MATRIX FACTORIZATIONS AND REPRESENTATIONS OF QUIVERS I

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Dedicated to Professor Kyoji Saito on the occasion of his 60th birthday

Abstract. This paper introduces a mathematical definition of the category of D-branes in Landau-Ginzburg orbifolds in terms of A_∞-categories. Our categories coincide with the categories of equivariant matrix factorizations for quasi-homogeneous polynomials. After setting up the necessary definitions, we prove that our category for the polynomial x^{n+1} is equivalent to the derived category of representations of the Dynkin quiver of type A_n. We also construct a special stability condition for the triangulated category in the sense of T. Bridgeland, which should be the "origin" of the space of stability conditions.

1. Introduction

This paper introduces new triangulated categories associated to quasi-homogeneous polynomials which define isolated singularities only at the origin and relates those categories with the derived categories of representations of quivers. Our motivation comes from K. Saito's theory of primitive forms, especially from a problem in his study on regular weight systems and generalized root systems [Sa1]. We will explain the problem below.

Let $(a, b, c; h)$ be a quadruple of positive integers such that the function

$$\chi(T) := \frac{(T^{h-a}-1)(T^{h-b}-1)(T^{h-c}-1)}{(T^a-1)(T^b-1)(T^c-1)}$$

has no poles. Such a quadruple $W := (a, b, c; h)$ is called a regular weight system. It is known that W is a regular weight system if and only if we have at least one polynomial $f(x, y, z) \in \mathbb{C}[x, y, z]$ such that

$$ax\frac{\partial f}{\partial x} + by\frac{\partial f}{\partial y} + cz\frac{\partial f}{\partial z} = hf$$

and

$$X_0 := \{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0\}$$

has an isolated singularity only at the origin. Note that the (restricted) map

$$f : \mathbb{C}^3 \backslash f^{-1}(0) \to \mathbb{C}\backslash\{0\}$$

is a topologically locally trivial fiber bundle, and the general fiber $X_1 := f^{-1}(1)$ (called the Milnor fiber) is an open 2-dimensional complex manifold whose second homology group

$H_2(X_1,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $\mu := (h-a)(h-b)(h-c)/abc = \lim_{t\to 1}\chi(T)$. Since X_1 is real 4-dimensional, $H_2(X_1,\mathbb{Z})$ has an intersection form

$$I_{H_2(X_1,\mathbb{Z})} : H_2(X_1,\mathbb{Z})\times H_2(X_1,\mathbb{Z})\to\mathbb{Z}.$$

If f is a defining polynomial of a simple (ADE) singularity, then $(H_2(X_1,\mathbb{Z}), -I_{H_2(X_1,\mathbb{Z})})$ gives the root lattice of the finite root system corresponding to the singularity. In [Sa1], it is shown that $(H_2(X_1,\mathbb{Z}), -I_{H_2(X_1,\mathbb{Z})})$ with the set of vanishing cycles (which corresponds to the set of roots) and the Milnor monodromy (which corresponds to the Coxeter transformation) satisfies the axioms of the generalized root system which naturally extends the classical (finite) root systems. Since both weight systems and generalized root systems are combinatorial, it is natural to propose the following problem.

Problem 1.1. ([Sa1])
Construct directly from a regular weight system W, without passing through the homology group $H_2(X_1,\mathbb{Z})$ of the Milnor fiber, arithmetically or combinatorially, the generalized root system of the vanishing cycles.

The purpose of this paper is to develop a necessary tools in terms of A_∞-categories and to give a partial answer to the above problem. Let k be a field of characteristic zero. First, we introduce a notion of $\mathbb{Q}$-graded A_∞-categories over k (Definition 2.1) in order to consider the polynomial ring $\mathbb{C}[x_1,\dots x_n]$ with a polynomial f satisfying the quasi-homogeneous condition

$$\sum_{i=1}^{n}\frac{2a_i}{h}\cdot x_i\frac{\partial f}{\partial x_i}=2f, \tag{1.1}$$

where $a_1,\dots,a_n$ and h are positive integers such that the greatest common divisor of them is 1, as a usual $\mathbb{Z}_2$-graded A_∞-category with $m_0(1)=f$ and an "extra $\frac{2}{h}\cdot\mathbb{Z}$-grading". We shall denote the $\mathbb{Q}$-graded A_∞-category defined by $f\in\mathbb{C}[x_1,\dots,x_n]$ by $\mathcal{A}_f$ (Example-Definition 2.3).

Next, we consider the category of twisted complexes over $\mathbb{Q}$-graded A_∞-categories (Proposition 2.15) and the derived category of $\mathbb{Q}$-graded A_∞-categories (Definition 2.17). The important fact is that the twisted complexes over $\mathcal{A}_f$ coincide with matrix factorizations of f introduced by Eisenbud [E] in his study of maximal Cohen-Macauley modules. Since we consider the quasi-homogeneous polynomial f, we have a group action ($\mathbb{Z}$-action) on the category of matrix factorizations. Inspired by the work by Hori and Walcher [HW], we introduce the $\mathbb{Z}$-equivariant derived category of $\mathcal{A}_f$ denoted by $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ (Definition 2.23). We can now propose a conjecture to K. Saito's problem.

Conjecture 1.2. *Let W be a regular weight system and f be a quasi-homogeneous polynomial attached to W. Assume W has a dual regular weight system $W^*=(a^*,b^*,c^*;h)$ in the sense* *of* [Sa2] *and let* f^* *be a quasi-homogeneous polynomial attached to* W^*. *Then the following should hold.*

(i) $D^b_{\mathbb{Z}}(\mathcal{A}_{f^*})$ *is generated as a triangulated category by objects* $\{E_1, \ldots, E_\mu\}$ *such that*

$$\mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A}_{f^*})}(E_i, E_j) = 0, \quad \text{if } i > j, \quad \mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A}_{f^*})}(E_i, E_j[k]) = 0, \quad k \neq 0, {}^\forall i, j. \tag{1.2}$$

That is to say, $D^b_{\mathbb{Z}}(\mathcal{A}_{f^*})$ *is generated by a strongly exceptional collection.*

(ii) $D^b_{\mathbb{Z}}(\mathcal{A}_{f^*})$ *has the Serre functor* S *such that* $S^h \simeq [3h - 2a^* - 2b^* - 2c^*]$ *where* $[1]$ *is the shift functor on* $D^b_{\mathbb{Z}}(\mathcal{A}_{f^*})$.

(iii) *Let* $a_{ij} := \chi(E_i, E_j) = \dim_k \mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A}_{f^*})}(E_i, E_j)$. *Put* $A := (a_{ij})$ *and* $I_{K_0(D^b_{\mathbb{Z}}(\mathcal{A}_{f^*}))} = A^{-1} + {}^tA^{-1}$. *Then* $(K_0(D^b_{\mathbb{Z}}(\mathcal{A}_{f^*})), I_{K_0(D^b_{\mathbb{Z}}(\mathcal{A}_{f^*}))})$ *is isomorphic to* $(H_2(X_1, \mathbb{Z}), -I_{H_2(X_1,\mathbb{Z})})$ *as a lattice.*

This conjecture is based on the relation between the duality of regular weight systems and the mirror symmetry of Landau-Ginzburg orbifolds (see [T]). We do not discuss this background in detail here but we write the following diagram for reader's convenience.

$$\begin{array}{ccc}
\text{Quasi-homogeneous polynomial } f \text{ for } W & \overset{Milnor fiber}{\Longrightarrow} & \{\text{Vanishing cycles in } X_1 = f^{-1}(1)\} \\
\text{Duality of weights + Orbifold} \downarrow & & || \\
\{\text{B-branes in LG orbifold } W^*//(\mathbb{Z}/h\mathbb{Z})\} & \overset{Mirror Symmetry}{\simeq} & \{\text{A-branes in LG model for } W\}
\end{array}$$

For ADE singularities, we know that $W \simeq W^*$ and the generalized root systems for them are the classical finite root systems. Therefore, we may expect that the following conjecture should hold.

Conjecture 1.3. *Let* W *be a regular weight system corresponding to an ADE singularity and* f *be a quasi-homogeneous polynomial attached to* W. *Then* $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ *is equivalent as a triangulated category to the bounded derived category of representations of Dynkin quiver corresponding to the type of singularity of* f.

