Greedy Fans:
A geometric approach to dual greedy algorithms

Hiroshi HIRAI
Research Institute for Mathematical Sciences,
Kyoto University, Kyoto 606-8502, Japan
hirai@kurims.kyoto-u.ac.jp

July 2005

Abstract

The purpose of this paper is to understand greedily solvable linear programs in a geometric way. Such linear programs have recently been considered by Queyranne, Spieksma and Tardella, Faigle and Kern, and Krüger for antichains of posets, and by Frank for a class of lattice polyhedra, and by Kashiwabara and Okamoto for extreme points of abstract convex geometries. Our guiding principle is that solving linear programs is equivalent to finding a normal cone of a polyhedron which contains a given cost vector. Motivated by this observation, we introduce and investigate a class of simplicial subdivisions, called *greedy fans*, whose membership problem can be greedily solvable. Our approach sheds a new perspective on greediness and submodularity in terms of theory of regular triangulations. Furthermore we introduce a well-behaved special class of greedy fans, named *acyclic greedy fans*, which can be obtained by some poset. In particular, its close relationship to reverse lexicographic triangulations is revealed. We show that the set of acyclic greedy fans on fixed vertices can be naturally regarded as a certain kind of a polyhedral subdivision like secondary fans. We establish the relationship between our approach and Frank's and Kashiwabara and Okamoto's models.

Keywords: submodularity, greedy algorithms, regular triangulations

1 Introduction

In this paper, given a finite set V, a nonempty family $\mathcal{A} \subseteq 2^V$ and a function $f : \mathcal{A} \to \mathbf{R}$, we consider the following dual pair of linear programs

$$
\begin{array}{lll|lll}
\mathrm{P}_{(\mathcal{A},f,w)}: & & & \mathrm{D}_{(\mathcal{A},f,w)}: & & \\
\max. & \displaystyle\sum_{e\in V} w(e)x(e) & & \min. & \displaystyle\sum_{A\in\mathcal{A}} \lambda(A)f(A) & \\
\text{s.t.} & \displaystyle\sum_{e\in A} x(e) \leq f(A) \ (A \in \mathcal{A}), & & \text{s.t.} & \displaystyle\sum_{A\in\mathcal{A}:e\in A} \lambda(A) = w(e) \ (e \in V), & \\
 & x \in \mathbf{R}^V, & & & \lambda(A) \geq 0 \ (A \in \mathcal{A}). &
\end{array}
\tag{1.1}
$$

Since many combinatorial optimization problems can be reduced to this form, it is important to characterize efficiently solvable classes of LPs of this type. One of fundamental examples of such classes is a *polymatroid* [3]; linear programs over polymatroids can be greedily solved.

Since recent works by Queyranne, Spieksma and Tardella [27] and Faigle and Kern [5], several researchers [7], [21], [6], [1], [19], [10] have investigated greedily solvable linear programs, where so-called *dual greedy algorithms* construct a dual optimal solution in a greedy way. Such a system of linear inequalities is called a *dual greedy system* and a polyhedron associated with a dual greedy system is called a *dual greedy polyhedron* [10].

In particular, Krüger [21] extended Faigle and Kern's dual greedy algorithm [5] for antichains of arbitrary posets and gave a kind of the submodularity condition (see also [1]). Kashiwabara and Okamoto [19] extended Krüger's dual greedy system for extreme points of abstract convex geometries (see also [10]). These frameworks can be understood as a common generalization of the dual greedy algorithm for a submodular polyhedron [23] and the Monge algorithm for the assignment problem with Monge cost matrices [15].

On the other hand, Frank [7] considered a similar dual greedy algorithm for a class of lattice polyhedra [16]. Frank's dual greedy algorithm can be understood as an extension of the two-phase greedy algorithm for the minimum spanning arborescence problem by Fulkerson [11].

A common feature of these dual greedy algorithms is to construct a dual optimal solution $\lambda : \mathcal{A} \to \mathbf{R}$ by the following simple greedy procedure using some oracle $\Phi : 2^V \to 2^{\mathcal{A}}$:

> Set $n = \#V$, $X \leftarrow V$, $\lambda(A) \leftarrow 0$ $(A \in \mathcal{A})$.
> For each $i = 1, \ldots, n$, repeat the following process:
> Take $A_i \in \Phi(X)$ and $e_i \in \mathrm{Argmin}\{w(e) \mid e \in A_i\}$.
> Set $\lambda(A_i) \leftarrow w(e_i)$, $X \leftarrow X \setminus \{e_i\}$ and $w(e) \leftarrow w(e) - \lambda(A_i)$ for $e \in A_i$.

The main purpose of this paper is to understand these dual greedy systems in a *geometric way.* Our guiding principle is the following fundamental fact.

> Solving this dual linear program $\mathrm{D}_{(\mathcal{A},f,w)}$ is equivalent to finding a normal cone of the primal feasible polyhedron which contains a given cost vector w.

This is a membership problem of the normal fan of a polyhedron. Motivated by this observation, we introduce a special class of simplicial fans whose membership problem can be greedily solved. We call such a simplicial fan a *greedy fan.* Our approach is closely related to theory of *regular triangulations* [13, Chapter 7]. From this view point, we shed a new light on these dual greedy systems and submodularity conditions.

This paper is organized as follows. In Section 2, we introduce the concept of greedy fans and investigate its properties. In Subsection 2.2, we try to construct a polyhedron whose normal fan coincides with a given greedy fan Δ. This problem is closely related to the *regularity* of simplicial subdivision Δ, where Δ is *regular* if there exists some polyhedron whose normal fan coincides with Δ. We define Δ-*submodular inequalities* and show that a function f satisfies Δ-submodular inequalities strictly if and only if the normal fan of its associated polyhedron $P(f)$ coincides with Δ. Consequently, linear programs over this polyhedron $P(f)$ can be greedily solved. In Subsection 2.3, we discuss algebraic meaning of Δ-submodularity inequalities with connection to toric ideals and Gröbner bases. This argument throws a new perspective to the submodularity. In Subsection 2.4, we give a nontrivial characterization of greedy fans, using a certain *multiple-choice function*, which is a natural extension of *choice functions* [24].

In Section 3, we introduce and investigate a well-behaved special class of greedy fans, called *acyclic greedy fans*, which can be represented by some posets. We show that every acyclic greedy fan is *reverse lexicographic triangulation* and therefore is regular (Theorem 3.5). This argument clarifies a geometric structure of dual greedy polyhedra

(Figure 4). Furthermore we give a remarkable structure theorem that the set of all acyclic greedy fans on a fixed set of vertices forms a certain kind of a polyhedral fan (Theorem 3.12), which is an analogue of the *secondary fan* of regular triangulations [13, Chapter 7]. We call this polyhedral fan the *secondary greedy fan.* Furthermore, we see that if this secondary greedy fan is regular, it coincides with the normal fan of the base polyhedron associated with some (ordinary) submodular function (Corollary 3.15). In Subsections 3.2, 3.3, and 3.4, we investigate special classes of acyclic greedy fans. In particular, we establish the relationship between our approach and dual greedy systems by Kashiwabara and Okamoto [19] and by Frank [7], and give another systematic proof of the validity of their greedy algorithms.

It should be noted that a similar approach is given in Sohoni's Ph.D. thesis "*shapes of polyhedra in combinatorial optimization*" [29], where Sohoni considered a certain kind of the normal fan of polyhedra, called *shape*, in a more general setting. We emphasize that we are interested in the interplay between the structure of normal fans and the dual greediness of algorithms.

2 Greedy Fans

In this section, motivated by dual greedy algorithms, we introduce the concept of a greedy fan. We need some basic notation. Let V be a (nonempty) finite set with $\#V = n$. $\mathbf{R}$ and $\mathbf{R}_+$ denote the set of real numbers and of nonnegative real numbers, respectively. Similarly, let $\mathbf{Z}$ and $\mathbf{Z}_+$ denote the set of integer numbers and of nonnegative integer numbers, respectively For a function $f : \mathcal{A} \to \mathbf{R}$ on a set $\mathcal{A}$, the *support* $\operatorname{supp} f$ is defined by $\{A \in \mathcal{A} \mid f(A) \neq 0\}$. For a subset $A \subseteq V$, the characteristic vector $\chi_A \in \mathbf{R}^V$ is defined as

$$\chi_A(e) = \begin{cases} 1 & \text{if } e \in A, \\ 0 & \text{otherwise,} \end{cases} \qquad (e \in V). \tag{2.1}$$

For a set of vectors $\mathcal{V} \subseteq \mathbf{R}^V$, the *conical hull* $\operatorname{cone} \mathcal{V}$ of $\mathcal{V}$ is defined by

$$\operatorname{cone} \mathcal{V} = \{\sum_{i=1}^{m} \lambda_i v_i \mid v_1, v_2, \ldots, v_m \in \mathcal{V},\ \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbf{R}_+\}. \tag{2.2}$$

We need to recall the basic definitions of polyhedral geometry; see [36] for details. A set of polyhedral cones Δ is said to be a *polyhedral fan* if every face of every $P \in \Delta$ is in Δ, and the intersection of each two members $P, Q \in \Delta$ is the common face of P and Q. We denote by $|\Delta|$ the union of all members of Δ. Δ is also called a *polyhedral subdivision* of $|\Delta|$. If every member of Δ is a simplicial cone, we call Δ a *simplicial subdivision* or a *triangulation.* For a polyhedron $P \subseteq \mathbf{R}^n$ and a point $x \in P$, the *normal cone* of P at x is defined to be the set of vector $\{w \in \mathbf{R}^n \mid x \in \operatorname{Argmax}_{y \in P} \langle w, y \rangle\}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbf{R}^n$. The *normal fan* of P is the collection of normal cones of P. A simplicial subdivision Δ is said to be *regular* if there exists a polyhedron P such that the normal fan of P coincides with Δ. For two polyhedral subdivisions Δ and Δ', Δ is said to be a *coarsening* of Δ' if for each member $C \in \Delta$, $\{C' \in \Delta' \mid C' \subseteq C\}$ is a polyhedral subdivision of C.

2.1 Greedy Fans and Dual Greedy Algorithms

We consider a simplicial subdivision Δ of $\mathbf{R}_+^V$ with the following additional property, where we call a 1-dimensional cone a *vertex.*

Each vertex of Δ can be expressed by $\mathbf{R}_+ \chi_A$ for some nonempty set $A \subseteq V$.

Let $\mathcal{A} = \mathcal{A}_\Delta \subseteq 2^V$ be a nonempty family defined as

$$\mathcal{A} = \{A \subseteq V \mid \ \mathbf{R}_+\chi_A \text{ is a vertex of } \Delta\ \}.$$

In particular, Δ can be regarded as an *abstract simplicial complex* on the vertex set $\mathcal{A}$, which is denoted by $\hat{\Delta} \subseteq 2^{\mathcal{A}}$. We shall often identify Δ with $\hat{\Delta}$. For a nonempty subset $X \subseteq V$, we define the restriction Δ^X by $\{\mathcal{C} \in \Delta \mid \mathcal{C} \subseteq \mathbf{R}_+^X\}$. Note that Δ^X is a simplicial subdivision of $\mathbf{R}_+^X$.

We define *greedy fans* recursively. If $\#V = 1$, the trivial simplicial subdivision of $\mathbf{R}_+^V$ is defined to be greedy. Now we suppose that we have already defined the set of all greedy fans of $\mathbf{R}_+^U$ with $\#U < \#V$. A simplicial subdivision Δ of $\mathbf{R}_+^V$ is said to be *greedy* if there exists a nonempty subset $A \subseteq V$ such that

(G1) every maximal cone of Δ contains $\mathbf{R}_+\chi_A$ as a vertex and

(G2) for any $e \in A$, a restriction $\Delta^{V\setminus\{e\}}$ is a greedy fan of $\mathbf{R}_+^{V\setminus\{e\}}$.

We call a vertex satisfying (G1) and (G2) of a greedy fan Δ a *center vertex*. We consider the following membership problem:

Given a nonnegative vector w, find $\mathcal{C} \in \Delta$ with $w \in \mathcal{C}$.

In fact, this membership problem can be solved in a greedy way as follows.

Theorem 2.1. *For a greedy fan Δ and a nonnegative vector w, consider the following process:*

Set $X \leftarrow V$ and $w' \leftarrow w$.
For each $i = 1, \dots, n$, repeat the following process:
  Take a center vertex A_i of Δ^X and $e_i \in \operatorname{Argmin}\{w'(e) \mid e \in A_i\}$.
  Set $\lambda_i = w'(e_i)$, $w' \leftarrow w' - \lambda_i\chi_{A_i}$ and $X \leftarrow X \setminus \{e_i\}$.

Then we have $w = \sum_{i=1}^n \lambda_i\chi_{A_i}$ and $w \in \operatorname{cone}\{\chi_{A_i}\}_{i=1}^n \in \Delta$.

Proof. By construction, we have $\lambda_{A_i} \geq 0$ and $w = \sum_{i=1}^n \lambda_{A_i}\chi_{A_i}$. Hence it suffices to show $\operatorname{cone}\{\chi_{A_i}\}_{i=1}^n \in \Delta$. This immediately follows from the definition of a greedy fan. $\square$

Example 2.2. We draw greedy fans of dimension 2 and 3 in Figure 1, where each triangulation is the intersection of a greedy fan and the hyperplane $\{x \in \mathbf{R}^V \mid \sum_{e\in V} x(e) = 1\}$, where we simply denote $\{1, 2\}$ by 12. The center vertices for each 3-dimensional example are given as follows: (a) $1, 2, 3$, (b) $3, 12$, (c) 12, (d) 123, (f) 123, (g) 123.

Every restriction of a greedy fan is greedy.

Lemma 2.3. *For a greedy fan Δ of $\mathbf{R}_+^V$ and a nonempty subset $X \subseteq V$, the restriction Δ^X is greedy.*

Proof. We use induction on $\#V$. Suppose that this statement holds for any greedy fan on V with $\#V < k$. Now consider a greedy fan Δ on V with $\#V = k$. By definition, there exists a center vertex A of Δ such that $\Delta^{V\setminus\{e\}}$ is greedy for $e \in A$. By induction, any subset $U \subseteq V \setminus \{e\}$ is greedy. Hence it suffices to show that for any $e' \in V \setminus A$ $\Delta^{V\setminus\{e'\}}$ is greedy. Clearly A is also contained by every maximal cone of $\Delta^{V\setminus\{e'\}}$ as a vertex. Since $V \setminus \{e, e'\} \subseteq V \setminus \{e\}$, $\Delta^{V\setminus\{e,e'\}}$ is greedy for $e \in A$. Hence A satisfies (G1) and (G2) for $\Delta^{V\setminus\{e'\}}$. $\square$

From this lemma, we have the following nonrecursive characterization of greedy fans.

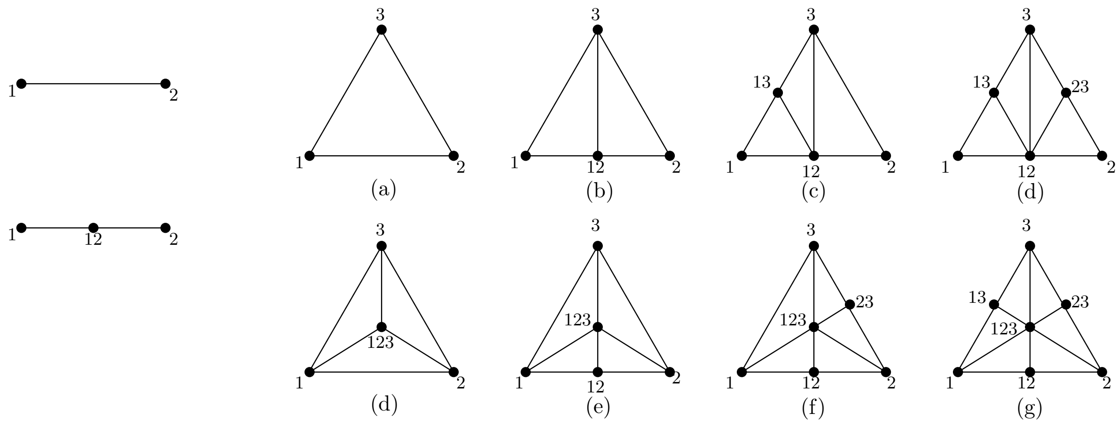

Figure 1: Greedy fans of dimension 2 (left) and dimension 3 (right)

Proposition 2.4. *A simplicial subdivision Δ of $\mathbf{R}^V_+$ is greedy if and only if for each nonempty subset $X \subseteq V$ there exists $A \subseteq X$ such that every maximal cone of Δ^X contains A as a vertex.*

We define a map $\Phi_\Delta : 2^V \to 2^{\mathcal{A}}$ as

$$\Phi_\Delta(X) = \{A \subseteq V \mid A \text{ is a center vertex of } \Delta^X\} \quad (X \subseteq V,\ X \neq \emptyset) \tag{2.3}$$

and $\Phi_\Delta(\emptyset) = \emptyset$ for convenience. Using this map $\Phi = \Phi_\Delta$, the process in Theorem 2.1 can be rephrased as follows.

