ON SOME NONCOMMUTATIVE ALGEBRAS RELATED WITH K-THEORY OF FLAG VARIETIES, I

ANATOL N. KIRILLOV AND TOSHIAKI MAENO

Abstract

For any Lie algebra of classical type or type G_2 we define a K-theoretic analog of Dunkl's elements, the so-called truncated *Ruijsenaars-Schneider-Macdonald elements*, RSM-elements for short, in the corresponding *Yang-Baxter group*, which form a commuting family of elements in the latter. For the root systems of type A we prove that the subalgebra of the *bracket algebra* generated by the RSM-elements is isomorphic to the Grothendieck ring of the flag variety. In general, we prove that the subalgebra generated by the *images* of the RSM-elements in the corresponding *Nichols-Woronowicz algebra* is canonically isomorphic to the Grothendieck ring of the corresponding flag varieties of classical type or of type G_2. In other words, we construct the "Nichols-Woronowicz algebra model" for the Grothendieck Calculus on Weyl groups of classical type or type G_2, providing a partial generalization of some recent results by Y. Bazlov. We also give a conjectural description (theorem for type A) of a commutative subalgebra generated by the *truncated RSM-elements* in the bracket algebra for the classical root systems. Our results provide a proof and generalizations of recent conjecture and result by C. Lenart and A. Yong for the root system of type A.

1 Introduction

In the paper [2] S. Fomin and the first author have introduced a model for the cohomology ring of flag varieties of type A as a commutative subalgebra generated by the so-called truncated Dunkl elements in a certain (noncommutative) quadratic algebra. This construction has been generalized to other root systems in [5]. The main purpose of the present paper is to construct a K-theoretic analog of these constructions. More specifically, we introduce certain families of pairwise commuting elements in the Yang-Baxter group $\mathcal{GK}(B_n)$ or in the bracket algebra $B\mathcal{E}(B_n)$, which conjecturally generate commutative subalgebras in the bracket algebra $B\mathcal{E}(B_n)$ isomorphic to the Grothendieck ring of the flag varieties of type B_n. The corresponding results/conjectures for the flag varieties of other classical type root systems can be obtained from those for the type B after certain specializations. There exists the natural surjective homomorphism [1] from the algebra $B\mathcal{E}(B_n)$ to the Nichols-Woronowicz

[1]It is believed that for a simply–laced (finite) Coxeter system (W, S) the corresponding bracket algebra $BE(W, S)$ and the Nichols–Woronowicz algebra $\mathcal{B}_{W,S}$ are isomorphic as braided Hopf algebras. However, this is not the case in a non simply–laced case. For example, if $n \geq 3$, the natural epimorphism $B\mathcal{E}(B_n) \longrightarrow \mathcal{B}_{B_n}$ has a non-trivial kernel in degree 6. In fact, $Hilb(B\mathcal{E}(B_n), t) - Hilb(\mathcal{B}_{B_n}, t) = 4t^6 + \cdots$.

algebra $\mathcal{B}_{B_n}$ of type B. One of our main results of the present paper states that the *image* of our construction in the Nichols-Woronowicz algebra $\mathcal{B}_{B_n}$ is indeed isomorphic to the Grothendieck ring of the flag variety of type B_n. We also present a similar construction for the root system of type G_2. These results can be viewed as a multiplicative analog/generalization for classical root systems and for G_2 of the "Nichols-Woronowicz algebra model" for the cohomology ring of flag varieties which has been constructed recently by Y. Bazlov [1].

In a few words the main idea behind the constructions of the paper can be described as follows. As it was mentioned, in [2] for type A and in [5] for other root systems, a realization of the (small quantum) cohomology ring of flag varieties has been invented. More specifically, the papers mentioned above present a model for the cohomology ring of flag varieties as a commutative subalgebra generated by the so-called Dunkl elements in a certain (noncommutative) algebra. The main ingredient of this construction is based on some very special solutions to the *classical* Yang-Baxter equation (for type A) and *classical* reflection equations (for types B, C and G_2). Our original motivation was to study the related algebras and groups which correspond to the "quantization" of the solutions to the classical Yang-Baxter type equations mentioned above, in connection with classical and quantum Schubert and Grothendieck Calculi. In more detail, we define the group of "local set-theoretical solutions" to the quantum Yang-Baxter equations of type B_n or of type G_2, together with the distinguished set of pairwise commuting elements in the former, the so-called truncated *Ruijsenaars-Schneider-Macdonald elements*. The latter is a relativistic or multiplicative generalization of the Dunkl elements. For applications to the K-theory, we specialize the general construction to the bracket algebra $B\mathcal{E}(B_n)$ and algebra $B\mathcal{E}(G_2)$.

Summarizing, the main construction of our paper presents a conjectural description of the Grothendieck ring $K(G/B)$ corresponding to flag varieties G/B of classical types (or G_2-type) to be a commutative subalgebra in the corresponding bracket algebra generated by the truncated RSM-elements. To be more specific, we construct in the algebra $B\mathcal{E}(B_n)$ a pairwise commuting family of elements, *multiplicative or relativistic Dunkl elements*, and state a conjecture about the complete list of relations among the latter.

Using some properties of the Chern homomorphism, we prove our conjecture for the root systems of type A. To our best knowledge, for the root systems of type A a similar description of the Grothendieck ring was given by C. Lenart and A. Yong [9], [10], however without reference to the Yang-Baxter theory.

The main problem to prove relations between the RSM-elements in the bracket algebra $B\mathcal{E}(B_n)$ appears to be show that the intersection of kernels of all the "braided derivations" $\Delta_{ij}, \Delta_{\overline{ij}}$ $1 \le i < j \le n$, and Δ_i, $1 \le i \le n$, acting on the algebra $B\mathcal{E}(B_n)$ contains only constants, see Section 5. At this point we pass to the Nichols-Woronowicz algebra $\mathcal{B}_{B_n}$ where the corresponding property of the braided derivations is guaranteed, [1]. Since as mentioned above, there exists the natural epimorphism of braided Hopf algebras $\mathcal{B}_{B_n} \longrightarrow B\mathcal{E}(B_n)$, to check the corresponding relations in the Nichols-Woronowicz algebra $\mathcal{B}_{B_n}$ seems to be a good step to confirm our conjectures. To prove the needed relations in the algebra $\mathcal{B}_{B_n}$, we develop a multiplicative analog/generalization of the Nichols-Woronowicz algebra model for cohomology ring of flag varieties recently introduced by Y. Bazlov [1].

Let us describe briefly the content of our paper.

Section 2 is devoted to a general construction of commuting family of elements in the group $\mathcal{GK}(B)$ generated by local set-theoretical solutions to the family of quantum Yang-Baxter equations of type B, see Definition 2.1 for precise formulation. This construction lies at the heart of our approach. In the case of type A (i.e. if $g_{ij} = h_i = 1$ for all i and j) and the Calogero-Moser representation (i.e. $h_{ij} = 1 + \partial_{ij}$) of the bracket algebra $BE(A_{n-1})$, the elements $\Theta_1^{A_{n-1}}, \cdots, \Theta_n^{A_{n-1}}$ correspond to the (rational) truncated (i.e. without *differential* part) Ruijsenaars-Schneider-Macdonald operators. It seems an interesting problem to classify all irreducible finite dimensional representations of the groups $\mathcal{GK}(X)$, $(X = A_{n-1}, B_n, ...)$ together with a simultaneous diagonalization of the operators $\Theta_1^X, \cdots, \Theta_n^X$ in these representations.

In Section 3 we apply the result of Section 2 (Key Lemma) to construct some distinguished multiplicative analogue $\Theta_j^{A_{n-1}} := \Theta_j^{A_{n-1}}(x)$ of the Dunkl elements $\theta_j^{A_{n-1}}$, $1 \le j \le n$, in the bracket algebra $BE(A_{n-1})$. It happened that our elements $\Theta_j^{A_{n-1}}$ coincide with the K-theoretic Dunkl elements $1 - \kappa_j$ introduced by C. Lenart and A. Yong in [9]. Proof of the statement that the elements $\kappa_1, \cdots, \kappa_n$ form a family of pairwise commuting elements in the algebra $BE(A_{n-1})$ given in [9], appears to be quite long and involved. On the other hand, our "Yang-Baxter approach" enables us to give a simple and transparent proof that the elements $\Theta_j^{A_{n-1}}$ mutually commute, as well as to describe relations among these commuting elements in the algebra $BE(A_{n-1})$. On this way we come to the main result of Section 3, namely

Theorem A *The subalgebra in $BE(A_{n-1})$ generated by the elements $\Theta_j^{A_{n-1}}$, $1 \le j \le n$, is isomorphic to the Grothendieck ring of the flag varieties of type A.*

In particular,

Theorem B *The following identity in the algebra $BE(A_{n-1})$ holds:*

$$\sum_{j=1}^{n} (\Theta_j^{A_{n-1}}(x))^k = n$$

for any $k \in \mathbb{Z}$.

Theorem A was stated as Conjecture 3.4 in [9]. We also state a positivity conjecture as Conjecture 3.13, which relates the elements $\Theta_j^{A_{n-1}}$ to the Grothendieck Calculus on the group GL_n. This conjecture is a restatement of non-negativity conjectures from [2], Conjecture 8.1, and [9], Conjecture 3.2., in our setting.

It should be emphasized that there are a lot of possibilities to construct a mutually commuting family of elements in the algebra $BE(A_{n-1})$ which generate a subalgebra isomorphic to the Grothendieck ring $K(\mathcal{F}l_n)$. For example one can take the elements $E_1 := \exp(\theta_1^{A_{n-1}}), \cdots, E_n := \exp(\theta_n^{A_{n-1}})$. It is easy to see that $E_j \ne \Theta_j$ for all j, however connections between Grothendieck polynomials and the elements $E_1, \cdots, E_n$ are not clear for the authors.

Our method to describe the relations between the elements $\Theta_j^{A_{n-1}}$ is based on the study of the Chern homomorphism which relates the K-theory to the cohomology theory of flag varieties, and moreover, on description of the *commutative quotient* of the algebra $BE(A_{n-1})$, see Subsection 3.2.