In this paper, we will prove the conjecture for A_n-singularities (Theorem 3.1), where we reduce to the case $f := x^{n+1} \in \mathbb{C}[x]$ by Knörrer's periodicity [K] (see also [O]). We will give a proof of the above conjecture for general cases in a separate paper [KST].

Finally, we will construct a special stability condition for the triangulated category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ in the sense of T. Bridgeland [B] for $f = x^h$. We can naturally introduce in our formulation the phase of objects (Definition 4.1) and the central charge Z_ω (Definition 4.3).

While our preparation of this paper, the paper [W] by J. Walcher appeared where he is studying from physical point of view the similar categories and the stability conditions on them (his notion of "R-stability").

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2. $\mathbb{Q}$-graded A_∞-categories

In this section, we set up several definitions which we will use in the later sections. Let k be a field of characteristic zero.

Definition 2.1. *Let h be a positive number. A $\mathbb{Q}$-graded A_∞-category $\mathcal{A}$ of index h is a collection of the following data.*

(i) *A set of objects $Ob(\mathcal{A})$,*

(ii) *A set of homomorphisms, a $\mathbb{Q}\times\mathbb{Z}_2$ graded k-linear vector space for each $a,b\in Ob(\mathcal{A})$*

$$\mathcal{A}(a,b)=\bigoplus_{q\in\mathbb{Q}}\mathcal{A}^q(a,b)_+\oplus\mathcal{A}^q(a,b)_-,$$

such that

$$\mathcal{A}^q(a,b)_+=0,\quad q\notin\frac{2}{h}\mathbb{Z},\quad \mathcal{A}^q(a,b)_-=0,\quad q-1\notin\frac{2}{h}\mathbb{Z}.$$

We call the subspaces

$$\mathcal{A}(a,b)_+:=\bigoplus_{q\in\frac{2}{h}\mathbb{Z}}\mathcal{A}^q(a,b)_+,\quad \mathcal{A}(a,b)_-:=\bigoplus_{q-1\in\frac{2}{h}\mathbb{Z}}\mathcal{A}^q(a,b)_-$$

the even and the odd subspaces.

(iii) *for $n\geq 0$, k-multilinear maps*

$$m_n^{\mathcal{A}}:\mathcal{A}(a_{n-1},a_n)\otimes\cdots\otimes\mathcal{A}(a_0,a_1)\to\mathcal{A}(a_0,a_n),\quad a_i\in Ob(\mathcal{A}),$$

of degree $2-n$ with respect to the $\mathbb{Q}$-grading which is even (odd) with respect to the $\mathbb{Z}_2$-grading when n is even (odd), where $m_0^{\mathcal{A}}$ is a map

$$m_0^{\mathcal{A}}:k\to\mathcal{A}(a,a).$$

The multilinear maps satisfy the following (A_∞-relation). For fixed n, we have

$$\sum_{r+s+t=n}\sum_{r+1+t=u}(-1)^{|x_1|+\cdots+|x_r|+r}m_u^{\mathcal{A}}(x_{r+s+t}\otimes\cdots\otimes x_{r+s+1}\otimes m_s^{\mathcal{A}}(x_{r+s}\otimes\cdots\otimes x_{r+1})\otimes x_r\otimes\cdots\otimes x_1)=0,\quad (2.1)$$

where $|x_i|$ is the parity of the morphism defined by

$$|x_i| := \begin{cases} 0, & x_i \in \mathcal{A}(a_{i-1}, a_i)_+, \\ 1, & x_i \in \mathcal{A}(a_{i-1}, a_i)_- \end{cases}. \tag{2.2}$$

Remark 2.2. *$\mathbb{Q}$-graded A_∞-category of index 1 is nothing but an A_∞-category with the usual $\mathbb{Z}$-grading, we call it a $\mathbb{Z}$-graded $\mathcal{A}_\infty$-category or simply an $\mathcal{A}_\infty$-category. See* [F] *and* [Se] *for details of homological algebra of $\mathcal{A}_\infty$-categories.*

We write down explicitly the relation (2.1) when $m_n^{\mathcal{A}} = 0$ for $n \geq 3$. For $x, y, z \in \oplus_{a,b}\mathcal{A}(a, b)$, we have

$$\begin{gathered} m_1^{\mathcal{A}}(m_0^{\mathcal{A}}(1)) = 0, \\ m_1^{\mathcal{A}}(m_1^{\mathcal{A}}(x)) = (-1)^{|x|} m_2^{\mathcal{A}}(m_0^{\mathcal{A}}(1) \otimes x) - m_2^{\mathcal{A}}(x \otimes m_0^{\mathcal{A}}(1)), \\ m_1^{\mathcal{A}}(m_2^{\mathcal{A}}(x \otimes y)) = (-1)^{|y|} m_2^{\mathcal{A}}(m_1^{\mathcal{A}}(x) \otimes y) - m_2^{\mathcal{A}}(x \otimes m_1^{\mathcal{A}}(y)), \\ m_2^{\mathcal{A}}(m_2^{\mathcal{A}}(x \otimes y) \otimes z) = (-1)^{|z|} m_2^{\mathcal{A}}(x \otimes m_2^{\mathcal{A}}(y \otimes z)). \end{gathered}$$

Put $u := -m_0^{\mathcal{A}}(1)$, $d(x) := (-1)^{|x|+1} m_1^{\mathcal{A}}(x)$ and $x \cdot y := (-1)^{|y|} m_2^{\mathcal{A}}(x \otimes y)$. Then a triple $(u, d, \cdot)$ defines on $\oplus_{a,b}\mathcal{A}(a, b)$ a curved differential graded (CDG) algebra structure [KL1].

Example-Definition 2.3. *Let $f \in \mathbb{C}[x_1, \ldots x_n]$ be a polynomial which satisfies the following quasi-homogeneous condition:*

$$\sum_{i=1}^{n} \frac{2a_i}{h} \cdot x_i \frac{\partial f}{\partial x_i} = 2f, \tag{2.3}$$

where $a_1, \ldots, a_n$ and h are positive integers such that the greatest common divisor of them is 1. We denote by $\mathcal{A}_f$ the $\mathbb{Q}$-graded A_∞-category of index h defined as follows:

$$Ob(\mathcal{A}_f) = \{a\},$$

$$\begin{gathered} \mathcal{A}_f(a, a) := \mathbb{C}[x_1, \ldots x_n], \quad \mathcal{A}_f(a, a)_- := 0, \\ m_0^{\mathcal{A}_f}(1) := -f \in \mathcal{A}_f^2(e, e)_+, \quad m_1^{\mathcal{A}_f} := 0, \\ m_2^{\mathcal{A}_f}(\alpha \otimes \beta) := \alpha \cdot \beta, \quad \alpha, \beta \in \mathbb{C}[x_1, \ldots x_n], \end{gathered}$$

where $\cdot$ is the usual product on $\mathbb{C}[x_1, \ldots x_n]$.

In the above example, we have the special element $1 \in \mathcal{A}_f^0(a, a)_+$ which defines a unit of the algebra $\mathbb{C}[x_1, \ldots x_n]$. It is well-known that the notion of units in A_∞-categories can be introduced as follows:

Definition 2.4. *Let $a \in Ob(\mathcal{A})$. $e_a \in \mathcal{A}^0(a,a)_+$ is called a unit if*

$$m_2^{\mathcal{A}}(x, e_a) = y, \quad m_2^{\mathcal{A}}(e_a, y) = (-1)^{|y|} y, \tag{2.4}$$

for $x \in \mathcal{A}(a,b)$ and $y \in \mathcal{A}(b,a)$, and for $n \neq 2$

$$m_n^{\mathcal{A}}(x_1, \dots, x_n) = 0, \tag{2.5}$$

if one of x_i coincides with e_a.

