_0(F) \to \mathbb{Z}_{\geq 0}$ the degree of f.

In Section A.1.11, the degree is defined using the functor ω instead of $\omega^\dagger$. The two definitions coincide for fibrations. The following two lemmas follow from Lemmas A.13 and A.14.

Lemma 2.4. *Let*

$$\begin{tikzcd} F_1' \arrow[r, "g_1"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & F_1 \arrow[d, "f"] \\ F_2' \arrow[r, "g_2"] & F_2 \end{tikzcd}$$

be a cartesian diagram in $\widetilde{\mathcal{FC}}^{\dagger,d}$ where all morphisms are fibrations. Then for any $y \in \pi_0(F_2')$, we have $(\deg f')(y) = (\deg f)(\pi_0(g_2)(y))$.

Lemma 2.5. *Let $N = \coprod_{i \in \pi_0(N)} N_i$ be an object in $\mathcal{FC}^d$ and H be a subgroup of $\mathrm{Aut}_{\mathcal{FC}^d}(N)$. Suppose that, for each $i \in \pi_0(N)$, the stabilizer $H_i \subset H$ of i acts faithfully on N_i. Then the canonical quotient map $f : (N/\{\mathrm{id}_N\})^\dagger \to (N/H)^\dagger$ is a Galois covering, and we have $(\deg f)(i) = \#H$ for any $i \in \pi_0((N/H)^\dagger)$.*

2.2. Presheaves with transfers.

Definition 2.6. An *abelian presheaf with transfers* on $\widetilde{\mathcal{FC}}^d$ (resp. *with semi-transfers* on $\widetilde{\mathcal{FC}}^{\dagger,d}$) is a presheaf F of abelian groups on $\widetilde{\mathcal{FC}}^d$ (resp. $\widetilde{\mathcal{FC}}^{\dagger,d}$) equipped with, for each morphism (resp. each fibration) $f : N \to N'$ in $\widetilde{\mathcal{FC}}^d$ (resp. in $\widetilde{\mathcal{FC}}^{\dagger,d}$), a homomorphism $f_* : F(N) \to F(N')$ satisfying the following properties:

(1) For any two composable morphisms (resp. fibrations) f and f', $(f \circ f')_* = f_* \circ f'_*$.

(2) For any cartesian diagram

$$\begin{tikzcd} N'_1 \arrow[r, "g_1"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & N_1 \arrow[d, "f"] \\ N'_2 \arrow[r, "g_2"] & N_2 \end{tikzcd}$$

in $\widetilde{\mathcal{FC}}^d$ (resp. in $\widetilde{\mathcal{FC}}^{\dagger,d}$, with f, f' fibrations), we have $g_2^* \circ f_* = f'_* \circ g_1^*$.

(3) The composite $f_* \circ f^*$ is the multiplication by $\deg f$.

2.2.1. Any abelian sheaf on $\widetilde{\mathcal{FC}}^d$ (resp. semi-sheaf on $\widetilde{\mathcal{FC}}^{\dagger,d}$) has a unique structure of abelian presheaf with transfers on $\widetilde{\mathcal{FC}}^d$ (resp. semi-transfers on $\widetilde{\mathcal{FC}}^{\dagger,d}$).

2.2.2. A homomorphism of abelian presheaves with transfers (resp. semi-transfers) is a homomorphism of abelian presheaves compatible with f_*. If F is an abelian sheaf (resp. semi-sheaf) on $\widetilde{\mathcal{FC}}^d$ (resp. on $\widetilde{\mathcal{FC}}^{\dagger,d}$), any homomorphism of abelian presheaves from an abelian presheaf with transfers (resp. semi-transfers) to F is compatible with f_*.

2.2.3. *Hecke operators.* Let N be an object in $\mathcal{C}^d$, and N' be an object in $\widetilde{\mathcal{FC}}^d$. Suppose that N' is of the form $\coprod_j N'_j/H_j$ such that $N'_j \oplus N$ is an object in $\mathcal{C}^d$ for every j. We define an object $N' \oplus [N]$ by

$$N' \oplus [N] = \coprod_j (N'_j \oplus N)/(H_j \times \mathrm{Aut}_{\mathcal{O}_X}(N)).$$

The two morphisms $N'_j = N'_j \hookrightarrow N'_j \oplus N$ and $N'_j \twoheadleftarrow N'_j \oplus N = N'_j \oplus N$ induce the morphisms

$$r_{N'\oplus[N],N'}, m_{N'\oplus[N],N'} : N' \oplus [N] \to N'$$

in $\widetilde{\mathcal{FC}}^d$.

Let F be an abelian presheaf with transfers on $\widetilde{\mathcal{FC}}^d$ (resp. semi-transfers on $\widetilde{\mathcal{FC}}^{\dagger,d}$). The composite

$$(m_{N'\oplus[N],N'})_* r^*_{N'\oplus[N],N'} : F(N') \to F(N')$$

is called the Hecke operator for $[N]$ (resp. $[N]^\dagger$) and is denoted by $T_{[N]}$ (resp. $T_{[N]^\dagger}$).

Definition 2.7. A *presheaf of rings with transfers* on $\widetilde{\mathcal{FC}}^d$ (resp. *semi-transfers* on $\widetilde{\mathcal{FC}}^{\dagger,d}$) is a presheaf G of rings on $\widetilde{\mathcal{FC}}^d$ (resp. on $\widetilde{\mathcal{FC}}^{\dagger,d}$) equipped with a structure of abelian presheaf with transfers (resp. semi-transfers) satisfying the following property:

- For any morphism (resp. fibration) $f : N \to N'$, $x \in G(N)$ and $y \in G(N')$, we have $f_*(x \cdot f^*y) = f_*(x) \cdot y$ and $f_*(f^*y \cdot x) = y \cdot f_*(x)$.

Any sheaf of rings on $\widetilde{\mathcal{FC}}^d$ (resp. semi-sheaf of rings on $\widetilde{\mathcal{FC}}^{\dagger,d}$) has a unique structure of presheaf of rings with transfers on $\widetilde{\mathcal{FC}}^d$ (resp. semi-transfers on $\widetilde{\mathcal{FC}}^{\dagger,d}$).

2.3. Distributions.

2.3.1. We define two special abelian presheaves $\mathrm{SB}' = \mathrm{SB}'_d$, and $\mathrm{SB}^{*'} = \mathrm{SB}^{*'}_d$ on $\mathcal{FC}^d$. For an object N in $\mathcal{C}^d$, let $\mathrm{SB}'(N)$ (resp. $\mathrm{SB}^{*'}(N)$) be the free abelian group generated by the set $\Gamma(X,N)$ (resp. $\Gamma(X,N)\backslash\{0\}$). For a morphism $N' \overset{p}{\twoheadleftarrow} N'' \overset{i}{\hookrightarrow} N$ from N to N' in $\mathcal{C}^d$, define a homomorphism $\mathrm{SB}'(N') \to \mathrm{SB}'(N)$ (resp. $\mathrm{SB}^{*'}(N') \to \mathrm{SB}^{*'}(N)$) by sending $x \in \Gamma(X,N')$ (resp. $x \in \Gamma(X,N') \setminus \{0\}$) to the element $\sum_{p(y)=x} i(y)$. Finally, for an object $\coprod_j N_j$ in $\mathcal{FC}^d$, put $\mathrm{SB}'(\coprod_j N_j) = \prod_j \mathrm{SB}'(N_j)$ (resp. $\mathrm{SB}^{*'}(\coprod_j N_j) = \prod_j \mathrm{SB}^{*'}(N_j)$). We denote the sheaf (resp. semi-sheaf) associated to the presheaf SB', $\mathrm{SB}^{*'}$ by SB, SB^* (resp. $\mathrm{SB}^\dagger$, $\mathrm{SB}^{*\dagger}$) respectively and call them the Schwartz-Bruhat sheaf (of rank d), the punctured Schwartz-Bruhat sheaf (of rank d) (resp. the Schwartz-Bruhat semi-sheaf (of rank d), the punctured Schwartz-Bruhat semi-sheaf (of rank d)) respectively.

2.3.2. Under the equivalence described in Section A.1.9, the sheaf SB (resp. SB^*) corresponds to the smooth $\mathrm{GL}_d(\mathbb{A}_X)$-module $\mathcal{S}(\mathbb{A}_X^{\oplus d})$ of locally constant, compactly supported $\mathbb{Z}$-valued functions on $\mathbb{A}_X^{\oplus d}$ (resp. the submodule of $\mathcal{S}(\mathbb{A}_X^{\oplus d})$ of the functions f with $f(0) = 0$). Similarly, the semi-sheaf $\mathrm{SB}^\dagger$ corresponds to the submodule $\mathcal{S}(\widehat{\mathcal{O}}_X^{\oplus d})$ of $\mathcal{S}(\mathbb{A}_X^{\oplus d})$ which consists of the functions whose support is contained in $\widehat{\mathcal{O}}_X^{\oplus d}$.

Definition 2.8. Let F be an abelian sheaf on $\mathcal{FC}^d$. A *distribution* (resp. *punctured distribution*) with values in F is a homomorphism $\mathrm{SB} \to F$ (resp. $\mathrm{SB}^* \to F$) of abelian sheaves on $\mathcal{FC}^d$. For an abelian semi-sheaf F, the notions of semi-distribution and punctured semi-distribution are defined in a similar way.

2.4. Construction of Euler systems.

2.4.1. Let G be a presheaf of rings with semi-transfers on $\widetilde{\mathcal{FC}}^{\dagger,d}$. Suppose for $i = 1,\ldots,d$ the following data are given.

(1) An abelian semi-sheaf F_i on $\widetilde{\mathcal{FC}}^{\dagger,d}$.
(2) A homomorphism $\alpha_i : F_i \to G$ of presheaves with semi-transfers.
(3) An object N_i in $\mathcal{C}^1$ and a generator $b_i \in \Gamma(X,N_i)$ as $\widehat{\mathcal{O}}_X$-module.
(4) A quotient $\mathcal{O}_X$-module N_i' of N_i.

Let b_i' denote the image of b_i in N_i'. We write $\mathbf{N} = \bigoplus_{j=1}^d N_j$, $\mathbf{N}' = \bigoplus_j N_j'$ and $N_i'' = \mathrm{Ker}(N_i \twoheadrightarrow N_i')$.

We consider two settings which we call Situation I and Situation II. In Situation I, we have a semi-distribution $g_i : \mathrm{SB}^\dagger \to F_i$ with values in F_i for each i. In Situation II, we assume $N_i' \neq 0$ and we have a punctured semi-distribution $g_i : \mathrm{SB}^{\dagger *} \to F_i$ with values in F_i for each i. We put

$$\kappa_{\mathbf{N},(b_j)} = \prod_{j=1}^d \mathrm{pr}_j^*(\alpha_j g_j(N_j)[b_j]) \in G(\mathbf{N})$$

where pr_j $(j = 1,\ldots,d)$ is the morphism $N_j = N_j \hookrightarrow \mathbf{N}$ from $\mathbf{N}$ to N_j given by the inclusion of N_j into the j-th factor of $\mathbf{N}$.

Theorem 2.9. *Suppose that we are either in Situation I or in II.*
Then the following statements hold.

(1) *If* $\mathrm{Supp}\,(N_i'') \subset \mathrm{Supp}\,(N_j')$ *for any* $1 \le i, j \le d$, *then*

$$(m_{\mathbf{N},\mathbf{N}'})_* \kappa_{\mathbf{N},(b_j)} = \kappa_{\mathbf{N}',(b_j')}.$$

(2) *Let* $\wp$ *be a closed point of* X. *Suppose that* $\mathrm{Supp}\,(N_i'') \subset \{\wp\} \subset \mathrm{Supp}\,(N_i)$ *for every* i. *Let* e *denote the number of* i*'s with* $\wp \notin \mathrm{Supp}\,(N_i')$. *Then*

$$(m_{\mathbf{N},\mathbf{N}'})_* \kappa_{\mathbf{N},(b_j)} = \sum_{r=0}^{e} (-1)^r q_\wp^{r(r-1)/2} T_{[\wp^{\oplus r}]^\dagger} \kappa_{\mathbf{N}',(b_j')},$$

where $q_\wp$ *is the cardinality of the residue field at* $\wp$.

Remark 2.10. There is a variant of Theorem 2.9 where the roles of $m_{\mathbf{N},\mathbf{N}'}$ and $r_{\mathbf{N},\mathbf{N}'}$ are interchanged. The association $N \mapsto \mathbb{D}(N) = \mathrm{Hom}_{\widehat{\mathcal{O}}_X}(N, \mathbb{A}_X/\widehat{\mathcal{O}}_X)$ gives a covariant auto-equivalence of the category $\mathcal{C}^d$. It is canonically extended to an auto-equivalence $\mathbb{D}$ of the category $\mathcal{FC}^d$. A presheaf F on $\mathcal{FC}^d$ is called a dual semi-sheaf if $F \circ \mathbb{D}$ is a semi-sheaf. We define the dual version of the category $\widetilde{\mathcal{FC}}^d$ and the notion of presheaves with dual semi-transfers. Let us change the notations and the situation in Theorem 2.9 at the following four points:

- The presheaf G is a presheaf of rings with dual semi-transfers.
- For $i = 1, \ldots, d$, F_i is a dual semi-sheaf on $\widetilde{\mathcal{FC}}^{\dagger,d}$.
- For $i = 1, \ldots, d$, N_i' is a submodule of N_i and $b_i' \in N_i'$ is an arbitrary generator,
- In Situation I (resp. II), $\mathrm{SB}^\dagger$ (resp. $\mathrm{SB}^{\dagger *}$) is the dual semi-sheaf associated to the presheaf SB' (resp. $\mathrm{SB}^{*'}$).

Then we have formulae describing the image of $\kappa_{\mathbf{N},(b_j)}$ under $(r_{\mathbf{N},\mathbf{N}'})_*$, which are proved in a manner similar to Theorem 2.9. This variant is not an immediate consequence of Theorem 2.9, since we do not replace pr_j's by their duals in the definition of $\kappa_{\mathbf{N},(b_j)}$. In [Ka], Kato states the norm relations concerning his system on elliptic modular curves in the style of this variant. However, we will not pursue this variant further since this has the disadvantage that extra parameters must enter in order to describe the relations between b_i's and b_i''s. Details are left to the reader.

2.5. Proof of Theorem 2.9 (1). We set $\mathbf{N}'' = \bigoplus_i N_i''$. By induction, we may assume that $\mathrm{length}_{\mathcal{O}_{X,x}} N_i' \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x} \ge \mathrm{length}_{\mathcal{O}_{X,x}} N_j'' \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}$ for any $x \in X$ and for any i, j.

The morphism $m_{\mathbf{N},\mathbf{N}'} : \mathbf{N} \to \mathbf{N}'$ is a Galois covering. Its Galois group is isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\mathbf{N}, \mathbf{N}'') \cong \mathrm{Aut}_{\mathbf{N}'}(\mathbf{N})$ via the map which sends $f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathbf{N}, \mathbf{N}'')$ to the $\mathcal{O}_X$-automorphism $n \mapsto n + f(n)$. By the assumption on the length of N_i' and of N_i'', the Galois group is also isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\mathbf{N}', \mathbf{N}'')$.

Given an object M in $\mathcal{C}^d$, by abuse of notation, we denote simply by M the semi-sheaf $(M/\{\mathrm{id}_M\})^\dagger$. We define the objects and the morphisms of the following diagram in $\widetilde{\mathcal{FC}}^{\dagger,d}$ and show that it is commutative and that the middle square is cartesian:

$$\begin{tikzcd}
\mathbf{N} \arrow[r, "\beta_i"] \arrow[rrr, bend left, "r_{\mathbf{N},N_i}"] \arrow[dr, "m_{\mathbf{N},\mathbf{N}'}"'] & \overline{N}_i \arrow[r, "\gamma_i"] \arrow[d, "\delta_i"] \arrow[dr, phantom, "\square"] & \widetilde{N}_i \arrow[r, "r_{\widetilde{N}_i,N_i}"] \arrow[d, "\delta_i'"] \arrow[dr, "m_{\widetilde{N}_i,N_i'}"] & N_i \\
& \mathbf{N}' \arrow[r, "\gamma_i'"] \arrow[rr, bend right, "r_{\mathbf{N}',N_i'}"'] & (\widetilde{N}_i/H_i)^\dagger \arrow[r, "\epsilon_i"] & N_i'.
\end{tikzcd}$$

Consider the subgroup $H_i' = \mathrm{Hom}_{\mathcal{O}_X}(\bigoplus_{j\neq i} N_j', \mathbf{N}'') \subseteq \mathrm{Hom}(\mathbf{N}', \mathbf{N}'')$ for each $i = 1, \ldots, d$. Let $\overline{N}_i = (\mathbf{N}/H_i')^\dagger$, and $\beta_i : \mathbf{N} \to \overline{N}_i$ be the Galois covering. Then the canonical morphism

$$\beta_1 \times \cdots \times \beta_d : \mathbf{N} \to \overline{N}_1 \times_{\mathbf{N}'} \cdots \times_{\mathbf{N}'} \overline{N}_d$$

is an isomorphism.

Let $\widetilde{N}_i$ denote the inverse image of $N_i' \subset \mathbf{N}'$ by the $\mathcal{O}_X$-homomorphism $\mathbf{N} \to \mathbf{N}'$. Let H_i be the group of the $\mathcal{O}_X$-module automorphisms on $\widetilde{N}_i$ which induce identities on both $\mathrm{Ker}(\widetilde{N}_i \to N_i')$ and N_i'. Then the morphism $m_{\widetilde{N}_i,N_i}$ factors through $(\widetilde{N}_i/H_i)^\dagger$ as in the diagram above since the morphism of presheaves factors as $\mathrm{Hom}_{\mathcal{FC}^d}(-, \widetilde{N}_i) \to \mathrm{Hom}_{\mathcal{FC}^d}(-, \widetilde{N}_i)/H_i \to \mathrm{Hom}_{\mathcal{FC}^d}(-, N_i)$.

Let us construct the morphism $\gamma_i' : \mathbf{N}' \to \widetilde{N}_i/H_i$. Let X be an object of $\mathcal{FC}^d$. We define the map $\mathbf{N}'(X) = \varinjlim_{X' \to X} \mathrm{Hom}(X', \mathbf{N}) \to (\widetilde{N}_i/H_i)^\dagger(X)$ as follows. Let $f \in \mathrm{Hom}_{\mathcal{FC}^d}(X', \mathbf{N})$ for some Galois covering $X' \to X$. By Lemma A.3, there exist an object $M \in \mathcal{FC}^d$ which is Galois over N_i' (in particular Galois over X'), and morphisms g and h which make the following diagram commutative:

$$\begin{tikzcd}
M \arrow[rr, "g"] \arrow[d, "h"] & & \widetilde{N}_i \arrow[d, "m_{\widetilde{N}_i,N_i'}"] \\
X' \arrow[r, "f"] & \mathbf{N}' \arrow[r, "r_{\mathbf{N}',N_i'}"] & N_i'.
\end{tikzcd}$$

We see that the map $\mathrm{Hom}_{\mathcal{FC}^d}(X', \mathbf{N}') \to \mathrm{Hom}_{\mathcal{FC}^d}(M, \widetilde{N}_i)/H_i$ sending f to g is well-defined, and that $r_{\mathbf{N}',N_i'} = \epsilon_i \circ \gamma_i'$.

Consider the morphism $r_{\mathbf{N},\widetilde{N}_i} : \mathbf{N} \to \widetilde{N}_i$. Since the morphism of presheaves factors as $\mathrm{Hom}_{\mathcal{FC}^d}(-, \mathbf{N}) \to \mathrm{Hom}_{\mathcal{FC}^d}(-, \mathbf{N})/H_i' \to \mathrm{Hom}_{\mathcal{FC}^d}(-, \widetilde{N}_i)$, $r_{\mathbf{N},\widetilde{N}_i}$ factors as in the diagram.

Note that $\mathrm{Hom}_{\mathcal{O}_X}(\mathbf{N}', \mathbf{N}'')/H_i' \to \mathrm{Aut}(\widetilde{N})$ which maps f to $(n \mapsto n + f(n))$ is an isomorphism onto H_i. One can check that the middle square is cartesian.

By the definition of $\kappa_{\mathbf{N},(b_j)}$, we have

$$\kappa_{\mathbf{N},(b_j)} = (\beta_1 \times \cdots \times \beta_d)^* \prod_{j=1}^{d} \check{\mathrm{pr}}_j^*(r_{\widetilde{N}_j,N_j}\gamma_j)^*(\alpha_j g_j(N_j)[b_j])$$

where $\check{\mathrm{pr}}_j : \overline{N}_1 \times_{\mathbf{N}'} \cdots \times_{\mathbf{N}'} \overline{N}_d \to \overline{N}_j$ denotes the j-th projection. Hence,

$$\begin{aligned}(m_{\mathbf{N},\mathbf{N}'})_*\kappa_{\mathbf{N},(b_j)} &= (\delta_1 \times \cdots \times \delta_d)_* \prod_{j=1}^{d} \check{\mathrm{pr}}_j^*(r_{\widetilde{N}_j,N_j}\gamma_j)^*(\alpha_j g_j(N_j)[b_j]) \\ &= \prod_{j=1}^{d} (\gamma_j')^*(\delta_j')_* r^*_{\widetilde{N}_j,N_j}(\alpha_j g_j(N_j)[b_j]).\end{aligned}$$

Let y be an element in $\Gamma(\widetilde{N}_i)$, and y' be its image in N_i'. Suppose that N_i' is generated by y'. Let $[y] \in \mathrm{SB}^{*\dagger}(\widetilde{N}_i) \subset \mathrm{SB}^{\dagger}(\widetilde{N}_i)$, $[y'] \in \mathrm{SB}^{*\dagger}(N_i') \subset \mathrm{SB}^{\dagger}(N_i')$ be the sections corresponding to y, y'. Then we have

$$\delta'^*\delta'_*([b]) = \sum_{x \in \mathrm{Ker}(\widetilde{N} \to N')} [b + x] = m^*_{\widetilde{N},N'}([b']).$$

The first equality is because the group H_i is isomorphic to $\mathrm{Hom}(N_i', \mathrm{Ker}(\widetilde{N}_i \to N_i'))$, and the second equality follows from the definition of the Schwartz-Bruhat sheaf. Since δ' is a fibration, it follows that $\delta'_*([y]) = \epsilon^*[y']$. Applying this, we then have

$$\begin{aligned}(m_{\mathbf{N},\mathbf{N}'})_*\kappa_{\mathbf{N},(b_j)} &= \textstyle\prod_{j=1}^{d} r^*_{\mathbf{N}',N_j'}(\alpha_j g_j(N_j')[b_j']) \\ &= \kappa_{\mathbf{N}',(b_j')}.\end{aligned}$$

□

2.6. Proof of Theorem 2.9 (2). By (1), we may and will assume that $X = \mathrm{Spec}\,(\mathcal{O}_{X,\wp})$, $e \geq 1$, $N_1 = \cdots = N_e = N \cong \kappa(\wp)$, $N_i' = 0$ for $i = 1, \ldots, e$, $N_i = N_i' \neq 0$ for $i = e+1, \ldots, d$, and we are in Situation I.

2.6.1. We set $\mathbf{N}'' = \bigoplus_{i=1}^{e} N_i \cong \kappa(\wp)^{\oplus e}$. For a sub $\mathcal{O}_X$-module M of $\mathbf{N}''$, let H_M denote the group of the automorphisms of $\mathbf{N}''$ which stabilize M. For $r = 0, \ldots, e$, we define two objects M_r, $\widetilde{M}_r$ in $\widetilde{\mathcal{FC}}^{\dagger,d}$ as follows. We set $M_r'' = \bigoplus_{i=1}^{r} N_i$. We define M_r to be $(\mathbf{N}/H_{M_r''})^\dagger$. The morphisms $r_{\mathbf{N},\mathbf{N}'}, m_{\mathbf{N},\mathbf{N}'} : \mathbf{N} \to \mathbf{N}'$ induce morphisms $M_r \to \mathbf{N}'$ which are denoted $r_{M_r,\mathbf{N}'}$, $m_{M_r,\mathbf{N}'}$ respectively.

Let $h \in H_{M_r''}$. We have two $\mathcal{O}_X$-automorphisms $h|_{M_r''} : M_r'' \to M_r''$ and $h \oplus \mathrm{id}_{\mathbf{N}'} : \mathbf{N} = \mathbf{N}'' \oplus \mathbf{N}' \to \mathbf{N}$. We define $\coprod_{\nu:N\to M_r''} \mathbf{N} \to \coprod_{\nu:N\to M_r''} \mathbf{N}$, where ν runs over the $\mathcal{O}_X$-homomorphisms from N to M_r'', to be the morphism which maps the α-th component to the $h|_{M_r''} \circ \alpha$-th component via the morphism $h \oplus \mathrm{id}_{\mathbf{N}'}$. This gives an action of $H_{Mr''}$ on $\coprod_{\nu:N\to M_r''} \mathbf{N}$. We let $\widetilde{M}_r = ((\coprod_{\nu:N\to M_r''} \mathbf{N})/H_{M_r''})^\dagger$ be the quotient by this action.

The morphism $\coprod_{\nu:N\to M_r''} \mathbf{N} \to \mathbf{N}$ given by the identity map on every component induces the morphism $\eta_r : \widetilde{M}_r \to M_r$ for each $r = 0, \ldots, e$. The morphism $\coprod_{\nu:N\to M_r''} \mathbf{N} \to \coprod_{\nu:N\to \mathbf{N}} \mathbf{N}$ given by the identity map from the ν-th component to the component associated to the map $N \to M_r'' \hookrightarrow \mathbf{N}$ induces $f_r : \widetilde{M}_r \to \widetilde{M}_e$ for each r. Let $f_r' : M_r \to M_e$ be the natural quotient map. Then the following is a commutative diagram in $\widetilde{\mathcal{FC}}^{\dagger,d}$ for each r:

$$\begin{array}{ccc} \widetilde{M}_r & \xrightarrow{\eta_r} & M_r \\ {\scriptstyle f_r}\big\downarrow & & \big\downarrow{\scriptstyle f_r'} \\ \widetilde{M}_e & \xrightarrow{\eta_e} & M_e. \end{array}$$

2.6.2. For each $i = 1, \dots, e$, let $\alpha_i : N \to N \oplus \cdots \oplus N = \mathbf{N}'' \hookrightarrow \mathbf{N}$ be the morphism given by the inclusion into the i-th factor. The morphism $\alpha_i' : \mathbf{N} \to \coprod_{\nu : N \to \mathbf{N}} \mathbf{N} \to M_e$, where the first morphism is the identity map to the α_i-th component and the second map is the natural quotient map, factors as $\mathbf{N} \to \widetilde{M}_e \to M_e$. We let $\mathbf{M}_e = \widetilde{M}_e \times_{M_e} \cdots \times_{M_e} \widetilde{M}_e$ be the e-fold fiber product of η_e, and $\iota : \mathbf{N} \to \mathbf{M}_e$ be the morphism $(\alpha_1', \dots, \alpha_e')$.

2.6.3. For each $r = 0, \dots, e$, we let $\mathbf{M}_r$ denote the e-fold fiber product $\widetilde{M}_r \times_{M_r} \cdots \times_{M_r} \widetilde{M}_r$ of $\widetilde{M}_r$ over M_r, and let $\widetilde{f}_r = f_r \times \cdots \times f_r : \mathbf{M}_r \to \mathbf{M}_e$.

Lemma 2.11. *Let the notations be as above. The morphisms ι and $\widetilde{f}_j$ for each $j = 0, \dots, d$ are fibrations, and we have*

$$\deg \iota = \sum_{r=0}^{e} (-1)^r q_\wp^{r(r-1)/2} \deg \widetilde{f}_{e-r}.$$

Proof of Lemma 2.11. We recall some notations in q-calculus. For non-negative integers $s, m \le n$, we let $[s] = \frac{{q_\wp}^s - 1}{q_\wp - 1}$, $[s]! = [s][s-1]\cdots[1]$, and $\begin{bmatrix} n \\ m \end{bmatrix} = \dfrac{[n]!}{[m]![n-m]!}$.

We pull everything back by the natural quotient map $\mathbf{N} \to M_e = \mathbf{N}/\mathrm{Aut}(\mathbf{N}'')$. Note that $M_r \times_{M_e} \mathbf{N} \cong \coprod_{\mathrm{Aut}(\mathbf{N}'')/H_{M_r''}} \mathbf{N}$, and $\widetilde{M}_r \times_{M_r} \mathbf{N} \cong \coprod_{\nu : N \to M_r''} \mathbf{N}$, hence $\mathbf{M}_r \times_{M_r} \mathbf{N} \cong \coprod_{\nu_1, \dots, \nu_e : N \to M_r''} \mathbf{N}$. Let $\mathrm{Gr}(\mathbf{N}'', r)$ be the set of r-dimensional subspaces of $\mathbf{N}''$. Then the map which sends $g \in \mathrm{Aut}(\mathbf{N}'')$ to $g(M_r'')$ induces an isomorphism $\mathrm{Aut}(\mathbf{N}'')/H_{M_r''} \cong \mathrm{Gr}(\mathbf{N}'', r)$. We may then express $\mathbf{M}_r \times_{M_e} \mathbf{N}$ as

$$\coprod_{W \in \mathrm{Gr}(\mathbf{N}'', r)} \ \coprod_{\nu_1 \dots, \nu_e : N \to W} \mathbf{N}.$$

When $r = e$, $\mathbf{M}_e \times_{M_e} \mathbf{N} = \coprod_{\nu_1, \dots, \nu_e : N \to \mathbf{N}''} \mathbf{N}$. The morphism $\widetilde{f}_r \times \mathrm{id}_\mathbf{N} : \mathbf{M}_r \times_{M_e} \mathbf{N} \to \mathbf{M}_e \times_{M_e} \mathbf{N}$ maps the component corresponding to $(W; \nu_1, \dots, \nu_e)$ to the component corresponding to $(i_W \circ \nu_1, \dots, i_W \circ \nu_e)$ where $i_W : W \hookrightarrow \mathbf{N}''$ is the canonical inclusion map.

With this explicit description of $\widetilde{f}_j \times \mathrm{id}_\mathbf{N}$ for each $j = 0, \dots, d$, we see that $\widetilde{f}_j$ for each j is a fibration, by using Lemma 2.3. We also find that $\iota \times \mathrm{id}_\mathbf{N}$ is a fibration; it implies that ι is a fibration.

Let $\nu = (\nu_1, \dots, \nu_e) \in \pi_0(\mathbf{M}_e \times_{M_e} \mathbf{N})$ and let $V = \sum_{i=1}^{e} \mathrm{Im}\, \nu_i$. The degree of $\widetilde{f}_r \times \mathrm{id}_\mathbf{N}$ on $\nu \in \pi_0(\mathbf{M}_e \times_{M_e} \mathbf{N})$ is then equal to the number of r-dimensional subspaces of $\mathbf{N}'$ containing V. Since $\begin{bmatrix} n \\ j \end{bmatrix}$ is the number of j-dimensional subspaces in an n-dimensional $\kappa(\wp)$-vector space, we have

$$\begin{aligned} &\deg(\widetilde{f}_r \times \mathrm{id}_\mathbf{N}) \text{ on } \nu \\ &= \begin{cases} \begin{bmatrix} e - \dim_{\kappa(\wp)} V \\ r - \dim_{\kappa(\wp)} V \end{bmatrix}, & \text{if } \dim_{\kappa(\wp)} V \le r \\ 0, & \text{if } \dim_{\kappa(\wp)} V > r. \end{cases} \end{aligned}$$

Applying Gauss' binomial formula [Ka-Ch, (5.5)], we have, on ν,

$$\sum_{r=0}^{e} (-1)^r q_\wp^{r(r-1)/2} \deg(\widetilde{f}_{e-r} \times \mathrm{id}_\mathbf{N}) = \begin{cases} 0, & \text{if } V \ne \mathbf{N}'' \\ 1, & \text{if } V = \mathbf{N}''. \end{cases}$$

On the other hand, the degree of $\iota \times \mathrm{id}_{\mathbf{N}}$ on ν is

$$\deg(\iota \times \mathrm{id}_{\mathbf{N}}) = \begin{cases} 0, & \text{if } V \neq \mathbf{N}'' \\ 1, & \text{if } V = \mathbf{N}''. \end{cases}$$

Hence

$$\sum_{r=0}^{e} (-1)^r q_{\wp}^{r(r-1)/2} \deg(\widetilde{f}_{e-r} \times \mathrm{id}_N) = \deg(\iota \times \mathrm{id}_N).$$

Let $\overline{\nu} \in \pi_0(M_e) = \pi_0(\mathbf{M}_e \times_{M_e} \mathbf{N})/\mathrm{Aut}(\mathbf{N}'')$ and $\nu \in \pi_0(\mathbf{M}_e \times_{M_e} \mathbf{N})$ be an element representing $\overline{\nu}$. By Lemma 2.4, $\deg(\widetilde{f}_{e-r} \times \mathrm{id}_{\mathbf{N}})(\nu) = \deg \widetilde{f}_{e-r}(\overline{\nu})$, and $\deg(\iota \times \mathrm{id}_N)(\nu) = \deg \iota(\overline{\nu})$. The assertion follows. $\square$

2.6.4. We return to the proof of Theorem 2.9 (2). For $j = 1, \ldots, e$, the element

$$([\nu(b_j)])_\nu \in \mathrm{SB}^\dagger(\coprod_{\nu : N \to \mathbf{N}''} \mathbf{N})$$

defines an element in $\mathrm{SB}^\dagger(\widetilde{M}_e)$, which we denote by $\widetilde{b_j}$. We put $\widetilde{g}_j = \alpha_j g_j(\widetilde{M}_e)(\widetilde{b}_j)$. We also set

$$\kappa' = \prod_{j=e+1}^{d} r^*_{\mathbf{N}', N_j} \alpha_j g_j(N_j)([b_j]).$$

Let $r_{\mathbf{M}_e, \mathbf{N}'}$, $m_{\mathbf{M}_e, \mathbf{N}'}$ denote the morphisms $r_{M_e, \mathbf{N}'} \circ (\eta_e \times \cdots \times \eta_e)$, $m_{M_e, \mathbf{N}'} \circ (\eta_e \times \cdots \times \eta_e) : \mathbf{M}_e \to M_e \to \mathbf{N}'$ respectively. We have

$$(m_{\mathbf{N}, \mathbf{N}'})_* \kappa_{\mathbf{N}, (b_j)} = (m_{\mathbf{M}_e, \mathbf{N}'})_* \iota_* \iota^* ((\prod_{j=1}^{e} \widetilde{\mathrm{pr}}_j^* \widetilde{g}_j) \cdot r^*_{\mathbf{M}_e, \mathbf{N}'} \kappa')$$

where for $j = 1, \ldots, e$, $\widetilde{\mathrm{pr}}_j : \mathbf{M}_e \to \widetilde{M}_e$ is the j-th projection.

Applying Lemma 2.11, it is equal to

$$\sum_{r=0}^{e} (-1)^r q_{\wp}^{r(r-1)/2} (m_{\mathbf{M}, \mathbf{N}'})_* (\widetilde{f}_{e-r})_* \widetilde{f}^*_{e-r} ((\prod_{j=1}^{e} \widetilde{\mathrm{pr}}_j^* \widetilde{g}_j) \cdot r^*_{\mathbf{M}_e, \mathbf{N}'} \kappa').$$

For each $r = 0, \ldots, e$ and each $i = 1, \ldots, d$, consider the following diagram:

$$\begin{tikzcd}
\widetilde{M}_r \arrow[d, "f_r"'] & \widetilde{M}_r \times_{M_r} \cdots \times_{M_r} \widetilde{M}_r \arrow[l, "\mathrm{pr}_i"'] \arrow[r, "\eta_r \times \cdots \times \eta_r"] \arrow[d, "\widetilde{f}_r"] & M_r \arrow[d, "f'_r"] & \\
\widetilde{M}_e & \mathbf{M}_e \arrow[l, "\widetilde{\mathrm{pr}}_i"] \arrow[r, "\eta_e \times \cdots \times \eta_e"] \arrow[rr, bend right, "m_{\mathbf{M}_e, \mathbf{N}'}"'] & M_e \arrow[r, "m_{M_e, \mathbf{N}'}"] & \mathbf{N}'
\end{tikzcd}$$

where $\dot{\mathrm{pr}}_i$ is the projection to the i-th factor, and η_r and f_r are as in Section 2.6.1. Since the diagram commutes, we have

$$
\begin{aligned}
&(m_{\mathbf{M},\mathbf{N}'})_*(\widetilde{f}_r)_*\widetilde{f}_r^*((\prod_{j=1}^{e}\widetilde{\mathrm{pr}}_j^*\widetilde{g}_j)\cdot r^*_{\mathbf{M},\mathbf{N}'}\kappa')\\
&=(m_{M_e,\mathbf{N}'})_*((((\widetilde{f}_r)_*\widetilde{f}_r^*\prod_{j=1}^{e}\widetilde{\mathrm{pr}}_j^*\widetilde{g}_j)\cdot r^*_{M_e,\mathbf{N}'}\kappa')\\
&=(m_{M_e,\mathbf{N}'})_*((((f'_r)_*(\eta_r\times\cdots\times\eta_r)_*\prod_{j=1}^{e}\dot{\mathrm{pr}}_j^*f_r^*\widetilde{g}_j)\cdot r^*_{M_e,\mathbf{N}'}\kappa')\\
&=(m_{M_e,\mathbf{N}'})_*((((f'_r)_*\prod_{j=1}^{e}(\eta_r)_*f_r^*\widetilde{g}_j)\cdot r^*_{M_e,\mathbf{N}'}\kappa')\\
&=(m_{M_r,\mathbf{N}'})_*((\prod_{j=1}^{e}(\eta_r)_*f_r^*\widetilde{g}_j)\cdot r^*_{M_r,\mathbf{N}'}\kappa').
\end{aligned}
$$

Next we consider the diagram

$$
\begin{tikzcd}
\widetilde{M}_e & \widetilde{M}_r \arrow[l, "f_r"'] \arrow[r, "\eta_r"] & M_r \arrow[r, "\delta_r"] \arrow[d, "r_{M_r,\mathbf{N}'}"'] \arrow[rr, bend left, "m_{M_r,\mathbf{N}'}"] & \mathbf{N}'\oplus[M_r'''] \arrow[r, "m_{\mathbf{N}'\oplus[M_r'''],\mathbf{N}'}"'] \arrow[d, "r_{\mathbf{N}'\oplus[M_r'''],\mathbf{N}'}"] & \mathbf{N}' \\
 & & \mathbf{N}' & \mathbf{N}' &
\end{tikzcd}
$$

where $M_r'''=\bigoplus_{i=r+1}^{e}N_i$ and $\delta_r : M_r \to \mathbf{N}'\oplus[M_r''']$ is the unique morphism satisfying $m_{M_r,\mathbf{N}'}=m_{\mathbf{N}'\oplus[M_r'''],\mathbf{N}'}\circ\delta_r$. By Lemma 2.12 (1) below, we have

$$
\begin{aligned}
&(m_{M_r,\mathbf{N}'})_*((\prod_{j=1}^{e}(\eta_r)_*f_r^*\widetilde{g}_j)\cdot r^*_{M_r,\mathbf{N}'}\kappa')\\
&=(m_{\mathbf{N}'\oplus[M_r'''],\mathbf{N}'})_*((\prod_{j=1}^{e}\alpha_j g_j(\mathbf{N}'\oplus[M_r'''])(r^*_{\mathbf{N}'\oplus[M_r'''],\mathbf{N}'}[0]))\cdot(\delta_r)_*r^*_{M_r,\mathbf{N}'}\kappa').
\end{aligned}
$$

Let G be the subgroup of $\mathrm{Aut}_{\mathcal{O}_X}(\mathbf{N})$ of elements g such that $g|_{\mathbf{N}'}=\mathrm{id}_{\mathbf{N}'}$, $g(M_r'')=M_r''$, and $g \bmod M_r''|_{\mathbf{N}'}=\mathrm{id}_{\mathbf{N}'}$. Here, $g \bmod M_r''$ is the map $\mathbf{N}/M_r''\to\mathbf{N}/M_r''$ induced by $g:\mathbf{N}\to\mathbf{N}$, and we use the canonical isomorphism $\mathbf{N}'\oplus M_r'''\cong\mathbf{N}/M_r''$. Let G' be the subgroup of $\mathrm{Aut}_{\mathcal{O}_X}(\mathbf{N})$ of elements g' such that $G'|_{\mathbf{N}}=\mathrm{id}_{\mathbf{N}}$, $g(\mathbf{N}'')=\mathbf{N}''$, and $g(M_r'')=M_r''$. Then G' is a subgroup of G and G/G' is isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\mathbf{N}',M_r'')$. Note that $M_r''=\mathbf{N}/G'$, $\mathbf{N}'\oplus[M_r''']=\mathbf{N}/G$ and $\delta_r : M_r''\to\mathbf{N}'\oplus[M_r''']$ is the morphism induced from the canonical map of presheaves $\mathrm{Hom}_{\mathcal{FC}^d}(-,\mathbf{N})/G'\to\mathrm{Hom}_{\mathcal{FC}^d}(-,\mathbf{N})/G$.