In Section 4 we study the B_n-case. First of all we introduce a modified version $B\mathcal{E}(B_n)$ of the algebra $BE(B_n)$, which was introduced in our paper [5]. Namely, we add additional relations in degree four, see Definition 4.1, (6). In fact we have no need to use these relations in order to describe relations between Dunkl elements $\theta_1^{B_n}, \cdots, \theta_n^{B_n}$ in the algebra $BE(B_n)$. However, to ensure that the B_2 Yang-Baxter relations $h_{ij}\ h_i\ g_{ij}\ h_j = h_j\ g_{ij}\ h_i\ h_{ij}$ are indeed satisfied, the relations (6) are necessary. Another reason to add relations (6) is that these relations are satisfied in the Nichols-Woronowicz algebra $\mathcal{B}_{B_n}$. However, we would like to repeat again that if $n \geq 3$, then the natural homomorphism of algebras $B\mathcal{E}(B_n) \to \mathcal{B}_{B_n}$ has a non-trivial kernel.

The main results of Section 4 are:

(1) construction of a multiplicative analogue $\Theta_j^{B_n}$ of the B_n-Dunkl elements $\theta_j^{B_n}$, see Definition 4.4;

(2) proof of the fact that the RSM-elements $\Theta_j^{B_n}$, $1 \leq j \leq n$, form a pairwise commuting family of elements in the algebra $B\mathcal{E}(B_n)$.

Finally we give a conjectural description of all relations between the elements $\Theta_j^{B_n}$. Here we state this conjecture in the following form.

Conjecture The following identity in the algebra $B\mathcal{E}(B_n)$ holds:

$$\sum_{j=1}^{n} (\Theta_j(x,y)^{B_n} + (\Theta_j(x,y)^{B_n})^{-1})^k = n\ 2^k$$

for all $k \in \mathbb{Z}_{\geq 0}$.

In Section 5 we discuss on a model for the Grothendieck ring of flag varieties in terms of the Nichols-Woronowicz algebra for the classical root systems. Our construction is a K-theoretic analogue of Bazlov's result [1]. Our second main result proved in Section 5 is:

Theorem C *Let $\varphi : B\mathcal{E}(B_n) \to \mathcal{B}_{B_n}$ be a natural homomorphism of algebras. Then*

$$\varphi(F(\Theta_1^{B_n}(x,y), \cdots, \Theta_n^{B_n}(x,y))) = 0$$

for any Laurent polynomial F from the defining ideal of the Grothendieck ring of the flag variety of type B_n.

Theorem C implies the corresponding results for other classical root systems after some specializations. The Nichols-Woronowicz algebra $\mathcal{B}_{\mathcal{X}}$ treated in this paper is a quotient of the algebra $GK(\mathcal{X})$ for the classical root system $\mathcal{X}$. In particular, the result for A_{n-1} is a consequence of Theorem A, but the argument in Section 5 is another approach based on the property of the Nichols-Woronowicz algebra, which works well for the root systems other than of type A_{n-1}. The idea of the proof is to construct the operators on the Nichols-Woronowicz algebra which induce isobaric divided difference operators on the commutative subalgebra generated by the RSM-elements.

The main interest of this paper is concentrated on the classical root systems, for which we can use advantages of explicit handling, particularly in order to construct the RSM-elements.

Though most of the ideas in this paper are expected to be applicable to an arbitrary root system, to develop the general framework including the exceptional root systems is a matter of concern for the forthcoming work. However, the simplest exceptional root system G_2 can be dealt with in similar manner to the case of the classical root systems. In the last section, we formulate the Yang-Baxter relations and define the RSM-elements for the root system of type G_2. The argument in Section 5 again works well, so the Nichols-Woronowicz model for the Grothendieck ring of the flag variety of type G_2 is presented.

2 Key Lemma

Definition 2.1 *Let $\mathcal{GK}(B_n)$ be a group generated by the elements $\{h_{ij}, g_{ij} \mid 1 \leq i < j \leq n\}$ and $\{h_i \mid 1 \leq i \leq n\}$, subject to the following set of relations:*

- $h_{ij}\ h_{kl} = h_{kl}\ h_{ij}, \quad g_{ij}\ g_{kl} = g_{kl}\ g_{ij}, \quad h_k\ h_{ij} = h_{ij}\ h_k, \quad h_k\ g_{ij} = g_{ij}\ h_k,$ *if all i, j, k, l are distinct;*

  $h_i\ h_j = h_j\ h_i$, *if* $1 \leq i, j \leq n$; $\quad h_{ij}\ g_{ij} = g_{ij}\ h_{ij}$, *if* $1 \leq i < j \leq n$;
- (Mixed Yang–Baxter relations)

$$(1) \quad h_{ij}\ h_{ik}\ h_{jk} = h_{jk}\ h_{ik}\ h_{ij},$$

$$(2) \quad h_{ij}\ g_{ik}\ g_{jk} = g_{jk}\ g_{ik}\ h_{ij},$$

$$(3) \quad h_{ik}\ g_{ij}\ g_{jk} = g_{jk}\ g_{ij}\ h_{ik},$$

$$(4) \quad h_{jk}\ g_{ij}\ g_{ik} = g_{ik}\ g_{ij}\ h_{jk},$$

if $1 \leq i < j < k \leq n$;

- (B_2 quantum Yang-Baxter relation)

$$h_{ij}\ h_i\ g_{ij}\ h_j = h_j\ g_{ij}\ h_i\ h_{ij},$$

if $1 \leq i < j \leq n$.

Definition 2.2 *Define the following elements in the group $\mathcal{GK}(B_n)$:*

$$\Theta_j = (\prod_{i=j-1}^{1} h_{ij}^{-1})\ h_j\ (\prod_{i=1, i\neq j}^{n} g_{ij})\ h_j\ (\prod_{k=n}^{j+1} h_{jk}), \tag{2.1}$$

for $1 \leq j \leq n$. In the RHS of (2.1) *it is assumed that $g_{ij} = g_{ji}$.*

Theorem 2.3 (Key Lemma)

$\Theta_i \Theta_j = \Theta_j \Theta_i$ *for all* $1 \leq i, j \leq n$.

Proof. Induction plus a masterly use of the Yang-Baxter relations, see defining relations in the definition of the group $\mathcal{GK}(B_n)$. See the proof of Corollary 3.3 and Example 2.5 (2) below.

Remark 2.4 It's not difficult to see that

$$\prod_{1\le j\le k}\Theta_j=\prod_{j=1}^{k}(h_j\prod_{s=j+1}^{n}g_{js}\prod_{s=1}^{j-1}g_{sj}\ h_j)\quad\prod_{j=1}^{k}(\prod_{s=n}^{k+1}h_{js}).$$

In particular,

$$\prod_{j=1}^{n}\Theta_j=(\prod_{k=1}^{n}(\prod_{j\le k}g_{jk})\ h_k)^2.$$

Example 2.5 (1) Take $n=2$. Then $\Theta_1=h_1\ g_{12}\ h_1\ h_{12}$ and $\Theta_2=h_{12}^{-1}\ h_2\ g_{12}\ h_2$. Let us check that Θ_1 and Θ_2 commute. Indeed, using the B_2-quantum Yang-Baxter relation $h_{12}\ h_1\ g_{12}\ h_2=h_2\ g_{12}\ h_1\ h_{12}$ and the commutativity relation $h_1\ h_2=h_2\ h_1$, we see that

$$\Theta_1\ \Theta_2=h_1\ g_{12}\ h_1\ h_2\ g_{12}\ h_2=h_{12}^{-1}\ (h_{12}\ h_1\ g_{12}\ h_2)h_1\ g_{12}\ h_2$$

$$=h_{12}^{-1}\ h_2\ g_{12}\ h_1(h_{12}\ h_1\ g_{12}\ h_2)=h_{12}^{-1}\ h_2\ g_{12}\ h_1\ h_2\ g_{12}\ h_1\ h_{12}=\Theta_2\ \Theta_1=(h_1\ g_{12}\ h_2)^2.$$

(2) Take $n=3$. Then we have

$$\Theta_1=h_1\ g_{12}\ g_{13}\ h_1\ h_{13}\ h_{12},\ \Theta_2=h_{12}^{-1}\ h_2\ g_{12}\ g_{23}\ h_2\ h_{23},\ \Theta_3=h_{23}^{-1}\ h_{13}^{-1}\ h_3\ g_{13}\ g_{23}\ h_3,$$

and

$$\Theta_1\ \Theta_2\ \Theta_3=(h_1\ g_{12}\ h_2\ g_{13}\ g_{23}\ h_3)^2.$$

Let us illustrate the main ideas behind the proof of Key Lemma by the following example.

$\Theta_1\ \Theta_3\ \Theta_1^{-1}=h_1\ g_{12}\ g_{13}\ h_1\ h_{13}\ \mathbf{h_{12}}\ \mathbf{h_{23}^{-1}}\ \mathbf{h_{13}^{-1}}\ h_3\ \mathbf{g_{13}}\ \mathbf{g_{23}}\ h_3\ \mathbf{h_{12}^{-1}}\ h_{13}^{-1}\ h_1^{-1}\ g_{13}^{-1}\ g_{12}^{-1}\ h_1^{-1}$
$=h_1\ \mathbf{g_{12}}\ \mathbf{g_{13}}\ h_1\ \mathbf{h_{23}^{-1}}\ h_3\ g_{23}\ \mathbf{g_{13}}\ \mathbf{h_3}\ \mathbf{h_{13}^{-1}}\ \mathbf{h_1^{-1}}g_{13}^{-1}\ g_{12}^{-1}\ h_1^{-1}$ (by (1) and (2))
$=h_1\ h_{23}^{-1}\ g_{13}\ \mathbf{g_{12}}\ h_3\ \mathbf{g_{23}}\ \mathbf{h_{13}^{-1}}\ h_3\ g_{12}^{-1}\ h_1^{-1}$ (by (4) and B_2-YBE)
$=h_1\ h_{23}^{-1}\ \mathbf{g_{13}}\ \mathbf{h_3}\ \mathbf{h_{13}^{-1}}\ g_{23}\ h_3\ \mathbf{h_1^{-1}}$ (by (3))
$=\Theta_3$ (by B_2-YBE).

We define the groups $\mathcal{GK}(A_{n-1})$ and $\mathcal{GK}(D_n)$ to be the quotients of that $\mathcal{GK}(B_n)$ by the normal subgroups generated respectively by the elements $\{h_i, g_{ij}, 1\le i<j\le n\}$ and $\{h_i, 1\le i\le n\}$. The group $\mathcal{GK}(G_2)$ will be defined in Section 6. We expect that the subgroup in $\mathcal{GK}(B_n)$ generated by the elements $\Theta_1^{B_n},\cdots,\Theta_n^{B_n}$ is isomorphic to the free abelian group of rank n. It seems an interesting problem to construct analogues of the group $\mathcal{GK}(B_n)$ and the elements $\Theta_1^{B_n},\cdots,\Theta_n^{B_n}$ for any (finite) Coxeter group.

