PRINCIPAL Γ-CONE FOR A TREE

KYOJI SAITO

With an Appendix by Yoshio Sano

ABSTRACT. A Γ-cone for a graph Γ is, by the definition in §1, a connected component of the complement of a union of a system of hyperplanes attached to the edges of Γ in the real vector space of type $A_{\#\Gamma-1}$. It is subdivided into chambers of type $A_{\#\Gamma-1}$. If Γ is a tree, we introduce the *principal* Γ*-cone* E_Γ and characterize it by the maximality of the number of chambers contained in it. A formula for the maximal number is obtained by a *finite sum of hook length formulae*, and is explained by the *block decomposition* of the principal Γ-cone. The generating functions of the maximal numbers for the series A_l, D_l and E_l are given in Appendix.

The principal Γ-cone, when Γ is a Coxeter diagram $\Gamma(W)$ of a finite Coxeter group W, is introduced in [S1] in the study of real bifurcation set. The principal Γ-cone for any tree Γ, introduced in the present paper, is its generalization. As we shall see, the characterization of the principal Γ-cone (§3 Theorem), the enumeration of the chambers in the principal Γ-cone (§4 Theorem) and the block decomposition of the principal Γ-cone (§5 Theorem) can be formulated and proven only in terms of the tree Γ but not of the group W. Even a classical principal $\Gamma(W)$-cone decomposes into blocks which are non-classical (i.e. non-Coxeter) Γ-cones (e.g. §5 *Example*). Therefore, we publish the general combinatorial frame work separately from [S1] in the present paper.

The contents of the present paper are as follows. In §1, we fix the basic notation related to Γ-cones for oriented graphs. In §2, we prepare two assertions to count the number of chambers in a Γ-cone. In §3, we introduce the principal Γ-cone E_Γ for a tree Γ, and prove the first main Theorem of the present paper which states that the principal Γ-cone is the Γ-cone containing strictly maximal number of chambers. In §4, as the second main Theorem, we give the formula enumerating chambers in the principal Γ cone in terms of Γ. The formula is a finite sum of terms where each term resembles the classical hook length formula for a rooted tree even though the cone E_Γ does not corresponds to a rooted tree. This is explained in §5 by decomposing the principal Γ-cone E_Γ into blocks, where each block is a cone attached to a rooted tree and the hook length formula is available. In §6, we explain the motivation for the study of principal Γ-cones, which arises from a study of bifurcation set [S1]. At the end of §6, we compare Γ-cones with somewhat similar concept: Springer cones [Ar1][Sp], and clarify the relationship between them. The generating functions for the series of types A_l ($l \geq 1$), D_l ($l\geq 3$) and E_l ($l\geq 4$) are explicitly calculated by Sano in Appendix.

The author is grateful to Timothy Logvinenko for a careful reading of the manuscript and for several useful advises of improvements. He thanks also to Victor Reiner for suggestions to adjust the original version of the paper to standard terminology in combinatorics and for information on their references.

1. The Γ-cones and their chamber decomposition

For a finite graph Γ, we introduce Γ-cones in the real vector space of type $A_{\#\Gamma-1}$, which are subdivided into *chambers* of the type $A_{\#\Gamma-1}$ (see [B, chap.5] for terminologies). These geometric objects naturally correspond to some combinatorial structures on the graph Γ (e.g. [G-Z], [St1]). We fix notation and dictionary between the two subjects.

Let Π be a finite set with $\#\Pi = l \in \mathbb{Z}_{\geq 1}$. Consider a vector space:

$$V_\Pi := \oplus_{\alpha\in\Pi}\mathbb{R}v_\alpha/\mathbb{R}\cdot v_\Pi \tag{1}$$

of rank $l-1$, where $\{v_\alpha\}_{\alpha\in\Pi}$ is a generator system of V_Π satisfying a single relation $v_\Pi = 0$ with $v_\Pi := \sum_{\alpha\in\Pi} v_\alpha$. The permutation group $\mathfrak{S}(\Pi)$ acts on $\{v_\alpha\}_{\alpha\in\Pi}$ fixing v_Π, and, hence, the action extends linearly on V_Π (the reflection group action of type A_{l-1}). Let $\{\lambda_\alpha\}_{\alpha\in\Pi}$ be the dual basis of $\{v_\alpha\}_{\alpha\in\Pi}$, so that the difference $\lambda_{\alpha\beta} := \lambda_\alpha - \lambda_\beta$ for $\alpha, \beta \in \Pi$ is a well defined linear form on V_Π, forming the root system of type A_{l-1}. The zero locus $H_{\alpha\beta}$ of $\lambda_{\alpha\beta}$ ($\alpha\neq\beta$) in V_Π is a reflection hyperplane of the reflection action induced by the transposition (α, β). The union of $H_{\alpha\beta}$ for all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$ cuts V_Π into $(\#\Pi)!$ connected components, called *chambers* of type A_{l-1}. The set of chambers is naturally bijective to the set $Ord(\Pi)$ of all *linear orderings* on the set Π:

$$c := \{\alpha_1 <_c \ldots <_c \alpha_l\} \in Ord(\Pi) \;\leftrightarrow\; C_c := \cap_{i=1}^{l-1}\{v \in V_\Pi \mid \lambda_{\alpha_i\alpha_i+1}(v) < 0\}.$$

Here, the order-relation with respect to c is denoted by $<_c$, and the corresponding chamber is denoted by C_c. If we denote by $-c$ the reversed ordering of c, then one has $C_{-c} = -C_c$.

A *graph* Γ on Π is a one-dimensional simplicial complex whose set of vertices is Π. An edge connecting vertices α and β (if it exists) is denoted by $\overline{\alpha\beta} = \overline{\beta\alpha}$. The set of all edges of Γ is denoted by $Edge(\Gamma)$. By an abuse of notation, we shall sometimes denote the set of vertices by $|\Gamma|$, and write "a vertex $\alpha \in |\Gamma|$" instead of "a vertex $\alpha \in \Pi$".

Definition. A Γ-*cone* is a connected component of $V_\Pi \setminus \cup_{\overline{\alpha\beta}\in Edge(\Gamma)} H_{\alpha\beta}$.

The Γ-cones and the set of chambers contained in a Γ-cone are described in terms of combinatorics on Γ as follows: by an *orientation* o on Γ, we mean a collection of orientations $\alpha <_o \beta$ for all edges $\overline{\alpha\beta} \in Edge(\Gamma)$ such that *the oriented graph* (Γ, o) *must not contain an oriented cycle (in natural sense)*. Such orientation is called *acyclic* (c.f. [St2]). Put

$$Or(\Gamma) := \{\text{all acyclic orientations on } \Gamma\}. \tag{2}$$

The following dictionary is an immediate consequence of the definition.

Assertion 1.1. *1. For an orientation $o \in Or(\Gamma)$, define a cone:*

$$E_o := \cap_{\overline{\alpha\beta} \in Edge(\Gamma) \text{ oriented as } \alpha <_o \beta} \{v \in V_\Pi \mid \lambda_{\alpha\beta}(v) < 0\}. \tag{3}$$

Then E_o is a Γ-cone. The correspondence $o \mapsto E_o$ induces a bijection

$$Or(\Gamma) \simeq \{\Gamma\text{-cones}\}. \tag{4}$$

2. A chamber C_c for $c \in Ord(\Pi)$ is contained in the Γ-cone E_o for $o \in Or(\Gamma)$ if and only if c is a linear extension of o, i.e. $o = c|_{Edge(\Gamma)}$.

Proof. 1. For any $o \in Or(\Gamma)$, let us show that $E_o \neq \emptyset$, that is: there exists a map $v : \Pi \to \mathbb{R}$ such that $v(\alpha) < v(\beta)$ if $\alpha <_o \beta$. This is achieved by an induction on $\#\Pi$. Since there is no oriented cycle in (Γ, o), there exists a *minimal* vertex $\alpha \in \Pi$, that is: for any edge $\overline{\alpha\beta} \in Edge(\Gamma)$, one has $\alpha <_o \beta$. Put $\Pi' := \Pi \setminus \{\alpha\}$. Then clearly $o' := o|_{\Pi'}$ is an orientation on the graph $\Gamma' := \Gamma|_{\Pi'}$. Therefore, by the induction hypothesis, there exists a map $v' : \Pi' \to \mathbb{R}$ preserving the sub-orientation o'. Then, v is defined by an extension of v' by choosing the value $v(\alpha)$ from the non-empty set $\mathbb{R} \setminus \cup_{\beta \in \Pi', \overline{\alpha\beta} \in Edge(\Gamma)} [v'(\beta), \infty)$.

Conversely, for a given Γ-cone E, define the orientation $\alpha <_E \beta$ on the edge $\overline{\alpha\beta} \in Edge(\Gamma)$ if $\lambda_{\alpha\beta}|_E < 0$. This defines the orientation o_E on Γ. These establish the bijection (3).

2. The inclusion $C_c \subset E_o$ is equivalent to the inclusions $C_c \subset \{\lambda_{\alpha\beta} < 0\} \Leftrightarrow \alpha <_c \beta$ for any oriented edge $\overline{\alpha\beta}$ with $\alpha <_o \beta$. □

According to the previous assertion, we put

$$\Sigma(o) := \{\ c \in Ord(\Pi)\ \mid\ o = c|_{Edge(\Gamma)}\ \}, \tag{5}$$

and identify $\Sigma(o)$ with the set of chambers contained in E_o. Let us introduce a numerical invariant for the orientation $o \in Or(\Gamma)$:

$$\sigma(E_o) := \sigma(o) := \#\Sigma(o) = \#\{\text{chambers contained in } E_o\}. \tag{6}$$

If we denote by $-o$ the reversed orientation of o, one has $E_{-o} = -E_o$ and, therefore, $\Sigma(-o) = -\Sigma(o)$ and $\sigma(-o) = \sigma(o)$.

If $\#\Pi = 1$, then $V_\Pi = \{0\}$ has only one chamber $O := \{0\}$. There is only one graph (tree) structure on Π, denoted by $\Gamma(A_1)$, which admits only a trivial orientation denoted by o_{A_1}: $\Sigma(o_{A_1}) = \{O\}$ and $\sigma(o_{A_1}) = 1$.

In order to obtain the smallest oriented graph giving the same cone, we introduced the *reduced oriented graph* o_{red}, or *Hasse diagram*: an oriented edge $\alpha <_o \beta$ of the oriented graph o is called *removable* if there is a sequence $\alpha_0 = \alpha, \alpha_1, \cdots, \alpha_{k-1}, \alpha_k = \beta \in \Pi$ for some $k \in \mathbb{Z}_{>1}$ such that $\alpha_{i-1} <_o \alpha_i$ for $i = 1, \cdots, k$. We define the *reduction of* o by

o_{red} := the oriented graph obtained from o by deleting all removable edges = Hasse diagram of o.

One easily observes that i) *the associated cones coincides:* $E_o = E_{o_{red}}$, ii) *there is a natural 1:1 correspondence of the edges of* o_{red} *and the* l–2-*dimensional faces of* $E_o = E_{o_{red}}$. As a consequence of them, one has: iii) $E_o = E_{o'}$ *for oriented graphs* o *and* o' *on* Π, *if and only if* $o_{red} = o'_{red}$.

