Counterexample of Kodaira's vanishing and Yau's inequality in higher dimensional variety of characteristic $p > 0$

By

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In his paper [3], Raynaud has constructed algebraic surfaces on which the Kodaira's vanishing does not hold. In this article, generalizing his method, we shall show

Theorem. *Let p be a prime number and $n \geq 2$ an integer. Then there exists an n-dimensional smooth projective variety X of characteristic p and an ample line bundle L such that*

(a) $H^1(X, L^{-1}) \neq 0$

(b) *the canonical class K of X is ample and the intersection number $(c_i.K^{n-i})$ is negative for every $i \geq 2$, where c_i is the i-th Chern class of X.*
and

(c) *there is a finite cover G of X isomorphic to a $(\mathbb{P}^1)^{n-1}$-bundle over a nonsingular curve C. The Euler characteristic $e(X)$ of X is equal to $e(G) = 2^{n-1}e(C)$.*

Put $X' = X \times \mathbb{P}^m$ and $L' = p_1^*L \otimes p_2^*\mathcal{O}(m+1)$. Then we have, by Künneth formula,

$$H^{m+1}(X', L'^{-1}) \supseteq H^1(X, L^{-1}) \otimes H^m(\mathbb{P}^m, \mathcal{O}(-m-1)) \neq 0.$$

Therefore we have

Corollary 1. *For every pair of integers n' and i with $0 < i < n'$, there exist an n'-dimensional nonsingular projective variety X' of characteristic p and an ample line bundle L' on X' such that $H^i(X', L'^{-1}) \neq 0$.*

In [5], Yau has proved the inequality $(c_2.K^{n-2}) \geq \frac{n}{2(n+1)}(K^n) > 0$ for a complex manifold whose canonical class is ample. Since the Chern number $(c_2.K^{n-2})$ and the ampleness of K are stable under generalization, we have

Corollary 2. *The variety X in the theorem is not liftable to a variety of characteristic 0.*

Hence the following problem is still open.

Problem. * Assume that a variety X (resp. a polarized variety (X, L)) is liftable to a variety (resp. a polarized variety) of characteristic 0. Does the Kodaira's vanishing hold on X ? (resp. Does $H^1(X, L^{-1})$ vanish ?)

In §1, we shall construct counterexamples by Proposition 1.4, 1.7 and 1.8, and in §2, we shall prove (b) of the theorem in Proposition 2.6 and 2.14.

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§1 Construction of counterexamples

We begin with a geometric interpretation of the injectivity of the Frobenius map. Let X be a smooth variety of characteristic $p > 0$ and L a line bundle on X. Let $F : L^{-1} \to L^{-p}(F(a) = a^p)$ be the Frobenius map of L^{-1}. If L is ample, $H^i(X, L^{-p^n})$ vanishes for sufficiently large n and every $i < \dim X$. Hence Kodaira's vanishing in characteristic p is equivalent to the injectivity of the Frobenius map $H^i(F)$.

Proposition 1.1. *Assume that $H^0(F) : H^0(X, L^{-1}) \to H^0(X, L^{-p})$ is an isomorphism. Then the following are equivalent :*

(1) *The Frobenius map $H^1(F) : H^1(X, L^{-1}) \to H^1(X, L^{-p})$ is not injective.*

(2) *There exist an $\mathbb{A}^1$-bundle $f : A \to X$ and a reduced irreducible effective divisor G on A such that $\tau = f|_G$ is a purely inseparable finite morphism of degree p and that the normal bundle of the ∞-section S is isomorphic to L.*

Proof. (1) $\Rightarrow$ (2) Let α be a nonzero element of the kernel of $H^1(F)$. Since $H^1(L^{-1})$ is canonically isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_X}(L, \mathcal{O}_X)$, α defines the exact sequence

$$0 \to \mathcal{O}_X \to E \to L \to 0. \tag{1.2}$$

In other words, α defines the $\mathbb{P}^1$-bundle $P = \mathbb{P}(E)$ and its section S such that $N_{S/P} \cong L$. Let $\varphi : \mathbb{P}(E) \to \mathbb{P}(E^{(p)})$ be the relative Frobenius morphism. Since $H^1(F)(\alpha) = 0, E^{(p)}$ is isomorphic to $\mathcal{O}_X \oplus L^{\otimes p}$, that is, $\mathbb{P}(E^{(p)})$ has a section T disjoint from the section $\varphi(S)$. Put $A = P - S$ and $G = \varphi^{-1}(T)$. Since φ is everywhere ramified, so is $G \to X$. If G were not reduced, then G would be linearly equivalent to pG' and G' would be a section of $f : A \to S$, which contradicts to $\alpha \neq 0$. Hence G is reduced. The other requirements are easily verified.

*If a variety lifts to the Witt ring modp^2, then Kodaira's vanishing holds.
Deligne, P. and Illusie, L.: Relèvements modulo p^2 et décomposition du complexe de de Rham, Invent. Math. (1987), 247-270.

(2) $\Rightarrow$ (1) The $\mathbb{P}^1$-bundle $P = A \cup S$ is isomorphic to $\mathbb{P}(E), E$ being of the form (1.2). Let $\alpha \in H^1(L^{-1})$ be an extension class of E, which is unique up to constant multiplications. Let $\varphi : A \to A^{(p)}$ be the relative Frobenius morphism. Since $G \to X$ is purely inseparable and of degree p, $\varphi(G \cup f^{-1}(x))$ is a reduced point for every $x \in X$. Hence $\varphi(G)$ is a section of $A^{(p)}$ and α is contained in the kernel of $H^1(F) : H^1(L^{-1}) \to H^1(L^{-p})$. We show $\alpha \neq 0$. If A has a section U, then G is linearly equivalent to pU on P. By our assumption, G is equal to pU' for a section U', which contradicts our assumption. Hence A has no section and α is nonzero. $\square$

As in the proof of the proposition, we can associate for an element of $\operatorname{Ker} H^1(F)$, a purely inseparable covering $\tau : G \to X$ embeddable in an $\mathbb{A}^1$-bundle A over X.