Question 2.6 Does there exist a finite-dimensional *faithful* representation of the group $\mathcal{GK}(X)$, $X=A_{n-1},B_n,...$?

3 Algebras $GK(A_{n-1})$ and $BE(A_{n-1})$

3.1 Definitions and main results

(i) **Algebra** $GK(A_{n-1})$

Definition 3.1 *Let R be a $\mathbb{Q}$-algebra. Define the algebra $GK_R(A_{n-1})$ as an associative algebra over R generated by the elements $h_{ij}(x)$, $1 \le i \ne j \le n$, $x \in R$, subject to the relations* $(0) - (4)$:

(0) $h_{ij}(x)h_{ji}(x) = 1$,

(1) $h_{ij}(x)h_{ij}(y) = h_{ij}(x+y)$; *in particular,* $h_{ij}(x)h_{ij}(-x) = 1$,

(2) $h_{ij}(x)h_{kl}(y) = h_{kl}(y)h_{ij}(x)$, *if* i,j,k,l *are distinct,*

(3) $h_{ij}(x)h_{jk}(y) + h_{ik}(x+y) = h_{jk}(y)h_{ik}(x) + h_{ik}(y)h_{ij}(x)$,
$h_{jk}(y)h_{ij}(x) + h_{ik}(x+y) = h_{ik}(x)h_{jk}(y) + h_{ij}(x)h_{ik}(y)$,
if $1 \le i < j < k \le n$,

(4) $h_{ik}(x)\ (h_{ij}(x) - h_{ik}(y))\ h_{ij}(y) = h_{ij}(y)\ (h_{ij}(x) - h_{ik}(y))\ h_{ik}(x)$,
if $1 \le i < j < k \le n$.

For any element $z \in R$ we denote by $GK(A_{n-1})[z]$ (resp. $GK(A_{n-1})$) the algebra over $\mathbb{Q}$ generated by the elements $h_{ij}(z)$ and $h_{ij}(-z)$, (resp. $h_{ij}(1)$ and $h_{ij}(-1)$), $1 \le i \ne j \le n$.

Lemma 3.2 (Quantum Yang-Baxter equation)
The following relations in the algebra $GK(A_{n-1})[z]$

$$h_{ab}(z)h_{ac}(z)h_{bc}(z) = h_{bc}(z)h_{ac}(z)h_{ab}(z), \quad 1 \le a < b < c \le n, \tag{3.2}$$

are a consequence of the relations $(0) - (4)$ *in the algebra* $GK(A_{n-1})[z]$.

Corollary 3.3 *Define elements $\Theta_j^{A_{n-1}}(z)$, $j = 1, \ldots, n$, in the algebra $GK(A_{n-1})[z]$ as follows:*

$$\Theta_j^{A_{n-1}}(z) = h_{j-1,j}^{-1}(z) \cdots h_{1j}^{-1}(z)\ h_{jn}(z) \cdots h_{j,j+1}(z), \quad 1 \le j \le n. \tag{3.3}$$

Then

$$\Theta_j^{A_{n-1}}(z)\Theta_k^{A_{n-1}}(z) = \Theta_k^{A_{n-1}}(z)\Theta_j^{A_{n-1}}(z), \quad \text{for all } 1 \le j,k \le n.$$

Proof. It is enough to check that if $1 \le i \le j \le n$, then

$$\Theta_i\ \Theta_j\ \Theta_i^{-1} = \Theta_j.$$

By definition,

$$\Theta_i\ \Theta_j\ \Theta_i^{-1} = h_{i-1,i}^{-1} \cdots h_{1,i}^{-1}\ h_{i,n} \cdots h_{i,i+1}\ h_{j-1,j}^{-1} \cdots \mathbf{h_{i+1,j}^{-1}}\ \mathbf{h_{i,j}^{-1}}\ h_{i-1,j}^{-1} \cdots h_{1,j}^{-1}$$
$$h_{j,n} \cdots h_{j,j+1}\ \mathbf{h_{i,i+1}^{-1}} h_{i,i+2}^{-1} \cdots h_{i,n}^{-1}\ h_{1,i} \cdots h_{i-1,i}.$$

Using local commutativity relations, see Definition 3.1 (2), we can move the factor $\mathbf{h_{i,i+1}^{-1}}$ to the left till we have touched on the factor $\mathbf{h_{i,j}^{-1}}$. As a result, we will come up with the triple product:

$$\mathbf{h_{i+1,j}^{-1}}\ \mathbf{h_{i,j}^{-1}}\ \mathbf{h_{i,i+1}^{-1}},$$

which is equal, according to the Yang-Baxter relation (3.2), to the product

$$\mathbf{h_{i,i+1}^{-1}}\ \mathbf{h_{i,j}^{-1}}\ \mathbf{h_{i+1,j}^{-1}}.$$

Now we can move the factor $\mathbf{h_{i,i+1}^{-1}}$ to the left to cancel it with the term $h_{i,i+1}$, which comes from the rightmost factor in the element Θ_i.

As a result, we will have

$$\Theta_i\ \Theta_j\ \Theta_i^{-1} = h_{i-1,i}^{-1} \cdots h_{i,i+2}\ h_{j-1,j}^{-1} \cdots \mathbf{h_{i+2,j}^{-1}}\ \mathbf{h_{i,j}^{-1}} \cdots h_{j,j+1}^{-1}\ \mathbf{h_{i,i+2}^{-1}} \cdots h_{i-1,i}.$$

Now we can move to the left the factor $\mathbf{h_{i,i+2}^{-1}}$ till we have touched on the factor $\mathbf{h_{i,j}^{-1}}$ to give the triple product

$$\mathbf{h_{i+2,j}^{-1}}\ \mathbf{h_{i,j}^{-1}}\ \mathbf{h_{i,i+2}^{-1}},$$

which is equal to $\mathbf{h_{i,i+2}^{-1}}\ \mathbf{h_{i,j}^{-1}}\ \mathbf{h_{i+2,j}^{-1}}$. Now we can move the factor $\mathbf{h_{i,i+2}^{-1}}$ to the left to cancel it with the corresponding factor $h_{i,i+2}$, and so on.

It is readily seen that finally we will come to the element Θ_j. ■

It is clear that $\prod_{j=1}^{n} \Theta_j^{A_{n-1}}(z) = 1$.

Remark 3.4 Let $\Theta_j(z) := \Theta_j^{A_{n-1}}(z)$. Then it is not true that $\Theta_j(x)\Theta_k(y) = \Theta_k(y)\Theta_j(x)$, if $j \neq k$, $x \neq y$.

Remark 3.5 Though the algebra $GK(A_{n-1})[z]$ can be constructed as a quotient of the group algebra $\mathbb{Q}\langle \mathcal{GK}(A_{n-1})\rangle$, they are not isomorphic.

Theorem 3.6 (Main theorem, the case of algebra $GK(A_{n-1})[z]$)

$$\prod_{j=1}^{n}(1 + (1 - \Theta_j^{A_{n-1}}(z))t) = 1. \quad \textit{Equivalently,} \quad \prod_{j=1}^{n}(1 + \Theta_j^{A_{n-1}}(z)t) = (1+t)^n. \tag{3.4}$$

This theorem is equivalent to:

Theorem 3.7 *Let $G_j^{A_{n-1}} = \Theta_j^{A_{n-1}}(z) - 1$, $1 \leq j \leq n$. Then, after the substitution $z = 1$,*

$$e_j(G_1^{A_{n-1}}, \ldots, G_n^{A_{n-1}}) = 0, \quad 1 \leq j \leq n$$

is the complete list of relations in the algebra $GK(A_{n+1})$ among the elements $G_1^{A_{n-1}}, \ldots, G_n^{A_{n-1}}$. Here, e_j is the j-th elementary symmetric polynomial.

The proof is given in Subsection 3.2. It is based on the properties of the *Chern homomorphism*.

Corollary 3.8 *The algebra over $\mathbb{Z}$ generated by the elements $G_1^{A_{n-1}}|_{z=1},\ldots,G_n^{A_{n-1}}|_{z=1}$, is canonically isomorphic to the integral Grothendieck ring $K(\mathcal{F}l_n)$ of the flag manifold of type A_{n-1}.*

(ii) **Algebra** $BE(A_{n-1})$

Definition 3.9 ([2]) *Define algebra $BE(A_{n-1})$ (denoted by $\mathcal{E}_n$ in [2]) as an associative algebra over $\mathbb{Z}$ with generators x_{ij}, $1 \le i \ne j \le n$, subject to the following relations*

(0) $x_{ij}+x_{ji}=0$, $1 \le i \ne j \le n$,
(1) $x_{ij}^2=0$, $1 \le i \ne j \le n$,
(2) $x_{ij}\ x_{jk}+x_{jk}\ x_{ki}+x_{ki}\ x_{ij}=0$, *if all i,j,k are distinct.*

The Dunkl elements θ_j, $j=1,\ldots,n$, in the algebra $BE(A_{n-1})$ are defined by $\theta_j := \theta_j^{A_{n-1}} = \sum_{i \ne j} x_{ij}$.

The Dunkl elements form a pairwise commuting family of elements in the algebra $BE(A_{n-1})$, [2], and generate a commutative subalgebra in $BE(A_{n-1})$, which is canonically isomorphic to the cohomology ring $H^*(\mathcal{F}l_n)$ of the flag variety $\mathcal{F}l_n$ of type A_{n-1}, [2].

For an element t of a $\mathbb{Q}$-algebra R, define $h_{ij}(t)=1+tx_{ij}=\exp(tx_{ij}) \in BE(A_{n-1}) \otimes R$.

Lemma 3.10 *The elements $h_{ij}(t), 1 \le i,j \le n$, satisfy the all relations $(0)-(4)$ of the definition of the algebra $GK_R(A_{n-1})$.*

We will use the same notation $\Theta_j^{A_{n-1}}$, $1 \le j \le n$, to denote the elements in the algebra $BE(A_{n-1})$ defined by the formula (3.3). It follows from Corollary 3.3 that they form a pairwise commuting family of elements in the algebra $BE(A_{n-1})$.