For each $i=e+1,\dots,d$, let G_i be the subgroup of G generated by G' and the representatives of the subgroup $\bigoplus_{j\neq i}\mathrm{Hom}_{\mathcal{O}_X}(N_j',M_r'')\subset\mathrm{Hom}_{\mathcal{O}_X}(\mathbf{N}',M_r'')=G/G'$. We let $M_{r,i}=(\mathbf{N}/G_i)^\dagger$ and let $\beta_i' : M_r\to M_{r,i}$ be the canonical quotient map. Then $\beta'_{e+1}\times\cdots\times\beta'_d : M_r\to M_{r,e+1}\times_{\mathbf{N}'\oplus[M_r''']}\cdots\times_{\mathbf{N}'\oplus[M_r''']}M_{r,d}$ is an isomorphism.

Note that the submodule $N_i'\hookrightarrow\mathbf{N}$ is fixed under the action of G_i, hence we have a morphism $M_{r,i}=(\mathbf{N}/G_i)^\dagger\to N_i'$ which we denote h_i.

We have a commutative diagram

$$\begin{tikzcd}
M_r \arrow[r, "\beta'_i"] \arrow[d, "r_{M_r,\mathbf{N}'}"'] & M_{r,i} \arrow[r, "\epsilon'_i"] \arrow[d, "h_i"] & \mathbf{N}' \oplus [M'''_r] \arrow[r, "r_{\mathbf{N}'\oplus[M'''_r],\mathbf{N}'}"] & \mathbf{N}' \arrow[d, "r_{\mathbf{N}',N_i}"'] \\
\mathbf{N}' \arrow[r, "r_{\mathbf{N}',N'_i}"'] & N'_i & & N'_i.
\end{tikzcd}$$

By Lemma 2.12 (2) below, we have

$$\begin{aligned}
&(\delta_r)_* r^*_{M_r,\mathbf{N}'}\kappa' \\
&= (\epsilon'_{e+1} \times \cdots \times \epsilon'_d)_*(\beta'_{e+1} \times \cdots \times \beta'_d)_* r^*_{M_r,\mathbf{N}'}\kappa' \\
&= \prod_{i=e+1}^{d} (\epsilon'_i)_* h^*_i(\alpha_i g_i(N'_i)(b_i)) \\
&= \prod_{i=e+1}^{d} r^*_{\mathbf{N}'\oplus[M'''_r],\mathbf{N}'} r^*_{\mathbf{N}',N'_i}(\alpha_i g_i(N'_i)(b_i)) \\
&= r^*_{\mathbf{N}'\oplus[M'''_r],\mathbf{N}'}\kappa'.
\end{aligned}$$

Hence the assertion follows. $\square$

Lemma 2.12. *Let the notations be as above.*

(1) *For* $j = 1, \dots, e$, *we have* $(\gamma_r)_* {f_r}^*(\widetilde{b_j}) = \delta^*_r r^*_{\mathbf{N}'\oplus[M'''_r],\mathbf{N}'}([0])$ *in* $\mathrm{SB}^\dagger(M_r)$.

(2) *For* $i = e+1, \dots, d$, *we have* $(\epsilon'_i)_* h^*_i([b_i]) = r^*_{\mathbf{N}'\oplus[M'''_r],\mathbf{N}'} r^*_{\mathbf{N}',N'_i}([b_i])$ *in* $\mathrm{SB}^\dagger(\mathbf{N}' \oplus [M'''_r])$.

Proof. (1) The following is a cartesian diagram in $\widetilde{\mathcal{FC}}^{\dagger,d}$

$$\begin{tikzcd}
\coprod_{\alpha:N\to M''_r} \mathbf{N} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbf{N} \arrow[d, "h"] \\
\widetilde{M}_r \arrow[r, "\gamma_r"'] & M_r.
\end{tikzcd}$$

We have

$$\begin{aligned}
h^*(\gamma_r)_* f^*_r(\widetilde{b_j}) &= \sum_{\alpha:N\to M''_r} [\alpha(b_j)] = \sum_{x\in M''_r} [x] \\
&= m^*_{\mathbf{N},M'''_r\oplus\mathbf{N}'}([0]) = h^*\delta^*_r r^*_{\mathbf{N}'\oplus[M'''_r],\mathbf{N}'}([0]).
\end{aligned}$$

Since h is a fibration, we have the assertion.

(2) Let H denote the group of the automorphisms $g \in \mathrm{Aut}(\mathbf{N})$ which induce identities on both $\bigoplus_{j\neq i} N_j$ and $\mathbf{N}/M''_r$. Let $\beta_{r,i} : \mathbf{N} \to M_{r,i}$ and $\epsilon' : M_{r,i} \to \mathbf{N}' \oplus [M'''_r]$ be the canonical quotient morphisms induced by the identity map $\mathbf{N} \to \mathbf{N}$. Then the following diagram is cartesian:

$$\begin{tikzcd}
\mathbf{N} \arrow[r] \arrow[d, "\beta_{r,i}"'] \arrow[dr, phantom, "\square"] & \mathbf{N}/H \arrow[d] \\
M_{r,i} \arrow[r, "\epsilon'_i"'] & \mathbf{N}' \oplus [M'''_r].
\end{tikzcd}$$

Hence

$$\begin{aligned}\beta^*_{r,i}(\epsilon'_i)^*(\epsilon'_i)_*h^*_i([b_i]) &= \sum_{g\in H} g^* r_{\mathbf{N},N_i}[b_i] \\ &= m^*_{\mathbf{N},\mathbf{N}/M''_r} r^*_{\mathbf{N}/M''_r,N_i}([b_i]) \\ &= \beta^*_{r,i}(\epsilon'_i)^* r^*_{\mathbf{N}'\oplus[M'''_r],\mathbf{N}'} r^*_{\mathbf{N}',N_i}([b_i]).\end{aligned}$$

Since $\epsilon'_i \circ \beta_{r,i}$ is a fibration, we have the assertion. $\square$

3. Euler systems in the K-theory of Drinfeld modular varieties

The main purpose of this section is to prove Theorem 3.7. In Section 3.1, we recall some facts on Drinfeld modular varieties. The function field analogue of Siegel units and theta functions are defined in Sections 3.2 and 3.3. In the case of elliptic modular curves, the algebraic construction of theta functions is due to Kato ([Ka]). The construction of special elements follows the idea of Beilinson. The main result is a rather direct consequence of the result in Section 2.

3.1. Drinfeld modular varieties.

3.1.1. *Notations.* Let C be a smooth projective geometrically irreducible curve over the finite field $\mathbb{F}_q$ of q elements. Let F denote the function field of C. Fix a closed point ∞ of C. Let q_∞, F_∞, $|\ |_\infty : F_\infty \to q_\infty^{\mathbb{Z}} \cup \{\infty\}$ denote the cardinality of the residue field of C at ∞, the completion of F at ∞, and absolute value at ∞, respectively. Let $A = \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$ be the coordinate ring of the affine $\mathbb{F}_q$-scheme.

3.1.2. We fix an integer $d \geq 1$.

Definition 3.1 ([Dr])**.** Let S be an A-scheme. A *Drinfeld module* of rank d over S is an A-module scheme E over S satisfying the following conditions:

(1) Zariski locally on S, E is isomorphic to $\mathbb{G}_a$ as a commutative group scheme.
(2) If we denote the A-action on E by $\varphi : A \to \mathrm{End}_{S\text{-group}}(E)$, then, for every $a \in A\setminus\{0\}$, the a-action $\varphi(a) : E \to E$ on E is finite, locally free of constant degree $|a|_\infty$.
(3) The A-action on $\mathrm{Lie}\, E$ induced by φ coincide with the A-action on $\mathrm{Lie}\, E$ which comes from the structure homomorphism $A \to \Gamma(S, \mathcal{O}_S)$.

3.1.3. Let N be a torsion A-module. Let $U = U_N := \mathrm{Spec}\, A \setminus \mathrm{Supp}\, N$ be the spectrum of the localization of A by the elements in A which is invertible on $\mathrm{Spec}\, A \setminus \mathrm{Supp}\, N$. Let S be a U-scheme, and (E, φ) be a Drinfeld module of rank d over S.

Definition 3.2. A *level N-structure* on (E, φ) is a monomorphism $\psi : N_S \hookrightarrow E$ from the constant group scheme N_S to E in the category of A-modules schemes over S.

3.1.4. Let us consider the sheaf $\mathcal{M}^d_N$ of groupoids which associates, to a U-scheme S, the groupoid of triples (E, φ, ψ) where (E, φ) is a Drinfeld module over S and ψ is a level N-structure. If $N \neq 0$ (resp. if N is of finite length), the functor $\mathcal{M}^d_N$ is representable by an affine U-scheme (resp. by a smooth Deligne-Mumford U-stack). The proof of Proposition 5.3 [Dr] (see also Theorem 1.4.1 [La]), where the case $N = (I^{-1}/A)^{\oplus d}$ is considered, may be applied to our case.

3.1.5. Let S be a finite set of closed points of C, and

$$X = \varprojlim_{S' \subset \operatorname{Spec} A \setminus S} (\operatorname{Spec} A) \setminus S'$$

be the localization of $\operatorname{Spec} A$ at S, where in the projective limit S' runs over the finite sets of closed points of $\operatorname{Spec} A \setminus S$. We consider the categories $\mathcal{C}^d$, $\mathcal{FC}^d$ and $\widetilde{\mathcal{FC}}^d$ introduced in Section 2.

To each object $N = \coprod_{i \in \pi_0(N)} N_i$ in $\mathcal{FC}^d$, we associate the moduli stack $\mathcal{M}^d_N \times_{U_N} (C \setminus S)$. Thus we obtain a covariant functor $\mathcal{M}^{d,S}$ from $\mathcal{FC}^d$ to the category of Deligne-Mumford stacks over A. For each morphism $f : M \to N$ in $\mathcal{FC}^d$, the morphism $\mathcal{M}^{d,S}(f) : \mathcal{M}^{d,S}(M) \to \mathcal{M}^{d,S}(N)$ is a finite étale morphism.

Lemma 3.3. *The functor $\mathcal{M}^{d,S}$ preserves fiber products. In particular, if $M \to N$ is a Galois covering in $\mathcal{FC}^d$ with Galois group G, then $\mathcal{M}^{d,S}(M)/G \to \mathcal{M}^{d,S}(N)$ is an isomorphism.*

Proof. For a Drinfeld module E over T and an ideal I of A, let $E[I] \subset E$ denote the subgroup scheme of I-division points of E. We say that an isogeny $u : E \to E'$ of Drinfeld modules of rank d over T is an S-isogeny if the kernel of u is contained in $E[I]$ for some ideal $I \subset A$ with $\operatorname{Supp}(A/I) \subset S$. We denote by $(\mathrm{DrMod}_T/\sim_S)$ the category obtained from the category of Drinfeld modules over T by inverting S-isogenies. We define the S-Tate module $T_S(E)$ of E to be $T_S(E) = \varprojlim_I E[I]$, which we understand as an object in the 2-projective limit $\mathrm{Et}(T, \mathbb{A}_X) = 2\text{-}\varprojlim_I \mathrm{Et}(T, A/I)$ of the category of étale A/I-sheaves over T, where I runs over the ideals of A with $\operatorname{Supp}(A/I) \subset S$. We denote by $\mathrm{Et}(T, \mathbb{A}_X) \otimes_A F$ the F-linearization of the category $\mathrm{Et}(T, \mathbb{A}_X)$. For an object $\mathcal{F}$ in $\mathrm{Et}(T, \mathbb{A}_X)$, we denote by $\mathcal{F} \otimes_A F$ the corresponding object in $\mathrm{Et}(T, \mathbb{A}_X) \otimes_A F$. Then the association $E \mapsto T_S(E) \otimes_A F$ defines a functor from the category $(\mathrm{DrMod}_T/\sim_S)$ to the category $\mathrm{Et}(T, \mathbb{A}_X) \otimes_A F$. We denote this functor by V_S.

Let $\overline{E}$ be an object in $(\mathrm{DrMod}_T/\sim_S)$. For an object $M = \coprod_{i \in \pi_0(M)} M_i$ in $\mathcal{FC}^d$, we define an étale sheaf $h(V_S(\overline{E}), M)$ of sets on T in the following way. For an étale T-scheme T' and for $i \in \pi_0(M)$, let $h'(V_S(\overline{E}), M_i)(T')$ be the set of equivalence classes of triples $(\mathcal{F}, \iota, p)$, where $\mathcal{F} = (\mathcal{F}_I)_I$ is an object in $\mathrm{Et}(T', \mathbb{A}_X)$ such that for every I, the stalk of $\mathcal{F}_I$ at every geometric point is a free A/I-module of rank d, $\iota : \mathcal{F} \otimes_A F \cong V_S(\overline{E} \times_T T') \otimes_A F$ is an isomorphism in the category $\mathrm{Et}(T', \mathbb{A}_X) \otimes_A F$, and $p : \mathcal{F} \twoheadrightarrow (M_i \otimes_A A/I)_I$ is an epimorphism in the category $\mathrm{Et}(T', \mathbb{A}_X)$ from $\mathcal{F}$ to the system of constant sheaves $(M_i \otimes_A A/I)_I$. Then we define $h(V_S(\overline{E}), M)$ to be the sheaf associated to the presheaf $T' \mapsto \coprod_i h'(V_S(\overline{E}), M_i)(T')$. At every geometric point x of T, the stalk of $h(V_S(\overline{E}), M)$ is isomorphic to the set $\omega(M)$. This implies that the functor $M \mapsto h(V_S(\overline{E}), M)$ from $\mathcal{FC}^d$ to the category of étale sheaves on T preserves fiber products.

The stack $\mathcal{M}^d_M$ is canonically identified with the functor which associates, to a U_N-scheme T, the groupoid of pairs $(\overline{E}, \alpha)$ where $\overline{E}$ is an object in $(\mathrm{DrMod}_T/\sim_S)$ and α is an element in $h(V_S(\overline{E}), M)(T)$. Therefore, if $N_1 \to N_3 \leftarrow N_2$ is a diagram in $\mathcal{FC}^d$ such that the fiber product $N_1 \times_{N_3} N_2$ exists, then the canonical functor $\mathcal{M}^d_{N_1 \times_{N_3} N_2}(T) \to \mathcal{M}^d_{N_1} \times_{\mathcal{M}^d_{N_3}} \mathcal{M}^d_{N_2}(T)$ gives an equivalence of groupoids. Hence the assertion follows. $\square$

Let us extend the functor $\mathcal{M}^{d,S}$ to a functor from $\widetilde{\mathcal{FC}}^d$, which we also denote by $\mathcal{M}^{d,S}$. For an object N/H in $\widetilde{\mathcal{FC}}^d$ as in Lemma A.5, we put $\mathcal{M}^{d,S}(N/H) = \mathcal{M}^{d,S}(N)/H$. Let $f : N_1/H_1 \to N_2/H_2$ be a morphism in $\widetilde{\mathcal{FC}}^d$ where, for $i = 1, 2$, N_i/H_i is an object as in Lemma A.5. We take an object M in $\mathcal{FC}^d$ and two morphisms $M \to N_1$, $M \to N_2$ in $\mathcal{FC}^d$ such that the composition $M \to N_1 \xrightarrow{m_{N_1,0}} 0$ is a Galois covering and the diagram

$$\begin{tikzcd} N_1 \arrow[d] & M \arrow[l] \arrow[r] & N_2 \arrow[d] \\ N_1/H_1 \arrow[rr] & & N_2/H_2 \end{tikzcd}$$

is commutative. Then by Lemma 3.3, the composition $\mathcal{M}^{d,S}(M) \to \mathcal{M}^{d,S}(N_2) \to \mathcal{M}^{d,S}(N_2/H_2)$ canonically descends to a morphism $\mathcal{M}^{d,S}(f) : \mathcal{M}^{d,S}(N_1/H_1) \to \mathcal{M}^{d,S}(N_2/H_2)$.

Let $\overline{N}$ be an object in $\widetilde{\mathcal{FC}}^d$ and let H be a finite group which acts freely on $\overline{N}$ (see Section 2.1.5 for the definition). Then, by the definition of $\mathcal{M}^{d,S}$, we have $\mathcal{M}^{d,S}(\overline{N}/H) \cong \mathcal{M}^{d,S}(\overline{N})/H$.

Lemma 3.4. *The extended functor $\mathcal{M}^{d,S}$ preserves fiber products.*

Proof. Let $\overline{N}_1 \to \overline{N}_3 \leftarrow \overline{N}_2$ be a diagram in $\widetilde{\mathcal{FC}}^d$. Take an object $\overline{M}$ in $\widetilde{\mathcal{FC}}^d$ and two morphisms $\overline{M} \to \overline{N}_1$, $\overline{M} \to \overline{N}_2$ such that the diagram

$$\begin{tikzcd} \overline{M} \arrow[r] \arrow[d] & \overline{N}_1 \arrow[d] \\ \overline{N}_2 \arrow[r] & \overline{N}_3 \end{tikzcd}$$

is commutative and that for $i = 1, 2, 3$, the morphism $\overline{M} \to \overline{N}_i$ is a Galois covering. For $i = 1, 2, 3$, we denote by H_i the Galois group of $\overline{M}$ over $\overline{N}_i$. Then we have

$$\begin{array}{cl} & \mathcal{M}^{d,S}(\overline{N}_1 \times_{\overline{N}_3} \overline{N}_2) \\ \cong & \mathcal{M}^{d,S}((\overline{M} \times_{\overline{N}_3} \overline{M})/(H_1 \times H_2)) \\ \cong & \mathcal{M}^{d,S}(\overline{M} \times_{\overline{N}_3} \overline{M})/(H_1 \times H_2) \\ \cong & (\coprod_{h \in H_3} \mathcal{M}^{d,S}(\overline{M}))/(H_1 \times H_2) \\ \cong & (\mathcal{M}^{d,S}(\overline{M}) \times_{\mathcal{M}^{d,S}} \mathcal{M}^{d,S}(\overline{M}))/(H_1 \times H_2) \\ \cong & \mathcal{M}^{d,S}(\overline{N}_1) \times_{\mathcal{M}^{d,S}(\overline{N}_3)} \mathcal{M}^{d,S}(\overline{N}_2). \end{array}$$

□

Let N, N', N'' be objects in $\mathcal{C}^d$. The morphism $N = N \hookrightarrow N'$ is sent via this functor to the morphism $(E, \phi, \psi) \mapsto (E, \phi, \psi|_N)$ where $\psi|_N$ is the restriction to the submodule N. The morphism $N \twoheadleftarrow N'' = N''$ is sent to the morphism $(E, \phi, \psi) \mapsto (E'', \phi'', \psi'')$ where $E'' = E/\phi(\mathrm{Ker}(N'' \twoheadrightarrow N))$, and ϕ'', ψ'' are those induced by the quotient map ([La, Lemma 1.4.1]). We use the notations such as $r_{N',N}$ and $m_{N'',N}$ to denote the corresponding morphisms in the category of stacks via this functor.

3.2. Theta functions.

3.2.1. Let (E,φ) be a Drinfeld module of rank d over a *reduced* A-scheme S. Let $\pi : E \to S$ denote the structure morphism. We regard S as a closed subscheme of E via the zero section $S \hookrightarrow E$.

Lemma 3.5. *Let the notations be as above. There exists an element $f \in \Gamma(E \setminus S, \mathcal{O}_E^\times)$ satisfying the following properties:*

(1) *For $a \in A \setminus \{0\}$, let $N_a : \mathcal{O}^\times_{E\setminus \mathrm{Ker}\varphi(a)} \to \mathcal{O}^\times_{E\setminus S}$ denote the norm map with respect to the finite flat morphism $\varphi(a) : E \setminus \mathrm{Ker}\varphi(a) \to E \setminus S$. Then $N_a(f) = f$ for any $a \in A \setminus \{0\}$.*

(2) *The order $\mathrm{ord}_S(f)$ of zero of f at the closed subscheme S is equal to $q_\infty^d - 1$.*

Proof. Let us consider the exact sequence

$$0 \to \mathcal{O}_S^\times \to \pi_*\mathcal{O}_{E\setminus S} \xrightarrow{\mathrm{ord}_S} \mathbb{Z}_S \to 0$$

of Zariski-sheaves on S. The multiplicative monoid $A \setminus \{0\}$ acts on $\mathcal{O}_{E\setminus S}$ by the norm map N_a for $a \in A \setminus \{0\}$. The above exact sequence induces the structure of $A \setminus \{0\}$-module on $\mathbb{Z}_S$ and on $\mathcal{O}_S^\times$, becomes an exact sequence of $A \setminus \{0\}$-modules, and defines an element of the extension module $\mathrm{Ext}^1_{\mathbb{Z}[A\setminus\{0\}]_S}(\mathbb{Z}_S, \mathcal{O}_S^\times)$ in the abelian category of Zariski sheaves of $A \setminus \{0\}$-modules on E. Since $A \setminus \{0\}$ acts trivially on $\mathbb{Z}_S$ and via the character $|\ |_\infty^d : A \setminus \{0\} \to q_\infty^{d\mathbb{Z}_{\geq 0}}$ on $\mathcal{O}_S^\times$, we have $(|a|_\infty^d - 1)\mathrm{Ext}^1_{\mathbb{Z}[A\setminus\{0\}]_S}(\mathbb{Z}_S, \mathcal{O}_S^\times) = 0$ for any $|a|_\infty \in A \setminus \{0\}$. Since the greatest common divisor of $|a|_\infty^d - 1$ as a runs through $A \setminus \{0\}$ is $q_\infty^d - 1$, the extension group $\mathrm{Ext}^1_{\mathbb{Z}[A\setminus\{0\}]_S}(\mathbb{Z}_S, \mathcal{O}_S^\times)$ is annihilated by $q_\infty^d - 1$. In particular, the above exact sequence splits after pulling back by $q_\infty^d - 1 : \mathbb{Z}_S \to \mathbb{Z}_S$. Now let f be the image of $1 \in \mathbb{Z}_S$ by the section which gives the splitting. □

3.2.2. The choice of f is unique up to $\mathrm{Hom}_{\mathbb{Z}[A\setminus\{0\}]_S}(\mathbb{Z}_S, \mathcal{O}_S^\times) \cong \mu_{q_\infty^d - 1}(S)$. Hence $f^{q_\infty^d - 1}$ does not depend on the choice of f. We denote it by $\theta_{E/S} \in \Gamma(E \setminus S, \mathcal{O}_E^\times)$ and call it the *theta function* of (E,φ).

Zariski locally on S, the function $\theta_{E/S}$ is explicitly calculated in the following way: Take an S-local defining equation f' of the divisor $S \hookrightarrow E$ of the zero section. Then for any $a \in A \setminus \{0\}$, we have

$$\theta_{E/S}^{\frac{|a|_\infty^d - 1}{q_\infty^d - 1}} = \left(\frac{f'^{|a|_\infty^d}}{N_a(f')} \right)^{q_\infty^d - 1}.$$

The following properties are easily checked:

Proposition 3.6. (1) *Let $g : S' \to S$ be a morphism from another reduced scheme S' to S, $g_E : E \times_S S' \to S$ be the morphism induced by g. Then we have $g_E^* \theta_{E/S} = \theta_{E\times_S S'/S'}$.*

(2) *Let $h : E \to E'$ be an isogeny (that is, a morphism of A-module schemes with finite kernel) from another Drinfeld module E' of rank d over S to E. Then $N_h \theta_{E/S} = \theta_{E'/S}$.*

□

3.3. Siegel units. Let N be a torsion A-module, and let $U_N := \mathrm{Spec}\, A \setminus \mathrm{Supp}\, N$ be the spectrum of the localization of A by the elements in A which is invertible on $\mathrm{Spec}\, A \setminus \mathrm{Supp}\, N$. We let $E_N^d \to \mathcal{M}_N^d$ denote the universal Drinfeld module, and $\psi : N_{\mathcal{M}^d_N} \hookrightarrow E^d_N$ the universal level structure. For $b \in N \setminus \{0\}$, we let $g_{N,b} = \psi_b^* \theta_{E^d_N/\mathcal{M}^d_N} \in \mathcal{O}(\mathcal{M}^d_N)^*$ and call such elements Siegel units.

Let N be an A-module of finite length generated by at most d elements, and N' be a sub A-module of N. By Proposition 3.6(1), we have $r^*_{N,N'} g_{N',b} = g_{N,b}$ for any $b \in N' \setminus \{0\} \subset N \setminus \{0\}$. Let $\psi : N \to N''$ be a quotient A-module of N. It follows from Proposition 3.6(1)(2) that, for any $b'' \in N'' \setminus \{0\}$, $m^*_{N,N''} g_{N'',b''} = \prod_{b \in N, \psi(b)=b''} g_{N,b}$.

3.4. Euler systems in K-theory.

3.4.1. *Elements in K-theory.* For $i = 1, \dots, d$, let N_i be a non-zero finite abelian group which is generated by one element as an A-module. Let b_i be an element of $N_i \setminus \{0\}$. Put $\mathbf{N} = \bigoplus_{i=1}^d N_i$. For $i = 1, \dots, d$, let $\iota_i : N_i \hookrightarrow \mathbf{N}$ be the canonical i-th inclusion. Each induces $f_i : \mathcal{M}^d_{\mathbf{N}} \to \mathcal{M}^d_{N_i}$. Let $\kappa^K_{\mathbf{N},(b_i)} = f_1^* g_{N_1,b_1} \otimes \cdots \otimes f_d^* g_{N_d,b_d} \in \mathcal{O}(\mathcal{M}^d_{\mathbf{N}})^{\otimes d}$. We consider $\kappa^K_{\mathbf{N},(b_i)}$ also as an element in $K_d(\mathcal{M}_{\mathbf{N}})$ via the symbol map $\mathcal{O}(\mathcal{M}^d_{\mathbf{N}})^{*\otimes d} \to K_d(\mathcal{M}^d_{\mathbf{N}})$. Here $K_d(\mathcal{M}^d_{\mathbf{N}})$ is the Quillen K-group of the scheme $\mathcal{M}^d_{\mathbf{N}}$.

3.4.2. *Main theorem.*

Theorem 3.7. *Let N'_i be a quotient $\mathcal{O}_C$-module of N_i for $i = 1, \dots, d$. Let b'_i denote the image of b_i in N'_i. We write $\mathbf{N}' = \bigoplus_j N'_j$ and $N''_i = \mathrm{Ker}(N_i \twoheadrightarrow N'_i)$. Let*

$$m : \mathcal{M}^d_{\mathbf{N}} \to \mathcal{M}^d_{\mathbf{N}'} \times_{U_{\mathbf{N}'}} U_{\mathbf{N}}$$

be the morphism induced by $m_{\mathbf{N},\mathbf{N}'} : \mathcal{M}^d_{\mathbf{N}} \to \mathcal{M}^d_{\mathbf{N}'}$. Since m is finite étale, we can consider the transfer map $m_* : K_d(\mathcal{M}^d_{\mathbf{N}}) \to K_d(\mathcal{M}^d_{\mathbf{N}} \times_{U_{\mathbf{N}'}} U_{\mathbf{N}})$ between K-groups. Let ${\kappa'}^K_{\mathbf{N}',(b'_j)}$ denote the image of $\kappa^K_{\mathbf{N}',(b'_j)}$ in $K_d(\mathcal{M}^d_{\mathbf{N}'} \times_{U_{\mathbf{N}'}} U_{\mathbf{N}})$. Then the following statements hold.

(1) *If $\mathrm{Supp}\, N''_i \subset \mathrm{Supp}\, N'_j$ for any $1 \le i, j \le d$, then*

$$m_* \kappa^K_{\mathbf{N},(b_j)} = {\kappa'}^K_{\mathbf{N}',(b'_j)}.$$

(2) *Let $\wp$ be a closed point of C. Suppose that $\mathrm{Supp}\, N''_i \subset \{\wp\} \subset \mathrm{Supp}\, N_i$ for every i. Let e denote the number of i's with $\wp \notin \mathrm{Supp}\, N'_i$. Then*

$$m_* \kappa^K_{\mathbf{N},(b_j)} = \sum_{r=0}^{e} (-1)^r q_\wp^{r(r-1)/2} T_{[\wp^{\oplus r}]} {\kappa'}^K_{\mathbf{N}',(b'_j)}.$$

Proof. We set $S = \mathrm{Supp}\, \mathbf{N}$. We consider the functor $\mathcal{M}^{d,S}$ introduced in Section 3.1.5. Let G be a presheaf of rings on $\mathcal{FC}^d$ defined by setting $G(\overline{N}) = \bigoplus_{j \ge 0} K_j(\mathcal{M}^{d,S}(\overline{N}))$ for an object $\overline{N}$ in $\widetilde{\mathcal{FC}}^d$. Here $K_j(\mathcal{M}^{d,S}(\overline{N}))$ denotes the j-th algebraic K-group of the Deligne-Mumford stack $\mathcal{M}^{d,S}(\overline{N})$ defined in [Gi2]. Let $\overline{N} \to \overline{N}'$ be a morphism in $\widetilde{\mathcal{FC}}^d$. Since the morphism $\mathcal{M}^{d,S}(\overline{N}) \to \mathcal{M}^{d,S}(\overline{N}')$ of Deligne-Mumford stacks is finite flat, we have a transfer map $K_j(\mathcal{M}^{d,S}(\overline{N}')) \to K_j(\mathcal{M}^{d,S}(\overline{N}))$ between K-groups. The projection formula holds for this transfer map. It follows formally from the definition of the product of the K-groups in [Wa, §9]. Hence G is equipped with a structure of a presheaf of rings with transfers.

Let H be a sheaf on $\mathcal{FC}^d$ defined by $H(N) = \mathcal{O}(\mathcal{M}^d_N \otimes_A U_{\mathbf{N}})^*$ for an object N in $\mathcal{C}^d$. We define a morphism of presheaves $\bar{g} : \mathrm{SB}^{*\prime} \to H$. Given a non-zero object N in $\mathcal{C}^d$ and an element b in $N \setminus \{0\}$, we define $\bar{g}(N)(b)$ to be the image of $g_{N,b}$ in

$\mathcal{O}(\mathcal{M}_N^d \otimes_A U_{\mathbf{N}})^*$. This induces a morphism of sheaves $\alpha : \mathrm{SB}^* \to H$ since H is a sheaf.

We have a morphism $\beta : H \to G$ of abelian presheaves with transfers given by $\mathcal{O}(\mathcal{M}_N^d \otimes_A F)^* \to K_1(\mathcal{M}_N^d \otimes_A F) \to \bigoplus_{j\geq 0}^d K_j(\mathcal{M}_N^d \otimes_A F)$, for each object N in $\mathcal{C}^d$. Now the assertions follow immediately from Theorem 2.9. □

3.4.3. Given two non-zero ideals $I \subsetneq A$, $J \subsetneq A$, we set $\mathbf{N}_{I,J} = (A/I)^{\oplus d-1} \oplus A/J$ and put

$$\kappa_{I,J}^K = \kappa_{\mathbf{N}_{I,J},(1)} \in K_d(\mathcal{M}_{\mathbf{N}_{I,J}}^d).$$

The following corollary is a special case of the theorem.

Corollary 3.8. *The system of elements $(\kappa_{I,J}^K)_{I,J}$ indexed by two non-zero ideals I, J is an Euler system. That is, the following statement holds.*

Let $I' \subset I, J' \subset J$ be ideals of A. We let $\wp$ be a prime ideal dividing $I' + J'$, and assume that $\mathrm{Supp}\,(A/I') = \mathrm{Supp}\,(A/I\wp)$, $\mathrm{Supp}\,(A/J') = \mathrm{Supp}\,(A/J\wp)$. *Let*

$$m : \mathcal{M}_{\mathbf{N}_{I',J'}}^d \to \mathcal{M}_{\mathbf{N}_{I,J}}^d \times_{U_{\mathbf{N}_{I,J}}} U_{\mathbf{N}_{I',J'}}$$

be the morphism induced by $m_{\mathbf{N}_{I',J'},\mathbf{N}_{I,J}} : \mathcal{M}_{\mathbf{N}_{I',J'}}^d \to \mathcal{M}_{\mathbf{N}_{I,J}}^d$. Let ${\kappa'}_{I,J}^K$ denote the image of $\kappa_{I,J}^K$ in $K_d(\mathcal{M}_{\mathbf{N}_{I,J}}^d \times_{U_{\mathbf{N}_{I,J}}} U_{\mathbf{N}_{I',J'}})$.

We let

$$e_\wp = \begin{cases} 0 & \text{if } \wp | I, \quad \wp | J, \\ 1 & \text{if } \wp | I, \quad \wp \nmid J, \\ d-1 & \text{if } \wp \nmid I, \quad \wp | J, \\ d & \text{if } \wp \nmid I, \quad \wp \nmid J. \end{cases}$$

Then

$$m_* \kappa_{I',J'}^K = \sum_{r=0}^{e_\wp} (-1)^r q_\wp^{r(r-1)/2} T_{[\wp^{\oplus r}]} {\kappa'}_{I,J}^K.$$

□

3.4.4. *Variant with characters.* The elements $\kappa_{I,J}^K$, which we have constructed in the previous section, are related to the special values of L-functions of automorphic forms (Section 6). As a variant, we give here a similar system of elements in the d-th K-group of Drinfeld modular varieties which is related to the L-functions with twists by an idele class character.

Let $I \subsetneq A$, $J \subsetneq A$ be two non-zero ideals of A. We let $\mathbb{A}$ denote the ring of adeles of F. Let $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be a character of the idele class group of F whose conductor $\mathrm{cond}^\infty(\chi)$ divides I and whose ∞-component $\chi|_{F_\infty^\times}$, is trivial.

Let $F_{I,J}$ be the field of constants of the F-scheme $\mathcal{M}_{\mathbf{N}_{I,J}}^d \times F$. Then $F_{I,J}$ is the maximal abelian extension of F unramified outside I and completely split at ∞. Let $A_{I,J}$ denote the normalization of A in $F_{I,J}$. Then there is a canonical isomorphism

$$\mathcal{M}_{\mathbf{N}_{I,J}} \otimes_A A_{I,J} \cong \coprod_{\sigma \in \mathrm{Gal}(F_{I,J}/F)} \mathcal{M}_{\mathbf{N}_{I,J}}.$$

We define an element $\kappa_{I,J,\chi}^K \in K_d(\mathcal{M}_{\mathbf{N}_{I,J}} \otimes_A A_{I,J}) \otimes_{\mathbb{Z}} \mathbb{Z}[\chi]$ to be $\kappa_{I,J,\chi}^K = (\chi^{-1}(\sigma)\kappa_{I,J}^K)_{\sigma \in \mathrm{Gal}(F_{I,J}/F)}$.

4. Kronecker limit formula

We prove a function field analogue of the Kronecker limit formula. The case $d = 1$ is due to Gross and Rosen [Gr-Ro]. The second author follows the same line to prove the general case [Ko]. Here we give a simpler, more conceptual proof. First, we recall the analytic study at infinity of Drinfeld modular varieties. The reader is referred to [De-Hu] for more details. We then give the analytic description of theta functions and Siegel units which were defined in Section 3. In Section 4.2, Eisenstein series with complex parameter s are defined. The limit as s tends to 0 is expressed in terms of those analytic functions (Proposition 4.4).

4.1. Generalities.

4.1.1. *Notations.* In this section, we use the notations C, F, ∞, q_∞, A, and $\mathbb{A}$ introduced in Sections 3.1.1 and 3.4.4. We also let $\mathcal{O}_\infty$ denote the ring of integers in F_∞ and $\widehat{A}$ denote the profinite completion of A.

Let us consider the d-dimensional vector space $V = F^{\oplus d}$ over F. We regard it as the set of row vectors. We write $V_\infty = V \otimes_F F_\infty$, $\mathcal{O}_{V_\infty} = \mathcal{O}_\infty^{\oplus d} \subset V_\infty$, $V^\infty = V \otimes_F \mathbb{A}$, and $\mathcal{O}_{V^\infty} = \widehat{A}^{\oplus d} \subset V^\infty$. Given a sub A-module $\Lambda \subset V$, we put $\widehat{\Lambda} = \Lambda \otimes_A \widehat{A} \subset V^\infty$. We let V^* denote the dual of V; the elements are regarded as column vectors in F. We define similarly V^*_∞, $\mathcal{O}_{V_\infty{}^*}$, $V^{*\infty}$, and $\mathcal{O}_{V^{*\infty}}$.