Remark 1. As we have described, the geometry of the chambers and the Γ-cones in V_Π has natural correspondence with the combinatorics (partially ordered structures) on the set Π. The problem of enumeration of $\sigma(o)$ is *the basic problem of enumeration of linear extension of a partially ordered set* in combinatorics (e.g., see [St1]).

On the other hand, the Γ-cone with the subdivision into chambers appears naturally in the study of finite Coxeter group W as follows, where Π stands for a *simple generator system* of W and a linear ordering $\alpha_1 <_c \alpha_2 <_c \cdots <_c \alpha_l$ of Π defines a *Coxeter element* $\alpha_1 \cdots \alpha_l \in W$. Two Coxeter elements coincide if the corresponding chambers belong to the same $\Gamma(W)$-cone for the Coxeter-Dynkin diagram $\Gamma(W)$ on Π. The principal $\Gamma(W)$-cone $E_{\Gamma(W)}$, which we shall introduce in §3, has the particular geometric significance, for which we refer to [S1] (see §7).

It may be also worthwhile to mention that a choice of the orientations on $\Gamma(W)$ plays often an important role in the studies related to the reflection group W or the Artin group and their representations (e.g. Auslander-Reiten quiver [K-S-T], the quantized Toda equations [E]).

Remark 2. Two chambers are said to be *adjacent* if they have a common $l-2$-dimensional face. The reflection hyperplane containing the face is called a *wall* of the chambers. The adjacency relation defines a graph structure on the set $Ord(\Pi)$ (i.e. two elements are connected by an edge if the corresponding two chambers are adjacent). The set $\Sigma(o)$($\simeq$ the set of chambers contained in E_o) naturally inherit the graph structure. The graph structure is important in the application (see §7 or [S1]). However in the present paper, we shall not go into any details of the subject, except for the following trivial description:

Assertion 1.2. *Let* $c, c' \in Ord(\Pi)$ *such that* $\alpha_1 <_c \alpha_2 <_c \cdots <_c \alpha_l$ *and* $\beta_1 <_{c'} \beta_2 <_{c'} \cdots <_{c'} \beta_l$. *Then the chambers* C_c *and* $C_{c'}$ *are adjacent if and only if there exists* $1 \le i < l$ *such that* $\alpha_i = \beta_{i+1}$, $\alpha_{i+1} = \beta_i$ *and* $\alpha_j = \beta_j$ *for* $j \ne i, i+1$. *The common face is supported in the wall* $H_{\alpha_i \alpha_{i+1}}$.

2. A decomposition formula

We prepare two Assertions which are to calculate $\sigma(o)$. They are used in the proof of the Theorems in §3,4 and 5. The idea is to divide (Γ, o) into the right and left sides of a base point $\alpha\in\Gamma$. Some readers might consider postponing reading this section until it becomes necessary.

For any $o \in Or(\Gamma)$, $\alpha \in \Pi$ and $r \in \mathbb{Z}_{\geq 0}$, we put

$$\Sigma(o,\alpha,r) := \{c \in \Sigma(o) \mid \#\{\beta \in \Pi \mid \alpha <_c \beta\} = r\}, \tag{7}$$

$$\sigma(o,\alpha,r) := \#\Sigma(o,\alpha,r). \tag{8}$$

Obviously, one has the disjoint decomposition $\Sigma(o) = \coprod_{r=0}^{l-1}\Sigma(o,\alpha,r)$ for any $\alpha \in \Pi$ so that $\sigma(o) = \sum_{r=0}^{l-1}\sigma(o,\alpha,r)$.

1. Suppose that the complement $\Gamma \setminus \{\alpha\}$ of Γ at a vertex $\alpha \in \Pi$ decomposes into components. More precisely, let $\Gamma_1, \cdots, \Gamma_k$ be graphs, which contain the same named vertex α. Let us denote by

$$\Gamma_1 \textstyle\coprod_\alpha \cdots \coprod_\alpha \Gamma_k, \tag{9}$$

a graph obtained by the disjoint union of the graphs Γ_i $(i=1,\cdots,k)$ up to an identification of the common vertex α.

Assertion 2.1. *Let $\Gamma = \Gamma_1 \coprod_\alpha \cdots \coprod_\alpha \Gamma_k$ be a decomposition as above. For an orientation $o\in Or(\Gamma)$, put $o_i := o|_{\Gamma_i} \in Or(\Gamma_i)$ $(i = 1,\cdots,k)$ and $l_i := \#\Gamma_i$ $(i=1,\cdots,k)$. Then, for any $r\in\mathbb{Z}_{\geq 0}$, one has a formula:*

$$\sigma(o,\alpha,r) = \sum_{\substack{r_1,\cdots,r_k\in\mathbb{Z}_{\geq 0}\\ r_1+\cdots+r_k=r}} \sigma(o_1,\alpha,r_1)\cdots\sigma(o_k,\alpha,r_k)\tbinom{r_1+\cdots+r_k}{r_1,\cdots,r_k}\tbinom{l-1-r_1-\cdots-r_k}{l_1-r_1-1,\cdots,l_k-r_k-1}, \tag{10}$$

where $\binom{r_1+\cdots+r_k}{r_1,\cdots,r_k} := (r_1+\cdots+r_k)!/r_1!\cdots r_k!$ is the multinomial coefficient. By summing the formula (10) *for all $r \in \mathbb{Z}_{\geq 0}$, one has a formula:*

$$\sigma(o) = \sum_{r_1,\cdots,r_k\in\mathbb{Z}_{\geq 0}} \sigma(o_1,\alpha,r_1)\cdots\sigma(o_k,\alpha,r_k)\tbinom{r_1+\cdots+r_k}{r_1,\cdots,r_k}\tbinom{l-1-r_1-\cdots-r_k}{l_1-r_1-1,\cdots,l_k-r_k-1}. \tag{11}$$

Proof. It is sufficient to prove only (10).

Consider the projection $\Sigma(o) \to \Sigma(o_1)\times\cdots\times\Sigma(o_k)$, $c \mapsto (c|_{\Gamma_i})_{i=1,\cdots,k}$. The projection decomposes into projections

$$\Sigma(o,\alpha,r) \to \coprod_{\substack{r_1,\cdots,r_k\in\mathbb{Z}_{\geq 0}\\ r_1+\cdots+r_k=r}} \Sigma(o_1,\alpha,r_1)\times\cdots\times\Sigma(o_k,\alpha,r_k) \tag{12}$$

for $r \in \mathbb{Z}_{\geq 0}$. Let us see that the cardinality of the inverse image of a point $(c_1,\cdots,c_k) \in \Sigma(o_1,\alpha,r_1)\times\cdots\times\Sigma(o_k,\alpha,r_k)$ depends only on $(r_1,\cdots,r_k)$. Put $\Gamma_i^+ := \{\beta \in |\Gamma_i| \mid \alpha <_{c_i} \beta\}$ and $\Gamma_i^- := \{\beta \in |\Gamma_i| \mid$

$\beta <_{c_i} \alpha\}$. Then, an ordering $c \in \Sigma(o,\alpha,r)$ is in the inverse image, if c defines the ordering of $r = r_1 + \cdots + r_k$ elements $\amalg_{i=1}^k \Gamma_i^+$ in RHS of α and the ordering of $l-r-1 = (l_1-r_1-1)+\cdots+(l_k-r_k-1)$ elements $\amalg_{i=1}^k \Gamma_i^-$ in LHS of α, which satisfies the condition: the sub-orderings on each $\Gamma_i^{\pm}$ is pre-fixed by $c_i^{\pm} := c_i|_{\Gamma_i^{\pm}}$ for $i = 1, \cdots, k$ where $c_i^{\pm}$ defines a linearly ordered graphs structure on $\Gamma_i^{\pm}$. The condition on c is equivalent to that c belongs to $\Sigma(\amalg_{i=1}^k c_i^+) \times \Sigma(\amalg_{i=1}^k c_i^-)$, where $\amalg_{i=1}^k c_i^{\pm}$ is the partial ordering structure on $\Gamma^{\pm} := \amalg_{i=1}^k \Gamma_i^{\pm}$. Since $\Sigma(\amalg_{i=1}^k c_i^+)$ is just the set of shuffles of the sets of elements of $r_1, \ldots, r_k$, its cardinality is given by the combination number: $\sigma(\amalg_{i=1}^k c_i^+) = \binom{r_1+\cdots+r_k}{r_1,\cdots,r_k}$. Similarly, one has $\sigma(\amalg_{i=1}^k c_i^-) = \binom{l-1-r_1-\cdots-r_k}{l_1-r_1-1,\cdots,l_k-r_k-1}$. Both are independent of $(c_1, \cdots, c_k)$. □

2. A vertex $\alpha \in \Pi$ is called *maximal* (resp. *minimal*) with respect to $o \in Or(\Gamma)$, if $\beta <_o \alpha$ (resp. $\alpha <_o \beta$) for any edge $\overline{\alpha\beta} \in Edge(\Gamma)$ at α.

Assertion 2.2. *If α is maximal with respect to o, then one has*

$$\sigma(o,\alpha,0) \geq \sigma(o,\alpha,1) \geq \cdots \geq \sigma(o,\alpha,l-2) \geq \sigma(o,\alpha,l-1).$$

If α is minimal with respect to o, then one has

$$\sigma(o,\alpha,0) \leq \sigma(o,\alpha,1) \leq \cdots \leq \sigma(o,\alpha,l-2) \leq \sigma(o,\alpha,l-1).$$

If α is non-isolated in Γ, the smallest terms in the sequences are zero.

Proof. We show only the first case. The latter case is shown similarly.

It is sufficient to show that there is an injection map $\Sigma(o,\alpha,r) \to \Sigma(o,\alpha,r-1)$ for $r > 0$. In fact the map is constructed as follows: let $c = \{A <_c \alpha <_c \beta <_c B\} \in \Sigma(o,\alpha,r)$ where $\beta \in \Pi$ and A and B are such linear sequence of inequalities of elements of Π that the length of B is equal to $r-1$ (this is possible since $r \geq 1$). Then we set $c' := \{A <_c \beta <_c \alpha <_c B\} \in \Sigma(o,\alpha,r-1)$ where c' is well defined since α is maximal. The correspondence $c \mapsto c'$ is clearly injective.

If α is non-isolated, then the set $\Sigma(o,\alpha,l-1)$ is empty, since there exists a vertex $\beta \in \Pi$ such that $\overline{\beta\alpha} \in Edge(\Gamma)$ and $\beta <_o \alpha$ and hence for any $c \in \Sigma(o)$ one has $\beta <_c \alpha$ and $c \notin \Sigma(o,\alpha,l-1)$. □

3. Principal Γ-cones

A graph Γ is called a *tree* if it is connected and simply connected. For a tree Γ, we introduce particular Γ-cones (unique up to a sign), called the principal Γ-cones. The first main result of the present paper, formulated in Theorem (3.2), is to characterize the principal Γ-cones.