It's clear that $\Theta_j^{A_{n-1}}(z)=1+z\ \theta_j^{A_{n-1}}+\cdots$, and the product in the RHS of (3.3) may be written as follows:

$$\Theta_j^{A_{n-1}}(z)=\sum(-1)^s x_{b_1,j}\ x_{b_2,j}\cdots x_{b_s,j}\ x_{j,a_1}\ x_{j,a_2}\cdots x_{j,a_r}\ z^{r+s}, \qquad (3.5)$$

where the sum runs over the all sequences of integers $(a_1>a_2>\cdots>a_r)$ and $(b_1>b_2>\cdots>b_s)$ such that $n \ge a_1 > a_r > j > b_1 > b_s \ge 1$; cf. [9, Section 2].

Remember that $G_j^{A_{n-1}} := \Theta_j^{A_{n-1}}-1$, $1 \le j \le n$.

Definition 3.11 *Let $w \in S_n$ be a permutation. Define the Grothendieck polynomial $\mathcal{G}_w(X_n) \in \mathbb{Z}\ [X_n]$ to be a unique polynomial of the form $\mathcal{G}_w(X_n)=\sum_{\alpha \subset \delta_n} c_\alpha(w)\ x^\alpha$ such that*

$$\mathcal{G}_w(G_1^{A_{n-1}},\ldots,G_n^{A_{n-1}}) \cdot id = w \qquad (3.6)$$

in the Bruhat representation of the algebra $BE(A_{n-1})$ (see [2, Section 3.1]), where $\delta_n := (n-1,n-2,\ldots,1,0)$ and $X_n := (x_1,\ldots,x_n)$.

It is not difficult to see that the Grothendieck polynomials defined here coincide with those introduced in [6], see also [10].

Corollary 3.12 *Let $u \in S_n$ and $v \in S_n$ be two permutations. Assume that in the group ring $\mathbb{Z}\langle S_n\rangle$ of the symmetric group S_n we have the following equality:*

$$\mathcal{G}_u(G_1^{A_{n-1}}, \ldots, G_n^{A_{n-1}}) \cdot v = \sum_{w \in S_n} c_{u,v}^w \ w.$$

Then the coefficient $c_{u,v}^w$ is equal to the multiplicity of the Grothendieck polynomial $\mathcal{G}_w(X_n)$ in the product of $\mathcal{G}_u(X_n)$ and $\mathcal{G}_v(X_n)$:

$$\mathcal{G}_u(X_n) \ \mathcal{G}_v(X_n) = \sum_{w \in S_n} c_{u,v}^w \ \mathcal{G}_w(X_n)$$

in the Grothendieck ring $K(\mathcal{F}l_n)$ of the flag manifold of type A_{n-1}.

Conjecture 3.13 For any permutation $w \in S_n$ the value of the Grothendieck polynomial $\mathcal{G}_w(x_1, \ldots, x_n)$ after the substitution $x_1 := G_1^{A_{n-1}}, \ldots, x_n := G_n^{A_{n-1}}$, and $z = 1$, can be written as a linear combination of monomials in x_{ij}'s , $1 \le i < j \le n$, with **non-negative** integer coefficients.

Example 3.14 (Grothendieck-Pieri formula in the algebra $BE(A_{n-1})$, cf [10])

$$1 + \mathcal{G}_{(k,k+1)}(G_1, \ldots, G_n) = \prod_{1 \le j \le k} \Theta_j = \prod_{j=1}^{k} \prod_{s=n}^{k+1} h_{js} = \sum \prod_{j=1}^{r} x_{a_j, b_j},$$

where the sum runs over all sequences of integers $(1 \le a_1 \le \cdots \le a_r \le k)$ and $(b_1, \ldots, b_r)$ such that $k < b_j \le n$, $j = 1, \ldots, r$, and $a_i = a_{i+1} \Rightarrow b_i > b_{i+1}$.

Example 3.15 Take $n = 3$, then
$\Theta_1 := \Theta_1^{A_2}(1) = h_{13}(1) \ h_{12}(1) = 1 + x_{12} + x_{13} + x_{12} \ x_{13}$,
$\Theta_2 := \Theta_2^{A_2}(1) = h_{12}^{-1}(1) \ h_{23}(1) = 1 - x_{13} + x_{23} - x_{13} \ x_{12} - x_{23} \ x_{13}$,
$\Theta_3 := \Theta_3^{A_2}(1) = h_{23}^{-1}(1) \ h_{13}^{-1}(1) = 1 - x_{13} - x_{23} + x_{23} \ x_{13}$.
As a preliminary step, we compute the elementary symmetric polynomials $e_k(\Theta_1, \Theta_2, \Theta_3)$, $k = 1, 2, 3$. Indeed, it's easily seen from the formulae above that $\Theta_1 + \Theta_2 + \Theta_3 = 3$ and $\Theta_1 \ \Theta_2 \ \Theta_3 = 1$. To compute $e_2(\Theta_1, \Theta_2, \Theta_3)$, all one has to do is to apply the following relation

$$h_{12} \ h_{23}^{-1} = h_{23}^{-1} \ h_{13} + h_{13}^{-1} \ h_{12} - 1,$$

where we put by definition $h_{ij} := h_{ij}(1)$. The former equality follows from the relation (3) in Definition 3.1. Hence,

$$e_2(\Theta_1, \Theta_2, \Theta_3) = h_{13} \ h_{23} + h_{13} \ \mathbf{h_{12} \ h_{23}^{-1}} \ h_{13}^{-1} + h_{12}^{-1} \ h_{13}^{-1}$$

$$= h_{13} \ h_{23} + h_{13} \ h_{23}^{-1} + h_{12} \ h_{13}^{-1} - 1 + h_{12}^{-1} \ h_{13}^{-1} = 2h_{13} + 2h_{13}^{-1} - 1 = 3.$$

To continue, let us list the Grothendieck polynomials $\mathcal{G}_w(x)$ corresponding to the symmetric group S_3:

$$\mathcal{G}_{id}(x) = 1, \ \mathcal{G}_{s_1}(x) = x_1, \ \mathcal{G}_{s_2}(x) = x_1 + x_2 + x_1 x_2,$$

$$\mathcal{G}_{s_1s_2}(x)=x_1x_2,\ \mathcal{G}_{s_2s_1}(x)=x_1^2,\ \mathcal{G}_{w_0}(x)=x_1^2x_2.$$

Now let us consider the substitution $x_j = G_j = \Theta_j(1)-1$, $j = 1,2,3$. More explicitly, $G_1 = x_{12}+x_{13}+x_{13}\ x_{12}$ and $G_2 = -x_{12}+x_{23}-x_{13}\ x_{12}-x_{23}\ x_{13}$. Therefore,

$$\mathcal{G}_{s_2}(G_1,G_2)=x_{13}+x_{23}+x_{13}\ x_{23},\ \mathcal{G}_{s_1s_2}(G_1,G_2)=x_{13}\ x_{23}+x_{23}\ x_{13},$$

$$\mathcal{G}_{s_2s_1}(G_1,G_2)=x_{12}\ x_{13}+x_{13}\ x_{12},$$

$$\mathcal{G}_{w_0}(G_1,G_2)=x_{12}\ x_{13}\ x_{23}+x_{13}\ x_{12}\ x_{13}+x_{13}\ x_{23}\ x_{13}+x_{13}\ x_{12}\ x_{13}\ x_{23}.$$

Finally, let us consider the commutative subalgebra in $BE(A_2)\otimes\mathbb{Q}$ generated by the elements $E_j := \exp(\theta_j)$, $j=1,2,3$. It's not difficult to check that

$$2E_1 = h_{13}\ h_{12}+h_{12}\ h_{13},\ 2E_2 = h_{12}^{-1}\ h_{23}+h_{23}\ h_{12}^{-1},\ 2E_3 = h_{23}^{-1}\ h_{13}^{-1}+h_{13}^{-1}\ h_{23}^{-1}.$$

It is an easy matter as well to see that the subalgebra in $BE(A_2)\otimes\mathbb{Q}$ generated over $\mathbb{Q}$ by the elements E_i, $i=1,2,3$, is isomorphic to the algebra $\mathbb{Q}[\Theta_1,\Theta_2,\Theta_3]$. In particular, for all symmetric polynomials $f(x_1,x_2,x_3)$ we have

$$f(1-E_1,1-E_2,1-E_3)=0.$$

Proposition 3.16 *The subalgebra in $BE(A_{n-1})\otimes\mathbb{Q}$ generated by the elements $E_i := \exp(\theta_i)$, $1\le i\le n$, is isomorphic to the algebra over $\mathbb{Q}$ generated by the elements $\Theta_j^{A_{n-1}}$, $1\le j\le n$.*

In particular, the complete list of relations among the elements $1-E_1,\ldots,1-E_n$ in the quadratic algebra $BE(A_{n-1})$ is given by

$$e_i(1-E_1,\ldots,1-E_n)=0,$$

for $i=1,\cdots,n$. Thus the commutative subalgebra generated by the elements $\exp(\theta_1),\ldots,\exp(\theta_n)$ is isomorphic to the rational Grothendieck ring $K(\mathcal{F}l_n)\otimes\mathbb{Q}$ of the flag manifold $\mathcal{F}l_n$ of type A_{n-1}.

However, it seems that there are no direct connections of the elements E_j's with the Grothendieck Calculus.

Remark 3.17 More generally, let $Q(t)\neq 0$ be a polynomial such that $Q(0)=0$. Define the elements $q_i := 1+Q(\theta_i)$, $1\le i\le n$, in the algebra

$BE(A_{n-1})$. It's clear that the elements $q_1,\ldots,q_n$ pairwise commute, and

$$e_i(q_1-1,\ldots,q_n-1)=0,\quad 1\le i\le n.$$

Remark 3.18 (Quantum Grothendieck Calculus)
It is easy to see that the relations in Definition 3.1 are still true, if we replace the condition (1) in Definition 3.9 by the following one

$(1')$ $x_{ij}^2 = q_{ij}$, $1\le i<j\le n$, where the parameters q_{ij} are assumed to commute with all the generators x_{kl}, $1\le k<l\le n$.

The algebra over $\mathbb{Z}[\ q_{ij}\mid 1\le i<j\le n]$ generated by the elements x_{ij}, $1\le i\neq j\le n$, subject to the relations (0), $(1')$ and (2), is called the *quantized bracket algebra* and denoted by $qBE(A_{n-1})$, cf. [2, Section 15] and [3].

As a corollary we see that the elements Θ_j^q, $1\le j\le n$, defined by the formula (3.2), form a pairwise commuting family of elements in the algebra $qBE(A_{n-1})$.