4.1.2. *Drinfeld symmetric space.* We let $\widetilde{\mathfrak{X}} = V^*_{C_\infty} \setminus \bigcup_H H$ where $V^*_{C_\infty} = V^* \otimes_F C_\infty$, and H runs over the F_∞-rational hyperplanes. Dividing out by the similitudes gives the Drinfeld symmetric space: $\mathfrak{X} = \widetilde{\mathfrak{X}}/C_\infty^\times$. The sets $\widetilde{\mathfrak{X}}$, $\mathfrak{X}$ are canonically regarded as the sets of C_∞-valued points of certain rigid analytic varieties over F_∞ which, by abuse of notation, are also denoted by the same symbols $\widetilde{\mathfrak{X}}$, $\mathfrak{X}$.

4.1.3. *Bruhat-Tits building.* For $1 \leq i \leq d$, we let $\widetilde{\mathcal{T}_i} = \{\cdots \supsetneq L_{-1} \supsetneq L_0 \supsetneq L_1 \supsetneq \ldots | \pi_\infty L_i = L_{j+i+1}$ for all $j \in \mathbb{Z}\}$, where $L_k (k \in \mathbb{Z})$ are $\mathcal{O}_\infty$-lattices in V_∞ and $\pi_\infty \in F_\infty$ is a uniformizer. We also let $\mathcal{T}_i$ denote the quotient $\widetilde{\mathcal{T}_i}/F_\infty^\times$. In particular, $\widetilde{\mathcal{T}_0}$ is the set of $\mathcal{O}_\infty$-lattices in V_∞, which we also denote $\mathrm{Lat}_{\mathcal{O}_\infty}(V_\infty)$. We identify the set $\widetilde{\mathcal{T}_0}$ with the coset $\mathrm{GL}_d(F_\infty)/\mathrm{GL}_d(\mathcal{O}_\infty)$ by associating to an element $g \in \mathrm{GL}_d(F_\infty)/\mathrm{GL}_d(\mathcal{O}_\infty)$ the lattice $\mathcal{O}_{V_\infty} g^{-1}$. Similarly, we identify the set $\widetilde{\mathcal{T}}_{d-1}$ with the coset $\mathrm{GL}_d(F_\infty)/\mathcal{I}$, where $\mathcal{I} = \{(a_{ij}) \in \mathrm{GL}_d(\mathcal{O}_\infty) \,|\, a_{ij} \bmod \pi_\infty = 0 \text{ if } i > j\}$ is the Iwahori subgroup, by associating to an element $g \in \mathrm{GL}_d(F_\infty)/\mathcal{I}$ the chain of lattices $(L_i)_{i \in \mathbb{Z}}$ characterized by $L_i = \mathcal{O}_{V_\infty} \Pi_i g^{-1}$ for $i = 0, \ldots, d$. Here, for $i = 0, \ldots, d$, we let Π_i denote the diagonal $d \times d$ matrix $\Pi_i = \mathrm{diag}(\pi_\infty, \ldots, \pi_\infty, 1, \ldots, 1)$ with π_∞ appearing i times and 1 appearing $d - i$ times.

4.1.4. *The order on lattices of rigid analytic functions.* Let L be an $\mathcal{O}_\infty$-lattice in V_∞. Take $g \in \mathrm{GL}_d(F_\infty)$ such that $L = \mathcal{O}_{V_\infty} g^{-1}$. We let $U_L = \{\boldsymbol{\tau} \in \widetilde{\mathfrak{X}} \,|\, \mathbf{v} g^{-1} \boldsymbol{\tau} \in \mathcal{O}^*_{C_\infty}$ for all $\mathbf{v} \in \mathcal{O}_{V_\infty} \setminus \pi_\infty \mathcal{O}_{V_\infty}\}$. Let h be a rigid analytic function on $\widetilde{\mathfrak{X}}$. We define ord_L by $\mathrm{ord}_L h = \inf_{\boldsymbol{\tau} \in U_L} |h(\boldsymbol{\tau})|$.

Given a lattice $L \in \widetilde{\mathcal{T}_0}$ and a row vector $\mathbf{a} \in V_\infty$, we let $\mathrm{ord}_L(\mathbf{a}) = \sup\{n \in \mathbb{Z} | \mathbf{a} \in \pi_\infty^n L\}$, and $|\mathbf{a}|_L = q_\infty^{-\mathrm{ord}_L(\mathbf{a})}$. Note that $|\mathbf{a}|_L = 1$ if and only if $\mathbf{a} \in L \setminus \pi_\infty L$.

Proposition 4.1. *Given a lattice $L \in \widetilde{\mathcal{T}_0}$ and a row vector $\mathbf{a} \in V_\infty$, let $f_{\mathbf{a}}$ be the rigid analytic function on $\widetilde{\mathfrak{X}}$ characterized by $f_{\mathbf{a}}(\boldsymbol{\tau}) = \mathbf{a}\boldsymbol{\tau}$ for every column vector $\boldsymbol{\tau} \in \widetilde{\mathfrak{X}}$. Then we have* $\mathrm{ord}_L f_{\mathbf{a}} = \mathrm{ord}_L(\mathbf{a})$.

Proof. Translating by an element $g \in \mathrm{GL}_d(F_\infty)$, we may and will assume $L = \mathcal{O}_\infty^{\oplus d}$. The set $U_{\mathcal{O}_{V_\infty}}$ is equal to the set of column vectors $\boldsymbol{\tau} \in \mathcal{O}_\infty^{\oplus d}$ such that $\boldsymbol{\tau}$ modulo the maximal ideal of $\mathcal{O}_\infty$ does not belong to a $\kappa(\infty)$-rational hyperplane. Hence, if $\mathbf{a} = (a_1, \dots, a_d), a_i \in F_\infty (1 \le i \le d)$, then $\mathrm{ord}_L f_{\mathbf{a}} = \inf_{1 \le i \le d} \mathrm{ord}\, a_i$. The claim follows. □

4.2. Theta functions and Siegel units. For an A-lattice $\Lambda \subset V$, and $\boldsymbol{\tau} \in \widetilde{\mathfrak{X}}$, we let $\Lambda_{\boldsymbol{\tau}} = \{\mathbf{x}\boldsymbol{\tau} | \mathbf{x} \in \Lambda\}$. We define $\sigma(z)$ to be the rigid analytic function on $z \in C_\infty$: $\sigma(z) = z \prod_{\lambda \in \Lambda_{\boldsymbol{\tau}} \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right)$. We note that $\sigma(z)$ defines a structure of a Drinfeld module over C_∞ on $C_\infty / \Lambda_{\boldsymbol{\tau}}$.

The theta function defined in Section 3.2.2 on $C_\infty/\Lambda_{\boldsymbol{\tau}}$ has the following description. For any $a \in A \setminus \{0\}$, we have

$$\theta_{C_\infty/\Lambda_{\boldsymbol{\tau}}}(z)^{\frac{|a|^d - 1}{q_\infty^d - 1}} = \left(\frac{\sigma_{\Lambda_{\boldsymbol{\tau}}}}{\mathrm{N}_a(\sigma_{\Lambda_{\boldsymbol{\tau}}})}\right)^{q_\infty^d - 1} = \left(\frac{\sigma_{\Lambda_{\boldsymbol{\tau}}}(z)}{\prod_{\mathbf{a} \in \Lambda/a} \sigma_{\Lambda_{\boldsymbol{\tau}}}\left(\frac{z}{a} + \frac{\mathbf{a}\boldsymbol{\tau}}{a}\right)}\right)^{q_\infty^d - 1}.$$

Given $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$, we let $g_{\Lambda,\mathbf{b}} = \theta_{C_\infty/\Lambda_{\boldsymbol{\tau}}}(\mathbf{b}\boldsymbol{\tau})$. It is an invertible rigid analytic function on $\mathfrak{X}$ over F_∞.

4.3. Eisenstein series.

4.3.1. We define $\mathbb{C}((q_\infty^{-s}))$-valued functions on the set $\widetilde{\mathcal{T}_0}$ of lattices in V_∞. (Here, q_∞^{-s} is regarded as a variable.) Given an A-lattice $\Lambda \subset V$ and $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$, we let

$$E_{\Lambda,\mathbf{b}}(L) = \sum_{\mathbf{x} \in V, \mathbf{x} \bmod \Lambda = \mathbf{b}} |\mathbf{x}|_L^{-s}.$$

The sum is convergent in the (q_∞^{-s})-adic topology.

The following lemma is easily checked.

Lemma 4.2. *Let $\Lambda \supset \Lambda'$ be two A-lattices in V, and $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$. Then*

(1) $E_{\Lambda,\mathbf{b}} = \sum_{\mathbf{b}' \in V/\Lambda', \mathbf{b}' \bmod \Lambda = \mathbf{b}} E_{\Lambda',\mathbf{b}'}$.
(2) *If $a \in A \setminus \{0\}$, then $E_{a\Lambda,a\mathbf{b}} = E_{\Lambda,\mathbf{b}} |a|^{-s}$.*

□

4.3.2. Let $\mathbb{A}^\infty$ denote the ring of finite adeles of F. There are canonical isomorphisms $\mathrm{GL}_d(\mathbb{A}^\infty)/\mathrm{GL}_d(\widehat{A}) \cong \mathrm{Lat}_{\widehat{A}}(V^\infty) \cong \mathrm{Lat}_A(V)$, where $g^\infty \in \mathrm{GL}_d(\mathbb{A}^\infty)/\mathrm{GL}_d(\widehat{A})$ is sent to $\mathcal{O}_{V^\infty} g^{\infty -1}$ in $\mathrm{Lat}_{\widehat{A}}(V^\infty)$, and to $\mathcal{O}_{V^\infty} g^{\infty -1} \cap V$ in $\mathrm{Lat}_A(V)$. Then, given $\Lambda \in \mathrm{Lat}_A(V)$ and $g^\infty \in \mathrm{GL}_d(\mathbb{A}^\infty)$, the lattice $\Lambda g^{\infty -1} \in \mathrm{Lat}_A(V)$ is defined. An element $g^\infty \in \mathrm{GL}_d(\mathbb{A}^\infty)$ induces an isomorphism $V/\Lambda \xrightarrow{\sim} V/(\Lambda g^{\infty -1})$. We denote by $\mathbf{b} g^{\infty -1} \in V/(\Lambda g^{\infty -1})$ the image of $\mathbf{b} \in V/\Lambda$ via this isomorphism.

4.3.3. *Convention.* Given an element $g \in \mathrm{GL}_d(\mathbb{A})$, we always denote by g_∞ the component at infinity, and g^∞ the finite part. Given a function f on $\mathrm{GL}_d(\mathbb{A})$, we write $f(g) = f(g_\infty, g^\infty)$ for $g = (g_\infty, g^\infty) \in \mathrm{GL}_d(\mathbb{A})$.

4.3.4. Given an A-lattice $\Lambda \subset V$ and $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$, we let

$$\mathbb{E}_{\Lambda,\mathbf{b}}(g_\infty, g^\infty) = E_{\Lambda g^{\infty -1}, \mathbf{b} g^{\infty -1}}(\mathcal{O}_{V_\infty} g_\infty^{-1}),$$

for $(g_\infty, g^\infty) \in \mathrm{GL}_d(\mathbb{A})$.

We note that $\mathbb{E}_{\Lambda,\mathbf{b}}$ is a $\mathbb{C}((q_\infty^{-s}))$-valued function

$$\mathbb{E}_{\Lambda,\mathbf{b}} : \mathrm{GL}_d(F) \backslash \mathrm{GL}_d(\mathbb{A}) / \mathrm{GL}_d(\mathcal{O}_\infty) \mathbb{K}_{\Lambda,\mathbf{b}} \to \mathbb{C}((q_\infty^{-s}))$$

on the double coset space $\mathrm{GL}_d(F)\backslash\mathrm{GL}_d(\mathbb{A})/\mathrm{GL}_d(\mathcal{O}_\infty)\mathbb{K}_{\Lambda,\mathbf{b}}$ where

$$\mathbb{K}_{\Lambda,\mathbf{b}} = \{g^\infty \in \mathrm{GL}_d(\mathbb{A}^\infty) \mid \Lambda g^{\infty-1} = \Lambda \,\text{and}\, \mathbf{b}g^{\infty-1} = \mathbf{b}\}$$

is a compact open subgroup of $\mathrm{GL}_d(\mathbb{A}^\infty)$.

4.3.5. We let $\Lambda \subset V$ be an A-lattice, $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$, $\widehat{\Lambda} = \Lambda \otimes_A \widehat{A} \subset V^\infty$, and $V_\mathbb{A} = V \otimes_F \mathbb{A} = V_\infty \times V^\infty$.

We define a $\mathbb{C}((q_\infty^{-s}))$-valued function $\phi_{\Lambda,\mathbf{b}}$ on $V_\mathbb{A}$. For $\mathbf{x} = (\mathbf{x}_\infty, \mathbf{x}^\infty) \in \mathbf{V}_\mathbb{A}$ where $\mathbf{x}_\infty$ (resp. $\mathbf{x}^\infty$) denotes the component at ∞ (resp. the finite part) of $\mathbf{x}$. we put

$$\phi_{\Lambda,\mathbf{b}}(\mathbf{x}) = \phi_\infty(\mathbf{x}_\infty)\phi^\infty_{\mathbf{\Lambda},\mathbf{b}}(\mathbf{x}^\infty)$$

where $\phi^\infty_{\Lambda,\mathbf{b}}$ is defined to be the characteristic function on $\mathbf{b} + \widehat{\Lambda} \subset V^\infty$, and $\phi_\infty(\mathbf{x}_\infty) = |\mathbf{x}_\infty|^{-s}_{\mathcal{O}_{V_\infty}}$.

Proposition 4.3. *If $g \in \mathrm{GL}_d(\mathbb{A})$, then $\mathbb{E}_{\Lambda,\mathbf{b}}(g) = \sum_{\mathbf{x}\in V} \phi_{\Lambda,\mathbf{b}}(\mathbf{x}g)$.*

Proof. This is immediate from the definition of $\mathrm{GL}_d(\mathbb{A})$ and of $\phi_{\Lambda,\mathbf{b}}$. □

4.4. Limit formula. We give a short proof of th e function field analogue of the Kronecker limit formula proved in [Gr-Ro] and [Ko].

Proposition 4.4. *Let $\Lambda \subset V$ be an A-lattice, $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$, and $(g_\infty, g^\infty) \in \mathrm{GL}_d(\mathbb{A})$. Then*

$$\mathrm{ord}_{\mathcal{O}_{V_\infty} g_\infty^{-1}} g_{\Lambda g^{\infty-1}, \mathbf{b}g^{\infty-1}} = (1 - q_\infty^d)\, \frac{1 - q_\infty^{d-s}}{1 - q_\infty^{-s}} \mathbb{E}_{\Lambda,\mathbf{b}}(g_\infty, g^\infty)\bigg|_{s=0}.$$

Proof. Let $L \in \mathrm{Lat}_{\mathcal{O}_\infty}(V_\infty)$. Using Proposition 4.1, we have

$$\begin{aligned}
&\mathrm{ord}_L g_{\Lambda,\mathbf{b}} \\
&= \frac{(q_\infty^d - 1)^2}{|a|^d - 1} \times \left[|a|^d \left\{ \mathrm{ord}_L(\mathbf{b}) + \sum_{\boldsymbol{\lambda}\in\Lambda\setminus\{0\}} (\mathrm{ord}_L(\boldsymbol{\lambda} - \mathbf{b}) - \mathrm{ord}_L(\boldsymbol{\lambda})) \right\} \right. \\
&\left. - \sum_{\mathbf{a}\in\Lambda/a} \left\{ \mathrm{ord}_L\left(\frac{\mathbf{a}+\mathbf{b}}{a}\right) + \sum_{\boldsymbol{\lambda}\in\Lambda\setminus\{0\}} \left(\mathrm{ord}_L\left(\boldsymbol{\lambda} - \frac{\mathbf{a}+\mathbf{b}}{a}\right) - \mathrm{ord}_L(\boldsymbol{\lambda}) \right) \right\} \right]
\end{aligned}$$

for any $a \in A \setminus \{0\}$. We note that the summands $\mathrm{ord}_L(\boldsymbol{\lambda} - \mathbf{b}) - \mathrm{ord}_L(\boldsymbol{\lambda})$ and $\mathrm{ord}_L(\boldsymbol{\lambda} - \frac{\mathbf{a}+\mathbf{b}}{a}) - \mathrm{ord}_L(\boldsymbol{\lambda})$ are zero for almost all $\boldsymbol{\lambda} \in \Lambda \setminus \{0\}$.

By the definition of the Eisenstein series, the expression above is equal to

$$\frac{(q_\infty^d - 1)^2}{|a|^d - 1} \frac{1}{\log q_\infty} \frac{\partial}{\partial s} \left\{ |a|^d E_{\Lambda,\mathbf{b}}(L) - \sum_{\mathbf{a}\in\Lambda/a} E_{\Lambda,(\mathbf{a}+\mathbf{b})/a}(L) \right\}.$$

From Lemma 4.2, we have

$$\sum_{\mathbf{a}\in\Lambda/a} E_{\Lambda,(\mathbf{a}+\mathbf{b})/a}(L) = E_{\Lambda/a,\mathbf{b}/a}(L) = E_{\Lambda,\mathbf{b}}(L)|a|^s.$$

Applying this, the expression above is equal to

$$-(q_\infty^d - 1) \left[\frac{q_\infty^d - q_\infty^s}{1 - q_\infty^s} E_{\Lambda,\mathbf{b}}(L) \right]_{s=0}.$$

The proposition now follows from the definition of adelic Eisenstein series $\mathbb{E}_{\Lambda,\mathbf{b}}$. □

5. Zeta Integral

We recall the definition of automorphic forms in Section 5.1. Godement and Jacquet [Go-Ja] first defined the L-function of automorphic representations of GL_d. We define the L-function of automorphic cusp forms explicitly in terms of Hecke operators. The compatibility of the two definitions is given in Appendices B and C.

We compute the pairing of a cusp form and a certain product of Eisenstein series (Theorem 5.1). The integral is expressed as the product of L-function and a simpler integral. The key idea is to use the norm property of the Euler system of the product of Eisenstein series.

In this section, we use the notations C, F, ∞, q_∞, A, and $\mathbb{A}$ introduced in Sections 3.1.1 and 3.4.4, and V, $\mathcal{O}_{V_\infty}$, $|\ |_{\mathcal{O}_{V_\infty}}$, and $\mathbb{A}^\infty$ introduced in Sections 4.1.1 and 4.3.2.

5.1. Automorphic forms. In Sections 5 and 6, we use the term "automorphic form" in a more brutal sense than usual. Let R be a commutative ring. An R-valued automorphic form for the general linear group $\mathrm{GL}_{d,F}$ over F is just an R-valued function on $\mathrm{GL}_d(F)\backslash\mathrm{GL}_d(\mathbb{A})$ which is invariant under right translation by an open compact subgroup of $\mathrm{GL}_d(\mathbb{A})$. We often omit the words "for $\mathrm{GL}_{d,F}$". An R-valued automorphic form f is called a cusp form if there exists an open compact subset $\mathbb{K}$ of $\mathrm{GL}_d(\mathbb{A})$ such that the support of f is contained in $Z(\mathbb{A})\mathbb{K}$ where Z denotes the center of GL_d. The set of R-valued automorphic forms (resp. cusp forms) is an R-algebra on which the group $\mathrm{GL}_d(\mathbb{A})$ acts by right translation.

Let χ be a continuous R-valued character (we endow R with the discrete topology) of the group $Z(F)\backslash Z(\mathbb{A})$. We say that an R-valued automorphic form f for $\mathrm{GL}_{d,F}$ has central character χ if the subgroup $Z(\mathbb{A})$ of $\mathrm{GL}_d(\mathbb{A})$ acts on f via χ. We often identify Z with $\mathbb{G}_m$ and regard χ as a character of the idele class group $F^\times\backslash\mathbb{A}^\times$.

For an R-valued character χ_∞ of the ∞-component $Z(F_\infty)$ of $Z(\mathbb{A})$, let $\mathcal{A}_R(\chi_\infty)$ (resp. $\mathcal{A}_R^{\mathrm{cusp}}(\chi_\infty)$) denote the R-algebra of R-valued automorphic forms (resp. cusp forms) on which $Z(F_\infty)$ acts via χ_∞. For two non-zero ideals I, J of$A$, let $\mathcal{A}_R(I,J,\chi_\infty)$ (resp. $\mathcal{A}_R^{\mathrm{cusp}}(I,J,\chi_\infty)$) denote the $\mathbb{K}^\infty_{I,J}$-invariant part of $\mathcal{A}_R(\chi_\infty)$ (resp. $\mathcal{A}_R^{\mathrm{cusp}}(\chi_\infty)$), where

$$\begin{aligned}\mathbb{K}^\infty_{I,J} = \ & \{(g_{ij}) \in \mathrm{GL}_d(\widehat{A}) \mid (g_{ij})_{1\le j\le d} \equiv (\delta_{ij})_{1\le j\le d} \bmod I \text{ for } 1 \le i \le d-1 \\ & \text{ and } \bmod J \text{ for } i = d\}.\end{aligned}$$

5.2. L-functions.

5.2.1. *Hecke operators.* We write $\mathrm{diag}(a_1,\dots,a_d)$ for the diagonal $(d\times d)$-matrix whose diagonal entries are $a_1,\dots,a_d$. Let $\pi_\wp$ denote the element in $\mathbb{A}^\times$ whose component at $\wp$ is a uniformizer and whose components at other places are 1.

Let $I, J \subset A$ be non-zero ideals. Let $\wp$ be a prime ideal, and let $e_\wp$ be the integer defined in Corollary 3.8. We define the Hecke operators $T_{\wp,r}$ and the dual Hecke operators $T^*_{\wp,r}$ for each $r = 0,\dots,e_\wp$.

If $\wp \nmid I$, we define $T_{\wp,r}$ (resp. $T^*_{\wp,r}$) to be the operator given by the double coset

$$\begin{gathered}\mathbb{K}^\infty_{I,J}\mathrm{diag}(\varpi_\wp,\dots,\varpi_\wp,1,\dots,1)\mathbb{K}^\infty_{I,J} \\ (\mathit{resp.}\ \mathbb{K}^\infty_{I,J}\mathrm{diag}(\varpi_\wp^{-1},\dots,\varpi_\wp^{-1},1,\dots,1)\mathbb{K}^\infty_{I,J})\end{gathered}$$

where $\varpi_\wp$ (resp. $\varpi_\wp^{-1}$) appears r times.

If $\wp|I$, we define $T_{\wp,0}$ and $T^*_{\wp,0}$ to be the identity. If moreover $\wp \nmid J$, we define $T_{\wp,1}$ (resp. $T^*_{\wp,1}$) to be the operator given by the double coset

$$\mathbb{K}^\infty_{I,J}\mathrm{diag}(1,\ldots,1,\varpi_\wp)\mathbb{K}^\infty_{I,J}$$
$$(\textit{resp. } \mathbb{K}^\infty_{I,J}\mathrm{diag}(1,\ldots,1,\varpi_\wp^{-1})\mathbb{K}^\infty_{I,J}).$$

5.2.2. Let f be a $\mathbb{C}$-valued automorphic form. Suppose that f satisfies the following conditions for some non-zero ideals $I \subsetneq A$, $J \subsetneq A$ of A.

(1) The open compact subgroup $\mathbb{K}^\infty_{I,J}$ of $\mathrm{GL}_d(\mathbb{A}^\infty)$ acts trivially on f.
(2) Let $\wp$ be a prime ideal of A not dividing $I+J$, and define the integer $e_\wp$ as in Corollary 3.8. Then f is an eigenform with respect to the operator $T_{\wp,r}$ for all $r \le e_\wp$.

These conditions imply that f has a certain central character χ with $\mathrm{cond}^\infty(\chi)$ dividing $I \cap J$. Here $\mathrm{cond}^\infty(\chi)$ denotes the finite part of the conductor of χ.

5.2.3. Let $a_{\wp,r}$ denote the eigenvalue of the operator $T_{\wp,r}$ on f.

For a $\mathbb{C}$-valued (quasi-)character χ' of $F^\times\backslash\mathbb{A}^\times$ with $\mathrm{cond}^\infty(\chi')$ prime to I, we define the L-function $L^{I,J}(f,s,\chi')$ of f twisted by χ' to be the infinite product

$$L^{I,J}(f,s,\chi') = \prod_{\wp\nmid I+J}\left[\sum_{r=0}^{e_\wp}(-1)^r\chi'(\wp)^{-r}a_{\wp,r}q_\wp^{\frac{r(r-1)}{2}-r(s+\frac{d-1}{2})}\right]^{-1}$$

in $\mathbb{C}((q_\infty^{-s}))$ where $\wp$ runs through the prime ideals of A prime to I. The infinite product $L^{I,J}(f,s,\chi')$ is convergent for the (q_∞^{-s})-adic topology. The compatibility of the above definition of $L^{I,J}(f,s,\chi)$ with the usual definition of L-function is explained in Proposition B.1 of Appendix B and Corollary C.7 of Appendix C.

5.3. Zeta Integral. We let $\mathbf{1}_j$ denote the row vector $(0,\ldots,0,\overset{j}{1},0,\ldots,0) \in V = F^{\oplus d}$ for each $j=1,\ldots,d$. Set $R_d = \mathbb{C}((q_\infty^{-s_1},\ldots,q_\infty^{-s_d}))$. Given two non-zero ideals I', J' of A and an element $\mathbf{h} = (h_1,\ldots,h_d) \in \mathrm{GL}_d(F_\infty)\times\cdots\times\mathrm{GL}_d(F_\infty)$ (d times), we consider the R_d-valued automorphic form

$$\mathcal{E}_{I',J',\mathbf{h}} = \prod_{j=1}^d \mathbb{E}_{I'^{\oplus d-1}\oplus J',\mathbf{1}_j}(gh_j)(s_j),$$

where $\mathbb{E}_{I'^{\oplus d-1}\oplus J',\mathbf{1}_j}$ is as in Section 4.3.4. Then $\mathcal{E}_{I',J',\mathbf{h}}$ is an element in $\mathcal{A}_{R_d}(I',J',|\ |_\infty^{-(s_1+\cdots+s_d)})$, where $|\ |_\infty : F_\infty^\times \to \mathbb{R}^\times$ denotes the norm at ∞.

Let χ_∞ be a continuous $\mathbb{C}$-valued (quasi-)character of the multiplicative group $F_\infty^\times$. Let I,J be two non-zero ideals of A. We fix a non-unit $a_\infty \in F_\infty^\times$, $a_\infty \notin \mathcal{O}_{F_\infty}^\times$ of F_∞ and take a continuous (quasi-)character χ' of $F^\times\backslash\mathbb{A}^\times$ satisfying $\mathrm{cond}^\infty(\chi')|(I\cap J)$ and $\chi_\infty(a_\infty) = \chi'(a_\infty)^d$.

We fix a Haar measure $dg_\wp$ of $\mathrm{GL}_d(F_\wp)$ for each place $\wp$ of F such that $\prod_\wp dg_\wp$ defines a Haar measure of $\mathrm{GL}_d(\mathbb{A})$ with $\mathrm{vol}(\mathrm{GL}_d(\widehat{\mathcal{O}}_C)) = 1$. Let us consider the $\mathbb{C}$-bilinear map

$$\langle\ ,\ \rangle_{\chi'} : \mathcal{A}^{\mathrm{cusp}}_{\mathbb{C}}(I,J,\chi_\infty)\times\mathcal{A}_{R_d}(I,J,|\ |_\infty^{-(s_1+\cdots+s_d)}) \to R_d[q_\infty^{-\frac{s_1+\cdots+s_d}{d}}]$$

defined by

$$\langle f_1,f_2\rangle_{\chi'} = \int_{a_\infty^{\mathbb{Z}}\mathrm{GL}_d(F)\backslash\mathrm{GL}_d(\mathbb{A})} f_1(g)f_2(g)|\det g|^{\frac{s_1+\cdots+s_d}{d}}\chi'(\det g)^{-1}dg,$$

where $|\ |$ denotes the idelic norm.

Theorem 5.1. *Let f be an element in $\mathcal{A}_{\mathbb{C}}^{\mathrm{cusp}}(I, J, \chi_\infty)$. Suppose that, for every prime ideal $\wp$ of A prime to $I+J$, and for every integer r with $0 \le r \le e_\wp$ (see Section 5.2.1 for the definition of $e_\wp$), f is an eigenform with respect to the operator $T_{\wp,r}$. Then for any element $\mathbf{h} = (h_1, \ldots, h_d) \in \mathrm{GL}_d(F_\infty) \times \cdots \times \mathrm{GL}_d(F_\infty)$ (d times), we have*

$$\langle f, \mathcal{E}_{I,J,\mathbf{h}} \rangle_{\chi'} = L^{I,J}(f, \frac{s_1 + \cdots + s_d}{d} - \frac{d-1}{2}, \chi'^{-1}) \mathrm{vol}(\mathbb{K}_{I,J}^\infty) \mathbb{I}_{\infty,\mathbf{h}}(f, \chi'),$$

where $\mathbb{I}_{\infty,\mathbf{h}}(f, \chi')$ is the integral

$$\begin{aligned}&\mathbb{I}_{\infty,\mathbf{h}}(f,\chi')\\ &= \textstyle\int_{a_\infty^{\mathbb{Z}} \backslash \mathrm{GL}_d(F_\infty)} f(g_\infty, 1) \prod_{j=1}^d |\mathbf{1}_j g_\infty h_j|_{\mathcal{O}_{V_\infty}}^{-s_j} |\det g_\infty|_\infty^{\frac{s_1+\cdots+s_d}{d}} \chi'(\det g_\infty)^{-1} dg_\infty.\end{aligned}$$

5.3.1. Proof of Theorem 5.1: Step 1. Application of Euler systems. Set $R = \mathbb{C}((q_\infty^{-s}))$. By considering the R-algebra $\mathcal{A}_R(|\ |_\infty^{-s})$ as a representation of $\mathrm{GL}_d(\mathbb{A}^\infty)$, we have (canonically up to canonical isomorphisms) a sheaf $\widetilde{\mathcal{A}}_R(|\ |_\infty^{-s})$ of R-algebras on $\mathcal{FC}^d$ for $X = \operatorname{Spec} A$, using Lemma 2.2.

For two non-zero ideals I, J of A, the R-algebra $\mathcal{A}_R(I, J, |\ |_\infty^{-s})$ is canonically identified with the R-algebra of the sections $\widetilde{\mathcal{A}}_R(|\ |_\infty^{-s})((A/I)^{\oplus d-1} \oplus A/J)$ of the sheaf $\widetilde{\mathcal{A}}_R(|\ |_\infty^{-s})$.

Given an A-lattice $\Lambda \subset V$ and $\mathbf{b} \in (V/\Lambda) \setminus \{0\}$, the function $\mathbb{E}_{\Lambda,\mathbf{b}}$ is an element in $\mathcal{A}_R(|\ |_\infty^{-s})$. For each j with $1 \le j \le d$, by assigning $h_j \mathbb{E}_{\Lambda,b}$ to the characteristic function of $\mathbf{b} + \widehat{\Lambda}$, we obtain a punctured distribution $\mathrm{SB}^* \to \mathcal{A}_R(|\ |_\infty^{-s})$ with values in $\mathcal{A}_R(|\ |_\infty^{-s})$.

Proposition 5.2. *The system of automorphic forms $(\mathcal{E}_{I,J,\mathbf{h}})_{I,J}$ indexed by two non-zero ideals is an Euler system. That is, the following statement holds.*

Let $I' \subset I$, $J' \subset J$ be ideals of A. We let $\wp$ be a prime ideal dividing $I' + J'$, and assume that $\operatorname{Supp}(A/I') = \operatorname{Supp}(A/I\wp)$, $\operatorname{Supp}(A/J') = \operatorname{Supp}(A/J\wp)$. *Let $e_\wp$ be as in Corollary 3.8. Then*

$$\mathrm{Tr}_{I,J}^{I',J'} \mathcal{E}_{I',J',\mathbf{h}} = \sum_{r=0}^{e_\wp} (-1)^r q_\wp^{r(r-1)/2} T_{\wp,r}^* \mathcal{E}_{I,J,\mathbf{h}}.$$

Here $T_{\wp,r}^*$ is the dual Hecke operator defined in Section 5.2.1, and $\mathrm{Tr}_{I,J}^{I',J'} : \mathcal{A}_{R_d}(I', J', |\ |_\infty^{-(s_1+\cdots+s_d)}) \to \mathcal{A}_{R_d}(I, J, |\ |_\infty^{-(s_1+\cdots+s_d)})$ is the trace map.

Proof. We apply Theorem 2.9 to the punctured distribution $\mathrm{SB}^* \to \mathcal{A}_R(|\ |_\infty^{-s})$ defined above. The assertion follows by noting that $T_{[\wp^{\oplus r}]}$ corresponds to the dual Hecke operator $T_{\wp,r}^*$ for each r. □

For any non-zero ideal I' of A with $I' \subset I \cap J$, we consider the element

$$\mathcal{E}_{I,J,I',\mathbf{h}} = \mathrm{Tr}_{I,J}^{I',I'} (\mathcal{E}_{I',I',\mathbf{h}})$$

in $\mathcal{A}_{R_d}(I, J, |\ |_\infty^{-(s_1+\cdots+s_d)})$. By Proposition 5.2, we have

$$\mathcal{E}_{I,J,I',\mathbf{h}} = \prod_{\wp | I',\, \wp \nmid I+J} \left(\sum_{r=0}^{e_\wp} (-1)^r q_\wp^{r(r-1)/2} T_{\wp,r}^* \right) \mathcal{E}_{I,J,\mathbf{h}},$$

where $e_\wp$ is as in Corollary 3.8 with $J' = I'$. Thus

$$\langle f, \mathcal{E}_{I,J,I',\mathbf{h}}\rangle_{\chi'} = \langle \prod_{\wp | I',\, \wp\nmid I+J} (\sum_{r=0}^{e_\wp} (-1)^r \chi'(\pi_\wp)^r q_\wp^{\frac{r(r-1)}{2} - \frac{r(s_1+\cdots+s_d)}{d}} T_{\wp,r}) f, \mathcal{E}_{I,J,\mathbf{h}}\rangle_{\chi'}$$

and hence

$$\langle f, \mathcal{E}_{I,J,\mathbf{h}}\rangle_{\chi'} = \prod_{\wp | I',\, \wp\nmid I+J} \left[\sum_{r=0}^{e_\wp} (-1)^r \chi'(\pi_\wp)^r a_{\wp,r} q_\wp^{\frac{r(r-1)}{2} - \frac{r(s_1+\cdots+s_d)}{d}}\right]^{-1} \langle f, \mathcal{E}_{I,J,I',\mathbf{h}}\rangle_{\chi'}.$$

Next we consider the limit of $\mathcal{E}_{I,J,I',\mathbf{h}}$ with respect to I'. We note that for all $I' \subset I \cap J$, the function $\mathcal{E}_{I,J,I',\mathbf{h}}$ is invariant under the action of $K_\infty \times \mathbb{K}_{I,J}^\infty$, where $K_\infty = \bigcap_{j=1}^d h_j \mathrm{GL}_d(\mathcal{O}_{F_\infty}) h_j^{-1}$. It is easily checked that for any $g \in \mathrm{GL}_d(\mathbb{A})$, the value $\mathcal{E}_{I,J,I',\mathbf{h}}(g) \in R_d$ converges (and hence uniformly converges on the coset $g(K_\infty \times \mathbb{K}_{I,J}^\infty)$) to

$$\mathcal{E}_{I,J,\lim,\mathbf{h}}(g) = \sum_{X \in \mathrm{GL}_d(F), X g^\infty \in \mathbb{K}_{I,J}^\infty} \prod_{j=1}^d |\mathbf{1}_j X g_\infty h_j|_{\mathcal{O}_{V_\infty}}^{-s_j}$$

with respect to the $(q_\infty^{-s_1}, \ldots, q_\infty^{-s_d})$-adic topology. Since f is a cusp form, the support of the function $f(g)\chi'(\det g)^{-1}$ on $a_\infty^{\mathbb{Z}} \backslash \mathrm{GL}_d(\mathbb{A})$ is compact. Hence

$$\langle f, \mathcal{E}_{I,J,\mathbf{h}}\rangle_{\chi'} = L^{I,J}(f, \frac{s_1 + \cdots + s_d}{d} - \frac{d-1}{2}, \chi'^{-1}) \langle f, \mathcal{E}_{I,J,\lim,\mathbf{h}}\rangle_{\chi'}.$$

5.3.2. Proof of Theorem 5.1: Step 2. Unfolding the integral. Now to prove the theorem, it suffices to prove the following proposition.

Proposition 5.3. *Let the notations be as above. We have*

$$\langle f, \mathcal{E}_{I,J,\lim,\mathbf{h}}\rangle_{\chi'} = \mathrm{vol}(\mathbb{K}_{I,J}^\infty) \mathbb{I}_{\infty,\mathbf{h}}(f, \chi').$$

Proof. Given two non-zero ideals I, J of A, we define a function $\widetilde{\phi}_{I,J,\mathbf{h}}$ on $\mathrm{GL}_d(\mathbb{A})$ as follows. For $g = (g_\infty, g^\infty) \in \mathrm{GL}_d(\mathbb{A})$, we let

$$\widetilde{\phi}_{I,J,\mathbf{h}}(g) = \widetilde{\phi}_{I,J}^\infty(g^\infty) \widetilde{\phi}_{\infty,\mathbf{h}}(g_\infty),$$

where $\widetilde{\phi}_{\infty,\mathbf{h}}(g_\infty) = \prod_{j=1}^d |\mathbf{1}_j g_\infty h_j|_{\mathcal{O}_{V_\infty}}^{-s_j}$, and $\widetilde{\phi}_{I,J}^\infty$ is the characteristic function of $\mathbb{K}_{I,J}^\infty$ We have

$$\mathcal{E}_{I,J,\lim,\mathbf{h}} = \sum_{\gamma \in \mathrm{GL}_d(F)} \widetilde{\phi}_{I,J,\mathbf{h}}(\gamma g).$$

Hence $\langle f, \mathcal{E}_{I,J,\lim,\mathbf{h}}\rangle_{\chi'}$ is equal to

$$\begin{aligned}
& \int_{a_\infty^{\mathbb{Z}} \mathrm{GL}_d(F) \backslash \mathrm{GL}_d(\mathbb{A})} f(g) \sum_{\gamma \in \mathrm{GL}_d(F)} \widetilde{\phi}_{I,J,\mathbf{h}}(\gamma g) |\det g|^{\frac{s_1+\cdots+s_d}{d}} \chi'(\det g)^{-1} dg \\
= & \int_{a_\infty^{\mathbb{Z}} \backslash \mathrm{GL}_d(\mathbb{A})} f(g) \widetilde{\phi}_{I,J,\mathbf{h}}(g) |\det g|^{\frac{s_1+\cdots+s_d}{d}} \chi'(\det g)^{-1} dg \\
= & \ \mathrm{vol}(\mathbb{K}_{I,J}^\infty) \int_{a_\infty^{\mathbb{Z}} \backslash \mathrm{GL}_d(F_\infty)} f(g_\infty, 1) \widetilde{\phi}_{\infty,\mathbf{h}}(g_\infty) |\det g_\infty|_\infty^{\frac{s_1+\cdots+s_d}{d}} \chi'(\det g_\infty)^{-1} dg_\infty \\
= & \ \mathrm{vol}(\mathbb{K}_{I,J}^\infty) \mathbb{I}_\infty(f, \chi').
\end{aligned}$$

$\square$

6. Regulators and special values of L-functions

6.1. Regulator. We will construct, in the spirit of [Con] and of [Sr], regulator maps from the d-th K-group of Drinfeld modular varieties for GL_d to the groups of $\mathbb{Z}$-valued harmonic cochains. Because of a lack of a satisfactory theory of higher Chow groups in the context of rigid geometry, we give a somewhat ad-hoc method to construct regulator maps. The reader is referred to [Bo-Gu-Re] for the basics on rigid analysis. Using the Kronecker limit formula, we express the image under the regulator map of the special elements in K-groups as the limit of the determinant of a matrix whose entries are Eisenstein series.