Remark 2.5. *It is easy to check that if a unit exists then it is unique.*

Definition 2.6. $\mathbb{Q}$*-graded A_∞-category is called unital if each object has a unit.*

Let $\mathcal{A}$ be a $\mathbb{Q}$-graded A_∞-category of index h. We can construct another $\mathbb{Q}$-graded A_∞-category $\overline{\mathcal{A}}$ of index h from $\mathcal{A}$ as follows:

Definition 2.7. *Let $\mathcal{A}$ be a unital $\mathbb{Q}$-graded A_∞-category of index h.*

(i) *a set of objects $Ob(\overline{\mathcal{A}})$ is given by*

$$Ob(\overline{\mathcal{A}}) := \left\{ a\{\frac{2k}{h}\}, a \in Ob(\mathcal{A}), k \in \mathbb{Z}, \quad b\{\frac{2l}{h}\}[-1], b \in Ob(\mathcal{A}), l \in \mathbb{Z} \right\}, \tag{2.6}$$

(ii) *a set of homomorphisms is given by*

$$\overline{\mathcal{A}}^q \left(a\{\frac{2k}{h}\}, b\{\frac{2l}{h}\} \right)_\pm := \mathcal{A}^{q + \frac{2(l-k)}{h}}(a,b)_\pm, \tag{2.7}$$

$$\overline{\mathcal{A}}^q \left(a\{\frac{2k}{h}\}, b\{\frac{2l}{h}\}[-1] \right)_\pm := \mathcal{A}^{q + \frac{2(l-k)}{h} - 1}(a,b)_\mp, \tag{2.8}$$

$$\overline{\mathcal{A}}^q \left(a\{\frac{2k}{h}\}[-1], b\{\frac{2l}{h}\} \right)_\pm := \mathcal{A}^{q + \frac{2(l-k)}{h} + 1}(a,b)_\mp, \tag{2.9}$$

$$\overline{\mathcal{A}}^q \left(a\{\frac{2k}{h}\}[-1], b\{\frac{2l}{h}\}[-1] \right)_\pm := \mathcal{A}^{q + \frac{2(l-k)}{h}}(a,b)_\pm. \tag{2.10}$$

(iii) *k-multilinear maps $m_n^{\overline{\mathcal{A}}}$ are defined using those on $\mathcal{A}$ with additional signs as follows. For $x_1 \in \overline{\mathcal{A}}(a_0, a_1), \dots, x_n \in \overline{\mathcal{A}}(a_{n-1}, a_n)$,*

$$m_n^{\overline{\mathcal{A}}}(x_n \otimes \cdots \otimes x_1) := (-1)^{|a_0|} m_n^{\mathcal{A}}(x_n \otimes \cdots \otimes x_1),$$

where we regard x_i in the right hand side as a homomorphism in $\mathcal{A}$ by the above definition (ii) and $|a_0|$ is the parity of a_0 defined by

$$|a_0| := \begin{cases} 0, & a_0 = a\{\frac{2k}{h}\}, \quad a \in Ob(\mathcal{A}), k \in \mathbb{Z} \\ 1, & a_0 = a\{\frac{2l}{h}\}[-1], \quad a \in Ob(\mathcal{A}), l \in \mathbb{Z} \end{cases}. \tag{2.11}$$

A_∞-functors for $\mathbb{Q}$-graded A_∞-categories can be defined in an obvious way. There are the "translation" functor $\{\frac{2}{h}\}$ and the shift functor $[1]$ on $\overline{\mathcal{A}}$.

Proposition 2.8. *The following functors* $\{\frac{2}{h}\}$ *and* $[1]$ *define autoequivalences of* $\overline{\mathcal{A}}$ *:*

$$\{\frac{2}{h}\}\left(a\{\frac{2k}{h}\}\right) := a\{\frac{2(k+1)}{h}\}, \quad \{\frac{2}{h}\}\left(a\{\frac{2k}{h}\}[-1]\right) := a\{\frac{2(k+1)}{h}\}[-1], \tag{2.12}$$

and

$$[1]\left(a\{\frac{2k}{h}\}\right) := a\{\frac{2(k+h)}{h}\}[-1], \quad [1]\left(a\{\frac{2k}{h}\}[-1]\right) := a\{\frac{2k}{h}\}. \tag{2.13}$$

Put $\{\frac{2k}{h}\} := \{\frac{2}{h}\}^k$ *and* $[l] := [1]^l$ *for* $k, l \in \mathbb{Z}$*. We have the relation* $\{\frac{2h}{h}\} = [2]$*.* □

Let $\overline{\mathcal{A}}$ be an $\mathbb{Q}$-graded A_∞-category of index h. Consider the $\mathbb{Q}$-graded A_∞-category $\widetilde{\mathcal{A}}$ of index h whose set of objects is the set of finite (formal) direct sums of objects of $\overline{\mathcal{A}}$,

$$Ob(\widetilde{\mathcal{A}}) := \left\{ a = \bigoplus_i a_i\{\frac{2k_i}{h}\} \oplus \bigoplus_j a_j\{\frac{2l_j}{h}\}[-1], \quad a_i, a_j \in Ob(\overline{\mathcal{A}}), \quad k_i, l_j \in \mathbb{Z} \right\}, \tag{2.14}$$

whose set of homomorphisms is

$$\begin{aligned}\widetilde{\mathcal{A}}(a,b) := &\bigoplus_{i_1,i_2} \overline{\mathcal{A}}(a_{i_1}\{\frac{2k_{i_1}}{h}\}, b_{i_2}\{\frac{2k_{i_2}}{h}\}) \oplus \bigoplus_{i_1,j_2} \overline{\mathcal{A}}(a_{i_1}\{\frac{2k_{i_1}}{h}\}, b_{j_2}\{\frac{2l_{j_2}}{h}\}[-1]) \\ &\oplus \bigoplus_{j_1,i_2} \overline{\mathcal{A}}(a_{j_1}\{\frac{2l_{j_1}}{h}\}[-1], b_{i_2}\{\frac{2k_{i_2}}{h}\}) \oplus \bigoplus_{j_1,j_2} \overline{\mathcal{A}}(a_{j_1}\{\frac{2l_{j_1}}{h}\}, b_{j_2}\{\frac{2l_{j_2}}{h}\}[-1]), \end{aligned} \tag{2.15}$$

and whose k-linear maps are defined by those on $\overline{\mathcal{A}}$ using the natural "matrix multiplication" rule.

Definition 2.9. *Take an object* $a \in Ob(\widetilde{\mathcal{A}})$ *and* $Q \in \widetilde{\mathcal{A}}(a,a)_-$*.* $(a;Q)$ *is called a twisted complex if* Q *satisfies the Maurer-Cartan equation*

$$\sum_{n\geq 0} m_n^{\widetilde{\mathcal{A}}}(Q^{\otimes n}) = 0. \tag{2.16}$$

The set of all twisted complexes is denoted by $Ob(Tw(\mathcal{A}))$*. If* $Q \in \widetilde{\mathcal{A}}^1(a,a)_-$ *in addition, then* $(a;Q)$ *is called a graded twisted complex and we denote the set of all graded twisted complexes by* $Ob(Tw_{\mathbb{Z}}(\mathcal{A}))$*.*

An assumption is necessary for the equation (2.16) to make sense. If $m_0^{\mathcal{A}} = 0$, then it is usually introduced that the notion of one-sided twisted complexes which makes the sum in the equation (2.16) finite. However we study in this paper the case when $m_0^{\mathcal{A}} \neq 0$, we shall assume that our A_∞-categories have no higher product, in other words, $m_n^{\mathcal{A}} = 0$ for all $n \geq 3$.