Problem 3.19 Describe the commutative subalgebras in the quantized algebra $qGK(A_{n-1})$ generated by

(1) $\Theta_1^q(1), \ldots, \Theta_n^q(1)$,

(2) $\widetilde{E}_1 := \exp(\theta_1), \ldots, \widetilde{E}_n := \exp(\theta_n)$.

3.2 Chern homomorphism

Denote by $\mathcal{H} := BE(A_{n-1})^{ab} \otimes \mathbb{Q}$ the quotient of the algebra $BE(A_{n-1})$ by its commutant. It is known, [2, Proposition 4.2], that the algebra $BE(A_{n-1})^{ab}$ has dimension $n!$, and its Hilbert polynomial is given by

$$Hilb(BE(A_{n-1})^{ab}, t) = (1+t)(1+2t)\cdots(1+(n-1)t).$$

Denote by $1 + \mathcal{H}^+$ the multiplicative monoid generated by the elements of the form $1 + h$, where $h \in \mathcal{H}$ does not have the term of degree zero.

Proposition 3.20 *Let $R^{(n-1)}$ be the subspace of the commutative subalgebra $R = \mathbb{Q}[\theta_1, \ldots, \theta_n] \subset BE(A_{n-1}) \otimes \mathbb{Q}$ whose elements are of degree $\leq n-1$. Then the subspace $R^{(n-1)}$ is injectively mapped into $\mathcal{H}$ by the quotient homomorphism $BE(A_{n-1}) \otimes \mathbb{Q} \to \mathcal{H}$.*

Proof. Since the algebra R is isomorphic to the coinvariant algebra of the symmetric group, the monomials

$$\theta_1^{i_1} \cdots \theta_{n-1}^{i_{n-1}}, \quad 0 \leq i_k \leq n-k,$$

form a linear basis of R. The linear map $R^{(n-1)} \to \mathcal{H}$ induced by the quotient homomorphism is a homomorphism between S_n-modules. Hence, it is enough to show the images of the monomials $\theta_1^{i_1} \cdots \theta_{n-1}^{i_{n-1}}$ do not vanish in $\mathcal{H}$ for $(i_1, \ldots, i_{n-1})$ such that $\sum_{k=1}^{n-1} i_k = n-1$ and $i_1 \geq i_2 \geq \cdots \geq i_{n-1}$. We expand the monomials $\theta_1^{i_1} \cdots \theta_{n-1}^{i_{n-1}}$ of this form in the algebra $BE(A_{n-1}) \otimes \mathbb{Q}$ by using the Pieri formula proved by Postnikov [11], (first conjectured in [2]). The Pieri formula shows that

$$e_k(\theta_1, \ldots, \theta_m) = \widetilde{\sum} [i_1 j_1] \cdots [i_k j_k],$$

where $\widetilde{\sum}$ stands for the multiplicity-free sum, and $(i_1, j_1), \ldots, (i_k, j_k)$ run over all pairs such that $i_a \leq m < j_a \leq n$, $a = 1, \ldots, k$, and all i_a's are distinct.

On the other hand, the monomials of form

$$[i_1 j_1] \cdots [i_k j_k], \quad i_a < j_a \ (a = 1, \ldots, k), \ \ j_1 < j_2 < \cdots < j_k,$$

give a linear basis of $\mathcal{H}$ ([3, Corollary 10.3]). By the involution $\omega : [ij] \mapsto [n+1-j \ \ n+1-i]$, we have a linear basis of form

$$[i_1 j_1] \cdots [i_k j_k], \quad i_a < j_a \ (a = 1, \ldots, k), \ \ i_1 < i_2 < \cdots < i_k. \tag{3.7}$$

For each monomial expression $[i_1 j_1] \cdots [i_k j_k]$ in $\mathcal{H}$, we define

$$\mu([i_1 j_1] \cdots [i_k j_k]) := \sum_{m=1}^{k} (j_m - i_m).$$

Every element in $\mathcal{H}$ can be expressed as a linear combination of the monomials listed in (3.7) by repeatedly applying the substitution $[ab][ac] \to [ab][bc] - [ac][bc]$ with $a < b < c$. On each step of the procedure, the monomials of minimal μ appearing in the expression of $\theta_1^{i_1} \cdots \theta_{n-1}^{i_{n-1}}$ with $i_1 + \cdots + i_{n-1} = n - 1$ are not cancelled or are replaced by new ones. So one can check the image of $\theta_1^{i_1} \cdots \theta_{n-1}^{i_{n-1}}$ in $\mathcal{H}$ is not zero. ■

Definition 3.21 *Define the Chern homomorphism (to the commutative quotient)*

$$c' : GK(A_{n-1}) \to 1 + \mathcal{H}^+$$

by the following rules:

- $c'(f + g) = c'(f)c'(g)$, *if* $f, g \in GK(A_{n-1})$,
- $c'(\prod_{i<j} h_{ij}^{n_{ij}}) = 1 + \sum_{i<j} n_{ij}\ x_{ij}$.

It is clear that $c'(\Theta_j) = 1 + \theta_j$, $\forall j$.

Remark 3.22 We can also define the homomorphism

$$c : \mathbb{Q}[\Theta_1, \ldots, \Theta_n] \to \mathbb{Q}[\theta_1, \ldots, \theta_n]$$

by the conditions $c(f + g) = c(f)c(g)$ and $c(\Theta_j) = 1 + \theta_j$, $j = 1, \ldots, n$, which is compatible with the Chern homomorphism (in the usual sense)

$$c : K(Fl_n) \to 1 + H^+(Fl_n).$$

However, the homomorphism c' defined above does not coincide with c in the part of degree $\geq n$. Indeed, the maximal degree of the commutative quotient $\mathcal{H}$ is $n - 1$.

Proposition 3.23 (cf. [7, Section 5]) *For any permutation $w \in S_n$,*

$$c(1 + \mathcal{G}_w(G_1, \ldots, G_n)) = 1 - (-1)^{l(w)}(l(w) - 1)!\ \mathfrak{S}_w(\theta_1, \ldots, \theta_n) + \sum_u a_u(w)\ \mathfrak{S}_u(\theta_1, \ldots, \theta_n),$$

where the sum ranges over all permutations $u \in S_n$ such that $l(u) > l(w)$.

Proof of Theorem 3.7. Note that the commutative quotient $GK(A_{n-1})^{ab}$ is isomorphic to the algebra $\mathcal{H}$. Moreover, The subspace of polynomials of degree $\leq n - 1$ in the RSM-elements $\Theta_1^{A_{n-1}}, \ldots, \Theta_n^{A_{n-1}}$ in $GK(A_{n-1})$ is also injectively mapped into $\mathcal{H}$ from Proposition 3.20. We regard $1 + \mathcal{H}^+$ as an $(n! - 1)$-dimensional $\mathbb{Q}$-linear space so that the homomorphism $\bar{c} : \mathcal{H}^+ \to 1 + \mathcal{H}^+$ induced by the Chern homomorphism c' is a $\mathbb{Q}$-linear map. The image of the linear basis (3.7) of $\mathcal{H}^+$ by the homomorphism $\bar{c}$ is linearly independent. Hence, $\bar{c} : \mathcal{H}^+ \to 1 + \mathcal{H}^+$ is an isomorphism between linear spaces. Since it is easy to see

$$c'(e_j(\Theta_1^{A_{n-1}}, \ldots, \Theta_n^{A_{n-1}})) = 1 \in 1 + \mathcal{H}^+, \quad 1 \leq j \leq n - 1,$$

one can conclude that

$$e_j(G_1^{A_{n-1}}, \ldots, G_n^{A_{n-1}}) = 0, \quad 1 \le j \le n-1.$$

The equality

$$\prod_{i=1}^{n} \Theta_i^{A_{n-1}} = 1$$

in the algebra $GK(A_{n-1})$ can be obtained by direct computation. ■

Problem 3.24 Construct a lift of c' to

$$GK(A_{n-1}) \to 1 + BE(A_{n-1})^{+}$$

in some suitable sense.

4 Algebras $B\mathcal{E}(B_n)$ and $GK(B_n)$

(i) **Algebra** $B\mathcal{E}(B_n)$ (cf. [5])

Definition 4.1 *Define the algebra $B\mathcal{E}(B_n)$ as the algebra (say, over $\mathbb{Q}$) with generators*

$$[i,j],\ \overline{[i,j]},\ 1 \le i \ne j \le n, \quad \text{and} \quad [i],\ 1 \le i \le n,$$

subject to the following relations:
(0) $[i,j] = -[j,i]$, $\overline{[i,j]} = \overline{[j,i]}$,
(1) $[i,j]^2 = 0$, $\overline{[i,j]}^2 = 0$, $1 \le i < j \le n$, *and* $[i]^2 = 0$, $1 \le i \le n$,
(2) $[i,j][k,l] = [k,l][i,j]$, $\overline{[i,j]}[k,l] = [k,l]\overline{[i,j]}$, $\overline{[i,j]}\overline{[k,l]} = \overline{[k,l]}\overline{[i,j]}$,
if $\{i,j\} \cap \{k,l\} = \emptyset$,
(3) $[i][j] = [j][i]$, $[i,j]\overline{[i,j]} = \overline{[i,j]}[i,j]$, $[i,j][k] = [k][i,j]$, *if* $k \ne i,j$,
(4) $$[i,j][j,k] + [j,k][k,i] + [k,i][i,j] = 0,$$

$$\overline{[i,k]}[i,j] + [j,i]\overline{[j,k]} + \overline{[k,j]}[i,k] = 0,$$

$$[i,j][i] + [j][j,i] + [i]\overline{[i,j]} + \overline{[i,j]}[j] = 0,$$
if all i, j and k are distinct,
(5) $[i,j][i]\overline{[i,j]}[i] + \overline{[i,j]}[i][i,j][i] + [i][i,j][i]\overline{[i,j]} + [i]\overline{[i,j]}[i][i,j] = 0$, *if* $i < j$,

(6) $[i,j][i]\overline{[i,j]}[j] = [j]\overline{[i,j]}[i][i,j]$, *if* $i < j$.

Remark 4.2 (a) In the definition of the algebra $BE(B_n)$, see [5, Section 9.1], the condition (6) is absent. In fact, there is no need to use the latter condition for the purposes of [5]. However, we need the condition (6) to ensure the B_2 quantum Yang-Baxter relation, which is necessary for our construction of a commutative family of elements in the algebra $GK(B_n)$, see (ii) below.