The following is a characterization of trees in terms of Γ-cones.

Assertion 3.1. *Let Γ be a graph on Π. Then, $\{H_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ forms a system of coordinate hyperplanes of V_Π if and only if Γ is a tree.*

Proof. For each edge $\overline{\alpha\beta}$ of Γ, we choose one of $\lambda_{\alpha\beta}$ or $\lambda_{\beta\alpha}$. Then, it is immediate that i) $\{\lambda_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ is linearly independent if and only if Γ does not contain a cycle, and ii) $\{\lambda_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ spans the dual space of V_Π if and only if Γ is connected. □

From now on in the present paper, we shall assume that Γ is a tree on Π. Then the system of coordinate hyperplanes $\{H_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ cuts the vector space V_Π into $2^{\#\Pi-1}$-number of quadrants, each of which is a Γ-cone. Therefore, the Γ-cones are *simplicial* (i.e. are cones over simplices). The size of the decomposition vector for a tree Γ is equal to 2^{l-1}. The other distinguishing property of the decomposition vector for a tree is that it contains a unique (up to an involution, c.f. below) maximal entry (=the principal entry), which we explain now.

If Γ is a tree, which is not of type A_1, then there is a decomposition, unique up to a transposition, of the set Π of vertices into two parts:

$$\Pi = \Pi_1 \amalg \Pi_2 \tag{13}$$

such that each Π_i is totally disconnected in Γ.

Definition. Let Γ be a tree. If $\Gamma \neq \Gamma(A_1)$, a *principal orientation* on Γ is an element in $Or(\Gamma)$ which is either equal to

$$o_{\Pi_1,\Pi_2} := \{\ \alpha <_{o_{\Pi_1,\Pi_2}} \beta \ \text{ for } \overline{\alpha\beta} \in Edge(\Gamma) \text{ with } \alpha\in\Pi_1, \beta\in\Pi_2\}. \tag{14}$$

or to $o_{\Pi_2,\Pi_1} = -o_{\Pi_1,\Pi_2}$. A *principal* Γ-*cone* is the Γ-cone attached to a principal orientation. That is: it is one of the following two cones:

$$E_{\Pi_1,\Pi_2} := E_{o_{\Pi_1,\Pi_2}} \quad \text{and} \quad E_{\Pi_2,\Pi_1} := E_{o_{\Pi_2,\Pi_1}}. \tag{15}$$

Since $o_{\Pi_2,\Pi_1} = -o_{\Pi_1,\Pi_2}$ and $E_{\Pi_2,\Pi_1} = -E_{\Pi_1,\Pi_2}$, two principal Γ-cones are isomorphic to each other as abstract cones. The isomorphisms class:

$$E_\Gamma := E_{\Pi_1,\Pi_2} \simeq E_{\Pi_2,\Pi_1} \tag{16}$$

is called the *principal* Γ-*cone.*

If $\Gamma = \Gamma(A_1)$, the trivial orientation o_{A_1} on $\Gamma(A_1)$ is called the *principal orientation on* $\Gamma(A_1)$. Thus the sole principal cone $E_{\Gamma(A_1)} = \{0\}$ consists of a single chamber O, i.e. $\Sigma(o_{A_1}) = \{O\}$ and $\sigma(o_{A_1}) = 1$.

The following first main theorem of the present paper characterizes the principal Γ-cone.

Theorem 3.2. *Let* Γ *be a tree on* Π. *The principal* Γ-*cone is the* Γ-*cone which contains strictly maximal number of chambers. That is: a* Γ-*cone* E_o *for* $o \in Or(\Gamma)$ *is principal if and only if* $\sigma(o) = \sigma(\Gamma)$, *where*

$$\sigma(\Gamma) := \max\{\sigma(p) \mid p \in Or(\Gamma)\}. \tag{17}$$

Proof. With the results established in §2, the proof is straight forward: suppose $o \in Or(\Gamma)$ is not principal, that is: there exist $\alpha, \beta, \gamma \in \Pi$ with $\gamma <_o \alpha <_o \beta$. Actually, Γ decomposes as $\Gamma = \Gamma_+ \coprod_\alpha \Gamma_-$, where Γ_+ (resp. Γ_-) is a full subgraph of Γ containing α and any connected component of $\Gamma \setminus \{\alpha\}$ which contains a vertex β s.t. $\alpha <_o \beta$ (resp. $\alpha >_o \beta$). By the assumption on o, one has $\Gamma_\pm \neq \emptyset$.

Put $o_+ := o|_{\Gamma_+} \in Or(\Gamma_+)$ and $o_- := o|_{\Gamma_-} \in Or(\Gamma_-)$.

Assertion 3.3. *Define a new orientation $\tilde{o} \in Or(\Gamma)$ by the following rule: $\tilde{o}$ agrees with o_+ on Γ_+ and with $-o_-$ on Γ_-. Then $\sigma(\tilde{o}) > \sigma(o)$.*

Proof. For a proof of the Assertion, we apply the formula (11) in Assertion 2.1 to the decomposition $\Gamma = \Gamma_+ \coprod_\alpha \Gamma_-$and to $o, \tilde{o} \in Or(\Gamma)$:

$$\sigma(o) = \sum_{r_+=0}^{l_+} \sum_{r_-=0}^{l_-} \sigma(o_+, \alpha, r_+)\sigma(o_-, \alpha, r_-) C_{r_+, r_-} C_{l_+ - r_+, l_- - r_-},$$

$$\sigma(\tilde{o}) = \sum_{r_+=0}^{l_+} \sum_{r_-=0}^{l_-} \sigma(o_+, \alpha, r_+)\sigma(o_-, \alpha, r_-) C_{r_+, l_- - r_-} C_{l_+ - r_+, r_-},$$

where $l_+ := \#\Gamma_+ - 1 > 0$ and $l_- := \#\Gamma_- - 1 > 0$.

We want to calculate the difference $\sigma(\tilde{o}) - \sigma(o)$ term-to-term. Observe that the terms for $r_+ = l_+/2$ (if l_+ is even) and the terms for $r_- = l_-/2$ (if l_- is even) in the two formulae give the same value and so cancel each other in the difference. Therefore, we decompose the region $[0, l_+] \times [0, l_-]$ of the summation index (r_+, r_-) into 4 regions according to whether r_+ is larger or less than $l_+/2$ and whether r_- is larger or less than $l_-/2$.

For an index (r_+, r_-) in the region $[0, l_+/2) \times [0, l_-/2)$, we consider 4 indices (r_+, r_-), (r_+, r_-^*), (r_+^*, r_-) and (r_+^*, r_-^*) in the 4 regions simultaneously, where $r_+^* := l_+ - r_+$ and $r_-^* := l_- - r_-$. Let us explicitly write down the difference between these 4 terms in $\sigma(\tilde{o})$ and in $\sigma(o)$:

$$\begin{aligned} &\sigma(r_+)\sigma(r_-)C_{r_+, r_-^*} C_{r_+^*, r_-} + \sigma(r_+)\sigma(r_-^*)C_{r_+, r_-} C_{r_+^*, r_-^*} \\ &+ \sigma(r_+^*)\sigma(r_-)C_{r_+^*, r_-^*} C_{r_+, r_-} + \sigma(r_+^*)\sigma(r_-^*)C_{r_+^*, r_-} C_{r_+, r_-^*} \\ &- \sigma(r_+)\sigma(r_-)C_{r_+, r_-} C_{r_+^*, r_-^*} - \sigma(r_+)\sigma(r_-^*)C_{r_+, r_-^*} C_{r_+^*, r_-} \\ &- \sigma(r_+^*)\sigma(r_-)C_{r_+^*, r_-} C_{r_+, r_-^*} - \sigma(r_+^*)\sigma(r_-^*)C_{r_+^*, r_-^*} C_{r_+, r_-}, \end{aligned}$$

where we used the simplified notation $\sigma(r_+) := \sigma(o_+, \alpha, r_+)$, $\sigma(r_+^*) := \sigma(o_+, \alpha, r_+^*)$, $\sigma(r_-) := \sigma(o_-, \alpha, r_-)$ and $\sigma(r_-^*) := \sigma(o_-, \alpha, r_-^*)$.

Miraculously, one can factorize this difference as follows:

$$(\sigma(r_+) - \sigma(r_+^*))(\sigma(r_-^*) - \sigma(r_-))(C_{r_+, r_-} C_{r_+^*, r_-^*} - C_{r_+, r_-^*} C_{r_+^*, r_-}).$$

Let us examine the sign of the factors and demonstrate that the product turns out to be non-negative. First, recall that the vertex α

is minimal in Γ_+ and maximal in Γ_- by definition. Note also that $r_+ < l_+/2 < r_+^*$ and $r_- < l_-/2 < r_-^*$. Therefore, applying §2 Assertion 2.2, we observe that $(\sigma(r_+*) - \sigma(r_+))(\sigma(r_-) - \sigma(r_-^*)) \geq 0$. Next, let us examine the last factor. For this purpose, we use the proportion of the two terms in the last factor:

$$\frac{C_{r_+,r_-}C_{r_+^*,r_-^*}}{C_{r_+,r_-^*}C_{r_+^*,r_-}} = \frac{(r_+^* + r_-^*)!}{(r_+ + r_-^*)!} \cdot \frac{(r_+ + r_-)!}{(r_+^* + r_-)!}.$$

Using the fact that $r_+ < r_+^*$, one has $r_+^* + r_-^* > r_+ + r_-^*$ and $r_+ + r_- < r_+^* + r_-$. Hence, the expression can be reduced to $\prod_{k=r_++1}^{r_+^*} \frac{r_-^*+k}{r_-+k}$, where each factor is larger than 1 since $r_- + k < r_-^* + k$ and the number of the factors is $r_+^* - r_+ > 0$ so that the result is always larger than 1. These together imply that the difference of the 4 terms is non-negative.

By summing up terms for all indices (r_+, r_-) in the region $[0, l_+/2) \times [0, l_-/2)$, we see that the difference $\sigma(\tilde{o}) - \sigma(o)$ is non-negative. To show that it is strictly positive, let us calculate the term for $(r_+, r_-) = (0, 0)$. Then, §2, Assertion 2.2 again, one has $\sigma(r_+) = \sigma(r_-^*) = 0$. Since $l_+, l_- > 0$ (non-principality of σ), one obtains a rather big number:

$$\sigma(o_+, \alpha, l_+)\sigma(o_-, \alpha, 0)(C_{l_+,l_-} - 1) \neq 0.$$

This completes the proof of the Assertion. □

The Assertion says that if an orientation o on Γ is not principal, it can not attain the maximal value $\sigma(\Gamma)$ of $\sigma(o)$ for $o \in Or(\Gamma)$. In fact, starting from any orientation $o \in Or(\Gamma)$, and by a successive application of the construction in the Assertion, one arrives at one of the principal orientations. Since $E_{\Pi_1,\Pi_2} \simeq E_{\Pi_1,\Pi_2}$, one has $\sigma(o_{\Pi_1,\Pi_2}) = \sigma(o_{\Pi_1,\Pi_2})$. This number gives the maximal value $\sigma(\Gamma)$.