We often write Q in the following form:

$$Q = \begin{pmatrix} Q_{++} & Q_{-+} \\ Q_{+-} & Q_{--}, \end{pmatrix} \tag{2.17}$$

where

$$Q_{\pm\pm} \in \widetilde{\mathcal{A}}(a_\pm, a_\pm)_-, Q_{\pm\mp} \in \widetilde{\mathcal{A}}(a_\pm, a_\mp)_+, \tag{2.18}$$

and $a_\pm$ are given by the following decomposition

$$a = a_+ + a_-[-1], \quad a_+ = \bigoplus_i a_{+,i}\{\frac{2k_i}{h}\}, \quad a_- = \bigoplus_i a_{-,i}\{\frac{2l_i}{h}\}. \tag{2.19}$$

Remark 2.10. *If $\mathcal{A}$ is a unital $\mathbb{Q}$-graded A_∞-category of index h with $m_n^{\mathcal{A}} = 0$, $n \geq 3$, then there exists at least one twisted complex for each object $a \in Ob(\widetilde{\mathcal{A}})$. Indeed,*

$$Q|_{ij} = \begin{cases} 0, & \text{for } i,j \text{ such that } a_{+,i} \neq a_{-,j} \text{ and } a_{+,i}\{\frac{2\cdot h}{h}\} \neq a_{-,j}, \\ \begin{pmatrix} 0 & m_0^{\mathcal{A}}(1) \\ e_{a_{+,i}} & 0 \end{pmatrix}, & \text{for } i,j \text{ such that } a_{+,i} = a_{-,j}, \\ \begin{pmatrix} 0 & e_{a_{+,i}} \\ m_0^{\mathcal{A}}(1) & 0 \end{pmatrix}, & \text{for } i,j \text{ such that } a_{+,i}\{\frac{2\cdot h}{h}\} = a_{-,j}, \end{cases} \tag{2.20}$$

is a twisted complex.

Example 2.11. *Since $\mathcal{A}_f$ has no odd homomorphisms, each twisted complex $(a = a_+ \oplus a_-[-1]; Q_a)$ has the following form*

$$Q_a := \begin{pmatrix} 0 & Q_{-+} \\ Q_{+-} & 0 \end{pmatrix}, \quad Q_{+-} \in \widetilde{\mathcal{A}}(a_+, a_-)_+, \quad Q_{-+} \in \widetilde{\mathcal{A}}(a_-, a_+)_+.$$

The Maurer-Cartan equation (2.16) *becomes*

$$-f \cdot \mathrm{Id} + Q_a^2 = 0. \tag{2.21}$$

This is exactly the same equation which first studied by Eisenbud [E] *in his work on maximal Cohen-Macauley modules. Q_a is called a matrix factorization of f.*

Let $\mathcal{A}$ be a unital $\mathbb{Q}$-graded A_∞-category of index h with $m_n^{\mathcal{A}} = 0$, $n \geq 3$.

Definition 2.12. *Let $\alpha := (a; Q_a)$ and $\beta := (b; Q_b)$ be twisted complexes. We first put*

$$Tw(\mathcal{A})(\alpha, \beta) := \widetilde{\mathcal{A}}(a,b)_+ \oplus \widetilde{\mathcal{A}}(a,b)_-. \tag{2.22}$$

We define a k-multilinear maps $m_n^{Tw(\mathcal{A})} (n = 0,1,2)$ by

$$m_0^{Tw(\overline{\mathcal{A}})}(1) := 0, \tag{2.23}$$

$$m_1^{Tw(\mathcal{A})}(\Phi) := m_1^{\widetilde{\mathcal{A}}}(\Phi) + m_2^{\widetilde{\mathcal{A}}}(Q_b \otimes \Phi) + m_2^{\widetilde{\mathcal{A}}}(\Phi \otimes Q_a), \tag{2.24}$$

where $\Phi \in Tw(\mathcal{A})(a,b)$ and

$$m_2^{Tw(\mathcal{A})}(\Psi_2 \otimes \Psi_1) := m_2^{\widetilde{\mathcal{A}}}(\Psi_2 \otimes \Psi_1), \tag{2.25}$$

for $\Psi_1 \in Tw(\mathcal{A})(a_0,a_1) = \widetilde{\mathcal{A}}(a_0,a_1)$ and $\Psi_2 \in Tw(\mathcal{A})(a_1,a_2) = \widetilde{\mathcal{A}}(a_1,a_2)$.

We often write the spaces of morphisms in the matrix form:

$$Tw(\mathcal{A})(\alpha,\beta)_\pm = \begin{pmatrix} \widetilde{\mathcal{A}}(a_+,b_+)_\pm & \widetilde{\mathcal{A}}(a_-,b_+)_\mp \\ \widetilde{\mathcal{A}}(a_+,b_-)_\mp & \widetilde{\mathcal{A}}(a_-,b_-)_\pm \end{pmatrix}.$$

Lemma 2.13. $(m_1^{Tw(\mathcal{A})})^2 = 0.$

Proof. *For $\Phi_\pm \in Tw(\mathcal{A})(a,b)_\pm$, we have*

$$\begin{aligned}(m_1^{Tw(\mathcal{A})})^2(\Phi_\pm) =&(m_1^{\widetilde{\mathcal{A}}})^2(\Phi_\pm) + m_1^{\widetilde{\mathcal{A}}}(m_2^{\widetilde{\mathcal{A}}}(Q_b\otimes\Phi_\pm)) + m_1^{\widetilde{\mathcal{A}}}(m_2^{\widetilde{\mathcal{A}}}(\Phi_\pm\otimes Q_a))\\ &+ m_2^{\widetilde{\mathcal{A}}}(Q_b\otimes(m_1^{\widetilde{\mathcal{A}}}(\Phi_\pm) + m_2^{\widetilde{\mathcal{A}}}(Q_b\otimes\Phi_\pm) + m_2^{\widetilde{\mathcal{A}}}(\Phi_\pm\otimes Q_a)))\\ &+ m_2^{\widetilde{\mathcal{A}}}((m_1^{\widetilde{\mathcal{A}}}(\Phi_\pm) + m_2^{\widetilde{\mathcal{A}}}(Q_b\otimes\Phi_\pm) + m_2^{\widetilde{\mathcal{A}}}(\Phi_\pm\otimes Q_a))\otimes Q_a)\\ =&\pm m_2^{\widetilde{\mathcal{A}}}(m_0^{\widetilde{\mathcal{A}}}(1)\otimes\Phi_\pm) - m_2^{\widetilde{\mathcal{A}}}(\Phi_\pm\otimes m_0^{\widetilde{\mathcal{A}}}(1)) \pm m_2^{\widetilde{\mathcal{A}}}(m_1^{\widetilde{\mathcal{A}}}(Q_b)\otimes\Phi_\pm)\\ &- m_2^{\widetilde{\mathcal{A}}}(\Phi_\pm\otimes m_1^{\widetilde{\mathcal{A}}}(Q_a)) \pm m_2^{\widetilde{\mathcal{A}}}(m_2^{\widetilde{\mathcal{A}}}(Q_b^{\otimes 2})\otimes\Phi_\pm) - m_2^{\widetilde{\mathcal{A}}}(\Phi_\pm\otimes m_2^{\widetilde{\mathcal{A}}}(Q_a^{\otimes 2}))\\ =&\pm m_2^{\widetilde{\mathcal{A}}}((m_0^{\widetilde{\mathcal{A}}}(1) + m_1^{\widetilde{\mathcal{A}}}(Q_b) + m_2^{\widetilde{\mathcal{A}}}(Q_b^{\otimes 2})\otimes\Phi_\pm)\\ &- m_2^{\widetilde{\mathcal{A}}}(\Phi_\pm\otimes(m_0^{\widetilde{\mathcal{A}}}(1 + m_1^{\widetilde{\mathcal{A}}}(Q_a) + m_2^{\widetilde{\mathcal{A}}}(Q_a^{\otimes 2}))\\ =&0.\end{aligned}$$

□

Lemma 2.14. *For $\Phi \in Tw(\mathcal{A})(a_0,a_1)$ and $\Psi \in Tw(\mathcal{A})(a_1,a_2)$, we have*

$$m_1^{Tw(\mathcal{A})}(m_2^{Tw(\mathcal{A})}(\Psi\otimes\Phi)) = (-1)^{|\Phi|}m_2^{Tw(\mathcal{A})}(m_1^{Tw(\mathcal{A})}(\Psi)\otimes\Phi) - m_2^{Tw(\mathcal{A})}(\Psi\otimes m_1^{Tw(\mathcal{A})}(\Phi)). \quad (2.26)$$

Proof.

