Log Minimal Model Program and Affine Algebraic Threefolds

By

Takashi KISHIMOTO

Abstract

This article is intended to a refinement and a generalization of the paper [Ki05]. We shall investigate the (biregular) structures of affine algebraic threefolds X from a point of view of 3-dimensional Log Minimal Model Program $(\mathsf{LMMP})_3$. For this purpose, we need to take a compactification $X \hookrightarrow (V, D)$ and perform $(\mathsf{LMMP})_3$, which starts with the dlt pair (V, D), describing how the inside affine part X changes via the process of $(\mathsf{LMMP})_3$. We succeed in this attempt in the case where the linear system $|D|$ contains a Du Val member.

1. Introduction

Throughout the present article we work over the field of complex numbers $\mathbb{C}$. The theory of affine algebraic surfaces has been developed around 1980's due to S. Iitaka, M. Miyanishi, T. Sugie, Y. Kawamata, T. Fujita, F. Sakai and so on (cf. [Kaw79], [Mi-Su80], [Fuj79, Fuj82], [Sak87], [Miy81, Miy01]). This development is indebted to the concept of log Kodaira dimension defined by S. Iitaka (cf. [Ii77]) and making use of the classification theory of projective surfaces. More precisely to say, in order to investigate the biregular structure on a smooth affine algebraic surface Y, taking a compactification $Y \hookrightarrow (W, B)$ into a smooth projective surface W with a simple normal crossing (= SNC, for short) boundary divisor B, and applying the minimal model theory of surfaces to obtain the data on W and B are the crucial steps. On the other hand, since there exists a powerful theory (Mori theory, (Log) Minimal Model Program) for the birational classification of projective threefolds (with boundaries) with certain kinds of singularities (cf. [Mori88], [Sho93], [FA], [Ko-Mo98]), it seems to be natural that 3-dimensional Log Minimal Program (= $(\mathsf{LMMP})_3$, for short) plays the substantial roles for the study of affine algebraic threefolds. Namely, the framework of the idea is stated as follows:

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Let X be a smooth affine algebraic threefold. We embed X into a smooth projective threefold V in such a way that the (reduced) boundary divisor D is SNC. By virtue of the framework of $(\mathsf{LMMP})_3$ (cf. [Sho93],[FA],[Ko-Mo98]), starting with this dlt pair $(V, D) \in (\mathsf{Dlt})_3$, we perform $(\mathsf{LMMP})_3$, say:

$$(*) \quad \phi : (V, D) \overset{\phi^0}{\cdots\to} (V^1, D^1) \overset{\phi^1}{\cdots\to} \cdots\cdots \to (V^{s-1}, D^{s-1}) \overset{\phi^{s-1}}{\cdots\to} (V^s, D^s),$$

where the final object $(V', D') := (V^s, D^s)$ is either a Log Mori fiber space (= $(\mathsf{LMfs})_3$, for short) or a log minimal model, i.e., $K_{V'} + D'$ is nef, according to the value of log Kodaira dimension $\overline{\kappa}(X)$ of X. Set $X' := V' \backslash \mathrm{Supp}\,(D')$. Usually, since (V', D') has a distinguished simple structure compared with the original pair (V, D), we expect that we are able to analyze the structure of X' in detail. Hence, if we can compare X with X', then we obtain the data on X from those on X'. But, in general, there occur some crucial obstacles in this strategy. These obstacles are caused by the difference of the points of view of Birational Geometry and Affine Algebraic Geometry. Namely, since the main interest in Birational Geometry lies in birational properties of algebraic varieties, the existence and termination of flips and the Abundance are the most important problems. Once these three are established, the birational properties on projective varieties can be reduced to those on more simple varieties (i.e., (log) Mori fiber spaces or (log) minimal models). On the other hand, since the main interest in Affine Algebraic Geometry lies in *biregular* properties of affine algebraic varieties, we have to investigate all divisorial contractions and (log) flips, which do not take place in the boundary parts, appearing in the process of (Log) Minimal Model Program, that is, we have to describe how exceptional divisors and flipping/flipped curves intersect the boundary parts. Thus, the obstacles occuring when we try to compare X with X' are summarized as follows:

<u>OBSTACLE</u> Each of the birational maps $\phi^i : (V^i, D^i) \cdots \to (V^{i+1}, D^{i+1})$ appearing in $(*)$ above is either a log-divisorial contraction or a log-flip. Let D^i denote the proper transform of D on V^i, and let $X^i := V^i \backslash \mathrm{Supp}\,(D^i)$ be the complement $(0 \leq i \leq s)$. Then:

(1) In the case where ϕ^i is of log-divisorial type and the exceptional divisor $E^i := \mathrm{Exc}(\phi^i)$ is NOT contained in the boundary Supp (D^i), then X^{i+1} is strictly smaller than the previous one X^i. If we were to describe the contraction ϕ^i and how E^i meets D^i explicitly, then we can recover the data on X^i from those on X^{i+1} in principle. But this seems to be hopeless in general.

(2) In the case where ϕ^i is of log-flipping type, if we were to know that all the flipping curves (resp. flipped curves) are contained in the boundary $\mathrm{Supp}\,(D^i)$ (resp. $\mathrm{Supp}\,(D^{i+1})$), then there exists no difference between X^i and X^{i+1}. But, this expectation does not always hold true, namely, some of flipping curves or flipped curves may not be contained in the boundary. As a result, we can not compare X^i and X^{i+1} explicitly and, in addition to this, X^i (resp. X^{i+1}) may be no longer affine even if X^{i+1} (resp. X^i) is affine.

This is why there seems to be no clear principle to compare X^i with X^{i+1}. Hence, even if we can analyze X' concretely, it is usually impossible to recover the data on the original X from those on X'.

The main interest in this article lies in how to control the process $(*)$ of $(\mathsf{LMMP})_3$ starting with (V, D) being anxious about the changes of the inside affine threefold X under a certain condition $(\natural)$. Our condition is geometric in nature:

$(\natural)$ *The linear system $|D|$ contains a Du Val member, say $S \in |D|$.*

Remark 1. In the previous paper [Ki05], we investigated the process $(*)$ of $(\mathsf{LMMP})_3$ starting with (V, D) under the more specific assumption that *the linear system $|D|$ is nef and contains a smooth member*. In fact, under these numerical and geometrical assumptions, we can describe explicitly how inside affine threefold X changes via the process $(*)$ by making use of the theory of $\sharp$-MMP due to M. Mella [Me02]. Moreover, we are able to choose the process $(*)$ in such way that the first half process of $(*)$ are composite of (ordinary) flips, and the latter half are composed of terminal divisorial contractions (cf. [Ki05]). Whereas, as the present condition $(\natural)$ above is more general compared with that in [Ki05], we can not apply the $\sharp$-MMP and the consideration makes more complicated. Indeed, the log-flips may occur in $(*)$, and the varieties V^i are no longer in the category of terminal singularities in general.

As said in Remark 1, the process $(*)$ of $(\mathsf{LMMP})_3$ starting with (V, D) is difficult to handle in general. In fact, we do not know about the explicit description of $(*)$ itself. Nevertheless, we can understand the description as long as the process $(*)$ is restricted onto the complements to the divisors D^i. Namely, our main result is stated as in the following fashion:

Theorem 1.1. *Let X be a smooth affine algebraic threefold. Suppose that X is embedded into a pair (V, D) consisting of a smooth projective threefold V containing X as a Zariski open subset and the SNC (reduced) boundary divisor $D = V \backslash X$ satisfying the condition $(\natural)$. Then, for* any $(\mathsf{LMMP})_3$ *starting with (V, D) (i.e., $(K_V + D)$-MMP), say:*

$$(*) \quad \phi : (V, D) \overset{\phi^0}{\cdots\rightarrow} (V^1, D^1) \overset{\phi^1}{\cdots\rightarrow} \cdots\cdots \rightarrow (V^{s-1}, D^{s-1}) \overset{\phi^{s-1}}{\cdots\rightarrow} (V^s, D^s),$$

the following properties $(1) \sim (6)$ hold true.

Notation. *Let $R^i = \mathbb{R}_+[l^i] \subset \overline{\mathrm{NE}}\,(V^i)$ be an extremal ray associated to which the birational map $\phi^i : (V^i, D^i) \cdots\rightarrow (V^{i+1}, D^{i+1})$ is obtained. Let D^i and S^i be the proper transforms of D and S on V^i respectively, and let $X^i := V^i \backslash D^i$ denote the complement for $0 \le i \le s$.*

(1) *D^i and S^i are Cartier divisors on V^i and $D^i \sim S^i$. Moreover, S^i is a Du Val member of $|D^i|$.*

(2) *If $(K_{V^i} \cdot l^i) \ge 0$, then φ^i is a log-flip. All the flipping curves (resp. flipped curves) are contained in the boundary D^i (resp. D^{i+1}). In particular, we have $X^i \cong X^{i+1}$.*

(3) *If* $(K_{V^i} \cdot l^i) < 0$ *and* ϕ^i *is a flip, then all the flipping curves (resp. flipped curves) are contained in the boundary* D^i *(resp.* D^{i+1}*). In particular, we have* $X^i \cong X^{i+1}$.

Assume that ϕ^i *is a divisorial contraction (hence then* $(K_{V^i} \cdot l^i) < 0$ *by (2)), and let* E^i *denote the exceptional divisor. Then:*

(4) *If* E^i *is contained in* $\mathrm{Supp}\,(D^i)$*, we have* $X^i \cong X^{i+1}$.

(5) *If* E^i *is not contained in* $\mathrm{Supp}(D^i)$*, then* ϕ^i *contracts* E^i *onto a smooth point* $P := \phi^i(E^i) \in V^{i+1}$*, and* ϕ^i *is realized as a weighted blow-up at* P *with respect to* $\mathrm{wts} = (1,1,b)$ *for some* $b \in \mathbb{N}$. *Moreover,* X^i *is obtained as the half-point attachment to* X^{i+1} *of type* (b,k) *for some* $1 \le k \le b$ *(cf. Definition 1.1). In particular,* X^{i+1} *is an open affine subset of* X^i *such that* $X^i \backslash X^{i+1} \cong \mathbb{C}^{(k-1)*} \times \mathbb{A}^1$.