(b) In [5], the authors has introduced the quantum deformation $qBE(B_n)$ of the bracket algebra. Similarly we introduce the quantum deformation of the algebra $qB\mathcal{E}(B_n)$ which is generated by the same symbols as in $B\mathcal{E}(B_n)$ and is obtained by replacing the relation in (1) corresponding to the simple roots by

$[i, i+1]^2 = q_i$, $1 \le i \le n-1$, and $[n]^2 = q_n$.

In the subsequent construction, we can work in the quantum bracket algebra $qB\mathcal{E}(B_n)$ instead of $B\mathcal{E}(B_n)$. The RSM-elements in Definition 4.4 also form a commuting family of elements in $qB\mathcal{E}(B_n)$. Though it is expected that the RSM-elements in the quantum setting should describe the quantum Grothendieck Calculus in B_n-case, it is not clear to see the relations satisfied by them in the algebra $qB\mathcal{E}(B_n)$.

The Dunkl elements [5] are given by

$$\theta_i := \theta_i^{B_n} = \sum_{j \neq i} ([i,j] + \overline{[i,j]}) + 2[i], \quad 1 \le i \le n. \tag{4.8}$$

Note that the Dunkl elements $\tilde{\theta}_i$ correspond to the Monk type formula in the cohomology ring of the flag variety of type B.

(ii) **Algebra** $GK(B_n)$

Let x and y be elements in a $\mathbb{Q}$-algebra R. Define the algebra $GK(B_n)$ as a subalgebra in $B\mathcal{E}(B_n) \otimes R$ generated over R by the elements:

$h_{ij} := \exp(x[i,j]) = 1 + x[i,j]$, $\quad g_{ij} := \exp(x\overline{[i,j]}) = 1 + x\overline{[i,j]}$, $1 \le i < j \le n$,
and $h_j := \exp(y[j]) = 1 + y[j]$, $1 \le j \le n$.

Proposition 4.3 *The elements h_{ij}, g_{ij} and h_k, $1 \le i < j \le n$, $1 \le k \le n$, satisfy the all relations listed in Definition 2.1.*

Definition 4.4 *Define*

$$\Theta_j^{B_n}(x,y) = (\prod_{i=j-1}^{1} h_{ij}(x)^{-1}) \ h_j(y) \ (\prod_{i=1, i \neq j}^{n} g_{ij}(x)) \ h_j(y) \ (\prod_{k=n}^{j+1} h_{jk}(x)),$$

for $1 \le j \le n$.

Corollary 4.5 *The elements $\Theta_j^{B_n}(x,y)$ commute pairwise.*

Remark 4.6 It is not difficult to see that

$$\Theta_j^{B_n}(1,1) \neq \exp(\theta_j^{B_n}),$$

where $\theta_j^{B_n}, 1 \le j \le n$, denote the B_n-Dunkl elements in the algebra $B\mathcal{E}(B_n)$. The commuting family of elements $\exp(\theta_j^{B_n}), 1 \le j \le n$, also generate a (finite dimensional) commutative subalgebra in $B\mathcal{E}(B_n) \otimes \mathbb{Q}$. However, we don't know the complete list of relations among these elements.

Conjecture 4.7 (The case of algebra $GK(B_n)$)
In the algebra $GK(B_n)$ we have the following identity

$$\prod_{j=1}^{n}(1+(\Theta_j^{B_n}(x,y)+(\Theta_j^{B_n}(x,y))^{-1})t)=(1+2t)^n. \tag{4.9}$$

Equivalently,

$$\prod_{j=1}^{n}(1+\Theta_j^{B_n}(x,y)t)(1+(\Theta_j^{B_n}(x,y))^{-1}t)=(1+t)^{2n}. \tag{4.10}$$

This conjecture is equivalent to:

Conjecture 4.8 *Let $G_{j,\alpha}^{B_n}=(\Theta_j^{B_n}(x,y))^{\alpha}-(\Theta_j^{B_n}(x,y))^{-\alpha}$, $1\le j\le n$, $\alpha\in\mathbb{Q}$. Then*

$$e_j((G_{1,\alpha}^{B_n})^2,\ldots,(G_{n,\alpha}^{B_n})^2)=0,\ 1\le j\le n. \tag{4.11}$$

Remark 4.9 Theorem 3.5, i.e. the equality

$$\prod_{j=1}^{n}(1+\Theta_j^{A_{n-1}}\ t)=(1+t)^n. \tag{4.12}$$

follows from Conjecture 4.7.

Proof. The multiplicative Dunkl elements $\Theta_j^{A_{n-1}}(x)$ can be obtained from those $\Theta_j^{B_n}(x,y)$ after the specialization $y:=0$ and $g_{ij}:=1$. Since $\prod_{j=1}^{n}\Theta_j^{A_{n-1}}=1$, it follows from Conjecture 4.7 that if we denote by $P_n(t)$ the LHS of (4.11) then $P_n(t)P_n(t^{-1})=(1+t)^n(1+t^{-1})^n$. Therefore, $P_n(t)=(1+t)^n$. ∎

Remark 4.10 The algebra $GK(C_n)$ can be naturally identified with the algebra $GK(B_n)$. The corresponding RSM-elements relate via

$$\Theta_j^{C_n}(x,y)=\Theta_j^{B_n}(x,y/2).$$

5 Nichols-Woronowicz model for Grothendieck ring of flag varieties

Let us consider the Nichols-Woronowicz algebra $\mathcal{B}_{\mathcal{X}}$ obtained from the Yetter-Drinfeld module

$$V=\bigoplus_{\alpha\in\Psi}\mathbb{Q}[\alpha]/([\alpha]+[-\alpha])_{\alpha\in\Psi}$$

for the root system Ψ of classical type $\mathcal{X}$, ($\mathcal{X}=A_{n-1},\ B_n,\ C_n,\ D_n$). Let $W(\mathcal{X})$ be the corresponding Weyl group. The $W(\mathcal{X})$-action on V is given by $w([\alpha])=[w(\alpha)]$, and the $W(\mathcal{X})$-degree of $[\alpha]$ is the reflection s_α. The structure of the braided vector space on V is given by the braiding $\psi([\alpha]\otimes[\beta]) = [s_\alpha(\beta)]\otimes[\alpha]$. For the details on the definition of the algebra $\mathcal{B}_\mathcal{X}$, see [1]. The algebra $\mathcal{B}_\mathcal{X}$ is a quotient of the algebra $GK(\mathcal{X})$.

The Weyl group $W(B_n)$ acts on the algebra $GK(B_n)$. Denote by $s_1 = s_{12}, \ldots, s_{n-1} = s_{n-1\,n}$, and s_n the simple reflections. The subgroup $S_n = W(A_{n-1}) \subset W(B_n)$ acts on $GK(B_n)$ via the permutation of the indices of h_{ij}, g_{ij} and h_i. The action of the simple reflection s_n is given as follows:

$$s_n(h_{ij}) = h_{ij},\ s_n(g_{ij}) = g_{ij},\ s_n(h_i) = h_i, \quad \text{for } i,j \neq n,$$

$$s_n(h_{in}) = g_{in},\ s_n(g_{in}) = h_{in},\ s_n(h_n) = h_n^{-1}.$$

Define the twisted derivations Δ_{ij} $(i<j)$ and Δ_i on $GK(B_n)$ by

$$\Delta_{ij}(h_{kl}) = \begin{cases} 1, & \text{if } i=k \text{ and } j=l, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Delta_{ij}(g_{ij}) = \Delta_{ij}(h_k) = 0,$$

$$\Delta_i(h_j) = \begin{cases} 1, & \text{if } i=j, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Delta_i(h_{jk}) = \Delta_i(g_{jk}) = 0,$$

and the twisted Leibniz rule

$$\Delta_{ij}(xy) = \Delta_{ij}(x)y + s_{ij}(x)\Delta_{ij}(y),$$

$$\Delta_i(xy) = \Delta_i(x)y + s_i(x)\Delta_i(y).$$

Let us consider the operators $\mathcal{Q}_i := h_{i\,i+1}^{-1}\circ\Delta_{i\,i+1}$ $(i<n)$ and $\mathcal{Q}_n := h_n^{-1}\circ\Delta_n$ on $GK(B_n)$.

Lemma 5.1 *Let $\Theta_j := \Theta_j^{B_n}(1,1)$. We have*

$$\mathcal{Q}_i(\Theta_j) = \begin{cases} \Theta_{i+1}, & \text{if } j=i, \\ -\Theta_{i+1} & \text{if } j=i+1, \\ 0, & \text{otherwise,} \end{cases}$$

for $i<n$, and

$$\mathcal{Q}_n(\Theta_j) = \begin{cases} 1+\Theta_n^{-1}, & \text{if } j=n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It is clear that $\mathcal{Q}_i(\Theta_j) = h_{i\,i+1}^{-1}\Delta_{i\,i+1}(\Theta_j) = 0$ $(i<n)$ for $j\neq i, i+1$ and $\mathcal{Q}_n(\Theta_j) = h_n^{-1}\Delta_n(\Theta_j) = 0$ for $j\neq n$. We have by direct computation

$$\mathcal{Q}_i(\Theta_i) = h_{i\,i+1}^{-1}\Delta_i\left(\prod_{k=i-1}^{1} h_{ki}^{-1}\cdot h_i \prod_{k=1,k\neq i}^{n} g_{ki}\cdot h_i\cdot\prod_{k=n}^{i+1} h_{ik}\right)$$

$$= h_{i\,i+1}^{-1}\cdot\left(\prod_{k=i-1}^{1} h_{k\,i+1}^{-1}\cdot h_{i+1}\prod_{k=1,k\neq i+1}^{n} g_{k\,i+1}\cdot h_{i+1}\cdot\prod_{k=n}^{i+2} h_{i+1\,k}\right) = \Theta_{i+1},$$

$$\mathcal{Q}_i(\Theta_{i+1}) = h_{i\,i+1}^{-1}\Delta_i\left(\prod_{k=i}^{1} h_{k\,i+1}^{-1}\cdot h_{i+1}\prod_{k=1,k\neq i+1}^{n} g_{k\,i+1}\cdot h_{i+1}\cdot\prod_{k=n}^{i+2} h_{i+1\,k}\right)$$

$$= h_{i\,i+1}^{-1}\cdot\left(-\prod_{k=i-1}^{1} h_{k\,i+1}^{-1}\cdot h_{i+1}\prod_{k=1,k\neq i+1}^{n} g_{k\,i+1}\cdot h_{i+1}\cdot\prod_{k=n}^{i+2} h_{i+1\,k}\right) = -\Theta_{i+1}.$$