Procedure: `Dual_Greedy`

Input: A vector $w \in \mathbf{R}^V_+$.

Output: $\lambda \in \mathbf{R}^{\mathcal{A}}_+$ with $w = \sum_{A \in \mathcal{A}} \lambda(A)\chi_A$.

Initialization: $w' \leftarrow w$, $X \leftarrow V$, $\lambda(A) \leftarrow 0$ $(\forall A \in \mathcal{A})$.

> **step1:** If $X = \emptyset$, then stop.
> **step2:** Pick arbitrary $A^* \in \Phi(X)$ and $e^* \in \mathrm{Argmin}\{w'(e) \mid e \in A^*\}$.
> **step3:** Put $\lambda(A^*) \leftarrow w'(e^*)$ and $w' \leftarrow w' - w'(e^*)\chi_{A^*}$.
> **step4:** Put $X \leftarrow X \setminus \{e^*\}$ and go to **step1**.

It is convenient for subsequent arguments to consider the following variant of `Dual_Greedy`.

Procedure: `Dual_Greedy*`

Input: A vector $w \in \mathbf{R}^V_+$.

Output: $\lambda \in \mathbf{R}^{\mathcal{A}}_+$ with $w = \sum_{A \in \mathcal{A}} \lambda(A)\chi_A$.

Initialization: $w' \leftarrow w$, $\lambda(A) \leftarrow 0$ $(\forall A \in \mathcal{A})$.

> **step1:** If $w' = 0$, then stop.
> **step2:** Pick arbitrary $A^* \in \Phi(\mathrm{supp}\, w')$.
> **step3:** Put $\lambda(A^*) \leftarrow \min(w'(e) \mid e \in A^*)$ and $w' \leftarrow w' - \lambda(A^*)\chi_{A^*}$.
> **step4:** Go to **step1**.

For a greedy fan Δ, `Dual_Greedy` and `Dual_Greedy*` return the same output.

2.2 Δ-Submodular Functions Associated with Greedy Fan Δ

Here, we try to construct a polyhedron whose normal fan coincides with a given greedy fan Δ with vertex set $\mathcal{A} \subseteq 2^V \setminus \{\emptyset\}$. In the argument of this subsection, the greediness of Δ is not essential. By slight modification, we can apply subsequent arguments (except Corollary 2.10) to any simplicial subdivision.

For a function $f : \mathcal{A} \to \mathbf{R}$, we define a polyhedron $P(f)$ as

$$P(f) = \{x \in \mathbf{R}^V \mid \sum_{e \in A} x(e) \leq f(A) \ (A \in \mathcal{A})\}. \tag{2.4}$$

Note that $P(f)$ is the feasible region of $\mathrm{P}_{(\mathcal{A},f,w)}$ in (1.1). Let $(\hat{\Delta})^* \subseteq 2^{\mathcal{A}}$ be a family defined as

$$(\hat{\Delta})^* = \{\mathcal{F} \subseteq \mathcal{A} \mid \mathcal{F} \notin \hat{\Delta},\ \mathcal{F}' \subset \mathcal{F} \Rightarrow\ \mathcal{F}' \in \hat{\Delta}\}. \tag{2.5}$$

Namely, $(\hat{\Delta})^*$ is the set of minimal nonmembers of $\hat{\Delta}$. For $\mathcal{F} \in (\hat{\Delta})^*$, let $\lambda_{\mathcal{F}} : \mathcal{A} \to \mathbf{Z}_+$ be defined by the output of `Dual_Greedy` for input vector $w = \sum_{A \in \mathcal{F}} \chi_A$, or equivalently, $\lambda_{\mathcal{F}}$ is a nonnegative vector satisfying $\operatorname{supp} \lambda_{\mathcal{F}} \in \hat{\Delta}$ and $\sum_{A \in \mathcal{F}} \chi_A = \sum_{A \in \mathcal{A}} \lambda_{\mathcal{F}}(A) \chi_A$. Note that such $\lambda_{\mathcal{F}}$ is uniquely determined. We define Δ-*submodularity inequalities* as

$$\sum_{A \in \mathcal{F}} f(A) \geq \sum_{A \in \mathcal{A}} \lambda_{\mathcal{F}}(A) f(A) \quad (\mathcal{F} \in (\hat{\Delta})^*). \tag{2.6}$$

Note that LHS of (2.6) depends only on abstract simplicial complex $\hat{\Delta}$ and RHS of (2.6) depends on its geometric realization Δ. A function $f : \mathcal{A} \to \mathbf{R}$ is said to be Δ-*submodular* if it satisfies Δ-submodularity inequalities (2.6). f is said to be *strictly* Δ-*submodular* if it satisfies Δ-submodularity inequalities (2.6) with strict inequality.

Theorem 2.5. *Suppose that there exists a strict Δ-submodular function. Then, f is Δ-submodular if and only if* `Dual_Greedy` *produces an optimal dual solution of* $\mathrm{D}_{(\mathcal{A},f,w)}$ *for every nonnegative cost vector w.*

Proof. The if part follows from the definition of Δ-submodularity inequalities (2.6). We show the only-if part. We can take a strict Δ-submodular function g. Consider linear program $\mathrm{D}_{(\mathcal{A},f,w)}$ with a Δ-submodular function f and a cost vector $w \in \mathbf{R}^V_+$. Since both $\mathrm{P}_{(\mathcal{A},f,w)}$ and $\mathrm{D}_{(\mathcal{A},f,w)}$ are feasible, $\mathrm{D}_{(\mathcal{A},f,w)}$ has an optimal solution.

We take an optimal solution λ^* of $\mathrm{D}_{(\mathcal{A},f,w)}$ which minimizes the value $\sum_{A \in \mathcal{A}} \lambda^*(A) g(A)$. We claim $\operatorname{supp} \lambda^* \in \hat{\Delta}$. If so, λ^* must be the output of `Dual_Greedy`. Suppose that $\operatorname{supp} \lambda^* \notin \hat{\Delta}$. Then there exists $\mathcal{F} \in (\hat{\Delta})^*$ such that $\mathcal{F} \subseteq \operatorname{supp} \lambda^*$. Let $\tilde{\lambda}$ be defined as

$$\tilde{\lambda}(A) = \begin{cases} \lambda^*(A) - \mu & \text{if } A \in \mathcal{F}, \\ \lambda^*(A) + \mu \lambda_{\mathcal{F}}(A) & \text{if } A \in \operatorname{supp} \lambda^{\mathcal{F}}, \\ \lambda^*(A) & \text{otherwise,} \end{cases} \qquad (A \in \mathcal{A}), \tag{2.7}$$

where $\mu = \min\{\lambda^*(A) \mid A \in \mathcal{F}\} > 0$. From $\sum_{A \in \mathcal{F}} \chi_A = \sum_{A \in \mathcal{A}} \lambda_{\mathcal{F}}(A) \chi_A$, $\tilde{\lambda}$ is also feasible to $\mathrm{D}_{(\mathcal{A},f,w)}$. Furthermore, by Δ-submodularity of f, the objective value of $\mathrm{D}_{(\mathcal{A},f,w)}$ for $\tilde{\lambda}$ is given by

$$\begin{aligned} \sum_{A \in \mathcal{A}} \tilde{\lambda}(A) f(A) &= \sum_{A \in \mathcal{A}} \lambda^*(A) f(A) - \mu \left(\sum_{A \in \mathcal{F}} f(A) - \sum_{A \in \mathcal{A}} \lambda_{\mathcal{F}}(A) f(A) \right) \\ &\leq \sum_{A \in \mathcal{A}} \lambda^*(A) f(A). \end{aligned} \tag{2.8}$$

Hence $\tilde{\lambda}$ is also optimal to $\mathrm{D}_{(\mathcal{A},f,w)}$. Similarly, we have

$$\begin{aligned}\sum_{A\in\mathcal{A}} \tilde{\lambda}(A)g(A) &= \sum_{A\in\mathcal{A}} \lambda^*(A)g(A) - \mu\left(\sum_{A\in\mathcal{F}} g(A) - \sum_{A\in\mathcal{A}} \lambda_{\mathcal{F}}(A)g(A)\right) \\ &< \sum_{A\in\mathcal{A}} \lambda^*(A)g(A). \end{aligned} \tag{2.9}$$

This contradicts the definition of λ^*. □

This is a standard proof technique in combinatorial optimization; see Remark 2.15 for further discussion. It follows from the proof of Theorem 2.5 that $\mathrm{D}_{(\mathcal{A},f,w)}$ has the unique optimum if and only if f is strict Δ-submodular. From this we have the following.

Corollary 2.6. *A function f is a strict Δ-submodular if and only if the normal fan of $P(f)$ coincides with Δ. In particular, Δ is regular if and only if there exists a strict Δ-submodular function.*

Corollary 2.7. *Suppose that Δ is regular. If f is Δ-submodular, the normal fan of $P(f)$ is a coarsening of Δ.*

A primal optimal solution can be obtained by the backward iteration as follows.

Proposition 2.8. *Suppose that Δ is regular and $f : \mathcal{A} \to \mathbf{R}$ is Δ-submodular. For a nonnegative cost vector $w \in \mathbf{R}^V_+$, let $\{(A_i, e_i)\}_{i=1}^n$ be the sequence of pairs (A^*, e^*) chosen by* **step 2** *of* `Dual_Greedy` *for input vector w. Then, $x^* \in \mathbf{R}^V$ defined by*

$$x^*(e_i) = f(A_i) - \sum_{k:k>i,e_k\in A_i} x^*(e_k) \quad (i = 1, \dots, n) \tag{2.10}$$

is optimal to $\mathrm{P}_{(\mathcal{A},f,w)}$.

Proof. The following argument is essentially the same as the proof of [10, Theorem 2.1]. For any nonnegative vector w, we define a continuous piecewise linear function $\hat{f}$ as

$$\hat{f}(w) = \sum_{A\in\mathcal{A}} \lambda^w(A)f(A), \tag{2.11}$$

where λ^w is the output of `Dual_Greedy` for w. Note that $\hat{f}$ is well-defined and coincides with the interpolation of f with respect to Δ. By Theorem 2.5 and the duality theorem of linear programming, $\hat{f}$ coincides with the support function of $P(f)$, which is convex. Then $x^* \in \mathbf{R}^V$ defined by (2.10) coincides with the gradient vector of $\hat{f}$ at $\mathrm{cone}\{\chi_{A_i}\}_{i=1}^n \in \Delta$. Convexity of $\hat{f}$ implies that $\sum_{e\in V} x^*(e)w'(e) \le \hat{f}(w')$ for any $w' \in \mathbf{R}^V_+$. In particular, we have $\sum_{e\in V} x^*(e)\chi_A \le \hat{f}(\chi_A) = f(A)$ for $A \in \mathcal{A}$. This implies that x^* is feasible to $P_{(\mathcal{A},f,w)}$. Furthermore, by construction, x^* has the same objective value of $\mathrm{D}_{(\mathcal{A},f,w)}$ at λ^w. Hence x^* is optimal to $\mathrm{P}_{(\mathcal{A},f,w)}$. □

Remark 2.9. If we use `Dual_Greedy*`, we cannot directly obtain a primal optimal solution.

Recall that a dual optimal solution can be taken as integral for any integral cost vector by the construction of `Dual_Greedy`.

Corollary 2.10. *For a Δ-submodular function f of a regular greedy fan Δ with vertex set $\mathcal{A}$, the system of linear inequalities $\sum_{e\in A} x(e) \le f(A)$ $(A \in \mathcal{A})$ is totally dual integral.*

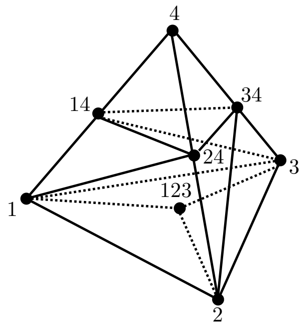

Figure 2: A nonregular greedy fan

Unfortunately, not every greedy fan is regular.

Example 2.11. We give an example of a 4-dimensional nonregular greedy fan. Figure 2 illustrates (the boundary complex of) the hyperplane section of a 4-dimensional greedy fan. A center vertex is given by 123. The set of strict submodularity inequalities for this greedy fan contains

$$\begin{aligned} f(14) + f(2) &> f(24) + f(1), \\ f(24) + f(3) &> f(34) + f(2), \\ f(34) + f(1) &> f(14) + f(3). \end{aligned}$$

Summing these three inequalities leads to a contradiction. Hence, this greedy fan is nonregular.

Remark 2.12. For a Δ-submodular function f with a regular greedy fan Δ, we have $f(A) = \max\{\sum_{e \in A} x(e) \mid x \in P(f)\}$ for $A \in \mathcal{A}$. In particular, f is *polyhedral tight* in the sense of Narayanan [25].

Remark 2.13. For a simplicial subdivision Δ with vertex set $P \subseteq \mathbf{R}^n$, the *secondary cone* for Δ is defined by

$$\{f : P \to \mathbf{R} \mid \hat{f} \text{ is convex }\}, \tag{2.12}$$

where $\hat{f}$ is a piecewise linear function interpolating f with respect to Δ. Sohoni [29] call (2.12) *the set of functions admitting* for Δ. Usually, the defining inequalities (*admittance inequalities* for Δ in Sohoni's sense) of this secondary cone are given by

$$\det \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \det \begin{pmatrix} a_1 & a_2 & \cdots & a_n & b \\ f(a_1) & f(a_2) & \cdots & f(a_n) & f(b) \end{pmatrix} \geq 0 \tag{2.13}$$

for $\{a_1, a_2, \ldots, a_n\} \in \hat{\Delta}$ and $b \in P \backslash \{a_1, a_2, \ldots, a_n\}$; see [29], [13, Chapter 7], [18, Lemma 14]. For a regular greedy fan Δ, the set of Δ-submodular function is the same as the secondary cone for Δ by Theorem 2.5. However, our defining set of inequalities (2.6) is different to (2.13). In the case of a nonregular greedy fan, we do not know whether the cone of Δ-submodular functions coincides with the secondary cone for Δ. In particular, we do not know whether Theorem 2.5 holds without the regularity assumption.

2.3 Algebraic Meaning of Δ-Submodularity

In this subsection, we give an algebraic meaning of Δ-submodularity inequalities. Throughout this subsection, we assume the regularity of greedy fan Δ. Similarly as in the previous subsection, greediness of Δ is not essential. By slight modifications, we can apply the subsequent arguments to any simplicial subdivision each of whose members with maximal dimension has unimodular integral vectors as vertices.

We use basic terminology of *Gröbner bases* and *toric ideals*; see [33] for details. Let k be any field and $k[\mathbf{u}] = k[u_A : A \in \mathcal{A}]$ the polynomial ring in $\#\mathcal{A}$ indeterminates. The monomials in $k[\mathbf{u}]$ are denoted by $\mathbf{u}^\lambda = \prod_{A\in\mathcal{A}} u_A^{\lambda(A)}$ for $\lambda \in \mathbf{Z}_+^{\mathcal{A}}$. Every vector $\lambda \in \mathbf{Z}^{\mathcal{A}}$ can be written uniquely as $\lambda = \lambda^+ - \lambda^-$, where λ^+ and λ^- are nonnegative and have disjoint support. Consider the *toric ideal*

$$I_{\mathcal{A}} = \langle \mathbf{u}^{\lambda^+} - \mathbf{u}^{\lambda^-} : \lambda \in \mathbf{Z}^{\mathcal{A}},\ \sum_{A\in\mathcal{A}} \chi_A \lambda(A) = 0\rangle \tag{2.14}$$

of the vectors $\{\chi_A \mid A \in \mathcal{A}\}$. We can take a sufficiently generic, positive, strict Δ-submodular function f. Then, f induces a *term order* $<_f$ of the monomials in $k[\mathbf{u}]$ as

$$\mathbf{u}^\lambda <_f \mathbf{u}^\mu \overset{\text{def}}{\Longleftrightarrow} \sum_{A\in\mathcal{A}} f(A)\lambda(A) < \sum_{A\in\mathcal{A}} f(A)\mu(A) \quad (\lambda, \mu \in \mathbf{Z}_+^{\mathcal{A}}). \tag{2.15}$$

For any non-zero polynomial $p \in k[\mathbf{u}]$, we define the *initial monomial* $\mathrm{in}_{<_f} p$ by the maximum monomial in p with respect to $<_f$. The *initial ideal* $\mathrm{in}_{<_f} I_{\mathcal{A}}$ is defined by $\langle \mathrm{in}_{<_f} p : p \in I_{\mathcal{A}}\rangle$. A finite subset of polynomials $\mathcal{G} \subseteq I_{\mathcal{A}}$ is called a *Gröbner base* for $I_{\mathcal{A}}$ with respect to $<_f$ if $\mathrm{in}_{<_f} I_{\mathcal{A}}$ is generated by $\{\mathrm{in}_{<_f}(g) \mid g \in \mathcal{G}\}$. Gröbner base $\mathcal{G}$ is said to be *reduced* if, for any two distinct elements $g, g' \in \mathcal{G}$, no term of g' is divisible by $\mathrm{in}_{<_f}(g)$. Then the following holds.