(6) *The final object* $(V', D') := (V^s, D^s)$ *in the process* $(*)$ *satisfies one of the following according to the log Kodaira dimension* $\overline{\kappa}(X)$:

- (i) *If* $\overline{\kappa}(X) = -\infty$*, then* (V', D') *is a* $(\mathsf{LMfs})_3$*, say* $(V', D')/W$*. Moreover, we can describe* $(V', D')/W$ *in detail (see §3 for the detailed description of* $(V', D')/W$*).*
- (ii) *If* $\overline{\kappa}(X) \ge 0$*, then* (V', D') *is a Log Minimal Model, i.e.,* $K_{V'} + D'$ *is nef and* $\kappa(V'; K_{V'} + D') = \overline{\kappa}(X)$.

We shall prepare the definiton of half-point attachments used in Theorem 1.1 (5).

Definition 1.1 (cf. [Ki05]). Let $\overline{Z}$ be a normal quasi-projective threefold and let $\overline{Z} \hookrightarrow \overline{V}$ be a compactification into a normal projective threefold $\overline{V}$ with the boundary $\overline{V} \backslash \overline{Z} = \mathrm{Supp}\,(\overline{B})$. Let $P \in \mathrm{Supp}\,(\overline{B})$ be a point where $\overline{V}$ is smooth. Let $f : V \to \overline{V}$ be the weighted blow-up at the point $P \in \overline{V}$ with weights $\mathrm{wts} = (1,1,b)$, where $b \in \mathbb{N}$. Let $E \cong \mathbb{P}(1,1,b) \cong \mathbb{F}_{b,0}$ denote the exceptional divisor of f. *Assume* that the proper transform B of $\overline{B}$ by f intersects E in such a way that $B|_E = \sum_{j=1}^{k} m_j l_j$, where l_j's are the mutually distinct generators of rulings on $E \cong \mathbb{F}_{b,0}$ and $m_j \in \mathbb{N}$ with $\sum_{j=1}^{k} m_j = b$. Then the complement $Z := V \backslash \mathrm{Supp}\,(B)$ is said to be a half-point attachment to $\overline{Z}$ of (b,k)-type. It follows, by the definition, that $Z \backslash \overline{Z} \cong \mathbb{C}^{(k-1)*} \times \mathbb{A}^1$.

By making use of Theorem 1.1, the classification of certain kinds of polarized $\mathbb{Q}$-Fano threefolds with $\varrho = 1$ (cf. [C-F93]) and Log Abundance Theorem of dimension three (cf. [K-M-M94]), we can investigate the structure of the original affine algebraic threefold X in detail for the cases $\overline{\kappa}(X) = -\infty$, 1 or 2 (see §3 and §4).

This article is organized as follows: In §2, we shall give the proof of Theorem 1.1. As said in Remark 1, in the previous paper [Ki05], we have succeeded in the explicit description of the process $(*)$ under the assumption that $|D|$ is nef and contains a smooth member. Indeed, we then are able to apply the theory of $\sharp$-MMP (cf. [Me02]). Roughly speaking, the $\sharp$-MMP guarantees the existence of a "good" (ordinary) $(\mathsf{MMP})_3$ which starts with V. Here "good" means that we are able to describe the process $(\mathsf{MMP})_3$ very explicitly in neighborhoods of the proper transforms of a smooth member of $|D|$. In this strategy, we need the nefness and the existence of a smooth member of $|D|$ in the crucial parts (see [Me02] for the detailed mechanism for $\sharp$-MMP). On the other hand, in our present condition ($\natural$), we assume neither the nefness nor the existence of a smooth member of $|D|$. Instead of them, we assume the existence of a Du Val member $S \in |D|$. Distinct from the case mentioned above (cf. [Ki05]), the situation makes more subtle and it is hard to handle the process $(*)$ of (K_V+D)-MMP itself. But, once we restrict ourselves to the inside affine parts, we can obtain the explicit description. The key points are investigating : what are the proper transforms of the normal member $S \in |D|$ like, and how appearing exceptional divisors and flipping/flipped curves intersect the boundary parts via the process $(*)$, in the inductive way.

It is well known, in the theory of affine surfaces, that a smooth affine algebraic surface Y with log Kodaira dimension $\overline{\kappa}(X) = -\infty$ (resp. 1) has a structure of an $\mathbb{A}^1$-fibration (resp. a $\mathbb{C}^*$-fibration) (cf. [Mi-Su80, Miy81, Miy01], [Kaw79]). In the sections §3 and §4, we shall consider the three-dimensional version of these results, namely, we investigate the structures of affine algebraic threefolds X with log Kodaira dimension $\overline{\kappa}(X) = -\infty$, 1 or 2 under the condition ($\natural$) by applying Theorem 1.1 (cf. Theorems 3.1 and 4.1).

We employ the following notation in this article.

Notation and Convention.

- $\sim$: linear equivalence;
- $\equiv$: numerical equivalence;
- $\mathbb{A}^n$: the n-dimensional affine space;
- $\mathbb{P}^n$: the n-dimensional projective space;
- $\mathbb{P}(a,b,c)$: the weighted projective plane with weights wts $= (a,b,c)$;
- $\mathbb{F}_b$: the Hirzebruch surface of degree b $(b \geq 0)$;
- $\mathbb{F}_{b,0}$: the normal surface obtained from $\mathbb{F}_b$ by contracting the minimal section;
- $\mathbb{C}^{(k)*}$: the affine line with k-point(s) punctured. We write $\mathbb{C}^{(0)*} = \mathbb{A}^1$ and $\mathbb{C}^{(1)*} = \mathbb{C}^*$ for the simplicity;
- $\mathrm{Exc}(f)$: the exceptional set of a given birational morphism f;
- $\overline{\mathrm{NE}}\,(V)$: the closure of the cone of effective 1-cycles on V modulo $\equiv$;
- $\mathrm{NonSing}(V)$: the smooth locus of V;
- $\mathrm{Diff}_*(\cdot)$: the difference (see [FA, Chapters 16 and 17]);
- $(\textbf{Terminal})_d$ (resp. $(\textbf{Canonical})_d$, $(\textbf{Dlt})_d$) : the set of all pairs (W, B) consisting of a normal projective variety W of dimension d and the the boundary B on W (B may be empty) such that (W, B) has terminal singularities (resp. canonical singularities, divisorial log-terminal singularities). See [FA] or [Ko-Mo98, Chapters 2 and 5] for the definitions and the some (local and global) properties on these classes of singularities.

The projective birational morphism $f : V \to W$ from a normal threefold V with only $\mathbb{Q}$-factorial singularities is said to be a divisorial contraction if its exceptional set $E := \mathrm{Exp}(f)$ is a prime divisor on V. f is said to be of $(2,0)$-type (resp. $(2,1)$-type) if E is contracted to a point (resp. to a curve).

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2. The Proof of Theorem 1.1

2.1. Strategy of the Proof and Preliminaries

In this section, we shall give the proof of Theorem 1.1. Let X be an affine algebraic threefold, and $\iota : X \hookrightarrow (V, D)$ an SNC compactification of X, that is, V is a smooth projective threefold containing X as a Zariski open subset such that the boundary $D := V \backslash \iota(X)$ is a reduced SNC divisor on V. Assume that (V, D) satisfies ($\natural$), i.e., the linear system $|D|$ associated with D contains a Du Val member, say $S \in |D|$. As mentioned in §1, the essential for investigating the structure on X from a point of view of Birational Geometry lies in how to understand the behavior about changes of the inside affine threefold X via the process of $(\mathsf{LMMP})_3$ starting with the dlt pair $(V, D) \in (\mathbf{Dlt})_3$ (that is $(K + D)$-MMP), say:

$$(*) \quad \phi : (V, D) \overset{\phi^0}{\cdots\rightarrow} (V^1, D^1) \overset{\phi^1}{\cdots\rightarrow} \cdots\cdots \rightarrow (V^{s-1}, D^{s-1}) \overset{\phi^{s-1}}{\cdots\rightarrow} (V^s, D^s),$$

where D^i are the proper transforms of D on V^i, and the final object $(V', D') := (V^s, D^s)$ is either a $(\mathsf{LMfs})_3$ or a log minimal model. We need to observe how the exceptional divisors of divisorial contractions (resp. flipping/flipped curves of log-flips) appearing in the process $(*)$ intersect the proper transforms $D^i \subset V^i$ of the boundary D. More precisely to say, we shall proceed inductively with emphasis on the following matters in the process $(*)$:

(A) In the case where $\phi^i : (V^i, D^i) \cdots\rightarrow (V^{i+1}, D^{i+1})$ is a (ordinary) flip, we have to understand where all the flipping/flipped curves are located.

(B) In the case where $\phi^i : (V^i, D^i) \cdots\rightarrow (V^{i+1}, D^{i+1})$ is a log-flip (a flipping curve l^i intersects $K_{V^i} + D^i$ negatively, but does K_{V^i} non-negatively), we have to investigate not only the location of flipping/flipped curves but also the analytic types of singularities along flipped curves for the subsequent inductive arguments. For this purpose, we shall cite the result in [Ki].

(C) In the case where $\phi^i : (V^i, D^i) \rightarrow (V^{i+1}, D^{i+1})$ is a divisorial contraction that do not take place in the boundary part D^i, we have to investigate what kind of exceptional divisor is and how an exceptional divisor intersects the boundary D^i explicitly. For this purpose, we shall make use of the result in [Ki05] concerning special kinds of terminal divisorial contractions. This needs the result due to M. Kawakita crucially (cf. [Ka01]).

(D) Via the (log-)flips and divisorial contractions appearing in $(*)$, we shall observe how inside affine threefold X changes in consideration of the data obtained in (A), (B) and (C) above. In this strategy, we use the notion of half-point attachments (cf. Definition 1.1).

Remark 2.

(1) Distinct from the case treated in the previous paper [Ki05] (cf. Remark 1), since we only assume the existence of a Du Val member of $|D|$ at present, we can not apply the $\sharp$-MMP to our situation. This obstacle predicts that we can no longer stay in $(\mathbf{Terminal})_3$ in general. In addition to this, we have to take good care of log-flips.

(2) The $\sharp$-MMP is useful to *find* a good $(\mathsf{MMP})_3$ starting with V provided that $|D|$ is nef and contains a smooth member (cf. [Me02], [Ki05]). Meanwhile, although we can not make use of the $\sharp$-MMP now, our argument performed below in this section shows that we obtain an explicit description as stated in Theorem 1.1 for *every* $(\mathsf{LMMP})_3$ starting with $(V, D) \in (\mathbf{Dlt})_3$.