This completes the proof of the Theorem. □

Remark 3. Some particular cases of Theorem 3.2 was known already.

If Γ is a linear graph of type A_l, then the (principal) Γ-cones coincide with the (principal) Springer cone of type A_{l-1} (see [Ar],[Sp] and the latter half of §7 of the present paper). In that case, the result is shown [Sp, Prop.3]. [S-Y-Z, Theo.1.2., (2)], [N],[B]

Remark 4. Let $\Gamma = \amalg_{i=1}^k \Gamma_i$ be the decomposition of a forest into trees. For an orientation o on Γ, put $o_i := o|_{\Gamma_i}$. Since $\sigma(o) = \Pi_i \sigma(o_i)\binom{\sum \#\Gamma_i}{\#\Gamma_1, \cdots, \#\Gamma_k}$, the maximal number of chambers in a Γ-cone is attained by the orientations o such that each o_i is a principal orientation on Γ_i.

A question of interest is a characterization of the *decomposition vector*: $(\sigma(o))_{o \in Or(\Gamma)}$ for a tree Γ. One, obviously, has $\sum_{o \in Or(\Gamma)} \sigma(o) = l!$. Even though the vector is algorithmically determined from the graph Γ, it is non-trivial to calculate the vector in general.

Example. The principal cone $E_{\Gamma(D_4)}$ consists of 6 chambers forming a hexagon. The decomposition vector is $(\sigma(o))_{o\in Or(\Gamma(D_4))} = 2(6,2,2,2)$.

The principal cone $E_{\Gamma(A_4)}$ consists of 5 chambers forming a spoon graph. The decomposition vector is $(\sigma(o))_{o\in Or(\Gamma(A_4))} = 2(5,3,3,1)$.

Let Γ be a cyclic graph of 4 vertices. Even though Γ is not a tree, the decomposition vector contains the maximal entry: $(\sigma(o))_{o\in Or(\Gamma)} = 2(4,2,2,1,1,1,1)$, where the maximal is attained by the Γ-cone corresponding to the partial ordering defined by the decomposition of Π of the form (13). Conjecturely, *this may happen for any connected graph* Γ *which admits the principal decomposition* (13) (c.f. *Remark* below).

Remark 5. Let us call the decomposition (13) for a graph (which may not necessarily be a tree) the *principal decomposition*. Then, we have the following [S2]: assume that a connected graph Γ admits a principal decomposition. consider the lattice L_Γ spanned by Π with the symmetric bilinear form as in the usual convention in a theory of root systems. Then the "Coxeter element"defined as the product of reflections attached to the vertex in the order of a principal order (recall *Remark* 1.) is i) semi-simple of finite order, or ii) quasi-unipotent if and only if i) Γ is one of the Coxeter-Dynkin diagram for a finite Coxeter group or ii) Γ is either i) or one of the affine Coxeter-Dynkin diagram, respectively.

4. Enumeration of chambers in the principal Γ cone

As the second main result of the present paper, we give an enumeration formula (19) for the principal number $\sigma(\Gamma)$. It is formulated as i) a sum whose summation index runs over certain equivalence classes $\widetilde{Ord}(\Pi_1)$ of all linear orderings on Π_1 and ii) each summand is the quotient of $(\#\Pi)!$ by a product of cardinalities of certain subgraphs. Thus, each summand resembles the hook length formula of Knuth [K2,p70]. This may cause a puzzle since the hook length formula is an enumeration of chambers in a Γ-cone for rooted forests, but the principal orientations and the rooted forests are, in some sense, the most contrasting orientations on forests. We shall find an answer in §5 that the principal Γ-cone decomposes into a union of Γ-cones for certain rooted trees. Thus, we present two proofs of the formula (19): the proof in this section is based on the principal Γ-ordering on Π and the proof in §5 is based on the newly introduced rooted tree structures on Π.

We start with the definition of the equivalence $\sim$ on the set $Ord(\Pi_1)$.

Let $d \in Ord(\Pi_1)$ be an ordering on Π_1. For $v \in \Pi_1$, put

$$\Gamma_{d,v} := \text{the connected component of } \Gamma\setminus\{w\in\Pi_1 \mid w<_d v\} \text{ containing } v. \tag{18}$$

In particular, one has $\Gamma_{d,v} = \Gamma$ for the smallest element v of Π_1.

Definition. Two orderings $d, d' \in Ord(\Pi_1)$ are called *equivalent* if $\Gamma_{d,v} = \Gamma_{d',v}$ for all $v \in \Pi_1$. The equivalence class of d is denoted by $\tilde{d}$ and the set of all equivalence classes is denoted by $\widetilde{Ord}(\Pi_1)$.

Theorem 4.1. *Let Γ be a tree on Π. Choose the decomposition* (13). *Then the principal number $\sigma(\Gamma)$* (17) *is given by*

$$\sigma(\Gamma) = (\#\Gamma)! \sum_{\tilde{d} \in \widetilde{Ord}(\Pi_1)} \frac{1}{\prod_{v \in \Pi_1} \#\Gamma_{d,v}}, \tag{19}$$

where the terms in RHS is well-defined since $\Gamma_{d,v}$ depends only on the equivalence class $\tilde{d}$ of $d \in Ord(\Pi_1)$ and on $v \in \Pi_1$.

Proof. Before we start with the proof of the formula, we reformulate the equivalence $\sim$ in terms of the partial orderings on the set Π_1.

Fact i) *For any two indices $v, v' \in \Pi_1$, one has three cases:*

$$\Gamma_{d,v} \cap \Gamma_{d,v'} = \begin{cases} \emptyset \\ \Gamma_{d,v} \\ \Gamma_{d,v'}. \end{cases}$$

Fact ii) *If $\Gamma_{d,v} \cap \Gamma_{d,v'} = \Gamma_{d,v}$ then $v' \leq_d v$.*

Fact iii) *The next three conditions are equivalent:*
a) $\Gamma_{d,v} \cap \Gamma_{d,v'} = \Gamma_{d,v}$, b) $\Gamma_{d,v} \subset \Gamma_{d,v'}$, c) $v \in \Gamma_{d,v'}$.

Proof. i) Since d is a linear ordering, we may assume $v' <_d v$. The fact that $\Gamma \setminus \{w \in \Pi_1 \mid w <_d v\} \subset \Gamma \setminus \{w \in \Pi_1 \mid w <_d v'\}$ implies that the component $\Gamma_{d,v}$ is either contained in the component $\Gamma_{d,v'}$ or they are disjoint. Accordingly, the intersection is either $\Gamma_{d,v}$ or an empty set.

ii) Suppose the contrary $v' \not\leq_d v$. Then the totally orderedness of d implies $v' >_d v$. Then, by the construction, $\Gamma_{d,v'}$ cannot contain v. This contradicts to the assumption $\Gamma_{d,v} \cap \Gamma_{d,v'} = \Gamma_{d,v}$.

iii) The implications: a) $\Rightarrow$ b) $\Rightarrow$ c) are trivial. Assume c). This implies $\Gamma_{d,v} \cap \Gamma_{d,v'} \neq \emptyset$. Suppose, further, $\Gamma_{d,v} \cap \Gamma_{d,v'} \neq \Gamma_{d,v}$. Then i) implies $\Gamma_{d,v} \cap \Gamma_{d,v'} = \Gamma_{d,v'} \neq \Gamma_{d,v}$, and, hence, $\Gamma_{d,v'} \subset\neq \Gamma_{d,v}$. This mean that $\Gamma_{d,v'}$ is a connected component by deleting strictly more vertices than those for $\Gamma_{d,v}$. This is possible only when $v <_d v'$. Then, $v \notin \Gamma_{d,v'}$. A contradiction to the assumption c) ! □

Definition. To the equivalence class $\tilde{d}$ in $\widetilde{Ord}(\Pi_1)$ of $d \in Ord(\Pi_1)$, we attach a *partial ordering* on Π_1: for $v, v' \in \Pi_1$, put

$$\begin{aligned} & v' \leq_{\tilde{d}} v \\ \overset{\text{def}}{\Leftrightarrow} \quad & \text{the three equivalent conditions a), b) and c) in Fact iii).} \end{aligned} \tag{20}$$

In the other words, *there is no order relation between* $v, v' \in \Pi_1$ *if* $\Gamma_{d,v} \cap \Gamma_{d,v'} = \emptyset$, *otherwise the order relation of* $\tilde{d}$ *agrees with* d.

Fact iv) *Consider the partial ordering on* Π_1 *for* $\tilde{d} \in \widetilde{Ord}(\Pi_1)$. *For any* $v \in \Pi_1$, *the set of predecessors* $\{w \in \Pi_1 \mid w <_{\tilde{d}} v\}$ *is totally ordered by* $\tilde{d}$.

Proof. Suppose $w_i <_{\tilde{d}} v$ $(i = 1, 2)$. This means $v \in \Gamma_{d,w_i}$ (Fact iii) c)), and hence $\Gamma_{d,w_1} \cap \Gamma_{d,w_2} \neq \emptyset$. Then, Fact i) implies that either $w_1 \leq_{\tilde{d}} w_2$ or $w_1 \geq_{\tilde{d}} w_2$ occurs. □

We obtain the following characterization of the partial ordering $<_{\tilde{d}}$.

Assertion 4.2. *For two orderings* $d, d' \in Ord(\Pi_1)$, *the following two conditions are equivalent.*

a) One has the equality $\Gamma_{d,v} = \Gamma_{d',v}$ *for all* $v \in \Pi_1$, *i.e.* $d \sim d'$.

b) The partial orderings $<_{\tilde{d}}$ *and* $<_{\tilde{d}'}$ *on the set* Π_1 *coincide.*

Proof. We have only to show that the partial ordering $<_{\tilde{d}}$ determine the set $\Gamma_{d,v}$ for $v \in \Pi_1$. First, we show that the set $\Gamma_{d,v}$ is given by

$$\Gamma_{d,v} = \big(\Gamma_{d,v} \cap \Pi_1\big) \cup \bigcup_{w \in \Gamma_{d,v} \cap \Pi_1} Nbd(w)$$

from the set $\Gamma_{d,v} \cap \Pi_1$, where $Nbd(w) := \{u \in \Pi \mid^{\exists}\ \overline{wu} \in Edge(\Gamma)\} \subset \Pi_2$.

(*Proof.* The inclusion $\subset$ follows from the connectivity of $\Gamma_{d,v}$. The opposite inclusion $\Gamma_{d,v} \supset Nbd(w)$ for $w \in \Gamma_{d,v} \cap \Pi_1$ follows also from the connectivity of $\Gamma_{d,v}$.□) On the other hand, due to Fact iii) c), one has

$$\Gamma_{d,v} \cap \Pi_1 := \{w \in \Pi_1 \mid v \leq_{\tilde{d}} w\}.$$

Thus, $\Gamma_{d,v}$, as a set, is determined from the partial ordering $\tilde{d}$. □

Finally, let us show $\Gamma_{d,v} = \Gamma_{\tilde{d},v}$ for $v \in \Pi_1$, where

$$(21) \qquad \Gamma_{\tilde{d},v} \ := \ \text{the connected component of } \Gamma \setminus \{w \in \Pi_1 \mid w <_{\tilde{d}} v\} \text{ containing } v.$$

Proof. Since the partial ordering $\tilde{d}$ is rough than the total ordering d, one has the inclusion $\Gamma_{d,v} \subset \Gamma_{\tilde{d},v}$. To show the opposite inclusion, it is sufficient to show that if $v \not\leq_{\tilde{d}} w$, then w does not belong to $\Gamma_{\tilde{d},v}$.