Similarly,

$$\mathcal{Q}_n(\Theta_n) = h_n^{-1}\Delta_n\left(\prod_{k=n-1}^{1} h_{kn}^{-1}\cdot h_n\prod_{k=1}^{n-1} g_{kn}\cdot h_n\right)$$

$$= h_n^{-1}\left(h_n + \prod_{k=n-1}^{1} g_{kn}^{-1}\cdot h_n^{-1}\prod_{k=1}^{n-1} h_{kn}\right) = 1+\Theta_n^{-1}. \quad \blacksquare$$

Lemma 5.2 *The simple reflections act on the elements* $\Theta_1,\ldots,\Theta_n$ *as follows.*

$$h_{i\,i+1}^{-1}\cdot s_i(\Theta_j)\cdot h_{i\,i+1} = \Theta_{s_i(j)}, \quad \text{for } i=1,\ldots,n-1,$$

$$h_n^{-1}\cdot s_n(\Theta_j)\cdot h_n = \begin{cases}\Theta_n^{-1}, & \text{if } j=n,\\ \Theta_j, & \text{otherwise.}\end{cases}$$

Proof. If $j\neq i,i+1$, then the equality

$$h_{i\,i+1}^{-1}\cdot s_i(\Theta_j)\cdot h_{i\,i+1} = \Theta_j$$

follows from the relations

$$h_{i\,i+1}^{-1}h_{ji}h_{j\,i+1}h_{i\,i+1} = h_{j\,i+1}h_{ji}$$

and

$$h_{i\,i+1}^{-1}g_{i+1\,j}g_{ji}h_{i\,i+1} = g_{ij}g_{i+1\,j}.$$

We also have

$$\begin{aligned} &h_{i\,i+1}^{-1}\cdot s_i(\Theta_i)\cdot h_{i\,i+1}\\ &= h_{i\,i+1}^{-1}\left(\prod_{k=i-1}^{1} h_{k\,i+1}^{-1}\cdot h_{i+1}\prod_{k=1,k\neq i+1}^{n} g_{k\,i+1}\cdot h_{i+1}\cdot\prod_{k=n}^{i+2} h_{i+1\,k}\cdot h_{i+1\,i}\right)\cdot h_{i\,i+1}\\ &= \Theta_{i+1}, \end{aligned}$$

and this completes the proof of the first equality.

We can obtain

$$h_n^{-1}h_{jn}h_jg_{jn}h_n = g_{jn}h_jh_{jn}$$

from the B_2 Yang-Baxter relation. This shows the equality

$$h_n^{-1}\cdot s_n(\Theta_j)\cdot h_n = \Theta_j$$

for $j \neq n$. Since

$$s_n(\Theta_n) = \prod_{k=n-1}^{1} g_{kn}^{-1} \cdot h_n^{-1} \prod_{k=1}^{n-1} h_{kn} \cdot h_n^{-1},$$

we have

$$h_n^{-1} \cdot s_n(\Theta_n) \cdot h_n = \Theta_n^{-1}. \quad \blacksquare$$

Consider the action of $W(B_n)$ on the ring of Laurent polynomials $\mathbb{Q}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ via

$$(wf)(X_1, \ldots, X_n) := f(X_{w(1)}, \ldots, X_{w(n)}), \quad w \in S_n = W(A_{n-1}),$$

and

$$(s_n f)(X_1, \ldots, X_n) := f(X_1, \ldots, X_{n-1}, X_n^{-1}).$$

Lemma 5.3 *Let $F(\Theta)$ and $G(\Theta)$ be Laurent polynomials in $\Theta_1, \ldots, \Theta_n$. Then,*

$$\mathcal{Q}_i(F(\Theta)G(\Theta)) = \mathcal{Q}_i(F(\Theta))G(\Theta) + (s_i F)(\Theta)\mathcal{Q}_i(G(\Theta)), \quad i = 1, \ldots, n.$$

Proof. The equalities in Lemma 5.2 imply

$$h_{i\,i+1}^{-1} \cdot s_i(F(\Theta)) \cdot h_{i\,i+1} = (s_i F)(\Theta),$$

so

$$\begin{aligned}
\mathcal{Q}_i(F(\Theta)G(\Theta)) &= h_{i\,i+1}^{-1}\Delta_{i\,i+1}(F(\Theta)G(\Theta)) \\
&= h_{i\,i+1}^{-1}\Delta_{i\,i+1}(F(\Theta))G(\Theta) + h_{i\,i+1}^{-1} s_i(F(\Theta)) h_{i\,i+1} \cdot h_{i\,i+1}^{-1}\Delta_{i\,i+1}(G(\Theta)) \\
&= \mathcal{Q}_i(F(\Theta))G(\Theta) + (s_i F)(\Theta)\mathcal{Q}_i(G(\Theta))
\end{aligned}$$

for $i < n$. The equality

$$\mathcal{Q}_n(F(\Theta)G(\Theta)) = \mathcal{Q}_n(F(\Theta))G(\Theta) + (s_n F)(\Theta)\mathcal{Q}_n(G(\Theta))$$

is proved in the same way. $\blacksquare$

Define the operators $\tau_1, \ldots, \tau_{n-1}$ and $\tau_n := \tau_n^{B_n}$ on $\mathbb{Q}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ by

$$(\tau_i f)(X) := X_{i+1} \frac{f(X) - (s_i f)(X)}{X_i - X_{i+1}}, \quad i = 1, \ldots, n-1,$$

$$(\tau_n f)(X) := \frac{f(X) - (s_n f)(X)}{X_n - 1}.$$

The operator corresponding to τ_n in the case of type C_n is given by

$$(\tau_n^{C_n} f)(X) := \frac{f(X) - (s_n f)(X)}{X_n^2 - 1}.$$

We consider the group $W(D_n)$ as the subgroup of $W(B_n)$. Let $\tau_n^{D_n} := \tau_n^{B_n} \tau_{n-1} \tau_n^{B_n}$. Then we have

$$(\tau_n^{D_n} f)(X_1, \cdots, X_{n-1}, X_n) = \frac{f(X_1, \cdots, X_{n-1}, X_n) - f(X_1, \cdots, X_n^{-1}, X_{n-1}^{-1})}{X_{n-1}X_n - 1}.$$

Proposition 5.4 *Let* $\Theta_j := \Theta_j^{B_n}(1,1),\ \ 1 \leq j \leq n$, *then*

$$\mathcal{Q}_i(F(\Theta_1, \ldots, \Theta_n)) = (\tau_i F)(\Theta_1, \ldots, \Theta_n).$$

Proof. This follows from Lemmas 5.1 and 5.3. ■

Remark 5.5 One can obtain the corresponding results for A_{n-1} (resp. D_n) after specialization $g_{ij} = h_i = 1$ (resp. $h_i = 1$), $\forall i, j$.

Remark 5.6 All the construction in this section till Proposition 5.4 can be done on the level of the group algebra $\mathbb{Q}\langle \mathcal{GK}(B_n)\rangle$.

We have the homomorphisms

$$\varphi : GK(A_{n-1}) \to BE(A_{n-1}) \to \mathcal{B}_{A_{n-1}},$$

$$\varphi : GK(D_n) \to BE(D_n) \to \mathcal{B}_{D_n},$$

$$\varphi : GK(B_n) \to BE(B_n) \to \mathcal{B}_{B_n},$$

given by $h_{ij} \mapsto 1 + [ij]$, $g_{ij} \mapsto 1 + \overline{[ij]}$ and $h_i \mapsto 1 + [i]$.
Conjecturally, the homomorphism between the quartic algebra $B\mathcal{E}(B_n)$ and the Nichols-Woronowicz algebra $\mathcal{B}_{B_n}$ is an isomorphism. If so, the quadratic algebras $BE(A_{n-1})$ and $BE(D_n)$ [5] have to be isomorphic respectively to the Nichols-Woronowicz algebras $\mathcal{B}_{A_{n-1}}$ and $\mathcal{B}_{D_n}$.

The Nichols-Woronowicz algebra is equipped with the duality pairing

$$\langle\ ,\ \rangle : \mathcal{B}_{\mathcal{X}} \otimes \mathcal{B}_{\mathcal{X}} \to \mathbb{Q}$$

and naturally defined braided derivations acting on it. Here we are interested in the derivations $\overline{D}_{[\alpha]}$ given by the formula

$$\overline{D}_{[\alpha]}(\xi) = (\mathrm{id}_{\mathcal{B}} \otimes \langle\ ,\ \rangle)(\psi_{V,\mathcal{B}} \otimes \mathrm{id}_{\mathcal{B}})([\alpha] \otimes \xi_{(1)} \otimes \xi_{(2)}),$$

where $\psi_{V,\mathcal{B}} : V \otimes \mathcal{B} \to \mathcal{B} \otimes V$ is the braiding induced by ψ, and we use Sweedler's notation $\triangle(\xi) = \xi_{(1)} \otimes \xi_{(2)}$ for the coproduct $\triangle$ of the Nichols-Woronowicz algebra. The twisted derivations Δ_{ij}, $\Delta_{\overline{ij}}$ and Δ_i are corresponding to the derivations on the Nichols algebras, namely $\varphi(\Delta_{ij}(x)) = \overline{D}_{ij}(\varphi(x))$, $\varphi(\Delta_{\overline{ij}}(x)) = \overline{D}_{\overline{ij}}(\varphi(x))$, $\varphi(\Delta_i(x)) = \overline{D}_i(\varphi(x))$.

Let P be the root lattice associated to some root system and $\mathbb{Q}[P] = \mathbb{Q}[e^\lambda | \lambda \in P]$ its group algebra. Denote by $\epsilon : \mathbb{Q}[P] \to \mathbb{Q}$ the algebra homomorphism given by $e^\lambda \mapsto 1$, $\forall \lambda \in P$. The Grothendieck ring of the corresponding flag variety can be expressed as a quotient algebra $\mathbb{Q}[P]/I$, where the ideal I is generated by the W-invariant elements of form $f - \epsilon(f)$.

Theorem 5.7 *Let* F *be a Laurent polynomial in the defining ideal of the Grothendieck ring of the flag variety of classical type* $\mathcal{X}$, *and* $\Theta_j := \Theta_j^{\mathcal{X}}$. *Then,*

$$\varphi(F(\Theta_1, \ldots, \Theta_n)) = 0$$

in the corresponding Nichols-Woronowicz algebra $\mathcal{B}_{\mathcal{X}}$ ($\mathcal{X} = A_n$, B_n, C_n *or* D_n).