Let S^i be the proper transform of S on V^i for $0 \leq i \leq s$. First of all, we note the following:

Lemma 2.1. *We have the following concerning the singularities of V^i, S^i and irreducible components of D^i for $0 \leq i \leq s$.*

(1) *(V^i, S^i) has canonical singularities, and V^i has terminal singularities along S^i.*

(2) *$\mathrm{Diff}_{S^i}(0) = 0$ and S^i has at Du Val singularities.*

(3) *For each irreducible component D^i_j of D^i, we have $(D^i_j, \Delta^i_j) \in (\mathbf{Dlt})_2$, where $\Delta^i_j := \mathrm{Diff}_{D^i_j}(D^i - D^i_j)$.*

Proof. Since S has at most Du Val singularities in a smooth threefold V, it is easy to see that $(V, S) \in (\mathbf{Canonical})_3$. On the other hand, the process $(*)$ of $(K + D)$-MMP coincides with that of $(K + S)$-MMP as S is linearly equivalent to D. Hence $(V^i, S^i) \in (\mathbf{Canonical})_3$. This implies the assertion (1). Since $(V^i, S^i) \in (\mathbf{Canonical})_3$, we know that S^i is normal and $(S^i, \mathrm{Diff}_{S^i}(0)) \in (\mathbf{Canonical})_2$ (cf. [Ko-Mo98, Propositions 5.46, 5.51]). Since V^i is terminal along S^i as seen just above, we have $\mathrm{Diff}_{S^i}(0) = 0$ (cf. [FA, Chapters 16 and 17]). Hence S^i has at most Du Val singularities. Since $(V^i, D^i) \in (\mathbf{Dlt})_3$, the assertion (3) is obtained by [Ko-Mo98, Prop. 5.59]. □

Lemma 2.2. *If $\overline{\kappa}(X) = -\infty$, then $K_V + D$ is not nef.*

Proof. Assume to the contrary that $K_V + D$ is nef, i.e., (V, D) is a log minimal model in $(\mathbf{Dlt})_3$. Hence, we have $\mathrm{Bs}\,|m(K_V + D)| = \emptyset$ for a sufficiently large $m >> 0$ (cf. [K-M-M94]). This is obviously a contradiction. □

On the other hand, in case of $\overline{\kappa}(X) \geq 0$, we ask whether or not $K_V + D$ is nef. If $K_V + D$ is already nef, then we have nothing to do in order to obtain Theorem 1.1. Hence, in what follows, we may and shall assume that $K_V + D$ is not nef. Since $(V, D) \in (\mathbf{Dlt})_3$ and $K_V + D$ is not nef, there exists an extremal ray, say $R^0 = \mathbb{R}_+[l^0]$, contained in $\overline{\mathrm{NE}}(V)_{(K_V+D)<0}$, and we denote by $\phi^0 : V \cdots \to V^1$ the rational map associated to the contraction of R^0 (cf. [Ko-Mo98]). ϕ^0 is either a $(\mathsf{LMfs})_3$ or a birational map.

Lemma 2.3. *If $\overline{\kappa}(X) \geq 0$, then ϕ^0 is birational.*

Proof. Assume to the contrary that ϕ^0 gives rise to a $(\mathsf{LMfs})_3$ provided that $\overline{\kappa}(X) \geq 0$. Since $\overline{\kappa}(X) = \kappa(V; K_V + D) \geq 0$ and ϕ^0 is a fibration onto a lower dimensional variety, we have $(K_V + D \cdot R^0) \geq 0$. This is a contradiction as R^0 is $(K_V + D)$-negative. □

If ϕ^0 gives rise to a $(\mathsf{LMfs})_3$ (this never occurs when $\overline{\kappa}(X) \geq 0$ by Lemma 2.3), then we have nothing to do with in order to obtain Theorem 1.1. Hence we may and shall assume that $\phi^0 : V \cdots \to V^1$ is birational in the sequel.

2.2. The first step of the process of $(\mathsf{LMMP})_3$

From now on for the time being, we shall observe this birational map $\phi^0 : (V, D) \cdots \to (V^1, D^1)$ with emphasis on (A), (B), (C) and (D) in detail. We consider according as $(K_V \cdot l^0) \geq 0$ or $(K_V \cdot l^0) < 0$, separately.

2.2.1. CASE (I) $(K_V \cdot l^0) \geq 0$.

At first, we consider the case of $(K_V \cdot l^0) \geq 0$. Then:

Lemma 2.4. *ϕ^0 is a log-flip.*

Proof. In fact, assume that ϕ^0 is of divisorial type, and let E denote the exceptional divisor. Since R^0 is $(K_V + D)$-negative and K_V-nonnegative, we have $(D \cdot l^0) < 0$. This implies that $E \subset \mathrm{Bs}\,|D|$, in particular, E is contained in the fixed part of the linear system $|D|$. But, since $|D|$ contains a Du Val member S, the possibility is that E and S coincide with D, $|D|$ consists of D only and D itself is Du Val. Note that since $V \backslash D$ is affine, D is connected. Hence, D is irreducible. After the divisorial contraction $\phi^0 : V \to V^1$, we have a new compactification $X \hookrightarrow V^1$ such that $V^1 \backslash X = \phi^0(E)$. As X is affine and $\mathrm{codim}_{V^1} \phi^0(E) \geq 2$, this is a contradiction. □

Thus we may assume that ϕ^0 is a log-flip. Since R^0 is $(K_V + D)$-negative, it follows that $(D \cdot l^0) < 0$, so that $l^0 \subset D$. Simultaneously, since $S \sim D$, we have $(S \cdot l^0) < 0$ and $l^0 \subset S$. Concerning the explicit description of the log-flip ϕ^0, we have the following:

Lemma 2.5 (cf. [Ki]). *Assume that $\phi^0 : (V, D) \cdots \to (V^1, D^1)$ is a log-flip with $(K_V \cdot l^0) \geq 0$. Then we have:*

(1) *There exists exactly one flipping curve l^0.*

(2) *The flipping curve l^0 is a double curve of two irreducible components of D, i.e., $l^0 = D_{j_0} \cap D_{j_1}$ for suitable components D_{j_0} and D_{j_1} of D.*

(3) *Put $d_0 := -(l^0)^2_{D_{j_0}}$ and $d_1 := -(l^0)^2_{D_{j_1}}$. Then $d_0, d_1 > 0$.*

(4) *We may assume that $d_0 \geq d_1$. Then $\phi^0 : V \cdots \to V^1$ is obtained as the Euclidean log-flips associated to (d_0, d_1) (see [Ki, §2] for the construction of Euclidean log-flips). In particular, for a general point P on the flipped curve l^{0+}, we have:*

$$(P \in l^{0+} \subset V^1) \simeq o \in (x = y = 0) \subset \left(\mathbb{C}^2\,(x, y)/\mathbb{Z}_d(1, 1)\right) \times \mathbb{C},$$

where $d := \gcd(d_0, d_1)$.

(5) *The flipped curve l^{0+} is contained in the boundary D^1.*

Proof. See [Ki]. □

In order to obtain more detailed information about the log-flip ϕ^0, we need to investigate the birational map $\psi^0 := \phi^0|_S : S \cdots \to S^1$ obtained by restricting ϕ^0 onto S.

Lemma 2.6. *Let the notation be the same as above. Then:*

(1) *$\psi^0 : S \cdots \to S^1$ contracts l^0 to a smooth point $Q := l^{0+} \cap S^1$ on S^1.*

(2) *Q is a smooth on V^1.*

(3) *S^1 is contained in the smooth locus of V^1. In particular, S^1 (and D^1 also) is a Cartier divisor on V^1.*

Proof. Our Proof consists of several steps.

Step 1. First of all, we claim the following:

Claim 1. *$l^{0+} \not\subset S^1$.*

Proof of Claim 1. Assume to the contrary that $l^{0+} \subset S^1$. Since $\mathrm{Diff}_{S^1}(0) = 0$ (cf. Lemma 2.1), we have $(K_{V^1} + S^1 \cdot l^{0+}) = (K_{S^1} \cdot l^{0+})_{S^1}$. Then we have $(K_S \cdot l^0)_S < 0$ and $(K_{S^1} \cdot l^{0+})_{S^1} > 0$, hence it is obvious that ψ^0 is not an isomorphism. Let $S \xleftarrow{p} \widehat{S} \xrightarrow{q} S^1$ be the common resolution. Let $\{\widehat{e}_k\}$ exhaust all the p and q-exceptional curves. Then $\mathrm{Exc}(p) = (\cup \widehat{e}_k) \cup \widehat{l^0}^+$ and $\mathrm{Exc}(q) = (\cup \widehat{e}_k) \cup \widehat{l}^0$, where $\widehat{l}^0$ (resp. $\widehat{l^0}^+$) is the proper transform on $\widehat{S}$ of l^0 (resp. l^{0+}). Write:

$$K_{\widehat{S}} \equiv p^*(K_S) + \sum \alpha_k \widehat{e}_k + \alpha \widehat{l^0}^+ \equiv q^*(K_{S^1}) + \sum \beta_k \widehat{e}_k + \beta \widehat{l}^0,$$

where all of the α, β, α_k's and β_k's are non-negative as S and S^1 has at most Du Val singularities (cf. Lemma 2.1 (2)). Rewrite the above relation as:

$$q^*(K_{S^1}) - p^*(K_S) + \beta \widehat{l}^0 \ \equiv \ \sum (\alpha_k - \beta_k) \widehat{e}_k + \alpha \widehat{l^0}^+.$$

Note that $q^*(K_{S^1}) - p^*(K_S)$ is p-nef and $\beta \widehat{l}^0$ is p-effective. Moreover, $q^*(K_{S^1}) - p^*(K_S)$ is positive along the curve $\widehat{l^0}^+$. Hence, we have $\alpha_k \le \beta_k$ and $\alpha < 0$ by the Nagativity Lemma (cf. [FA],[Cor94]). This contradicts to the fact $\alpha \ge 0$. □

Step 2. Since $(S^1 \cdot l^{0+}) > 0$ and $l^{0+} \not\subset S^1$, the flipped curve l^{0+} intersects S^1. Moreover, $\psi^0 : S \to S^1$ is just the contraction of l^0 to a point $Q := l^{0+} \cap S^1$. In fact, we have the following:

Claim 2. *Q is a smooth point of S^1.*

Proof of Claim 2. If S is smooth along l^0, then l^0 is a (-1)-curve, and the assertion is obvious to see. Hence we may and shall assume that S has Du Val singularities on l^0. Let $p : \widehat{S} \to S$ be the minimal resolution with exceptional curves $\{\widehat{e}_k\}$, and let $q := \psi^0 \circ p : \widehat{S} \to S^1$ be the induced morphism. It is clear that $\mathrm{Exc}(q)$ coincides with $\mathrm{Exc}(p) \cup \widehat{l}^0$, where $\widehat{l}^0$ is the proper transform on $\widehat{S}$ of l^0. Since $(K_{\widehat{S}} \cdot \widehat{l}^0) = (K_S \cdot l^0) < 0$, $\widehat{l}^0$ is a (-1)-curve on $\widehat{S}$. Thus $\mathrm{Exc}(q)$ is composed of the (-1)-curve $\widehat{l}^0$ and the (-2)-curves $\{\widehat{e}_k\}$. In order to contract $\mathrm{Exc}(q)$ to a point $Q \in S^1$, the only possibility of the configuration of $\mathrm{Exc}(q)$ is that $\mathrm{Exc}(p) = \cup \widehat{e}_k$ is a linear chain of (-2)-curves and $\widehat{l}^0$ intersects one of the terminal components of $\mathrm{Exc}(p)$ at a single point transversally. As a consequence, Q is a smooth point of S^1 as desired. □

Step 3. Since V^1 has at most canonical singularities (cf. Lemma 2.1), one of the following three cases (a), (b) and (c) occurs concerning the singularities of V^1 near the point $Q \in V^1$ (cf. [Ko-Mo98, Chapter 5]). Let $(\widetilde{Q} \in \widetilde{V}^1) \to (Q \in V^1)$ denote the index one covering:

(a) $(\widetilde{Q} \in \widetilde{V}^1)$ is not cDV.