Theorem 2.14. *For a regular greedy fan Δ, the set of polynomials*

$$\mathcal{G} = \left\{ \mathbf{u}^{\chi_{\mathcal{F}}} - \mathbf{u}^{\lambda_{\mathcal{F}}} \;\middle|\; \mathcal{F} \in (\hat{\Delta})^* \right\} \tag{2.16}$$

is the reduced Gröbner base for $I_{\mathcal{A}}$ with respect to $<_f$ for a generic positive strict Δ-submodular function f.

Proof. Since $\lambda_{\mathcal{F}}$ is a nonnegative integral vector, $\mathbf{u}^{\lambda_{\mathcal{F}}}$ is well-defined. From $\sum_{A\in\mathcal{F}} \chi_A = \sum_{A\in\mathcal{A}} \lambda_{\mathcal{F}}(A)\chi_A$, we have $\mathcal{G} \subseteq I_{\mathcal{A}}$ and $\langle \mathrm{in}_{<_f} \mathcal{G}\rangle \subseteq \mathrm{in}_{<_f} I_{\mathcal{A}}$. Suppose that this inclusion is strict. Then there exists $\mathbf{u}^{\tilde{\lambda}} - \mathbf{u}^{\tilde{\mu}} \in I_{\mathcal{A}}$ such that $\sum_{A\in\mathcal{A}} \tilde{\lambda}(A)f(A) > \sum_{A\in\mathcal{A}} \tilde{\mu}(A)f(A)$ and $\mathbf{u}^{\tilde{\lambda}} \notin \langle \mathrm{in}_{<_f} \mathcal{G}\rangle$. The latter implies $\mathrm{supp}\,\tilde{\lambda} \in \hat{\Delta}$. This implies that $\tilde{\lambda}$ is optimal to the linear program $\mathrm{D}_{(\mathcal{A},f,w)}$ for $w = \sum_{A\in\mathcal{A}} \chi_A \tilde{\lambda}(A) = \sum_{A\in\mathcal{A}} \chi_A \tilde{\mu}(A)$. This contradicts $\sum_{A\in\mathcal{A}} \tilde{\lambda}(A)f(A) > \sum_{A\in\mathcal{A}} \tilde{\mu}(A)f(A)$. Hence (2.16) is a Gröbner base. By construction, (2.16) is reduced. □

Remark 2.15. The proof of Theorem 2.5 is based on a standard technique to show TDI-ness of linear inequality systems related to submodular functions; see [28, Chapter 60] for example. We point out that the updating λ^* to $\hat{\lambda}$ in (2.7) corresponds to the division of $\mathbf{u}^\lambda$ by the Gröbner base $\mathbf{u}^{\chi_{\mathcal{F}}} - \mathbf{u}^{\lambda_{\mathcal{F}}}$. In addition, the totally dual integrality of the dual greedy system (Corollary 2.10) corresponds to the square-freeness of the Gröbner base (2.16); see [17] for the relationship between totally dual integrality and square-free Gröbner base.

Example 2.16 (The barycentric subdivision). Consider the following simplicial subdivision

$$\Delta = \{\mathrm{cone}\{\chi_{A_1}, \chi_{A_2}, \ldots, \chi_{A_m}\} \mid \emptyset \neq A_1 \subset A_2 \subset \cdots \subset A_m \subseteq V\}. \tag{2.17}$$

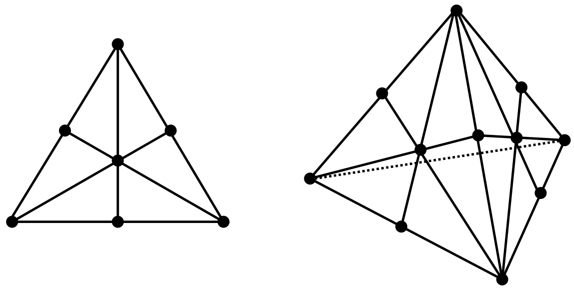

Figure 3: Barycentric subdivisions of 2-simplex(left) and 3-simplex(right)

Then Δ is the *barycentric subdivision* of simplex; see Figure 3. We can easily verify that

$$\begin{aligned}\mathcal{A} &= 2^V \setminus \{\emptyset\}, && (2.18)\\ \Phi_\Delta(X) &= \{X\} \quad (X \subseteq V), && (2.19)\\ \hat{\Delta} &= \{\{A_1, A_2, \ldots, A_m\} \mid \emptyset \neq A_1 \subset A_2 \subset \cdots \subset A_m \subseteq V\}, && (2.20)\\ \hat{\Delta}^* &= \{\{A, B\} \mid A \not\subseteq B, B \not\subseteq A\}. && (2.21)\end{aligned}$$

By $\chi_A + \chi_B = \chi_{A\cup B} + \chi_{A\cap B}$, Δ-submodularity inequalities are given as

$$\begin{array}{ll} f(A) + f(B) \geq f(A\cup B) + f(A\cap B) & (\{A, B\} \in (\hat{\Delta})^* : A\cap B \neq \emptyset),\\ f(A) + f(B) \geq f(A\cup B) & (\{A, B\} \in (\hat{\Delta})^* : A\cap B = \emptyset). \end{array} \tag{2.22}$$

(2.22) coincides with ordinary submodularity inequalities by putting $f(\emptyset) = 0$ and $P(f)$ coincides with a *submodular polyhedron* [9]. In particular, Δ is regular greedy; see Example 3.26. `Dual_Greedy` using this Φ coincides with the dual greedy algorithm for a submodular polyhedron [23],[9]. The corresponding Gröbner base is given as

$$\begin{array}{ll} u_A u_B - u_{A\cup B} u_{A\cap B} & (\{A, B\} \in (\hat{\Delta})^* : A\cap B \neq \emptyset),\\ u_A u_B - u_{A\cup B} & (\{A, B\} \in (\hat{\Delta})^* : A\cap B = \emptyset). \end{array} \tag{2.23}$$

2.4 Greedy Multiple-Choice Functions

In this subsection, we discuss properties of the map Φ_Δ, defined by (2.3), associated with greedy fan Δ and try to define greedy fans by means of a certain map $\Phi : 2^V \to 2^{2^V}$. Our main purpose here is to derive conditions of Φ which determine a greedy fan. First, we see that Φ_Δ has the following properties.

Proposition 2.17. *For a greedy fan Δ with vertex set $\mathcal{A}$, the map $\Phi = \Phi_\Delta : 2^V \to 2^{\mathcal{A}}$ has the following properties.*

(C1) *for a nonempty $X \subseteq V$, $\Phi(X)$ is nonempty and $\emptyset \notin \Phi(X)$.*

(C2) *for $X \subseteq V$ and $A \in \Phi(X)$, we have $A \subseteq X$.*

(M1) *for $X, Y \subseteq V$ and $A \in \Phi(X)$, if $A \subseteq Y \subseteq X$, then we have $A \in \Phi(Y)$.*

(M2) *for $X \subseteq V$ and $A, B \in \Phi(X)$, we have $A = B$ or $A \cap B = \emptyset$.*

Proof. (C1) and (C2) follow from the definition of Φ. (M1) follows from the observation that if A is a center vertex of the greedy fan Δ^X, A is also a center of Δ^Y for $A \subseteq Y \subseteq X$. We show (M2). Suppose that A and B have nonempty intersection. Consider `Dual_Greedy` for the greedy fan Δ^X and input vector $\chi_{A\cup B}$. Then the solution is not unique. This is a contradiction. ◻

If a function $\Phi : 2^V \to 2^{2^V}$ satisfies (C1) and (C2), we call $\Phi : 2^V \to 2^{2^V}$ a *multiple-choice function*. Given a multiple-choice function $\Phi : 2^V \to 2^{2^V}$, we can apply `Dual_Greedy` for any nonnegative input vector. However, the output λ depends on the choices of A^* and e^* in **step 2**. Next we discuss the uniqueness of outputs of `Dual_Greedy` for general multiple-choice functions. For a multiple-choice function Φ and an input vector $w \in \mathbf{R}_+^V$, a sequence $\{(A_i, e_i)\}_{i=1}^n \subseteq 2^V \times V$ is said to be *feasible* to w if $A_i = A^*$ and $e_i = e^*$ can be chosen in **step 2** of the ith iteration step for input vector w. Then we see the following.

Lemma 2.18. *If a sequence $\{(A_i, e_i)\}_{i=1}^n$ is feasible to some input vector, then it is also feasible to any vector in* $\mathrm{cone}\{\chi_{A_i}\}_{i=1}^n$.

Let Δ_Φ be a set of simplicial cones defined as

$$\Delta_\Phi = \{\mathrm{cone}\{\chi_A\}_{A \in \mathrm{supp}\,\lambda} \mid \lambda \text{ is an output for some } w \in \mathbf{R}_+^V\}. \tag{2.24}$$

Lemma 2.18 above implies that any face of any member of Δ_Φ is also contained by Δ_Φ. Hence, if the output λ is uniquely determined for any $w \in \mathbf{R}_+^V$, Δ_Φ forms a simplicial subdivision of $\mathbf{R}_+^V$. In fact, the conditions (M1) and (M2) are sufficient for this uniqueness as follows.

Proposition 2.19. *If a multiple-choice function $\Phi : 2^V \to 2^{2^V}$ satisfies the conditions* (M1) *and* (M2)*, for any nonnegative input vector w, the solution of* `Dual_Greedy` *is determined independently of the choices A^* and e^* in* **step 1**.

Proof. We use induction on the number of the nonzero support of nonnegative input vectors. For any $w \in \mathbf{R}_+^V$ with $\#\,\mathrm{supp}\,w \leq 1$, the statement clearly holds. Consider a nonnegative input vector w with $\#\,\mathrm{supp}\,w > 1$. Let $\{(A_i, e_i)\}_{i=1}^n$ and $\{(B_i, d_i)\}_{i=1}^n$ be two feasible sequences to w. Let λ and μ be outputs for w using feasible sequences $\{A_i, e_i\}_{i=1}^n$ and $\{B_i, d_i\}_{i=1}^n$, respectively. We define X_i and Y_i as

$$X_1 = Y_1 = V, \quad X_{i+1} = X_i \setminus \{e_i\},\ Y_{i+1} = Y_i \setminus \{d_i\} \tag{2.25}$$

for $i = 1, \ldots, n$. We show $\lambda = \mu$. Let i be the smallest number satisfying $\lambda(A_i) > 0$. Similarly, let j be the smallest number satisfying $\mu(B_j) > 0$. Then we have $A_i \subseteq \mathrm{supp}\,w \subseteq X_i$ and $B_j \subseteq \mathrm{supp}\,w \subseteq Y_j$. By (M1), we have $A_i, B_j \in \Phi(\mathrm{supp}\,w)$. Hence, it follows from (M2) that $A_i = B_j$ or $A_i \cap B_j = \emptyset$. If $A_i = B_j$, then we have $\lambda(A_i) = \mu(B_j)$. If $A_i \cap B_j = \emptyset$, then we have $A_i \subseteq \mathrm{supp}\,w \setminus \{d_j\} \subseteq X_i$ and $B_{j+1} \subseteq \mathrm{supp}\,w \setminus \{d_j\} \subseteq Y_{j+1}$. Similarly we have $A_i = B_{j+1}$ or $A_i \cap B_{j+1} = \emptyset$. Repeating this process, there exists some number k such that $A_i = B_{j+k}$ or $A_i = \mathrm{supp}\,w \setminus \{d_j, d_{j+1}, \ldots d_{j+k-1}\}$. In the latter case, we have $B_{j+k} \in \Phi(A_i)$. Hence $B_{j+k} = A_i$ holds by Lemma 2.20 below. Since $A_i \cap B_{j+1} = A_i \cap B_{j+2} = \cdots = A_i \cap B_{j+k-1} = \emptyset$, we have $\lambda(A_i) = \mu(B_{j+k})$. We define a modified input vector $w' = w - \lambda(A_i)\chi_{A_i} = w - \mu(B_{j+k})\chi_{B_{j+k}}$. Then we have $\#\,\mathrm{supp}\,w' < \#\,\mathrm{supp}\,w$. Two sequences $\{(A_i, e_i)\}_{i=1}^n$ and $\{(B_i, d_i)\}_{i=1}^n$ are also feasible to the modified input vector w' by Lemma 2.18. Let λ' and μ' be two outputs of `Dual_Greedy` for input vector w' using feasible sequences $\{(A_i, e_i)\}_{i=1}^n$ and $\{(B_i, d_i)\}_{i=1}^n$, respectively. Then we have

$$\lambda'(A) = \begin{cases} 0 & \text{if } A = A_i = B_{j+k} \\ \lambda(A) & \text{otherwise} \end{cases}, \quad \mu'(A) = \begin{cases} 0 & \text{if } A = A_i = B_{j+k} \\ \mu(A) & \text{otherwise} \end{cases}$$

for $A \subseteq V$. By induction, we have $\lambda' = \mu'$. This implies $\lambda = \mu$. $\square$

For a function $\Phi : 2^V \to 2^{2^V}$, we define the *image* $\mathrm{Im}\,\Phi \subseteq 2^V$ as

$$\mathrm{Im}\,\Phi = \{A \subseteq V \mid \exists \text{ nonempty } X \subseteq V,\ A \in \Phi(X)\}. \tag{2.26}$$

Lemma 2.20. *Let Φ be a multiple-choice function Φ satisfying* (M1) *and* (M2). *For $A \in \mathrm{Im}\,\Phi$, we have $\{A\} = \Phi(A)$.*

Proof. If $A \in \mathrm{Im}\,\Phi$, then there exists $X \subseteq V$ such that $A \in \Phi(X)$ and $A \subseteq X$ by (C2). By (M1), we have $A \in \Phi(A)$. By (C1) and (M2), we have $\{A\} = \Phi(A)$. ☐

A multiple-choice function is said to be *greedy* if it satisfies (M1) and (M2). Hence, we obtain a greedy fan by a greedy multiple-choice function as follows.

Theorem 2.21. *For a greedy multiple-choice function $\Phi : 2^V \to 2^{2^V}$, Δ_Φ is a greedy fan of $\mathbf{R}^V_+$ with vertex set $\mathrm{Im}\,\Phi$ and satisfies*

$$\Phi(X) \subseteq \{\text{ center vertices of } \Delta^X_\Phi \text{ }\} \quad (X \subseteq V). \tag{2.27}$$

Conversely, every greedy fan Δ can be represented as $\Delta = \Delta_\Phi$ for some greedy multiple-choice function Φ.