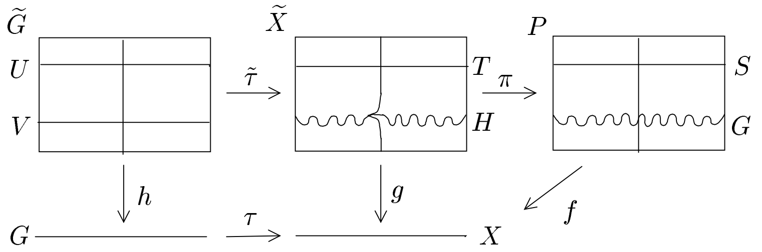
claim. The normal bundle $N_{G/A}$ is isomorphic to $\tau^* L^{-p}$.

Since τ is of degree $p, \mathcal{O}_P(pS - G)$ is isomorphic to the pull back of a line bundle on X by f. Since $N_{S/P} \cong L$ and $S \cap G = \emptyset$, $\mathcal{O}_P(pS - G) \cong f^* L^{\otimes p}$. Hence we have $N_{G/A} \cong N_{G/P} \cong \mathcal{O}_G(G) \cong \mathcal{O}_G(G - pS) \cong \tau^* L^{-p}$, which proves our claim.

By the claim. if L is ample, then $N_{G/A}$ is negative.

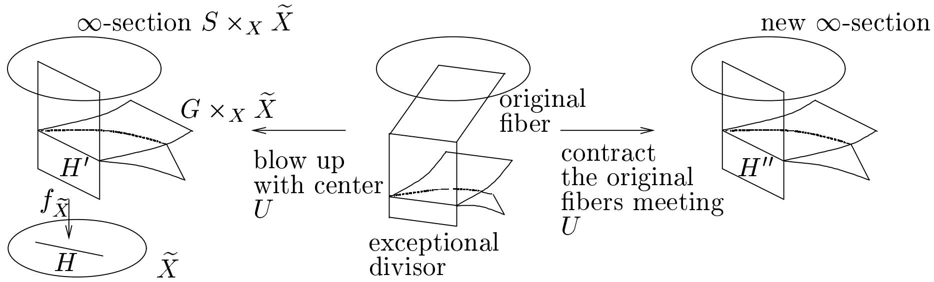
Definition 1.3. An element $\alpha \in \operatorname{Ker} H^1(F)$ is *special* if G is smooth. A pair of a smooth variety X and an ample line bundle L is called a *special counterexample* of I.F. if $\operatorname{Ker} H^1(F)$ contains a special element. (I.F. means the injectivity of the Frobenius map.)

Let (X, L) be a special counterexample of I.F. and A, P, S and G as in Proposition 1.1. Assume that L is isomorphic to $M^{\otimes k}$ for some line bundle M and positive integer k prime to p. Let m be a positive integer such that $p + m$ is divisible by k. Since $\mathcal{O}_P(G - pS) \cong f^* L^{\otimes p}$, we have $\mathcal{O}_P(G + mS) \cong (\mathcal{O}_P(\frac{p+m}{k} S) \otimes_{\mathcal{O}_X} M)^{\otimes k}$, that is, $G + mS$ is the zero locus of a global section of $(\mathcal{O}_P(\frac{p+m}{k} S) \otimes_{\mathcal{O}_X} M^{\otimes p})^{\otimes k}$. In the wellknown manner, we can construct a cyclic k-fold covering of P ramifying exactly on $G + mS$. If $m \geq 2$, then the covering has a singularity along S. Let $\widetilde{X}$ be its normalization. Since G and S are smooth, so is $\widetilde{X}$. Let $\pi : \widetilde{X} \to P$ be the covering morphism. There exist effective divisors T and H such that $\pi^* S = kT$ and $\pi^* G = kH$. H is isomorphic to G and every fiber of $g = f \circ \pi : \widetilde{X} \to X$ has a singularity of the form $Y^k = Z^p$ at its intersection with H. The following is an essential step of our construction of counterexamples.



Proposition 1.4. *If (X, L) is a special counterexample of I.F., then so is the pair of $\widetilde{X}$ and $\widetilde{L} \stackrel{\text{def}}{=} \mathcal{O}_{\widetilde{X}}((k-1)T) \otimes_{\mathcal{O}_X} M$.*