We may assume $w \not\leq_{\tilde{d}} v$, otherwise $w \notin \Gamma_{\tilde{d},v}$ is trivial. Let u_0 be the unique maximal element of the (non-empty by assumptions) totally ordered set $\{u \in \Pi_1 \mid u <_{\tilde{d}} w,\ u <_{\tilde{d}} v\}$ (c.f. Fact iv)). By definition, v and w belong to the connected component $\Gamma_{d,u_0} \subset \Gamma_{\tilde{d},u_0}$. However, since they belong to different components of $\Gamma_{d,u_0} \setminus \{u_0\}$, they also belong to different components of $\Gamma_{\tilde{d},u_0} \setminus \{u_0\}$ (since $\Gamma_{\tilde{d},u_0}$ is a tree). □

Let us return to the proof of the Theorem. The formula (19) is shown by the induction on $\#\Gamma$. We first prepare an induction formula.

Let Γ be a tree. For a given decomposition $\{\Pi_1, \Pi_2\}$ and an attached principal orientation o_{Π_1,Π_2}, we want to enumerate the set $\Sigma(o_{\Pi_1,\Pi_2})$.

By definition, for any total ordering $d \in \Sigma(o_{\Pi_1,\Pi_2})$, the smallest element belongs to Π_1. Therefore, we have a decomposition:

$$\Sigma(o_{\Pi_1,\Pi_2}) = \coprod_{v\in\Pi_1} \Sigma(o_{\Pi_1,\Pi_2}, v <)$$

where $\Sigma(o_{\Pi_1,\Pi_2}, v <) := \{d \in \Sigma(o_{\Pi_1,\Pi_2}) \mid v \text{ is the smallest element in } d.\}$. Put $\sigma(o_{\Pi_1,\Pi_2}, v <) := \#\Sigma(o_{\Pi_1,\Pi_2}, v <)$ so that one has

$$\sigma(\Gamma) := \sigma(o_{\Pi_1,\Pi_2}) = \sum_{v\in\Pi_1} \sigma(o_{\Pi_1,\Pi_2}, v <).$$

For $w \in Nbd(v)$, let us denote by Γ_{vw} the connected component of $\Gamma \setminus \{v\}$ containing w. One has the decomposition $\Gamma\setminus\{v\} = \coprod_{w\in Nbd(v)} \Gamma_{vw}$.

Applying (10) in Assertion 2.1 for $\alpha = v$ and $r = \#\Gamma - 1 = \sum_{w\in Nbd(v)} r_w$, $r_w := \#\Gamma_{vw}$, we obtain: $\sigma(o_{\Pi_1,\Pi2}, v <) = (\#\Gamma - 1)! \prod_{w\in Nbd(v)} \frac{\sigma(\Gamma_{vw})}{(\#\Gamma_{vw})!}$. Summing over all vertices $v \in \Pi_1$, we obtain the induction formula:

$$\frac{\sigma(\Gamma)}{(\#\Gamma)!} = \frac{1}{\#\Gamma} \sum_{v\in\Pi_1} \prod_{w\in Nbd(v)} \frac{\sigma(\Gamma_{vw})}{(\#\Gamma_{vw})!}. \tag{22}$$

By the induction hypothesis, for any $v \in \Pi_1$ and $w \in Nbd(v)$, we have already the formula for Γ_{vw}:

$$*) \qquad \frac{\sigma(\Gamma_{vw})}{(\#\Gamma_{vw})!} = \sum_{\tilde{d}_w \in \widetilde{Ord}(\Gamma_{vw}\cap\Pi_1)} \frac{1}{\prod_{u\in\Gamma_{vw}\cap\Pi_1} \#(\Gamma_{vw})_{\tilde{d}_w,u}}.$$

The substitution of $*)$ into RHS of (22) gives a formula summing the terms: $\frac{1}{\#\Gamma} \prod_{w\in Nbd(v)} \frac{1}{\prod_{u\in\Gamma_{vw}\cap\Pi_1} \#(\Gamma_{vw})_{\tilde{d}_w,u}}$, where the summation index $v \times \{\tilde{d}_w\}_{w\in Nbd(v)}$ runs in the set $\bigcup_{v\in\Pi_1} \big(v \times \prod_{w\in Nbd(v)} (\widetilde{Ord}(\Gamma_{vw} \cap \Pi_1))\big)$.

For the index $v \times \{\tilde{d}_w\}_{w\in Nbd(v)}$, we attach the partial ordering $\tilde{d}$ of the set Π_1 defined by the rule a) v is the smallest element, b) $\tilde{d}$ agrees with $\tilde{d}_w$ on the set $\Gamma_{vw} \cap \Pi_1$ for $w \in Nbd(v)$, and c) there is no order relation between $\Gamma_{vw} \cap \Pi_1$ and $\Gamma_{vw'} \cap \Pi_1$ for different $w, w' \in Nbd(v)$.

This correspondence $v \times \{\tilde{d}_w\}_{w\in Nbd(v)} \mapsto \tilde{d}$ gives a bijection:

$$\bigcup_{v\in\Pi_1} \big(v \times \prod_{w\in Nbd(v)} (\widetilde{Ord}(\Gamma_{vw} \cap \Pi_1))\big) \;\simeq\; \widetilde{Ord}(\Pi_1),$$

where the opposite correspondence is given by the restriction map.

On the other hand, the term $\frac{1}{\#\Gamma}\prod_{w\in Nbd(v)}\frac{1}{\prod_{u\in\Gamma_{vw}\cap\Pi_1}\#(\Gamma_{vw})_{\tilde{d}_w,u}}$ for the index $v\times\{\tilde{d}_w\}_{w\in Nbd(v)}$ coincides with the term $\frac{1}{\prod_{u\in\Pi_1}\#\Gamma_{\tilde{d},u}}$ in (19) given by the corresponding partial ordering $\tilde{d}$. This means that the substitution of $*$) into RHS of (22) gives RHS of the formula (19).

This completes the proof of the Theorem. □

5. Block decomposition of the principal Γ-cone

As the third main result of the present paper, we introduce the *block decomposition* of a principal Γ-cone $E_{o_{\Pi_1,\Pi_2}}$, where each block is a simplicial cone associated to a *rooted tree*. Since the number of chambers in a cone associated to a rooted tree is well know by the hook length formula, this reproduce an alternative proof of the formula (19).

Definition. An oriented graph (Γ, o) is called a *rooted tree* if

i) There exists the unique minimal vertex $v_o\in\Gamma$ with respect to o.

ii) Any vertex $(\neq v_o)$ of Γ has a unique immediate predecessor.

The smallest vertex v_0 is called the *root* of (Γ, o). It is easy to see that *the definition implies that* (Γ, o) *is a tree* (hence, is reduced). On the contrary, a pair of a tree Γ and a vertex v of Γ determines a unique rooted tree structure (Γ, o_v) having v as its root.

We return to the setting in §4, where Γ is a tree on Π, Π_1 is the first component of the decomposition (13), and $\widetilde{Ord}(\Pi_1)$ is a set of partial orderings on Π_1 (recall §4 Definition (20) and the equivalence of a) and b) in Assertion 4.2). We identify the partial ordering $\tilde{d}\in\widetilde{Ord}(\Pi_1)$ with its naturally defined reduced oriented graph $\tilde{d}=\tilde{d}_{red}$ on the set Π_1 (= the Hasse diagram, whose oriented edges are primitive pairs $\alpha<_{\tilde{d}}\beta$ of the order relation $\tilde{d}$. See the last paragraph of §1).

Recollect some facts which were already implicitly used in §4.

Fact. a) *Any partial ordering $\tilde{d}\in\widetilde{Ord}(\Pi_1)$ defines a rooted tree structure on Π_1. We shall denote by $v_{\tilde{d}}$ its root.*

b) *For any total ordering $d\in Ord(\Pi_1)$, there exists an unique partial ordering $\tilde{d}\in\widetilde{Ord}(\Pi_1)$ such that d is a linear extension of $\tilde{d}$.*

c) *The system $\{E_{\tilde{d}}\}_{\tilde{d}\in\widetilde{Ord}(\Pi_1)}$ is a simplicial cone decomposition of V_{Π_1}.*

Proof. a) The smallest element $v_{\tilde{d}}$ exists because of the connectivity of Γ. The uniqueness of the predecessor follows from **Fact iv)** in §4.

b) This follows from the definition in §4 of $\widetilde{Ord}(\Pi_1)$, where any total ordering d belongs to the unique equivalence class $\tilde{d}$. □

We "sharpen" the Fact c) by "pull-back" of the decomposition to the principal Γ-cone E_{Π_1,Π_2}. Precisely, we mean the following Theorem.

Theorem 5.1. *Let the setting be as in Theorem* 4.1. *For any partial ordering* $\tilde{d} \in \widetilde{Ord}(\Pi_1)$*, consider the reduced oriented graph on* Π*:*

$$\tilde{od} := (o_{\Pi_1,\Pi_2} \cup \tilde{d})_{red}, \tag{23}$$

(see Proof. for a precise explanation of this notation). Then one has

1) *The* $\tilde{od}$ *defines a rooted tree structure on* Π *whose root is equal to* $v_{\tilde{d}}$*. Let us call the associated simplicial cone* $E_{\tilde{od}}$ *a block.*

2) *The closure of the principal cone* E_{Π_1,Π_2} *decomposes into a disjoint union (up to identifications of faces) of the closures of the blocks:*

$$\overline{E}_{\Pi_1,\Pi_2} = \coprod_{\tilde{d}\in\widetilde{Ord}(\Pi_1)} \overline{E}_{\tilde{od}}. \tag{24}$$

Proof. The notation $o_{\Pi_1,\Pi_2} \cup \tilde{d}$ means the oriented graph obtained by the union of oriented edges of o_{Π_1,Π_2} and $\tilde{d}$ (in order this to be well-defined, we check that it is acyclic (recall §1). But this is trivial, since any element of Π_2 is maximal with respect to $o_{\Pi_1,\Pi_2}\cup\tilde{d}$ so that it cannot be a part of any oriented cycle. The remaining part $\tilde{d}$ on Π_1 is a tree (**Fact a)**) and does not contain a cycle). The notation $(*)_{red}$ means the reduction of $*$, i.e. the oriented graph obtained by deleting all the removable edges from $*$ (called the Hasse diagram, recall §1).