6.1.1. Drinfeld symmetric space $\mathfrak{X}$ introduced in Section 4.1.2 is a rigid analytic space over F_∞. For each integer $m \geq 0$, let $K_m(\mathfrak{X})$ (resp. $G_m(\mathfrak{X})$) denote the m-th K-group (resp. the m-th G-group) constructed from the exact category of locally free coherent $\mathcal{O}_\mathfrak{X}$-modules (resp. coherent $\mathcal{O}_\mathfrak{X}$-modules).

6.1.2. For each integer m with $1 \leq m \leq d$, we will construct a $\mathrm{GL}_d(F_\infty)$-equivariant homomorphism

$$\mathrm{reg}_\mathfrak{X} : K_m(\mathfrak{X}) \to \mathrm{Map}(\mathcal{T}_{m-1}, \mathbb{Z}).$$

There is a canonical continuous map

$$\mathfrak{X} \to |\mathcal{T}|$$

from (the underlying topological space of) $\mathfrak{X}$ to the geometric realization of the Bruhat-Tits building $\mathcal{T}$. For each cell $\sigma \in \mathcal{T}_* = \coprod_{0 \leq i \leq d} \mathcal{T}_i$, let $\mathfrak{U}_\sigma = \mathrm{Spm}\,(A_\sigma)$ denote the open affinoid corresponding to the closure of σ. The group $\mathrm{PGL}_d(F_\infty)$ acts both on $\mathfrak{X}$ and on $\mathcal{T}_i$. The action of $g \in \mathrm{PGL}_d(F_\infty)$ induces for each $\sigma \in \mathcal{T}_*$ a canonical isomorphism $\mathfrak{U}_\sigma \xrightarrow{\cong} \mathfrak{U}_{g\sigma}$.

We have a canonical homomorphism

$$K_m(\mathfrak{X}) \to \varprojlim_{\sigma \in \mathcal{T}_*} K_m(\mathrm{Spec}\, A_\sigma)$$

of K-groups, where the inverse limit is taken with respect to the inclusions of the closure of cells. The group $\mathrm{GL}_d(F_\infty)$ canonically acts on $K_d(\mathfrak{X})$. We have a similar homomorphism also for G-groups.

For each cell $\sigma \in \mathcal{T}_*$, let $\mathcal{U}_\sigma = \mathrm{Spf}\, A^o_\sigma$ (resp. $V_\sigma = \mathrm{Spec}\, \overline{A}_\sigma$) denote the formal model (resp. the analytic reduction) of the affinoid $\mathfrak{U}_\sigma$. Since the valuation on F_∞ is discrete, A^o_σ is an adic noetherian ring of finite Krull dimension. Furthermore, it is known that $\overline{A}_\sigma = A^o_\sigma \otimes_{\mathcal{O}_{F_\infty}} \kappa(\infty)$ and that V_σ is a normal crossing variety of pure dimension $d-1$ over $\kappa(\infty)$.

The canonical homomorphism $K_m(A_\sigma) \to G_m(A_\sigma)$ from K-theory to G-theory combined with the localization sequence with respect to $\mathrm{Spec}\, \overline{A}_\sigma \subset \mathrm{Spec}\, A^o_\sigma \supset \mathrm{Spec}\, A_\sigma$ yields a canonical homomorphism $K_m(A_\sigma) \to G_{m-1}(\overline{A}_\sigma)$.

Let τ be a face of σ. We know that $\mathcal{U}_\sigma \to \mathcal{U}_\tau$ is an open immersion ([Ge, III,1]). In particular, the morphism $A^o_\tau \to A^o_\sigma$ is flat, which implies that the diagram

$$\begin{array}{ccc} K_m(A_\tau) & \longrightarrow & G_{m-1}(\overline{A}_\tau) \\ \downarrow & & \downarrow \\ K_m(A_\sigma) & \longrightarrow & G_{m-1}(\overline{A}_\sigma) \end{array}$$

is commutative. Thus we obtain a $\mathrm{GL}_d(F_\infty)$-equivariant homomorphism

$$K_m(\mathfrak{X}) \to \varprojlim_{\sigma \in \mathcal{T}_*} G_{m-1}(\overline{A}_\sigma).$$

For each $\sigma \in \coprod_{0 \le i \le d-1} \mathcal{T}_i$, let X_σ denote the intersection of all irreducible components in $\operatorname{Spec} \overline{A}_\sigma$. Then X_σ is a smooth variety over $\kappa(\infty)$ of dimension $d-1-i$. When $i = d-1$, X_σ is isomorphic to $\operatorname{Spec} \kappa(\infty)$. When $i = d-2$, X_σ is isomorphic to the projective line over $\kappa(\infty)$ minus all the $\kappa(\infty)$-rational points. Let $\sigma \in \mathcal{T}_i$ with $i \le m-1$ and take for $j = 0, \ldots, i$ a j-cell $\sigma_j \in \mathcal{T}_j$ such that $\sigma_i = \sigma$ and σ_j is a face of σ_{j+1} for $j = 0, \ldots, i-1$. Then the connecting homomorphisms in localization sequences yield a homomorphism

$$G_{m-1}(\overline{A}_\sigma) \to G_{m-1}(X_{\sigma_0}) \to G_{m-2}(X_{\sigma_1}) \to \cdots \to G_{m-1-i}(X_\sigma).$$

It is easily checked that this homomorphism depends only on σ and does not depend on the choice of σ_j's. Using this homomorphism, we obtain

$$K_d(\mathfrak{X}) \to \varprojlim_{\sigma \in \coprod_{0 \le i \le m-1} \mathcal{T}_i} G_{m-1-i}(X_\sigma).$$

Looking at the $\mathcal{T}_{m-1}$-component of this homomorphism, we obtain a $\mathrm{GL}_d(F_\infty)$-equivariant map

$$\mathrm{reg}_{\mathfrak{X},m} : K_m(\mathfrak{X}) \to \mathrm{Map}(\mathcal{T}_{m-1}, \mathbb{Z}).$$

If $m = d \ge 2$, by looking also at the $\mathcal{T}_{m-2}$-component, we see that the image of $\mathrm{reg}_{\mathfrak{X},d}$ lies in the space of $\mathbb{Z}$-valued harmonic $(d-1)$-cochains.

6.1.3. There exists a canonical symbol map

$$\{\ ,\ldots,\ \} : \mathcal{O}(\mathfrak{X})^{*\otimes m} \to K_m(\mathfrak{X}).$$

Let $\mathrm{reg}'_{\mathfrak{X},m} : \mathcal{O}(\mathfrak{X})^{*\otimes m} \to \mathrm{Hom}(\widetilde{\mathcal{T}}_{m-1}, \mathbb{Z})$ be the composition of the symbol map with $\mathrm{reg}_{\mathfrak{X},m}$. If $f_1, \ldots, f_m \in \mathcal{O}(\mathfrak{X})^*$ and $\sigma = (L_i)_{i \in \mathbb{Z}} \in \widetilde{\mathcal{T}}_{m-1}$, then

$$\mathrm{reg}'_{\mathfrak{X},m}(f_1 \otimes \cdots \otimes f_m)((L_i)_{i\in\mathbb{Z}}) = \det \begin{pmatrix} \mathrm{ord}_{L_0} f_1 & \cdots & \mathrm{ord}_{L_{m-1}} f_1 \\ \vdots & \ddots & \vdots \\ \mathrm{ord}_{L_0} f_m & \cdots & \mathrm{ord}_{L_{m-1}} f_m \end{pmatrix}.$$

Lemma 6.1. *The homomorphism* $\mathrm{reg}'_{\mathfrak{X},m}(f_1 \otimes \cdots \otimes f_m)$ *coincides with* $\mathrm{reg}_{\mathfrak{X},m}(\{f_1, \cdots, f_m\})$ *in* $\mathrm{Hom}(\widetilde{\mathcal{T}}_{m-1}, \mathbb{Z})$.

Proof. This follows from the computation of boundary maps in localization sequences described in [Gi1, 7.21]. □

6.1.4. Let $I \subsetneq A$, $J \subsetneq A$ be two non-zero ideals of A. We set $\mathbf{N}_{I,J} = (A/I)^{\oplus d-1} \oplus A/J$. We construct a regulator map

$$\mathrm{reg} : K_d(\mathcal{M}^d_{\mathbf{N}_{I,J}} \times F_\infty) \to \mathrm{Map}_{\mathrm{GL}_d(F)}(\mathcal{T}_{d-1} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{I,J}, \mathbb{Z}).$$

For $(\sigma, g^\infty) \in (\coprod_{0 \le i \le d-1} \mathcal{T}_i) \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{I,J}$, there is a canonical morphism from the affinoid $\mathfrak{U}_\sigma$ to the rigid analytic space over F_∞ associated to $\mathcal{M}^d_{\mathbf{N}_{I,J}} \times F_\infty$. This induces a homomorphism $K_d(\mathcal{M}^d_{\mathbf{N}_{I,J}} \times F_\infty) \to K_d(A_\sigma)$ of algebraic K-groups. Hence we obtain a homomorphism

$$K_d(\mathcal{M}^d_{\mathbf{N}_{I,J}} \times F_\infty) \to \mathrm{Map}_{\mathrm{GL}_d(F)}\left(\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{I,J},\ K_d(\mathfrak{X})\right).$$

Composing this homomorphism with $\mathrm{reg}_{\mathfrak{X}}$, we obtain the desired homomorphism

$$K_d(\mathcal{M}^d_{\mathbf{N}_{I,J}} \times F_\infty) \to \mathrm{Map}_{\mathrm{GL}_d(F)}\left(\mathcal{T}_{d-1} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{I,J}, \mathbb{Z}\right).$$

Here, since $\mathcal{T}_{d-1}$ is identified with the coset $\mathrm{GL}_d(F_\infty)/F_\infty^\times \mathcal{I}$ via the isomorphism in Section 4.1.3, an element in the target $\mathrm{Map}_{\mathrm{GL}_d(F)}\left(\mathcal{T}_{d-1} \times \mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}^\infty_{I,J}, \mathbb{Z}\right)$ is regarded as a $\mathbb{Z}$-valued function on $\mathrm{GL}_d(\mathbb{A})$.

Proposition 6.2. *Let $I \subsetneq A$, $J \subsetneq A$ be two non-zero ideals of A. For $g \in \mathrm{GL}_d(\mathbb{A})$, let $\widetilde{\mathcal{E}}_{I,J}$ denote the $d \times d$ matrix with entries in $\mathbb{C}((q_\infty^{-s}))$ whose (i,j)-component is $\mathbb{E}_{I^{\oplus d-1} \oplus J, \mathbf{1}_i}(g\Pi_{j-1})(s)$. Let $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be a (quasi-)character of the idele class group of F whose conductor $\mathrm{cond}^\infty(\chi)$ divides IJ and whose ∞-component $\chi|_{F_\infty^\times}$ is trivial. Then*

$$\mathrm{reg}(\kappa^K_{I,J,\chi})(g) = \chi(\det(g))(1-q_\infty^d)^{2d} \lim_{s\to 0} \frac{1}{(1-q_\infty^{-s})^d} \det \widetilde{\mathcal{E}}_{I,J}.$$

Proof. This follows from Proposition 4.4 and the isomorphism given in Section 4.1.3. □

6.2. Special values of L-functions. We prove the second of our main theorems (Theorem 6.3). This is implied by Lemma 6.4, which expresses the integral over $\mathrm{GL}_d(F_\infty)$ as the integral over the diagonal matrices.

6.2.1. Let $I \subsetneq A$,$J \subsetneq A$ be two non-zero ideals of A. Let us consider the $\mathbb{C}$-bilinear map

$$\langle\ ,\ \rangle : \mathcal{A}^{\mathrm{cusp}}_{\mathbb{C}}(I,J,1) \times \mathcal{A}_{\mathbb{C}}(I,J,1) \to \mathbb{C}$$

(where 1 denotes the trivial character of the multiplicative group $F_\infty^\times$) defined by the integral

$$\langle f_1, f_2\rangle = \int_{Z(F_\infty)\mathrm{GL}_d(F)\backslash \mathrm{GL}_d(\mathbb{A})} f_1(g)f_2(g)dg$$

similar to the one introduced in Section 5.3.

6.2.2. Given an element σ in the d-th symmetric group S_d, we denote by $w_\sigma = (w_{\sigma,\infty}, w_\sigma^\infty)$ the matrix $(\delta_{i\sigma(j)})_{1\le i,j\le d} \in \mathrm{GL}_d(F)$ diagonally embedded in $\mathrm{GL}_d(\mathbb{A})$. For $f \in \mathcal{A}^{\mathrm{cusp}}_{\mathbb{C}}(I,J,1)$, we define its period $P(f) \in \mathbb{C}$ by

$$P(f) = \sum_{\sigma \in S_d} \mathrm{sgn}(\sigma) \int_{Z(F_\infty)\backslash w_{\sigma,\infty} M(F_\infty)\mathcal{I}} f(g_\infty, 1)dg_\infty,$$

where M denotes the subgroup of diagonal matrices of GL_d.

Theorem 6.3. *Let $\chi : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be a (quasi-)character of the idele class group of F such that $\mathrm{cond}^\infty(\chi)$ divides $I \cap J$ and $\chi|_{F_\infty^\times}$ is trivial. Let $f \in \mathcal{A}^{\mathrm{cusp}}_{\mathbb{C}}(I,J,1)$ be a cusp form satisfying the conditions (1), (2) in Section 5.2.2. Then*

$$\langle f, \mathrm{reg}(\kappa^K_{I,J,\chi})\rangle = (1-q_\infty^d)^{2d} \log q_\infty \lim_{s\to 0} \frac{\partial}{\partial s} L^{I,J}(f, s - \frac{d-1}{2}, \chi)\mathrm{vol}(\mathbb{K}^\infty_{I,J})P(f).$$

Proof. For $g_\infty \in \mathrm{GL}_d(F_\infty)$, let $H(g_\infty)$ denote the $d \times d$ matrix with entries in $\mathbb{C}((q_\infty^{-s}))$ whose (i,j)-component is $H_{i,j}(g_\infty) = \left|\mathbf{1}_i g_\infty \Pi_{j-1}^{-1}\right|_{\mathcal{O}_{V_\infty}}^{-s}$. Combining Proposition 6.2 with Theorem 5.1, we have

$$\begin{aligned}
& \frac{1}{(1-q_\infty^d)^{2d}\mathrm{vol}(\mathbb{K}_{I,J}^\infty)}\langle f, \mathrm{reg}(\kappa_{I,J,\chi}^K)\rangle \\
= \; & \lim_{s\to 0} \frac{1}{(1-q_\infty^{-s})^d} L^{I,J}(f, s - \tfrac{d-1}{2}, \chi) \\
& \quad \times \int_{Z(F_\infty)\backslash \mathrm{GL}_d(F_\infty)} f(g_\infty, 1) \det H(g_\infty) |\det g_\infty|^s dg_\infty.
\end{aligned}$$

Hence the assertion follows from Lemma 6.4 below. $\square$

Lemma 6.4. *Let the notations be as above. We have*

$$\int_{Z(F_\infty)\backslash \mathrm{GL}_d(F_\infty)} f(g_\infty, 1) \det H(g_\infty) |\det g_\infty|^s dg_\infty = (1-q_\infty^{-s})^{d-1} P(f).$$

Proof. Let us fix g_∞ and consider the matrix $H(g_\infty)$. Then for each $i = 1, \ldots, d$, there exists a unique $n_i = n_i(g_\infty) \in \{1, \ldots, d\}$ such that $H_{i,j}(g_\infty) = H_{i,1}(g_\infty)$ for $1 \le j \le n_i$ and $H_{i,j}(g_\infty) = q_\infty^{-s} H_{i,1}(g_\infty)$ for $n_i + 1 \le j \le d$. If $n_{i_1} = n_{i_2}$ for some $i_i \neq i_2$, the i_1-th row and i_2-th row of $H(g_\infty)$ are linearly dependent and hence $\det H(g_\infty) = 0$. Suppose that $n_1, \ldots, n_d$ are distinct. This occurs exactly when there exists σ such that $g_\infty \in w_{\sigma,\infty} M(F_\infty)\mathcal{I}$. Then we have $\prod_{i=1}^d H_{i,1}(g_\infty) = |\det(g_\infty)|^{-s}$ and hence

$$\det H(g_\infty) = \mathrm{sgn}(\sigma) \det H(w_{\sigma,\infty}^{-1} g_\infty) = \mathrm{sgn}(\sigma)\, |\det(g_\infty)|^{-s} \det D(s),$$

where $D(s)$ is the $d \times d$ matrix

$$D(s) = \begin{pmatrix} 1 & q_\infty^{-s} & \cdots\cdots & & q_\infty^{-s} \\ 1 & 1 & q_\infty^{-s} & \cdots & q_\infty^{-s} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & q_\infty^{-s} \\ 1 & \cdots\cdots & & & 1 \end{pmatrix}.$$

Simple calculation shows that $\det D(s) = (1-q_\infty^{-s})^{d-1}$, whence the claim follows. $\square$

7. Diagonal periods of cusp forms on GL_d over function fields

In this section, we will do some computation concerning the period $P(f)$ of a cusp form f, which appeared in Theorem 6.3. When $d = 3$ and C is rational, imposing some conditions on the cusp form, we describe $P(f)$ in terms of the L-function associated to f.

7.1. Notations. In this section, we use the notations C, q, F, ∞, $\mathbb{F}_q$, q_∞, A, $\mathbb{A}$, $\widehat{A}$, $\mathcal{O}_\infty$ and $\mathbb{A}^\infty$ introduced in Section 3.1.1, 3.4.4, 4.1.1, and 4.3.2 We set $\widehat{\mathcal{O}} = \mathcal{O}_\infty \times \widehat{A} \subset \mathbb{A}$. Let $\kappa(\infty)$ denote the residue field of $\mathcal{O}_\infty$. For each place v of F, let F_v, $\mathcal{O}_v$ denote the completion of F at v, the ring of integers of F_v respectively. We denote by q_v the cardinality of the residue field of $\mathcal{O}_v$.

We denote by $\mathrm{Div}(C)$ the group of divisors on C. For $D = \sum_v n_v[v] \in \mathrm{Div}(C)$, we use the following standard notations: $\mathrm{mult}_v(D) = n_v$ for each place v of F,

$\deg D = \sum_v n_v[\kappa(v) : \mathbb{F}_q]$, $\mathrm{Supp}\,(D) = \{v \mid n_v \neq 0\}$. For $D_i = \sum_v n_{i,v}[v] \in \mathrm{Div}(C)$, $i = 1, 2$, we write $D_1 \geq D_2$ if $n_{1,v} \geq n_{2,v}$ for all v. For an open subscheme $U \subset C$, we denote by $\mathrm{Div}(U) \subset \mathrm{Div}(C)$ the subgroup of elements D with $\mathrm{Supp}\,(D) \subset U$. For $D \in \mathrm{Div}(C)$ and a subset $S \subset C$, the divisor $\sum_{v \in S} \mathrm{mult}_v(D)[v]$ is denoted by $D|_S$. We denote by $\mathrm{div} : \mathbb{A}^\times \to \mathrm{Div}(C)$ the group homomorphism which sends $a = (a_v)_v \in \mathbb{A}^\times$ to $\sum_v \mathrm{ord}_v(a_v)[v]$ (we mainly use this notation for $a \in \mathbb{A}^\times$).

Let $\mathrm{Pic}(A) = F^\times \backslash \mathbb{A}^\times / F_\infty^\times \widehat{\mathcal{O}}^{\infty\times}$ denote the ideal class group of A. It is a finite abelian group. For $m \in \mathbb{A}^\times$, let $\mathrm{cl}(m)$ denote the class of m in $\mathrm{Pic}(A)$. The canonical homomorphism $\mathrm{cl} : \mathbb{A}^\times \to \mathrm{Pic}(A)$ factors through both $\mathrm{div} : \mathbb{A}^\times \to \mathrm{Div}(C)$ and the projection $\mathbb{A}^\times \to \mathbb{A}^{\infty\times}$. We use the notation $\mathrm{cl}(D)$ for $D \in \mathrm{Div}(C)$ and $\mathrm{cl}(m^\infty)$ for $m^\infty \in \mathbb{A}^{\infty\times}$.

For $D \in \mathrm{Div}(C)$, let $\mathcal{L}(D) \subset \mathbb{A}$ denote the $\widehat{\mathcal{O}}$-lattice

$$\mathcal{L}(D) = \{(a_v) \in \mathbb{A} \mid \mathrm{ord}_v(a_v) + \mathrm{mult}_v D \geq 0 \text{ for all } v\}.$$

We also denote by $H^0(\mathcal{L}(D))$ (resp. $H^1(\mathcal{L}(D))$) the kernel (resp. cokernel) of the composition $F \hookrightarrow \mathbb{A} \twoheadrightarrow \mathbb{A}/\mathcal{L}(D)$. For a pair (D, D') of two divisors on C with $D' \geq 0$, there exists a canonical long exact sequence

$$\begin{aligned} 0 \quad &\to H^0(\mathcal{L}(D - D')) \to H^0(\mathcal{L}(D)) \to \mathcal{L}(D)/\mathcal{L}(D - D') \\ &\to H^1(\mathcal{L}(D - D')) \to H^1(\mathcal{L}(D)) \to 0. \end{aligned} \tag{7.1}$$

For such a pair (D, D'), we denote by $(\mathcal{L}(D)/\mathcal{L}(D - D'))^0$ the subset

$$\{(a_v)_{v \in \mathrm{Supp}\,(D')} \in \mathcal{L}(D)/\mathcal{L}(D-D') \mid \mathrm{ord}_v(a_v) + \mathrm{mult}_v(D) = 0 \text{ for all } v \in \mathrm{Supp}\,(D')\}$$

of $\mathcal{L}(D)/\mathcal{L}(D - D')$ (when $D' = 0$ we understand $(\mathcal{L}(D)/\mathcal{L}(D - D'))^0 = 0$). Then for $a \in (\mathcal{L}(D)/\mathcal{L}(D-D'))^0$, its inverse $a^{-1} \in (\mathcal{L}(-D)/\mathcal{L}(-D-D'))^0$ is well-defined.

7.2. Diagonal periods.

7.2.1. We fix a positive integer $d \geq 2$ and consider the group scheme $G = \mathrm{GL}_d$ over C. Let G' denote the group scheme $G' = \mathrm{GL}_{d-1}$ over C, considered as a subgroup of G via the embedding

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $M' \subset G'$ denote the subgroup of diagonal matrices, and let $\widetilde{W}' = N_{G'}(M')$ be the normalizer of M' in G'. If $W' \subset G'$ denotes the subgroup of the permutation matrices, then $\widetilde{W}'$ is equal to the semi-direct product $\widetilde{W}' = W' \ltimes M'$. For an element σ in the $(d-1)$-st symmetric group S_{d-1}, let $w_\sigma = (\delta_{\sigma(i)j}) \in W'(C)$ denote the permutation matrix corresponding to σ.

7.2.2. We define $\widetilde{\omega}_{s,\infty} : \widetilde{W}'(F_\infty) \to \mathbb{C}((q_\infty^{-s}))^\times$ to be the unique continuous character whose restriction to $M'(F_\infty)$ is equal to $|\det(\ )|_\infty^s$ and whose restriction to $W'(F_\infty) \cong S_{d-1}$ is equal to the signature character.

For each place v of F, we fix a unique Haar measure dm_v of $M'(F_v)$ satisfying $\mathrm{vol}(M'(\mathcal{O}_v)) = 1$. They induce a Haar measure on $M'(\mathbb{A}^\infty)$, $\widetilde{W}'(F_\infty)$ and $\widetilde{W}'(F_\infty) \times M'(\mathbb{A}^\infty)$.

Definition 7.1. For a cusp form $f : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$, we put

$$P_\infty(f, s) = \int_{\widetilde{W}'(F_\infty)} f(g_\infty) \widetilde{\omega}_{s,\infty}(g_\infty) dg_\infty.$$

Since the restriction of f to $G'(F_\infty)$ has compact support, $P_\infty(f,s)$ is an element in $\mathbb{C}((q_\infty^{-s}))$.

In the rest of this section, we concentrate on computing $P_\infty(f,s)$. In the course of computation, we use the following adelic version of $P_\infty(f,s)$. Let $\widetilde{\omega}_s^\infty : M'(\mathbb{A}^\infty) \to (\mathbb{C}[\mathrm{Pic}(A)^{\oplus d-1}]((q^{-s})))^\times$ denote the continuous group homomorphism which sends $m^\infty = \mathrm{diag}(m_1^\infty, \ldots, m_{d-1}^\infty) \in M'(\mathbb{A}^\infty)$ to $|\det(m^\infty)|^s(\mathrm{cl}(m_1), \ldots, \mathrm{cl}(m_{d-1})) \in \mathbb{C}[\mathrm{Pic}(A)^{\oplus d-1}]((q^{-s}))^\times$. Let $\widetilde{\omega}_s : \widetilde{W}'(F_\infty) \times M'(\mathbb{A}^\infty) \to (\mathbb{C}[\mathrm{Pic}(A)^{\oplus d-1}]((q^{-s})))^\times$ denote the continuous group homomorphism which sends $(g_\infty, m^\infty) \in \widetilde{W}'(F_\infty) \times M'(\mathbb{A}^\infty)$ to $\widetilde{\omega}_{s,\infty}(g_\infty)\widetilde{\omega}_s^\infty(m^\infty)$.

Definition 7.2. For a cusp form $f : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$, we define $P(f,s) \in \mathbb{C}[\mathrm{Pic}(A)^{\oplus d-1}]((q^{-s}))$ to be the integral

$$P(f,s) = \int_{M'(F)\backslash(\widetilde{W}'(F_\infty)\times M'(\mathbb{A}^\infty))} f(g)\widetilde{\omega}_s(g)dg.$$

Let $\mathrm{ev}_0 : \mathbb{C}[\mathrm{Pic}(A)^{\oplus d-1}]((q^{-s}))$ to $\mathbb{C}((q^{-s}))$ denote the $\mathbb{C}((q^{-s}))$-linear map which associates the coefficient of $0 \in \mathrm{Pic}(A)^{\oplus d}$. When f is $M'(\widehat{A})$-invariant, we have an equality $P_\infty(f,s) = q^{d-1}\mathrm{ev}_0(P(f,s))$ in $\mathbb{C}((q^{-s}))$.

7.3. Let the notations be as before. In this section, we consider the following conditions.

Conditions 7.3.
1. The $G(\mathbb{A})$-module generated by f is an irreducible cuspidal automorphic representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A})$.
2. f is factorizable, i.e. $f = \bigotimes_v' f_v$ for some $f_v \in \pi_v$.
3. π_∞ is isomorphic to the Steinberg representation of $G(F_\infty)$ with trivial central character, and $f_\infty \in \pi_\infty$ is an Iwahori-spherical vector.
4. There exists a place $o \neq \infty$ satisfying the following properties.
   - For $v \neq o, \infty$, $f_v \in \pi_v$ is a new vector ("vector essentielle") in the sense of [Ja-Pi-Sh] (in particular f_v is $G'(\mathcal{O}_v)$-invariant).
   - For $v = o$, π_o is isomorphic to an unramified twist of the Steinberg representation of $G(F_o)$, and $f_o \in \pi_o$ is an Iwahori-spherical vector.
5. $d = 3$ and the class of o in $\mathrm{Pic}(A)$ is trivial.
6. $C = \mathbb{P}^1_{\mathbb{F}_q}$, and ∞, o are the usual ones.

From now on we assume that Conditions 7.3 (1)-(3) are satisfied.

Remark 7.4. When $d = 2$, by the classical theory of Jacquet-Langlands, the integral $P(f,s)$ is related to the L-function $L(\pi, s, \chi)$ of π with twists by an unramified character χ of $\mathbb{A}_F$ whose ∞-component is trivial. If $d \geq 3$ and if f_v is a new vector for all $v \neq \infty$, then $P(f,s)$ vanishes for a trivial reason. This is a basic reason why we introduce another place o in the condition (4) above.

Remark 7.5. Let f be a cusp form satisfying the conditions (1)-(4) in Conditions 7.3. Let us describe the relation between the period in Section 6.2 and $P_\infty(f,s)$. Let us take a uniformizer ϖ_o of F_o. Define the cusp form f' by $f'(g) = f(g \cdot \mathrm{diag}(1, \ldots, 1, \varpi_o))$ where $\mathrm{diag}(1, \ldots, 1, \varpi_o)$ is the diagonal $d \times d$ matrix with the diagonal entries $1, \ldots, 1, \varpi_o$. Let J denote the the prime-to-∞ part of the conductor of π. We consider J as an ideal of A and write J as the product $J = J_o J^{\infty,o}$ of the o-part and the prime-to-o part. Then f' is an element in $\mathcal{A}_{\mathbb{C}}^{\mathrm{cusp}}(J_o, J^{\infty,o}, 1)$.

For $i = 0, \ldots, d-1$, let $h_i \in G(F_\infty)$ be as in Appendix C.3 (for $K = F_\infty$). Then we have

$$\begin{aligned} P(f') &= \sum_{\sigma \in S_{d-1}} \mathrm{sgn}(\sigma) \sum_{i=0}^{d-1} (-1)^{i(d-1)} \int_{M'(F_\infty)} f(m w_\sigma h_i) dm \\ &= d P_\infty(f, 0) \end{aligned}$$

by Lemma C.6.

Let $M \subset G$ denote the subgroup of diagonal matrices, $B \subset G$ (resp. $B' \subset G'$) denote the subgroup of upper triangular matrices, and $N \subset B$ (resp. $N' \subset B'$) denote its unipotent radical. We fix a non-trivial additive character $\psi : F \backslash \mathbb{A} \to \mathbb{C}$. For each finite place v of F, let ψ_v denote the v-component of ψ and $\mathrm{ord}_v \psi$ denote its conductor. We also set $\mathrm{ord}\psi = \sum_v \mathrm{ord}_v \psi [v] \in \mathrm{Div}(C)$. Let $\psi_N : N(\mathbb{A}) \to \mathbb{C}^\times$ denote the character which sends $n = (n_{ij}) \in N(\mathbb{A})$ to $\psi(\sum_{i=1}^{d-1} n_{i,i+1})$. The v-component of ψ_N is denoted by $\psi_{N,v}$. Let $\mathrm{Wh} = \prod_v \mathrm{Wh}_v : G(\mathbb{A}) \to \mathbb{C}$ denote the Whittaker function associated to f and ψ_N, where, for each v, Wh_v is the Whittaker function associated to f_v and $\psi_{f,v}$. We put $\mathrm{Wh}^\infty = \prod_{v \neq \infty} \mathrm{Wh}_v : \mathbb{A}^\infty \to \mathbb{C}$. We have a Fourier expansion

$$f(g) = \sum_{\gamma \in N'(F) \backslash G'(F)} \mathrm{Wh}(\gamma g),$$

where in the sum in the right hand side, $\mathrm{Wh}(\gamma g) = 0$ except for finitely many γ on any compact subset of $G(\mathbb{A})$.

Theorem 7.6. *Suppose that Conditions 7.3 (1)-(6) are satisfied. Let $\mu_o \in \mathbb{C}^\times$ denote the unique complex number such that π_o is isomorphic to the twist of the Steinberg representation of $G(F_o)$ by the unramified character of $F_o^\times$ which sends a uniformizer in F_o to μ_o. For each place v of F, take an element $a_v \in F_v$ with $\mathrm{ord}_v(a_v) = -\mathrm{ord}_v \psi$ and put $C_v = \mathrm{Wh}_v(\mathrm{diag}(a_v^2, a_v, 1))$. It is non-zero and does not depend on the choice of a_v. Finally we put $C(f) = \prod_v C_v$. Then we have*

$$P_\infty(f, s) = (1+q) q^{4-4s} \mu_o^{-1} C(f) L(\pi, s) L(\pi, s+1).$$

Corollary 7.7. *Let the notations be as in Theorem 7.6 and in Remark 7.5 above. We have*

$$P(f') = 3(1+q) q^4 \mu_o^{-1} C(f) L(f, 0) L(f, 1).$$

Let $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be a character of the idele class group of F such that $\mathrm{cond}^\infty(\chi)$ divides J and $\chi|_{F_\infty^\times}$ is trivial. Then we have

$$\begin{aligned} \langle f', \mathrm{reg}(\kappa^K_{J_o, J^{\infty,o}, \chi}) \rangle = \ & 3(1+q) q^4 (1 - q_\infty^d)^{2d} \mathrm{vol}(\mathbb{K}^\infty_{J_o, J^{\infty,o}}) \\ & \times \mu_o^{-1} C(f) L(\pi, -1, \chi) L(f, 0) L(f, 1). \end{aligned}$$

Proof. The first equation is an immediate consequence of Theorem 6.3 and the formula in Remark 7.5. The second equation follows from the first one combined with Theorem 6.3, Proposition B.1 of Appendix B, and Corollary C.7 of Appendix C. □

7.4. Strategy of proof of Theorem 7.6. Our proof of Theorem 7.6 consists of three steps. The first (resp. second) step is Proposition 7.8 (resp. Proposition 7.9), which is proved in Section 7.5 (resp. Section 7.8). The final step is given in Section 7.9.

7.4.1. Let $B'^{-} \subset G'$ denote the Borel subgroup of lower triangular matrices, and $N'^{-} \subset B'^{-}$ denote its unipotent radical. Let $\mathcal{I}' \subset G'(F_\infty)$ be the Iwahori subgroup, that is, the subgroup of $G'(\mathcal{O}_\infty)$ of elements whose image in $G'(\kappa(\infty))$ is an upper triangle matrix. For an invertible matrix $g \in G'(F_\infty)$ whose diagonal entries are all 1, let $\mathrm{IW}(g) \subset M'(F_\infty)$ denote the open subset defined by

$$\mathrm{IW}(g) = \{m \in M'(F_\infty) \mid m^{-1}gm \in \mathcal{I}'\}.$$

Given an element $b \in B'^{-}(F_\infty)$, we write it in the form $b = mn$ with $m \in M'(F_\infty)$, $n \in N'^{-}(F_\infty)$, and then define an element $D_{b,\sigma}$ in $\mathbb{C}((q_\infty^{-s}))$ to be

$$D_{b,\sigma} = \int_{\mathrm{IW}(w_\sigma^{-1} n w_\sigma)} \mathrm{Wh}_\infty(m w_\sigma m') |\det(m')|_\infty^s dm'.$$

When we write b in the form $b = n'm$ with $m \in M'(F_\infty)$, $n' \in N'^{-}(F_\infty)$, we have

$$D_{b,\sigma} = |\det(m)|_\infty^{-s} \int_{\mathrm{IW}(w_\sigma^{-1} n' w_\sigma)} \mathrm{Wh}_\infty(w_\sigma m') |\det(m')|_\infty^s dm' = |\det(m)|_\infty^{-s} D_{n',\sigma}.$$

Proposition 7.8. *Suppose that Conditions 7.3 (1)-(3) are satisfied.*

(1) *We have*

$$P_\infty(f,s) = \sum_{b \in B'^{-}(F)} \left(\sum_{\sigma \in S_{d-1}} \mathrm{sgn}(\sigma) D_{b,\sigma} \right) \left(\sum_{\sigma' \in S_{d-1}} \mathrm{sgn}(\sigma') \mathrm{Wh}^\infty(b w_{\sigma'}) \right).$$

(2) *We have*

$$P(f,s) = \sum_{n \in N'^{-}(F)} \left(\sum_{\sigma \in S_{d-1}} \mathrm{sgn}(\sigma) D_{n,\sigma} \right) \left(\sum_{\sigma' \in S_{d-1}} \mathrm{sgn}(\sigma') I^\infty_{n,\sigma'} \right),$$

where, for $\sigma' \in S_{d-1}$, $I^\infty_{n,\sigma'}$ is the integral

$$I^\infty_{n,\sigma'} = \int_{M'(\mathbb{A}^\infty)} \mathrm{Wh}^\infty(n w_{\sigma'} m^\infty) \widetilde{\omega}_s^\infty(m^\infty) dm^\infty.$$

7.4.2. In this paragraph we assume that Conditions 7.3 (1)-(5) are satisfied. For each place v of F, the value $\mathrm{Wh}_v(m)$ of Wh_v at $m = \mathrm{diag}(m_1, m_2) \in M'(F_v)$ depends only on $(\mathrm{ord}_v(m_1), \mathrm{ord}_v(m_2)) \in \mathbb{Z}^{\oplus 2}$. We denote the value by $\overline{\mathrm{Wh}}_v(\mathrm{ord}_v(m_1), \mathrm{ord}_v(m_2))$. We define a function $\overline{\mathrm{Wh}}'^{\infty,o} : \mathrm{Div}(C \setminus \{\infty, o\})^{\oplus 2} \to \mathbb{C}$ by $\overline{\mathrm{Wh}}'^{\infty,o}(D_1, D_2) = \prod_{v \neq \infty, o} \overline{\mathrm{Wh}}_v(\mathrm{mult}_v(D_1) - 2\,\mathrm{ord}_v \psi, \mathrm{mult}_v(D_2) - \mathrm{ord}_v \psi)$.