Proof. Indeed, Δ_Φ is a simplicial subdivision of $\mathbf{R}^V$ by Proposition 2.19. We show the greediness and (2.27). It suffices to show that for any $X \subseteq V$, $A \in \Phi(X)$, and $w \in \mathbf{R}^V_+$ with $\mathrm{supp}\, w = X$, the output λ of `Dual_Greedy` for w satisfies $\lambda(A) > 0$. Let $\{(A_i, e_i)\}_{i=1}^n$ be a feasible sequence to w and $\{X_i\}_{i=1}^n$ a sequence defined by (2.25). If $A_1 \not\subseteq X$, then $\lambda(A_1) = 0$ and $A \subseteq X \subseteq X_2$. If $A_1 \subseteq X$, then we have $A_1 \in \Phi(X)$ by (M1), and $A_1 = A$ or $A_1 \cap A = \emptyset$ by (M2). If $A_1 = A$, then $\lambda(A_1) = \lambda(A) > 0$ as desired. If $A_1 \cap A = \emptyset$, then for each $e \in A$, $w'(e)$ is invariant in **step 3** and we have $A \subseteq X_2$. Repeating this process, we have $A = A_k$ or $A = X_k$ for some k. In the latter case, we have $A = A_k$ by Lemma 2.20. Since $w'(e)$ is invariant for each $e \in A$ in this process, we obtain $\lambda(A) = \lambda(A_k) > 0$ as desired. Final part of this theorem follows from Proposition 2.17. ☐

Corollary 2.22. *For a greedy multiple-choice function Φ,* `Dual_Greedy` *and* `Dual_Greedy*` *return the same output.*

The proof of Proposition 2.19 gives the following useful criterion to test whether two greedy multiple-choice functions produce the same greedy fan.

Proposition 2.23. *For two greedy multiple-choice functions $\Phi_1, \Phi_2 : 2^V \to 2^{2^V}$, the following two statements are equivalent.*

(1) $\Delta_{\Phi_1} = \Delta_{\Phi_2}$.

(2) *for any $X \subseteq V$, $A \in \Phi_1(X)$, and $B \in \Phi_2(X)$, we have $A = B$ or $A \cap B = \emptyset$.*

Remark 2.24. A multiple-choice function Φ with $\#\Phi(X) = 1$ for $X \subseteq V$ can be regarded as a *choice function*; a choice function is a function $\phi : 2^V \to 2^V$ satisfying $\phi(X) \neq \emptyset$ if $X \neq \emptyset$ and $\phi(X) \subseteq X$ for $X \subseteq V$. See [24] for choice functions and also see [10] for the relationship to dual greedy algorithms.

3 Acyclic Greedy Fans

In this section, we investigate a certain class of greedy fans which can be represented by some posets. This approach is motivated by the dual greedy system by Frank [7]. In Subsection 3.1, we introduce acyclic greedy fans and investigate their geometric properties. In the subsequent subsections, we study some special cases of acyclic greedy fans.

3.1 Acyclic Greedy Fans

Here, we introduce a special class of greedy fans, named *acyclic greedy fans*, and study its geometric properties. Our main purpose here is to show the following:

- Acyclic greedy fans are regular and obtained by successive stellar subdivisions (Theorem 3.5).
- The set of all acyclic greedy fans on a fixed set of vertices forms a kind of a polyhedral fan, named the *secondary greedy fan* (Theorem 3.12).

Throughout this subsection, we assume that $\mathcal{A}$ is a subset of $2^V \setminus \{\emptyset\}$. A pair $A, B \subseteq V$ is said to be *intersecting* if it satisfies $A \cap B \neq \emptyset$, $A \not\subseteq B$, and $B \not\subseteq A$. Let $\mathcal{P} = (\mathcal{A}, \leq)$ be a poset on $\mathcal{A}$. We define a function $\Phi_\mathcal{P} : 2^V \to 2^\mathcal{A}$ associated with $\mathcal{P}$ as

$$\Phi_\mathcal{P}(X) = \{A \in \mathcal{A} \mid A \subseteq X, B > A \Rightarrow B \not\subseteq X \ (B \in \mathcal{A})\} \quad (X \subseteq V), \tag{3.1}$$

that is, $\Phi_\mathcal{P}(X)$ is the set of maximal members of $\mathcal{A}$ contained in X. Such a function $\Phi_\mathcal{P}$ was used by Frank [7] in his dual greedy algorithm. We easily see the following properties of $\Phi_\mathcal{P}$.

Lemma 3.1. *Let $\mathcal{P} = (\mathcal{A}, \leq)$ be a poset on $\mathcal{A}$. Then we have the following.*

(1) *$\Phi_\mathcal{P}$ is a multiple-choice function if and only if $\{e\} \in \mathcal{A}$ for $e \in V$.*

(2) *$\Phi_\mathcal{P}$ satisfies* (M1).

(3) $\mathrm{Im}\, \Phi_\mathcal{P} = \mathcal{A}$ *if and only if $A \not> B$ for each pair of $A, B \in \mathcal{A}$ with $A \subseteq B$.*

(4) *$\Phi_\mathcal{P}$ satisfies* (M2) *if and only if for each pair of $A, B \in \mathcal{A}$ having nonempty intersection, there exists $C \in \mathcal{A}$ with $C \subseteq A \cup B$ such that $A < C$ or $B < C$ (or both).*

A poset $\mathcal{P} = (\mathcal{A}, \leq)$ is said to be *greedy* if $\Phi_\mathcal{P}$ is a greedy multiple-choice function and satisfies $\mathrm{Im}\, \Phi_\mathcal{P} = \mathcal{A}$ (or $\mathrm{Im}\, \Phi_\mathcal{P} = \mathcal{A} \setminus \{\emptyset\}$ in the case $\{\emptyset\} \in \mathcal{A}$). Hence, from a greedy poset $\mathcal{P}$, we obtain a greedy fan, which is denoted by $\Delta_\mathcal{P}$. A greedy fan Δ is said to be *acyclic* if there exists a greedy poset $\mathcal{P}$ such that $\Delta = \Delta_\mathcal{P}$.

For two posets $\mathcal{P}_1 = (\mathcal{A}, \leq_1)$ and $\mathcal{P}_2 = (\mathcal{A}, \leq_2)$, $\mathcal{P}_2$ is a *refinement* of $\mathcal{P}_1$ if it satisfies

$$A \leq_1 B \Rightarrow A \leq_2 B \quad (A, B \in \mathcal{A}). \tag{3.2}$$

A greedy poset $(\mathcal{A}, \leq)$ is a refinement of poset $(\mathcal{A}, \subseteq)$ as follows.

Lemma 3.2. *Let $\mathcal{P} = (\mathcal{A}, \leq)$ be a greedy poset. For any $A, B \in \mathcal{A}$, if $A \subseteq B$, then $A \leq B$.*

Proof. $\mathrm{Im}\, \Phi_\mathcal{P} = \mathcal{A}$ and Lemma 2.20 imply $\{B\} = \Phi_\mathcal{P}(B)$. Suppose $A \not\leq B$. Then A and B are incomparable by Lemma 3.1 (3). Then there exists $C \in \mathcal{A}$ such that $C \neq B$ and $C \in \Phi_\mathcal{P}(B)$. This is a contradiction. □

Furthermore, from the condition of Proposition 2.23, we have the following.

Proposition 3.3. *Let $\mathcal{P}$ is a greedy poset. Then, any refinement $\mathcal{P}'$ of $\mathcal{P}$ is greedy and satisfies $\Delta_\mathcal{P} = \Delta_{\mathcal{P}'}$.*

Proof. It follows from $\Phi_{\mathcal{P}'}(X) \subseteq \Phi_\mathcal{P}(X)$ and Proposition 2.23. □

For a poset $\mathcal{P} = (\mathcal{A}, \leq)$, a *linear extension* $(\mathcal{A}, \leq^*)$ of $\mathcal{P}$ is a totally ordered set which refines $\mathcal{P}$. Then we have the following.

Lemma 3.4. *If $\mathcal{A}$ contains every singleton, then any linear extension of $(\mathcal{A}, \subseteq)$ is greedy.*

Next, we give a geometric interpretation of this linear extension. For this, we introduce *stellar operations* (or *pulling operations*) of simplicial fans; see [4], [22] for stellar/pulling operations. For a simplicial fan Δ and a point $p \in |\Delta|$, the stellar subdivision $\mathrm{st}_p\Delta$ by p is obtained by the following process:

Set $\mathrm{st}_p\Delta = \emptyset$.
For each member $C \in \Delta$, repeat the following:
  if $p \notin C$, then $\mathrm{st}_p\Delta \leftarrow \mathrm{st}_p\Delta \cup \{C\}$.
  if $p \in C$, then, for each face F of C not containing p,
    $\mathrm{st}_p\Delta \leftarrow \mathrm{st}_p\Delta \cup \{\mathrm{cone}(p \cup F)\}$.

Let Δ_0 be a simplicial fan consisting of the nonnegative orthant and its faces. Then the following theorem implies that acyclic greedy fans can be obtained by successive stellar operations from the trivial subdivision Δ_0 with respect to some order which refines $\subseteq$. This implies that every acyclic greedy fan is a *reverse lexicographic triangulation*; see [22], [32], [33] for reverse lexicographic triangulations.

Theorem 3.5. *Let $\mathcal{P} = (\mathcal{A}, \leq)$ be a greedy poset and $\leq^*$ an arbitrary linear extension of $\leq$. Then we have*

$$\Delta_{\mathcal{P}} = \mathrm{st}_{\chi_{A_m}} \circ \cdots \circ \mathrm{st}_{\chi_{A_1}}(\Delta_0), \tag{3.3}$$

where $\mathcal{A} = \{A_1, \ldots, A_m\}$ and $A_m <^* A_{m-1} <^* \cdots <^* A_1$.

Proof. First, we note that for any simplicial subdivision Δ' of $\mathbf{R}^V_+$ and any singleton $e \in V$ we have $\mathrm{st}_{\chi_e}(\Delta') = \Delta'$. We use induction on $\#\mathcal{A}$. Since $\mathcal{A}$ contains every singleton, $\#\mathcal{A} \geq \#V$ holds. If $\#\mathcal{A} = \#V$, we have $\Delta_{\mathcal{P}} = \Delta_0$. Since $\mathrm{st}_{\chi_{\{e\}}}(\Delta_0) = \Delta_0$ holds for every singleton $e \in V$, the statement is true. Suppose $\#\mathcal{A} > \#V$. Then there exists $A = \{e_1, e_2, \ldots, e_k\} \in \mathcal{A}$ with $\#A > 1$ such that any element $A' \in \mathcal{A}$ with $A' <^* A$ is a singleton. Consider the greedy poset $(\mathcal{A} \setminus \{A\}, \leq^*)$ and its associated greedy fan $\Delta_{(\mathcal{A}\setminus\{A\},\leq^*)}$. By induction, it suffices to show

$$\Delta_{(\mathcal{A},\leq^*)} = \mathrm{st}_{\chi_A}(\Delta_{(\mathcal{A}\setminus\{A\},\leq^*)}). \tag{3.4}$$

By `Dual_Greedy*` for $\Delta_{(\mathcal{A}\setminus\{A\},\leq^*)}$, the unique minimal member of $\hat{\Delta}_{(\mathcal{A}\setminus\{A\},\leq^*)}$ containing χ_A in the relative interior of its conical hull is $\{\{e_1\}, \{e_2\}, \ldots, \{e_k\}\}$. Hence the element F of $\hat{\Delta}_{(\mathcal{A}\setminus\{A\},\leq^*)}$ containing χ_A in its conical hull is given by

$$\{A_{i_1}, A_{i_2}, \ldots, A_{i_h}, \{e_1\}, \{e_2\}, \ldots, \{e_k\}\} \tag{3.5}$$

for some $A_{i_1}, A_{i_2}, \ldots, A_{i_h} \in \mathcal{A} \setminus \{A\}$. Then we show that both sides of (3.4) contain

$$\{A\} \cup F \setminus \{\{e_j\}\} \quad (1 \leq j \leq k). \tag{3.6}$$

Indeed, LHS of (3.4) contains (3.6) by the definition of the stellar operation. Similarly, we can show by `Dual_Greedy*` that RHS of (3.4) also contains (3.6); consider `Dual_Greedy*` for $\Delta_{(\mathcal{A},\leq^*)}$ and a cost vector $w = \chi_{A_{i_1}} + \cdots + \chi_{A_{i_h}} + b_1\chi_{e_1} + \cdots + b_k\chi_{e_k}$ for $b_1, \ldots, b_k > 0$. Finally, we verify that any $F' \in \Delta_{(\mathcal{A}\setminus\{A\},\leq^*)}$ not containing χ_A is contained in both sides of (3.4). This is also immediate from `Dual_Greedy*` and the definition of the stellar subdivision. $\square$

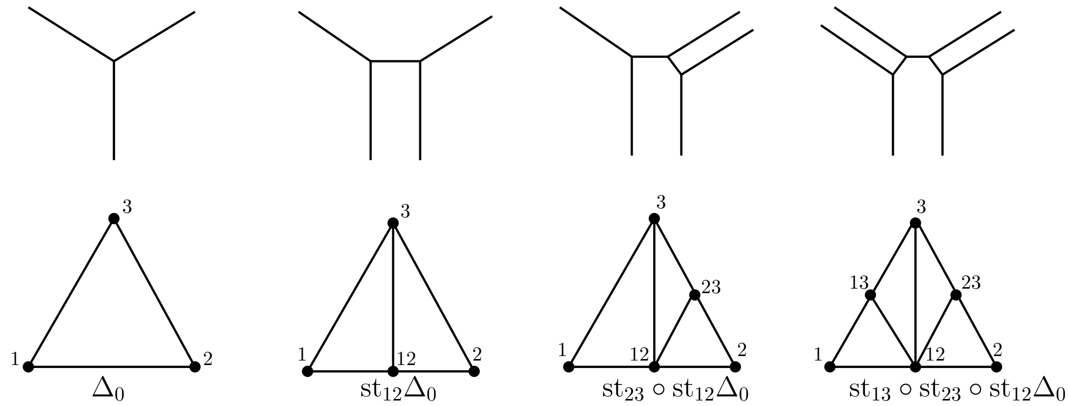

Figure 4: stellar subdivisions (down) and their dual (up)

Δ_0 is the normal fan of the nonpositive orthant. In particular, Δ_0 is regular. Recall that a stellar subdivision corresponds to *cutting* a corner of the dual polyhedron [4]. This implies that a polyhedron whose normal fan is acyclic greedy is obtained by successive cutting corners of nonpositive orthant according to a certain order; see Figure 4. This gives a construction of dual greedy polyhedra. Thus, we have the following.

Corollary 3.6. *Acyclic greedy fans are regular.*

Hence, we can apply arguments in Subsection 2.2 and 2.3 to acyclic greedy fans.

Remark 3.7. Regularity of acyclic greedy fan Δ can be shown by the existence of a strict Δ-submodular function. Indeed, for a greedy poset $(\mathcal{A}, \leq)$, consider a linear extension $\leq^*$ ordering $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ as

$$A_m <^* A_{m-1} <^* \cdots <^* A_1, \tag{3.7}$$

and define $f : \mathcal{A} \to \mathbf{R}$ as

$$f(A_i) = -\epsilon^i \quad (A_i \in \mathcal{A}), \tag{3.8}$$

where $\epsilon \in \mathbf{R}_+$ is a sufficiently small positive real. Then f satisfies Δ-submodularity inequalities strictly. In particular, the corresponding term order of the polynomial ring $k[u_A : A \in \mathcal{A}]$ coincides with the reverse lexicographic term order with respect to $<^*$, where the degree of each indeterminate u_A $(A \in \mathcal{A})$ is defined by $\#A$ (see [32], [33]).

Problem 3.8. We have

$$\{ \text{ acyclic greedy fans } \} \subseteq \{ \text{ regular greedy fans } \}. \tag{3.9}$$

Is the inclusion strict?

Next, we show that for fixed $\mathcal{A}$, the set of all acyclic greedy fans forms a kind of polyhedral fan. For two posets $\mathcal{P}_1 = (\mathcal{A}, \leq_1)$ and $\mathcal{P}_2 = (\mathcal{A}, \leq_2)$ with a common ground set $\mathcal{A}$, we define *meet* $\mathcal{P}_1 \wedge \mathcal{P}_2 = (\mathcal{A}, \leq_{1\wedge 2})$ as

$$A \leq_{1\wedge 2} B \overset{\text{def}}{\iff} A \leq_1 B \text{ and } A \leq_2 B \quad (A, B \in \mathcal{A}). \tag{3.10}$$

The next proposition shows that if two greedy posets define the same greedy fan, then their meet is also greedy and defines the same one.