1) First, we observe that any oriented edge in $\tilde{d}$ is un-removable in $o_{\Pi_1,\Pi_2} \cup \tilde{d}$, since any vertex($\neq v_{\tilde{d}}$) in Π_1 has only one predecessor.

In general, if an oriented graph o is connected, then o_{red} is connected. Since $o_{\Pi_1,\Pi_2} \cup \tilde{d}$ is connected then $\tilde{od}$ is connected. In particular, any vertex $\beta \in \Pi_2$ has at least one predecessor (in Π_1) with respect to $\tilde{od}$.

In fact, there is a unique predecessor, which we determine now.

$*)$ *For any element* $\beta \in \Pi_2$*, the set* $Nbd(\beta) := \{\alpha\in\Pi \mid \overline{\alpha\beta}\in Edge(\Gamma)\} = \{\alpha\in\Pi_1 \mid \alpha <_{o_{\Pi_1,\Pi_2}} \beta\} \subset \Pi_1$ *is totally ordered by the partial ordering* $\tilde{d}$.

Before the proof of $*$), recall the notation $\Gamma_{\tilde{d},v}$ (21) and the fact that *if* $\Gamma_{\tilde{d},v}$ *contains a vertex* $\alpha\in\Pi_1$*, then it contains also the neighborhood of* α*:* $Nbd(\alpha)\subset\Gamma_{\tilde{d},v}$ (the formula $*$) in the proof of §4 Assertion 4.2).
Proof of $*$). Consider $\alpha_1, \alpha_2 \in Nbd(\beta) \subset \Pi_1$ with $\alpha_1 \neq \alpha_2$. Since $\Gamma_{\tilde{d},\alpha_i}$ contains $Nbd(\alpha_i)$, one has $\beta \in \Gamma_{\tilde{d},\alpha_i}$ for $i = 1, 2$. That is $\Gamma_{\tilde{d},\alpha_1} \cap \Gamma_{\tilde{d},\alpha_2} \neq \emptyset$. Then, due to §4 Fact i) and ii), either $\Gamma_{\tilde{d},\alpha_1} \supset \Gamma_{\tilde{d},\alpha_2}$ or $\Gamma_{\tilde{d},\alpha_1} \subset \Gamma_{\tilde{d},\alpha_2}$ occurs, and, hence, one has either $\alpha_1 <_{\tilde{d}} \alpha_2$ or $\alpha_1 >_{\tilde{d}} \alpha_2$. $\square$ of $*$)

As a consequence of $*$), we obtain:

$**$) *Let* $\beta\in\Pi_2$*. If* $\alpha\in Nbd(\beta)$ *is not the largest with respect to* $\tilde{d}$*, then the oriented edge* $\alpha <_{o_{\Pi_1,\Pi_2}} \beta$ *is removable in the oriented graph* $o_{\Pi_1,\Pi_2}\cup\tilde{d}$.

Proof of $**$). Let $\alpha \in Nbd(\beta)$ be not the largest, i.e. $^\exists\alpha' \in Nbd(\beta)$ which is larger than α due to $*$). Then $\alpha <_{\tilde{d}} \alpha' <_{o_{\Pi_1,\Pi_2}} \beta$ implies that the oriented edge $\alpha <_{o_{\Pi_1,\Pi_2}} \beta$ is removable (recall §1). $\Box$ of $**$)

2) $E_{\tilde{od}}$ is simplicial since $\tilde{od}$ is a tree (Assertion 3.1). By definition (23), one has $E_{\tilde{od}} = E_{o_{\Pi_1,\Pi_2}\cup\tilde{d}} \subset E_{o_{\Pi_1,\Pi_2}} = E_{\Pi_1,\Pi_2}$. The decomposition (24) follows, since for any $d \in \Sigma(o_{\Pi_1,\Pi_2})$, there exists a unique $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ such that d is a linear extension of $o_{\Pi_1,\Pi_2} \cup \tilde{d}$ (**Fact** b) and c)). Thus,

$$\Sigma(o_{\Pi_1,\Pi_2}) \ = \coprod_{\tilde{d}\in\widetilde{Ord}(\Pi_1)} \Sigma(\tilde{od}). \tag{25}$$

$\Box$

Remark 6. The block decomposition (24) of the principal Γ-cone E_Γ depends on a choice of the principal ordering o_{Π_1,Π_2}. In fact, the block decompositions for o_{Π_1,Π_2} and for o_{Π_2,Π_1} are often quite different.

As a corollary to the block decomposition (24) of the principal cone, let us give an alternative proof of the formula (19). This is achieved by two steps. The first step is to recall the well known hook length formula of Knuth enumerating the chambers in a Γ-cone for a rooted tree (it is an immediate consequence of the decomposition formula (11)).

Lemma 5.2. (Knuth [K2,p70]) *Let* $(\Gamma, o \in Or(\Gamma))$ *be a rooted tree. Then one has*

$$\sigma(o) = \frac{(\#\Gamma)!}{\prod_{v\in\Pi} \#\Gamma_{o,v}} \tag{26}$$

where

$$\Gamma_{o,v} \ := \ \text{the connected component of } \Gamma_o \setminus \{w \in \Pi \mid w <_o v\} \text{ containing } v. \tag{27}$$

Note. There is an unfortunate discrepancy between the two notations $\Gamma_{\tilde{d},v}$ (18) and $\Gamma_{o,v}$ (27). The underlying oriented graph structure in $\Gamma_{\tilde{d},v}$ is the principal orientation o_{Π_1,Π_2} and that for $\Gamma_{o,v}$ is the rooted tree o. They are, in a sense, the most contrasting orientations. However, we show in the following a "numerical coincidence" of them.

The second step of the alternative proof of (19) is as follows. Apply (26) to $\sigma(\tilde{od})$ to count the number of chambers in $E_{\tilde{od}}$. Comparing (19) and (24), let us show the equality:

$$\frac{(\#\Gamma)!}{\prod_{v\in\Pi} \#\Gamma_{\tilde{od},v}} = \frac{(\#\Gamma)!}{\prod_{v\in\Pi_1} \#\Gamma_{\tilde{d},v}} \tag{28}$$

for $\tilde{d} \in \widetilde{Ord}(\Pi_1)$. We note that the region of the running index v in LHS of (28) can be shrunken from Π to $\Pi_1 = \Pi\setminus\Pi_2$, since for $v \in \Pi_2$

one has $\#\Gamma_{\tilde{od},v} = 1$ because of the fact that v is maximal with respect to $o_{\Pi_1,\Pi_2} \cup \tilde{d}$. Therefore, we have only to show the formula

$$\#\Gamma_{\tilde{d},v} = \#\Gamma_{\tilde{od},v} \tag{29}$$

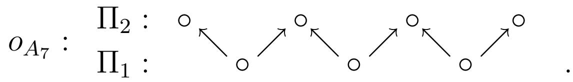
for $v \in \Pi_1$. We show that the vertex sets $|\Gamma_{\tilde{d},v}|$ and $|\Gamma_{\tilde{od},v}|$ coincide (even though the graph structures are quite different, see Example).

Note the inclusion relation: $o_{\Pi_1,\Pi_2} \subset (o_{\Pi_1,\Pi_2} \cup \tilde{d}) \supset \tilde{od}$ among oriented graphs and the equality among the vertex sets: $A := \{w \in \Pi_1 \mid w <_{\tilde{d}} v\} = \{w \in \Pi \mid w <_{o_{\Pi_1,\Pi_2} \cup \tilde{d}} v\} = \{w \in \Pi \mid w <_{\tilde{od}} v\}$. Thus, one has relation: $\Gamma_{\tilde{d},v} \subset \Gamma_{o_{\Pi_1,\Pi_2} \cup \tilde{d},v} \supset \Gamma_{\tilde{od},v}$ among the connected components of the complements of A containing v. The sets $|\Gamma_{\tilde{d},v}|$ and $|\Gamma_{o_{\Pi_1,\Pi_2} \cup \tilde{d},v}|$ coincide, since, if $v <_{\tilde{d}} w$ for $w \in \Pi_1$ then $w \in \Gamma_{\tilde{d},v}$. The sets $|\Gamma_{o_{\Pi_1,\Pi_2} \cup \tilde{d},v}|$ and $|\Gamma_{\tilde{od},v}|$ coincide, since, if an element $w \in \Pi_2$ is connected with v in $\Gamma_{o_{\Pi_1,\Pi_2} \cup \tilde{d},v}$ then w is connected with v by $\tilde{od}$.

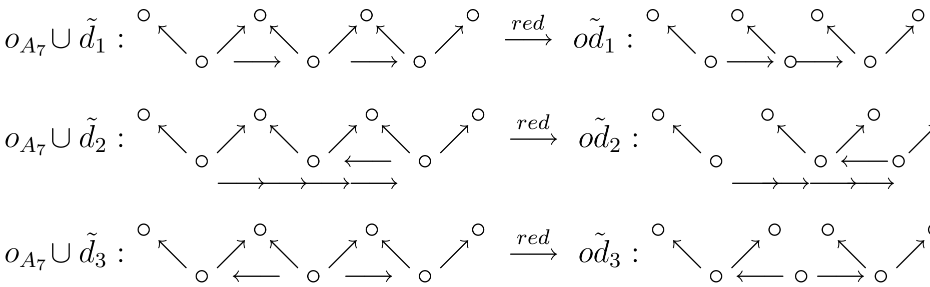
This completes the proof of the equality (29), and hence the alternative proof for (19) is completed.

Example. We illustrate the block decompositions of type A_7 and the calculations of the formula (19).