$$\begin{aligned}&m_1^{Tw(\mathcal{A})}(m_2^{Tw(\mathcal{A})}(\Psi\otimes\Phi))\\ =&m_1^{\widetilde{\mathcal{A}}}(m_2(\Psi\otimes\Phi)) + m_2^{\widetilde{\mathcal{A}}}(Q_{a_2}\otimes m_2^{\widetilde{\mathcal{A}}}(\Psi\otimes\Phi)) + m_2^{\widetilde{\mathcal{A}}}(m_2^{\widetilde{\mathcal{A}}}(\Psi\otimes\Phi)\otimes Q_{a_0})\\ =&(-1)^{|\Phi|}m_2^{\widetilde{\mathcal{A}}}(m_1^{\widetilde{\mathcal{A}}}(\Psi)\otimes\Phi)) - m_2^{\widetilde{\mathcal{A}}}(\Psi\otimes m_1^{\widetilde{\mathcal{A}}}(\Phi))\\ &(-1)^{|\Phi|}m_2^{\widetilde{\mathcal{A}}}(m_2^{\widetilde{\mathcal{A}}}(Q_{a_2}\otimes\Psi)\otimes\Phi) - m_2^{\widetilde{\mathcal{A}}}(\Psi\otimes m_2^{\widetilde{\mathcal{A}}}(\Phi\otimes Q_{a_0}))\\ =&(-1)^{|\Phi|}m_2^{Tw(\mathcal{A})}(m_1^{Tw(\mathcal{A})}(\Psi)\otimes\Phi) - (-1)^{|\Phi|}m_2^{\widetilde{\mathcal{A}}}(m_2^{\widetilde{\mathcal{A}}}(\Psi\otimes Q_{a_1})\otimes\Phi)\\ &- m_2^{Tw(\mathcal{A})}(\Psi\otimes m_1^{Tw(\mathcal{A})}(\Phi)) + m_2^{\widetilde{\mathcal{A}}}(\Psi\otimes m_2^{\widetilde{\mathcal{A}}}(Q_{a_1}\otimes\Phi))\\ =&(-1)^{|\Phi|}m_2^{Tw(\mathcal{A})}(m_1^{Tw(\mathcal{A})}(\Psi)\otimes\Phi) - m_2^{Tw(\mathcal{A})}(\Psi\otimes m_1^{Tw(\mathcal{A})}(\Phi)).\end{aligned}$$

□

By the above two Lemmas, we have the following.

Proposition 2.15. *Let $\mathcal{A}$ be a unital $\mathbb{Q}$-graded A_∞-category of index h with $m_n^{\mathcal{A}} = 0$, $n \geq 3$. A collection $Ob(Tw(\mathcal{A}))$, $Tw(\mathcal{A})(a,b)$ and $(m_0^{Tw(\mathcal{A})}, m_1^{Tw(\mathcal{A})}, m_2^{Tw(\mathcal{A})})$ given by Definition 2.9* *and Definition 2.12 determines a structure of a differential ($\mathbb{Z}_2$-)graded category. We denote it by $Tw(\mathcal{A})$.* □

Remark 2.16. *Note that the condition that $\mathcal{A}$ is $\mathbb{Q}$-graded is not necessary for the above definition of the category $Tw(\mathcal{A})$ and the category $D^b(\mathcal{A})$ below. We need only the $\mathbb{Z}_2$-grading.*

Definition 2.17. *Let $\mathcal{A}$ be a unital $\mathbb{Q}$-graded A_∞-category of index h with $m_n^{\mathcal{A}} = 0$, $n \geq 3$. We construct the category $D^b(\mathcal{A})$ called the bounded derived category of $\mathcal{A}$ as follows. The set of objects is given by*

$$Ob(D^b(\mathcal{A})) := Ob(Tw(\mathcal{A})), \tag{2.27}$$

and the set of homomorphisms is given by

$$\mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A})}(\alpha,\beta) := \mathrm{Ker}(m_1^{Tw(\mathcal{A})} : Tw(\mathcal{A})(\alpha,\beta)_+ \to Tw(\mathcal{A})(\alpha,\beta)_-) \tag{2.28}$$

$$\Big/ \mathrm{Im}(m_1^{Tw(\mathcal{A})} : Tw(\mathcal{A})(\alpha,\beta)_- \to Tw(\mathcal{A})(\alpha,\beta)_+)\,. \tag{2.29}$$

Let $T \in Tw(\mathcal{A})(\alpha,\beta)_+$ be a $m_1^{Tw(\mathcal{A})}$-closed homomorphism. We define a mapping cone $C(T)$ as an object

$$C(T) := (a[1] \oplus b; Q_{C(T)}), \quad Q_{C(T)} := \begin{pmatrix} Q_{a[1]} & 0 \\ T & Q_b \end{pmatrix}. \tag{2.30}$$

$C(T)$ is well-defined since the Maurer-Cartan equation (2.16) for $Q_{C(T)}$ is equivalent to the equation $m_1^{Tw(\mathcal{A})}(T) = 0$ and the Maurer-Cartan equation (2.16) for Q_a and Q_b. Note also that there are natural morphisms

$$\beta \to C(T), \quad C(T) \to \alpha.$$

We define an exact triangle in the category $D^b(\mathcal{A})$ as a triangle of the form

$$\alpha \xrightarrow{T} \beta \to C(T) \to \alpha[1], \tag{2.31}$$

for some $T \in \mathrm{Ker}(m_1^{Tw(\mathcal{A})} : Tw(\mathcal{A})(\alpha,\beta)_+ \to Tw(\mathcal{A})(\alpha,\beta)_-)$.

Theorem 2.18. *The category $D^b(\mathcal{A})$ endowed with a shift functor* [1] *and the class of exact triangles defined above becomes a triangulated category.*

Proof. *The proof is essentiality the same as the known results in the usual situation. See for example,* [BK],[GM], [KS], [O] *and* [Se]. □

Remark 2.19. *The twice of the shift functor* [2] *is isomorphic to the identity functor in $D^b(\mathcal{A})$.*

We shall add more objects to $D^b(\mathcal{A})$ following [Se].

Definition 2.20. *Consider the category $D^\pi(\mathcal{A})$ whose objects are pairs (X, p) where $X \in Ob(D^b(\mathcal{A}))$ and $p \in \mathrm{Hom}_{D^b(\mathcal{A})}(X, X)$ an idempotent endomorphism, and whose spaces of homomorphisms are $\mathrm{Hom}_{D^\pi(\mathcal{A})}((X_0, p_0), (X_1, p_1)) := p_1 \mathrm{Hom}_{D^b(\mathcal{A})}(X_0, X_1) p_0$. The category $D^\pi(\mathcal{A})$ is called the split-closed derived category of $\mathcal{A}$.*

It is known that $D^\pi(\mathcal{A})$ is again a triangulated category (see [BS]).

Remark 2.21. *Since any projective module over the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ is a free module, we have $D^b(\mathcal{A}_f) \simeq D^\pi(\mathcal{A}_f)$. Therefore, we shall study in this paper only the category $D^b(\mathcal{A}_f)$.*

It is not difficult to see that our category $D^b(\mathcal{A}_f)$ is equivalent to the category of matrix factorizations. Indeed, we can construct a functor from the category of matrix factorizations to $D^b(\mathcal{A}_f)$ once we choose a basis of the free module over the polynomial ring. Note that any object isomorphic to a direct sum of the following objects

$$\begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix}$$

becomes the zero object of the category $D^b(\mathcal{A}_f)$.

Next, we shall define the $\mathbb{Z}$-equivariant bounded derived category $D^b_{\mathbb{Z}}(\mathcal{A})$ of $\mathcal{A}$. Let $\alpha := (a; Q_a)$ and $\beta := (b; Q_b)$ be graded twisted complexes. We put

$$Tw_{\mathbb{Z}}(\mathcal{A})(\alpha, \beta) := \bigoplus_{q \in \mathbb{Z}} Tw^q_{\mathbb{Z}}(\mathcal{A})(\alpha, \beta),$$

where

$$Tw^q_{\mathbb{Z}}(\mathcal{A})(\alpha, \beta) := \begin{cases} \widetilde{\mathcal{A}}^q(a, b)_+, & q \in 2\mathbb{Z}, \\ \widetilde{\mathcal{A}}^q(a, b)_-, & q - 1 \in 2\mathbb{Z}. \end{cases} \tag{2.32}$$

Since $Tw^q_{\mathbb{Z}}(\mathcal{A})(\alpha, \beta) \subset Tw(\mathcal{A})(\alpha, \beta)$, we can define a k-multilinear maps $m_n^{Tw_{\mathbb{Z}}(\mathcal{A})}$ by restricting $m_n^{Tw(\mathcal{A})}$ to the subspaces.