Proof. In the following, we consider the root system of type B_n. The cases of type A, C, D can be obtained from this case by a certain specialization. For simplicity, we use the same symbol Θ_i for the corresponding element to the RSM-elements in $\mathcal{B}_{B_n}$. Let $\epsilon_j(X) := e_j(X_1 + X_1^{-1}, \dots, X_n + X_n^{-1}) - e_j(2, \dots, 2)$. Proposition 5.4 implies that

$$\varphi(\mathcal{Q}_i(\epsilon_j(\Theta))) = 0.$$

Hence, we have $\overline{D}_{i\,i+1}(\epsilon_j(\Theta)) = 0$ and $\overline{D}_n(\epsilon_j(\Theta)) = 0$. From the W-invariance of the polynomial ϵ_j and Lemma 5.2, it follows that $s_k(\epsilon_j(\Theta)) = h_{k\,k+1}\epsilon_j(\Theta)h_{k\,k+1}^{-1}$ and $s_n(\epsilon_j(\Theta)) = h_n\epsilon_j(\Theta)h_n^{-1}$. Thus, for $k \neq i$,

$$\Delta_{i\,i+1}(s_k(\epsilon_j(\Theta))) = \Delta_{i\,i+1}(h_{k\,k+1}\epsilon_j(\Theta)h_{k\,k+1}^{-1})$$

$$= s_i(h_{k\,k+1})\Delta_{i\,i+1}(\epsilon_j(\Theta))h_{k\,k+1}^{-1} = 0.$$

For $k = i$,

$$\Delta_{i\,i+1}(s_i(\epsilon_j(\Theta))) = \Delta_{i\,i+1}(h_{i\,i+1}\epsilon_j(\Theta)h_{i\,i+1}^{-1})$$

$$= (\epsilon_j(\Theta) - h_{i\,i+1}^{-1} \cdot s_i(\epsilon_j(\Theta)) \cdot h_{i\,i+1})h_{i\,i+1}^{-1} + h_{i\,i+1}^{-1}\Delta_{i\,i+1}(\epsilon_j(\Theta))h_{i\,i+1}^{-1} = 0.$$

More generally, one can show that if $\Delta_{kl}(\epsilon_j(\Theta)) = 0$, then $\Delta_{kl}(s_i(\epsilon_j(\Theta))) = 0$. Since $w \circ \Delta_{kl} \circ w^{-1} = \Delta_{w(k)\,w(l)}$ for $w \in W$, we can conclude that $\overline{D}_{kl}(\epsilon_j(\Theta)) = \overline{D}_{\overline{kl}}(\epsilon_j(\Theta)) = 0$, $\forall k, l$. Similarly, $\overline{D}_k(\epsilon_j(\Theta)) = 0$, $\forall k$. Since the constant term of $\epsilon_j(\Theta)$ considered as a polynomial in $[ab]$'s, $\overline{[ab]}$'s and $[a]$'s is zero, it follows that $\epsilon_j(\Theta) = 0$ in $\mathcal{B}_{B_n}$. ■

Finally, let us remark that it follows from the above considerations that in the case of D_n we have relations $e_k(\Theta_1^{D_n} + (\Theta_1^{D_n})^{-1}, \cdots, \Theta_n^{D_n} + (\Theta_n^{D_n})^{-1}) = 0$ for $1 \leq k < n$ and the additional relation $\prod_{j=1}^n((\Theta_j^{D_n})^{1/2} - (\Theta_j^{D_n})^{-1/2}) = 0$ in $\mathcal{B}_{D_n}$.

6 The case of root system of type G_2

Let us consider the root system of type G_2. Let

$$\Psi_+ = \{a,\ b,\ c,\ d,\ e,\ f\}$$

be the set of positive roots, where a and f are the simple roots and $b = 3a + f$, $c = 2a + f$, $d = 3a + 2f$, $e = a + f$.

Definition 6.1 *Denote by $\mathcal{GK}(G_2)$ the group generated by six elements $h_a, h_b, h_c, h_d, h_e, h_f$ subject to the following relations:*

- $h_a\ h_d = h_d\ h_a$, $\quad h_b\ h_e = h_e\ h_b$, $\quad h_c\ h_f = h_f\ h_c$;
- (*A_2 - Yang-Baxter relation*) $\quad h_b\ h_d\ h_f = h_f\ h_d\ h_b$;
- (*G_2 - Yang-Baxter relation*)

$$h_a\ h_b\ h_c\ h_d\ h_e\ h_f = h_f\ h_e\ h_d\ h_c\ h_b\ h_a.$$

Proposition 6.2 *Define the RSM-elements of type G_2 in $\mathcal{GK}(G_2)$ as follows*

$$\Theta_1^{G_2} := h_d\ h_b\ h_c\ h_d\ h_e\ h_f, \quad \Theta_2^{G_2} := h_f^{-1}\ h_b\ h_d\ h_c\ h_b\ h_a.$$

Then we have $\Theta_1^{G_2}\ \Theta_2^{G_2} = \Theta_2^{G_2}\ \Theta_1^{G_2}$.

Let us consider the group algebra $\mathbb{Q}\langle\mathcal{GK}(G_2)\rangle$. The Weyl group $W(G_2)$ naturally acts on the algebra $\mathbb{Q}\langle\mathcal{GK}(G_2)\rangle$. The twisted derivations Δ_a and Δ_f determined by the conditions

$$\Delta_a(h_i) = \begin{cases} 1, & \text{if} \quad i = a, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Delta_f(h_i) = \begin{cases} 1, & \text{if} \quad i = f, \\ 0, & \text{otherwise,} \end{cases}$$

and the twisted Leibniz rule are well-defined on $\mathbb{Q}\langle\mathcal{GK}(G_2)\rangle$. Let $\mathcal{Q}_a := h_a^{-1} \circ \Delta_a$ and $\mathcal{Q}_f := h_f^{-1} \circ \Delta_f$. The action of the simple reflections s_a and s_f on the Laurent polynomial ring $\mathbb{Q}[X_1^{\pm 1}, X_2^{\pm 1}]$ is given by

$$s_a(X_1) = X_1, \quad s_a(X_2) = X_1 X_2^{-1},$$

$$s_f(X_1) = X_2, \quad s_f(X_2) = X_1.$$

Define the operators $\tau_a^{G_2}$ and $\tau_f^{G_2}$ acting on $\mathbb{Q}[X_1^{\pm 1}, X_2^{\pm 1}]$ by

$$(\tau_a^{G_2} F)(X_1, X_2) := X_1 \frac{F(X_1, X_2) - (s_a F)(X_1, X_2)}{X_2^2 - X_1},$$

$$(\tau_f^{G_2} F)(X_1, X_2) := X_2 \frac{F(X_1, X_2) - (s_f F)(X_1, X_2)}{X_1 - X_2}.$$

The arguments as in the previous section show the following.

Proposition 6.3

$$\mathcal{Q}_a F(\Theta_1, \Theta_2) = (\tau_a^{G_2} F)(\Theta_1, \Theta_2), \quad \mathcal{Q}_f F(\Theta_1, \Theta_2) = (\tau_f^{G_2} F)(\Theta_1, \Theta_2).$$

Proposition 6.4 There exists a natural homomorphism from $\mathbb{Q}\langle\mathcal{GK}(G_2)\rangle$ to the Nichols algebra $\mathcal{B}_{G_2}$ obtained by $h_\alpha \mapsto 1 + [\alpha]$, $\alpha \in \Psi_+$. In other words, the G_2 Yang-Baxter relation holds in $\mathcal{B}_{G_2}$.

Proof. The Yang-Baxter relations give a set of relations among $[a], \ldots, [f]$ up to degree six. It is easy to check the compatibility for the quadratic relations and those from subsystems of type A_2. The rest of cubic relations and the ones of higher degree can be verified by direct computation with help of the factorization of the braided symmetrizer, [1]. ■

The independent $W(G_2)$-invariant Laurent polynomials are given by

$$\phi_1(X_1, X_2) = X_1 + X_1^{-1} + X_2 + X_2^{-1} + X_1 X_2^{-1} + X_1^{-1} X_2,$$

$$\phi_2(X_1, X_2) = X_1 X_2 + X_1^{-1} X_2^{-1} + X_1^2 X_2^{-1} + X_1^{-1} X_2^2 + X_1^{-2} X_2 + X_1 X_2^{-2}.$$

The propositions above imply:

Theorem 6.5 *We have* $\phi_1(\Theta_1, \Theta_2) = \phi_2(\Theta_1, \Theta_2) = 6$ *in the Nichols algebra* $\mathcal{B}_{G_2}$*, so the subalgebra of* $\mathcal{B}_{G_2}$ *generated by the images of the RSM-elements* $\Theta_1^{G_2}$ *and* $\Theta_2^{G_2}$ *is isomorphic to the Grothendieck ring of the flag variety of type* G_2*.*

Definition 6.6 *Define the algebra* $B\mathcal{E}(G_2)$ *as an associative algebra over* $\mathbb{Q}$ *with generators* $\{a, b, c, d, e, f\}$ *subject to the relations*

- (*Commutativity*) $ad = da, \quad be = eb, \quad cf = fc$;
- (*Quadratic relations*) $ae = ec + ca, \ ea = ce + ac, \ fb = df + bd, \ bf = fd + db$;

$$af = ba + cb + dc + ed + fe, \quad fa = ab + bc + cd + de + ef;$$

- (*Quartic relations*)

$$abac + acab + acbc = baca + cbca + caba, \ dfef + dedf + efdf = fded + fdfe + fefd,$$

$$abde + bcde + bcef + ecdb = cdbc + cdcd + decd + fdca,$$

$$bdce + edcb + edba + fecb = cbdc + dcdc + dcec + acdf,$$

- (G_2 *Yang–Baxter relation*) $abcdef = fedcba$.

Conjecture 6.7 *The relations* $\phi_1(\Theta_1, \Theta_2) = \phi_2(\Theta_1, \Theta_2) = 6$ *are still valid in the algebra* $B\mathcal{E}(G_2)$*.*

Remark 6.8 One can show that there exists the natural epimorphism of algebras $B\mathcal{E}(G_2) \longrightarrow \mathcal{B}_{G_2}$, which has a non-trivial kernel, however.

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Research Institute for Mathematical Sciences,
Kyoto University,
Sakyo-ku, Kyoto 606-8502, Japan
e-mail: kirillov@kurims.kyoto-u.ac.jp

Department of Mathematics,
Kyoto University,
Sakyo-ku, Kyoto 606-8502, Japan
e-mail: maeno@math.kyoto-u.ac.jp