(b) $(\widetilde{Q} \in \widetilde{V}^1)$ is cDV but not isolated.

(c) $(\widetilde{Q} \in \widetilde{V}^1)$ is cDV and isolated, i.e., $(Q \in V^1)$ is a terminal singularity.

In fact, we have the following:

Claim 3. *The case* (*c*) *occurs.*

Proof of Claim 3. Assume that the case (a) does occur. Then there exist a projective birational morphism $g : U \to V^1$ and an exceptional divisor $E \subseteq \mathrm{Exc}(g)$ such that $K_U \equiv g^*(K_{V^1})$ and $g(E) = \{Q\}$ (cf. [Ko-Mo98, Theorem 5.35]). Then we have $a(E; K_{V^1} + S^1) = -\mathrm{mult}_Q S^1$, which is negative as Q is contained in S^1. This is a contradiction because (V^1, S^1) has canonical singularities (cf. Lemma 2.1). Assume now that the case (b) occurs. Then V^1 is singular along the flipped curve l^{0+}. By [Ko-Mo98, Theorem 6.27], there exist a projective birational morphism $g : U \to V^1$ and an exceptional divisor $E \subseteq \mathrm{Exc}(g)$ such that $K_U \equiv g^*(K_{V^1})$ and $g(E) = l^{0+}$. Note that as S^1 does not contain l^{0+} it is generically Cartier along l^{0+}. Since $D^1 \equiv S^1$, it follows that D^1 is also generically Cartier along l^{0+}. On the other hand, we have $(V^1, D^1) \in (\mathbf{Dlt})_3$ (cf. Lemma 2.1). By the definition of divisorial log-terminal singularities (cf. [Ko-Mo98, Chapter 2]) and noting that V^1 has singularities along l^{0+}, we have $a(E; K_{V^1} + D^1) > -1$. But, since D^1 contains l^{0+} and is generically Cartier along l^{0+}, we have:

$$a(E; K_{V^1} + D^1) = a(E; K_{V^1}) - \mathrm{mult}_{l^{0+}} D^1 = -\mathrm{mult}_{l^{0+}} D^1 \leq -1,$$

which is a contradiction. Thus we have the case (c) as desired. □

Step 4. Thus $Q \in V^1$ is a terminal singularity. Since a $\mathbb{Q}$-Cartier divisor S^1 contains Q as a smooth point (cf. Claim 2), we know that Q is a smooth point of V^1 (cf. [FA]). Hence we have $S^1 \subset \mathrm{NonSing}(V^1)$ and, in particular, S^1 (hence D^1 also) is a Cartier divisor on V^1. □

2.2.2. CASE (II) $(K_V \cdot l^0) < 0$.

Next, we consider the case of $(K_V \cdot l^0) < 0$, i.e., R^0 is an extremal ray in an ordinary sense. Since V is a smooth threefold the case where ϕ^0 is a flip does not occur (cf. [C-K-M88]).[*1] Hence we may assume that $\phi^0 : V \to V^1$ is a (terminal) divisorial contraction, let E denote the exceptional divisor. Note that the classification of divisorial contractions from smooth threefolds are completed by S. Mori, which consists of five types (E1), (E2), (E3), (E4) and (E5) (cf. [Mori82], [Ko-Mo98]). The length of the extremal ray R^0:

$$\mu(R^0) := \min\{\ (-K_V \cdot l) \mid l \subset V \text{ and } [l] \in R^0\ \}$$

is equal to 1 (resp. 2) when ϕ^0 is of type (E1), (E3), (E4), (E5) (resp. (E2)). We may assume that this minimal value is obtained by l^0, that is, $\mu(R^0) = (-K_V \cdot l^0)$. Let $\psi^0 : S \to S^1$ denote the restriction of ϕ^0 onto S. We can easily verify the following:

Lemma 2.7. *Let the notation be the same as above. Then:*

(1) *If ϕ^0 is of type (E1), (E3), (E4) or (E5), then E is contained in D.*

(2) *If ϕ^0 is of type (E2) and E is not contained in D, then D intersects E in such a way that $D|_E \in |\mathcal{O}_{\mathbb{P}^2}(1)|$.*

(3) *The intersection $S \cap E$ is composed of mutually disjoint curves on S (if it is not empty at all), and $\psi^0 : S \to S^1$ contracts these curves to smooth points of S^1.*

(4) *S^1 is contained in the smooth locus of V^1. In particular, S^1 (hence D^1 also) is a Cartier divisor on V^1.*

Proof. Since R^0 is $(K_V + D)$-negative, we have $(D \cdot l^0) < \mu(R^0)$. Note that $(D \cdot l^0) \geq 0$. In fact, assume to the contrary that $(D \cdot l^0) < 0$, which implies that the exceptional divisor E of ϕ^0 is contained in the fixed part of the linear system $|D|$. Then we obtain a contradiction by the same argument as in the proof of Lemma 2.4. We consider according to the types of ϕ^0, separately.

Case (i): ϕ^0 is of $(2,0)$-type with $\mu(R^0) = 1$. This case corresponds to (E3), (E4) and (E5). Then we have $(D \cdot l^0) = (S \cdot l^0) = 0$. Since the complement $V \backslash D = X$ is affine, $(D \cdot l^0) = 0$ implies that E is contained in D. Moreover, $(S \cdot l^0)$ then implies that $S \cap E = \emptyset$. Hence the assertions are easy to see.

Case (ii): ϕ^0 is of $(2,1)$-type with $\mu(R^0) = 1$. This corresponds to (E1). As in Case (i) above, we then have $(D \cdot l^0) = (S \cdot l^0) = 0$ and E is contained in D. Moreover, since $\phi^0|_E : E \to \phi^0(E)$ is a $\mathbb{P}^1$-bundle, $(S \cdot l^0) = 0$ implies that $S|_E$ (which may be empty) is composed of several fibers of $\phi^0|_E$. Let $L \subseteq S \cap E$ be one of such fibers (provided that $S \cap E \neq \emptyset$). Let $p : \widehat{S} \to S$ be the minimal resolution of singularities on L, and let $q : \widehat{S} \to S^1$ be the induced morphism. Note that $\mathrm{Exc}(p)$ consists of (-2)-curves (unless S is smooth along L). Since $(K_{\widehat{S}} \cdot \widehat{L}) = (K_S \cdot L) = (K_V + S \cdot L) < 0$, we know that $\widehat{L}$ is a (-1)-curve on $\widehat{S}$, where $\widehat{L}$ is the proper transform on $\widehat{S}$ of L. On the other hand, since q contracts $\mathrm{Exc}(p) \cup \widehat{L}$ to a point $\phi^0(L) \in S^1$, the only possibility of the configuration of $\mathrm{Exc}(p)$ is a linear chain of (-2)-curves and the (-1)-curve curve $\widehat{L}$ intersects

[*1] But, for the inductive arguments in what follows, since the varieties V^i are not necessarily smooth, we need to consider the flip case.

one of the terminal components of $\mathrm{Exc}(p)$. Thus $\phi^0(L) \in S^1$ is a smooth point. (At the same time, we know that S has at most one singularity along L, and it is of A_*-type.) Then the assertions are not difficult to see.

Case (iii): $\mu(R^0) = 2$. This corresponds to (E2). We have $(D \cdot l^0) = (S \cdot l^0) = 0, 1$. If $(D \cdot l^0) = 0$, then we can verify the assertions by the same argument as in Case (i). Suppose that $(D \cdot l^0) = 1$. Then $(S \cdot l^0) = 1$ means that the intersection $S \cap E$ consists of exactly one curve which is a (-1)-curve on S. Hence $\psi^0 : S \to S^1$ contracts $S \cap E$ to a smooth point $\phi^0(E) \in S^1$. □

Summarizing the arguments performed in CASE (I) and (II), we have:

$(\mathsf{A})_1$ $(V^1, D^1) \in (\mathbf{Dlt})_3$, $(V^1, S^1) \in (\mathbf{Canonical})_3$ and S^1 has at most Du Val singularities (cf. Lemma 2.1).

$(\mathsf{B})_1$ $S^1 \subset \mathrm{NonSing}(V^1)$, S^1 and D^1 are linearly equivalent Cartier divisors on V^1 (cf. Lemmas 2.6 and 2.7).

$(\mathsf{C})_1$ The complement $X^1 := V^1 \backslash D^1$ is an open affine subset of X. If $X^1 \not\cong X$, then X is obtained from X^1 via the half-point attachment of $(1, 1)$-type (cf. Definition 1.1, Lemma 2.7).

$(\mathsf{D})_1$ $\overline{\kappa}(X) = \kappa(V^1; K_{V^1} + D^1)$.

2.3. The inductive strategy for the Proof of Theorem 1.1

We shall proceed in the inductive way from now on. Suppose that, after the i-th step $\phi^{i-1} : (V^{i-1}, D^{i-1}) \cdots \to (V^i, D^i)$ in the process $(*)$ of $(\mathsf{LMMP})_3$ starting with the pair $(V, D) \in (\mathbf{Dlt})_3$, we obtain the following data:

$(\mathsf{A})_i$ $(V^i, D^i) \in (\mathbf{Dlt})_3$, $(V^i, S^i) \in (\mathbf{Canonical})_3$ and S^i has at most Du Val singularities (cf. Lemma 2.1).