I. Let the principal decomposition (13) of $\Gamma(A_7)$ and the principal ordering $o_{A_7} := o_{\Pi_1,\Pi_2}$ be fixed by



There are 5 partial orderings $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ and, accordingly, 5 associated rooted trees $\tilde{od}$. Three of them are illustrated as follows:

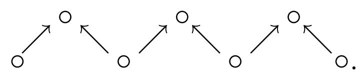


Two more rooted trees are obtained from $\tilde{od}_1$ and $\tilde{od}_2$ by the action of the left-right involutive diagram automorphism of $\Gamma(A_7)$. Therefore, the enumeration formula (19) for the type A_7 turns out to be

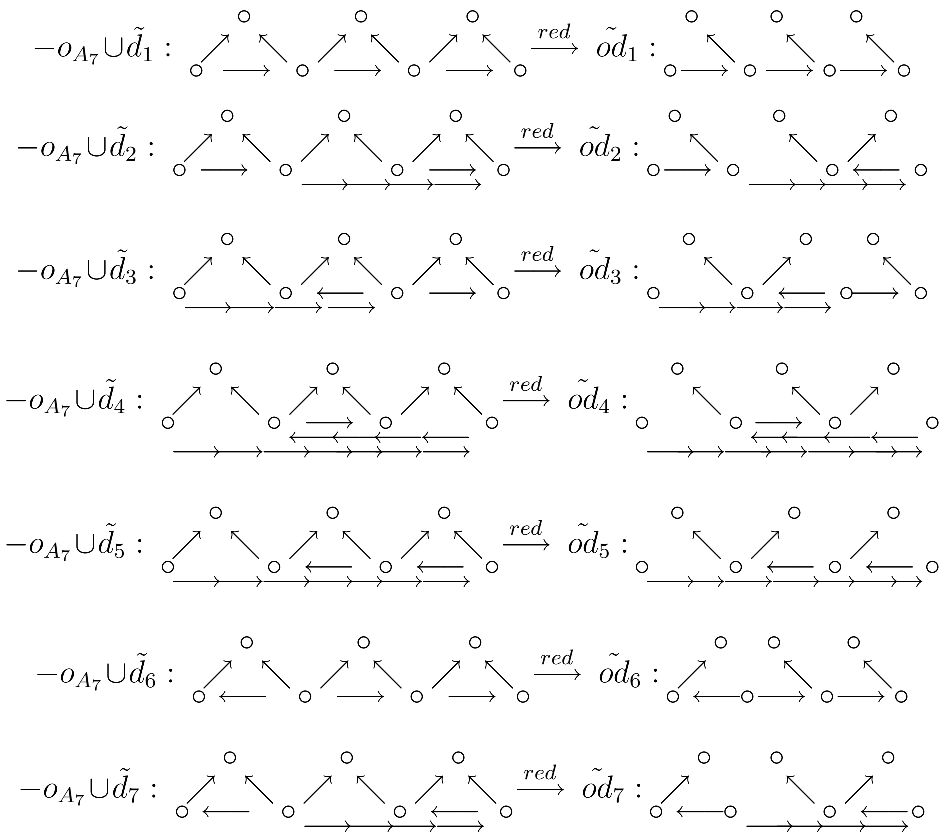
$$\begin{aligned}\sigma(A_7) &= 2\sigma(\tilde{od}_1) + 2\sigma(\tilde{od}_2) + \sigma(\tilde{od}_3) = 2\tfrac{7!}{7\cdot5\cdot3} + 2\tfrac{7!}{7\cdot5\cdot3} + \tfrac{7!}{7\cdot3\cdot3} \\ &= 272.\end{aligned}$$

II. The principal decomposition of $\Gamma(A_7)$ opposite to (13) and the opposit principal ordering $-o_{A_7} := o_{\Pi_2,\Pi_1}$ are given by

$-o_{A_7}$: Π_1 : Π_2 :



There are 14 partial orderings $\tilde{d} \in \widetilde{Ord}(\Pi_2)$ and, accordingly, 14 associated rooted trees $\tilde{od}$. Seven of them are illustrated as follows:



The remaining seven more rooted trees are obtained from the above seven trees by the action of the left-right involutive diagram automorphism of $\Gamma(A_7)$. Therefore, the enumeration formula (19) for the type A_7 turns out to be

$$\begin{aligned}\sigma(A_7) &= 2\big(\sigma(\tilde{od}_1)+\sigma(\tilde{od}_2)+\sigma(\tilde{od}_3)+\sigma(\tilde{od}_4)+\sigma(\tilde{od}_5)+\sigma(\tilde{od}_6)+\sigma(\tilde{od}_7)\big)\\ &= 2\big(\tfrac{7!}{7\cdot6\cdot4\cdot2}+\tfrac{7!}{7\cdot6\cdot4\cdot3}+\tfrac{7!}{7\cdot6\cdot3\cdot2}+\tfrac{7!}{7\cdot6\cdot5\cdot3}+\tfrac{7!}{7\cdot6\cdot5\cdot3}+\tfrac{7!}{7\cdot4\cdot2\cdot2}+\tfrac{7!}{7\cdot4\cdot3\cdot2}\big)\\ &= 272.\end{aligned}$$

Note. Even the starting diagram $\Gamma(A_7)$ is linear, the blocks of the principal cones correspond to non-linear (non-Coxeter) diagrams.

6. Geometric backgrounds

We recall briefly a theorem [S1§3], which combines the principal Γ-cones with some geometry of real bifurcation set in case of Γ is a Coxeter graph of finite type. For details, one is refereed to [ibid].

Let W be a finite reflection group acting irreducibly on an $\mathbb{R}$-vector space V of rank l. Due to Theorem of Chevalley, the quotient variety $S_W := V/\!\!/W$, as a scheme, is a smooth affine variety,which contains the discriminant divisor D_W consisting of irregular orbits. The integration $\tau = \exp(D)$ of the lowest degree vector field D on S_W, which is unique up to a constant factor and is called the *primitive vector field*, defines a $\mathbb{G}_a$-action on S_W. The quotient $T_W := S_W /\!\!/ \tau(\mathbb{G}_a)$ is an $l-1$-dimensional affine variety. The restriction to D_W of the projection map $S_W \to T_W$ is a l-fold flat covering, whose ramification divisor in T_W is denoted by B_W and called the *bifurcation divisor*. The B_W decomposes into the ordinary part $B_{W,2}$ and the higher part $B_{W,\geq 3}$ according to the ramification index.

Depending on $\varepsilon \in \{\pm 1\}$, there are real forms $T^{\varepsilon}_{W,\mathbb{R}}, B^{\varepsilon}_{W,2,\mathbb{R}}$ and $B^{\varepsilon}_{W,\geq 3,\mathbb{R}}$ of these schemes. There is a distinguished real half axis $AO^{\varepsilon} \simeq \mathbb{R}_{>0}$ (arising from eigenspaces of Coxeter elements, see [S1] for details) embedded in $T^{\varepsilon}_{W,\mathbb{R}} \setminus B^{\varepsilon}_{W,\geq 3,\mathbb{R}}$. The connected component of $T^{\varepsilon}_{W,\mathbb{R}} \setminus B^{\varepsilon}_{W,\geq 3,\mathbb{R}}$ containing AO^{ε} is denoted by E^{ε}_W and is called the *central region*.

Let P_l be a largest degree coordinate of S_W. Consider the l-valued algebraic correspondence $T_W \to D_W \overset{P_l|_{D_W}}{\to} \mathbb{A}$. Its l branches at the base point AO^{ε} can be indexified by the set of a simple generator system Π of W. Let us denote them by $\{\varphi_\alpha\}_{\alpha\in\Pi}$ as a system of algebroid functions on T_W (which are branching along $B_{W,\geq 3}$). Then, one has:

Theorem 6.1. *The correspondence* $b_W := \sum_{\alpha\in\Pi} \varphi_\alpha \cdot v_\alpha$ *induces a semi-algebraic homeomorphism:*

$$b_W : \overline{E}^{\varepsilon}_W \simeq \overline{E}_{\Gamma(W)} \tag{30}$$

from the closure of the central region of W *to the closure of the principal cone for the Coxeter graph* $\Gamma(W)$ *of* W *on* Π*, and a homeomorphism:*

$$b_W \ : \ \overline{E}^{\varepsilon}_W \cap B_{W,2,\mathbb{R}} \ \simeq \ \overline{E}_{\Gamma(W)} \cap \big(\cup_{\alpha\beta\in\Pi} H_{\alpha\beta} \big). \tag{31}$$

That is: the central region E^{ε}_W *is a simplicial cone isomorphic to the principal* $\Gamma(W)$*-cone and the connected components of* $E^{\varepsilon}_W \setminus B_{W,2,\mathbb{R}}$ *are in one to one correspondence with the set* $\Sigma(\Gamma(W))$ *of chambers contained in the principal* $\Gamma(W)$*-cone* $E_{\Gamma(W)}$*.*

The theorem (in a more precise form) has several important implications in the study of the topology of the configuration space S_W.

Note. 1. *The correspondence* b_W *is, up to a scaling factor, unique and does not depend on a choice of* P_l (a largest degree coordinate of S_W).
Proof. Since the largest exponent of W is unique, any other largest degree coordinate $\tilde{P}_l$ of S_W is of the form $a\cdot P_l+Q$ for a scaling constant a and a polynomial Q of lower degree coordinates. Then, $\tilde{\varphi}_\alpha = a\cdot\varphi_\alpha + Q$ $(\alpha\in\Pi)$, whose second term is independent of α, and, so, $\tilde{b}_W = a\cdot b_W$. □

2. The principal cone in RHS of (*) depends only on the graph structure of the diagram $\Gamma(W)$ and not on the labels on the edges. The graphs $\Gamma(W)$ (forgetting the labels) of types A_l, B_l, C_l, F_4, G_2, H_3, H_4 and $I_2(p)$ are linear. Hence, the central regions E_W for them are homeomorphic to the principal cones of type A.

Finally in the present paper, we compare the concept of Γ-cones with somewhat similar concept, the *Springer cones*, which we explain below.
Definition ([Ar1]). Let V_W be a real vector space with an irreducible action of a finite reflection group W. The reflection hyperplanes of W divides V_W into chambers. Let $\{H_\alpha\}_{\alpha\in\Pi}$ be the system of the walls of a chamber. A connected component of $V_W \setminus \cup_{\alpha\in\Pi} H_\alpha$ is called a *Springer cone*. A Springer cone containing the maximal number of chambers (unique up to sign [Sp1]) is called a *principal Springer cone*. This maximal number is called the *Springer number*. The Springer number has been calculated by the authors ([So],[Sp1], [Ar1]).

There are some formal similarities between the (principal) Springer cones in V_W and the (principal) Γ-cones in V_Π (see Table below). A result similar to Theorem 3.2 is proven for Springer cones [Sp1, Prop.3].

	Springer cone	Γ-cone
The ambient vector space	V_W with W-chambers (depending on the group W)	V_Π with $A_{\#\Pi-1}$-chambers (depending on the set Π)
The cutting hyperplanes	$\{H_\alpha\}_{\alpha\in\Pi}$ (indexed by the vertices of $\Gamma(W)$)	$\{H_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ (indexed by the edges of the tree Γ)

Roughly and symbolically speaking, the principal Springer cones deal with the *generators* of the Artin groups, whereas the principal Γ-cones deal with the *(non-commutative) braid relations* of the Artin groups.

The only cases when a Γ-cone decomposition is simultaneously a Springer cone decomposition are listed by the following.

Assertion 6.2. *For a forest* Γ, *the following i)–iii) are equivalent.*

i) The Γ*-cone decomposition of* V_Π *is isomorphic to the Springer cone decomposition of* V_W *for some finite Coxeter group* W.

ii) The smallest number of chambers contained in a Γ*-cone is equal to 1, i.e.* $\inf\{\sigma(o) \mid o\in Or(\Gamma)\} = 1$.

iii) The Γ *is a linear graph of type* A_l, *and* $W = W(A_{l-1})$ *for* $l>1$.

Proof. i) $\Rightarrow$ ii): This follows from the definition of the Springer cone.

ii) $\Rightarrow$ iii): if a Γ-cone consists of a single chamber $\overline{C} := \{\lambda_{\alpha_1} \leq \cdots \leq \lambda_{\alpha_l}\}$, then Γ is a linear graph α_1-α_2-$\cdots$-α_l (of type A_l) on Π.

iii) $\Rightarrow$ i): If Γ is a linear graph α_1-α_2-$\cdots$-α_l, then the orientation $\alpha_1 < \alpha_2 < \cdots < \alpha_l$ on Γ corresponds to the Γ-cone consisting only of a single chamber $\overline{C} := \{\lambda_{\alpha_1} \leq \cdots \leq \lambda_{\alpha_l}\}$ of type A_{l-1} in $V_\Pi = V_{A_{l-1}}$. □

Remark 7. Assertion is not true if Γ is not a forest (see §3 Example), since the argument ii) $\Rightarrow$ iii) fails.