Proposition 7.9. *Suppose that Conditions 7.3 (1)-(5) are satisfied. Let Δ_ψ denote the set of pairs $(D_1, D_2) \in \mathrm{Div}(C \setminus \{\infty, o\})^{\oplus 2}$ satisfying $D_1 \geq D_2 \geq 0$ and $\mathrm{cl}(D_1 + D_2) = 3\,\mathrm{cl}(\mathrm{ord}\,\psi)$. For $(D_1, D_2) \in \Delta_\psi$, let Σ_{D_1,D_2} denote the set of pairs $(D, m) \in \mathrm{Div}(C \setminus \{\infty, o\}) \times F^\times$ satisfying the following conditions.*

- $D \geq 0$.
- $\mathrm{cl}(D + D_1|_{C \setminus \mathrm{Supp}\,(D)} + D_2|_{\mathrm{Supp}\,D}) = \mathrm{cl}(2\mathrm{ord}\,\psi - (\mathrm{ord}\,\psi)|_{\mathrm{Supp}\,(D)})$.
- $D_1 - D_2 + \mathrm{div}(m)|_{C \setminus \{\infty, o\}} \geq \mathrm{ord}\,\psi|_{C \setminus \{\infty, o\}}$.
- $D = (D_1 - D_2 + \mathrm{div}(m) - \mathrm{ord}\,\psi)|_{\mathrm{Supp}\,(D)}$.

For each $(D_1, D_2) \in \Delta_\psi$, define $a_\psi(D_1, D_2) \in \mathbb{C}$ to be

$$a_\psi(D_1, D_2) = a^{(0)}(D_1, D_2) + \frac{1}{2} a^*_\psi(D_1, D_2),$$

where

$$a^{(0)}(D_1,D_2)=\begin{cases}1, & \text{if } \mathrm{cl}(D_1)=\mathrm{cl}(2\,\mathrm{ord}\,\psi) \text{ and } \mathrm{cl}(D_2)=\mathrm{cl}(\mathrm{ord}\,\psi),\\ 0, & \text{otherwise}\end{cases}$$

and

$$a^*_\psi(D_1,D_2)=\sum_{(D,m)\in\Sigma_{D_1,D_2}}(1-\psi_\infty(\frac{1}{m}))(1-\psi_o(\frac{1}{m}))\prod_{v\in\mathrm{Supp}\,(D)}\psi_v(\frac{1}{m}).$$

Then $P_\infty(f,s)$ *is equal to*

$$q^{d-1+3\deg(\mathrm{ord}\psi)s}\widetilde{C}_\infty(s)\widetilde{C}_o(s)\sum_{(D_1,D_2)\in\Delta_\psi}a_\psi(D_1,D_2)q^{-\deg(D_1+D_2)\cdot s}\overline{\mathrm{Wh}}'^{\infty,o}(D_1,D_2),$$

where

$$\widetilde{C}_\infty(s)=\frac{C_\infty q_\infty^{1+s}}{(1-q_\infty^{-1-s})(1-q_\infty^{-2-s})}$$

and

$$\widetilde{C}_o(s)=\frac{C_o\mu_o^{-1}q_o^{1+s}}{(1-\mu_o q_o^{-1-s})(1-\mu_o q_o^{-2-s})}.$$

7.5. Proof of Proposition 7.8. Before proving Proposition 7.8, we need some preparations. Iwahori factorization $G'(F_\infty)=\coprod_{\sigma\in S_{d-1}}N'(F_\infty)w_\sigma M'(F_\infty)\mathcal{I}'$ ([Iw], [Br-Ti]) yields a map $\beta: N'(F_\infty)\backslash G'(F_\infty)\to S_{d-1}$ of sets.

Lemma 7.10. *For* $\sigma\in S_{d-1}$*, the set* $w_\sigma M'(F_\infty)\mathcal{I}'\cap B'^-(F_\infty)w_\sigma$ *forms a complete system of representatives of* $\beta^{-1}(\sigma)=N'(F_\infty)\backslash(N'(F_\infty)w_\sigma M'(F_\infty)\mathcal{I}')$.

Proof. We will prove that the composition

$$w_\sigma M'(F_\infty)\mathcal{I}'\cap B'^-(F_\infty)w_\sigma\hookrightarrow N(F_\infty)w_\sigma M'(F_\infty)\mathcal{I}'\twoheadrightarrow\beta^{-1}(\sigma)$$

is bijective. The injectivity follows from $B'^-(F_\infty)\cap N'(F_\infty)=\{1\}$. For the surjectivity, it suffices to prove that any element in $N'(\mathcal{O}_\infty)\backslash(N'(\mathcal{O}_\infty)w_\sigma\mathcal{I}')$ is represented by an element in $w_\sigma\mathcal{I}'\cap B'^-(\mathcal{O}_\infty)w_\sigma$. By Bruhat decomposition for $G'(\kappa(\infty))$, any element in $N'(\mathcal{O}_\infty)\backslash(N'(\mathcal{O}_\infty)w_\sigma\mathcal{I}')$ is represented by an element in $w_\sigma\mathcal{I}'\cap\mathcal{I}'^-(\mathcal{O}_\infty)w_\sigma$, where $\mathcal{I}'^-\subset G'(\mathcal{O}_\infty)$ denotes the subgroup of elements whose images in $G'(\kappa(\infty))$ lie in $N'^-(\kappa(\infty))$. It is easily checked that $\mathcal{I}'^-=N'(\mathcal{O}_\infty)^{(1)}\cdot B'^-(\mathcal{O}_\infty)$, where $N'(\mathcal{O}_\infty)^{(1)}\subset N'(\mathcal{O}_\infty)$ denotes the subgroup of elements whose images in $N'(\kappa(\infty))$ are equal to 1. Since $w_\sigma^{-1}N'(\mathcal{O}_\infty)^{(1)}w_\sigma\subset\mathcal{I}'$, we have $w_\sigma\mathcal{I}'\cap\mathcal{I}'^-(\mathcal{O}_\infty)w_\sigma=N'(\mathcal{O}_\infty)^{(1)}\cdot(w_\sigma\mathcal{I}'\cap B'^-(\mathcal{O}_\infty)w_\sigma)$. Hence the surjectivity follows. □

For $\sigma\in S_{d-1}$, we say that an element $g\in N'(F_\infty)\backslash G'(F_\infty)$ is *σ-generic* if there exists an element there exists a unique element $g^{(\sigma)}\in B'^-(F_\infty)$ such that g is represented by $g^{(\sigma)}w_\sigma$. When g is σ-generic, such $g^{(\sigma)}$ is unique. If further $g\in N'(F)\backslash G'(F)\subset N'(F_\infty)\backslash G'(F_\infty)$, then we have $g^{(\sigma)}\in B'^-(F)$.

Corollary 7.11. *Suppose that* $\sigma\in S_{d-1}$ *and* $g\in N'(F_\infty)\backslash G'(F_\infty)$ *satisfy* $\beta(g)=\sigma$*. Then* g *is* σ*-generic and* $w_\sigma^{-1}g^{(\sigma)}w_\sigma\in M'(F_\infty)\mathcal{I}'$.

Proof. We take an element $k\in w_\sigma M'(F_\infty)\mathcal{I}'$ representing g. By Lemma 7.10, we may assume that $k\in w_\sigma M'(F_\infty)\mathcal{I}'\cap B'^-(F_\infty)w_\sigma$. Hence g is σ-generic and we have $k=g^{(\sigma)}w_\sigma$ and $w_\sigma^{-1}g^{(\sigma)}w_\sigma=w_\sigma^{-1}k\in M'(F_\infty)\mathcal{I}$. □

Proof of Proposition 7.8. First we prove (1). For $\sigma \in S_{d-1}$ and $g \in N'(F_\infty)\backslash G'(F_\infty)$, let us define a map $H_{\sigma,g} : M'(F_\infty) \to S_{d-1}$ by putting $H_{\sigma,g}(m) = \beta(g w_\sigma m)$. Then $P_\infty(f,s)$ is equal to

$$\sum_{\sigma \in S_{d-1}} \operatorname{sgn}(\sigma) \sum_{\gamma \in N'(F)\backslash G'(F)} \sum_{\sigma' \in S_{d-1}} \int_{H_{\sigma,\gamma}^{-1}(\sigma')} \operatorname{Wh}(\gamma w_{\sigma,\infty} m) |\det(m)|_\infty^s dm,$$

where $w_{\sigma,\infty}$ is an element in $G'(\mathbb{A})$ whose ∞-component is w_σ and whose components at the places other than ∞ are 1.

Lemma 7.12. *Suppose that two elements $\sigma, \sigma' \in S_{d-1}$ are given. Then for $\gamma \in N'(F)\backslash G'(F)$ and $m \in M'(F_\infty)$, we have $H_{\sigma,\gamma}(m) = \sigma'$ if and only if there exists an element $b \in B'^-(F)$ satisfying the following two properties.*

1. *The class $\gamma \in N'(F)\backslash G'(F)$ is represented by $b w_{\sigma'\sigma^{-1}} \in G'(F)$.*
2. *If we write b in the form $b = m'n'$ with $m' \in M'(F)$, $n' \in N'^-(F)$, then $m \in \operatorname{IW}(w_{\sigma'}^{-1} n' w_{\sigma'})$.*

Proof of Lemma 7.12. Suppose that $H_{\sigma,\gamma}(m) = \sigma'$. Let us apply Corollary 7.11 for $g = \gamma w_\sigma m$. Since $\beta(g) = \sigma'$, g is σ'-generic and $w_{\sigma'}^{-1} g^{(\sigma')} w_{\sigma'} \in M'(F_\infty)\mathcal{I}'$. Hence γ is $\sigma'\sigma^{-1}$-generic and $g^{(\sigma')} = \gamma^{(\sigma'\sigma^{-1})} w_{\sigma'} m w_{\sigma'}^{-1}$. Hence $b = \gamma^{(\sigma'\sigma^{-1})}$ satisfies (1). Since $g^{(\sigma')} = b w_{\sigma'} m w_{\sigma'}^{-1}$, we have $m^{-1} w_{\sigma'}^{-1} b w_{\sigma'} m = m^{-1} w_{\sigma'}^{-1} g^{\sigma'} w_\sigma \in M'(F_\infty)\mathcal{I}'$. Write $b = m'n'$ as in (2). Then since $m^{-1} w_{\sigma'}^{-1} m' w_{\sigma'} m \cdot m^{-1} w_{\sigma'}^{-1} n' w_{\sigma'} m \in M'(F_\infty)\mathcal{I}'$, we have $m^{-1} w_{\sigma'}^{-1} n' w_{\sigma'} m \in \mathcal{I}'$. Hence (2) follows.

Conversely, suppose that there exists $b \in B'^-(F)$ satisfying (1) and (2). Then by (2), we have $m^{-1} w_{\sigma'}^{-1} b w_{\sigma'} m \in M'(F_\infty)\mathcal{I}'$, so that $b w_{\sigma'\sigma^{-1}} \cdot w_\sigma m \in w_{\sigma'} M'(F_\infty)\mathcal{I}'$. Hence, by (1), we have $H_{\sigma,\gamma}(m) = \sigma'$. □

We return to the proof of Proposition 7.8. By Lemma 7.12, $P_\infty(f,s)$ is equal to

$$\sum_{\sigma,\sigma' \in S_{d-1}} \operatorname{sgn}(\sigma) \sum_{\substack{b \in B'^-(F) \\ b = m'n'}} \int_{\operatorname{IW}(w_{\sigma'}^{-1} n' w_{\sigma'})} \operatorname{Wh}(b w_{\sigma'\sigma^{-1}} w_{\sigma,\infty} m) |\det(m)|_\infty^s dm.$$

Since

$$\begin{aligned} \operatorname{Wh}(b w_{\sigma'\sigma^{-1}} w_{\sigma,\infty} m) &= \operatorname{Wh}_\infty(b w_{\sigma'} m) \operatorname{Wh}^\infty(b w_{\sigma'\sigma^{-1}}) \\ &= \operatorname{Wh}_\infty(m' w_{\sigma'} m) \operatorname{Wh}^\infty(b w_{\sigma'\sigma^{-1}}) \end{aligned}$$

for $b = m'n' \in B'^-(F)$ and $m \in \operatorname{IW}(w_{\sigma'}^{-1} n' w_{\sigma'})$, we have,

$$\begin{aligned} &\sum_{\sigma,\sigma' \in S_{d-1}} \operatorname{sgn}(\sigma) \int_{\operatorname{IW}(w_{\sigma'}^{-1} n' w_{\sigma'})} \operatorname{Wh}(b w_{\sigma'\sigma^{-1}} w_{\sigma,\infty} m) |\det(m)|_\infty^s dm \\ = &\left(\sum_{\sigma' \in S_{d-1}} \operatorname{sgn}(\sigma') D_{b,\sigma'} \right) \left(\sum_{\sigma \in S_{d-1}} \operatorname{sgn}(\sigma) \operatorname{Wh}^\infty(b w_\sigma) \right), \end{aligned}$$

whence the assertion follows.

Next we prove (2). Take a complete system of representatives $S \subset \operatorname{Div}(A) = \mathbb{A}^{\infty\times}/\widehat{\mathcal{O}}^{\infty\times}$ of the quotient $\operatorname{Pic}(A) = F^\times\backslash\operatorname{Div}(A)$. Let $\widetilde{S} \subset M'(\mathbb{A}^\infty)$ denote the subset $\{\operatorname{diag}(m_1^\infty, \ldots, m_{d-1}^\infty) \mid \operatorname{div}(m_i^\infty) \in S \text{ for all } i\}$. Then the composition $M'(F_\infty) \times \widetilde{S} \hookrightarrow M'(\mathbb{A}) \twoheadrightarrow M'(F)\backslash M'(\mathbb{A})$ induces a bijection $(k^\times)^{\oplus d}\backslash(M'(F_\infty) \times$

$\widetilde{S}) \xrightarrow{\cong} M'(F)\backslash M'(\mathbb{A})$. Hence

$$P(f,s) = \frac{1}{q^{d-1}} \sum_{\sigma \in S_{d-1}} \mathrm{sgn}(\sigma) \int_{M'(F_\infty)\times \widetilde{S}} f(w_\sigma m)\widetilde{\omega}_s(m)dm.$$

By (1), we have

$$\begin{aligned}
&\sum_{\sigma \in S_{d-1}} \mathrm{sgn}(\sigma) \int_{M'(F_\infty)} f(w_\sigma(m_\infty, m^\infty))\widetilde{\omega}_s((m_\infty, m^\infty))dm_\infty \\
= &\sum_{b \in B'^-(F)} \left(\sum_{\sigma \in S_{d-1}} \mathrm{sgn}(\sigma) D_{b,\sigma} \right) \left(\sum_{\sigma' \in S_{d-1}} \mathrm{sgn}(\sigma')\mathrm{Wh}^\infty(b w_{\sigma'} m^\infty)\widetilde{\omega}_s^\infty(m^\infty) \right)
\end{aligned}$$

for any $m^\infty \in M'(\mathbb{A}^\infty)$. Hence $P(f,s)$ is equal to

$$\begin{aligned}
&\frac{1}{q^{d-1}} \sum_{n' \in N'^-(F)} \left(\sum_{\sigma \in S_{d-1}} \mathrm{sgn}(\sigma) D_{n',\sigma} \right) \sum_{m' \in M'(F)} |\det(m')|_\infty^{-s} \\
&\quad \times \left(\sum_{\sigma' \in S_{d-1}} \mathrm{sgn}(\sigma') \int_{\widetilde{S}} \mathrm{Wh}^\infty(n'm'w_{\sigma'}m^\infty)\widetilde{\omega}_s^\infty(m^\infty)dm^\infty \right) \\
= &\sum_{n' \in N'^-(F)} \left(\sum_{\sigma \in S_{d-1}} \mathrm{sgn}(\sigma) D_{n',\sigma} \right) \\
&\quad \times \left(\sum_{\sigma' \in S_{d-1}} \mathrm{sgn}(\sigma') \int_{M'(\mathbb{A}^\infty)} \mathrm{Wh}^\infty(n'w_{\sigma'}m^\infty)\widetilde{\omega}_s^\infty(m^\infty)dm^\infty \right).
\end{aligned}$$

This completes the proof. □

7.6. Computation of $D_{n,\sigma}$. We compute the integral $D_{n,\sigma}$ for $n = (n_{ij})_{1\le i,j\le d-1} \in N'^-(F)$ and $\sigma \in S_{d-1}$. Let $\widetilde{\sigma} \in S_d$ denote the element defined by $\widetilde{\sigma}(i) = \sigma(i)$ for $i = 1, \ldots, d-1$ and $\widetilde{\sigma}(d) = d$. Take an element $a \in F_\infty$ with $\mathrm{ord}_\infty(a) = -\mathrm{ord}_\infty\psi$ and put $m_a = \mathrm{diag}(a^{d-1}, a^{d-2}, \ldots, a) \in M'(F_\infty)$. We define a function $\mathrm{Wh}'_\infty : G(F_\infty) \to \mathbb{C}$ by $\mathrm{Wh}'_\infty(g) = \mathrm{Wh}_\infty(m_a g)$.

Applying Proposition C.3 to Wh'_∞, we have, for $m \in M'(F_\infty)$,

$$\begin{aligned}
&\mathrm{Wh}_\infty(w_\sigma m) = \mathrm{Wh}'_\infty(m_a^{-1}m'w_\sigma) \\
&= \begin{cases} \mathrm{sgn}(\sigma)q_\infty^{-\ell(\sigma)}\delta_B(m_a^{-1}m')\mathrm{Wh}'_\infty(1), & \text{if } m_a^{-1}m' \in M'(F_\infty)\cap M(F_\infty)^-_{\widetilde{\sigma}}, \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

where $m' = w_\sigma m w_\sigma^{-1}$ and $\delta_B : B(F_\infty) \to \mathbb{R}^\times$ is the modular character. We put $X_{n,\sigma} = w_\sigma \mathrm{IW}(w_\sigma^{-1} n w_\sigma) w_\sigma^{-1} \cap (M'(F_\infty) \cap m_a M(F_\infty)^-_\sigma)$. Since $\delta_B(m_a) = q_\infty^{\frac{(d-1)d(d+1)}{6}\mathrm{ord}_\infty\psi}$, we have

$$D_{n,\sigma} = \mathrm{sgn}(\sigma)q_\infty^{-\ell(\sigma)-\frac{(d-1)d(d+1)}{6}\mathrm{ord}_\infty\psi}\mathrm{Wh}_\infty(m_a)\int_{X_{n,\sigma}} \delta_B(m)|\det(m)|_\infty^s dm.$$

We note that for $m = \mathrm{diag}(m_1, \ldots, m_{d-1}) \in M'(F_\infty)$, $m \in X_{n,\sigma}$ if and only if the following three conditions are satisfied.

- For every $1 \le i,j \le d-1$ with $i < j$, we have

  $$\mathrm{ord}_\infty(m_i) - \mathrm{ord}_\infty(m_j) + \min(\mathrm{ord}_\infty(n_{ji}), (j-i)(1+\mathrm{ord}_\infty\psi)) \ge 0.$$

- For every $1 \leq i, j \leq d-1$ with $i < j$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$, we have
$$\mathrm{ord}_\infty(m_i) - \mathrm{ord}_\infty(m_j) + \min(\mathrm{ord}_\infty(n_{ji}), (j-i)(1+\mathrm{ord}_\infty\psi)) \geq 1.$$
- For every $1 \leq i \leq d-1$, we have $\mathrm{ord}_\infty(m_i) + (d-i)(1+\mathrm{ord}_\infty\psi) \geq 1$.

7.7. Computation of $I^\infty_{n,\sigma}$. From now on we assume that Conditions 7.3 (1)-(4) are satisfied. For $n \in N'^-(F)$ and $\sigma \in S_{d-1}$, we consider the integral
$$I^\infty_{n,\sigma} = \int_{M'(\mathbb{A}^\infty)} \mathrm{Wh}^\infty(nw_{\sigma'}m^\infty)\widetilde{\omega}^\infty_s(m^\infty)dm^\infty$$
appearing in Proposition 7.8(2).

The group S_{d-1} acts on the $\mathbb{C}((q^{-s}))$-algebra $\mathbb{C}[\mathrm{Pic}(A)^{\oplus d-1}]((q^{-s}))$ from the right via permutations on $\mathrm{Pic}(A)^{\oplus d-1}$. Since $W'(F_v)$ acts trivially on f_v for each $v \neq \infty, o$, we have
$$I^\infty_{n,\sigma} = \prod_{v\neq\infty} I_{n,\sigma,v},$$
where for $v \neq \infty, o$, $I_{n,\sigma,v}$ is the integral
$$I_{n,\sigma,v} = \int_{M'(F_v)} \mathrm{Wh}_v(nm_v)(\widetilde{\omega}^\infty_s(m_v))^\sigma dm_v = (I_{n,1,v})^\sigma$$
and
$$I_{n,\sigma,o} = \int_{M'(F_o)} \mathrm{Wh}_o(nm_ow_{\sigma'})(\widetilde{\omega}^\infty_s(m_o))^\sigma dm_o.$$

7.8. More computation in $d = 3$ case and proof of Proposition 7.9. From now on we assume that Conditions 7.3 (1)-(5) are satisfied.

7.8.1. For $n \in N'^-(F)$, we set $c_\infty(n) = \min(0, \mathrm{ord}_\infty(n_{21}) - \mathrm{ord}_\infty\psi - 1)$. Then the two subsets $X_{n,1}$, $X_{n,(12)}$ of $M'(F_\infty)$ are expressed as follows:
$$X_{n,1} = \left\{ \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \;\middle|\; \mathrm{ord}_\infty(\frac{m_1}{m_2}) + c_\infty(n) \geq -\mathrm{ord}_\infty\psi, \mathrm{ord}_\infty(m_2) \geq -\mathrm{ord}_\infty\psi \right\},$$
$$X_{n,(12)} = \left\{ \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \;\middle|\; \mathrm{ord}_\infty(\frac{m_1}{m_2}) + c_\infty(n) \geq -\mathrm{ord}_\infty\psi, \mathrm{ord}_\infty(m_2) \geq -\mathrm{ord}_\infty\psi \right\}.$$
Thus we have
$$D_{n,1} = \frac{C_\infty q_\infty^{2c_\infty(n)+(c_\infty(n)+3\,\mathrm{ord}_\infty\psi)s}}{(1-q_\infty^{-2-2s})(1-q_\infty^{-2-s})}$$
and
$$D_{n,(12)} = -\frac{C_\infty q_\infty^{2c_\infty(n)+1+(c_\infty(n)+3\,\mathrm{ord}_\infty\psi+1)s}}{(1-q_\infty^{-2-2s})(1-q_\infty^{-2-s})}.$$
Therefore,
$$D_{n,1} - D_{n,(12)} = \frac{C_\infty q_\infty^{2c_\infty(n)+1+(c_\infty(n)+3\,\mathrm{ord}_\infty\psi+1)s}}{(1-q_\infty^{-1-s})(1-q_\infty^{-2-s})}$$

7.8.2.

Lemma 7.13. *For $x, y \in \mathbb{Z}$, we have $\overline{\mathrm{Wh}}_v(x,y) = 0$ unless $x + 2\,\mathrm{ord}_v\psi \geq y + \mathrm{ord}_v\psi \geq 0$.*

Proof. Suppose that $x, y \in \mathbb{Z}$ do not satisfy $x + 2\,\mathrm{ord}_v\psi \geq y + \mathrm{ord}_v\psi \geq 0$. Take an element $m = \mathrm{diag}(m_1, m_2) \in M'(F_v)$ with $(\mathrm{ord}_v(m_1), \mathrm{ord}_v(m_2)) = (x, y)$. Then there exists an element $n \in N(\mathcal{O})$ satisfying $\psi_{N,v}(mnm^{-1}) \neq 1$. Since $\mathrm{Wh}_v(m) = \mathrm{Wh}_v(mn) = \psi_{N,v}(mnm^{-1})\mathrm{Wh}_v(m)$, we have $\mathrm{Wh}_v(m) = 0$. □

We introduce the following notations. For a place $v \neq \infty$ of F, let $X_v, Y_v \in \mathbb{C}[\mathrm{Pic}(A)^{\oplus 2}]((q^{-s}))$ denote the two elements $X_v = q_v^{-s}(\mathrm{cl}([v]), 0)$, $Y_v = q_v^{-s}(0, \mathrm{cl}([v]))$.

Lemma 7.14. *For $n \in N'^{-}(F)$, we have the following formulae describing $I_{n,\sigma,v}$.*

(1) *For $v \neq \infty, o$, we have*

$$\begin{aligned} I_{n,1,v} = & \sum_{\substack{x,y\in\mathbb{Z} \\ x+2\mathrm{ord}_v\psi\geq y+\mathrm{ord}_v\psi\geq 0 \\ x-y+\mathrm{ord}_v(n_{21})\geq 0}} \overline{\mathrm{Wh}}_v(x,y) X_v^x Y_v^y \\ & + \psi_v(\frac{1}{n_{21}}) \sum_{\substack{x,y\in\mathbb{Z} \\ x+2\mathrm{ord}_v\psi\geq y+\mathrm{ord}_v\psi\geq 0 \\ x-y+\mathrm{ord}_v(n_{21})\geq 1}} \overline{\mathrm{Wh}}_v(x,y) X_v^{y-\mathrm{ord}_v(n_{21})} Y_v^{x+\mathrm{ord}_v(n_{21})}. \end{aligned}$$

(2) *Put $c_o(n) = \min(0, \mathrm{ord}_o(n_{21}) - \mathrm{ord}_o\psi - 1)$. Then we have*

$$I_{n,1,o} = C_o \frac{\mu_o^{-c_o(n)} q_o^{2c_o(n)+(c_o(n)+3\mathrm{ord}_o\psi)s}}{(1-\mu_o^2 q_o^{-2-2s})(1-\mu_o q_o^{-2-s})} \left(1 - \psi_o(\frac{1}{n_{21}})\mu_o^{-1} q_o^s\right),$$

and

$$I_{n,(12),o} = C_o \frac{\mu_o^{-c_o(n)} q_o^{2c_o(n)+(c_o(n)+3\mathrm{ord}_o\psi)s}}{(1-\mu_o^2 q_o^{-2-2s})(1-\mu_o q_o^{-2-s})} \left(-\mu_o^{-1} q_o^s + \psi_o(\frac{1}{n_{21}})\right).$$

Here we understand $\psi_v(\frac{1}{n_{21}}) = 0$ when $n = 1$.

Proof. (1) We may assume that $n \neq 1$. For a given $m = \mathrm{diag}(m_1, m_2) \in M(F_v)$, we have

$$nm \in \begin{cases} mG'(\mathcal{O}_v), & \text{if } \mathrm{ord}_v(\frac{m_1}{m_2}) + \mathrm{ord}_v(n_{21}) \geq 0, \\ \begin{pmatrix} 1 & \frac{1}{n_{21}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{m_2}{n_{21}} & 0 \\ 0 & m_1 n_{21} \end{pmatrix} G'(\mathcal{O}_v) & \text{if } \mathrm{ord}_v(\frac{m_1}{m_2}) + \mathrm{ord}_v(n_{21}) \leq 0. \end{cases}$$

Since f_v is $G'(\mathcal{O}_v)$-invariant, we have

$$\mathrm{Wh}_v(nm) = \begin{cases} \mathrm{Wh}_v(m), & \text{if } \mathrm{ord}_v(\frac{m_1}{m_2}) + \mathrm{ord}_v(n_{21}) \geq 0, \\ \psi_v(\frac{1}{n_{21}})\mathrm{Wh}_v(\begin{pmatrix} \frac{m_2}{n_{21}} & 0 \\ 0 & m_1 n_{21} \end{pmatrix}), & \text{if } \mathrm{ord}_v(\frac{m_1}{m_2}) + \mathrm{ord}_v(n_{21}) \geq -1, \end{cases}$$

whence, with Lemma 7.13, the claim follows via simple calculation.

(2) We assume that $n \neq 1$. (In the $n = 1$ case, the claim follow from similar, less complicated arguments). Let $\mathcal{I}'_o \subset G'(\mathcal{O}_o)$ denote the Iwahori subgroup. For a given $m = \mathrm{diag}(m_1, m_2) \in M(F_v)$, we have

$$nm \in \begin{cases} m\mathcal{I}'_o, & \text{if } \mathrm{ord}_o(\frac{m_1}{m_2}) + \mathrm{ord}_o(n_{21}) \geq 1, \\ \begin{pmatrix} 1 & \frac{1}{n_{21}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{m_2}{n_{21}} & 0 \\ 0 & m_1 n_{21} \end{pmatrix} w_{(12)}\mathcal{I}'_o, & \text{if } \mathrm{ord}_v(\frac{m_1}{m_2}) + \mathrm{ord}_v(n_{21}) \leq 0, \end{cases}$$

and

$$nmw_{(12)} \in \begin{cases} mw_{(12)}\mathcal{I}'_o, & \text{if } \mathrm{ord}_o(\frac{m_1}{m_2}) + \mathrm{ord}_o(n_{21}) \geq 0, \\ \begin{pmatrix} 1 & \frac{1}{n_{21}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{m_2}{n_{21}} & 0 \\ 0 & m_1 n_{21} \end{pmatrix} \mathcal{I}'_o, & \text{if } \mathrm{ord}_v(\frac{m_1}{m_2}) + \mathrm{ord}_v(n_{21}) \leq -1. \end{cases}$$

Let us apply Proposition C.3. We put $C'_o = C_o \mu_o^{3\,\mathrm{ord}_o \psi + \mathrm{ord}_o(\det m)}$. Then we have

$$\mathrm{Wh}_o(nm) = \begin{cases} C'_o |m_1 \varpi_o^{2\mathrm{ord}_o \psi}|_o^2, & \begin{array}{l} \text{if } \mathrm{ord}_o(\frac{m_1}{m_2}) + \mathrm{ord}_o(n_{21}) \geq 1 \text{ and} \\ \quad \mathrm{ord}_o(m_1) + 2\mathrm{ord}_o\psi \\ \quad \geq \mathrm{ord}_o(m_2) + \mathrm{ord}_o\psi \geq 0, \end{array} \\ -C'_o \psi_o(\frac{1}{n_{21}}) q_o^{-1} |\frac{m_2}{n_{21}} \varpi_o^{2\mathrm{ord}_o \psi}|_o^2, & \begin{array}{l} \text{if } \mathrm{ord}_o(\frac{m_1}{m_2}) + \mathrm{ord}_o(n_{21}) \leq 0 \text{ and} \\ \quad \mathrm{ord}_o(\frac{m_2}{n_{21}}) + 2\mathrm{ord}_o\psi + 1 \\ \quad \geq \mathrm{ord}_o(m_1 n_{21}) + \mathrm{ord}_o\psi \geq 0, \end{array} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathrm{Wh}_o(nmw_{(12)}) = \begin{cases} -C'_o q_o^{-1} |m_1 \varpi_o^{2\mathrm{ord}_o \psi}|_o^2, & \begin{array}{l} \text{if } \mathrm{ord}_o(\frac{m_1}{m_2}) + \mathrm{ord}_o(n_{21}) \geq 0 \text{ and} \\ \quad \mathrm{ord}_(m_1) + 2\mathrm{ord}_o\psi + 1 \\ \quad \geq \mathrm{ord}_o(m_2) + \mathrm{ord}_o\psi \geq 0, \end{array} \\ C'_o \psi_o(\frac{1}{n_{21}}) |\frac{m_2}{n_{21}} \varpi_o^{2\mathrm{ord}_o \psi}|_o^2, & \begin{array}{l} \text{if } \mathrm{ord}_o(\frac{m_1}{m_2}) + \mathrm{ord}_o(n_{21}) \leq -1 \text{ and} \\ \quad \mathrm{ord}_(\frac{m_2}{n_{21}}) + 2\mathrm{ord}_o\psi \\ \quad \geq \mathrm{ord}_o(m_1 n_{21}) + \mathrm{ord}_o\psi \geq 0, \end{array} \\ 0, & \text{otherwise} \end{cases}$$

Substituting these into the definitions of $I_{n,1,o}$ and $I_{n,(12),o}$, we have the desired formulae. □

Corollary 7.15. *When $n \neq 1$, we have*

$$I_{n,1,v} = \psi_v(\frac{1}{n_{21}}) X_v^{-\mathrm{ord}_v(n_{21})} Y_v^{\mathrm{ord}_v(n_{21})} I_{n^{-1},(12),v}$$

for $v \neq \infty$. In particular, we have $I_{n,1}^\infty = \psi_\infty(\frac{1}{n_{21}})^{-1} I_{n^{-1},(12)}^\infty$. □

Proof of Proposition 7.9. Using the above corollary and noting that $c_\infty(n) = 0$ for $1 \neq n \in N'^-(F)$ with $\psi(\frac{1}{n_{21}}) \neq 1$, we have

$$\begin{aligned} P(f,s) = \quad & (D_{1,1} - D_{1,(1,2)})(I_{1,1}^\infty - I_{1,(12)}^\infty) \\ & + \frac{1}{2} \sum_{n \in N'^-(F), n \neq 1} (D_{n,1} - D_{n,(12)})(1 - \psi(\frac{1}{n_{21}}))(I_{n,1}^\infty - I_{n,(12)}^\infty) \\ = \quad & \frac{C_\infty q_\infty^{1+(3\,\mathrm{ord}_\infty \psi + 1)s}}{(1 - q_\infty^{-1-s})(1 - q_\infty^{-2-s})} \\ & \times \left(I_{1,1}^\infty - I_{1,(12)}^\infty + \frac{1}{2} \sum_{n \in N'^-(F), n \neq 1} (1 - \psi(\frac{1}{n_{21}}))(I_{n,1}^\infty - I_{n,(12)}^\infty) \right). \end{aligned}$$

For $n \in N'^-(F)$, let $I_{n,1}^{\infty,o}$ denote $\prod_{v \neq \infty, s} I_{n,1,v}$. Since

$$\mathrm{ev}_0(I_{n,(12)}^\infty) = \mathrm{ev}_0((I_{n,(12)}^\infty)^{(12)}) = \mathrm{ev}_0(I_{n,1}^{\infty,o}(I_{n,(12),o})^{(12)}),$$

we have

$$\begin{aligned}
& P_\infty(f,s) \\
= \; & \frac{q^{d-1} C_\infty q_\infty^{1+(3\,\mathrm{ord}_\infty \psi+1)s}}{(1-q_\infty^{-1-s})(1-q_\infty^{-2-s})} \\
& \times \mathrm{ev}_0 \left(I_{1,1}^\infty - (I_{1,(12)}^\infty)^{(12)} + \frac{1}{2} \sum_{n \in N'^-(F), n \neq 1} (1-\psi(\frac{1}{n_{21}}))(I_{n,1}^\infty - (I_{n,(12)}^\infty)^{(12)}) \right) \\
= \; & \frac{q^{d-1} C_\infty q_\infty^{1+(3\,\mathrm{ord}_\infty \psi+1)s}}{(1-q_\infty^{-1-s})(1-q_\infty^{-2-s})} \\
& \times \mathrm{ev}_0 \left(\begin{array}{l} I_{1,1}^{\infty,o}(I_{1,(12),o}^{\infty,o} - (I_{1,(12),o})^{(12)}) \\ +\frac{1}{2} \displaystyle\sum_{n \in N'^-(F), n \neq 1} (1-\psi(\frac{1}{n_{21}})) I_{n,1}^{\infty,o}(I_{n,1,o} - (I_{n,(12),o})^{(12)}) \end{array} \right).
\end{aligned}$$

For $n \in N'^-(F)$ with $n \neq 1$, we have

$$\begin{aligned}
I_{n,1,o} - (I_{n,(12),o})^{(12)} &= (1-\psi_o(\frac{1}{n_{21}})) C_o \frac{\mu_o^{-c_o(n)-1} q_o^{2c_o(n)+1+(c_0(n)+3\,\mathrm{ord}_o\psi+1)s}}{(1-\mu_o q_o^{-1-s})(1-\mu_o q_o^{-2-s})} \\
&= (1-\psi_o(\frac{1}{n_{21}})) C_o \frac{\mu_o^{-1} q_o^{1+(3\,\mathrm{ord}_o\psi+1)s}}{(1-\mu_o q_o^{-1-s})(1-\mu_o q_o^{-2-s})}.
\end{aligned}$$

Hence the assertion follows from Lemma 7.14. □

7.9. More computation of $a_\psi(D_1,D_2)$ and proof of Theorem 7.6. Let the notations and the assumption be as in Section 7.8. In this section, we compute $a_\psi(D_1,D_2)$ for $(D_1,D_2) \in \Delta_\psi$.

For $m \in F^\times$, we have $(1-\psi_\infty(\frac{1}{m}))(1-\psi_o(\frac{1}{m})) = 0$ unless $\mathrm{div}(m)|_{\{\infty,o\}} \geq (\mathrm{ord}\psi)|_{\{\infty,o\}} + [\infty] + [o]$. We set $D_3 = D_1 - D_2 - \mathrm{ord}\psi - [\infty] - [o]$ and define Σ'_{D_1,D_2} to be the set of triples (S,D,m) satisfying the following conditions.

- $D \in \mathrm{Div}(C)$ with $D \geq 0$, $\mathrm{cl}(D+D_1) = 2\mathrm{cl}(\mathrm{ord}\psi) + \mathrm{cl}(D_3|_{\mathrm{Supp}\,(D)})$.
- $S \subset C$ is a finite closed subset with $\mathrm{Supp}\,(D) \subset S \subset \mathrm{Supp}\,(D) \cup \{\infty, o\}$.
- $m \in F^\times$ with $D_3 + \mathrm{div}(m) \geq 0$.
- $D = (D_3 + \mathrm{div}(m))|_S$.