$(\mathsf{B})_i$ $S^i \subset \mathrm{NonSing}(V^i)$, S^i and D^i are linearly equivalent Cartier divisors on V^i.

$(\mathsf{C})_i$ The complement $X^i := V^i \backslash D^i$ is an open affine subset of $X^{i-1} := V^{i-1} \backslash D^{i-1}$. If $X^i \not\cong X^{i-1}$, then X^{i-1} is obtained from X^i via the half-point attachment of (b^i, k^i)-type for some positive integers $1 \le k^i \le b^i$ (cf. Definition 1.1).

$(\mathsf{D})_i$ $\overline{\kappa}(X) = \kappa(V^i; K_{V^i} + D^i)$.

First of all, we ask whether or not $K_{V^i} + D^i$ is nef. (Note that it is never nef in the case of $\overline{\kappa}(X) = -\infty$.) If it is already nef, then we have nothing to do for the proof of Theorem 1.1. Hence we may and shall assume that $K_{V^i} + D^i$ is not nef in what follows, and we consider the $(i+1)$-th step $\phi^i : (V^i, D^i) \cdots \to (V^{i+1}, D^{i+1})$ in the process $(*)$ of $(K + D)$-MMP. We shall prove the following:

Proposition 2.1. *Assuming the properties $(\mathsf{A})_i, (\mathsf{B})_i, (\mathsf{C})_i$ and $(\mathsf{D})_i$, we obtain those $(\mathsf{A})_{i+1}, (\mathsf{B})_{i+1}, (\mathsf{C})_{i+1}$ and $(\mathsf{D})_{i+1}$ concerning V^{i+1}, D^{i+1} and S^{i+1} via the $(i+1)$-th step $\phi^i : (V^i, D^i) \cdots \to (V^{i+1}, D^{i+1})$ of the process $(*)$ of $(K + D)$-MMP.*

Let $R^i = \mathbb{R}_+[l^i] \subset \overline{\mathrm{NE}}(V^i)$ be the $(K_{V^i} + D^i)$-negative extremal ray associated to which the birational map ϕ^i is obtained. We prove Proposition 2.1 according to $(K_{V^i} \cdot l^i) \geq 0$ or not, separately. The argument is somewhat similar to that performed in Lemmas 2.5 and 2.6. But, since V^i is not smooth in general, we need more subtle treatments about appearing (log-)flips and divisorial contractions.

2.3.1. CASE $(\mathrm{I})_i$ $(K_{V^i} \cdot l^i) \geq 0$.

At first, we shall deal with the case of $(K_{V^i} \cdot l^i) \geq 0$. Then we have:

Lemma 2.8. *ϕ^i is a log-flip.*

Proof. We can verify the assertion by the same argument as in Lemma 2.4. □

Let $l_1, \cdots, l_t$ (resp. $l'_1, \cdots, l'_r$) exhaust all the flipping curves (resp. flipped curves)[*2]. Since R^i is $(K_{V^i} + D^i)$-negative, we have $(D^i \cdot l_k) = (S^i \cdot l_k) < (-K_{V^i} \cdot l_k) \leq 0$. This means that the flipping curves l_k's are contained in $\mathrm{Bs}|D^i|$. Since $S^i \in |D^i|$ is contained in $\mathrm{NonSing}(V^i)$, it follows that $l_k \subset \mathrm{NonSing}(V^i)$. Now we wish to get the information about the log-flip ϕ^i. But, as remaked, the variety V^i has canonical singularities and is no longer smooth in general. This is why it seems to be hopeless to obtain the descriptions as explicitly as in Lemma 2.5. However, the most necessary information for our inductive procedure lies in investigating where the flipping/flipped curves are located, more precisely to say, whether or not flipping/flipped curves are contained in the boundary D^i / D^{i+1}. We have already seen that all the flipping curves are contained in D^i. Concerning the location of flipped curves, we can easily obtain the following:

Lemma 2.9. *All the flipped curves are contained in the boundary D^{i+1}. In particular, we have $X^i \cong X^{i+1}$.*

Proof. Assume to the contrary that some of flipped curves, say $l'_{j_1}, \cdots, l'_{j_a}$, are not contained in D^{i+1}. Note that since the ray R^i, which is composed of flipping curves, is nagative on D^i, the corresponding flipped curves are positive on D^{i+1}. Thus $l'_{j_1}, \cdots, l'_{j_a}$ intersect D^{i+1}. Then we have $X^i = V^i \backslash D^i \cong V^{i+1} \backslash (D^{i+1} \cup l'_{j_1} \cup \cdots \cup l'_{j_a})$. Thus the affine algebraic threefold X^i is embedded into a normal projective threefold V^{i+1} as the complement of $D^{i+1} \cup l'_{j_1} \cup \cdots \cup l'_{j_a}$, which is not of pure codimension one. This is a contradiction. □

Let $\psi^i := \phi^i|_{S^i} : S^i \cdots \to S^{i+1}$ denote the restriction of ϕ^i onto S^i. Then we have the following result which is one of the core part of our inductive procedure:

Lemma 2.10. *Let the notation be the same as above. Then:*

(1) *The curves $l_1, \cdots, l_t$ are mutually disjoint, and $\psi^i : S^i \cdots \to S^{i+1}$ contracts these $l_1, \cdots, l_t$ to smooth points on S^{i+1}.*

[*2] The number t of flipping curves may be different from that r of flipped curves.

(2) *The points on S^{i+1}, which are obtained by the contraction of the curves $l_1, \cdots, l_t$, are smooth points on V^{i+1}.*

(3) *S^{i+1} is contained in the smooth locus of V^{i+1}. In particular, S^{i+1} (and hence D^{i+1} also) is a Cartier divisor on V^{i+1}.*

Proof. Our proof consists of several steps and is somewhat similar to that in Lemma 2.6. But different from the description of the first log-flip (cf. Lemma 2.5), since we do not get the explicit description about the log-flip ϕ^i, we have to take care of a little bit.

Step 1. First of all, we claim the following:

Claim 1. *All the flipped curves are not contained in S^{i+1}.*

Proof of Claim 1. Assume to the contrary that some of flipped curves are contained in S^{i+1}, say $l'_1, \cdots, l'_a$. Note that we have $\mathrm{Diff}_{S^{i+1}}(0) = 0$ (cf. Lemma 2.1). Let $S^i \xleftarrow{p} \widehat{S}^i \xrightarrow{q} S^{i+1}$ denote the common resolution, and let $\{\widehat{e}_k\}$ exhaust all the p and q-exceptional curves. As $\mathrm{Diff}_{S^{i+1}}(0) = 0$, we have $(K_{V^{i+1}} + S^{i+1} \cdot l'_j) = (K_{S^{i+1}} \cdot l'_j)_{S^{i+1}} > 0$. Hence it is not difficult to see that the proper transform of at least one of the curves $l_1, \cdots, l_t$ (resp. the curves $l'_1, \cdots, l'_a$) is q-exceptional (resp. p-exceptional). We may assume that $l_1, \cdots, l_{b_0}$ (resp. $l'_1, \cdots, l'_{a_0}$) exhaust the flipping curves (resp. the flipped curves) such that their proper transforms on $\widehat{S}^i$ are q-exceptional (resp. p-exceptional). Write:

$$K_{\widehat{S}^i} \equiv p^*(K_{S^i}) + \sum \alpha_l \widehat{e}_l + \sum_{j=1}^{a_0} \gamma_j \widehat{l}'_j \equiv q^*(K_{S^{i+1}}) + \sum \beta_l \widehat{e}_l + \sum_{k=1}^{b_0} \delta_k \widehat{l}_k,$$

where $\widehat{l}_k$ (resp. $\widehat{l}'_j$) is the proper transform on $\widehat{S}^i$ of l_k (resp. l'_j). Since S^i and S^{i+1} has at most Du Val singularities (cf. Lemma 2.1 (2)), all of the α_l's, β_l's, γ_j's and δ_k's are non-negative. Rewrite as:

$$q^*(K_{S^{i+1}}) - p^*(K_{S^i}) + \sum_{k=1}^{b_0} \delta_k \widehat{l}_k \equiv \sum (\alpha_l - \beta_l) \widehat{e}_l + \sum_{j=1}^{a_0} \gamma_j \widehat{l}'_j.$$

Note that $q^*(K_{S^{i+1}}) - p^*(K_{S^i})$ is p-nef (and positive along $\widehat{l}'_1, \cdots, \widehat{l}'_{a_0}$), and $\sum_{k=1}^{b_0} \delta_k \widehat{l}_k$ is p-effective. Therefore we have $\alpha_l \leq \beta_l$ and $\gamma_l < 0$ by the Nagativity Lemma (cf. [FA], [Cor94]). This is absurd. □

Step 2. Thus we may assume that all the flipped curves $l'_1, \cdots, l'_r$ intersect S^{i+1} but not contained in S^{i+1} by noting that $(S^{i+1} \cdot l'_j) > 0$ and Claim 1. Hence $\psi^i : S^i \cdots \to S^{i+1}$ is just the contraction of the flipping curves $l_1, \cdots, l_t$. More precisely to say, we have the following:

Claim 2. *The curves l_k's are mutually disjoint, and ψ^i contracts each l_k to a smooth point of S^{i+1}.*

Proof of Claim 2. Let $p : \widehat{S}^i \to S^i$ be the minimal resolution with the exceptional curves $\{\widehat{e}_l\}$ (which are (-2)-curves as S^i has at most Du Val singularities), and let $q := \psi^i \circ p : \widehat{S^i} \to S^{i+1}$ denote the induced morphism. Let $\widehat{l}_k$ be the proper transform on $\widehat{S}^i$ of l_k. Since $(K_{\widehat{S}^i} \cdot \widehat{l}_k) = (K_{S^i} \cdot l_k) < 0$, we know that l_k's are (-1)-curves on $\widehat{S}^i$. On the other hand, q contracts $(\cup \widehat{e}_l) \cup (\cup \widehat{l}_k)$ to several points. This is possible only if the flipping curves l_k's are mutually disjoint, and S^i has at most one A_*-type Du Val singularity along l_k such that the dual graph of the connected component of $\mathrm{Exc}(p)$ which meets $\widehat{l}_k$ is a linear chain of (-2)-curves and $\widehat{l}_k$ meets one of the terminal components of this chain. Thus we have the assertion as desired. □

Step 3. Let $Q_k := \psi^i(l_k)$ be the point on S^{i+1} obtained by the contraction of the curve l_k. From now on we shall investigate the class of singularities Q_k's as points in the ambient variety V^{i+1}. Anyway, we know that V^{i+1} has at most canonical singularities (cf. Lemma 2.1 (1)). Hence one of the following three cases (a), (b) and (c) occurs concerning the index one covering $(\widetilde{Q}_k \in \widetilde{V}^{i+1}) \to (Q_k \in V^{i+1})$ (cf. [Ko-Mo98, Chapter 5]):

(a) $(\widetilde{Q}_k \in \widetilde{V}^{i+1})$ is not cDV.