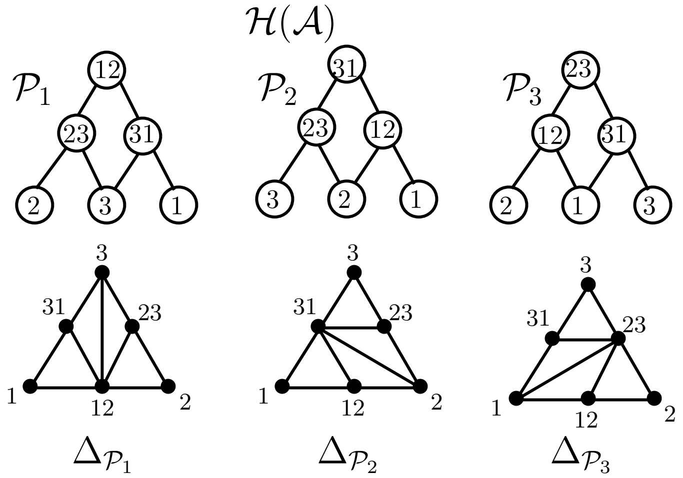

Figure 5: $\mathcal{H}(\mathcal{A})$ for $\mathcal{A} = \{12, 23, 31, 1, 2, 3\}$ and corresponding acyclic greedy fans

Proposition 3.9. *Let $\mathcal{P}_1 = (\mathcal{A}, \leq_1)$ and $\mathcal{P}_2 = (\mathcal{A}, \leq_2)$ be greedy posets on $\mathcal{A}$. If $\Delta_{\mathcal{P}_1} = \Delta_{\mathcal{P}_2}$, $\mathcal{P}_1 \wedge \mathcal{P}_2$ is greedy and satisfies $\Delta_{\mathcal{P}_1 \wedge \mathcal{P}_2} = \Delta_{\mathcal{P}_1} = \Delta_{\mathcal{P}_2}$.*

Proof. From the definitions of $\Phi_{\mathcal{P}}$ and the meet $\mathcal{P}_1 \wedge \mathcal{P}_2$, we see

$$\Phi_{\mathcal{P}_1 \wedge \mathcal{P}_2}(X) \supseteq \Phi_{\mathcal{P}_1}(X) \cup \Phi_{\mathcal{P}_2}(X) \quad (X \subseteq V). \tag{3.11}$$

By Proposition 2.23, it suffices to show that $\Phi_{\mathcal{P}_1 \wedge \mathcal{P}_2}$ satisfies (M2). Suppose that there exist distinct $A, B \in \Phi_{\mathcal{P}_1 \wedge \mathcal{P}_2}(X)$ such that $A \cap B$ is nonempty. Then we have $A, B \in \Phi_{\mathcal{P}_1 \wedge \mathcal{P}_2}(A \cup B)$ by (M1) and Lemma 3.1 (2). Then A or B is not contained by $\Phi_{\mathcal{P}_1}(A \cup B) \cup \Phi_{\mathcal{P}_2}(A \cup B)$. We assume $A \notin \Phi_{\mathcal{P}_1}(A \cup B) \cup \Phi_{\mathcal{P}_2}(A \cup B)$. Then there exists $C \in \Phi_{\mathcal{P}_1}(A \cup B)$ such that $C \geq_1 A$. We claim that A and C are disjoint. Suppose that A and C intersect. By (M1) we have $C \in \Phi_{\mathcal{P}_1}(A \cup C)$. We show $A \notin \Phi_{\mathcal{P}_2}(A \cup C)$ and $C \not\geq_2 A$. The former follows from $\Delta_{\mathcal{P}_1} = \Delta_{\mathcal{P}_2}$ and Proposition 2.23. The latter follows from $C \not\geq_{1 \wedge 2} A$. Hence there exists $D \in \Phi_{\mathcal{P}_2}(A \cup C)$ such that $D \geq_2 A$ and $D \neq C$. Then $D \cap C = \emptyset$ and hence $D \subset A$ (strict inclusion). By Lemma 3.2, we have $D <_2 A$, which contradicts $D \in \Phi_{\mathcal{P}_2}(A \cup C)$. Hence we have $A \cap C = \emptyset$ and $C \subset B$ (strict inclusion). By Lemma 3.2, we have $C <_1 B$. This contradicts $C \in \Phi_{\mathcal{P}_1}(A \cup B)$. $\square$

From this proposition, we have the following.

Theorem 3.10. *For a family $\mathcal{A}$ which contains every singleton, there exists a set of greedy posets $\mathcal{H}(\mathcal{A})$ satisfying the following properties.*

(1) *for $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{H}(\mathcal{A})$, $\Delta_{\mathcal{P}_1} = \Delta_{\mathcal{P}_1}$ if and only if $\mathcal{P}_1 = \mathcal{P}_2$.*

(2) *for any greedy poset $\mathcal{P}'$ on $\mathcal{A}$, there uniquely exists $\mathcal{P} \in \mathcal{H}(\mathcal{A})$ such that $\mathcal{P}'$ is a refinement of $\mathcal{P}$.*

Example 3.11. Figure 5 illustrates $\mathcal{H}(\mathcal{A})$ for $\mathcal{A} = \{12, 23, 31, 1, 2, 3\}$. $\mathcal{H}(\mathcal{A})$ consists of three greedy posets.

For a poset $(\mathcal{A}, \leq)$ we define an *order cone* $\mathcal{C}_{(\mathcal{A}, \leq)} \subseteq \mathbf{R}^{\mathcal{A}}$ as

$$\mathcal{C}_{(\mathcal{A}, \leq)} = \{z \in \mathbf{R}^{\mathcal{A}} \mid z(A) \leq z(B) \quad (A, B \in \mathcal{P},\ A \leq B)\}. \tag{3.12}$$

The order cone is a conical version of the *order polytope*; see [31], [8, Section 3.3] for order polytopes. The set of polyhedral cones consisting of order cones $\{\mathcal{C}_{\mathcal{P}} \mid \mathcal{P} \in \mathcal{H}(\mathcal{A})\}$ and their faces is denoted by $\mathcal{N}(\mathcal{A})$. In fact, $\mathcal{N}(\mathcal{A})$ forms a polyhedral fan as follows.

Theorem 3.12. *For a family $\mathcal{A}$ which contains every singleton, $\mathcal{N}(\mathcal{A})$ is a polyhedral subdivision of order cone $\mathcal{C}_{(\mathcal{A},\subseteq)}$.*

To prove this theorem, we use the following lemma, where we simply denote $\Delta_{(\mathcal{A},\leq)}$, $\Phi_{(\mathcal{A},\leq)}$ and $\mathcal{C}_{(\mathcal{A},\leq)}$ by $\Delta_{\leq}$, $\Phi_{\leq}$, and $\mathcal{C}_{\leq}$, respectively, and $A \lessdot B$ means that $A < B$ and there is no element C such that $A < C < B$.

Lemma 3.13. *Let $(\mathcal{A}, \leq)$ be a greedy poset. Let $A, B \in \mathcal{A}$ be a pair satisfying $A \lessdot B$, $A \not\subseteq B$, and $B \not\subseteq A$. For two linear extensions $\leq_1$ and $\leq_2$ of $\leq$ such that $A \lessdot_1 B$ and $A \lessdot_2 B$, consider the modified linear extensions $\leq_1^*$ and $\leq_2^*$ obtained by interchanging A and B in $\leq_1$ and $\leq_2$, respectively. Then we have $\Delta_{\leq_1^*} = \Delta_{\leq_2^*}$.*

Proof. We prove two claims given below.
(Claim 1). We have

$$\Phi_{\leq_1^*}(X) = \begin{cases} \{A\} & \text{if } A \cup B \subseteq X \text{ and } \{B\} = \Phi_{\leq_1}(X), \\ \Phi_{\leq_1}(X) & \text{otherwise,} \end{cases} \qquad (X \subseteq V). \tag{3.13}$$

Consider $X \subseteq V$ with $\{B\} = \Phi_{\leq_1}(X)$. Note that $\Phi_{\leq_1^*}(X) = \{A\}$ or $\{B\}$. Suppose $A \cup B \not\subseteq X$. Then $A \not\subseteq X$ implies $\Phi_{\leq_1^*}(X) = \{B\} = \Phi_{\leq_1}(X)$. Suppose $A \cup B \subseteq X$. Then we have $\Phi_{\leq_1^*}(X) = \{A\}$. Hence we obtain (3.13).
(Claim 2). If $A \cup B \subseteq X$ and $\{B\} = \Phi_{\leq_1}(X)$, for C with $\{C\} = \Phi_{\leq_2}(X)$ we have $B = C$ or $A \cap C = B \cap C = \emptyset$.

Suppose $B \neq C$. Then $B \cap C = \emptyset$ follows from Proposition 2.23. We show $A \cap C = \emptyset$. Suppose $A \cap C \neq \emptyset$. Consider $\Phi_{\leq}(A \cup C)$. If $A \in \Phi_{\leq}(A \cup C)$, then $A > C$ holds and contradicts $\{C\} = \Phi_{\leq_2}(X)$. Hence, there exists $D \in \Phi_{\leq}(A \cup C)$ such that $D \neq B$ and $A < D$. By $A \lessdot B$, we have $A <_1 B <_1 D$. This contradicts $\{B\} = \Phi_{\leq_1}(X)$.

From Claims 1 and 2 above, we can verify $\Delta_{\leq_1^*} = \Delta_{\leq_2^*}$ by Proposition 2.23. □

Proof of Theorem 3.12. The order cone $\mathcal{C}_{(\mathcal{A},\leq)}$ has the following properties [31] (see also [8, Section 3.3]):

(1) $\mathcal{C}_{(\mathcal{A},\leq)}$ is triangulated by order cones of linear extensions

$$\{\mathcal{C}_{(\mathcal{A},\leq^*)} \mid \leq^* \text{ is a linear extension of } \leq\}. \tag{3.14}$$

(2) Each facet defining inequality of $\mathcal{C}_{(\mathcal{A},\leq)}$ is given by $z(A) \leq z(B)$ for $A, B \in \mathcal{A}$ with $A \lessdot B$.

Theorem 3.10 and (1) imply $\mathcal{C}_{(\mathcal{A},\subseteq)} = \bigcup\{\mathcal{C}_{\mathcal{P}} \mid \mathcal{P} \in \mathcal{H}(\mathcal{A})\}$ and that for each pair of $(\mathcal{A}, \leq_1), (\mathcal{A}, \leq_2) \in \mathcal{H}(\mathcal{A})$, $\mathcal{C}_{\leq_1}$ and $\mathcal{C}_{\leq_2}$ have no common interior points. By the Gruber-Ryshkov theorem [14], it suffices to show that if $\mathcal{C}_{\leq_1} \cap \mathcal{C}_{\leq_2}$ has codimension one, i.e., $\mathcal{C}_{\leq_1}$ and $\mathcal{C}_{\leq_2}$ are adjacent, then $\mathcal{C}_{\leq_1} \cap \mathcal{C}_{\leq_2}$ is the common facet of $\mathcal{C}_{\leq_1}$ and $\mathcal{C}_{\leq_2}$.

Suppose that $\mathcal{C}_{\leq_1} \cap \mathcal{C}_{\leq_2}$ has codimension one. By (2), there uniquely exists the pair of $A, B \in \mathcal{A}$ with $A \lessdot_1 B$ and $B \lessdot_2 A$ such that the linear hull of $\mathcal{C}_{\leq_1} \cap \mathcal{C}_{\leq_2}$ is given by $H_{A,B} := \{z \in \mathbf{R}^{\mathcal{A}} \mid z(A) = z(B)\}$. By (1) and (2), the corresponding facets of $\mathcal{C}_{\leq_1}$ and $\mathcal{C}_{\leq_2}$ are given as

$$\begin{aligned} H_{A,B} \cap \mathcal{C}_{\leq_1} &= H_{A,B} \cap \bigcup\{\mathcal{C}_{\leq_1^*} \mid \leq_1^* \text{ is a linear extension of } \leq_1 \text{ with } A \lessdot_1^* B\}, \\ H_{A,B} \cap \mathcal{C}_{\leq_2} &= H_{A,B} \cap \bigcup\{\mathcal{C}_{\leq_2^*} \mid \leq_2^* \text{ is a linear extension of } \leq_2 \text{ with } B \lessdot_2^* A\}. \end{aligned}$$

Take a sufficiently generic point $z \in \mathcal{C}_{\leq_1} \cap \mathcal{C}_{\leq_2}$, then z is ordered as

$$z(A_1) < z(A_2) < \ldots < z(A_{k-1}) < z(A) = z(B) < z(A_{k+2}) < \ldots < z(A_m).$$

From this, we define two linear orders $\leq_1^*$ and $\leq_2^*$ as

$$A_1 <_1^* A_2 <_1^* \ldots <_1^* A_{k-1} <_1^* A <_1^* B <_1^* A_{k+2} <_1^* \ldots <_1^* A_m,$$
$$A_1 <_1^* A_2 <_2^* \ldots <_2^* A_{k-1} <_2^* B <_2^* A <_2^* A_{k+2} <_2^* \ldots <_2^* A_m.$$

Then $\leq_1^*$ and $\leq_2^*$ are linear extensions of $\leq_1$ and $\leq_2$, respectively. By Lemma 3.13, for any linear extension $\leq_1^{**}$ of $\leq_1$ with $A \lessdot_1^{**} B$, the modified linear extension $\leq_2^{**}$ obtained by interchanging A and B of $\leq_1^{**}$ is a linear extension of $\leq_2$. This implies $H_{A,B} \cap \mathcal{C}_{\leq_1} \subseteq H_{A,B} \cap \mathcal{C}_{\leq_2}$. Similarly, we have $H_{A,B} \cap \mathcal{C}_{\leq_2} \subseteq H_{A,B} \cap \mathcal{C}_{\leq_1}$. Hence $\mathcal{C}_{\leq_1} \cap \mathcal{C}_{\leq_2} = H_{A,B} \cap \mathcal{C}_{\leq_1} = H_{A,B} \cap \mathcal{C}_{\leq_2}$ is the common facet of $\mathcal{C}_{\leq_1}$ and $\mathcal{C}_{\leq_2}$.

□

We call this polyhedral fan $\mathcal{N}(\mathcal{A})$ the *secondary greedy fan* of $\mathcal{A}$, which is an analogue of the *secondary fan* [13]. So it is natural to ask the following question.

Problem 3.14. Dose there exist some polyhedron $P \subseteq \mathbf{R}^{\mathcal{A}}$ whose normal fan coincides with $\mathcal{N}(\mathcal{A})$?

This problem is open. If such a polyhedron P exists, each edge vector of P is parallel to $\chi_{\{A\}} - \chi_{\{B\}}$ for some $A, B \in \mathcal{A}$. A well-known characterization of base polyhedra by edge directions [34], [12] implies that P is a base polyhedron associated with some (ordinary) submodular function defined on the set of upper ideals of the poset $(\mathcal{A}, \subseteq)$ (see [9]).

Corollary 3.15. *If $\mathcal{N}(\mathcal{A})$ is regular, then it is the normal fan of the base polyhedron with respect to some submodular function defined on the set of upper ideals of the poset $(\mathcal{A}, \subseteq)$.*

Remark 3.16. A set of posets on a common ground set whose associated set of order cones forms a polyhedral subdivision is called a *holometry*, which was introduced by Tomizawa [35] in 1983 as a combinatorial abstraction of normal fans of base polyhedra. Therefore $\mathcal{H}(\mathcal{A})$ is a holometry. In other words, $\mathcal{N}(\mathcal{A})$ is a coarsening of a subfan of the braid arrangement. It is shown in [30] that not every coarsening of the braid arrangement is regular.

Example 3.17. We give an example of holometry $\mathcal{H}(\mathcal{A})$ for $\mathcal{A} = \{12, 23, 34, 41, 1, 2, 3, 4\}$ and its realization of a base polyhedron. We consider the projection of $\mathcal{N}(\mathcal{A})$ into $\mathbf{R}^{\tilde{\mathcal{A}}}$ for $\tilde{\mathcal{A}} = \{12, 23, 34, 41\}$ by deleting every singleton, and draw its dual polyhedron in Figure 6. Note that this operation does not lose the original information of $\mathcal{N}(\mathcal{A})$.

We give a characterization of a member of $\mathcal{H}(\mathcal{A})$.

Proposition 3.18. *Let $\mathcal{P} = (\mathcal{A}, \leq)$ be a greedy poset. Then the following conditions are equivalent.*

(1) $\mathcal{P} \in \mathcal{H}(\mathcal{A})$.

(2) $\{$*linear extensions of* $\mathcal{P}\} = \{$*linear extension* $\mathcal{P}^*$ *of* $(\mathcal{A}, \subseteq) \mid \Delta_{\mathcal{P}^*} = \Delta_{\mathcal{P}}\}$.

(3) *Each pair $A, B \in \mathcal{A}$ with $A \lessdot B$ satisfies $A \cap B \neq \emptyset$ and $\{B\} = \Phi_{\mathcal{P}}(A \cup B)$.*

Proof. (1) $\Leftrightarrow$ (2) follows from Theorem 3.10.