For $\delta = (S,D,m) \in \Sigma'_{D_1,D_2}$, we set $c_\psi(\delta) = (-1)^{\sharp S \cap \{\infty,o\}} \prod_{v \in S} \psi_v(\frac{1}{m})$. Then we have

$$a_\psi^*(D_1,D_2) = \sum_{\delta \in \Sigma'_{D_1,D_2}} c_\psi(\delta).$$

For $\delta = (S,D,m) \in \Sigma'_{D_1,D_2}$, we set $\deg \delta = \deg D$ and $\iota(\delta) = ((\mathrm{Supp}\,(D_3 + \mathrm{div}(m)) \cup \{\infty,o\}) \backslash S, D_3 + \mathrm{div}(m) - D, -m)$. Then $\iota(\delta)$ is also in Σ'_{D_1,D_2} and we have $\iota(\iota(\delta)) = \delta$, $\deg(\delta) + \deg(\iota(\delta)) = \deg(D_3)$, and $c_\psi(\delta) = c_\psi(\iota(\delta))$. Let Θ_{D_1,D_2} denote the set of pairs (S,D) such that the set $V_0(S,D) = \{m \in F^\times \mid (S,D,m) \in \Sigma'_{D_1,D_2}\}$ is non-empty. The map $\delta \mapsto c_\psi(\delta)$ induces a map $c_{\psi,S,D} : V_0(S,D) \to \mathbb{C}$. Let $\Theta'_{D_1,D_2} \subset \Theta_{D_1,D_2}$ denote the subset of elements $(S,D) \in \Theta_{D_1,D_2}$ satisfying either $2 \deg D < \deg D_3$ or both $\infty \notin S$ and $2 \deg D = \deg D_3$. Then we have

$$a_\psi^*(D_1,D_2) = 2 \sum_{(S,D) \in \Theta'_{D_1,D_2}} \sum_{m \in V_0(S,D)} c_{\psi,S,D}(m).$$

The set $V_0(S, D)$ is a subset of the $\mathbb{F}_q$-vector space $H^0(\mathcal{L}(D_3 - D))$. For $m \in V_0(S, D)$, the value $c_{\psi,S,D}(m)$ depends only on the image of m in the quotient

$$\frac{H^0(\mathcal{L}(D_3 - D))}{H^0(\mathcal{L}(D_3 - D)) \cap H^0\mathcal{L}(D_3 - D + (D_3 - D + \mathrm{ord}\psi)|_S))}.$$

In particular, if we put $\widetilde{D} = D + ([\infty] + [o])|_S$, the map $c_{\psi,S,D}$ factors through the canonical map

$$\Phi_{S,D} : V_0(S, D) \hookrightarrow H^0(\mathcal{L}(D_3 - D)) \twoheadrightarrow \frac{H^0(\mathcal{L}(D_3 - D))}{H^0(\mathcal{L}(D_3 - D - \widetilde{D}))}.$$

Let $\overline{c}_{\psi,S,D} : \mathrm{Image}(\Phi_{S,D}) \to \mathbb{C}$ denote the induced map. Then the image of $\Phi_{S,D}$ is a subset of $(\mathcal{L}(D_3 - D)/\mathcal{L}(D_3 - D - \widetilde{D}))^0$. For any $m \in \mathrm{Image}(\Phi_{S,D}) \subset (\mathcal{L}(D_3 - D)/\mathcal{L}(D_3 - D - \widetilde{D}))^0$, we have $\overline{c}_{\psi,S,D}(m) = (-1)^{\sharp S \cap \{\infty,o\}} \prod_{v\in S} \psi_v(m^{-1})$.

Proof of Theorem 7.6. We assume that Conditions 7.3 (1)-(6) are satisfied. We may assume that ψ satisfies $\mathrm{ord}\,\psi = -[\infty] - [o]$. Since $\mathrm{Pic}(A) = 0$, we have $\Delta_\psi = \{(D_1, D_2) \in \mathrm{Div}(\mathbb{G}_m)^{\oplus 2} \mid D_1 \geq D_2 \geq 0\}$. Fix an element $(D_1, D_2) \in \Delta_\psi$. Then we have $D_3 = D_1 - D_2$, and Θ_{D_1,D_2} is the set of pairs (S, D) satisfying $D \geq 0$, $\deg D \leq \deg(D_1 - D_2)$, and $\mathrm{Supp}\,(D) \subset S \subset \mathrm{Supp}\,(D) \cup \{\infty, 0\}$.

Let $(S, D) \in \Theta'_{D_1,D_2}$ be an element. Then we have $H^1(\mathcal{L}(D_3 - D - \widetilde{D})) = 0$ since $\deg(D_3 - D - \widetilde{D}) \geq -1$. Hence $\mathrm{Image}(\Phi_{S,D})$ equals $\mathcal{O}^\times_{\widetilde{D}}$. For any $m \in \mathrm{Image}(\Phi_{S,D})$, the cardinality of $\Phi^{-1}_{S,D}(m)$ is equal to

$$\sharp(H^0(D_3 - D - \widetilde{D})) = q^{\deg(D_3 - 2D) - \sharp(S \cap \{\infty,o\}) + 1}.$$

if $(S, D) \neq (\emptyset, 0)$. Then we have

$$\sum_{m\in(\mathcal{L}(D_3-D)/\mathcal{L}(D_3-D-\widetilde{D}))^0} \prod_{v\in S} \psi_v(m^{-1}) = \prod_{v\in S} c_{\widetilde{D},v}$$

where, for $v \in S$, $c_{\widetilde{D},v}$ is defined to be

$$c_{\widetilde{D},v} = \begin{cases} q_v^{\mathrm{mult}_v \widetilde{D}} - q_v^{\mathrm{mult}_v \widetilde{D} - 1}, & \text{if } \mathrm{mult}_v(D_3 - \widetilde{D}) \geq 0 \\ -q_v^{\mathrm{mult}_v \widetilde{D} - 1}, & \text{if } \mathrm{mult}_v(D_3 - \widetilde{D}) = -1 \\ 0, & \text{otherwise.} \end{cases}$$

Hence we have

$$\sum_{m\in V_0(S,D)} c_{\psi,S,D}(m) = \begin{cases} q^{\deg\, D_3 + 1} - 1, & \text{if } (S, D) = (\emptyset, 0), \\ (-q)^{-\sharp S \cap \{\infty,o\}} q^{\deg(D_3 - 2D) + 1} \prod_{v\in S} c_{\widetilde{D},v}. & \text{otherwise.} \end{cases}$$

For each place v of F, we define a polynomial $P_v(T) \in \mathbb{C}[T]$ in the following way.

- When $v \neq \infty, o$, then

$$\begin{aligned} P_v(T) &= 1 + \sum_{1\leq i\leq \mathrm{mult}_v(D_3)} (q_v^{-i} - q_v^{-i-1}) T^{\deg[v]\cdot i} - q_v^{-1}(q_v^{-1} T^{\deg[v]})^{\mathrm{mult}_v(D_3)+1} \\ &= \frac{(1 - q_v^{-2} T^{\deg[v]})(1 - (q_v^{-1} T^{\deg[v]})^{\mathrm{mult}_v(D_3)} + 1)}{1 - q_v^{-1} T^{\deg[v]}}. \end{aligned}$$

- When $v \in \{\infty, o\}$, then

$$P_v(T) = q^{-1}.$$

Then $a_\psi(D_1, D_2) = 1 + \sum_{(S,D)\in\Theta'_{D_1,D_2}} \sum_{m\in V_0(S,D)} c_{\psi,S,D}(m)$ is equal to the sum of coefficients in degree $\leq \frac{1}{2} \deg D_3$ of the formal power series

$$F_{D_3}(T) = q^{\deg\, D_3+1}\left(1 + P_\infty(T) + T^{\frac{1}{2}}P_o(T) + T^{\frac{1}{2}}P_\infty(T)P_o(T)\right) \prod_{v\neq\infty,o} P_v(T)$$

in $\mathbb{C}[[T^{\frac{1}{2}}]]$. Since $\prod_{v\neq\infty,o}(1 - q_v^{-2}T^{\deg[v]}) = \frac{1-q^{-1}T}{1-q^{-2}T}$, we have

$$F_{D_3}(T) = \frac{q^{\deg\, D_3}(1+q)(1-q^{-1}T)}{1-q^{-1}T^{\frac{1}{2}}} \prod_{v\in\mathrm{Supp}\,(D_3)} \frac{1-(q_v^{-1}T^{\deg[v]})^{\mathrm{mult}_v(D_3)+1}}{1-q_v^{-1}T^{\deg[v]}}.$$

Since the sum of coefficients in degree $\leq \frac{i}{2}$ of $\frac{1-q^{-1}T}{1-q^{-1}T^{\frac{1}{2}}}$ is equal to $1 + q^{-i}$ (resp. 1) if $i \geq 1$ (resp. $i = 0$), we have

$$\begin{aligned} a_\psi(D_1, D_2) &= q^{\deg D_3}(1+q) \sum_{0\leq D\leq D_3} q^{-degD} \\ &= (1+q) \prod_{v\in\mathrm{Supp}\,(D_3)} \frac{q_v^{\mathrm{mult}_v(D_3)+1} - 1}{q_v - 1}. \end{aligned}$$

For $v \neq \infty, o$, let Wh_v^0 denote the class one Whittaker function with $\mathrm{Wh}_v^0(1) = 1$ with respect to ψ_v^{-1} of the (reducible) principal series $\mathrm{Ind}_{B'(F_v)}^{G'(F_v)}\delta_{B'}^{\frac{1}{2}}$ of $G'(F_v)$ (here $\delta_{B'} : B'(F_v) \to \mathbb{R}^\times$ denotes the modular character). By [Shi] we have

$$a_\psi(\mathrm{div}(m_1), \mathrm{div}(m_2)) = (1+q) \prod_{v\neq\infty,o} \mathrm{Wh}_v^0(\mathrm{diag}(m_{1,v}, m_{2,v}))q_v^{\mathrm{ord}_v(m_{1,v})-\mathrm{ord}_v(m_{2,v})}$$

for $m_1 = (m_{1,v}), m_2 = (m_{2,v}) \in \mathbb{A}^{\infty,o}$. Hence $P_\infty(f, s)$ equals

$$(1+q)q^{2-6s}\widetilde{C}_\infty(s)\widetilde{C}_o(s) \prod_{v\neq\infty,o} \int_{M'(F_v)} \mathrm{Wh}_v^0(m_v)\mathrm{Wh}_v(m_v) \left|\frac{m_{1,v}}{m_{2,v}}\right|^{-1} |\det m_v|^s dm_v.$$

By [Ja-Pi-Sh, §4, Théorème], we have

$$\begin{aligned} &\int_{M'(F_v)} \mathrm{Wh}_v^0(m_v)\mathrm{Wh}_v(m_v) \left|\frac{m_{1,v}}{m_{2,v}}\right|^{-1} |\det m_v|^{s-\frac{1}{2}} dm_v \\ =& \int_{N'(F_v)\backslash G'(F_v)} \mathrm{Wh}_v^0(g_v)\mathrm{Wh}_v(g_v)|\det g_v|^{s-\frac{1}{2}} dg_v \\ =& \ \mathrm{Wh}_v(1)L(\pi_v, s-\tfrac{1}{2})L(\pi_v, s+\tfrac{1}{2}) \end{aligned}$$

for all $v \neq \infty, o$. Hence

$$P_\infty(f, s) = (1+q)q^{4-4s}C_\infty C_o \mu_o^{-1} \prod_{v\neq\infty,o} \mathrm{Wh}_v(1)L(\pi, s)L(\pi, s+1),$$

which completes the proof. □

Appendix A. The Proofs of the material in Section 2

by Seidai Yasuda

A.1. Let $d \geq 1$ be a positive integer. Let X be a regular noetherian scheme of Krull dimension one such that the residue field at each closed point is finite.

A.1.1. We define the category $\mathcal{C}^d = \mathcal{C}^d_X$ as follows. An object in $\mathcal{C}^d$ is a coherent $\mathcal{O}_X$-module of finite length which admits a surjection from $\mathcal{O}_X^{\oplus d}$. For two objects N and N' in $\mathcal{C}^d$, the set $\mathrm{Hom}_{\mathcal{C}^d}(N, N')$ of morphisms from N to N' is the set of isomorphism classes of diagrams

$$N' \twoheadleftarrow N'' \hookrightarrow N$$

in the category of coherent $\mathcal{O}_X$-modules where the left arrow is surjective and the right arrow is injective. This definition of morphisms is due to Quillen ([Qu]) except that here we take morphisms in the opposite direction.

We often consider the following two types of morphisms in $\mathcal{C}^d$. Let N be an object in $\mathcal{C}^d$. For a sub $\mathcal{O}_X$-module N' of N, the morphism $N' = N' \hookrightarrow N$ in $\mathcal{C}^d$ is denoted by $r_{N,N'} : N \to N'$. For a quotient $\mathcal{O}_X$-module N'' of N, the morphism $N'' \twoheadleftarrow N = N$ in $\mathcal{C}^d$ is denoted by $m_{N,N''} : N \to N'$.

A.1.2. Let $\mathcal{FC}^d$ denote the category of finite families of objects in $\mathcal{C}^d$. An object in $\mathcal{FC}^d$ is a pair $(J, (N_j)_{j\in J})$ where J is a finite set and $(N_j)_{j\in J}$ is a family of objects in $\mathcal{C}^d$ indexed by J. We denote the object $(J, (N_j)_{j\in J})$ by $\coprod_{j\in J} N_j$. We regard $\mathcal{C}^d$ as a full subcategory of $\mathcal{FC}^d$. We define $\pi_0(\coprod_{j\in J} N_j)$ to be the set J.

A.1.3. A morphism $f : M \to M'$ in the category $\mathcal{FC}^d$ is said to be a *covering* if the underlying morphism $\pi_0(M) \to \pi_0(M')$ is surjective.

Definition A.1. A *presheaf* on $\mathcal{FC}^d$ is a contravariant functor from $\mathcal{FC}^d$ to the category of sets. A presheaf F on $\mathcal{FC}^d$ is a *sheaf* if it satisfies the following conditions (1), (2) and (3):

(1) The image of the empty set $F(\emptyset)$ is the set of one element.
(2) For two objects N and N' in $\mathcal{FC}^d$, the canonical map $F(N \amalg N') \to F(N) \times F(N')$ is an isomorphism.
(3) Let $N \to N'$ be a covering in $\mathcal{FC}^d$. If the fiber product $N \times_{N'} N$ exists in $\mathcal{FC}^d$, then $F(N')$ is canonically isomorphic to the difference kernel of $F(N) \rightrightarrows F(N \times_{N'} N)$ where the maps are induced by the first and the second projections.

We note that a representable presheaf is not necessarily a sheaf.

A.1.4. *Variant.* A morphism in $\mathcal{C}^d$ is called a *fibration* if it is isomorphic to a morphism of the form $m_{N,N'}$. A morphism $f : M \to M'$ in the category $\mathcal{FC}^d$ is said to be a fibration if it is a fibration in $\mathcal{C}^d$ on each component of M.

A presheaf F on $\mathcal{FC}^d$ is a *semi-sheaf* if it satisfies the conditions (1), (2) in Definition A.1 and the following condition (3)$'$:

(3)$'$ If $N \to N'$ is a covering in $\mathcal{FC}^d$ which is a fibration, and if the fiber product $N \times_{N'} N$ exists, then $F(N')$ is canonically isomorphic to the difference kernel of $F(N) \rightrightarrows F(N \times_{N'} N)$ where the maps are induced by the first and the second projections.

We remark that the requirement of the existence of the fiber product $N \times_{N'} N$ in (3)$'$ is superfluous, since in the category $\mathcal{FC}^d$, the fiber product of two fibrations always exists.

A.1.5. Let $f : N' \to N$ be a covering in $\mathcal{FC}^d$. We let $\mathrm{Aut}_N(N')$ denote the group of automorphisms σ in $\mathcal{FC}^d$ of N' such that $f \circ \sigma = f$.

Let $f : N' \to N$ be a morphism in $\mathcal{FC}^d$, and let G be a subgroup of $\mathrm{Aut}_N(N')$. We say that f is a *Galois covering* of Galois group G if the fiber product $N' \times_N N'$ exists, and the morphism $\coprod_{g \in G}(g, \mathrm{id}) : \coprod_{g\in G} N' \to N' \times_N N'$ is an isomorphism.

If $f : N' \to N$ be a Galois covering with N' and N in $\mathcal{C}^d$, then the standard argument in the theory of Galois categories shows that its Galois group equals $\mathrm{Aut}_N(N')$.

Lemma A.2. *Let $f : N' \to N$ be a morphism in $\mathcal{C}^d$ given by the diagram $N \overset{p}{\twoheadleftarrow} N'' \overset{i}{\hookrightarrow} N'$. Suppose there exists a sub $\mathcal{O}_X$-module N_1 of N such that $p^{-1}(N_1) \cong M_1^{\oplus d}$ and $N'/i(p^{-1}(N_1)) \cong M_2^{\oplus d}$ for some M_1, M_2 in $\mathcal{C}^1$. Then f is a Galois covering.*

Proof. Let M be an object in $\mathcal{C}^d$. It suffices to show the map $\alpha_M : \mathrm{Hom}_{\mathcal{FC}^d}(M, N') \to \mathrm{Hom}_{\mathcal{FC}^d}(M, N)$ induced by f is an $\mathrm{Aut}_N(N')$-torsor over the set $\mathrm{Hom}_{\mathcal{FC}^d}(M, N)$.

Since M_1 and M_2 are generated by one element, there exist sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ of ideals such that $M_1 \cong \mathcal{O}_X/\mathcal{I}_1$ and $M_2 \cong \mathcal{O}_X/\mathcal{I}_2$.

Take an element $x \in \mathrm{Hom}_{\mathcal{FC}^d}(M, N)$ and let us consider the set $\alpha_M^{-1}(x)$. Suppose $y \in \alpha_M^{-1}(x)$ is given by the diagram $N' \overset{s'}{\twoheadleftarrow} F \overset{s}{\hookrightarrow} M$. We let $F' = s'^{-1}(i(p^{-1}(N_1)))$ and $F'' = \mathrm{Ker}\, s'$.

Since $F'/F'' \cong (\mathcal{O}_X/\mathcal{I}_1)^{\oplus d}$, $F/F' \cong (\mathcal{O}_X/\mathcal{I}_2)^{\oplus d}$, and M is generated by d elements, it follows that $F'/F'' = F'/\mathcal{I}_1 F'$ and F/F' is the set of elements z in M/F' such that $I_2 z = 0$. Hence $F'' = \mathcal{I}_1 F'$ and F is the set of elements z in M such that $\mathcal{I}_2 z \subset F'$. In particular, $s(F)$ and $s(F'')$ as sub $\mathcal{O}_X$-modules of M are uniquely determined independent of the choice of y. Note that y is the composition of the canonical morphism $s(F)/s(F'') \twoheadleftarrow s(F) \hookrightarrow M$ and an isomorphism $s(F)/s(F'') \cong N'$. Thus the set $\alpha_M^{-1}(x)$ is canonically isomorphic to the subset of the set of isomorphisms $s(F)/s(F'') \cong N'$ such that the composition $M \to s(F)/s(F'') \cong N' \to N$ equals the morphism x. Hence the set $\alpha_M^{-1}(x)$ is an $\mathrm{Aut}_N(N')$-torsor. $\square$

Let $\mathbb{A}_X$ denote the ring of adeles on X, and $\widehat{\mathcal{O}}_X \subset \mathbb{A}_X$ its ring of integers.

Lemma A.3 (cofinality)**.** *Let k be a positive integer. Let N_i $(1 \le i \le k)$ and N be objects in $\mathcal{C}^d$, and $g_i : N_i \to N$ be morphisms. Then there exist an object M in $\mathcal{C}^d$ and morphisms $f_i : M \to N_i$ such that $g_i \circ f_i = g_j \circ f_j$ for any $1 \le i, j \le k$ and $g_1 \circ f_1$ is a Galois covering. Moreover if g_i is a fibration for all i, then one can take M and f_i as above such that f_i is a fibration for all i.*

Proof. We take $\widehat{\mathcal{O}}_X$-lattices L_{N_i}, L'_{N_i} $(1 \le i \le k)$, L_N, and L'_N in $\mathbb{A}_X^{\oplus d}$ such that

- $L_{N_i} \supset L_N \supset L'_N \supset L'_{N_i}$ for $1 \le i \le k$,
- $L_{N_i}/L'_{N_i} \cong N_i$ for all i and $L_N/L'_N \cong N$ as $\widehat{\mathcal{O}}_X$-modules,
- $L_N/L'_N \twoheadleftarrow L_N/L'_{N_i} \hookrightarrow L_{N_i}/L'_{N_i}$ is identified with g_i for all i.

There exists an integral ideal $\widehat{I} \subset \widehat{\mathcal{O}}$ such that $\widehat{I}^{-1} L_N \supset L_{N_i}$ and $L'_{N_i} \subset \widehat{I} L'_N$ for all i. Set $M = \widehat{I}^{-1} L_N / \widehat{I} L_N$ and define the morphisms $M \to N_i$ by $N_i \cong L_{N_i}/L'_{N_i} \twoheadleftarrow L_{N_i}/\widehat{I} L_N \hookrightarrow \widehat{I}^{-1} L_N/\widehat{I} L_N$ for all i. Then by Lemma A.2, the morphism $M \to N$ and the morphisms $M \to N_i$ for all i are Galois. $\square$

Lemma A.4. *A presheaf F on $\mathcal{FC}^d$ is a sheaf if and only if it satisfies conditions (1) and (2) in Definition A.1 and (3)$''$ below.*

(3)$''$ *For any Galois covering $N \to N'$ in $\mathcal{C}^d$, $F(N')$ is canonically isomorphic to the $\mathrm{Aut}_{N'}(N)$-fixed part $F(N)^{\mathrm{Aut}_{N'}(N)}$ of $F(N)$.*

Proof. The implication (3) to (3)$''$ is trivial. We prove (3)$''$ implies (3). Let $f : M \to N$ be a morphism in $\mathcal{FC}^d$ such that $M \times_N M$ exists. We write $M = \coprod_{i\in\pi_0(M)} M_i$, $N = \coprod_{i\in\pi_0(N)} N_i$ with M_i and N_i in $\mathcal{C}^d$. Denote by $\pi_0(f) : \pi_0(M) \to \pi_0(N)$ the morphism induced by f.

For any i, j such that $j = \pi_0(f)(i)$, let $f_{(i,j)} : M_i \to N_j$ be the morphisms induced by f. For each $j \in \pi(N)$, there exist, by Lemma A.3, a Galois covering $f'_j : M'_j \to N_j$ in $\mathcal{C}^d$ and morphisms $f'_{(j,i)} : M'_j \to M_i$ for any $i \in \pi_0(M)$ with $j = \pi_0(f)(i)$, such that $f_j = f_{(i,j)} \circ f'_{(j,i)}$. Set $M' = \coprod_{i\in\pi_0(M)} M'_{\pi_0(f)(i)}$ and $g = \coprod f'_{(j,i)}$. By condition (3)$''$, we have $F(M_i) \cong F(M'_{\pi_0(f)(i)})^{\mathrm{Gal}(M'_{\pi_0(f)(i)}/M_i)}$.

Then $\mathrm{Ker}[F(M) \rightrightarrows F(M \times_N M)]$ injects into

$$\begin{aligned}
&\mathrm{Ker}[F(M') \rightrightarrows F(M' \times_N M')] \\
= \ &\mathrm{Ker}\left[\prod_{i\in\pi_0(M)} F(M'_{\pi_0(f)(i)}) \rightrightarrows \prod_{\substack{i_1,i_2\in\pi_0(M),j\in\pi_0(N)\\ j=\pi_0(f)(i_1)=\pi_0(f)(i_2)}} F(M'_j \times_{N_j} M'_j)\right] \\
= \ &\prod_{j\in\pi_0(N)} F(M'_j)^{\mathrm{Gal}(M'_j/N_j)}
\end{aligned}$$

which, by condition (3)$''$, equals $F(N)$. One can see that this map gives the inverse of the map $F(N) \to \mathrm{Ker}[F(M) \rightrightarrows F(M \times_N M)]$. $\square$

A.1.6. The inclusion of the category of sheaves (resp. semi-sheaves) on $\mathcal{FC}^d$ into the category of presheaves on $\mathcal{FC}^d$ has a right adjoint $(-)^a$ (resp. $(-)^{a\dagger}$). Let us describe the construction. Given a presheaf $F : \mathcal{FC}^d \to (\mathrm{Sets})$, we define the functor $F^a : \mathcal{FC}^d \to (\mathrm{Sets})$ (resp. $F^{a\dagger}$). Let N be an object in $\mathcal{C}^d$. Then the section $F^a(N)$ (resp. $F^{a\dagger}(N)$) is given by

$$\varinjlim_{M\to N} \mathrm{Ker}[F(M) \rightrightarrows F(M \times_N M)] = \varinjlim_{M\to N} F(M)^{\mathrm{Gal}(M/N)}$$

where the limit is taken over (a small skeleton of the category of) all Galois coverings $M \to N$ (resp. all Galois coverings $M \to N$ which is a fibration) in $\mathcal{C}^d$. To check that F^a satisfies (1)(2) and (3)$''$, one uses Lemma A.3. One can also check that $F^{a\dagger}$ is a semi-sheaf. The details are omitted. Note that since $F^a(N)$ and $F^{a\dagger}(N)$ are expressed as filtered inductive limit, the functors $(-)^a$ and $(-)^{a\dagger}$ commute with finite (projective) limits ([Ma] Ch. IX).

A.1.7. Let N be an object in $\mathcal{FC}^d$ and H be a subgroup of $\mathrm{Aut}_{\mathcal{FC}^d}(N)$. We denote by N/H the sheaf associated to the presheaf $\mathrm{Hom}_{\mathcal{FC}^d}(\ ,N)/H$.

Let $\widetilde{\mathcal{FC}}^d$ denote the full subcategory of the category of sheaves on $\mathcal{FC}^d$ whose objects are sheaves of the form N/H with N in $\mathcal{FC}^d$ and H a subgroup of $\mathrm{Aut}(N)$. There is a canonical functor $\mathcal{FC}^d \to \widetilde{\mathcal{FC}}^d$ which sends an object N in $\mathcal{FC}^d$ to the sheaf $N/\{\mathrm{id}_N\}$. It commutes with finite (projective) limits.

We note that the canonical functor $\mathcal{FC}^d \to \widetilde{\mathcal{FC}}^d$ which sends N to the sheaf $N/\{\mathrm{id}_N\}$ is not fully faithful. For example, the endomorphisms $\mathrm{End}_{\mathcal{FC}^d}(0)$ consists of one element $\{\mathrm{id}_0\}$, while the endomorphisms $\mathrm{End}_{\widetilde{\mathcal{FC}}^d}(0/\{\mathrm{id}_0\})$ is isomorphic to the group of divisors on X. By the adjointness of $(-)^a$ and by the definition of $(-)^a$, we have $\mathrm{Hom}_{\widetilde{\mathcal{FC}}^d}(0/\{\mathrm{id}_0\}) = \varinjlim_{M\to 0} \mathrm{Hom}_{\mathcal{FC}^d}(M,0)$ where the limit is over all Galois coverings. Using the argument as in the proof of the cofinality lemma, and taking the automorphisms of M into consideration, we see that it equals $\varinjlim_{I_1\subset I_2} \mathrm{Hom}_{\mathcal{FC}^d}((I_2/I_1)^{\oplus d},0)^{\mathrm{Aut}((I_2/I_1)^{\oplus d})}$ where the limit is over the $\widehat{\mathcal{O}}_X$-lattices I_1, I_2 of $\mathbb{A}_X$ with $I_1 \subset I_2$. Now $\mathrm{Hom}_{\mathcal{FC}^d}((I_2/I_1)^{\oplus d},0)^{\mathrm{Aut}((I_2/I_1)^{\oplus d})}$ is the set of sub $\widehat{\mathcal{O}}_X$-modules of $(I_2/I_1)^{\oplus d}$ which is stable under the action of $\mathrm{GL}_d(I_2/I_1)$. By Morita equivalence (cf. [An-Fu]), it is isomorphic to the set of sub $\widehat{\mathcal{O}}_X$-modules of I_2/I_1. Hence the limit is isomorphic to the set of $\widehat{\mathcal{O}}_X$-lattices in $\mathbb{A}_X$.

Let I be a finite set, N_i be an object in $\mathcal{C}^d$, and H_i be a subgroup of $\mathrm{Aut}(N_i)$ for each $i \in I$. We write $\coprod_{i\in I} N_i/H_i$ for the object $(\coprod_{i\in I} N_i)/(\prod_i H_i)$ in $\widetilde{\mathcal{FC}}^d$; it is isomorphic to the sheaf associated to the presheaf $\coprod_{i\in I}\mathrm{Hom}(-,N_i)/H_i$. Any object in $\widetilde{\mathcal{FC}}^d$ is isomorphic to an object of the form above.

The notions of π_0 and covering are canonically extended to the category $\widetilde{\mathcal{FC}}^d$ by putting $\pi_0(N/H) = \pi_0(N)/H$. We say that an object in $\widetilde{\mathcal{FC}}^d$ is connected if its π_0 consists of one element. We define sheaves on $\widetilde{\mathcal{FC}}^d$ and Galois coverings in $\widetilde{\mathcal{FC}}^d$ in a similar manner.

Lemma A.5. *Let $N = \coprod_{i\in\pi_0(N)} N_i$ be an object in $\mathcal{FC}^d$ and H be a subgroup of $\mathrm{Aut}_{\mathcal{FC}^d}(N)$. Suppose that, for each $i \in \pi_0(N)$, the stabilizer $H_i \subset H$ of i acts faithfully on N_i. Then the quotient morphism $N/\{\mathrm{id}_N\} \to N/H$ in $\widetilde{\mathcal{FC}}^d$ is a Galois covering whose Galois group is canonically isomorphic to H.*

Proof. For any $M \in \mathcal{C}^d$ such that $\mathrm{Hom}_{\mathcal{FC}^d}(M,N)$ is non-empty, H acts freely on $\mathrm{Hom}_{\mathcal{FC}^d}(M,N)$. Hence we have an isomorphism of presheaves

$$\coprod_{h\in H}\mathrm{Hom}(-,N) \to \mathrm{Hom}(-,N)\times_{\mathrm{Hom}(-,N)/H}\mathrm{Hom}(-,N).$$

Since the functor $(-)^a$ preserves fiber products, we have the assertion. □

We remark that any object in $\widetilde{\mathcal{FC}}^d$ is isomorphic to N/H for some N and H satisfying the assumption in Lemma A.5.

We have an analogue of Lemma A.4 for sheaves on $\widetilde{\mathcal{FC}}^d$.

Lemma A.6. *A presheaf F on $\widetilde{\mathcal{FC}}^d$ is a sheaf if and only if it satisfies the conditions analogous to (1) and (2) in Definition A.1 and (3)$'''$ below.*

(3)$'''$ *For any Galois covering $N/\{\mathrm{id}_N\} \to N/H$ in $\widetilde{\mathcal{FC}}^d$ as in Lemma A.5, $F(N/H)$ is canonically isomorphic to the H-fixed part $F(N/\{\mathrm{id}_N\})^H$ of $F(N/\{\mathrm{id}_N\})$.*

Proof. We prove this lemma using Lemma A.7.

We proceed as in the proof of Lemma A.4. If G is a sheaf, then G obviously satisfies (3)$'''$. We show the other direction. Let $f : F' \to F$ be a covering in $\widetilde{\mathcal{FC}}^d$

such that the fiber product $F' \times_F F'$ exists. Take N and H as in Lemma A.7. If G is a presheaf satisfying $(3)'''$, then

$$\begin{aligned}&\mathrm{Ker}(G(F') \rightrightarrows G(F' \times_F F'))\\ &\to \mathrm{Ker}(G(N/\{\mathrm{id}_N\}) \rightrightarrows G(N/\{\mathrm{id}_N\} \times_{N/H} N/\{\mathrm{id}_N\})) \cong G(N/H)\end{aligned}$$

gives the inverse of $G(F') \to \mathrm{Ker}(G(F') \rightrightarrows G(F' \times_F F'))$, proving that it is an isomorphism. □

Lemma A.7. *Let $f : F' \to F$ be a covering in $\widetilde{\mathcal{FC}}^d$. Then there exist an object N in $\mathcal{FC}^d$, a subgroup H of $\mathrm{Aut}(N)$, and a covering $g : N/\{\mathrm{id}_N\} \to F'$ such that the composition $N/\{\mathrm{id}_N\} \to F' \to F$ induces an isomorphism $N/H \cong F$.*

Proof. We easily reduce to the case where F is connected. We write $F = M/G$ where M is an object in $\mathcal{C}^d$ and G is a subgroup of $\mathrm{Aut}(M)$. We further reduce to the case where F' is an object of the form $M'/\{\mathrm{id}_{M'}\}$ with M' in $\mathcal{FC}^d$. Let $\mathrm{Presh}(\mathcal{FC}^d)$ denote the category of presheaves on $\mathcal{FC}^d$. We have

$$\begin{aligned}\mathrm{Hom}_{\widetilde{\mathcal{FC}}^d}(M'/\{\mathrm{id}_{M'}\}, M/G) &= \mathrm{Hom}_{\mathrm{Presh}(\mathcal{FC}^d)}(\mathrm{Hom}_{\mathcal{FC}^d}(-, M'), M/G)\\ &= M/G(M')\\ &= \varinjlim_{M'' \to M'} \mathrm{Hom}_{\mathcal{FC}^d}(M'', M)/G\end{aligned}$$

where the limit is taken over (a small skeleton of the category of) all Galois coverings of M' in $\mathcal{FC}^d$. Hence there exist a Galois covering $M'' \xrightarrow{f''} M'$ in $\mathcal{FC}^d$ and a morphism $M'' \xrightarrow{f'} M$ in $\mathcal{FC}^d$ such that the diagram

$$\begin{array}{ccc} M''/\{\mathrm{id}_{M''}\} & \xrightarrow{f'} & M/\{\mathrm{id}_M\} \\ {\scriptstyle f''}\downarrow & & \downarrow \\ M'/\{\mathrm{id}_{M'}\} & \xrightarrow{f} & M/G \end{array}$$

is commutative.

It suffices to show that there exist an object N in $\mathcal{FC}^d$, a subgroup $H \subset \mathrm{Aut}(N)$ and a covering $N \to M''$ in $\mathcal{FC}^d$ such that the composition $N/\{\mathrm{id}_N\} \to M''/\{\mathrm{id}_{M''}\} \to M/G$ induces an isomorphism $N/H \cong M/G$.

Let $M'' = \coprod_{j \in J} M''_j$ where, for each $j \in J$, M''_j is an object in $\mathcal{C}^d$. Let $f'_j : M''_j \to M$ denote the morphism induced by f' for each j. Applying Lemma A.3 to the morphisms

$$M''_j \xrightarrow{f'_j} M \xrightarrow{m_{M,0}} 0$$

for $j \in J$, there exist an object N' in $\mathcal{FC}^d$ and a morphism $g_j : N' \to M''_j$ in $\mathcal{FC}^d$ for each $j \in J$ such that $f'_j \circ g_j = f'_{j'} \circ g_{j'}$ for all $j, j' \in J$ and $m_{M,0} \circ f'_j \circ g_j$ is a Galois covering.

Suppose the morphism $N' \to M$ is given by the diagram $M \overset{p}{\twoheadleftarrow} M_0 \overset{i}{\hookrightarrow} N'$. Let H' denote the subgroup of elements h' in $\mathrm{Aut}(N')$ such that $h'(\mathrm{Im}\, i) = \mathrm{Im}\, i$, $h'(i(\mathrm{Ker}\, p)) = i(\mathrm{Ker}\, p)$, and the action on $(\mathrm{Im}\, i)/i(\mathrm{Ker}\, p) \cong M$ induced by h' is in G. By the short exact sequence $1 \to \mathrm{Gal}(N'/M) \to G \to H' \to 1$, we have $N'/H' \cong M/G$.

Let H'' be a group whose cardinality is equal to the cardinality of the set $\pi_0(M'')$. We fix a bijection $H'' \xrightarrow{\cong} \pi_0(M'')$ of sets. Let $N = \coprod_{j \in \pi_0(M'')} N'$. We let $H = H'' \times H'$ act on N as follows. The element $(h'', 1) \in H'' \times H'$ acts on N via the translation of index set using the isomorphism $H'' \cong \pi_0(M'')$. The element $(1, h') \in H'' \times H'$ acts via the diagonal action of H'. Then N, H, and the covering

$$\coprod_{j \in \pi_0(M'')} g_j : N = \coprod_{j \in \pi_0(M'')} N' \to M'' = \coprod_{j \in \pi_0(M'')} M''_j$$

have the desired property. $\square$

Let F be a sheaf on $\mathcal{FC}^d$. We can construct a sheaf $\widetilde{F}$ on $\widetilde{\mathcal{FC}}^d$ by setting $\widetilde{F}(N/H) = F(N)^H$ for an object N/H in $\widetilde{\mathcal{FC}}^d$. Using Lemma A.6, we see that the functor $F \mapsto \widetilde{F}$ gives an equivalence of categories between the category of sheaves on $\mathcal{FC}^d$ and the category of sheaves on $\widetilde{\mathcal{FC}}^d$.

A.1.8. We define the functor $\omega : \mathrm{Presh}(\mathcal{FC}^d) \to (\mathrm{Sets})$ as follows. We consider $\mathbb{A}_X^{\oplus d}$ as the space of row vectors. Given a presheaf $F \in \mathrm{Presh}(\mathcal{FC}^d)$, we define $\omega(F)$ to be

$$\omega(F) = \varinjlim_{L_1 \subset L_2 \subset \mathbb{A}_X^{\oplus d}} F(L_2/L_1)$$

where the inductive limit is taken over the filtered ordered set of the pairs of two $\widehat{\mathcal{O}}_X$-lattices (L_1, L_2) in $\mathbb{A}_X^{\oplus}$ with $L_1 \subset L_2$. The order is defined as follows: for two such pairs (L_1, L_2) and (L'_1, L'_2), $(L_1, L_2) \geq (L'_1, L'_2)$ if and only if $L'_1 \subset L_1 \subset L_2 \subset L'_2$.

We have two functors $\mathcal{FC}^d \xrightarrow{f} \mathrm{Presh}(\mathcal{FC}^d) \xrightarrow{\omega} (\mathrm{Sets})$ and $\widetilde{\mathcal{FC}}^d \xrightarrow{g} \mathrm{Presh}(\mathcal{FC}^d) \xrightarrow{\omega}$ (Sets), where f is the functor $N \mapsto \mathrm{Hom}_{\mathcal{FC}^d}(-, N)$ and g is the functor induced by the inclusion of the category of sheaves into presheaves. We call them the canonical fiber functors, and denote them also by ω. It is easily checked that these canonical fiber functors are compatible with the canonical functor $\mathcal{FC}^d \to \widetilde{\mathcal{FC}}^d$ and preserve fiber products.

Lemma A.8. *Given a presheaf F, let F^a (resp. $F^{a\dagger}$) denote the associated sheaf (resp. semi-sheaf). Then the three sets $\omega(F)$, $\omega(F^a)$ and $\omega(F^{a\dagger})$ are canonically isomorphic.*

Proof. This follows from the explicit constructions of the associated sheaf functors $F \mapsto F^a$, $F \mapsto F^{a\dagger}$. $\square$

A.1.9. For a presheaf F on $\mathcal{FC}^d$, $\omega(F)$ admits a canonical functorial continuous left action of the adele group $\mathrm{GL}_d(\mathbb{A}_X)$. Hence the canonical fiber functors $\omega : \mathcal{FC}^d \to (\mathrm{Sets})$ and $\omega : \widetilde{\mathcal{FC}}^d \to (\mathrm{Sets})$ factor through the category of discrete sets with continuous left $\mathrm{GL}_d(\mathbb{A}_X)$-action.