Due to Assertion, $\sigma(A_l) := \sigma(\Gamma(A_l))$ is equal to the Springer number a_{l-1} of type A_{l-1}. Since the Springer number a_n of type A_n is given by the generating function: $1 + \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n = \frac{1}{1-\sin(x)}$ ([Sp1,3.]), one has

$$1 + \sum_{n=1}^{\infty} \frac{\sigma(A_n)}{n!} x^n = 1 + \int_0^x \frac{1}{1-\sin(x)} dx = \tan(\frac{x}{2} + \frac{\pi}{4}). \tag{32}$$

This formula was found repeatedly (e.g. [St2, Exercise 43(c)]). A direct proof is given in the Appendix.

Question. By an analogy to §3 Theorem, consider any system of l-reflection hyperplanes in V_W forming coordinate hyperplanes and ask a question: is there a unique (up to a sign) quadrangle of V_W, cut out by the hyperplanes, which contains the maximal number of chambers. The answer is apparently positive for the type A_l and $I_2(p)$ for odd $p \in 2\mathbb{Z}_{>0}$, and negative for the types B_l, C_l and $I_2(p)$ for even $p \in 2\mathbb{Z}_{>0}$.

7. Appendix: Generating functions for the series of types A_l, D_l and E_l

By Yoshio Sano

We give the generating functions for the series $\sigma(A_l)$, $\sigma(D_l)$ and $\sigma(E_l)$ of numbers of types A_l ($l \geq 1$), D_l ($l \geq 3$) and E_l ($l \geq 4$).

1. A_l-type.

Let $\Gamma(A_l)$ be the tree of type A_l ($l \geq 1$) given as follows:

where l is the number of vertices of the graph. Put $\sigma(A_n) := \sigma(\Gamma(A_n))$.

Formula.

$$1 + \sum_{n=1}^{\infty} \frac{\sigma(A_n)}{n!} x^n = \tan(\frac{x}{2} + \frac{\pi}{4}). \tag{33}$$

Proof. We put $\sigma(A_0) := 1$, and prove a formula:

$$\sigma(A_{n+1}) = \frac{1}{2}\sum_{i=0}^{n} \binom{n}{i} \sigma(A_i)\cdot\sigma(A_{n-i}) \qquad (n \geq 1) \tag{34}$$

Proof of (40). We apply the formula (10) in §3 and obtain, according to whether l is even or odd:

$$\sigma(A_{2k}) = \sum_{i=1}^{k} \binom{2k-1}{2i-1} \sigma(A_{2i-1})\cdot\sigma(A_{2k-2i}),$$

$$\sigma(A_{2k+1}) = \sum_{i=1}^{k} \binom{2k}{2i-1} \sigma(A_{2i-1})\cdot\sigma(A_{2k-2i+1}) \quad \square.$$

Put $f_A(x) := \sum_{n=0}^{\infty} \frac{\sigma(A_n)}{n!} x^n$. Then, (40) implies the differential equation

$$f_A' = \frac{1}{2}(f_A^2 + 1).$$

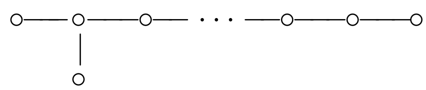
Given the initial condition: $f_A(0) = 1$, the solution is:

$$f_A(x) = \tan(\frac{x}{2} + \frac{\pi}{4}). \qquad \square$$

Remark 8. The formula (39) agrees with the formula (38) in §4.

2. D_l-type.

Let $\Gamma(D_l)$ be the tree of type D_l ($l \geq 3$) given as follows:



where l is the number of vertices of the graph. Put $\sigma(D_n) := \sigma(\Gamma(D_n))$.

Formula.

$$\sum_{n=3}^{\infty} \frac{\sigma(D_n)}{n!} x^n = 2(x-1)\tan(\frac{x}{2} + \frac{\pi}{4}) + 2 - 2x^2. \tag{35}$$

Proof. We put $\sigma(D_2) := 2$, $\sigma(D_1) := 0$ and $\sigma(D_0) := 2$, and prove a formula:

$$\sigma(D_{n+1}) = \frac{1}{2}\sum_{i=0}^{n} \binom{n}{i} \sigma(A_i)\cdot\sigma(D_{n-i}) \qquad (n \geq 2) \tag{36}$$

Proof of (42). We apply the formula (10) in §3 and obtain, according to whether l is even or odd:

$$\sigma(D_{2k}) = \sum_{i=1}^{k-1} \binom{2k-1}{2k-2i-1} \sigma(D_{2i})\cdot\sigma(A_{2k-2i-1}),$$

$$\sigma(D_{2k+1}) = \sum_{i=1}^{k} \binom{2k}{2k-2i} \sigma(D_{2i})\cdot\sigma(A_{2k-2i}) \qquad \square.$$

Put $f_D(x) := \sum_{n=0}^{\infty} \frac{\sigma(D_n)}{n!} x^n$. Then, (42) implies the differential equation

$$f_D' = \frac{1}{2} f_D f_A + x - 1.$$

Given the initial condition: $f_D(0) = 2$, the solution is:

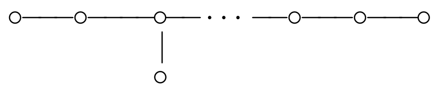
$$f_D(x) = 2(x-1)\tan(\frac{x}{2} + \frac{\pi}{4}) + 4. \qquad \square$$

Remark 9. Using the relation: $f_D(x) = 2(x-1)f_A(x) + 4$, one obtains

$$\sigma(D_n) = 2(n\sigma(A_{n-1}) - \sigma(A_n)) \qquad (n \geq 1). \tag{37}$$

3. E_l-type.

Let $\Gamma(E_l)$ be the tree of type E_l ($l \geq 4$) given as follows:



where l is the number of vertices of the graph. Put $\sigma(E_n) := \sigma(\Gamma(E_n))$.

Formula.

$$\sum_{n=4}^{\infty} \frac{\sigma(E_n)}{n!} x^n = (\frac{1}{2}x^2 - 2x + 3)\tan(\frac{x}{2} + \frac{\pi}{4}) - 3x^3 - x - 3. \tag{38}$$

Proof. We put $\sigma(E_3) := 3$, $\sigma(E_2) := 0$, $\sigma(E_1) := 3$ and $\sigma(E_0) := -1$, and prove a formula:

$$\sigma(E_{n+1}) = \frac{1}{2}\sum_{i=0}^{n} \binom{n}{i} \sigma(E_i) \cdot \sigma(A_{n-i}) \qquad (n \geq 5). \tag{39}$$

Proof of (45). Apply the formula (10) in §3 and obtain according to l is even or odd:

$$2\sigma(E_{2k}) = \sigma(A_{2k-1}) + \sigma(D_{2k-1}) + \binom{2k-1}{1}\sigma(A_{2k-2}) + \sum_{i=3}^{2k-1} \binom{2k-1}{i} \sigma(E_i) \cdot \sigma(A_{2k-i-1}),$$

$$2\sigma(E_{2k+1}) = \sigma(A_{2k}) + \sigma(D_{2k}) + \binom{2k}{1}\sigma(A_{2k-1}) + \sum_{i=3}^{2k} \binom{2k}{i} \sigma(E_i) \cdot \sigma(A_{2k-i})$$

Eliminate the $\sigma(D_n)$ term by the use of , we obtain (45). $\square$.

Put $f_E(x) := \sum_{n=0}^{\infty} \frac{\sigma(E_n)}{n!} x^n$. Then, (45) implies the differential equation

$$f_E' = \frac{1}{2} f_E f_A + \frac{1}{4}x^2 - x + \frac{7}{2}.$$

Given the initial condition: $f_E(0) = -1$, the solution is:

$$f_E(x) = (\frac{1}{2}x^2 - 2x + 3)\tan(\frac{x}{2} + \frac{\pi}{4}) + 2x - 4. \qquad \square$$

Remark 10. Using the relation: $f_E(x) = (\frac{1}{2}x^2 - 2x + 3)f_A(x) + 2x - 4$, one obtains

$$\sigma(E_n) = \frac{n(n-1)}{2}\sigma(A_{n-2}) - 2n\sigma(A_{n-1}) + 3\sigma(A_n) \qquad (n \geq 2). \tag{40}$$

References

[Ar] Arnol'd, Vlasimir I.: Springer numbers and morsification spaces, J. Alg. Geom. **1** (1992), 197-214.

[B] Bourbaki, N.:Elements de mathematique, Fasc. XXXIV, Groupes et algebres de Lie, Chs. 4-6, Hermann. Paris 1968.

[B] de Bruijn, N.G.: Permutations with given ups and downs, Nieuw Arch. Wisk. **18** (1970), 61-65.

[E] Etingof, Pavel: Whittaker Functions on Quantum Groups and Q-deformed Toda Operators, Ams Trans Ser. 2 **194** (1999).

[G-Z] Green, Curtis & Zaslavsky, Thomas: On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partions and orientations of graphs, Trans. Amer. Math. Soc. **280** (1983), mo. 1, 97-126.

[K1] Knuth, D. E.: A note on solid partitions, Math. Comp. **24** (1970), 955-962.

[K2] Knuth, D. E.: Sorting and searching, in "The Art of Computer Programming," Vol. 3, Adson-Wesley, Reading, MA, 1973.

[K-S-T] Kajiura, Hiroshige, Saito, Kyoji & Takahashi, Atshushi: Matrix factorizations and representations of quivers, in preparation.

[N] Niven, I.: A combinatorial problem of finite sequences, Nieuw Arch. Wisk. **16**(1968), 116-123.

[S-Y-Z] Sagan, Bruce E., Yeh, Yeong Nan & Ziegler Günter M.: Maximaizing Möbius functions on subsets of Boolean algebra, Discrete Math. **126** (1994), no. 1-3, 293-311.

[S1] Saito, Kyoji: Polyhedra dual to Weyl chamber decomposition: A Precis, Publ. RIMS, Kyoto Univ. **40** (2004), 1337-1384.

[S2] Saito, Kyoji: Exceptional root systems, unpublished.

[So] Solomon, Luis: A decomposition of the group algebra of a finite Coxeter group, J.Alg., **9** (1968), 220-239.

[Sp] Springer, Tony A.: Remarks on a combinatorial problem, Nieuw Arch. Wisk.(1) **19** (1971), 30-36.

[St1] Stanley, Richard P.: Ordered structures and partitions, Mem. Amer. Math. Soc. **119** (1972).

[St2] Stanley, Richard P.: Acyclic orientations of graphs, Discrete Math. **5** (1973), 171-178.

[St3] Stanley, Richard P.: Binomial Posets, Möbius Inversion and Permutation Enumeration, J. of Combinatorial Theory (A) **20** (1976), 336-356.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.