(b) $(\widetilde{Q}_k \in \widetilde{V}^{i+1})$ is cDV but not isolated.

(c) $(\widetilde{Q}_k \in \widetilde{V}^{i+1})$ is cDV and isolated, i.e., $(Q_k \in V^{i+1})$ is a terminal singularity.

In fact, we have the following:

Claim 3. *The case* (*c*) *occurs.*

Proof of Claim 3. We can verify the assertion by the same argument as in the proof of Claim 3 in Lemma 2.6, hence we shall omit the detail. In this strategy, we have to note that the pair (V^{i+1}, D^{i+1}) is in $(\mathbf{Dlt})_3$ (cf. Lemma 2.1). □

Thus $Q_k \in V^{i+1}$ is a terminal singular point. On the other hand, since a $\mathbb{Q}$-Cartier divisor S^{i+1} contains Q_k as a smooth point (cf. Claim 2), it follows that Q_k is a smooth point of V^{i+1} (cf. [FA]). Hence the surface S^{i+1} is contained in the smooth locus of V^{i+1}, in particular, S^{i+1} is a Cartier divisor on V^{i+1}. Thus we complete the proof. □

Thus we can verify the assertion of Proposition 2.1 when $(K_{V^i} \cdot l^i) \geq 0$.

2.3.2. CASE $(\mathrm{II})_i$ $(K_{V^i} \cdot l^i) < 0$.

Next we shall consider the case where the ray R^i intersects the canonical class K_{V^i} negatively. Distinct from the first step $\phi^0 : V \cdots \to V^1$, since the variety V^i is no longer smooth in general, we need to deal with the flipping case also. We divide the situation into two sub-cases according as $(-K_{V^i} \cdot l^i) \leq 1$ or not.

Sub-Case $(\mathrm{II})_i$-(i): $(-K_{V^i} \cdot l^i) \leq 1$. Assume that there exists an extremal rational curve $l^i \in R^i$ such that $(-K_{V^i} \cdot l^i) \leq 1$. We can easily see the following results:

Lemma 2.11. *Let the assumptions be the same as above. Suppose that* $\phi^i : V^i \cdots \to V^{i+1}$ *is a flip. Then:*

(1) *All the flipping curves (resp. flipped curves) are contained in the boundary* D^i *(resp.* D^{i+1}*).*

(2) *The surface* S^i *is disjoint from flipping curves. In particular, the proper transform* S^{i+1} *on* V^{i+1} *is contained in the smooth locus of* V^{i+1}.

(3) *We have* $X^i \cong X^{i+1}$.

Lemma 2.12. *Let the assumptions be the same as above. Suppose that* $\phi^i : V^i \to V^{i+1}$ *is a divisorial contraction with the exceptional divisor* E*. Then:*

(1) E *is contained in the boundary* D^i.

(2) *If* ϕ^i *is of* $(2,0)$*-type, then* S^i *is disjoint from* E.

(3) *If* ϕ^i *is of* $(2,1)$*-type, then the intersection* $S^i \cap E$*, which may be empty, is composed of several smooth fibers of the* $\mathbb{P}^1$*-fibration* $\phi^i|_E : E \to \phi^i(E)$.

(4) *The proper transform* S^{i+1} *on* V^{i+1} *is contained in the smooth locus of* V^{i+1}.

(5) *We have* $X^i \cong X^{i+1}$.

Proof of Lemmas 2.11 *and* 2.12 Since the ray $R^i = \mathbb{R}_+[l^i]$ is $(K_{V^i} + D^i)$-negative, we have $(D^i \cdot l^i) = (S^i \cdot l^i) < (-K_{D^i} \cdot l^i) \le 1$. Since D^i is Cartier by $(\mathsf{B})_i$, this inequality implies that $(D^i \cdot l^i) = (S^i \cdot l^i) \le 0$. Assume now that we have $(D^i \cdot l^i) < 0$, which means that the exceptional set of ϕ^i (the union of flipping curves or the exceptional divisor) is contained in the base points set of the linear system $|D^i|$. In the case where ϕ^i is of divisorial type, we can deduce a contradiction by the same argument as in the proof of Lemma 2.4 in consideration of the affineness of X^i. On the other hand, if ϕ^i is a flip, then all the flipping curves are contained in S^i as $(S^i \cdot l^i) < 0$. Since $S^i \subset \mathrm{NonSing}(V^i)$ by $(\mathsf{B})_i$, the flipping curves are in the smooth locus of V^{i+1}. This is absurd (cf. [C-K-M88]). Thus we may and shall assume that $(D^i \cdot l^i) = 0$. Since the complement $X^i = V^i \backslash D^i$ is affine, we know that exceptional set of ϕ^i is contained in D^i. Moreover, in the case of a flip, we see that all the flipped curves are contained in D^{i+1} by the similar argument as in Lemma 2.9. Thus we obtain the assertions (1), (3) in Lemma 2.11, and (1), (5) in 2.12. For the assertions (2) in Lemma 2.11 and (2), (3), (4) in Lemma 2.12, we shall consider separately.

If ϕ^i is a flip, then $(S^i \cdot l^i) = 0$ implies that each flipping curve is either disjoint from S^i or contained in it. But, as the flipping curves pass through the singularities on V^i and $S^i \subset \mathrm{NonSing}(V^i)$, the former occurs. Hence it is obvious that the flipped curves are disjoint from the proper transform S^{i+1}.

Suppose that ϕ^i is a divisorial contraction of $(2,0)$-type. Then $(S^i \cdot l^i) = 0$ implies that $S^i \cap E = \emptyset$ certainly.

Let us consider the case that ϕ^i is a divisorial contraction of $(2,1)$-type. Note that the restriction $\phi^i|_E : E \to \phi^i(E)$ is a $\mathbb{P}^1$-bundel in a neighborhood of the surface S^i (unless $S^i \cap E = \emptyset$) because of $S^i \subset \mathrm{NonSing}(V^i)$. Hence the resulting threefold V^{i+1} is smooth at the points $\phi^i(S^i \cap E)$. Moreover, the equality $(S^i \cdot l^i) = 0$ means that $S^i \cap E$ is composed of several smooth fibers of $\phi^1|_E$ (unless $S^i \cap E = \emptyset$).

In any case as above, we can verify that the proper transform S^{i+1} on V^{i+1} is contained in $\mathrm{NonSing}(V^{i+1})$. We thus complete the proof. □

Thus we verify Proposition 2.1 in the case of $0 < (-K_{V^i} \cdot l^i) \le 1$.

Sub-Case $(\mathrm{II})_i$**-(ii):** $(-K_{V^i} \cdot l^i) > 1$**.** Assume that any extremal rational curve l^i in the ray R^i satisfies $(-K_{V^i} \cdot l^i) > 1$. Then we have the following:

Lemma 2.13. *ϕ^i is a divisorial contraction of $(2,0)$-type.*

Proof. We can see the assertion by the same argument as in [C-F93, Lemma 2.1]. Although the proof in [C-F93, Lemma 2.1] is performed under the assumption that the variety has $\mathbb{Q}$-factorial and terminal singularities, the same can be applied to the present situation that V^i is $\mathbb{Q}$-factorial and canonical. □

Let E denote the exceptional divisor of ϕ^i, and let $Q := \phi^i(E) \in V^{i+1}$ be the point obtained by the contraction of E.

Lemma 2.14. *Let the notation and the assumptions be the same as above. Then we have:*

(1) $(D^i \cdot l^i) = (S^i \cdot l^i) \ge 0$.

(2) *If $(D^i \cdot l^i) = 0$, then the exceptional divisor E is contained in the boundary D^i. Moreover, S^i is disjoint from E, and the proper transform S^{i+1} on V^{i+1} is contained in the smooth locus of V^{i+1}.*

Proof. Both assertions can be verified by the same argument as in the former half of the proof of Lemmas 2.11 and 2.12. □

In the case of $(D^i \cdot l^i) > 0$, we have to observe the restricted birational morphism $\psi^i := \phi^i|_{S^i} : S^i \to S^{i+1}$ in order to get the more detailed information about the divisorial contraction $\phi : V^i \to V^{i+1}$.

Lemma 2.15. *Let the notation be the same as above. Suppose that $(D^i \cdot l^i) > 0$. Then $S^i \cap E$ is an irreducible curve and $\psi^i : S^i \to S^{i+1}$ contracts $S^i \cap E$ to a smooth point of S^{i+1}.*

Proof. Since $(S^i \cdot l^i) > 0$, the surface S^i intersects E. Let $S^i \cap E = \cup C_l$ be a decomposition into irreducible components. Let $p : \widehat{S}^i \to S^i$ be the minimal resolution with an exceptional set $\mathrm{Exc}(p) = \cup \widehat{e}_k$, and let $q := \psi^i \circ p : \widehat{S}^i \to S^{i+1}$ denote the induced morphism. Since S^i has at most Du Val singularities, the curve e_k's are (-2)-curves. Since $(K_{\widehat{S}^i} \cdot \widehat{C}_l) = (K_{S^i} \cdot C_l) = (K_{V^i} + S^i \cdot C_l) < 0$, $\widehat{C}_l$ is a (-1)-curve on $\widehat{S}^i$, where $\widehat{C}_l$ is the proper transform on $\widehat{S}^i$ of C_l. Since $q : \widehat{S}^i \to S^{i+1}$ contracts $(\cup \widehat{C}_l) \cup (\cup \widehat{e}_k)$ to a point $Q := \phi^i(E)$, we know that $C := S^i \cap E$ is irreducible and the configuration of $\mathrm{Exc}(p)$ is a linear chain of (-2)-curves, furthermore, $\widehat{C}$ intersects one of the terminal components of $\mathrm{Exc}(p)$. As a result, Q is a smooth point of S^{i+1}, as desired. □

Lemma 2.16. *The point $Q = \phi^i(E)$ is a smooth point of V^{i+1}. In particular, the surface S^{i+1} is contained in the smooth locus* $\mathrm{NonSing}(V^{i+1})$.