Proof. We give a proof only in the case $k \equiv 1 \mod p$ because it is sufficient for our proof of the theorem and we know only a tedious computational proof in general case. Consider the scheme $\widetilde{X} \times_X G$. Let H' be the image of $(i, \pi|_H) : H \to \widetilde{X} \times_X G$, where $i : H \hookrightarrow \widetilde{X}$ is the inclusion map. H' is a section of the projection $p : \widetilde{X} \times_X G \to G$ and $\widetilde{X} \times_X G$ has a singularity along H'. Let $\nu : \widetilde{G} \longrightarrow \widetilde{X} \times_X G$ be the normalization and $\tilde{\tau}$ the composition of ν and the projection $\widetilde{X} \times_X G \longrightarrow \widetilde{X}$. It is easily seen that every fiber of $\tilde{\tau}$ is smooth. Since every fiber of g is a rational curve, the composition $h : \widetilde{G} \to \widetilde{X} \times_X G \to G$ is a $\mathbb{P}^1$-bundle and hence $\widetilde{G}$ is smooth. Obviously $\tilde{\tau} : \widetilde{G} \to \widetilde{X}$ is a purely inseparable covering of degree p. So it suffices to show that $\widetilde{G}$ can be negatively embedded into an $\mathbb{A}^1$-bundle over X. Now consider the $\mathbb{P}^1$-bundle $f_{\widetilde{X}} : P \times_X \widetilde{X} \to \widetilde{X}$. $P \times_X \widetilde{X}$ contains $G \times_X \widetilde{X}$ which has a singularity of the form $Y^k = Z^p$ along H'. Since H' is a section of $f_{\widetilde{X}}|_{P \times_X H} : P \times_X H \to H$ and H is a Cartier divisor on $\widetilde{X}$, we can consider the elementary transformation along H' :



Blow up $P \times_X \widetilde{X}$ with center H' and contract the proper transform of $P \times_X H$, (cf. [1].) Then we get a new $\mathbb{P}^1$-bundle over $\widetilde{X}$. The proper transform of $G \times_X \widetilde{X}$ has a singularity of the form $Y^{k-p} = Z^p$ along H'', the proper transform of H'. H'' is also a section of the restricted $\mathbb{P}^1$-bundle of $f_{\widetilde{X}}$ to $H \subset \widetilde{X}$ and is disjoint from the proper transform of the ∞-section $S \times_X \widetilde{X}$ of the $\mathbb{P}^1$-bundle over $\widetilde{X}$. If $k - p > 0$, make an elementary transformation along H''. Repeating this process $(k-1)/p$ times, we get a $\mathbb{P}^1$-bundle $\widetilde{f} : \widetilde{P} \to \widetilde{X}$ on which the proper transform of $G \times_X \widetilde{X}$ is nonsingular, isomorphic to $\widetilde{G}$ and disjoint from the proper transform $\widetilde{S}$ of the ∞-section $S \times_X \widetilde{X}$ of the original $\mathbb{P}^1$-bundle. One elementary transformation raises the normal bundle of the ∞-section by $\mathcal{O}_{\widetilde{X}}(H)$. Hence the normal bundle $N_{\widetilde{S}/\widetilde{P}}$ is isomorphic to $\mathcal{O}_{\widetilde{X}}(\frac{k-1}{p}H) \otimes N_{S\times_X \widetilde{X}/P\times_X \widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}(\frac{k-1}{p}H) \otimes_{\mathcal{O}_X} L$.Therefore $\widetilde{G}$ is embedded in the $\mathbb{A}^1$-bundle $\widetilde{P} - \widetilde{S}$ so that the normal bundle $N_{\widetilde{S}/\widetilde{P}}$ is isomorphic to $\mathcal{O}_{\widetilde{X}}(\frac{k-1}{p}H) \otimes L$. Hence it suffices to show □

Lemma 1.5. *$\mathcal{O}_{\widetilde{X}}(\frac{k-1}{p}H) \otimes_{\mathcal{O}_X} L$ is isomorphic to $\mathcal{O}_{\widetilde{X}}((k-1)T) \otimes_{\mathcal{O}_X} M$ and ample.*

Proof. Since $\mathcal{O}_P(pS - G) \cong f^*L^{\otimes p}$, $\pi^*G = kH$ and $\pi^*S = kT$, we have $\mathcal{O}_{\widetilde{X}}(pkT - kH) \cong g^*L^{\otimes p} \cong g^*M^{\otimes pk}$. By the construction of the covering $G \to X$, we have $\mathcal{O}_{\widetilde{X}}(pT - H) \cong g^*M^{\otimes p}$, from which the first assertion easily follows. The second assertion follows from □

Sublemma 1.6. *If $a > 0$ and N is ample, then $aT + g^*N$ is ample.*

Proof. The $\mathbb{P}^1$ -bundle $\widetilde{G}$ over G has two mutually disjoint sections U and V such that $\widetilde{\tau}^*T = U$ and $\widetilde{\tau}^*H = pV$. Since $\mathcal{O}_{\widetilde{X}}(pT - H) \cong g^*M^{\otimes p}$, we have $\mathcal{O}_{\widetilde{G}}(pU - pV) \cong h^*\tau^*M^{\otimes p}$. Hence $\widetilde{G}$ is isomorphic to $\mathbb{P}(\mathcal{O}_G \oplus M')$, $M'^{\otimes p} \cong (\tau^*M)^{\otimes p}$, and $\mathcal{O}_{\widetilde{G}}(U)$ is just its tautological line bundle. Since $L \cong M^{\otimes k}$ is ample and τ is finite, M' is ample. Since $\widetilde{\tau}$ is finite, it suffices to show that $aU + h^*\tau^*N$ is ample. Let $\varphi : G \to G^{(p)}$ be the Frobenius morphism. Since φ is finite, replacing M' by $(\varphi^n)^*M' \cong M'^{\otimes p^n}$, we may assume that M' is very ample. Then the linear system $|\mathcal{O}_X(U)|$ defines a natural morphism from X onto the cone over G, which contracts the negative section V and is an isomorphism outside V. Therefore, $aU + h^*\tau^*N$ is ample. □

Now we construct a special counterexample of Kodaira's vanishing of an arbitrary dimension not less than two. First we note