Lemma A.9. *Let $L_1 \subset L_2 \subset \mathbb{A}_X^{\oplus d}$ be two $\widehat{\mathcal{O}}_X$-lattices of $\mathbb{A}_X^{\oplus d}$. Let $\mathbb{K} \subset \mathrm{GL}_d(\mathbb{A}_X)$ denote the compact open subgroup of the elements $g \in \mathrm{GL}_d(\mathbb{A}_X)$ such that $L_i g = L_i$ for $i = 1, 2$ and the map induced by g on L_2/L_1 is the identity. Then the following assertions hold.*

1. *There is a canonical $\mathrm{GL}_d(\mathbb{A}_X)$-equivariant isomorphism $\omega(L_2/L_1) \cong \mathrm{GL}_d(\mathbb{A}_X)/\mathbb{K}$ which sends the element in $\omega(L_2/L_1)$ represented by the element id_{L_2/L_1} in $\mathrm{Hom}(L_2/L_1, L_2/L_1)$ to the class of the identity matrix $\mathrm{GL}_d(\mathbb{A}_X)/\mathbb{K}$.*
2. *For any presheaf F on $\mathcal{FC}^d$, the canonical map $F^a(L_2/L_1) \to \omega(F^a) \cong \omega(F)$ induces an isomorphism $F^a(L_2/L_1) \cong \omega(F)^{\mathbb{K}}$.*

Proof. We can identify $\omega(L_2/L_1)$ with the set of triples (L', L'', α) where $L' \subset L'' \subset \mathbb{A}_X^{\oplus d}$ are $\widehat{\mathcal{O}}_X$-lattices and $\alpha : L''/L' \to L_2/L_1$ is an isomorphism. The map from $\mathrm{GL}_d(\mathbb{A}_X)/\mathbb{K}$ which sends the coset $g\mathbb{K}$ to the triple $(L_1 g^{-1}, L_2 g^{-1}, g : L_2 g^{-1}/L_1 g^{-1} \to L_2/L_1)$ then induces the isomorphism in (1).

By definition, $F^a(L_2/L_1) = \varinjlim_{M \to L_2/L_1} F(M)^{\mathrm{Gal}(M/(L_2/L_1))}$ where the limit is taken over (a small skeleton of the category of) all Galois coverings of L_2/L_1 in $\mathcal{C}^d$. By the definition of ω, we have $\omega(F)^{\mathbb{K}} = \varinjlim_{L_1' \subset L_2' \subset \mathbb{A}_X^d} F(L_2'/L_1')^{\mathrm{Gal}((L_2'/L_1')/(L_2/L_1))}$ where the limit is taken over all Galois coverings of the form $L_2/L_1 \twoheadleftarrow L_2/L_1' \hookrightarrow L_2'/L_1'$. One sees that the two limits are equal using the argument in the proof of Lemma A.3. This shows (2). □

Corollary A.10. *Let L_1 and L_2 be as in Lemma A.9. Let H be a subgroup of $\mathrm{Aut}_{\mathcal{O}_X}(L_2/L_1)$. Let $\mathbb{K}_{L_1,L_2,H} \subset \mathrm{GL}_d(\mathbb{A}_X)$ denote the compact open subgroup of the elements $g \in \mathrm{GL}_d(\mathbb{A}_X)$ such that $L_i g = L_i$ for $i = 1, 2$ and the action of g on L_2/L_1 lies in H. Then the following assertions hold.*

1. *There is a canonical $\mathrm{GL}_d(\mathbb{A}_X)$-equivariant isomorphism $\omega((L_2/L_1)/H) \cong \mathrm{GL}_d(\mathbb{A}_X)/\mathbb{K}_{L_1,L_2,H}$ which sends the element in $\omega((L_2/L_1)/H)$ represented by the class of id_{L_2/L_1} in $\mathrm{Hom}(L_2/L_1, L_2/L_1)/H$ to the class of the identity matrix in $\mathrm{GL}_d(\mathbb{A}_X)/\mathbb{K}_{L_1,L_2,H}$.*
2. *For any presheaf F on $\mathcal{FC}^d$, the composition*
$$\mathrm{Hom}_{\mathrm{Presh}(\mathcal{FC}^d)}((L_2/L_1)/H, F^a) \to F^a(L_2/L_1) \to \omega(F^a) \cong \omega(F)$$
induces an isomorphism $\mathrm{Hom}_{\mathrm{Presh}(\mathcal{FC}^d)}((L_2/L_1)/H, F^a) \cong \omega(F)^{\mathbb{K}_{L_2,L_1,H}}$.

□

We define the category $(\mathrm{GL}_d(\mathbb{A}_X)\text{-sets}^*)$. An object S in $(\mathrm{GL}_d(\mathbb{A}_X)\text{-sets}^*)$ is a set with left $\mathrm{GL}_d(\mathbb{A}_X)$-action such that S has finitely many $\mathrm{GL}_d(\mathbb{A}_X)$-orbits, and for any $s \in S$, the stabilizer at s is a compact open subgroup of $\mathrm{GL}_d(\mathbb{A}_X)$.

Lemma A.11. *The canonical fiber functor $\omega : \widetilde{\mathcal{FC}}^d \to (\mathrm{Sets})$ gives an equivalence between the category $\widetilde{\mathcal{FC}}^d$ and the category $(\mathrm{GL}_d(\mathbb{A}_X)\text{-sets}^*)$.*

Proof. An object in $(\mathrm{GL}_d(\mathbb{A}_X)\text{-sets}^*)$ is isomorphic to an object of the form $\coprod_{i \in I} \mathrm{GL}_d(\mathbb{A}_X)/\mathbb{K}_i$ where I is a finite set and $\mathbb{K}_i$ is an open compact subgroup of $\mathrm{GL}_d(\mathbb{A}_X)$. Then, by Corollary A.10 (1), it follows that ω is essentially surjective on $(\mathrm{GL}_d(\mathbb{A}_X)\text{-sets}^*)$.

To prove that ω is fully faithful, it suffices to treat the connected objects in $\widetilde{\mathcal{FC}}^d$. Such an object is isomorphic to an object of the form $(L_2/L_1)/H$ where $L_1 \subset L_2$ are $\widehat{\mathcal{O}}_X$-lattices in $\mathbb{A}_X^{\oplus d}$, and H is a subgroup of $\mathrm{Aut}(L_2/L_1)$.

Let $L_1 \subset L_2, L_1' \subset L_2'$ be $\widehat{\mathcal{O}}_X$-lattices in $\mathbb{A}_X^{\oplus d}$, and H, H' be subgroups of $\mathrm{Aut}(L_2/L_1)$, $\mathrm{Aut}(L_2'/L_1')$ respectively. By (2) and (1) of Corollary A.10, we have

$$\begin{aligned}\mathrm{Hom}_{\widetilde{\mathcal{FC}}^d}((L_2/L_1)/H, (L_2'/L_1')/H') &= \omega((L_2'/L_1')/H')^{\mathbb{K}_{L_1,L_2,H}} \\ &= (\mathrm{GL}_d(\mathbb{A}_X)/\mathbb{K}_{L_1',L_2',H'})^{\mathbb{K}_{L_1,L_2,H}}\end{aligned}$$

where $\mathbb{K}_{L_1,L_2,H}$ and $\mathbb{K}_{L_1',L_2',H'}$ are defined as in Corollary A.10. We see it equals

$$\begin{aligned}&\mathrm{Hom}_{(\mathrm{GL}_d(\mathbb{A}_X)\text{-sets}^*)}(\mathrm{GL}_d(\mathbb{A}_X)/\mathbb{K}_{L_1,L_2,H}, \mathrm{GL}_d(\mathbb{A}_X)/\mathbb{K}_{L_1',L_2',H'}) \\ &= \mathrm{Hom}_{(\mathrm{GL}_d(\mathbb{A}_X)\text{-sets}^*)}(\omega(L_2/L_1)/H, \omega(L_2'/L_1')/H').\end{aligned}$$

□

Corollary A.12. (1) *For any object F in $\widetilde{\mathcal{FC}}^d$, the set $\pi_0(N)$ is canonically isomorphic to the set of* $\mathrm{GL}_d(\mathbb{A}_X)$*-orbits in* $\omega(F)$.

(2) *For any morphism $f : F \to F'$ in $\widetilde{\mathcal{FC}}^d$ and for any $x \in \omega(F')$, the fiber $\omega(f)^{-1}(x)$ of $\omega(f)$ at x is a finite set.*

(3) *Fiber products always exist in $\widetilde{\mathcal{FC}}^d$.*

□

A.1.10. Given a smooth representation V of $\mathrm{GL}_d(\mathbb{A}_X)$, we construct a presheaf, which is also denoted by V, on ($\mathrm{GL}_d(\mathbb{A}_X)$-sets*). For an object Y in ($\mathrm{GL}_d(\mathbb{A}_X)$-sets*), we let $V(Y)$ denote the set of morphisms $Y \to V$ of left $\mathrm{GL}_d(\mathbb{A}_X)$-sets. Let $f : Y_1 \to Y_2$ be a morphism in ($\mathrm{GL}_d(\mathbb{A}_X)$-sets*). We define $f^* : V(Y_2) \to V(Y_1)$ to be the composition with f. We thus have a presheaf $Y \mapsto V(Y)$ which may be checked to be an abelian sheaf.

This gives an equivalence of categories between the category of smooth representations of the locally profinite group $\mathrm{GL}_d(\mathbb{A}_X)$ and the category of abelian sheaves on ($\mathrm{GL}_d(\mathbb{A}_X)$-sets*).

A.1.11. Let $F = (\coprod_{j\in J} N_j)/H$ be an object in $\widetilde{\mathcal{FC}}^d$. Using Lemma A.8, we have a map

$$\omega(F) = (\coprod_{j\in J} \omega(\mathrm{Hom}(-, N_j)))/H \to J/H = \pi_0(F)$$

induced by the map which sends the elements in $\omega(\mathrm{Hom}(-, N_j))$ to j.

Let $f : F_0 \to F$ be a morphism in $\widetilde{\mathcal{FC}}^d$. The map $\omega(F) \to \mathbb{Z}_{\geq 0}$, which sends $x \in \omega(F)$ to $\#\omega(f)^{-1}(x)$, factors through $\pi_0(F)$. We call the induced map $\deg f : \pi_0(F) \to \mathbb{Z}_{\geq 0}$ the degree of f.

Lemma A.13. *Let*

$$\begin{tikzcd} F_1' \arrow[r, "g_1"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Box"] & F_1 \arrow[d, "f"] \\ F_2' \arrow[r, "g_2"] & F_2 \end{tikzcd}$$

be a cartesian diagram in $\widetilde{\mathcal{FC}}^d$. Then $(\deg f')(y) = (\deg f)(\pi_0(g_2)(y))$ for any $y \in \pi_0(F_2')$.

Proof. Recall that the canonical fiber functor $\omega : \widetilde{\mathcal{FC}}^d \to (\mathrm{Sets})$ preserves fiber products. Thus we have a cartesian diagram

$$\begin{tikzcd} \omega(F_1') \arrow[r, "\omega(g_1)"] \arrow[d, "\omega(f')"'] \arrow[dr, phantom, "\Box"] & \omega(F_1) \arrow[d, "\omega(f)"] \\ \omega(F_2') \arrow[r, "\omega(g_2)"] & \omega(F_2) \end{tikzcd}$$

in the category of sets. The assertion then follows easily. □

Lemma A.14. *Let $f : N/\{\mathrm{id}_N\} \to N/H$ be a Galois covering as in Lemma A.5, then $(\deg f)(i) = \#H$ for any $i \in \pi_0(N/H)$.*

Proof. We may assume that N/H is connected. Then the assertion follows from the equality $\varinjlim(\mathrm{Hom}_{\mathcal{FC}^d}(L_2/L_1,N)/H) = (\varinjlim \mathrm{Hom}_{\mathcal{FC}^d}(L_2/L_1,N))/H$ where the limit is taken as in the definition of ω. $\square$

A.1.12. *Variant.* Let N be an object in $\mathcal{FC}^d$ and H be a subgroup of $\mathrm{Aut}_{\mathcal{FC}^d}(N)$. We denote by $(N/H)^\dagger$ the semi-sheaf associated to the presheaf $\mathrm{Hom}_{\mathcal{FC}^d}(-,N)/H$. Let $\widetilde{\mathcal{FC}}^{\dagger,d}$ denote the full subcategory of the category of sheaves on $\mathcal{FC}^d$ whose objects are semi-sheaves of the form $(N/H)^\dagger$ with N in $\mathcal{FC}^d$.

The notions of π_0 and covering are canonically extended to the categories $\widetilde{\mathcal{FC}}^{\dagger,d}$.

We say that a morphism f in $\widetilde{\mathcal{FC}}^{\dagger,d}$ is a fibration if there exist two coverings g_1, g_2 in $\widetilde{\mathcal{FC}}^{\dagger,d}$ such that $g_2 \circ f \circ g_1$ is a fibration in $\mathcal{FC}^d$. We define semi-sheaves on $\widetilde{\mathcal{FC}}^{\dagger,d}$ in a similar way. The proof of the following lemma is omitted.

Lemma A.15. *A presheaf F on $\widetilde{\mathcal{FC}}^{\dagger,d}$ is a semi-sheaf if and only if it satisfies conditions analogous to (1) and (2) in Definition A.1 and (3)$'''$ in Lemma A.6.* $\square$

The following lemma is used in the proof of Lemma 2.11.

Lemma A.16. *Let $f : M \to N$ be a fibration in $\mathcal{FC}^d$. Suppose a finite group H acts equivariantly on M and N. Then the induced morphism $m : (M/H)^\dagger \to (N/H)^\dagger$ is a fibration in $\widetilde{\mathcal{FC}}^{\dagger,d}$.*

Proof. Let $f_1 : (M/\{\mathrm{id}_M\})^\dagger \to (M/H)^\dagger$ be the quotient map, and $f_2 : (N/H)^\dagger \to 0$ be the morphism induced by $m_{N,0} : N \to 0$. Then the composition $f_2 \circ m \circ f_1 = m_{M,0}$ is a fibration in $\mathcal{FC}^d$. $\square$

Let F be a semi-sheaf on $\mathcal{FC}^d$. We can construct a semi-sheaf on $\widetilde{\mathcal{FC}}^{\dagger,d}$ by setting $\widetilde{F}^\dagger((N/H)^\dagger) = F(N)^H$ for an object $(N/H)^\dagger$ in $\widetilde{\mathcal{FC}}^{\dagger,d}$. One can see, using the previous lemma, that the functor $F \mapsto \widetilde{F}$ gives an equivalence of categories between the category of semi-sheaves on $\mathcal{FC}^d$ and the category of semi-sheaves on $\widetilde{\mathcal{FC}}^{\dagger,d}$.

A.1.13. We define the functor $\omega^\dagger$ as follows. For a presheaf F on $\mathcal{FC}^d$, we let

$$\begin{aligned}\omega^\dagger(F) &= \varinjlim_{L_1 \subset L_2 \subset \widehat{\mathcal{O}}_X^{\oplus d}} F(L_2/L_1) \\ &= \varinjlim_{L_1 \subset \widehat{\mathcal{O}}_X^{\oplus d}} F(\widehat{\mathcal{O}}_X^{\oplus d}/L_1)\end{aligned}$$

where the first inductive limit is taken over the filtered ordered set of the pairs of two $\widehat{\mathcal{O}}_X$-lattices (L_1, L_2) in $\widehat{\mathcal{O}}_X^{\oplus d}$ with $L_1 \subset L_2$; the transition maps are defined in a manner similar to those in the definition of ω in Section A.1.8. The second inductive limit is taken over the $\widehat{\mathcal{O}}_X$-lattices in $\widehat{\mathcal{O}}_X^{\oplus d}$ ordered by inclusion.

Let Mat^- be the monoid consisting of elements $g \in \mathrm{GL}_d(\mathbb{A}_X)$ such that g^{-1} belongs to $\mathrm{Mat}_d(\widehat{\mathcal{O}}_X)$. We define the category (Mat^--sets*). An object S in (Mat^--sets*) is a set with left Mat^--action such that S is isomorphic to the disjoint union $\coprod_{i \in I} \mathrm{Mat}^-/\mathbb{K}_i$, where I is a finite set and $\mathbb{K}_i$ is an open subgroup of $\mathrm{GL}_d(\widehat{\mathcal{O}}_X)$. The proof of the following lemma is similar to that of Lemma A.11, hence is omitted.

Lemma A.17. *The functor $\omega^\dagger : \widetilde{\mathcal{FC}}^{\dagger,d} \to$ (Sets) gives an equivalence between the category $\widetilde{\mathcal{FC}}^{\dagger,d}$ and the category* (Mat^--sets*). $\square$

We remark that the argument similar to that in Section A.1.10 establishes an equivalence of categories between the category of smooth representations of the monoid Mat^- and the category of abelian sheaves on (Mat^--sets*). Here we say that a representation V of Mat^- is smooth if for each $v \in V$, there exists an open subgroup of $\mathrm{GL}_d(\widehat{\mathcal{O}}_X)$ which fixes v.

A.1.14. Let $S = \coprod_{i\in I} \mathrm{Mat}^-/\mathbb{K}_i$ be an object of (Mat^--sets*) where I is a finite set and $\mathbb{K}_i$ is an open subgroup of $\mathrm{GL}_d(\widehat{\mathcal{O}}_X)$.

Let Y be an object in (Mat^--sets*). We let $Y^* = Y \setminus \bigcup_{m\in \mathrm{Mat}^-\setminus \mathrm{GL}_d(\widehat{\mathcal{O}}_X)} m\cdot Y$. If $f : Y \to Z$ is a morphism in (Mat^--sets*), then $f^{-1}(Z^*) \subset Y^*$ holds in general. We say that a morphism $f : Y \to Z$ is a fibration if $f(Y^*) \subset Z^*$. The following two lemmas are easily checked.

Lemma A.18. *Let $f : Y = \coprod_{i\in I} \mathrm{Mat}^-/\mathbb{K}_i \to Z = \coprod_{j\in J} \mathrm{Mat}^-/\mathbb{K}_j$. The following conditions are equivalent.*

(1) $f(Y^*) = Z^*$.
(2) $f(Y^*) \subset Z^*$ *and* $\pi_0(f)$ *is surjective.*
(3) *For each $i \in I$, the restriction $f|_{\mathrm{Mat}^-/\mathbb{K}_i} : \mathrm{Mat}^-/\mathbb{K}_i \to \mathrm{Mat}^-/\mathbb{K}_{\pi_0(f)(i)}$ is a surjective map of sets, and $\pi_0(f)$ is surjective.*

□

Lemma A.19. *Let $Y \xrightarrow{f_1} Z \xrightarrow{f_2} W$ be morphisms in* (Mat^--sets*). *Then $f_2 \circ f_1$ satisfies the conditions in Lemma A.18 and $\pi_0(f_1) : \pi_0(Y) \to \pi_0(Z)$ is surjective if and only if both f_1 and f_2 satisfy the conditions in Lemma A.18.* □

Lemma A.20. *Let $M \to N$ be a fibration in $\mathcal{FC}^d$. Then $\omega^\dagger(M) \to \omega^\dagger(N)$ is a fibration in* (Mat^--sets*).

Proof. We may assume $M, N \in \mathcal{C}^d$. It suffices to prove that $\omega^\dagger(M) \to \omega^\dagger(N)$ is surjective. By definition, the elements of $\omega^\dagger(M)$ are represented by diagrams of the form $M \twoheadleftarrow L \hookrightarrow \widehat{\mathcal{O}}_X$ where L is an $\widehat{\mathcal{O}}_X$-lattice, and there is a similar expression for those of $\omega^\dagger(N)$. Since $M \to N$ is a fibration in $\mathcal{FC}^d$, it is induced by a surjective map $M \twoheadrightarrow N$ of $\widehat{\mathcal{O}}_X$-modules. The induced map $\mathrm{Surj}(L, M) \to \mathrm{Surj}(L, N)$, where $\mathrm{Surj}(-,-)$ denotes the set of surjective maps, is surjective. We see this by reducing to the case where X is the spectrum of the ring of integers of a local field, and using Nakayama's lemma. Hence the assertion follows. □

Proposition A.21. *The fiber product of two fibrations in* (Mat^--sets*) *exists.*

In view of Lemmas A.18 and A.19, we have the following corollary.

Corollary A.22. *The fiber product of two fibrations in $\widetilde{\mathcal{FC}}^{\dagger,d}$ exists.*

Proof of Proposition A.21. Let $M_1 \xrightarrow{f_1} M_2$ and $M_3 \xrightarrow{f_2} M_2$ be two fibrations in (Mat^--sets*). We construct the fiber product of f_1 and f_2. We may assume that each of $\pi_0(M_1)$, $\pi_0(M_2)$, and $\pi_0(M_3)$ consists of one element. Take open subgroups $\mathbb{K}_i \subset \mathrm{GL}_d(\widehat{\mathcal{O}}_X)$ such that $M_i \cong \mathrm{Mat}^-/\mathbb{K}_i$ for each $i = 1, 2, 3$ and $\mathbb{K}_1 \subset \mathbb{K}_3 \supset \mathbb{K}_2$. Then the set theoretic fiber product $\mathrm{Mat}^-/\mathbb{K}_1 \times_{\mathrm{Mat}^-/\mathbb{K}_3} \mathrm{Mat}^-/\mathbb{K}_2 \cong \coprod_{g\in \mathbb{K}_1\setminus\mathbb{K}_3/\mathbb{K}_2} \mathrm{Mat}^-/(\mathbb{K}_1 \cap g\mathbb{K}_2 g^{-1})$ is an object in (Mat^--sets*), and hence is the fiber product in (Mat^--sets*). □

Appendix B. Ramified local L-factors

by Seidai Yasuda

The aim of this section is to derive a formula (Proposition B.1) which expresses the local L-factor of any ramified generic irreducible admissible representation of GL_d over a local field in terms of the eigenvalues of certain Hecke operators acting on the space of new vectors. A similar formula for unramified representations is well-known and is found, for example, in [Cog, Lecture 7].

B.1. Review of functors between categories of presheaves (cf. [SGA4, Expose I]).

B.1.1. For two categories $\mathcal{A}$, $\mathcal{C}$, let $\mathrm{Presh}(\mathcal{C},\mathcal{A})$ denote the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$. In this section, we assume that any category denoted by a letter $\mathcal{C}$ with some subscripts has a small skeleton.

B.1.2. Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be a covariant functor. Then the pull-back functor $f^* : \mathrm{Presh}(\mathcal{C}_2,\mathcal{A}) \to \mathrm{Presh}(\mathcal{C}_1,\mathcal{A})$ is canonically defined.

B.1.3. If the category $\mathcal{A}$ has a limit (= projective limit), there is a right adjoint functor of f^*, which we denote by $f_* : \mathrm{Presh}(\mathcal{C}_1,\mathcal{A}) \to \mathrm{Presh}(\mathcal{C}_2,\mathcal{A})$. The functor f_* can be explicitly given as follows. Let F be a presheaf on $\mathcal{C}_1$, and X be an object in $\mathcal{C}_2$. Then $(f_*F)(X)$ is a limit of $F(Y)$. Here the limit is taken over (a small skeleton of) the category of pairs (Y,α) of an object Y in $\mathcal{C}_1$ and a morphism $\alpha : f(Y) \to X$ in $\mathcal{C}_2$. When $g : \mathcal{C}_2 \to \mathcal{C}_3$ is another covariant functor, we have $g_*f_* = (gf)_*$.

B.2. Ramified local L-factors.

Let K be a non-Archimedean local field, $\mathcal{O}_K$ be its ring of integers, and ϖ be a uniformizer. Let (π, V) be an irreducible admissible representation of $G = \mathrm{GL}_d(K)$.

B.2.1. For an integer $n \geq 0$, let $\mathbb{K}_n \subset G$ be the open compact subgroup consisting of the elements in $\mathrm{GL}_d(\mathcal{O}_K)$ whose last row is congruent to $(0,\ldots,0,1)$ modulo ϖ^n. Let $\mathcal{H}(G,\mathbb{K}_n)$ be the Hecke algebra consisting of the bi-$\mathbb{K}_n$-invariant functions on G with compact supports. Then $\mathcal{H}(G,\mathbb{K}_n)$ is a convolution algebra with respect to the Haar measure of G satisfying $\mathrm{vol}(\mathbb{K}_n) = 1$, whose unit is the characteristic function of $\mathbb{K}_n$. For $r = 0,\ldots,d-1$, let $T_{n,r} = T_{n,r}^{(d)} \in \mathcal{H}(G,\mathbb{K}_n)$ denote the characteristic function of the double coset

$$\mathbb{K}_n \begin{pmatrix} \varpi & & & & & \\ & \ddots & & & & \\ & & \varpi & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \mathbb{K}_n$$

where in the above diagonal matrix ϖ appears r times and 1 appears $d-r$ times.

B.2.2. From now on we assume that π is generic and is not an unramified principal series. Let c denote the conductor of π. Then $c \geq 1$ and $V^{\mathbb{K}_c}$ is one-dimensional. The action of $\mathcal{H}(G, \mathbb{K}_c)$ on $V^{\mathbb{K}_c}$ defines an algebra homomorphism $\chi_V : \mathcal{H}(G, \mathbb{K}_c) \to \mathbb{C}$.

Proposition B.1. *Let notations and assumptions be as above. Let $L(\pi, s)$ be the local L-factor of π. Then we have*

$$L(\pi, s)^{-1} = \sum_{r=0}^{d-1} (-1)^r \chi_V(T_{c,r}) q^{\frac{r(r-1)}{2} - r(\frac{d-1}{2}+s)},$$

where q is the cardinality of the residue field of $\mathcal{O}_K$.

It is easy to check Proposition B.1 when π is supercuspidal. Suppose that π is supercuspidal. Since every matrix coefficient of π has a compact support modulo center, we have $(\chi_V(T_{c,r}))^n = 0$ for sufficiently large n when $r = 1, \ldots, d-1$. Hence $\chi_V(T_{c,r}) = 0$ for $r = 1, \ldots, d-1$. Thus we have $\sum_{r=0}^{d-1} \chi_V(T_{c,r}) q^{-rs} = 1 = L(\pi, s)^{-1}$.

To prove Proposition B.1 in general case, we use the classification of generic representations given in [Be-Ze], [Ze].

Remark B.2. Perhaps Proposition B.1 is a consequence of [Ja-Pi-Sh, p. 208, Théorème].

B.3. Categorical description of parabolic inductions. Let us consider the category $\mathcal{C}^d$ for $X = \mathrm{Spec}\,(\mathcal{O}_K)$.

B.3.1. For a partition $\mathbf{d} = (d_1, \ldots, d_m)$, $d = d_1 + \cdots + d_m$, $d_1, \ldots, d_m \geq 1$ of d, let $\mathcal{E}_0^{\mathbf{d}}$ denote the following category. An object in $\mathcal{E}^{\mathbf{d}}$ is an object M in $\mathcal{C}^d$ endowed with a decreasing filtration

$$M = \mathrm{Fil}^1 M \supset \mathrm{Fil}^2 M \supset \cdots \supset \mathrm{Fil}^{m+1} M = 0$$

of M by sub $\mathcal{O}_K$-modules such that for each $i = 1, \ldots, m$, $\mathrm{Gr}^i M = \mathrm{Fil}^i M / \mathrm{Fil}^{i+1} M$ is an object in $\mathcal{C}^{d_i}$. For two objects $(M, \mathrm{Fil}^\bullet)$, $(N, \mathrm{Fil}^\bullet)$ in $\mathcal{E}^{\mathbf{d}}$, a morphism from $(M, \mathrm{Fil}^\bullet M)$, $(N, \mathrm{Fil}^\bullet)$ is a Q-morphism from N to M such that the filtration $\mathrm{Fil}^\bullet N$ coincides with the the filtration induced from $\mathrm{Fil}^\bullet M$. We have the following diagram of categories

$$\mathcal{C}^{d_1} \times \cdots \times \mathcal{C}^{d_m} \xleftarrow{\mathrm{gr}} \mathcal{E}^{\mathbf{d}} \xrightarrow{\mathrm{for}} \mathcal{C}^d,$$

where, gr (resp. for) denotes the functor which sends an object $(M, \mathrm{Fil}^\bullet)$ in $\mathcal{E}^{\mathbf{d}}$ to the object $(\mathrm{Gr}^1 M, \ldots, \mathrm{Gr}^m M)$ in $\mathcal{C}^{d_1} \times \cdots \times \mathcal{C}^{d_m}$ (resp. the object M in $\mathcal{C}^d$).

Lemma B.3. *For $i = 1, \ldots, m$, let (π_i, V_i) be a smooth representation of $G_i = \mathrm{GL}_{d_i}(K)$, and F_i be the sheaf (with values in complex vector spaces) on $\mathcal{FC}^{d_i}$ corresponding to π_i. Let $\pi' = \mathrm{Ind}(\pi_1 \times \cdots \times \pi_m)$ denote the algebraic parabolic induction of $\pi_1 \times \cdots \times \pi_m$ to $\mathrm{GL}_d(K)$ (here the word algebraic means that we do not make any modification by a modular character). Let F' be the sheaf on $\mathcal{FC}^d$ corresponding to π'. Then we have a canonical isomorphism*

$$F'|_{\mathcal{C}^d} = \mathrm{for}_* \mathrm{gr}^* (F_1|_{\mathcal{C}^{d_1}} \boxtimes \cdots \boxtimes F_m|_{\mathcal{C}^{d_m}})$$

of presheaves on $\mathcal{C}^d$.

B.3.2. Let $\mathcal{F}(\mathcal{E}^{\mathbf{d}})$, $\mathcal{F}(\mathcal{C}^{d_1}\times\cdots\times\mathcal{C}^{d_m})$ denote the category of finite families of objects in $\mathcal{E}^{\mathbf{d}}$, $\mathcal{C}^{d_1}\times\cdots\times\mathcal{C}^{d_m}$, respectively. In a manner similar to that in Definition A.1, we define sheaves on these categories. Then we can check that the category of abelian sheaves on $\mathcal{F}(\mathcal{E}^{\mathbf{d}})$ (resp. on $\mathcal{F}(\mathcal{C}^{d_1}\times\cdots\times\mathcal{C}^{d_m})$) is canonically equivalent to the category of smooth representation of the standard parabolic subgroup of $\mathrm{GL}_d(K)$ corresponding to the partition $\mathbf{d}$ (resp. the group $\mathrm{GL}_{d_1}(K)\times\cdots\mathrm{GL}_{d_m}(K)$).

The category $\mathcal{F}(\mathcal{C}^{d_1}\times\cdots\times\mathcal{C}^{d_m})$ contains the direct product category $\mathcal{F}\mathcal{C}^{d_1}\times\cdots\times\mathcal{F}\mathcal{C}^{d_m}$ as a full subcategory. When an abelian sheaf F_i on $\mathcal{F}\mathcal{C}^{d_i}$ is given for each $i=1,\ldots,m$, they canonically produce an abelian presheaf, which we denote by $[F_1\boxtimes\cdots\boxtimes F_m]$, on $\mathcal{F}(\mathcal{C}^{d_1}\times\cdots\times\mathcal{C}^{d_m})$ whose restriction to $\mathcal{F}\mathcal{C}^{d_1}\times\cdots\times\mathcal{F}\mathcal{C}^{d_m}$ is equal to the presheaf $F_1\boxtimes\cdots\boxtimes F_m$. For any morphism $\varphi: M\to N$ in $\mathcal{F}\mathcal{C}^{d_1}\times\cdots\times\mathcal{F}\mathcal{C}^{d_m}$, the push-forward map $\varphi_*:[F_1\boxtimes\cdots\boxtimes F_m](M)\to[F_1\boxtimes\cdots\boxtimes F_m](N)$ is canonically defined.

We define a covariant functor $h:\mathcal{F}\mathcal{C}^d\to\mathcal{F}(\mathcal{C}^{d_1}\times\cdots\times\mathcal{C}^{d_m})$ in the following way. For an object M in $\mathcal{C}^d$, let $\mathrm{Flag}^{\mathbf{d}}(M)$ denote the set of decreasing filtrations

$$M=\mathrm{Fil}^1M\supset\mathrm{Fil}^2M\supset\cdots\supset\mathrm{Fil}^{m+1}M=0$$

of M by sub $\mathcal{O}_K$-modules such that for each $i=1,\ldots,m$, $\mathrm{Gr}^iM=\mathrm{Fil}^iM/\mathrm{Fil}^{i+1}M$ is an object in $\mathcal{C}^{d_i}$. We define $h(M)$ to be the disjoint sum

$$h(M)=\coprod_{\mathrm{Flag}^{\mathbf{d}}(M)}(\mathrm{Gr}^1M,\ldots,\mathrm{Gr}^mM).$$

For an object $M=\coprod_j M_j$ in $\mathcal{C}^d$, we set $h(M)=\coprod_j h(M_j)$.

Corollary B.4. *In the notation of Lemma B.3, the sheaf F' is given by the pull-back*

$$F'=h^*[F_1\boxtimes\cdots\boxtimes F_m].$$

□

B.3.3.

Proof of Lemma B.3. By the adjointness property of for_*, it suffices to prove that $F''=\mathrm{for}_*\mathrm{gr}^*(F_1|_{\mathcal{C}^{d_1}}\boxtimes F_m|_{\mathcal{C}^{d_m}})$ is the restriction of a sheaf on $\mathcal{F}\mathcal{C}^d$. Let $f:M\to N$ be a Galois covering in $\mathcal{C}^d$ with Galois group G. We set $h(M)=\coprod_{x\in\pi_0(h(M))}h(M)_x$ and $h(N)=\coprod_{y\in\pi_0(h(N))}h(N)_y$. Then for any $y\in\pi_0(h(N))$, the morphism $\coprod_{\pi_0(h(f))(x)=y}h(M)_x\to h(N)_y$ is a "Galois covering" in $\mathcal{F}(\mathcal{C}^{d_1}\times\cdots\times\mathcal{C}^{d_m})$ whose Galois group G_y is a quotient of G. Hence $F''(N)$ is isomorphic to the G-invariant part of $F''(M)$, whence the assertion follows. □

B.4. Description of push-outs.

B.4.1. For a morphism $f:M\to N$ in $\mathcal{F}\mathcal{C}^d$ and for $x\in\pi_0(h(M))$, we define the multiplicity $\mathrm{mult}_x(f)$ of f at x which is a power of q.

B.4.2. An element $x\in\pi_0(h(M))$ corresponds to a pair $(M_0,\mathrm{Fil}^\bullet M_0)$ of a connected component M_0 of M and a decreasing filtration

$$M_0=\mathrm{Fil}^1M_0\supset\cdots\supset M_{m+1}=0$$

such that for $i=1,\ldots,d$, Gr^iM_0 is an object in $\mathcal{C}^{d_i}$. Let $N_0\twoheadleftarrow M_0'\hookrightarrow M_0$ be the restriction of f to M_0, where N_0 is an appropriate connected component of N. The filtration $\mathrm{Fil}^\bullet M_0$ on M_0 induces a filtration $\mathrm{Fil}^\bullet M_0'$ on M_0' and a filtration $\mathrm{Fil}^\bullet N_0$ on N_0. We define $\mathrm{mult}_x(f)$ to be

$$\mathrm{mult}_x(f) = \sharp(M_0/M_0')^d \prod_{j=1}^{m} \left(\frac{(\sharp \mathrm{Fil}^{j+1} M_0')^2}{\sharp \mathrm{Fil}^{j+1} M_0 \cdot \sharp \mathrm{Fil}^{j+1} N_0} \right)^{d_j}.$$

B.4.3.