Proposition 2.22. *Let $\mathcal{A}$ be a unital $\mathbb{Q}$-graded A_∞-category of index h with $m_n^{\mathcal{A}} = 0$, $n \geq 3$. A collection $Ob(Tw_{\mathbb{Z}}(\mathcal{A}))$, $Tw_{\mathbb{Z}}(\alpha, \beta)$ and $m_n^{Tw_{\mathbb{Z}}(\mathcal{A})}$ given above determines a $\mathbb{Z}$-graded A_∞-category with $m_n \neq 0$ only if $n = 1, 2$, a differential graded (DG) category in the usual sense. We denote it by $Tw_{\mathbb{Z}}(\mathcal{A})$.* □

Definition 2.23. *Let $\mathcal{A}$ be a unital $\mathbb{Q}$-graded A_∞-category of index h with $m_n^{\mathcal{A}} = 0$, $n \geq 3$. We call the cohomology category of $Tw_{\mathbb{Z}}(\mathcal{A})$ the $\mathbb{Z}$-equivariant bounded derived category of $\mathcal{A}$ and denote by $D^b_{\mathbb{Z}}(\mathcal{A})$. More precisely, the set of objects is given by*

$$Ob(D^b_{\mathbb{Z}}(\mathcal{A})) := Ob(Tw_{\mathbb{Z}}(\mathcal{A})), \tag{2.33}$$

and the set of homomorphisms is given by

$$\mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A})}(\alpha,\beta) := \mathrm{Ker}(m_1^{Tw_{\mathbb{Z}}(\mathcal{A})} : Tw_{\mathbb{Z}}(\mathcal{A})^0(\alpha,\beta) \to Tw_{\mathbb{Z}}(\mathcal{A})^1(\alpha,\beta)) \tag{2.34}$$

$$\Big/ \mathrm{Im}(m_1^{Tw_{\mathbb{Z}}(\mathcal{A})} : Tw_{\mathbb{Z}}(\mathcal{A})^{-1}(\alpha,\beta) \to Tw_{\mathbb{Z}}(\mathcal{A})^0(\alpha,\beta)) \,. \tag{2.35}$$

Let $T \in Tw_{\mathbb{Z}}(\mathcal{A})^0(\alpha,\beta)$ be a $m_1^{Tw_{\mathbb{Z}}(\mathcal{A})}$-closed homomorphism. As in the case for $D^b(\mathcal{A})$, we define a mapping cone $C(T)$ as an object

$$C(T) := (a[1] \oplus b; Q_{C(T)}), \quad Q_{C(T)} := \begin{pmatrix} Q_{a[1]} & 0 \\ T & Q_b \end{pmatrix}. \tag{2.36}$$

We define an exact triangle in the category $D^b_{\mathbb{Z}}(\mathcal{A})$ as a triangle of the form

$$\alpha \xrightarrow{T} \beta \to C(T) \to \alpha[1], \tag{2.37}$$

for some $T \in \mathrm{Ker}(m_1^{Tw_{\mathbb{Z}}(\mathcal{A})} : Tw_{\mathbb{Z}}(\mathcal{A})^0(\alpha,\beta) \to Tw_{\mathbb{Z}}(\mathcal{A})^1(\alpha,\beta))$.

Theorem 2.24. *The category $D^b_{\mathbb{Z}}(\mathcal{A})$ endowed with a shift functor* $[1]$ *and the class of exact triangles defined above becomes a triangulated category.*

Proof. *As in the case for $D^b(\mathcal{A})$, the proof is essentiality the same as the known results in the usual situation.* □

Remark 2.25. *The twice of the shift functor* $[2]$ *is not isomorphic to the identity functor in $D^b_{\mathbb{Z}}(\mathcal{A})$.*

Consider the functor Tot

$$\mathrm{Tot} : Tw_{\mathbb{Z}}(Tw_{\mathbb{Z}}(\mathcal{A})) \to Tw_{\mathbb{Z}}(\mathcal{A}), \quad (\bigoplus_{i=1}^{k}(a_i; Q_{a_i}); T) \mapsto (\bigoplus_{i=1}^{k} a_i; Q + T), \tag{2.38}$$

where

$$Q := \begin{pmatrix} Q_{a_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_{a_k} \end{pmatrix}, \tag{2.39}$$

and $T \in Tw_{\mathbb{Z}}(\mathcal{A})(\oplus_{i=1}^k(a_i; Q_{a_i}), \oplus_{i=1}^k(a_i; Q_{a_i}))$ $(\subset \widetilde{\mathcal{A}}(\oplus_{i=1}^k a_i, \oplus_{i=1}^k a_i))$ satisfies

$$m_1^{Tw_{\mathbb{Z}}(\mathcal{A})}(T) + m_2^{Tw_{\mathbb{Z}}(\mathcal{A})}(T^{\otimes 2}) = 0.$$

It is well-defined since

$$\begin{aligned}
&m_0^{\widetilde{\mathcal{A}}}(1)+m_1^{\widetilde{\mathcal{A}}}(Q+T)+m_2^{\widetilde{\mathcal{A}}}((Q+T)^{\otimes 2})\\
=&m_0^{\widetilde{\mathcal{A}}}(1)+\sum_{i=1}^{k}m_1^{\widetilde{\mathcal{A}}}(Q_{a_i})+\sum_{i=1}^{k}m_2^{\widetilde{\mathcal{A}}}(Q_{a_i}^{\otimes 2})+m_1^{\widetilde{\mathcal{A}}}(T)+m_2^{\widetilde{\mathcal{A}}}(T\otimes Q)+m_2^{\widetilde{\mathcal{A}}}(Q\otimes T)+m_2^{\widetilde{\mathcal{A}}}(T^2)\\
=&m_0^{\widetilde{\mathcal{A}}}(1)+\sum_{i=1}^{k}m_1^{\widetilde{\mathcal{A}}}(Q_{a_i})+\sum_{i=1}^{k}m_2^{\widetilde{\mathcal{A}}}(Q_{a_i}^{\otimes 2})+m_1^{Tw_{\mathbb{Z}}(\mathcal{A})}(T)+m_2^{Tw_{\mathbb{Z}}(\mathcal{A})}(T^{\otimes 2}).
\end{aligned}$$

Now the following statement is easily shown as in [BK] where they consider the case when $\mathcal{A}$ is a DG category, i.e., the case when $m_0^{\mathcal{A}}=0$ and $h=1$ in our terminology.

Proposition 2.26. Tot *is an equivalence of DG categories.* □

Corollary 2.27. $D^b_{\mathbb{Z}}(\mathcal{A})$ *is an enhanced triangulated category in the sense of Bondal-Kapranov* [BK]. □

Let us consider our category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ a little bit in detail.