Proposition 1.7. *A complete nonsingular curve X of genus ≥ 2 is a special counterexample of I.F. if there is a nonzero rational function u on X such that $(du) = pD$ for some divisor D.*

Proof. By virtue of Tango's theorem ([4]), if $(du) = pD$, the Frobenius map $H^1(X, \mathcal{O}_X(-D)) \to H^1(X, \mathcal{O}_X(-pD))$ is not injective. Hence, by Proposition 1.1, a purely inseparable cover G of X is embedded into an $\mathbb{A}^1$-bundle A so that $N_{G/A} \cong \tau^*\mathcal{O}_X(-pD)$. There are two natural exact sequences

$$N_{G/A}^\vee \xrightarrow{\alpha} \Omega_A|_G \to \Omega_G \to 0$$

and

$$0 \to f^*\Omega_X|_G \to \Omega_A|_G \to \Omega_{A/X}|_G \cong \mathcal{O}_X(D)|_G \to 0.$$

Since deg $\tau^*\mathcal{O}_X(D)$ is smaller than deg $N_{G/A}^\vee = p\cdot$ deg $\tau^*\mathcal{O}_X(D)$, we have $\mathrm{Hom}_{\mathcal{O}_G}(N_{G/A}^\vee, \Omega_{A/X}|_G) = 0$ and hence $\alpha(N_{G/A}^\vee)$ is contained in $f^*\Omega_X|_G$. Since

α is nonzero and $N^\vee_{G/A}$ and $f^*\Omega_X|_G$ are of the same degree, $\alpha : N^*_{G/A} \to f^*\Omega_X|_G$ is an isomorphism. Hence Ω_G is isomorphic to $\Omega_{A/X}|_G$ and in particular locally free, that is, G is nonsingular. $\square$

A curve as in the proposition is called a *Tango − Raynaud curve*, from which our counterexample will be constructed. Next we show that there are Tango-Raynaud curves enough for our purpose:

Proposition 1.8. *For every integer $e > 0$, there is a Tango-Raynaud curve X_1 such that $(dZ) = pD_1$ and $D_1 = eD_1'$ for some nonzero rational funciton Z and divisors D_1 and D_1' on X_1.*

Proof. Let Q be a polynomial of one variable of degree e. Consider the curve in $\mathbb{A}^2$ defined by the equation

$$Q(Y^p) - Y = Z^{pe-1}. \tag{1.9}$$

It is easy to see that this curve is nonsingular. The closure X_1 of the curve in $\mathbb{P}^2$ has only one point ∞ on the ∞-line and X_1 is nonsingular at the point ∞. By (1.9), we have $-dY = -Z^{pe-2}dZ$. Hence the differential dZ of the rational function Z is a generator of $\Omega_{X_1\cap\mathbb{A}^2}$, that is, dZ has no zeros or poles on $X_1\cap\mathbb{A}^2$. Since deg $\Omega_X = pe(pe-3)$, we have $(dZ) = pe(pe-3)(\infty)$. Divisors $D_1 = e(pe-3)(\infty)$ and $D_1' = (pe-3)(\infty)$ satisfy our requirement. $\square$

Fix a positive integer $m > 0$. Define k_i $(1 \le i \le n)$ inductively so that $k_1 = 1 + mp$ and $k_i = 1 + mp\prod_{j=1}^{i-1} k_j$ $(2 \le j \le n-1)$ and put $e = \prod_{i=1}^{n-1} k_i$. Take a curve X_1 in Proposition 1.8 for this e. The pair (X_1, L_1), L_1 being $\mathcal{O}_{X_1}(D_1)$, is a special counterexample of I.F. by Proposition 1.7. Since $L_1 \cong M_1^{\otimes k_{n-1}}$ for $M_1 = \mathcal{O}_{X_1}(D_1')^{\otimes e'}$ and $e' = \prod_{i=1}^{n-2} k_i$, taking k_{n-1} as the k in Proposition 1.4, we can construct a special counterexample $(\widetilde{X}_1, \widetilde{L}_1)$ of I.F. of dimension 2, which we denote by (X_2, L_2). Since $L_2 \cong \mathcal{O}_{X_2}((k_{n-1}-1)T_1)\otimes_{\mathcal{O}_{X_1}} M_1 \cong (\mathcal{O}_{X_2}(mpT_1)\otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_1}(D_1'))^{\otimes e'} \cong M_2^{\otimes k_{n-2}}$ for $M_2 = (\mathcal{O}_{X_2}(mpT_1)\otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_1}(D_1'))^{\otimes e''}$ and $e'' = \prod_{i=1}^{n-3} k_i$, taking k_{n-2} as the k in Proposition 1.4, we can construct $(\widetilde{X}_2, \widetilde{L}_2) =: (X_3, L_3)$. Repeating this $n-1$ times, we obtain (X_n, L_n), an n-dimensional special counterexample of I.F. So we have proved (a) of the theorem.