Proof. Note that $(V^{i+1}, S^{i+1}) \in (\mathbf{Canonical})_3$, and so V^{i+1} is terminal along S^{i+1} (cf. Lemma 2.1 (1)), in particular, $Q \in V^{i+1}$ is a terminal singular point. As the surface S^{i+1} contains Q as a smooth point, it follows that Q is, in fact, a smooth point of V^{i+1} (cf. [FA]). □

By making use of Lemmas 2.15 and 2.16, we can describe ϕ^i explicitly as follows:

Lemma 2.17. *Let the notation be the same as above. Then we have:*

(1) $\phi^i : V^i \to V^{i+1}$ *is realized as the weighted blow-up at the smooth point* $Q \in V^{i+1}$ *(cf. Lemma 2.16) with weights* $\mathrm{wts} = (1, 1, b)$ *for some positive integer* $m > 0$.

(2) *The exceptional divisor* E *is isomorphic to* $\mathbb{F}_{b,0}$ *and* V^i *has one singular point* v *along* E*, which coincides with the vertex of* $E \cong \mathbb{F}_{b,0}$*, of analytic type* $(v \in V^i) \simeq \frac{1}{b}(1, 1, -1)$.

(3) $(D^i \cdot l^i) = (S^i \cdot l^i) = 1$*, where* l^i *is a generator of the rulings on the cone* $E \cong \mathbb{F}_{b,0}$.

(4) *If* $X^i \not\cong X^{i+1}$ *(i.e., if* E *is not contained in* D^i*), then* X^i *is obtained as a half-point attachment to* X^{i+1} *of* (b, k)*-type for some* $1 \le k \le b$ *(cf. Definition 1.1).*

Proof. Our proof consists of several steps.

Step 1. We have $(V^i, D^i) \in (\mathbf{Dlt})_3$, $(V^i, \emptyset) \in (\mathbf{Canonical})_3$ and D^i is a Cartier divisor on V^i (cf. Lemma 2.1 and $(\mathsf{B})_i$). Then the similar argument as in the proof of Claim 3 in Lemma 2.6 says that V^i has terminal singularities along the boundary D^i. Furthermore, since the complement $X^i = V^i \backslash D^i$ is an open affine subset of the original smooth affine algebraic threefold X by construction of our inductive procedure, X^i is smooth. Hence V^i itself is terminal.

Step 2. By Lemma 2.15, the exceptional divisor E of $\phi^i : V^i \to V^{i+1}$ is contracted to a smooth point $Q = \phi^i(E) \in V^{i+1}$. Then, by virtue of the remarkable result due to M. Kawakita (cf. [Ka01]), ϕ^i is obtained as the weighted blow-up at $Q \in V^{i+1}$ with weights $\mathrm{wts}(x, y, z) = (1, a, b)$, where (x, y, z) are the suitable system of local analytic coordinates at $Q \in V^{i+1}$ and $b \ge a > 0$ are positive integers such that $\gcd(a, b) = 1$. Note that the exceptional divisor E of ϕ^i is isomorphic to the weighted projective plane $\mathbb{P}(1, a, b)$ and $K_{V^i} \equiv {\phi^i}^*(K_{V^{i+1}}) + (a + b)E$. In fact, we can restrict the possibility of the weights $\mathrm{wts} = (1, a, b)$ as follows:

Claim. $a = 1$ *and* $(-K_{V^i} \cdot l^i) = 1 + (1/b)$*, where* l^i *is a generator of the rulings on the cone* $E \cong \mathbb{P}(1, 1, b) = \mathbb{F}_{b,0}$.

Proof of Claim. Let l^i be the curve on $E \cong \mathbb{P}(1, a, b)$ which is numerically equivalent to the class $\mathcal{O}_{\mathbb{P}}(1)$. Then we have $(-K_{V^i} \cdot l^i) = \frac{(a+b)}{ab}$. Noting that $(-K_{V^i} \cdot l^i) > 1$, we know that the integer a is equal to 1 as desired. □

Step 3. Since the ray $R^i = \mathbb{R}_+[l^i]$ is $(K_{V^i} + D^i)$-negative and D^i is Cartier, we have $(D^i \cdot l^i) = 1$. This means that $D^i|_E \equiv bl^i$. If the curve $D^i \cap E$ does not pass through the vertex v of the cone $E \cong \mathbb{F}_{b,0}$, then $X^i = V^i \backslash D^i$ contains v, which is of analytic type $(v \in V^i) \simeq \frac{1}{b}(1,1,-1)$. This is a contradiction as X^i is smooth. Thus the scheme-theoretic intersection $D^i|_E$ is written as $D^i|_E = \sum_{l=1}^k m_l C_l$, where C_l's are distinct generators of the rulings on $E \cong \mathbb{F}_{b,0}$ and m_l's are positive integers such that $\sum_{l=1}^k m_l = b$. Hence X^i is obtained as a half-point attachment to X^{i+1} of (b,k)-type (cf. Definition 1.1). Thus we complete the proof. □

Thus we can ascertain Proposition 2.1 in the case of $(-K_{V^i} \cdot l^i) > 1$.

2.3.3. Conclusion

Summarizing the arguments performed in **2.3.1** and **2.3.2** above, we obtain the assertions stated in Proposition 2.1. The inductive argument then concludes the assertions in Theorem 1.1. Note that our proof given above is valid for *every* process $(*)$ of $(\mathsf{LMMP})_3$ starting with the pair (V, D). Hence we obtain the desired explicit descriptions as in Theorem 1.1 for *every* $(\mathsf{LMMP})_3$ which starts with the dlt pair (V, D).

3. Affine Threefolds with $\overline{\kappa}(X) = -\infty$

In this section, we shall deal with the structure of smooth affine algebraic threefolds X with log Kodaira dimension $\overline{\kappa}(X) = -\infty$ satisfying the condition $(\natural)$. Since the statement and the proof for it are very similar to those in [Ki05, §3], we shall only sketch the proof. The essential lies in the combination of Theorem 1.1 with the result due to F. Campana and H. Flenner (cf. [C-F93]) concerning the classification of $\mathbb{Q}$-Fano threefolds containing certain kinds of surfaces having at most Du Val singularities in the smooth loci[*3]. Namely, our result is stated as follows:

Theorem 3.1 (cf. [C-F93])**.** *Let X be a smooth affine algebraic threefold with log Kodaira dimension $\overline{\kappa}(X) = -\infty$. Suppose that X can be embedded into a smooth projective threefold (V, D) satisfying the condition $(\natural)$. Then the process $(*)$ of $(\mathsf{LMMP})_3$ starting with the dlt pair (V, D) brings it to a $(\mathsf{LMfs})_3$, say $\pi : (V', D') \to W$, and X is obtained from $X' := V' \backslash D'$ via the composite of suitable half-point attachments unless $X \cong X'$ (cf.* Theorem 1.1, Definition 1.1*). More precisely, one of the following types occurs concerning $\pi : (V', D') \to W$:*

C_2-type: *$\pi : V' \cong \mathbb{P}(\mathcal{E}) \to W$ is a $\mathbb{P}^1$-bundle over a smooth projective surface W, where $\mathcal{E} := \pi_* \mathcal{O}_{V'}(D')$ is a rank 2 vector bundle on W, and $D' \sim \mathcal{O}(1)$.*

D'_2-type: *$\pi : V' \to W$ is a quadric bundle over a smooth curve W with a general fiber $(F, D'|_F) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$, and with at most finitely many singular fibers $G \cong \mathbb{Q}_0^2$ and the vertex of each G sits in a hypersurface singularity of*

[*3] This classification in [C-F93] is performed under the more specific assumption that a $\mathbb{Q}$-Fano threefold contains a *smooth* del Pezzo surface in the smooth locus. But, the same proof can be applied to the present situation. We shall comment on this matter in the sequel.

analytic type $o \in (xy + z^2 + t^k = 0) \subset \mathbb{C}^4 : (x, y, z, t)$ *for* $k \geq 1$, *where* $\mathbb{Q}_0^2 \hookrightarrow \mathbb{P}^3$ *is a quadric cone.* π *is obtained as the restriction of the* $\mathbb{P}^3$*-bundle* $V' \subset \mathbb{P}(\mathcal{E}) \to W$, *where* $\mathcal{E} := \pi_*\mathcal{O}_{V'}(D')$ *is a rank* 3 *vector bundle over* W.

D_3-type: $\pi : V' \cong \mathbb{P}(\mathcal{E}) \to W$ *is a* $\mathbb{P}^2$*-bundle over a smooth curve* W *with a fiber* $(F, D'|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, *where* $\mathcal{E} := \pi_*\mathcal{O}_{T^\sharp}(D^\sharp)$ *is a rank* 3 *vector bundle over* W.

D_3'-type: $\pi : V' \to W$ *is a* $\mathbb{P}^2$*-fibration over a smooth curve* W *with a general fiber* $(F, D'|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *and with at most finitely many singular fibers* $G \cong \mathbb{S}_4$ *and the vertex of* G *sits in a hyper-quotient singularity of analytic type* $o \in (xy + z^2 + t^k = 0) \subset \mathbb{C}^4 : (x, y, z, t)/\mathbb{Z}_2(1, 1, 1, 0)$ *for* $k \geq 1$, *where* $\mathbb{S}_4 \subset \mathbb{P}^5$ *is a cone over the quartic normal rational curve* $\subset \mathbb{P}^4$.[*4]

$\mathbb{Q}$-Fano: V' *is a* $\mathbb{Q}$*-Fano threefold with the Picard number* $\varrho(V') = 1$. *More precisely, the classification of the pair* (V', D') *up to deformations is given as in the following fashion:*

(i) $(\mathbb{P}(1, 1, 2, 3), \mathcal{O}(6))$;

(ii) $((6) \subset \mathbb{P}(1, 1, 2, 3, a), \{x_4 = 0\} \cap (6))$ *with* $a \in \{3, 4, 5\}$;

(iii) $((6) \subset \mathbb{P}(1, 1, 2, 2, 3), \{x_3 = 0\} \cap (6))$;

(iv) $((6) \subset \mathbb{P}(1, 1, 1, 2, 3), \{x_0 = 0\} \cap (6))$;

(v) $(\mathbb{P}(1, 1, 1, 2), \mathcal{O}(c))$ *with* $c \in \{2, 4\}$;

(vi) $((4) \subset \mathbb{P}(1, 1, 1, 1, 2), \{x_0 = 0\} \cap (4))$;