(2) $\Rightarrow$ (3). For $A, B \in \mathcal{A}$ with $A \lessdot B$, $A \not\subseteq B$, and $B \not\subseteq A$, we choose some linear extension $\leq^*$ of $\leq$ satisfying

$$\cdots <^* A <^* B <^* \cdots . \tag{3.15}$$

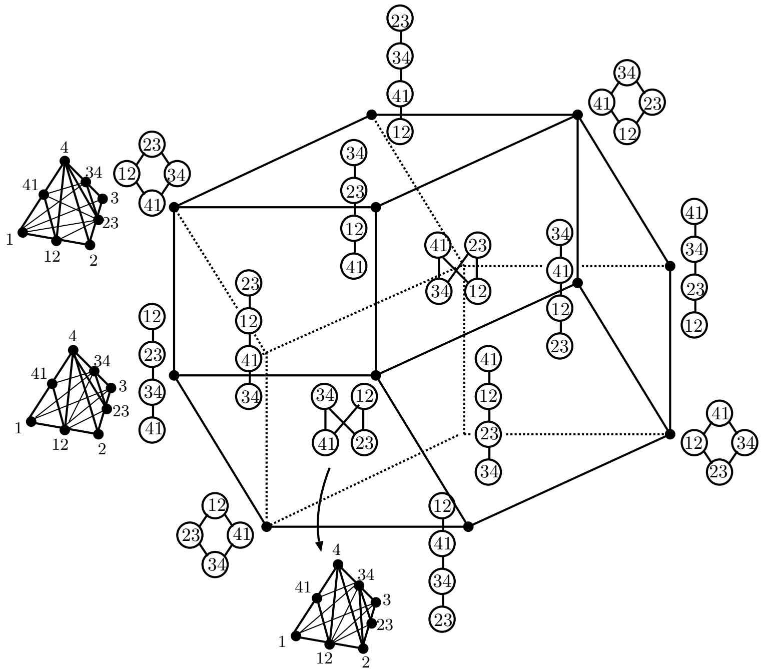

Figure 6: Holometry $\mathcal{H}(\mathcal{A})$ and its representation of a base polyhedron

Then we have $\Delta_{\leq} = \Delta_{\leq^*}$ by Proposition 3.3. Consider a linear order $\leq^{**}$ by interchanging A and B of $\leq^*$ as

$$\cdots <^{**} B <^{**} A <^{**} \cdots . \tag{3.16}$$

By construction, $\leq^{**}$ is not a linear extension of $\leq$. Suppose that A and B are disjoint. For any $X \subseteq V$ with $\{B\} = \Phi_{\leq^*}(X)$, we have $\Phi_{\leq^{**}}(X) = \{B\}$ or $\{A\}$. Hence, Proposition 2.23 implies $\Delta_{\leq^{**}} = \Delta_{\leq^*} = \Delta_{\leq}$. This is a contradiction. Suppose that A and B intersect and there exists $C \in \Phi_{\leq}(A \cup B)$ with $C \neq B$. Then C and B are incomparable or $B < C$. We can take the linear orders $<^*$ and $<^{**}$ above as

$$\cdots <^* A <^* B <^* \cdots <^* C <^* \cdots , \tag{3.17}$$

$$\cdots <^{**} B <^{**} A <^{**} \cdots <^{**} C <^{**} \cdots . \tag{3.18}$$

For any $X \subseteq V$ with $\{B\} = \Phi_{\leq^*}(X)$, we have $A \not\subseteq X$. Indeed, $A \subseteq X$ implies $C \subseteq X$ and this contradicts $\{B\} = \Phi_{\leq^*}(X)$. Hence we have $\{B\} = \Phi_{\leq^{**}}(X)$. From Proposition 2.23, we have $\Delta_{\leq^{**}} = \Delta_{\leq^*} = \Delta_{\leq}$. This is a contradiction.

(3) $\Rightarrow$ (2). Suppose that there exists a linear extension $\leq^*$ of $\subseteq$ such that $\Delta_{\leq} = \Delta_{\leq^*}$ and $\leq^*$ is not a linear extension of $\leq$. Then there exist a pair of $A, B \in \mathcal{A}$ such that $A \lessdot B$ and $B <^* A$. By the assumption, we have $A \cap B \neq \emptyset$ and $\{B\} = \Phi_{\leq}(A \cup B)$. Suppose $\{C\} = \Phi_{\leq^*}(A \cup B)$. Proposition 2.23 implies that $C = B$ or $C \cap B = \emptyset$. Furthermore, the nonemptiness of $A \cap B$ implies $A \neq C$, and therefore we have $A <^* C$. Hence, we have $C \cap B = \emptyset$ and $C \subset A$ (strict inclusion). However, $C \subset A$ contradicts $A <^* C$ (Lemma 3.2). $\square$

The condition (3) of Proposition 3.18 implies that each codimension 1 face of $\mathcal{N}(\mathcal{A})$ is given by the hyperplane $H_{A,B} := \{z \in \mathbf{R}^{\mathcal{A}} \mid z(A) = z(B)\}$ for some pair of $A, B \in \mathcal{A}$ having nonempty intersection. Consider the *intersection graph* $(\mathcal{A}, \mathcal{E})$ of $\mathcal{A}$ with its edge set $\mathcal{E}$ defined by $\{\{A, B\} \mid A, B \in \mathcal{A}, A \cap B \neq \emptyset\}$. Then the set of hyperplane $\{H_{A,B} \mid \{A, B\} \in \mathcal{E}\}$, so-called the *graphic arrangement* of $(\mathcal{A}, \mathcal{E})$, subdivides the order cone $\mathcal{C}_{(\mathcal{A},\subseteq)}$. This subdivision is denoted by $\mathcal{N}_0(\mathcal{A})$. Then there exists a bijection between the set of full dimensional members of $\mathcal{N}_0(\mathcal{A})$ and the set of acyclic orientations of $(\mathcal{A},\mathcal{E})$ satisfying the following condition:

$(*)$ for each $A, B \in \mathcal{A}$ with $A \subset B$, the orientation of the edge $\{A, B\}$ is $A \leftarrow B$.

See [26, Section 2.4] for graphic arrangements. From the arguments above, we have the following.

Corollary 3.19. *$\mathcal{N}(\mathcal{A})$ is a coarsening of $\mathcal{N}_0(\mathcal{A})$. In particular, the number of elements in $\mathcal{H}(\mathcal{A})$ is bounded by the number of acyclic orientations of $(\mathcal{A},\mathcal{E})$ satisfying the condition $(*)$ above.*

For subsequent subsections, we need the following lemma. A poset $(\mathcal{A}, \leq)$ has the *consecutive property* if for $A, B, C \in \mathcal{A}$ with $A < B < C$ we have $A \cap C \subseteq B$.

Lemma 3.20. *Let $\mathcal{P} = (\mathcal{A}, \leq)$ be a greedy poset. $\{A_1, A_2, \ldots, A_m\} \in (\hat{\Delta}_{\mathcal{P}})^*$ satisfies the following conditions.*

1. *There exists $C \in \mathcal{A}$ such that we have $C \subseteq \bigcup_{i=1}^m A_i$, $C > A_i$ and $A_i \cap C \neq \emptyset$ for $1 \leq i \leq m$.*
2. *If $\mathcal{P}$ has the consecutive property, $\{A_1, A_2, \ldots, A_m\}$ is pairwise incomparable.*

In particular, $(\hat{\Delta}_{\mathcal{P}})^*$ consists of minimal sets satisfying above conditions.

Proof. For $\{A_1, A_2, \ldots, A_m\} \subseteq \mathcal{A}$, a vector $\sum_{i=1}^m \chi_{A_i}$ can be uniquely represented as

$$\sum_{i=1}^{m} \chi_{A_i} = \sum_{A\in\mathcal{A}} \lambda(A)\chi_A \tag{3.19}$$

for nonnegative function $\lambda \in \mathbf{Z}_+^V$ satisfying $\operatorname{supp}\lambda \in \hat{\Delta}_{\mathcal{P}}$. Then $\{A_1, A_2, \ldots, A_m\}$ is not a member of $\hat{\Delta}_{\mathcal{P}}$ if and only if $\operatorname{supp}\lambda \neq \{A_1, A_2, \ldots, A_m\}$.

We show (1). Suppose that $\{A_1, A_2, \ldots, A_m\} \in (\hat{\Delta}_{\mathcal{P}})^*$. Take $C \in \Phi_{\mathcal{P}}(A_1 \cup A_2 \cup \cdots \cup A_m)$. If $C = A_k$ for some k, then we have $\lambda(A_k) > 1$ by `Dual_Greedy*`. Subtracting χ_{A_k} from both sides of (3.19), we see that $\{A_1, \ldots, A_m\} \setminus \{A_k\}$ is not a member of $\Delta_{\mathcal{P}}$. This contradicts the minimality of $(\hat{\Delta}_{\mathcal{P}})^*$. Hence $C \neq A_i$ holds for $i \in \{1, \ldots, m\}$. Next we show $C \cap A_i \neq \emptyset$ for $i \in \{1, \ldots, m\}$. If $C \cap A_k = \emptyset$ for some k, then we have $C \in \Phi_{\mathcal{P}}(\bigcup_{i\neq k} A_i)$ by (M1). Similarly, $\{A_1, \ldots, A_m\} \setminus \{A_k\}$ is not a member of $\Delta_{\mathcal{P}}$. This contradicts the minimality. Finally we show $C > A_i$ for $i \in \{1, \ldots, m\}$. Suppose that there exists k with $C \not> A_k$. Consider $\Phi_{\mathcal{P}}(C \cup A_k)$. Then, by (M1), we have $C \in \Phi_{\mathcal{P}}(C \cup A_k)$. The nonemptiness of $C \cap A_k$ implies that $A_k \notin \Phi_{\mathcal{P}}(C \cup A_k)$. By $C \not> A_k$, there exists $D \in \Phi_{\mathcal{P}}(C \cup A_k)$ such that $D \neq C$ and $D > A_k$. Hence, by (M1) we have $D \cap C \neq \emptyset$. This implies $D \subset A_k$. This is a contradiction. Hence we have (1).

Next we show (2). Suppose that $\mathcal{P}$ has the consecutive property. If $A_1 > A_2$ holds, then $C > A_1 > A_2$ implies $C \cap A_2 \subseteq A_1$. This implies $C \subseteq A_1 \cup A_3 \cup A_4 \cup \cdots \cup A_m$ and $C \in \Phi_{\mathcal{P}}(A_1 \cup A_3 \cup A_4 \cup \cdots \cup A_m)$. Hence $\{A_1 \cup A_3 \cup A_4 \cup \cdots \cup A_m\}$ is not a member of $\hat{\Delta}_{\mathcal{P}}$. This contradicts the minimality of $(\hat{\Delta}_{\mathcal{P}})^*$. $\square$

It would be interesting to characterize the case where $\hat{\Delta}_{\mathcal{P}}$ for a greedy poset $\mathcal{P}$ coincides with the *order complex* of $\mathcal{P}$. Recall that the order complex of $\mathcal{P}$ is the collection of chains of $\mathcal{P}$. Then, $\Delta_{\mathcal{P}}$ is a geometric realization of the order complex of $\mathcal{P}$. In this case, $\hat{\Delta}_{\mathcal{P}}$ and $(\hat{\Delta}_{\mathcal{P}})^*$ are given as

$$\hat{\Delta}_{\mathcal{P}} = \{\{A_1, A_2, \dots, A_m\} \subseteq \mathcal{A} \mid A_1 < A_2 < \cdots < A_m\}, \tag{3.20}$$
$$(\hat{\Delta}_{\mathcal{P}})^* = \{\{A, B\} \subseteq \mathcal{A} \mid A \text{ and } B \text{ are incomparable}\}. \tag{3.21}$$

In particular, LHS of $\Delta_{\mathcal{P}}$-submodularity inequalities consists of two terms.

Faigle and Kern [6] considered a general framework for dual greedy algorithms whose resulting output corresponds to a chain of a poset. The following characterization can be understood as an adaptation of [6, Theorem 6.1] to our approach.

Proposition 3.21. *Let $\mathcal{P}$ be a greedy poset on $\mathcal{A}$ satisfying the following condition:*

(1) $\#\Phi_{\mathcal{P}}(X) = 1$ *for any nonempty* $X \subseteq V$.

(2) $\mathcal{P}$ *has the consecutive property.*

Then $\hat{\Delta}_{\mathcal{P}}$ coincides with the order complex of $\mathcal{P}$.

Proof. We show (3.21). Since each element of $(\Delta_{\mathcal{P}})^*$ is pairwise incomparable by (2) and Lemma 3.20, it suffices to show that any incomparable pair $\{A, B\}$ is not a member of $\hat{\Delta}$. Consider $\Phi_{\mathcal{P}}(A \cup B)$. If $A \in \Phi_{\mathcal{P}}(A \cup B)$, then we have $\{A\} \in \Phi_{\mathcal{P}}(A \cup B)$ by (1). Hence we obtain $A > B$. This is a contradiction. Therefore we have $A, B \notin \Phi_{\mathcal{P}}(A \cup B)$. By an argument similar to that in the proof of Lemma 3.20, we conclude that $\{A, B\}$ is not a member of $\hat{\Delta}$. □

Remark 3.22. If $\mathcal{P}$ is a greedy poset satisfying the conditions (1) and (2) of Proposition 3.21, then any feasible sequence $\{(A_i, e_i)\}_{i=1}^n$ of `Dual_Greedy` using $\Phi_{\mathcal{P}}$ has the following property:

If $e_i \in A_j$ for $j \leq i$, then $e_i \in A_k$ for k with $j \leq k \leq i$.

This property is called the *consecutive* 1*'s property* of a *dual greedy basis matrix* $(\chi_{A_i} \mid 1 \leq i \leq n)$ in [10]. In this case, we see in Subsection 3.3 that this greedy poset $\mathcal{P}$ coincides with the poset of the set of extreme points of some convex geometry.

Remark 3.23. The converse direction of Proposition 3.21 is not true. Figure 7 illustrates an example of a nonconsecutive greedy poset $\mathcal{P}$ whose $\hat{\Delta}_{\mathcal{P}}$ coincides with the order complex of $\mathcal{P}$. Note that $13 < 34 < 1234$ violates the consecutive property.

3.2 Greedy Fans by Set Systems

Here, we discuss the case where $(\mathcal{A}, \subseteq)$ is a greedy poset, or equivalently, the associated holometry $\mathcal{H}(\mathcal{A})$ is a singleton, i.e., $\mathcal{H}(\mathcal{A}) = \{(\mathcal{A}, \subseteq)\}$. By Lemma 3.1, we obtain the following simple characterization.

Proposition 3.24. *$(\mathcal{A}, \subseteq)$ is greedy if and only if it satisfies the following two conditions:*

(S0) *for any* $e \in V$, *we have* $\{e\} \in \mathcal{A}$.

(S1) *for any intersecting pair* $A, B \in \mathcal{A}$, *we have* $A \cup B \in \mathcal{A}$.

The condition (S1) implies that $\Phi_{(\mathcal{A},\subseteq)}(X)$ forms the unique maximal partition of X, where "unique maximal" means that any partition $\Pi \subseteq 2^{\mathcal{A}}$ of X is a refinement of $\Phi_{(\mathcal{A},\subseteq)}(X)$, that is, for any $C \in \Pi$ there exists $C' \in \Phi_{(\mathcal{A},\subseteq)}(X)$ such that $C \subseteq C'$. Clearly $(\mathcal{A}, \subseteq)$ has the consecutive property. As an easy consequence of Lemma 3.20, the set of minimal nonmembers $(\hat{\Delta}_{(\mathcal{A},\subseteq)})^*$ and $\Delta_{(\mathcal{A},\subseteq)}$-submodularity inequalities are explicitly given as follows.