§2 Computation of Chern numbers

In this section we prove (b) and (c) of the theorem. Let X be a special counterexample of I.F. There is a smooth purely inseparable cover G of X embeddable into an $\mathbb{A}^1$-bundle A. Let P be the $\mathbb{P}^1$-bundle obtained from A by adding the ∞-section F. Since G is smooth, the sequence

$$0 \to T_G \to T_P|_G \to N_{G/P} \cong \tau^* L^{-p} \to 0 \tag{2.1}$$

is exact (see the claim below Proposition 1.1). On the other hand, restricting the natural exact sequence $0 \to T_{P/X} \to T_P \to f^*T_X \to 0$ to G, we have the exact sequence

$$0 \to \tau^*L^{-1} \to T_P|_G \to \tau^*T_X \to 0 \tag{2.2}$$

because the relative tangent bundle $T_{P/X}$ is isomorphic to $\mathcal{O}_P(2F) \otimes f^*L^{-1}$ and $\mathcal{O}_P(F)|_G$ is trivial. By these two exact sequences, we have $c(G).\tau^*c(L^{-p}) \sim \tau^*(c(L^{-1}).c(X))$, where $\sim$ denotes the rational equivalence of cycles. Hence we have

$$\begin{aligned} \tau^*c(X) &\sim c(G).(1 - p\tau^*c_1(L)).(1 - \tau^*c_1(L))^{-1} \\ &\sim c(G).(1 - (p-1)\sum_{i \geq 1} \tau^*c_1(L)^i). \end{aligned} \tag{2.3}$$

In particular, we have $\tau^*K_X \sim K_G + (p-1)\tau^*c_1(L)$.

In the exact sequences (2.1) and (2.2), τ^*L^{-1} is contained in T_G since $\mathrm{Hom}_{\mathcal{O}_G}(\tau^*L^{-1}, \tau^*L^{-p}) = 0$. Hence we have the exact sequence

$$0 \to \tau^*L^{-1} \xrightarrow{\alpha} T_G \to \tau^*T_X \xrightarrow{\beta} \tau^*L^{-p} \to 0. \tag{2.4}$$

Proposition 2.5. *$\tau^*c_n(X) \sim p \cdot c_n(G)$, where $n = \dim X$. In particular we have $e(X) = e(G)$.*

Proof. Let B be the kernel of β. Since B is a vector bundle of rank $n-1$ and since $B \cong \mathrm{Coker}\ \alpha$, we have $c_n(G) \sim \tau^*c_1(L^{-1}).c_{n-1}(B)$ and $\tau^*c_n(X) \sim \tau^*c_1(L^{-p}).c_{n-1}(B)$. Hence $\tau^*c_n(X)$ is rationally equivalent to $p \cdot c_n(G)$. □

So we have proved (c) of the theorem. The first half of (b) of the theorem follows from the following :

Proposition 2.6. *Let (X_n, L_n) be the special counterexample of I.F. constructed at the end of §1. If $\{p, k_{n-1}\} \neq \{2, 3\}$, then the canonical class K_{X_n} is ample.*

Proof. We put $D_i = c_1(L_i)$ for $1 \leq i \leq n$. Since $\tau_n : G_n \to X_n$ is finite, it suffices to show that $K_{G_n} + (p-1)\tau_n^*D_n$ is ample by (2.3). G_n is a $\mathbb{P}^1$-bundle over G_{n-1} and the natural projection $h_{n-1} : G_n \to G_{n-1}$ has two mutually disjoint sections U_{n-1} and V_{n-1} such that V_{n-1} is numerically equivalent to $U_{n-1} - k_{n-1}^{-1}h_{n-1}^*\tau_{n-1}^*D_{n-1}$ (see the proof of Proposition 1.4 and Sublemma 1.6). Hence we have

$$K_{G_n} \sim -U_{n-1} - V_{n-1} + h_{n-1}^*K_{G_{n-1}} \approx -2U_{n-1} + h_{n-1}^*(K_{G_{n-1}} + k_{n-1}^{-1}\tau_{n-1}^*D_{n-1}),$$

where $\approx$ denotes the numerical equivalence. Since $D_n \sim (k_{n-1} - 1)T_{n-1} + k_{n-1}^{-1}g_{n-1}^*D_{n-1}$, we have

$$
\begin{aligned}
&K_{Gn} + (p-1)\tau_n^* D_n \\
&\quad \approx \{-2U_{n-1} + h_{n-1}^*(K_{Gn-1} + k_{n-1}^{-1}\tau_{n-1}^* D_{n-1})\} \\
&\qquad + (p-1)\{(k_{n-1}-1)U_{n-1} + k_{n-1}^{-1}h_{n-1}^*\tau_{n-1}^* D_{n-1}\} \\
&\quad \approx (pk_{n-1} - p - k_{n-1} - 1)U_{n-1} + h_{n-1}^*(K_{Gn-1} + pk_{n-1}^{-1}\tau_{n-1}^* D_{n-1}).
\end{aligned}
$$

By Sublemma 1.6, it suffices to show that $K_{Gn-1} + pk_{n-1}^{-1}\tau_{n-1}^* D_{n-1}$ is ample because $pk_{n-1} - p - k_{n-1} - 1 > 0$ by our assumption. In the case $n = 2$, $K_{Gn-1} + pk_{n-1}^{-1}\tau_{n-1}^* D_{n-1}$ is ample because both K_{Gn-1} and D_{n-1} are ample. Hence we may assume that $n \geq 3$. Then we have

$$
\begin{aligned}
&K_{Gn-1} + pk_{n-1}^{-1}\tau_{n-1}^* D_{n-1} \\
&\approx \; -U_{n-2} - V_{n-2} + h_{n-2}^* K_{Gn-2} + pk_{n-1}^{-1}((k_{n-2}-1)U_{n-2} + k_{n-2}^{-1}\tau_{n-2}^* D_{n-2}) \\
&\approx \; \{pk_{n-1}^{-1}(k_{n-2}-1) - 2\}U_{n-2} + h_{n-2}^*(K_{Gn-2} + (1 + pk_{n-1}^{-1})k_{n-2}^{-1}\tau_{n-2}^* D_{n-2}).
\end{aligned}
$$

$$
\textit{claim.} \quad pk_{n-1}^{-1}(k_{n-2}-1) - 2 > 0.
$$

Since k_{n-1} divides $k_{n-2} - 1$, we have $pk_{n-1}^{-1}(k_{n-2}-1) - 2 \geq p - 2 \geq 0$. Since $(p, k_{n-2}) = (p, k_{n-1}) = 1$, we have either $k_{n-1} \neq k_{n-2} - 1$ or $p \neq 2$. Hence the two equalities do not hold at the same time, which shows our claim.