Example 2.28. *Let* $\alpha:=(a=a_+\oplus a_-[-1];Q_a)$ *and* $\beta:=(b=b_+\oplus b_-[-1]:Q_b)$ *be objects of* $Tw_{\mathbb{Z}}(\mathcal{A}_f)$. *Then the space of homomorphisms is of the following form:*

$$\Phi\in Tw^q_{\mathbb{Z}}(\alpha,\beta),\quad q\in 2\mathbb{Z}\Leftrightarrow \Phi=\begin{pmatrix}\Phi_{++} & 0\\ 0 & \Phi_{--}\end{pmatrix},\quad \Phi_{\pm\pm}\in\widetilde{\mathcal{A}_f}^q(a_\pm,b_\pm)_+,$$

$$\Phi\in Tw^q_{\mathbb{Z}}(\alpha,\beta),\quad q-1\in 2\mathbb{Z}\Leftrightarrow \Phi=\begin{pmatrix}0 & \Phi_{-+}\\ \Phi_{+-} & 0\end{pmatrix},\quad \Phi_{\pm\mp}\in\widetilde{\mathcal{A}_f}^{q\mp 1}(a_\pm,b_\mp)_+$$

and the coboundary operator $m_1^{Tw(\mathcal{A}_f)}$ *becomes*

$$m_1^{Tw(\mathcal{A}_f)}(\Phi):=(-1)^qQ_b\Phi-\Phi Q_a,\quad \Phi\in Tw^q_{\mathbb{Z}}(\alpha,\beta).$$

Note that if $\Phi\in\widetilde{\mathcal{A}_f}^q(\alpha,\beta)$, *then*

$$E\Phi+R_\beta\Phi-\Phi R_\alpha=q\Phi, \tag{2.40}$$

where we put

$$R_\alpha:=\mathrm{diag}(\frac{2k_1}{h},\dots,\frac{2k_m}{h},\frac{2l_1}{h}-1,\dots,\frac{2l_m}{h}-1),\quad a=\bigoplus_{i=1}^m a\{\frac{2k_i}{h}\}\oplus\bigoplus_{i=1}^m a\{\frac{2l_i}{h}\}[-1],$$

and

$$R_\beta:=\mathrm{diag}(\frac{2k'_1}{h},\dots,\frac{2k'_{m'}}{h},\frac{2l'_1}{h}-1,\dots,\frac{2l'_{m'}}{h}-1),\quad b=\bigoplus_{i=1}^{m'} a\{\frac{2k'_i}{h}\}\oplus\bigoplus_{i=1}^{m'} a\{\frac{2l'_i}{h}\}[-1].$$

By integrating the equation (2.40), *we get for* $\lambda\in\mathbb{C}$,

$$e^{\lambda R_\beta}\Phi(e^{\lambda\frac{2a_1}{h}}x_1,\dots,e^{\lambda\frac{2a_n}{h}}x_n)e^{-\lambda R_\alpha}=e^{q\lambda}\Phi(x_1,\dots,x_n). \tag{2.41}$$

This is the analogue of the homogeneity condition discussed in [HW].

Consider the $\mathbb{Z}$-action defined by $x_i \mapsto \exp(2\pi\sqrt{-1}p\cdot a_i/h)\cdot x_i$, $p\in\mathbb{Z}$. It is clear that f is invariant under this $\mathbb{Z}$-action. Note also that $\{\frac{2k}{h}\}$, $k\in\mathbb{Z}$ can be considered as the irreducible representations of $\mathbb{Z}$. For a graded twisted complex $\alpha := (a;Q_a)$, put

$$S_\alpha := \mathrm{diag}(\frac{2k_1}{h},\dots,\frac{2k_m}{h},\frac{2l_1}{h},\dots,\frac{2l_m}{h}),\quad a=\bigoplus_{i=1}^m a\{\frac{2k_i}{h}\}\oplus\bigoplus_{i=1}^m a\{\frac{2l_i}{h}\}[-1].$$

Since $Q_a\in\widetilde{\mathcal{A}}^1(a,a)_-$, the similar equation as (2.41) *shows that Q_a is equivariant with respect to the $\mathbb{Z}$-action, i.e., we have*

$$e^{\pi\sqrt{-1}S_\alpha}Q_a(e^{\frac{2\pi\sqrt{-1}a_1}{h}}x_1,\dots,e^{\frac{2\pi\sqrt{-1}a_n}{h}}x_n)e^{-\pi\sqrt{-1}S_\alpha}=Q_a(x_1,\dots,x_n). \tag{2.42}$$

One can show that there is also the $\mathbb{Z}$-action on the space of homomorphisms by (2.41)*. For $\Phi_\pm\in\widetilde{\mathcal{A}_f}^q(\alpha,\beta)_\pm$, we have*

$$e^{\pi\sqrt{-1}S_\beta}\Phi_\pm(e^{\frac{2\pi\sqrt{-1}a_1}{h}}x_1,\dots,e^{\frac{2\pi\sqrt{-1}a_n}{h}}x_n)e^{-\pi\sqrt{-1}S_\alpha}=\pm e^{\phi\sqrt{-1}q}\Phi_\pm(x_1,\dots,x_n). \tag{2.43}$$

Therefore, if Φ is even (*odd*)*, then Φ is $\mathbb{Z}$-invariant if and only if $q\in 2\mathbb{Z}$* ($q-1\in 2\mathbb{Z}$)*. These facts lead us to our definition of $\mathbb{Z}$-equivariant derived category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ of $\mathcal{A}_f$.*

Note that the above $\mathbb{Z}$-action on $\mathcal{A}_f$ factors through $\mathbb{Z}/h\mathbb{Z}$. The category whose set of objects is the set of $\mathbb{Z}/h\mathbb{Z}$-equivariant matrix factorizations and the space of morphisms is $\mathbb{Z}/h\mathbb{Z}$-invariant homomorphisms between matrix factorizations are called in physics the category of D-branes in Landau-Ginzburg ($\mathbb{Z}/h\mathbb{Z}$-)*orbifolds* (*see for example* [HW]) *We can construct it by considering the $\mathbb{Z}/h\mathbb{Z}$-equivariant version of $D^b(\mathcal{A}_f)$. Indeed, we can show that it is equivalent to $D^b_{\mathbb{Z}}(\mathcal{A}_f)/[2]$. In order to recover the $\mathbb{Z}$-grading by the shift functor, we introduced here the translation $\{2/h\}$ and defined a new category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$.*

3. $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ and representations of Dynkin quivers

The following is our main theorem in this paper.

Theorem 3.1. *Let us put $f(x):=x^h\in\mathbb{C}[x]$ for $h\geq 2$ and consider the unital $\mathbb{Q}$-graded A_∞-category $\mathcal{A}_f$ of index h. Then we have the following equivalence of triangulated categories*

$$D^b_{\mathbb{Z}}(\mathcal{A}_f)\simeq D^b(\mathrm{Mod}-B), \tag{3.1}$$

where B is the path algebra of the following Dynkin quiver of type A_{h-1} :

$$\bullet_1\to\bullet_2\to\cdots\to\bullet_{h-2}\to\bullet_{h-1}, \tag{3.2}$$

(*the algebra of upper triangular matrices over k*)*.*

Proof. *It is not difficult to see that our category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ is a Krull-Scmidt category (the spaces of homomorphisms are finite dimensional and the endomorphism rings of indecomposable objects are local rings). Therefore, we first study the set of isomorphism classes of indecomposable objects. We will use the fact that the Auslander-Reiten quiver of the category $D^b(\mathcal{A}_f)$ of matrix factorizations for f is given by*

$$[Q_1] \rightleftharpoons [Q_2] \rightleftharpoons \cdots \rightleftharpoons [Q_{h-2}] \rightleftharpoons [Q_{h-1}], \tag{3.3}$$

where

$$Q_l = \begin{pmatrix} 0 & x^{h-l} \\ x^l & 0 \end{pmatrix}, \quad l = 1, \dots, h-1, i \in \mathbb{Z},$$

and the morphisms from left to right are given by $\mathrm{diag}(1, x)$ *and the morphisms from right to left are given by* $\mathrm{diag}(x, 1)$. *See* [AR] *and also* [O]. *Hence we have the following.*

Lemma 3.2. *The set of isomorphism classes of all indecomposable objects of $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ is given by*

$$\{[M_{l,i}], \quad l = 1, \dots, h-1, i \in \mathbb{Z}\}, \tag{3.4}$$

where

$$M_{l,i} := \left(a\{\frac{2i}{h}\} \oplus a\{\frac{2(l+i)}{h}\}[-1]; Q_l \right). \tag{3.5}$$

We also have

$$\mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A}_f)}(M_{k,0}, M_{l,0}) = \begin{cases} \mathbb{C}, & \text{if } k \le l, \\ 0, & \text{if } k > l. \end{cases} \tag{3.6}$$

Proof. *One can easily show by direct computations.* □

Serre duality holds in our category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$.