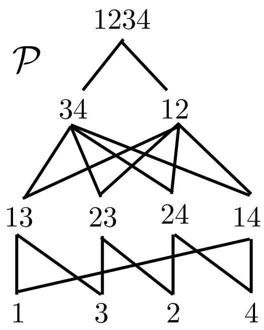

Figure 7: A nonconsecutive greedy poset $\mathcal{P}$ whose $\hat{\Delta}_{\mathcal{P}}$ coincides with the order complex of $\mathcal{P}$

Theorem 3.25. *Let $(\mathcal{A}, \subseteq)$ be a greedy poset. The set of minimal nonmembers $(\hat{\Delta}_{(\mathcal{A},\subseteq)})^*$ consists of*

$$\{\{A, B\} \subseteq \mathcal{A} \mid A \text{ and } B \text{ are intersecting}\} \tag{3.22}$$

and

$$\{\mathcal{F} \subseteq \mathcal{A} \mid \mathcal{F} \text{ is a minimal pairwise disjoint set satisfying } \textstyle\bigcup_{A \in \mathcal{F}} A \in \mathcal{A}\}. \tag{3.23}$$

$\Delta_{(\mathcal{A},\subseteq)}$-submodularity inequalities are given by

$$f(A) + f(B) \geq f(A \cup B) + \sum_{C \in \Phi_{(\mathcal{A},\subseteq)}(A \cap B)} f(C) \quad (A, B \in \mathcal{A} : \textit{intersecting}) \tag{3.24}$$

and

$$\sum_{A \in \mathcal{F}} f(A) \geq f(\bigcup_{A \in \mathcal{F}} A) \quad (\mathcal{F} \subseteq \mathcal{A} \textit{ in } (3.23)). \tag{3.25}$$

Example 3.26. In the case of $\mathcal{A} = 2^V \setminus \{\emptyset\}$, $\mathcal{A}$ clearly satisfies (S0),(S1). Then $\Delta_{(\mathcal{A},\subseteq)}$-submodularity inequalities coincide with the ordinary submodularity inequalities

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (A, B \in \mathcal{A}), \tag{3.26}$$

where we put $f(\emptyset) = 0$ for convenience; see Example 2.16.

In fact, $f : \mathcal{A} \to \mathbf{R}$ satisfying (3.24) and (3.25) in Theorem 3.25 can be extended to an ordinary submodular function $\tilde{f} : 2^V \to \mathbf{R}$. In addition, the polyhedron $P(f)$ coincides with a submodular polyhedron $P(\tilde{f})$. This implies that $\Delta_{(\mathcal{A},\subseteq)}$ is a coarsening of the barycentric subdivision; see Example 2.16.

Proposition 3.27. *Let $(\mathcal{A}, \subseteq)$ be a greedy poset and $f : \mathcal{A} \to \mathbf{R}$ a $\Delta_{(\mathcal{A},\subseteq)}$-submodular function. Then a function $\tilde{f} : 2^V \to \mathbf{R}$ defined as*

$$\tilde{f}(X) = \sum_{C \in \Phi_{(\mathcal{A},\subseteq)}(X)} f(C) \quad (X \in 2^V) \tag{3.27}$$

is an ordinary submodular function on 2^V. In particular, we have $P(f) = P(\tilde{f})$.

Proof. By definition, for $X, Y \subseteq V$, we have

$$\tilde{f}(X) + \tilde{f}(Y) = \sum_{C \in \Phi_{(\mathcal{A},\subseteq)}(X)} f(C) + \sum_{D \in \Phi_{(\mathcal{A},\subseteq)}(Y)} f(D). \tag{3.28}$$

Let $\mathcal{C}$ be a multiset which is the union (as a multiset) of $\Phi_{(\mathcal{A},\subseteq)}(X)$ and $\Phi_{(\mathcal{A},\subseteq)}(Y)$. If there exists an intersecting pair of $C', C'' \in \mathcal{C}$, by (3.24) we have

$$(3.28) = \sum_{C \in \mathcal{C}} f(C) \geq \sum_{C \in (\mathcal{C} \setminus \{C', C''\}) \cup \{C' \cup C''\} \cup \Phi(C' \cap C'')} f(C). \tag{3.29}$$

Put $\mathcal{C} \leftarrow (\mathcal{C} \setminus \{C', C''\}) \cup \{C' \cup C''\} \cup \Phi(C' \cap C'')$. Repeat this process to $\mathcal{C}$. After finitely many steps, there is no intersecting pair in $\mathcal{C}$. Then $\mathcal{C}$ is the union (as a multiset) of a partition $\mathcal{C}_1$ of $X \cap Y$ and a partition $\mathcal{C}_2$ of $X \cup Y$. By (3.25), we have

$$(3.28) \geq \sum_{C \in \mathcal{C}_1} f(C) + \sum_{D \in \mathcal{C}_2} f(D) \geq \tilde{f}(X \cap Y) + \tilde{f}(X \cup Y). \tag{3.30}$$

$\square$

Remark 3.28. The construction $f \to \tilde{f}$ of (3.27) can be understood as a variant of the *Dilworth truncation* for submodular functions; see [28, Chapter 48].

3.3 Greedy Fans by Abstract Convex Geometries

In this subsection, we establish the relationship between our acyclic greedy fan approach and dual greedy systems on convex geometry considered by Kashiwabara and Okamoto [19], and give another systematic proof of validity of their dual greedy algorithm.

First, we introduce some basic definitions of theory of convex geometries; see [2] and [20] for details. Let V be a finite set. A family $\mathcal{L} \subseteq 2^V$ is said to be a *convex geometry* if it satisfies

(CG1) $\emptyset, V \in \mathcal{L}$,

(CG2) $X, Y \in \mathcal{L} \Rightarrow X \cap Y \in \mathcal{L}$, and

(CG3) $X \in \mathcal{L} \setminus \{V\} \Rightarrow \exists e \in V \setminus X, X \cup \{e\} \in \mathcal{L}$.

From conditions (CG1) and (CG2), we can define the *closure operator* $\tau : 2^V \to \mathcal{L}$ as

$$\tau(X) = \bigcap \{Y \in \mathcal{L} \mid X \subseteq Y\} \quad (X \in 2^V). \tag{3.31}$$

The *extreme operator* $\mathrm{ex} : \mathcal{L} \to 2^V$ is defined as

$$\mathrm{ex}(X) = \bigcup \{e \in X \mid X \setminus \{e\} \in \mathcal{L}\} \quad (X \in \mathcal{L}). \tag{3.32}$$

Then the following property is fundamental; see [2] and [20].

Lemma 3.29. *$\tau \circ \mathrm{ex}(X) = X$ for $X \in \mathcal{L}$ and $\mathrm{ex} \circ \tau(A) = A$ for $A \in \mathrm{ex}(\mathcal{L})$.*

In particular, τ is a bijection between $\mathcal{L}$ and $\mathrm{ex}(\mathcal{L})$. Hence, we can define a poset $(\mathrm{ex}(\mathcal{L}), \leq)$ as

$$A \leq B \overset{\text{def}}{\iff} \tau(A) \subseteq \tau(B) \quad (A, B \in \mathrm{ex}(\mathcal{L})). \tag{3.33}$$

We give a characterization of the poset $(\mathrm{ex}(\mathcal{L}), \leq)$ in terms of greedy posets as follows, which can be understood as a refinement of [10, Theorem 3.2].

Proposition 3.30. *Let $\mathcal{P}$ be a poset on $\mathcal{A} \subseteq 2^V$ with $\{\emptyset\} \in \mathcal{A}$. Then $\mathcal{P}$ coincides with* $(\mathrm{ex}(\mathcal{L}), \leq)$ *for some convex geometry $\mathcal{L}$ if and only if $\mathcal{P}$ is a greedy poset satisfying the conditions* (1) *and* (2) *of Proposition* 3.21.

For the proof of this, we need some lemmas.

Lemma 3.31 ([19, Lemma 2.2]). *For $A \in \mathrm{ex}(\mathcal{L})$ and $B \subset A$, we have $B \in \mathrm{ex}(\mathcal{L})$ and $B < A$.*

From this lemma and (CG3), we see that $\mathrm{ex}(\mathcal{L})$ has every singleton of V. Furthermore, the multiple-choice function associated $(\mathrm{ex}(\mathcal{L}), \leq)$ is given as follows.

Lemma 3.32. $\Phi_{(\mathrm{ex}(\mathcal{L}),\leq)}(X) = \{\mathrm{ex} \circ \tau(X)\} \quad (X \subseteq V)$.

Proof. Take any $A \in \mathrm{ex}(\mathcal{L})$ with $A \subseteq X$. Then we have $\tau(A) \subseteq \tau(X)$. From Lemma 3.29, we have $A = \mathrm{ex} \circ \tau(A) \leq \mathrm{ex} \circ \tau(X) \subseteq X$. This implies $\Phi_{(\mathrm{ex}(\mathcal{L}),\leq)}(X) = \{\mathrm{ex} \circ \tau(X)\}$. $\square$

Lemma 3.33. $(\mathrm{ex}(\mathcal{L}), \leq)$ *has the consecutive property.*

Proof. Suppose that $A, B, C \in \mathcal{L}$ with $B \subseteq C \subseteq A$. We show that if $e \in \mathrm{ex}(A) \cap \mathrm{ex}(B)$, then $e \in \mathrm{ex}(C)$. By $e \in \mathrm{ex}(B)$, we have $e \in C$. From this, we have $(A \setminus e) \cap C = (A \cap C) \setminus e = C \setminus e \in \mathcal{L}$. Hence we obtain $e \in \mathrm{ex}(C)$. $\square$

Proof of Proposition 3.30. The only-if part has done. Since $\Phi_{\mathcal{P}}$ can be regarded as a choice function by Lemma 3.32, the if part follows from [10, Theorem 3.2] and the consecutive 1's property of $\Phi_{\mathcal{P}}$ (Remark 3.22). $\square$

In particular, for convex geometry $\mathcal{L}$, $(\mathrm{ex}(\mathcal{L}), \leq)$ is greedy. Therefore, $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$ is an acyclic greedy fan and `Dual_Greedy` using $\Phi_{(\mathrm{ex}(\mathcal{L}),\leq)}$ coincides with the dual greedy algorithm using $\mathrm{ex} \circ \tau$ considered by Kashiwabara and Okamoto [19]. Since $\hat{\Delta}_{(\mathrm{ex}(\mathcal{L}),\leq)}$ coincides with the order complex of $(\mathrm{ex}(\mathcal{L}), \leq)$, the set of minimal nonmembers $(\hat{\Delta}_{(\mathrm{ex}(\mathcal{L}),\leq)})^*$ and $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$-submodularity inequalities are given as follows.

Proposition 3.34. *The set of all minimal nonmembers $(\hat{\Delta}_{(\mathrm{ex}(\mathcal{L}),\leq)})^*$ is given as*

$$(\hat{\Delta}_{(\mathrm{ex}(\mathcal{L}),\leq)})^* = \{\{A, B\} \mid A, B \in \mathrm{ex}(\mathcal{L}), A \text{ and } B \text{ are incomparable } \} \tag{3.34}$$

and $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$-submodularity inequalities are given by

$$f(A) + f(B) \geq f(C_1) + \cdots + f(C_k) \quad (A, B \in \mathrm{ex}(\mathcal{L}) : \textit{incomparable}), \tag{3.35}$$

where $C_1, C_2, \ldots C_k \in \mathrm{ex}(\mathcal{L})$ are pairwise comparable and satisfy

$$\chi_A + \chi_B = \chi_{C_1} + \chi_{C_2} + \cdots + \chi_{C_k}. \tag{3.36}$$

In particular, `Dual_Greedy` works for $\mathrm{D}_{(\mathrm{ex}(\mathcal{L}),f,w)}$ if and only if f is $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$-submodular. In general, $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$-submodularity inequalities are redundant. Kashiwabara and Okamoto [19] gave a fewer number of inequalities which guarantee $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$-submodularity as follows, where we define three binary operators $A \vee B := \mathrm{ex}(\tau(A) \cup \tau(B))$, $A \sqcap B := (A \cup B) \cap \mathrm{ex}(\tau(A) \cap \tau(B))$, and $A \diamond B := (A \cap B) \setminus A \vee B$.

Theorem 3.35 ([19]). *For a function $f : \mathrm{ex}(\mathcal{L}) \to \mathbf{R}$,* `Dual_Greedy` *using $\mathrm{ex} \circ \tau$ solves the linear program $\mathrm{D}_{(\mathrm{ex}(\mathcal{L}),f,w)}$ for any nonnegative cost vector w if and only if f satisfies*

$$f(A) + f(B) \geq f(A \vee B) + f(A \sqcap B) + f(A \diamond B) \tag{3.37}$$

for $A, B \in \mathrm{ex}(\mathcal{L})$ satisfying $\chi_A + \chi_B = \chi_{A \vee B} + \chi_{A \sqcap B} + \chi_{A \diamond B}$, where $f(\emptyset) = 0$ is assumed.

Note that we have $A \vee B, A \sqcap B, A \diamond B \in \mathrm{ex}(\mathcal{L})$ by Lemma 3.31. Furthermore [19, Lemma 2.5] shows $A \vee B \geq A \sqcap B \geq A \diamond B$ for $A, B \in \mathrm{ex}(\mathcal{L})$. Therefore, the inequalities (3.37) are contained in (3.35). Here, we prove that a further fewer number of inequalities guarantee submodularity as follows.

Theorem 3.36. *$f : \mathrm{ex}(\mathcal{L}) \to \mathbf{R}$ is $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$-submodular if and only if f satisfies (3.37) for each pair $A, B \in \mathrm{ex}(\mathcal{L})$ satisfying*

$$A = \mathrm{ex}(X \cup \{i\}),\ B = \mathrm{ex}(X \cup \{j\}) \tag{3.38}$$

for some $X \in \mathcal{L}$ and $i, j \in V \setminus X$ with $X \cup \{i\}, X \cup \{j\}, X \cup \{i,j\} \in \mathcal{L}$.

Proof. For any function $f : \mathrm{ex}(\mathcal{L}) \to \mathbf{R}$, consider the continuous piecewise linear function $\hat{f}$ defined by (2.11). Recall that f is $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$-submodular if and only if `Dual_Greedy` using $\mathrm{ex} \circ \tau$ solves linear program $\mathrm{D}_{(\mathrm{ex}(\mathcal{L}),f,w)}$ for every nonnegative cost vector w. The latter condition is equivalent to the convexity of $\hat{f}$; see the proof of Proposition 2.8 and [10, Theorem 2.1]. The convexity of this piecewise linear function $\hat{f}$ can be guaranteed by the local convexity condition [18, Lemma 14] that for each two adjacent full dimensional simplices $\{A_1, A_2, \dots A_{n-1}, A_n\}, \{A_1, A_2, \dots, A_{n-1}, B\} \in \hat{\Delta}_{(\mathrm{ex}\,\mathcal{L},\leq)}$, f satisfies

$$\det \begin{pmatrix} \chi_{A_1} & \cdots & \chi_{A_n} \end{pmatrix} \det \begin{pmatrix} \chi_{A_1} & \cdots & \chi_{A_n} & \chi_B \\ f(A_1) & \cdots & f(A_n) & f(B) \end{pmatrix} \geq 0. \tag{3.39}$$

These inequalities (3.39) and $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$-submodular inequalities (3.35) define the same full dimensional polyhedral cone. Hence, the common inequalities contain all facets of the cone of $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$-submodular functions. Note that the coefficient of the inequality (3.39) coincides with a linear dependence vector of $\{\chi_{A_1}, \dots, \chi_{A_n}, \chi_B\}$, which is uniquely determined up to constant multiple.

The inequality (3.35) is contained in (3.39) if and only if $A, B, C_1, C_2, \dots$ in (3.35) are contained in the union of some two adjacent simplices of $\hat{\Delta}_{(\mathrm{ex}(\mathcal{L}),\leq)}$. Indeed, the only-if part is obvious. The if part follows from the fact that the coefficients of $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$-submodularity inequalities is also a linear dependence vector of $\{\chi_A, \chi_B, \chi_{C_1}, \chi_{C_2}, \dots\}$. Since two adjacent full dimensional simplices of $\Delta_{(\mathrm{ex}\,\mathcal{L},\leq)}$ correspond to two adjacent maximal chains of $(\mathrm{ex}(\mathcal{L}), \leq)$, the common inequalities of (3.35) and (3.39) are given by (3.35) for $A, B \in \mathrm{ex}(\mathcal{L})$ as (3.38).