By Sublemma 1.6, it suffices to show that $K_{Gn-2} + (1 + pk_{n-1}^{-1})k_{n-2}^{-1}\tau_{n-2}^* D_{n-2}$ is ample. Since

$$
(1 + pk_{n-1}^{-1})k_{n-2}^{-1} > (1 + k_{n-1}^{-1})k_{n-2}^{-1} \geq (1 + (k_{n-2}-1)^{-1})k_{n-2}^{-1} = (k_{n-2}-1)^{-1},
$$

our proposition follows from

$$
\textit{claim.} \quad H_i = K_{Gi} + (k_i - 1)^{-1}\tau_i^* D_i \quad (1 \leq i \leq n-2) \quad \textit{is ample.}
$$

We prove by induction on i. In the case $i = 1$, both K_{G1} and D_1 are ample. Hence H_1 is ample. Assume that $i \geq 2$. Then we have

$$
\begin{aligned}
H_i &\sim -U_{i-1} - V_{i-1} + h_{i-1}^* K_{Gi-1} \\
&\quad + (k_i - 1)^{-1}\{(k_{i-1}-1)U_{i-1} + k_{i-1}^{-1}h_{i-1}^*\tau_{i-1}^* D_{i-1}\} \\
&\sim \{(k_i-1)^{-1}(k_{i-1}-1) - 2\}U_{i-1} \\
&\quad + h_{i-1}^*\{K_{Gi-1} + (k_{i-1}^{-1} + k_{i-1}^{-1}(k_i-1)^{-1})\tau_{i-1}^* D_{i-1}\} \\
&\sim \{(k_i-1)^{-1}(k_{i-1}-1) - 2\}U_{i-1} \\
&\quad + h_{i-1}^*\{H_{i-1} + k_{i-1}^{-1}((k_i-1)^{-1} - (k_{i-1}-1)^{-1})\tau_{i-1}^* D_{i-1}\}.
\end{aligned}
$$

Since H_{i-1} is ample by induction hypothesis and $k_{i-1} > k_i$, $H_{i-1} + k_{i-1}^{-1}((k_i-1)^{-1} - (k_{i-1}-1)^{-1})\tau_{i-1}^* D_{i-1}$ is ample. Since $(k_{i-1}-1)/k_i$ is divisible by k_{i+1}, we have $(k_i-1)^{-1}(k_{i-1}-1) > k_i^{-1}(k_{i-1}-1) \geq 2$. Hence H_i is ample by Sublemma 1.6. □

Let X, $\widetilde{X}$, G and $\widetilde{G}$ be as in the proof of Proposition 1.4. We investigate the relation between the Chern numbers of G and $\widetilde{G}$. Let $c(G) = \sum_{i\geq 0} c_i(G)$ be the Chern class of G. $\widetilde{G}$ is a $\mathbb{P}^1$-bundle over G and has two mutually disjoint sections U and V. Hence $\Omega_{\widetilde{G}/G}$ is isomorphic to $\mathcal{O}_{\widetilde{G}}(-U-V)$ and we have

$$\begin{array}{ccc} \widetilde{X} & \xleftarrow{\tilde{\tau}} & \widetilde{G} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xleftarrow[\tau]{} & G \end{array} \qquad \begin{aligned} & c(\widetilde{G}) \sim (1+U+V).h^*c(G) \\ & c_i(\widetilde{G}) \sim h^*c_i(G) + h^*c_{i-1}(G).(U+V). \end{aligned} \tag{2.7}$$

Since $U - V \ \approx\ k^{-1}h^*\tau^*D$ and $V \cap U = \phi$, we have

$$\begin{aligned} & U.V \sim 0, \\ & U^2 \sim ((U-V)+V).U \ \approx\ k^{-1}h^*\tau^*D.U, \\ & V^2 \sim (U-(U-V)).V \ \approx\ -k^{-1}h^*\tau^*D.V, \end{aligned} \tag{2.8}$$

where $D = c_1(L)$. More generally, we have

$$U^m \ \approx\ k^{-m+1}h^*\tau^*D^{m-1}.U \quad \text{and} \quad V^m \ \approx\ (-k)^{-m+1}h^*\tau^*D^{m-1}.V \tag{2.9}$$

for every $m \geq 1$.