Proposition 3.3.

$$\mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A}_f)}(M_{k,i}, M_{l,j}) \simeq \mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A}_f)}(M_{l,j}, M_{k,i-1}[1])^*, \quad \text{for all} \quad 1 \le k, l \le h-1, \quad i, j \in \mathbb{Z},. \tag{3.7}$$

Proof. *There is a trace map* [KL2]

$$Tr_k : \widetilde{\mathcal{A}_f}^{1-\frac{2}{h}}(M_{k,i}, M_{k,i})_- \to \mathbb{C}, \quad \Phi \mapsto \frac{1}{h-1}\mathrm{Res}\left[\frac{Str(dQ_k \cdot \Phi)}{\frac{\partial f}{\partial x}}\right], \tag{3.8}$$

where

$$Str(dQ_k \cdot \Phi) := \left[(h-k)x^{h-k-1}\Phi_{+-} - kx^{k-1}\Phi_{-+}\right] dx, \quad \Phi = \begin{pmatrix} 0 & \Phi_{-+} \\ \Phi_{+-} & 0 \end{pmatrix}.$$

Note that

$$\Phi_k := \begin{pmatrix} 0 & -x^{h-k-1} \\ x^{k-1} & 0 \end{pmatrix} \in \widetilde{\mathcal{A}_f}^{1-\frac{2}{h}}(M_{k,i}, M_{k,i})_-, \quad Tr_k(\Phi_k) = 1.$$

It is not difficult to see that this Φ_k induces perfect pairings

$$\widetilde{\mathcal{A}_f}^{\frac{2m}{h}}(M_{k,i}, M_{l,j})_+ \otimes \widetilde{\mathcal{A}_f}^{1-\frac{2}{h}-\frac{2m}{h}}(M_{l,j}, M_{k,i})_- \to \mathbb{C}, \quad m \in \mathbb{Z},$$

and hence the perfect pairings which gives the duality

$$\mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A}_f)}(M_{k,i}, M_{l,j}) \otimes \mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A}_f)}(M_{l,j}, M_{k,i-1}[1]) \to \mathbb{C}.$$

□

Remark 3.4. $S := \{\frac{-2}{h}\} \circ [1]$ *is the Serre functor on* $D^b_{\mathbb{Z}}(\mathcal{A}_f)$. *In particular, we have* $S^h = [h-2]$. *Therefore* $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ *is a fractional noncommutative Calabi-Yau manifold of dimension* $1 - 2/h$ *in the sense of* [So].

Combining the above Serre duality and the Auslander-Reiten quiver of the category of matrix factorizations (3.3)*, we see that there are no "extensions" among* $\{M_{l,0}\}$.

Corollary 3.5. *For* $m \neq 0$*, we have*

$$\mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A})}(M_{k,0}, M_{l,0}[m]) = 0, \quad \textit{for all} \quad i, j = 1, \dots, h-1.$$

□

Corollary 3.6. $D^b(\mathrm{Mod}{-}B)$ *is a full triangulated subcategory of* $D^b_{\mathbb{Z}}(\mathcal{A}_f)$.

Proof. *Use the fact that* $(M_{1,0}, \dots, M_{h-1,0})$ *is a strongly exceptional collection and*

$$B \simeq \bigoplus_{i,j=1}^{h-1} \mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A})}(M_{k,0}, M_{l,0}).$$

Since $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ *is an enhanced triangulated category, we can apply the theorem by Bondal-Kapranov* ([BK] *Theorem 1*). □

Note that the number of indecomposable objects of $D^b_{\mathbb{Z}}(\mathcal{A}_f)/[2]$ *is* $h \cdot (h-1)$*, which coincides with the number of roots for the root system* A_{h-1}. *This number again coincides with the number of indecomposable objects of* $D^b(\mathrm{Mod}{-}B)/[2]$ *by Gabriel's theorem* [G]. *Therefore,* $D^b(\mathrm{Mod}{-}B)/[2] \simeq D^b_{\mathbb{Z}}(\mathcal{A}_f)/[2]$. *This proves Theorem 3.1.* □

Remark 3.7. *The similar proof can be applied for* D_n *and* E_6, E_7, E_8 *cases since the heart of our proof is to use the Auslander-Reiten quivers of* $D^b(\mathcal{A}_f)$*, the fact that any matrix factorization over ADE singularities is gradable, the Serre duality and the theorem by Gabriel on the number of indecomposables. They are well-known or can be shown by direct calculations with explicit presentations of matrix factorizations. We shall discuss this in detail in the next paper* [KST].

4. Stability condition on $D^b_{\mathbb{Z}}(\mathcal{A}_f)$

In this section, we will briefly discuss on a stability condition on $D^b_{\mathbb{Z}}(\mathcal{A}_f)$.

Definition 4.1. *Let* $\alpha := (\oplus_{i=1}^n a\{\frac{2k_i}{h}\} \oplus \oplus_{i=1}^n a\{\frac{2l_i}{h}\}[-1]; Q_a)$ *be an object of* $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ *such that* Q_a *is reduced, i.e., each matrix element of* Q_a *is in the maximal ideal generated by* $(x_1, \dots, x_n)$*. Then we call the real number*

$$\phi_\alpha := \frac{1}{2n}\mathrm{Tr}S_a - \frac{1}{2}, \quad S_a := \mathrm{diag}(\frac{2k_1}{h}, \dots, \frac{2k_n}{h}, \frac{2l_1}{h}, \dots, \frac{2l_n}{h}) \tag{4.1}$$

phase of the object α.

Example 4.2. *Let* $f := x^{n+1}$ *and consider the objects*

$$M_{l,i} := \left(a\{\frac{2i}{h}\} \oplus a\{\frac{2(l+i)}{h}\}[-1]; \begin{pmatrix} 0 & x^{h-l} \\ x^l & 0 \end{pmatrix} \right). \tag{4.2}$$

Then

$$\phi_{M_{l,i}} = \frac{l+2i}{h} - \frac{1}{2}.$$

Definition 4.3. *Let* $\omega := \exp 2\pi\sqrt{-1}/h$*. For* $\alpha = (\oplus_{i=1}^n a\{\frac{2k_i}{h}\} \oplus \oplus_{i=1}^n a\{\frac{2l_i}{h}\}[-1]; Q_a)$*, we define a* $\mathbb{C}$*-linear map* $Z_\omega : K_0(D^b_{\mathbb{Z}}(\mathcal{A}_f)) \to \mathbb{C}$ *as follows:*

$$Z_\omega(\alpha) := \sum_{i=1}^n (\omega^{k_i} - \omega^{l_i}). \tag{4.3}$$

Proposition 4.4. *For* $0 < -l + 2i \leq 1$*,*

$$\frac{1}{\pi}\arg(Z_\omega(M_{l,i})) = \phi_{M_{l,i}}. \tag{4.4}$$

□

Theorem 4.5. *Let* $f := x^{n+1}$ *and* $P(\phi)$ *be the full additive subcategory of* $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ *whose objects have phase* $\phi \in \mathbb{R}$*. Then* $P(\phi)$ *and* Z_ω *defines a stability condition on* $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ *in the sense of Bridgeland* [B].

More precisely, $P(\phi)$ *and* Z_ω *satisfy the following property:*

(i) *if* $M \in P(\phi)$*, then* $Z_\omega(M) = m(M)\exp(\sqrt{-1}\pi\phi)$ *for some* $m(M) \in \mathbb{R}_{\geq 0}$*,*
(ii) *for all* $\phi \in \mathbb{R}$*,* $P(\phi + 1) = P(\phi)[1]$*,*
(iii) *if* $\phi_1 > \phi_2$ *and* $M_i \in P(\phi_i)$*, then* $\mathrm{Hom}_{D^b_{\mathbb{Z}}(\mathcal{A}_f)}(M_1, M_2) = 0$*,*
(iv) *for each nonzero object* $M \in D^b_{\mathbb{Z}}(\mathcal{A}_f)$*, there is a finite sequence of real numbers*

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of exact triangles

$$M_{i-1} \to M_i \to N_i \to M_{i-1}[1], \quad M_n := M, \quad M_0 := 0$$

with $N_j \in P(\phi_j)$ *for all* j*.*

□

The space of stability conditions for $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ should be isomorphic to the base space of the universal unfolding of f by the mirror symmetry. Therefore we expect that there exists a natural Frobenius (K. Saito's flat) structure on the space of stability conditions and the stability condition constructed above should correspond to the origin of the base space of the universal unfolding. We shall study this in detail elsewhere.

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