Proposition B.5. *Let the notations be as above. Then for any morphism $f : M \to N$ in $\mathcal{FC}^d$, the push-forward map $f_* : F'(M) \to F'(N)$ is canonically identified with the map*

$$f'_* : [F_1 \boxtimes \cdots \boxtimes F_m](h(M)) \to [F_1 \boxtimes \cdots \boxtimes F_m](h(N))$$

which is defined as follows. We set $h(M) = \coprod_{x \in \pi_0(h(M))} h(M)_x$ and $h(N) = \coprod_{y \in \pi_0(h(N))} h(N)_y$. On each $x \in \pi_0(h(M))$, we define $h(f)_x : h(M)_x \to h(N)_{\pi_0(h(f))(x)}$ to be the restriction of the morphism $h(f) : h(M) \to h(N)$ to the component $h(M)_x$. Then f'_* is given as the direct sum of the morphisms

$$\mathrm{mult}_x(f)(h(f)_x)_* : [F_1 \boxtimes \cdots \boxtimes F_m](h(M)_x) \to [F_1 \boxtimes \cdots \boxtimes F_m](h(N)_{\pi_0(h(f))(x)}).$$

Proof. We easily reduce to the case where $f : M \to N$ is a Galois covering in $\mathcal{C}^d$. Moreover we may assume that $M = (\mathcal{O}_K/\varpi^n)^{\oplus d}$ for some n, and that for the Q-morphism $N \twoheadleftarrow M' \hookrightarrow M$ giving f, M' is equal to either M or N. Let G be the Galois group of M over N. We set $h(M) = \coprod_{x \in \pi_0(h(M))} h(M)_x$ and $h(N) = \coprod_{y \in \pi_0(h(N))} h(N)_y$. For $x \in \pi_0(h(M))$, let G_x denote the Galois group of $h(M)_x$ over $h(N)_{\pi_0(h(f))(x)}$. Then it is easily checked that the cardinality of the kernel of $G \twoheadrightarrow G_x$ is equal to $\mathrm{mult}_x(f)$. Hence the assertion follows. $\square$

B.4.4. Let V be a smooth representation of $\mathrm{GL}_d(F)$ and let F be the corresponding sheaf on $\mathcal{FC}^d$ with values in complex vector spaces. Let us consider the cyclic $\mathcal{O}_K$-module $N = \mathcal{O}_K/\varpi^n$ of length n. Then for $r = 1, \ldots, d$, Hecke operator $T_{n,r} : V^{\mathbb{K}_n} \to V^{\mathbb{K}_n}$ induces an endomorphism $F(N) \to F(N)$ which we also denote by $T_{n,r}$. For $i = 1, \ldots, d-1$, let $m_r = m_r^{(d)}$, $r_r = r_r^{(d)}$ denote the morphism from $(\mathcal{O}_K/\varpi)^{\oplus r} \oplus N \to N$ in $\mathcal{C}^d$ given by the canonical inclusion $N \hookrightarrow (\mathcal{O}_K/\varpi)^{\oplus r} \oplus N$, by quotient $(\mathcal{O}_K/\varpi)^{\oplus r} \oplus N \twoheadrightarrow N$ respectively. Then we have

$$T_{n,r} = \frac{1}{\sharp \mathrm{GL}_r(\mathcal{O}_K/\varpi)} (r_r)_* m_r^* : F(\mathcal{O}_K/\varpi^n) \to F(\mathcal{O}_K/\varpi^n).$$

Corollary B.6. *In the notation of Lemma B.3, suppose that π_i is generic of conductor n_i for each $i = 1, \ldots, m$. Put $N = \mathcal{O}_K/\varpi^{n_1 + \cdots + n_m}$. Then $F'(N)$ is one-dimensional and for $i = 0, \ldots, d-1$, the eigenvalue of T_r on $F'(N)$ is equal to the sum*

$$\sum_{\substack{r = r_1 + \cdots + r_m, \\ r_i \le \max(d_i - n_i, d_i - 1)}} \frac{\prod_{i=1}^m q^{r_i (\sum_{1 \le j < i} d_i)}}{q^{\sum_{1 \le i < j \le m} r_i r_j}} T^{(d_1)}_{n_1, r_1} \otimes \cdots \otimes T^{(d_m)}_{n_m, r_m}.$$

Proof. Let $\mathrm{Fil}^\bullet N$ be the decreasing filtration of N defined by $\mathrm{Fil}^i N = N$ for $i \le 1$, $\mathrm{Fil}^i N = \varpi^{n_1 + \cdots n_{i-1}} N$ for $2 \le i \le m$, and $\mathrm{Fil}^i N = 0$ for $i \ge m+1$. Let $x \in \pi_0(h(N))$ be the connected component corresponding to this filtration. Then it is easily checked that

$$[F_1 \boxtimes \cdots \boxtimes F_m](h(N)) = [F_1 \boxtimes \cdots \boxtimes F_m](\mathrm{Fil}^1 N / \mathrm{Fil}^2 N, \ldots, \mathrm{Fil}^m N / \mathrm{Fil}^{m+1} N).$$

Hence $F'(N) = F_1(\mathrm{Gr}^1 N) \otimes \cdots \otimes F(\mathrm{Gr}^m N)$ is one-dimensional. Now let us compute the eigenvalue of $T_{n,r} = \frac{1}{\sharp \mathrm{GL}_r(\mathcal{O}_K/\varpi)}(r_r)_* m_r^*$ on $F'(N)$. The only involved connected components $\widetilde{x} \in \pi_0(h((\mathcal{O}_K/\varpi)^{\oplus r} \oplus N))$ are those which satisfy $\pi_0(h(m_r))(\widetilde{x}) = \pi_0(h(r_r))(\widetilde{x}) = x$. For each such $\widetilde{x}$, the filtration on $(\mathcal{O}_K/\varpi)^{\oplus r} \oplus N$ corresponding to $\widetilde{x}$ is the direct sum of a filtration on $(\mathcal{O}_K/\varpi)^{\oplus r}$ and the filtration $\mathrm{Fil}^\bullet N$ on N. Hence $(m_r)_* r_r^*$ is equal to

$$\sum_{\substack{r = r_1 + \cdots + r_m, \\ r_i \leq \max(d_i - n_i, d_i - 1)}} \frac{\sharp \mathrm{GL}_d(\mathcal{O}_K/\varpi) \prod_{i=1}^m q^{d_i(\sum_{i<j\leq m} r_j)}}{\prod_{i=1}^m \sharp \mathrm{GL}_{r_i}(\mathcal{O}_K/\varpi) \cdot q^{r_i(\sum_{i<j\leq m} r_j)}} \times ((r_{r_1}^{(d_1)})_* m_{r_1}^{(d_1)*}) \otimes \cdots \otimes ((r_{r_m}^{(d_m)})_* m_{r_m}^{(d_m)*}).$$

The assertion follows. □

Corollary B.7. *In the situation of Corollary B.6, suppose that π_i is a discrete series for each $i = 1, \ldots, m$. Let $S \subset \{1, \ldots, m\}$ be the subset defined by $S = \{i \mid L(\pi_i, s) \neq 1\}$. For $i \in S$, let a_i denote the eigenvalue of $T_{n_i,1}^{(d_i)}$ on $F_i(\mathcal{O}_K/\varpi^{n_i})$. Then for $i = 0, \ldots, d-1$, the eigenvalue of $T_{n,r}$ on $F'(N)$ is equal to*

$$q^{-r(r-1)/2} \sum_{\substack{S' \subset S, \\ \sharp S' = r}} \prod_{i \in S'} a_i q^{\sum_{1 \leq j < i} d_i}$$

□

B.5. Proof of Proposition B.1. By Corollary B.6 and Corollary B.7, the proof of Proposition B.1 is easily reduced to the case where π is an unramified twist of Steinberg representation, that is, π is isomorphic (up to an appropriate unramified twist) to the quotient of the algebraic parabolic induction $\mathrm{Ind}(1^{(1)} \times \cdots \times 1^{(1)})$ of d trivial representations $1^{(1)}$ of $\mathrm{GL}_1(K)$ to $\mathrm{GL}_d(K)$ by the canonical image of the direct sum

$$\begin{aligned}&\mathrm{Ind}(1^{(2)} \times 1^{(1)} \times \cdots \times 1^{(1)}) \oplus \mathrm{Ind}(1^{(1)} \times 1^{(2)} \times 1^{(1)} \times \cdots \times 1^{(1)}) \\ &\oplus \cdots \oplus \mathrm{Ind}(1^{(1)} \times \cdots \times 1^{(1)} \times 1^{(2)}).\end{aligned}$$

Let $\mathbb{C}_{\mathcal{FC}^1}, \mathbb{C}_{\mathcal{FC}^2}$ denote the constant sheaves on $\mathcal{FC}^1, \mathcal{FC}^2$ respectively. Put $F' = h^*[\mathbb{C}_{\mathcal{FC}^1} \times \cdots \times \mathbb{C}_{\mathcal{FC}^1}]$ and $F_2 = h^*[\mathbb{C}_{\mathcal{FC}^2} \times \mathbb{C}_{\mathcal{FC}^1} \times \cdots \times \mathbb{C}_{\mathcal{FC}^1}]$, $F_3 = h^*[\mathbb{C}_{\mathcal{FC}^1} \times \mathbb{C}_{\mathcal{FC}^2} \times \mathbb{C}_{\mathcal{FC}^1} \times \cdots \times \mathbb{C}_{\mathcal{FC}^1}], \ldots, F_d = h^*[\mathbb{C}_{\mathcal{FC}^1} \times \cdots \times \mathbb{C}_{\mathcal{FC}^1} \times \mathbb{C}_{\mathcal{FC}^2}]$. Then $F', F_2, \ldots, F_d$ are the sheaves on $\mathcal{FC}^d$ corresponding to $\mathrm{Ind}(1^{(1)} \times \cdots \times 1^{(1)})$, $\mathrm{Ind}(1^{(2)} \times 1^{(1)} \cdots \times 1^{(1)})$, …, $\mathrm{Ind}(1^{(1)} \times \cdots \times 1^{(1)} \times 1^{(2)})$, respectively. Hence the sheaf on $\mathcal{FC}^d$ corresponding to the Steinberg representation is (an appropriate unramified twist of) the quotient sheaf F of

$$\bigoplus_{i=2}^{d} F_i \to F'.$$

Let F'' denote the quotient presheaf of $\oplus_{i=2}^d F_i \to F'$. Then F'' is a sub-presheaf of F. It is known that $\mathcal{O}_K/\varpi^{d-1}$ is one-dimensional. First we show that $F''(\mathcal{O}_K/\varpi^{d-1})$ is also one-dimensional, that is, $F(\mathcal{O}_K/\varpi^{d-1}) = F''(\mathcal{O}_K/\varpi^{d-1})$. We set $S = \{2, \ldots, d\}$. By definition, $F(\mathcal{O}_K/\varpi^{d-1})$ is canonically identified with the direct sum

$$F(\mathcal{O}_K/\varpi^{d-1}) = \bigoplus_{\alpha : S \to \{0, \ldots, d-1\}} \mathbb{C},$$

where α runs over the non-decreasing map from S to $\{0,\dots,d-1\}$. Similarly for $i \in S$, $F_i(\mathcal{O}_K/\varpi^{d-1})$ is canonically identified with the direct sum

$$F_i(\mathcal{O}_K/\varpi^{d-1}) = \bigoplus_{\alpha_i : S-\{i\} \to \{0,\dots,d-1\}} \mathbb{C},$$

where α_i runs over the non-decreasing map from $S-\{i\}$ to $\{0,\dots,d-1\}$. For a map $\epsilon : S \to \{0,1\}$, let $\alpha_\epsilon : S \to \{0,\dots,d-1\}$ denote the non-decreasing map defined by $\alpha_\epsilon(i) = i-2+\epsilon(i)$. We also set $s(\epsilon) = (-1)^{\sum_i \epsilon(i)}$. Define the $\mathbb{C}$-linear map $\beta : F''(\mathcal{O}_K/\varpi^{d-1}) \to \mathbb{C}$ by sending $(c_\alpha)_\alpha$ to $\sum_\epsilon s(\epsilon) c_{\alpha_\epsilon}$. Then it is easily checked that for each $i \in S$ the composition $F_i(\mathcal{O}_K/\varpi^{d-1}) \to F(\mathcal{O}_K/\varpi^{d-1}) \xrightarrow{\beta} \mathbb{C}$ is zero. Hence $F''(\mathcal{O}_K/\varpi^{d-1})$ is at least one-dimensional.

Let $\epsilon_0 : S \to \{0\} \subset \{0,1\}$ be the constant map on S. Let $v \in F'(\mathcal{O}_K/\varpi^{d-1})$ the element whose $\alpha = \alpha_{\epsilon_0}$-component is 1 and whose $\alpha \neq \alpha_{\epsilon_0}$-component is 0. For $r = 1,\dots,d-1$. set $T_{d-1,r}(v) = (w_{r,\alpha})_\alpha$. We compute

$$C_r = \beta(T_{d-1,r}(v)) = \sum_\epsilon s(\epsilon) w_{r,\alpha_\epsilon}.$$

Among the functions of the form α_ϵ, α_{ϵ_0} is the function which takes the minimal value at each point on S. It follows from this that $C_r = w_{r,\alpha_{\epsilon_0}}$. It is easily checked that $w_{r,\alpha_{\epsilon_0}} = 0$ for $r \geq 2$ and $w_{1,\alpha_{\epsilon_0}} = 1$. This completes the proof of Proposition B.1. □

Appendix C. The Steinberg representation of GL_d and Iwahori-spherical Whittaker functions

by Seidai Yasuda

In this appendix, after recalling several basic facts on the Steinberg representation of $\mathrm{GL}_d(K)$, we give an explicit formula of the Whittaker functions of Iwahori-spherical vectors of the Steinberg representation of GL_d over a non-Archimedean local field. The result is used in Section 7.

C.1. Notations. In this appendix, we fix a positive integer $d \geq 1$. Let G denote the group scheme GL_d over $\mathrm{Spec}\,(\mathbb{Z})$. We use the following notations. Let $B \subset G$ denote the Borel subgroup of upper triangular matrices, $N \subset B$ denote its unipotent radical, and $M \subset B$ denote the Levi subgroup of diagonal matrices. We also let $N^- \subset G$ denote the group of lower triangular matrices whose diagonal entries are 1.

Let $W \subset G$ denote the constant subgroup scheme of permutation matrices. For an element σ in the d-th symmetric group S_d, let $w_\sigma = (\delta_{\sigma(i)j}) \in W(\mathbb{Z})$ denote the permutation matrix corresponding to σ. For $\sigma \in S_d$ we set

$$\ell(\sigma) = \sharp\{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i < j \leq d,\ \sigma(i) > \sigma(j)\}.$$

C.2. Basic facts on the Steinberg representation of $\mathrm{GL}_d(K)$. Let K be a non-Archimedean local field, $\mathcal{O}_K$ be its ring of integers, k be its residue field, and q denote the cardinality of k. Let $|\ | : K \to \mathbb{C}$ (resp. $\mathrm{ord} : K^\times \to \mathbb{Z}$) denote the non-Archimedean absolute value (resp. the normalized valuation) of K.

C.2.1. Let us consider the Steinberg representation St of $G(K)$. First we recall its definition and its basic properties. Let $\delta_B : B(K) \to \mathbb{R}^\times$ denote the modular character of $B(K)$. Explicitly, the character δ_B sends $b = (b_{ij}) \in B(K)$ to $|b_{11}|^{d-1}|b_{22}|^{d-3} \cdots |b_{dd}|^{-d+1}$. By definition, St is a unique irreducible subrepresentation of

$$\mathrm{Ind}_{B(K)}^{G(K)} \delta_B^{\frac{1}{2}} = \{\phi : G(K) \to \mathbb{C} \mid \phi(bg) = \delta_B(b)\phi(g) \text{ for } b \in B(K), g \in G(K)\}.$$

Let $\mathcal{I} \subset G(\mathcal{O}_K)$ denote the Iwahori subgroup. It is well-known that $\mathrm{St}^{\mathcal{I}}$ is a one-dimensional $\mathbb{C}$-vector space.

C.2.2. By Iwasawa decomposition $G(K) = B(K)G(\mathcal{O}_K)$, the space $\mathrm{Ind}_{B(K)}^{G(K)} \delta_B^{\frac{1}{2}}$ is canonically isomorphic to the space of $\mathbb{C}$-valued functions on the coset $B(\mathcal{O}_K)\backslash G(\mathcal{O}_K)$ and this isomorphism is compatible with $G(\mathcal{O}_K)$-action. In particular we have an isomorphism $\Phi : (\mathrm{Ind}_{B(K)}^{G(K)} \delta_B^{\frac{1}{2}})^{\mathcal{I}} \xrightarrow{\cong} \mathrm{Map}(B(\mathcal{O}_K)\backslash G(\mathcal{O}_K)/\mathcal{I}, \mathbb{C})$. We note that $G(\mathcal{O}_K) = \coprod_{w \in W(K)} B(\mathcal{O}_K)w\mathcal{I}$ by Iwahori factorization.

Lemma C.1. *The image $\Phi(\pi^{\mathcal{I}})$ is, as a $\mathbb{C}$-vector space, generated by the function $\phi_0 : B(\mathcal{O}_K)\backslash G(\mathcal{O}_K)/\mathcal{I} \to \mathbb{C}$ which sends $B(\mathcal{O}_K)w_\sigma\mathcal{I}$ to $\mathrm{sgn}(\sigma)q^{-\ell(\sigma)}$ for each $\sigma \in S_d$.*

Proof. For $i = 1, \ldots, d-1$, let $P_i \supset B$ be the standard parabolic subgroup corresponding to the partition $(\overbrace{1, \ldots, 1}^{i-1}, 2, \overbrace{1, \ldots, 1}^{d-1-i})$ of d. Let $\delta_{P_i} : P_i(K) \to \mathbb{R}^\times$ denote the modular character of $P_i(K)$. Explicitly, δ_{P_i} sends $(p_{jk}) \in P_i(K)$ to

$$|p_{11}|^{d-1} \cdots |p_{i-1,i-1}|^{d-2i+1}|p_{ii}p_{i+1,i+1} - p_{i,i+1}p_{i+1,i}|^{d-2i}|p_{i+2,i+2}|^{d-2i-3} \cdots |p_{dd}|^{-d+1}.$$

It is known (cf. [Ca]) that St is equal to the kernel of the canonical homomorphism

$$\mathrm{Ind}_{B(K)}^{G(K)} \delta_B^{\frac{1}{2}} \to \bigoplus_{i=1}^{d-1} \mathrm{Ind}_{P_i(K)}^{G(K)} \delta_{P_i}^{\frac{1}{2}}.$$

Let $\mathcal{I}_1 \subset G(\mathcal{O}_K)$ denote the subgroup of the matrices which are congruent to 1 modulo the maximal ideal of $\mathcal{O}_K$. Given a function $f : B(\mathcal{O}_K)\backslash G(\mathcal{O}_K)/\mathcal{I}_1 \to \mathbb{C}$, we define for each $i = 1, \ldots, d-1$ a function $p_i(f) : P_i(\mathcal{O}_K)\backslash G(\mathcal{O}_K)/\mathcal{I}_1 \to \mathbb{C}$ as follows: for $y \in P_i(\mathcal{O}_K)\backslash G(\mathcal{O}_K)/\mathcal{I}_1$, we put

$$p_i(f)(y) = \sum_{x \in B(\mathcal{O}_K)\backslash G(\mathcal{O}_K)/\mathcal{I}_1, x \mapsto y} f(x).$$

By Corollary B.4, there is a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{Ind}_{B(K)}^{G(K)} \delta_B^{\frac{1}{2}} \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ind}_{P_i(K)}^{G(K)} \delta_{P_i}^{\frac{1}{2}} \arrow[d, "\cong"] \\
\mathrm{Map}(B(\mathcal{O}_K)\backslash G(\mathcal{O}_K)/\mathcal{I}_1, \mathbb{C}) \arrow[r, "p_i"] & \mathrm{Map}(P_i(\mathcal{O}_K)\backslash G(\mathcal{O}_K)/\mathcal{I}_1, \mathbb{C})
\end{tikzcd}$$

Therefore $\mathrm{St}^{\mathcal{I}}$ is isomorphic to the kernel of

$$(p_1, \ldots, p_{d-1}) : \mathrm{Map}(B(\mathcal{O}_K)\backslash G(\mathcal{O}_K)/\mathcal{I}, \mathbb{C}) \to \bigoplus_{i=1}^{d-1} \mathrm{Map}(P_i(\mathcal{O}_K)\backslash G(\mathcal{O}_K)/\mathcal{I}, \mathbb{C}).$$

which is, as is easily checked, generated by the function $B(\mathcal{O}_K)\backslash G(\mathcal{O}_K)/\mathcal{I} \to \mathbb{C}$ which sends $B(\mathcal{O}_K)w_\sigma G(\mathcal{O}_K)$ to $\frac{\mathrm{sgn}(\sigma)}{\sharp B(k)w_\sigma B(k)/B(k)}$. Since $\sharp B(k)w_\sigma B(k)/B(k) = q^{\ell(\sigma)}$ by Bruhat decomposition, we have the assertion. $\square$

C.2.3. *Action of the Iwahori-spherical Hecke algebra.* Let $\mathcal{H} = \mathcal{H}(G(K), \mathcal{I})$ denote the convolution algebra of the $\mathcal{I}$-biinvariant compactly supported $\mathbb{C}$-valued functions on $G(K)$. As a Haar measure of $G(K)$, we take the one with $\mathrm{vol}(\mathcal{I}) = 1$, so that the characteristic function of $\mathcal{I}$ is the unit of $\mathcal{H}$. For $g \in G(K)$, the characteristic function of the subset $\mathcal{I}g\mathcal{I} \subset G(K)$ is an element in $\mathcal{H}$ which, by abuse of notation, we also denote by $\mathcal{I}g\mathcal{I}$. The algebra $\mathcal{H}$ acts on the one-dimensional space $\mathrm{St}^{\mathcal{I}}$, which yields a $\mathbb{C}$-algebra homomorphism $\nu : \mathcal{H} \to \mathbb{C}$.

Lemma C.2 (cf. [Iw-Ma, §3], [Sh])**.** *For $\sigma \in S_d$, we have $\nu(\mathcal{I}w_\sigma\mathcal{I}) = \mathrm{sgn}(\sigma)$.*

Proof. Let $f \in \mathrm{St}^{\mathcal{I}}$ be the element corresponding to the function ϕ_0 in Lemma C.1 via the isomorphism Φ. Since ϕ_0 takes the constant value $\mathrm{sgn}(\sigma)q^{-\ell(\sigma)}$ on $\mathcal{I}w_\sigma\mathcal{I}$, we have

$$\Phi(\mathcal{I}w_\sigma\mathcal{I} \cdot f)(1) = \mathrm{vol}(\mathcal{I}w_\sigma\mathcal{I})sgn(\sigma)q^{-\ell(\sigma)} = sgn(\sigma).$$

Hence $\nu(\mathcal{I}w_\sigma\mathcal{I}) = \mathrm{sgn}(\sigma)$. $\square$

C.3. Explicit formula of the Iwahori-spherical Whittaker functions. From now on we fix a non-trivial additive character $\psi : K \to \mathbb{C}^\times$ of conductor 0. Let $N \subset B$ denote the unipotent radical of B. Let $\psi_N : N(K) \to \mathbb{C}^\times$ denote the character defined by $\psi_N((n_{ij}) = \psi(\sum_{i=1}^{d-1} n_{i,i+1})$.

It is well-known that there is an injective $G(K)$-homomorphism from $\mathrm{Ind}_{B(K)}^{G(K)}\delta_B^{\frac{1}{2}}$ to the space $\mathrm{Ind}_{N(K)}^{G(K)}\psi_N$ of functions $\varphi : G(K) \to \mathbb{C}$ satisfying $\varphi(ng) = \psi_N(n)\varphi(g)$ for any $n \in N(K)$, $g \in G(K)$, and such a homomorphism is unique up to a non-zero scalar. We say that a non-zero element $\mathrm{Wh} \in \mathrm{Ind}_{N(K)}^{G(K)}\psi_N$ is an *Iwahori-spherical Whittaker function* for St if it belongs to the image of $\mathrm{St}^{\mathcal{I}} \subset \mathrm{St} \subset \mathrm{Ind}_{B(K)}^{G(K)}\delta_B^{\frac{1}{2}}$.

For $\sigma \in S_d$, let $M(K)_\sigma^- \subset M(K)$ denote the subset of elements of the form $\mathrm{diag}(m_1, \ldots, m_d)$ with $\mathrm{ord}(m_i) + 1 \geq \mathrm{ord}(m_{i+1})$ for $1 \leq i \leq d$ and $\mathrm{ord}(m_i) \geq \mathrm{ord}(m_{i+1})$ for $1 \leq i \leq d$ with $\sigma(i) < \sigma(i+1)$.

Proposition C.3. *There is a unique Iwahori-spherical Whittaker function* $\mathrm{Wh}_1 \in \mathrm{Ind}_{N(K)}^{G(K)}\psi_N$ *for* St *with* $\mathrm{Wh}_1(1) = 1$. *For $m = \mathrm{diag}(m_1, \cdots, m_d) \in M(K)$ and for $\sigma \in S_d$, we have*

$$\mathrm{Wh}_1(mw_\sigma) = \begin{cases} \mathrm{sgn}(\sigma)q^{-\ell(\sigma)}\delta_B(m), & \textit{if } m \in M(K)_\sigma^-, \\ 0, & \textit{otherwise.} \end{cases} \tag{C.1}$$

There is another description of the set $M(K)_\sigma$. Put

$$S = \{(n_{ij}) \in N^-(K) \mid \mathrm{ord}(n_{ij}) \geq i - j \text{ for } 1 \leq j \leq i \leq d\}.$$

Then

$$M(K)_\sigma^- = \{m \in M(K) \mid (mw_\sigma)^{-1}S(mw_\sigma) \subset \mathcal{I}\}.$$

By [Li, Theorem 4.1], there is a unique Iwahori-spherical Whittaker function Wh_1 with $\mathrm{Wh}_1(1) = 1$ which satisfies the formula (C.1) for any $m \in M(K)$ and $\sigma = 1$. We will check the formula (C.1) for general σ in several steps.

Lemma C.4. *Let $m \in M(K)_1^-$ and let $\sigma \in S_d$. Then we have* $\mathrm{Wh}_1(mw_\sigma) = \mathrm{sgn}(\sigma)q^{-\ell(\sigma)}\delta_B(m)$, *that is, the formula (C.1) is valid for $m \in M(K)_1^-$.*

Proof. Since $m \in M(K)_1^-$, we have $mN(\mathcal{O}_K)m^{-1} \subset \mathrm{Ker}\,\psi_N$. Since $\mathcal{I}w_\sigma\mathcal{I} = N(\mathcal{O}_K)w_\sigma\mathcal{I}$, we have $\mathrm{Wh}_1(mg) = \mathrm{Wh}_1(mw_\sigma)$ for any $g \in \mathcal{I}w_\sigma\mathcal{I}$. Since $\nu(\mathcal{I}w_\sigma\mathcal{I}) = \mathrm{sgn}(\sigma)$ by Lemma C.2, we have

$$\mathrm{vol}(\mathcal{I}w_\sigma\mathcal{I})\mathrm{Wh}_1(mw_\sigma) = \mathrm{sgn}(\sigma)\mathrm{Wh}_1(m).$$

Since $\mathrm{vol}(\mathcal{I}w_\sigma\mathcal{I}) = \sharp(\mathcal{I}w_\sigma\mathcal{I}/\mathcal{I}) = \sharp(B(k)w_\sigma B(k)/B(k))$ and $\sharp(B(k)w_\sigma B(k)/B(k)) = q^{\ell(\sigma)}$ by Bruhat decomposition, the assertion follows. □

Let $\sigma_l \in S_d$ denote the longest element: $\sigma_l(i) = d+1-i$ for $i = 1, \ldots, d$.

Lemma C.5. *The formula (C.1) is valid if it is valid for $\sigma = \sigma_l$.*

Proof. Since $\nu(\mathcal{I}w_{\sigma^{-1}\sigma_l}\mathcal{I}) = \mathrm{sgn}(\sigma^{-1}\sigma_l)$ by Lemma C.2, we have

$$\int_{\mathcal{I}w_{\sigma^{-1}\sigma_l}\mathcal{I}} \mathrm{Wh}_1(mw_\sigma g)dg = \mathrm{sgn}(\sigma^{-1}\sigma_l)\mathrm{Wh}_1(mw_\sigma).$$

Let N_σ denote the subgroup $w_\sigma N w_\sigma^{-1} \cap N$ of G. By Bruhat decomposition, the left hand side is equal to

$$\sum_{n \in N_\sigma(k)} \mathrm{Wh}_1(ms(n)w_{\sigma_l})$$

for any set theoretic section s of the canonical surjection $N_\sigma(\mathcal{O}_K) \twoheadrightarrow N_\sigma(k)$. Since

$$\begin{aligned}\mathrm{Wh}_1(ms(n)w_{\sigma_l}) &= \mathrm{Wh}_1(ms(n)m^{-1} \cdot mw_{\sigma_l}) \\ &= \psi_N(ms(n)m^{-1})\mathrm{Wh}_1(mw_{\sigma_l}),\end{aligned}$$

we have

$$\int_{\mathcal{I}w_{\sigma^{-1}\sigma_l}\mathcal{I}} \mathrm{Wh}_1(mw_\sigma g)dg = \frac{\sharp N_\sigma(k)}{\mathrm{vol}(N_\sigma(\mathcal{O}_K))}\int_{N_\sigma(\mathcal{O}_K)} \psi_N(mnm^{-1})dn \cdot \mathrm{Wh}_1(mw_{\sigma_l}).$$

Let $M(K)'_\sigma \subset M(K)$ denote the subset

$$M(K)'_\sigma = \{m \in M(K) \mid mN_\sigma(\mathcal{O}_K)m^{-1} \subset \mathrm{Ker}(\psi_N)\}.$$

Then we have

$$\mathrm{sgn}(\sigma)\mathrm{Wh}_1(mw_\sigma) = \begin{cases} \mathrm{sgn}(\sigma_l)\sharp N_\sigma(k)\mathrm{Wh}_1(mw_{\sigma_l}), & \text{if } m \in M(K)'_\sigma \\ 0, & \text{otherwise.}\end{cases} \tag{C.2}$$

Hence the assertion follows from $\sharp N_\sigma(k) = q^{\ell(\sigma_l)-\ell(\sigma)}$ and $M(K)^-_{\sigma_l} \cap M(K)'_\sigma = M(K)^-_\sigma$, which are easily checked. □

Let $\sigma_c \in S_d$ be the cyclic permutation: $\sigma_c(i) = i+1$ for $i = 1, \ldots, d-1$ and $\sigma_c(d) = 1$. Take a uniformizer ϖ of K. For $i = 0, \cdots, d$, let $h_i \in G(K)$ be the element

$$h_i = w_{\sigma_c^i}\Pi_i = (w_{\sigma_c}\mathrm{diag}(\varpi, 1, \ldots, 1))^i,$$

where $\Pi_i = \mathrm{diag}(\overbrace{\varpi, \ldots, \varpi}^{i}, \overbrace{1, \ldots, 1}^{d-i}) \in M(K)$.

Lemma C.6. *We have $\nu(\mathcal{I}h_i\mathcal{I}) = (-1)^{i(d-1)}$.*

Proof. Since $h_i\mathcal{I}h_i^{-1} = \mathcal{I}$, the double coset $\mathcal{I}h_i\mathcal{I}$ consists of the single right coset $h_i\mathcal{I}$. Hence we have $\mathrm{Wh}_1(gh_i) = \nu(\mathcal{I}h_i\mathcal{I})\mathrm{Wh}_1(g)$ for any $g \in G(K)$. Substituting $g = w_{\sigma_c^{-i}}$, we have

$$\mathrm{Wh}_1(\Pi_i) = \nu(\mathcal{I}h_i\mathcal{I})\mathrm{Wh}_1(w_{\sigma_c^{-i}}).$$

Applying Lemma C.4, we have $\delta_B(\Pi_i) = \nu(\mathcal{I}h_i\mathcal{I})\mathrm{sgn}(\sigma_c^{-i})q^{-\ell(\sigma_c^{-i})}$, whence the assertion follows. □

The following corollary, together with Proposition B.1 is used in Section 5.2 and Corollary of Section 7.

Corollary C.7. *Let $\mathbb{K} \subset \mathcal{I}$ denote the subgroup of matrices $(g_{ij}) \in G(\mathcal{O}_K)$ satisfying $g_{ij} \bmod \varpi = \delta_{ij}$ for $1 \le i \le d$, $1 \le j \le d-1$. Let $\mathcal{H}(G(K), \mathbb{K})$ denote the convolution algebra (with respect to the Haar measure of $G(K)$ with $\mathrm{vol}(\mathbb{K}) = 1$) of the $\mathbb{K}$-biinvariant compactly supported $\mathbb{C}$-valued functions on $G(K)$. Let (π, V) be a smooth representation of $G(K)$ which is isomorphic to an unramified twist of* St. *Let $T \in \mathcal{H}(G(K), \mathbb{K})$ denote the characteristic function of $\mathbb{K}\mathrm{diag}(1, \ldots, 1, \varpi)\mathbb{K}$. Then for $v \in V^{\mathcal{I}} \subset V^{\mathbb{K}}$, we have*

$$(1 - q^{-s}T)v = L(\pi, s - \frac{d-1}{2})^{-1}v,$$

where $L(\pi, s)$ is the local L-factor of π.

Proof. We may assume that $V = \mathrm{St}$. Since $\mathbb{K}/(\mathbb{K} \cap \Pi_1\mathbb{K}\Pi_1^{-1}) \cong \mathcal{I}/(\mathcal{I} \cap \Pi_1\mathcal{I}\Pi_1^{-1}) \cong (\mathcal{O}_K/\varpi)^{\oplus d-1}$, the canonical map $\mathbb{K}\Pi_1\mathbb{K}/\mathbb{K} \to \mathcal{I}\Pi_1\mathcal{I}/\mathcal{I}$ is bijective. Hence we have

$$Tv = \mathcal{I}\Pi_1\mathcal{I}.v = \mathcal{I}w_{\sigma_c^{-1}}\mathcal{I}.h_1.v = v,$$

whence the assertion follows. □

Proof of Proposition C.3. By Lemma C.5, it suffices to prove the formula (C.1) for $\sigma = \sigma_l$.

Let $m = \mathrm{diag}(m_1, \ldots, m_d)$ be an element in $M(K)$. By Lemma C.6, we have $\mathrm{Wh}_1(mw_{\sigma_l}h_i) = (-1)\ i(d-1)\mathrm{Wh}_1(mw_{\sigma_l})$ for $i = 0, \ldots, d-1$. On the other hand, since $mw_{\sigma_l}h_i = m\Pi_i w_{\sigma_l\sigma_c^i}$, we have

$$\mathrm{Wh}_1(mw_{\sigma_l}h_i) = \begin{cases} \mathrm{sgn}(\sigma_c^i)\sharp N_{\sigma_l\sigma_c^i}(k)\mathrm{Wh}_1(m\Pi_i w_{\sigma_l}), & \text{if } m\Pi_i \in M(K)'_{\sigma_l\sigma_c^i}, \\ 0 & \text{otherwise,} \end{cases}$$

by (C.2), where $N_{\sigma_l\sigma_c^i}$ and $M(K)'_{\sigma_l\sigma_c^i}$ are as in the proof of Lemma C.5. Therefore,

$$\mathrm{Wh}_1(mw_{\sigma_l}) = \begin{cases} \delta_B(\Pi_i)^{-1}\mathrm{Wh}_1(m\Pi_i w_{\sigma_l}), & \text{if } \mathrm{ord}(m_i) + 1 \ge \mathrm{ord}(m_{i+1}), \\ 0 & \text{otherwise,} \end{cases}$$

whence the assertion inductively follows from Lemma C.4 for $\sigma = \sigma_l$. □

References

[An-Fu] F. W. Anderson, K. R. Fuller, *Rings and categories of modules*, Graduate Texts in Mathematics **13**, New York: Springer-Verlag, (1992)

[Be] A. A. Beilinson, *Higher regulators and values of L-functions*, in Current problems in mathematics **24**, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 181–238 (1984)

[Bo-Gu-Re] S. Bosch, U. Güntzer, R. Remmert, *Non-Archimedean analysis*, Grundlehren der Mathematischen Wissenschaften **261**, Berlin etc.: Springer Verlag. (1984)

[Br-Ti] F. Bruhat, J. Tits, *Groupes reductifs sur un corps local*, Publ. Math., Inst. Hautes Étud. Sci. **41**, 5–251 (1972)

[Be-Ze] I. N. Bernstein, A. V. Zelevinsky, *Induced representations of reductive p-adic groups. I*, Ann. Sci. Éc. Norm. Supér., IV. Sér. **10**, 441–472 (1977)

[Ca] W. Casselman, *The Steinberg character as a true character*, in Harmonic Analysis on homog. Spaces, Proc. Sympos. Pure Math. **26**, Williamstown 1972, 413–417 (1973)

[Cog] J. W. Cogdell, *Lectures on L-functions, converse theorems, and functoriality for* GL_n, in Lectures on Automorphic L-Functions, Fields Institute Monographs **20**, 1–96 (2004)

[Con] C. Consani, *Double complexes and Euler L-factors*, Compos. Math. **111**, No.3, 323-358 (1998)

[De-Hu] P. Deligne, D. Husemoller, *Survey of Drinfel'd modules*, in Current trends in arithmetical algebraic geometry, Proc. Summer Res. Conf., Arcata/Calif. 1985, Contemp. Math. **67** 25–91 (1987)

[Dr] V. G. Drinfeld, *Elliptic modules*, Math. USSR, Sb. **23**, 561–592 (1974); translation from Mat. Sb., n. Ser. 94(136), 594-627 (1974)

[Ge] A. Genestier, *Espaces symétriques de Drinfled*, Astérisque **234** Paris:Société Mathématique de France, (1996)

[Gi1] H. Gillet, *Riemann-Roch theorems for higher algebraic K-theory*, Adv. Math. **40**, 203–289 (1981)

[Gi2] H. Gillet, *Intersection theory on algebraic stacks and Q-varieties*, J. Pure Appl. Algebra **34**, 193–240 (1984)

[Go-Ja] R. Godement, H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Mathematics. **260**, Berlin-Heidelberg-New York: Springer-Verlag (1972)

[Gr-Ro] B. Gross, M. Rosen, *Fourier series and the special values of L-functions*, Adv. Math. **69**,1–31 (1988)

[Iw] N. Iwahori, *Generalized Tits system (Bruhat decompostition) on p-adic semisimple groups*, in Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math. **9**, 71–83 (1966)

[Iw-Ma] N. Iwahori, H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of $\mathfrak{p}$-adic Chevalley groups*, Publ. Math., Inst. Hautes Étud. Sci. **25**, 5–48 (1965)

[Ja-Pi-Sh] H. Jacquet, I. I. Piatestski-Shapiro, J. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. **256**, 199–214 (1981)

[Ka-Ch] V. Kac, P. Cheung, *Quantum calculus*, Universitext, New York, NY: Springer (2002)

[Ka] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, in Cohomologies p-adiques et applications arithmétiques (III), Astérisque **295**, 117–290 (2004)

[Ko] S. Kondo, *Kronecker limit formula for Drinfeld modules*, J. Number Theory **104**, No.2, 373-377 (2004)

[La] G. Laumon, *Cohomology of Drinfeld modular varieties. Part I*, Cambridge Studies in Advanced Mathematics. **41**, Cambridge: Cambridge University Press. (1996)

[Li] J.-S. Li, *Some results on the unramified principal series of p-adic groups*, Math. Ann. **292**, No.4, 747-761 (1992)

[Ma] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, **5**. New York etc.: Springer-Verlag. (1988)

[Qu] D. Quillen, *Higher algebraic K-theory. I*, Algebr. K-Theory I, Proc. Conf. Battelle Inst. 1972, Lect. Notes Math. 341, 85-147 (1973)

[Ru] K. Rubin, *Euler systems*, Annals of Mathematics Studies. **147**. Princeton, NJ: Princeton University Press. (2000)

[Sh] J. A. Shalika, *On the space of cusp forms of a $\mathfrak{p}$-adic Chevalley group*, Ann. of Math. (2) **92**, 262-278 (1970)

[Shi] T. Shintani, *On an explicit formula for class-1 "Whittaker functions" on GL_n over $\mathfrak{P}$-adic fields*, Proc. Japan Acad. **52**, 180-182 (1976)

[Sr] R. Sreekantan, *Non-Archimedean regulator maps and special values of L-functions*, preprint, math.NT/0309004.

[Wa] F. Waldhausen, *Algebraic K-theory of generalized free products. I, II*, Ann. of Math. (2) **108**, no. 1, 135-204 (1978)

[Ze] A. V. Zelevinsky, *Induced representations of reductive p-adic groups. II*, Ann. Sci. Éc. Norm. Supér., IV. Sér. **13**, 165-210 (1980)

[SGA1] A. Grothendieck, M. Raynaud, *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris) **3**, Paris: Société Mathématique de France. (2003)

[SGA4] M. Artin, A. Grothendieck, J. L. Verdier, *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, Lecture Notes in Mathematics **269**, Berlin-Heidelberg-New York: Springer-Verlag. (1972)

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