Proposition 2.10. *Let λ_i $(i = 1, \cdots, n)$ and μ be non-negative integers such that $\sum_{i=1}^n i\lambda_i + \mu = \dim \widetilde{G} = n$. Then we have*

$$\begin{aligned} & (c_1(\widetilde{G})^{\lambda_1}.\cdots.c_n(\widetilde{G})^{\lambda_n}.\tilde{\tau}^*\widetilde{D}^{\mu}) \\ & = \sum_{\substack{\alpha_1,\cdots,\alpha_{n-1} \\ l+\mu>0,\ 0\leq\alpha_i\leq\lambda_i}} k^{-l-\mu+1}(k^{\mu}+(-1)^{\mu}) \binom{\lambda_1}{\alpha_1}\cdots\binom{\lambda_{n-1}}{\alpha_{n-1}} (c_1(G)^{\lambda_1-\alpha_1+\alpha_2} \\ & \quad .\cdots.c_{n-2}(G)^{\lambda_{n-2}-\alpha_{n-2}+\alpha_{n-1}}.c_{n-1}(G)^{\lambda_{n-1}-\alpha_{n-1}+\lambda_n}.\tau^*D^{l+\mu-1}) \\ & = k(c_1(G)^{\lambda_1}.\cdots.c_{n-1}(G)^{\lambda_{n-1}}.\tau^*D^{\mu-1}) \\ & \quad + \sum_{\lambda'_1,\cdots,\lambda'_{n-1},\mu} f_{\lambda'_1,\cdots,\lambda'_{n-1},\mu}(k^{-1})(c_1(G)^{\lambda'_1}.\cdots.c_{n-1}(G)^{\lambda'_{n-1}}.\tau^*D^{\mu'}), \end{aligned}$$

where $l = \sum_{i=1}^{n-1}\alpha_i + \lambda_n$ and $f_{\lambda'_1,\cdots,\lambda'_{n-1},\mu}$ is a polynomial of one variable k^{-1} whose coefficients are integer and do not depend on D, G or k. If $\mu = 0$, then the first term of the last expression is understood to be zero.

Proof. Since $\widetilde{D} \sim (k-1)T + k^{-1}g^*D$ and $U - V \ \approx\ k^{-1}h^*\tau^*D$, we have $\tilde{\tau}^*\widetilde{D} \sim$

$(k-1)U + k^{-1}h^*\tau^*D \sim kU - V$. By (2.7), we have

$$
\begin{aligned}
&(c_1(\widetilde{G})^{\lambda_1}.\cdots.c_n(\widetilde{G})^{\lambda_n}.\tilde{\tau}^*\widetilde{D}^{\mu})\\
=&\Bigg(\Bigg(\sum_{\alpha_1=0}^{\lambda 1}\binom{\lambda_1}{\alpha_1}h^*c_1^{\lambda_1-\alpha_1}.h^*c_0^{\alpha_1}.(U+V)^{\alpha_1}\Bigg).\cdots.\Bigg(\sum_{\alpha_{n-1}=0}^{\lambda_{n-1}}\binom{\lambda_{n-1}}{\alpha_{n-1}}\\
&\quad h^*c_{n-1}^{\lambda_{n-1}-\alpha_{n-1}}.h^*c_{n-2}^{\alpha_{n-1}}.(U+V)^{\alpha_{n-1}}\Bigg).h^*c_{n-1}^{\lambda_n}.(U+V)^{\lambda_n}.(kU-V)^{\mu}\Bigg)\\
=&\sum_{\substack{\alpha_1,\cdots,\alpha_{n-1}\\ l+\mu>0}}\binom{\lambda_1}{\alpha_1}\cdots\binom{\lambda_{n-1}}{\alpha_{n-1}}(h^*(c_1^{\lambda_1-\alpha_1+\alpha_2}.\cdots.c_{n-2}^{\lambda_{n-2}-\alpha_{n-2}+\alpha_{n-1}}\\
&\quad .c_{n-1}^{\lambda_{n-1}-\alpha_{n-1}+\lambda_n}).(U+V)^{l}.(kU-V)^{\mu})\\
=&\sum_{\substack{\alpha_1,\cdots,\alpha_{n-1}\\ l+\mu>0}}\binom{\lambda_1}{\alpha_1}\cdots\binom{\lambda_{n-1}}{\alpha_{n-1}}(h^*(c_1^{\lambda_1-\alpha_1+\alpha_2}.\cdots.c_{n-2}^{\lambda_{n-2}-\alpha_{n-2}+\alpha_{n-1}}\\
&\quad .c_{n-1}^{\lambda_{n-1}-\alpha_{n-1}+\lambda_n}).(k^{\mu}U^{l+\mu}+(-1)^{\mu}V^{l+\mu})) \qquad (2.11)\\
=&\sum_{\substack{\alpha_1,\cdots,\alpha_{n-1}\\ l+\mu>0}}k^{-l-\mu+1}\binom{\lambda_1}{\alpha_1}\cdots\binom{\lambda_{n-1}}{\alpha_{n-1}}(h^*(c_1^{\lambda_1-\alpha_1+\alpha_2}.\cdots.c_{n-2}^{\lambda_{n-2}-\alpha_{n-2}+\alpha_{n-1}}\\
&\quad .c_{n-1}^{\lambda_{n-1}-\alpha_{n-1}+\lambda_n}.\tau^*D^{l+\mu-1}).(k^{\mu}U+(-1)^{\mu}V)) \qquad (2.12)
\end{aligned}
$$

where we put $c_i = c_i(G), i = 1,\cdots,n-1$. (If $l+\mu = 0$, then $\alpha_1 = \cdots = \alpha_{n-1} = \lambda_n = \mu = 0$ and $h^*(c_1^{\lambda_1}.\cdots.c_{n-2}^{\lambda_{n-2}}.c_{n-1}^{\lambda_{n-1}}) = 0$. Hence we may omit the terms for which $l+\mu = 0$.) Since both U and V are sections of $h : \widetilde{G} \to G$, we have $(h^*Z.U) = \deg Z$ for every 0-cycle Z on G. Therefore the proposition follows from the last expression. □