Finally we show $\chi_A + \chi_B = \chi_{A \vee B} + \chi_{A \sqcap B} + \chi_{A \diamond B}$. For this, we show the following.

$$\begin{aligned} A \vee B &= \mathrm{ex}(X \cup \{i,j\}), && (3.40) \\ (A \cap B) \cup \{i,j\} &\supseteq \mathrm{ex}(X \cup \{i,j\}), && (3.41) \\ (A \cup B) \setminus \{i,j\} &\subseteq \mathrm{ex}(X). && (3.42) \end{aligned}$$

Indeed, (3.40) is obvious. We show (3.41). If $f \in \mathrm{ex}(X \cup \{i,j\}) \setminus \{i,j\}$, we have $(X \cup \{i,j\} \setminus \{f\}) \cap (X \cup \{i\}) = (X \cup \{i\}) \setminus \{f\} \in \mathcal{L}$ and $(X \cup \{i,j\} \setminus \{f\}) \cap (X \cup \{j\}) = (X \cup \{j\}) \setminus \{f\} \in \mathcal{L}$. Hence we have $f \in A \cap B$. We show (3.42). Take $g \in A \cup B \setminus \{i,j\}$. If $g \in A \setminus B$, we have $(X \cup \{i\} \setminus \{g\}) \cap (X \cup \{j\}) = X \setminus \{g\} \in \mathcal{L}$. This implies $g \in \mathrm{ex}(X)$. Similarly, if $g \in B \setminus A$, we have $e \in \mathrm{ex}(X)$. If $g \in A \cap B$, $(X \cup \{i\} \setminus \{g\}) \cap (X \cup \{j\} \setminus \{g\}) = X \setminus \{g\} \in \mathcal{L}$. This implies $g \in \mathrm{ex}(X)$. Hence we have (3.42) and therefore $A \sqcap B = (A \cup B) \setminus \{i,j\}$. We obtain $\chi_A + \chi_B = \chi_{(A \cap B) \cup \{i,j\}} + \chi_{(A \cup B) \setminus \{i,j\}} = \chi_{A \vee B} + \chi_{(A \cap B) \cup \{i,j\} \setminus A \vee B} + \chi_{A \sqcap B} = \chi_{A \vee B} + \chi_{A \diamond B} + \chi_{A \sqcap B}$, where the second equality follows from (3.41) and $A \sqcap B = (A \cup B) \setminus \{i,j\}$, and the third follows from $i, j \in A \vee B$. ∎

If $\mathcal{L}$ is union closed, then $\mathcal{L}$ is a distributive lattice. In this case, (3.35) is explicitly written and coincides with the *b-submodularity inequality* in the sense of Krüger [21], which is obtained by the relation $\chi_A + \chi_B = \chi_{A\vee B} + \chi_{A\sqcap B}$; see [1, Fig.1] for binary operators $A \vee B$, $A \sqcap B$.

Proposition 3.37 ([21], see also [19]). *If $\mathcal{L}$ is a distributive lattice, $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$-submodularity inequalities* (3.35) *are given by*

$$f(A) + f(B) \geq f(A \vee B) + f(A \sqcap B) \tag{3.43}$$

for each incomparable pair $A, B \in \mathrm{ex}(\mathcal{L})$, where we assume $f(\emptyset) = 0$.

Remark 3.38. If $\mathcal{L}$ is a distributive lattice, it can be represented as the set of ideals of some poset. Then the resulting greedy fan $\Delta_{(\mathrm{ex}(\mathcal{L}),\leq)}$ is essentially the same as the canonical triangulation of the *chain polytope* of this poset given by Stanley [31, Section 5].

Remark 3.39. Consider the case $\mathcal{L} = 2^V$. Then $(\mathrm{ex}(\mathcal{L}), \leq) = (2^V, \subseteq)$ and (3.35) coincides with (ordinary) submodularity inequalities (3.26). In this case, the inequalities for (3.38) in Theorem 3.36 coincide with the *local submodularity inequalities*

$$f(X \cup \{i\}) + f(X \cup \{j\}) \geq f(X) + f(X \cup \{i, j\}) \tag{3.44}$$

for $X \subseteq V$ and $i, j \in V \setminus X$. It is well-known that the local submodularity guarantees submodularity. Hence, the condition (3.38) in Theorem 3.36 can be understood as a generalization of the local submodularity condition.

3.4 Dual Greedy System by Frank

In this subsection, we analyze Frank's dual greedy system [7], establish a connection to acyclic greedy fans and give another proof of the validity of his dual greedy algorithm.

First, we briefly summarize Frank's model [7]. Let $(\mathcal{F}, \preceq)$ be a poset on a nonempty finite set $\mathcal{F}$. A pair of $a, b \in \mathcal{F}$ is said to be *intersecting* if a and b are incomparable and there exists a member $c \in \mathcal{F}$ such that $c \prec a$ and $c \prec b$.

Two binary operations $\vee$, $\wedge$ are defined on comparable pairs and on intersecting pairs, with the following properties.

(F1) If $a \prec b$, then $a \wedge b = a$, $a \vee b = b$.

(F2) If a and b are intersecting, then $a \wedge b \prec a, b$ and $a \vee b \succ a, b$.

In addition, we are given a set V and a function $\phi : \mathcal{F} \to 2^V$ satisfying

(F3) If $a \prec b \prec c$, then $\phi(a) \cup \phi(b) \subseteq \phi(c)$,

(F4) If a, b are intersecting, and then $\phi(a \vee b) \cup \phi(a \wedge b) \subseteq \phi(a) \cup \phi(b)$.

(F5) If $\phi(a) \cap \phi(b) \neq \emptyset$, then A, B are intersecting or comparable.

A nonnegative function $f : \mathcal{F} \to \mathbf{R}_+$ is *intersecting supermodular* if f satisfies

$$f(a) + f(b) \leq f(a \vee b) + f(a \wedge b) \tag{3.45}$$

for intersecting pair of a, b with $f(a) > 0$, $f(b) > 0$. Function f is said to be *decreasing* if $a \preceq b$ implies $f(a) \geq f(b)$.

We consider the following dual pair of linear programs for a function f and a nonnegative cost vector $w \in \mathbf{R}^V_+$.

$$
\begin{array}{lll|lll}
\mathrm{P}': & & & \mathrm{D}': & & \\
\min. & \displaystyle\sum_{e \in V} w(e)x(e) & & \max. & \displaystyle\sum_{a \in \mathcal{F}} \lambda(a) f(a) & \\
\text{s.t.} & \displaystyle\sum_{e \in \phi(a)} x(e) \geq f(a) \ (a \in \mathcal{F}), & & \text{s.t.} & \displaystyle\sum_{a \in \mathcal{F}} \lambda(a)\chi_{\phi(a)} \leq w, & \\
& x(e) \geq 0 \ (e \in V), & & & \lambda(a) \geq 0 \ (a \in \mathcal{F}), &
\end{array}
\tag{3.46}
$$

where for feasibility of [P′] we assume that $\phi(a)$ is nonempty for $a \in \mathcal{F}$ with $f(a) > 0$. We define a multiple-choice function Φ associated with $(\mathcal{F}, \leq)$ and f as

$$\Phi(X) = \{\phi(a) \mid a \in \mathcal{F} \text{ is a minimal element satisfying } \phi(a) \subseteq X \text{ and } f(a) > 0\} \tag{3.47}$$

for $X \subseteq V$. Using this Φ, we modify **step1** of `Dual_Greedy` as

step1': If $\Phi(X)$ is empty, then stop.

Then Frank [7] shows the following.

Theorem 3.40 ([7]). *If f is intersecting supermodular and decreasing, the modified* `Dual_Greedy` *with* **step1'** *gives an optimal solution to* D′ *for any nonnegative cost vector $w \in \mathbf{R}^V_+$.*

In the following, we reduce Frank's model to our framework and give another proof of Theorem 3.40. For a decreasing function f, we define subsets $\mathcal{F}_+, \mathcal{F}' \subseteq \mathcal{F}$ as

$$
\begin{aligned}
\mathcal{F}_+ &= \{a \in \mathcal{F} \mid f(a) > 0\}, && (3.48)\\
\mathcal{F}' &= \{a \in \mathcal{F}_+ \mid \forall b \in \mathcal{F}_+ : b \prec a \Rightarrow \phi(b) \not\subseteq \phi(a)\}. && (3.49)
\end{aligned}
$$

Lemma 3.41. *The restriction $\phi|_{\mathcal{F}'}$ of ϕ to $\mathcal{F}'$ is injective.*

Proof. If there exist distinct $a, b \in \mathcal{F}'$ such that $\phi(a) = \phi(b)$, then by (F5) a and b are intersecting or comparable. If a and b are comparable, this contradicts the definition of $\mathcal{F}'$. If a and b are intersecting, then (F4), (F2), and the decreasing property of f imply $a, b \succ a \wedge b \in \mathcal{F}_+$ and $\phi(a \wedge b) \subseteq \phi(a) = \phi(b)$. This also contradicts the definition of $\mathcal{F}'$. □

From this, we can define a poset $\mathcal{P} = (\mathcal{A}, \leq)$ as

$$\mathcal{A} = \phi(\mathcal{F}'), \quad A \leq B \overset{\text{def}}{\Longleftrightarrow} (\phi|_{\mathcal{F}'})^{-1}(A) \succeq (\phi|_{\mathcal{F}'})^{-1}(B) \quad (A, B \in \mathcal{A}). \tag{3.50}$$

By construction, we have $\Phi = \Phi_{(\mathcal{A},\leq)}$. Furthermore, elements of $\Phi_{(\mathcal{A},\leq)}(X)$ are disjoint for $X \subseteq V$. Indeed, if $A, B \in \Phi_{(\mathcal{A},\leq)}(X)$ intersect, then (F4), (F5), and decreasing property of f imply that $a := (\phi|_{\mathcal{F}'})^{-1}(A)$ and $b := (\phi|_{\mathcal{F}'})^{-1}(B)$ satisfy $a \wedge b \in \mathcal{F}_+$ and $\phi(a \wedge b) \subseteq A \cup B$. Then there exists $c \in \mathcal{F}'$ such that $c \preceq a \wedge b \preceq a, b$ and $\phi(c) \subseteq \phi(a \wedge b) \subseteq A \cup B \subseteq X$. This inclusion contradicts $A, B \in \Phi_{(\mathcal{A},\leq)}(X)$.

Hence, extending $(\mathcal{A}, \leq)$ to $(\overline{\mathcal{A}}, \leq)$ by adding singletons not contained in $\mathcal{A}$ as

$$\overline{\mathcal{A}} = \mathcal{A} \cup \{\{e\} \mid e \in V, \{e\} \notin \mathcal{A}\}, \quad \{e\} < A \overset{\text{def}}{\Longleftrightarrow} \{e\} \notin \mathcal{A},\ A \in \mathcal{A} : e \in A, \tag{3.51}$$

we have the following.

Lemma 3.42. $(\overline{\mathcal{A}}, \leq)$ *is greedy.*

Let $\overline{f} : \overline{\mathcal{A}} \to \mathbf{R}$ be defined by

$$\overline{f}(A) = \begin{cases} f((\phi|_{\mathcal{F}'})^{-1}(A)) & \text{if } A \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.52}$$

Then we consider the linear programs $\mathrm{P}_{(\overline{\mathcal{A}},-\overline{f},w)}$ and $\mathrm{D}_{(\overline{\mathcal{A}},-\overline{f},w)}$. By construction and decreasing property of f, we can easily check that $\mathrm{P}_{(\overline{\mathcal{A}},-\overline{f},w)}$ and $\mathrm{D}_{(\overline{\mathcal{A}},-\overline{f},w)}$ are equivalent to P' and D'. Since $\Phi(X) \subseteq \Phi_{(\overline{\mathcal{A}},\leq)}(X)$ for nonempty $X \subseteq V$, `Dual_Greedy` with **step1'** using Φ works for D' if and only if `Dual_Greedy` using $\Phi_{(\overline{A},\leq)}$ works for $\mathrm{D}_{(\overline{\mathcal{A}},-\overline{f},w)}$. We show that $-\overline{f}$ satisfies $\Delta_{(\overline{A},\leq)}$-submodularity inequalities. For this, we investigate the set of minimal nonmembers $(\hat{\Delta}_{(\overline{A},\leq)})^*$.

Lemma 3.43. *The set of minimal nonmembers $(\hat{\Delta}_{(\overline{A},\leq)})^*$ consists of the following type of sets:*

type 1 $\{\phi(a), \phi(b)\}$ *for intersecting pair $a, b \in \mathcal{F}'$.*

type 2 $\{A, \{e_1\}, \ldots, \{e_j\}\}$ *for some $A \in \mathcal{A}$ and added singletons $\{e_i\} \notin \mathcal{A}$.*

type 3 $\{\{e_1\}, \ldots, \{e_k\}\}$ *for some added singletons $\{e_i\} \notin \mathcal{A}$.*

Proof. Note that by (F3) and the construction, $(\overline{A}, \leq)$ has the consecutive property. If $\mathcal{C} \in (\hat{\Delta}_{(\overline{A},\leq)})^*$ contains two elements $A, B \in \mathcal{A}$, then A, B are incomparable and have a common upper bound by Lemma 3.20. In particular, $a := (\phi|_{\mathcal{F}})^{-1}(A)$ and $b := (\phi|_{\mathcal{F}})^{-1}(B)$ are intersecting. By (F4) and the decreasing property of f, we have $a \wedge b \in \mathcal{F}_+$, $a \wedge b \prec a, b$ and $\phi(a \wedge b) \subseteq A \cup B$. Hence there exists $C \in \Phi_{(\overline{\mathcal{A}},\leq)}(A \cup B)$ such that $C \subseteq \phi(a \wedge b) \subseteq A \cup B$ and $C > A, B$. Hence $\{A, B\}$ is not a member of $\hat{\Delta}_{(\overline{A},\leq)}$ and coincides with $\mathcal{C}$. □

Finally, we verify that $-\overline{f}$ satisfies $\Delta_{(\overline{A},\leq)}$-submodularity inequalities (2.6) corresponding to type 1, 2, and 3. For $\Delta_{(\overline{A},\leq)}$-submodularity inequalities corresponding to type 2 and 3, we can easily check them by the nonnegativity and the decreasing property of f and $\overline{f}(\{e\}) = 0$ for added singleton $\{e\}$.

Consider $\Delta_{(\overline{A},\leq)}$-submodularity inequalities for type 1. The LHS of (2.6) is given by $-\overline{f}(\phi(a)) - \overline{f}(\phi(b)) = -f(a) - f(b)$ for intersecting pair $a, b \in \mathcal{F}'$. We show that there exist terms $\overline{f}(C')$ and $\overline{f}(C'')$ in RHS of (2.6) such that $f(a \wedge b) \geq \overline{f}(C')$ and $f(a \vee b) \geq \overline{f}(C'')$. By nonnegativity and intersecting supermodularity of f, this implies $\Delta_{(\overline{A},\leq)}$-submodularity of $-\overline{f}$.

By (F5), we have $\phi(a \wedge b) \subseteq \phi(a) \cup \phi(b)$. Then there exists $C' \in \Phi_{(\overline{\mathcal{A}},\leq)}(\phi(a) \cup \phi(b))$ such that $(\phi|_{\mathcal{F}'})^{-1}(C') \preceq a \wedge b$. By the consecutive property, we have $C' \subseteq \phi(a \wedge b)$. Consider `Dual_Greedy*` for $w = \chi_{\phi(a)} + \chi_{\phi(b)}$. Then in the first iteration, we can take $A^* = C'$ and $\lambda(C') = 1$ in **step 2**. Hence $-\overline{f}(C')$ appears in RHS of (2.6) and satisfies $\overline{f}(C') \leq f(a \wedge b)$ by the decreasing property.

By $C' \subseteq \phi(a \wedge b)$, the consecutive property, and (F5), we have $\chi_{\phi(a)} + \chi_{\phi(b)} - \chi_{C'} - \chi_{\phi(a \vee b)} \geq \chi_{\phi(a)} + \chi_{\phi(b)} - \chi_{\phi(a \wedge b)} - \chi_{\phi(a \vee b)} \geq 0$. In particular, $z := \chi_{\phi(a)} + \chi_{\phi(b)} - \chi_{C'}$ satisfies $\phi(a \vee b) \subseteq \operatorname{supp} z$. By consecutive property and the definition of $\mathcal{F}'$ there exists $c \in \mathcal{F}'$ with $c \preceq a \vee b$ such that $\phi(c) \subseteq \phi(a \vee b)$ and $\phi(c) \in \Phi_{(\overline{A},\leq)}(\operatorname{supp} z)$. In the second iteration of `Dual_Greedy*`, we can choose $\phi(c) = A^*$ in **step 2** such that $\lambda(\phi(c)) \geq 1$. Hence $-\overline{f}(\phi(c))$ appears in RHS of (2.6) and satisfies $\overline{f}(\phi(c)) = f(c) \geq f(a \vee b)$ by decreasing property of f.

Acknowledgements

The author thanks Satoru Fujishige for helpful comments and suggesting Tomizawa's work [35] and Sohoni's work [29], and H. Narayanan for helpful suggestion about polyhedral tightness. This work is supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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