(vii) $((4) \subset \mathbb{P}(1, 1, 1, 2, a), \{x_4 = 0\} \cap (4))$ *with* $a \in \{2, 3\}$;

(viii) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(c))$ *with* $c \in \{1, 2, 3\}$, $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}}(c))$ *with* $c \in \{1, 2\}$;

(ix) $((3) \subset \mathbb{P}(1, 1, 1, 1, 2), \{x_4 = 0\} \cap (3))$;

(x) $((3) \subset \mathbb{P}^4, \mathcal{O}(1))$;

(xi) $((2) \cap (2) \subset \mathbb{P}^5, \mathcal{O}(1))$;

(xii) $(B_5, \mathcal{O}(1))$, *where* $B_5 \hookrightarrow \mathbb{P}^6$ *is a linear section of the Grassmann variety* $\mathrm{Gr}(2, 5) \hookrightarrow \mathbb{P}^9$ *parametrizing lines in* $\mathbb{P}^4$;

Sketch of the Proof of Theorem 3.1 Let X be an affine algebraic threefold with log Kodaira dimension $\overline{\kappa}(X) = -\infty$. Assume that X has an SNC compactification $X \hookrightarrow (V, D)$ satisfying the condition ($\natural$). Then, by Theorem 1.1, we have the process $(*)$ of $(\mathsf{LMMP})_3$ starting with $(V, D) \in (\mathbf{Dlt})_3$:

$$(*) \quad \phi : (V, D) \overset{\phi^0}{\dashrightarrow} (V^1, D^1) \overset{\phi^1}{\dashrightarrow} \cdots\cdots \overset{\phi^{s-1}}{\dashrightarrow} (V^s, D^s) = (V', D'),$$

in such a way that $(V', D') \in (\mathbf{Dlt})_3$ has a structure of $(\mathsf{LMfs})_3$, say $\pi : (V', D') \to W$, and X is constructed from the complement $X' := V' \backslash D'$ via the composite of suitable half-point attachments (cf. Definition 1.1). More precisely to say, the morphism π is obtained as the contraction of an extremal ray $R' \subset \overline{\mathrm{NE}}\,(V')$ which intersects $K_{V'} + D'$ negatively. Since X' is affine by

[*4] In fact, a Mori fiber space of D_3'-type is constructed from that of D_3-type via the simple kinds of elementary links explicitly in the framework of Sarkisov Program (cf. [Me02]).

construction, the general curves l' from the ray R' intersects the boundary D' but are not contained in it. Hence $(D' \cdot l') \in \mathbb{N}$. (Note that D' is a Cartier divisor on V'.) Since $(K_{V'} + D' \cdot l') < 0$, we have $(-K_{V'} \cdot l') > 1$. Hence the type of this Mori fiber space $\pi : V' \to W$ and the value $(D' \cdot l')$ are described as one of the following:

C_2-type: $\pi : V' \to W$ is a conic bundle structure over a normal surface W and $(D' \cdot l') = 1$, where l' is a smooth fiber of π.

D'_2-type: $\pi : V' \to W$ is a quadric bundle over a smooth curve W and $(D' \cdot l') = 1$, where l' is a generator of rulings on a general fiber $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ of π.

D_3, D'_3-type: $\pi : V' \to W$ is a $\mathbb{P}^2$-fibration over a smooth curve W and $1 \leq (D' \cdot l') \leq 2$, where l' is a line on a general fiber $F \cong \mathbb{P}^2$ of π.

$\mathbb{Q}$-Fano: V' is a $\mathbb{Q}$-Fano threefold with $\varrho(V') = 1$ of Fano index greater than one. (Indeed, as D' is Cartier and $K_{V'} + D'$ is negative on V', the Fano index of V' is greater than one.)

Then, for the C_2, D'_2, D_3 and D'_3-types, we have the desired assertions by the same argument as in [Me02, 5.7.1, 5.7.2, 5.7.3, 5.7.4]. Hence we shall deal with the case of $\mathbb{Q}$-Fano in what follows. By construction, S' is a Cartier divisor on V' such that the anti-canonical divisor $-K_{V'}$ is written as $-K_{V'} \equiv \rho S'$ for some rational number $\rho > 1$. Then, by virtue of the result due to V.A. Alexeev [Al89], we can find a smooth member in the linear system $|D'|$, say T'. It is obvious that T' is contained in the smooth locus NonSing(V'). Hence, the argument in [C-F93, §4] then can be applied to the present situation without any modification to obtain the description of the pair (V', D') as stated in Theorem 3.1.

4. Affine Threefolds with $\overline{\kappa}(X) \geq 0$

In this section, we shall treat affine algebraic threefolds X with non-negative Kodaira dimension $\overline{\kappa}(X) \geq 0$ satisfying ($\natural$). Recall that a smooth affine algebraic surface Y with log Kodaira dimension $\overline{\kappa}(Y) = 1$ has the structure of a $\mathbb{C}^*$-fibration over a curve (cf. [Kaw79]). As the three-dimensional analogue of this result, we are especially interested in the case where $\overline{\kappa}(X)$ is of intermediate dimension, i.e., the case of $\overline{\kappa}(X) = 1$ or 2. The result is stated below (cf. Theorem 4.1). Once we obtain Theorem 1.1, it is not difficult to prove Theorem 4.1 by making use of Log Abundance Theorem (cf. [K-M-M94]).

Theorem 4.1. *Let X be a smooth affine algebraic threefold with non-negative log Kodaira dimension $\overline{\kappa}(X) \geq 0$. Suppose that X can be embedded into a smooth projective threefold (V, D) satisfying the condition ($\natural$). Then the process $(*)$ of* (LMMP)$_3$ *starting with the dlt pair (V, D) brings it to a log minimal model, say (V', D'), and X is obtained from $X' := V' \backslash D'$ via the composite of suitable half-point attachments unless $X \cong X'$ (cf. Theorem* 1.1*, Definition* 1.1*). Moreover, we have the following concerning the fibration structure on X in case of $\overline{\kappa}(X) = 1$ or* 2*.*

(1) *If* $\overline{\kappa}(X) = 1$, *then there exists a morphism* $\psi : X \to B$ *onto a smooth curve* B *such that a general fiber of* ψ *is an open algebraic surface with log Kodaira dimension* $\overline{\kappa} = 0$.

(2) *If* $\overline{\kappa}(X) = 2$, *then there exists a* $\mathbb{C}^*$*-fibration* $\psi : X \to B$ *onto a normal quasi-projective surface* B.

Proof of Theorem 4.1 Let X be an affine algebraic threefold with non-negative log Kodaira dimension $\overline{\kappa}(X) \geq 0$. Suppose that X has an SNC compactification $X \hookrightarrow (V, D)$ which satisfies ($\natural$). The former half of the assertions is contained in Theorem 1.1. Hence we have only to prove the assertions (1) and (2). Since $K_{V'} + D'$ is nef and $\overline{\kappa}(X) = \kappa(V'; K_{V'} + D')$ by Theorem 1.1, Log Abundance Theorem (cf. [K-M-M94]) implies that the morphism $\Phi : V' \to \mathbb{P}$ defined by the base point free complete linear system $|N(K_{V'} + D')|$ yields a morphism $\Psi : V' \to W$ with connected fibers onto a normal projective variety W of $\dim(W) = \overline{\kappa}(X)$ via the Stein factorization. We consider the restriction $\psi' := \Psi|_{X'} : X' \to B$ of Ψ onto the open affine subset $X' = V' \backslash D'$. On the other hand, as the original affine threefold X is constructed from X' via the composite of suitable half-point attachments, we may assume that X is obtained as the complement $X = \widetilde{V} \backslash \widetilde{D}$, where $\widetilde{V}$ is reached via the composite of suitable weighted blow-ups at smooth points (the weights of weighted blow-ups here are determined by the types of appearing half-point attachments), say $\sigma : \widetilde{V} \to V'$, with centers contained in the proper transforms of D', and $\widetilde{D}$ is the proper transform on $\widetilde{V}$ of D'. Let $\psi := (\Psi \circ \sigma)|_X : X \to B$ denote the restriction of $\Psi \circ \sigma$ onto the complement $X = \widetilde{V} \backslash \widetilde{D}$. It is then obvious that a general fiber of ψ coincides with that of ψ'. In what follows, we consider according to $\overline{\kappa}(X) = 1$ or 2, separately.

Case: $\overline{\kappa}(X) = 1$**.** Then W is a smooth projective curve and a general fiber F' of Ψ is a smooth projective surface with $(K_{V'} + D')|_{F'} = K_{F'} + D'|_{F'} = 0$. Since (V', D') is a three-dimensional dlt pair, we have $(D'_j, \Delta'_j) \in (\mathbf{Dlt})_2$ for an irreducible component D'_j of the boundary D', where $\Delta'_j := \mathrm{Diff}_{D'_j}(D' - D'_j)$ (cf. [Ko-Mo98, Chapter 5]). More precisely, as V' is terminal (see the argument in Step 1 in the proof of Lemma 2.17), we have $\Delta'_j = (\sum_{k \neq j} D'_k)|_{D'_j}$. In particular, the boundary components D'_j's are normal and intersect normally to each other. Hence we may assume that $D'|_{F'}$ is a reduced SNC divisor on F'. We then have $\overline{\kappa}(F'_0) = \kappa(F'; K_{F'} + D'|_{F'}) = 0$, where $F'_0 := F' \backslash (D' \cap F')$ is the complement. Thus it follows that a general fiber F'_0 of $\psi : X \to B$ is a smooth open algebraic surface with log Kodaira dimension $\overline{\kappa} = 0$.

Case: $\overline{\kappa}(X) = 2$**.** In this case, W is a normal projective surface. Let $C \subset W$ be a general smooth curve, $G := \Psi^*(C) \subset V'$ the pull-back of C, and l a general fiber of $\Psi|_G : G \to C$. If $(D' \cdot l) \leq 0$, then the affineness of X' implies that $l \subset D'$. This is absurd. Hence we have $(D' \cdot l) > 0$. Then $(-K_G \cdot l)_G = (-K_{V'} \cdot l) = (D' \cdot l) > 0$, thus we know that l is isomorphic to $\mathbb{P}^1$ and $(D' \cdot l) = 2$. Therefore, it follows that $\psi : X \to B$ gives rise to a $\mathbb{C}^*$-fibration over a normal quasi-projective surface as desired.

Thus we complete the proof of Theorem 4.1. □

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Department of Mathematics
Faculty of Science
Saitama University
Saitama, 338-8570
JAPAN
E-Mail: tkishimo@rimath.saitama-u.ac.jp