Since G_1 is a curve and $-\deg c_1(G_1) = \deg \tau_1^* D_1 = 2p_a(X_1) - 2$, applying the proposition successively for $G_n \xrightarrow{h_{n-1}} G_{n-1} \longrightarrow \cdot\cdot \xrightarrow{h_1} G_1, G_i = \widetilde{G}_{i-1}$ $(i = 2,\cdots,n)$, we have

Corollary 2.13. *Let (X_i, L_i) and D_i, $i = 1,\cdots,n$, be as at the end of the last section. Let λ_i and μ be non-negative integers such that $\sum_{i=1}^n i\lambda_i + \mu = n$. Then*

$$(c_1(G_n)^{\lambda_1}.\cdots.c_n(G_n)^{\lambda_n}.\tau_n^*D_n^{\mu}) = \sum_{i_1,\cdots,i_{n-1}\leq 1} a_{i_1,\cdots,i_{n-1}}k_1^{i_1}\cdots k_{n-1}^{i_{n-1}}(2p_a(X_1)-2)$$

where $a_{i_1,\cdots i_{n-1}}$ is an integer which does not depend on $k_1,\cdots,k_{n-1},X_1,G_1$ or D_1 for every $i_1,\cdots,i_{n-1}$. Moreover, $a_{1,1,\cdots,1}$ is equal to 1 if $\mu = n, -1$ if $\mu = n-1$ and 0 if $\mu < n-1$.

Now we investigate an asymptotic behavior of the Chern numbers when $k_1,\cdots,k_{n-1} \to \infty$.

Proposition 2.14. *Let λ_i $(i = 1, \cdots, n)$ be nonnegative integers such that $\sum_{i=1}^n i\lambda_i = n$. Then we have*

$$
\begin{aligned}
&p^n(c_1(X_n)^{\lambda_1}.\cdots.c_n(X_n)^{\lambda_n})\\
=&(2p_a(X_1)-2)\{(1-p)^{\sum\lambda_i-1}(1+\frac{\lambda_1}{p-1}-\sum_{j=2}^n\lambda_j)k_1\cdots k_{n-1}\\
&+\sum_{\substack{i_1,\cdots,i_{n-1}\leq 1\\(i_1,\cdots,i_{n-1})\neq(1,\cdots,1)}} a_{i_1,\cdots,i_{n-1}}k_1^{i_1}\cdots k_{n-1}^{i_{n-1}}\},
\end{aligned}
$$

where, for every $i_1, \cdots, i_{n-1}$, $a_{i_1,\cdots,i_{n-1}}$ is an integer independent of $k_1, \cdots, k_{n-1}, X_1$ and D_1.

Proof. By Proposition (2.3), $\tau_n^* c_i(X_n)$ is rationally equivalent to

$$
\begin{aligned}
&c_i(G_n)+(1-p)\sum_{j=1}^i c_{i-j}(G_n).\tau_n^*D_n^j\\
&\sim(1-p)\tau_n^*D_n^i+(1-p)c_1(G_n)\tau_n^*D_n^{i-1}+(\textit{lower terms on}\quad D_n)
\end{aligned}
$$

if $i \neq 1$ and to $(1-p)\tau_n^* D_n + c_1(G_n)$ if $i = 1$. Hence we have

$$
\begin{aligned}
&p^n(c_1(X_n)^{\lambda_1}.\cdots.c_n(X_n)^{\lambda_n})\\
&=(\tau_n^*c_1(X_n)^{\lambda_n}.\cdots.\tau_n^*c_n(X_n)^{\lambda_n})\\
&=(1-p)^{\Sigma\lambda_i}(\tau_n^*D_n^n)+(1-p)^{\Sigma\lambda_i-1}\lambda_1(c_1(G_n).\tau_n^*D_n^{n-1})\\
&\quad+\sum_{j=2}^n(1-p)^{\Sigma\lambda_i}\lambda_j(c_1(G_n).\tau_n^*D_n^{n-1})\\
&\quad+\sum_{\substack{\lambda_1',\cdots,\lambda_n'\\ \mu\leq n-1}}\text{const}.(c_1(G_n)^{\lambda_1'}.\cdots.c_n(G_n)^{\lambda_n'}.\tau_n^*D_n^{\mu'})\\
&=(2p_a(X_1)-2)\{(1-p)^{\Sigma\lambda_i}(1+\frac{\lambda_1}{p-1}-\sum_{j=2}^n\lambda_j)k_1\cdots k_n\\
&\quad+\sum_{\substack{i_1,\cdots,i_{n-1}\leq 1\\(i_1,\cdots,i_{n-1})\neq(1,\cdots,1)}}\text{const}.k_1^{i_1}\cdots k_{n-1}^{i_{n-1}}\}
\end{aligned}
$$

by Corollary 2.13. □

By the proposition, if $k_1, \cdots, k_{n-1}$ are sufficiently large, then the sign of $(c_1(X_n)^{\lambda_1}.\cdots.c_n(X_n)^{\lambda_n})$ is equal to the sign of $(-1)^{\Sigma\lambda_i}(1 + \frac{\lambda_1}{p-1} - \sum_{j=2}^n \lambda_j)$. Hence the sign of $(c_1(X_n)^{n-i}.c_i(X_n))$ is equal to the sign of $(-1)^{n-i+1}$, that is, $(K_X^{n-i}.c_i(X))$ is negative if $i \geq 2$ and $k_1, \cdots, k_n$ are sufficiently large. Hence if the m at the end of § 1 is sufficiently large, then $(K_X^{n-i}.c_i(X))$ is negative for every $2 \leq i \leq n$, which completes our proof of Theorem.

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