BORROMEAN SURGERY FORMULA FOR THE CASSON INVARIANT

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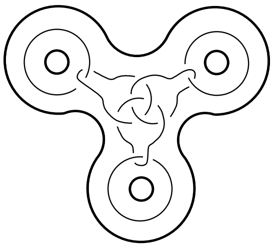
Abstract. Every oriented integral homology 3-sphere is obtained from S^3 by some Borromean surgery (or clasper surgery) moves. We give an explicit formula for the Casson invariant of an integral homology sphere given by such a surgery presentation. The formula involves simple classical invariants, namely the framing, linking number and Milnor's triple linking number.

1. Introduction

The notion of *Borromean surgery* was first introduced by S. Matveev [Mt] in the eighties. Roughly, a Borromean surgery on a 3-manifold $M = S^3_L$ is achieved by band-summing a copy of the Borromean link and the surgery link L in S^3. Matveev showed that two closed oriented 3-manifolds are *Borromean equivalent*, i.e. are related by a sequence of such surgery moves, if and only if they have the same homology and linking form. M. Goussarov and K. Habiro considered surgery moves along embedded graphs, called claspers, which refine the notion of Borromean surgery [G, H]. The Y_k-equivalence, which is generated by orientation-preserving diffeomorphisms and surgeries along claspers of degree k, coincides with the Borromean equivalence for $k = 1$ and becomes finer as k increases. Indeed, a surgery along a degree 1 clasper (or Y-graph) is defined as the surgery along a 6-component framed link obtained by embedding the link depicted below (we make use of the blackboard framing convention).



Borromean surgery is also the elementary move of the Goussarov-Habiro finite type invariant theory for closed oriented 3-manifolds, which essentially coincides with Ohtsuki's theory for integral homology spheres.

As a consequence of Matveev's result, every oriented integral homology 3-sphere is obtained from S^3 by surgery along claspers. It is thus a natural problem to give easily computable formulas for the variation of the Casson invariant λ under such a surgery move. Actually, $\lambda(M_G) = \lambda(M)$ for all clasper G of degree at least 3 in an integral homology sphere M. This is an immediate consequence of the fact that the Casson invariant is a finite type invariant of degree 2 in the Goussarov-Habiro sense. For claspers of degree 2, the formula is (at least implicitly) already known, from computations of the Kontsevich integral: it is given by a quadratic expression in the linking numbers of the clasper. We provide here a direct proof using only elementary clasper theory. Our main result is Theorem 4.1, which gives the formula for surgery along a connected degree 1 clasper. It expresses the variation of the

Casson invariant in terms of simple classical invariants, namely the framing, linking number and Milnor's triple linking number – see §4.1 for an explicit statement. A simple application of this formula is given.

The paper is organized as follows. In §2, we recall the definition and basic properties of the Casson invariant for integral homology spheres and give a short review of the theory of claspers and the Goussarov-Habiro finite type invariant theory. In §3 and 4, we state and prove the above-mentioned surgery formulas, respectively for the case of degree 2 and 1. §4 is completed by some remarks on the main theorem.

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2. Definitions and preliminary facts

In this paper, an *integral homology sphere* will always mean an oriented integral homology 3-sphere.

2.1. The Casson invariant of integral homology spheres.

2.1.1. *The Casson invariant.* We denote by $\mathbf{Z}HS$ the set of integral homology spheres, considered up to orientation-preserving diffeomorphisms.

Theorem 2.1 (A. Casson)**.** *There is a unique function*

$$\lambda : \mathbf{Z}HS \longrightarrow \mathbf{Z}$$

such that, for every $M, N \in \mathbf{Z}HS$ and for every knot K in M :

1. $\lambda(S^3) = 0$.
2. *For any $n \in \mathbf{Z}$, let M_{K_n} be the result of $\frac{1}{n}$-Dehn surgery on M along K. Then:*
$$\lambda(M_{K_{n+1}}) - \lambda(M_{K_n}) = \frac{1}{2}\Delta''_K(1),$$
where $\Delta_K(t)$ denotes the Alexander polynomial of K.
3. $\lambda(-M) = -\lambda(M)$, *where $(-M)$ denotes the homology sphere M with inverse orientation.*
4. $\lambda(M \sharp N) = \lambda(M) + \lambda(N)$, *where $\sharp$ denotes the connected sum.*
5. *The Rochlin invariant $\mu(M)$ is the mod 2 reduction of $\lambda(M)$. More precisely,*
$$\mu(M) \equiv \lambda(M) \ (mod\ 2).$$

This unique function is called the *Casson invariant* of integral homology spheres. Its existence was initially given by the construction of Casson. λ counts, in some sense, the conjugacy classes of irreducible representations of $\pi_1(M)$ in $SU(2)$.

2.1.2. *Johannes crossing change Formula for the Casson invariant.* In [J], J. Johannes expresses the difference of the Casson invariant of 3-manifolds presented by links in S^3 which differ by a crossing change within a component. This type of move on surgery links was studied in [Mt] under the name of *Whitehead move.*

Let $L^+ = L_1^+ \cup L_2 \cup ... \cup L_n$ and $L^- = L_1^- \cup L_2 \cup ... \cup L_n$ be two framed links in S^3, with same framing, which only differ by a crossing change on the first component. Let $L_a \cup L_b$ be the 2-component link obtained from $L_1^{\pm}$ by smoothing this crossing (see Fig.2.1).[1]

Denote by M the linking matrix of $L_{\pm}$, and consider the order n matrix

[1]Here and throughout this paper, blackboard framing convention is used.

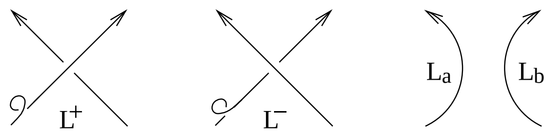


FIGURE 2.1

$$S(L) = \begin{pmatrix} l_{ab} & l_{a2} & \dots & l_{an} \\ l_{b2} & s_2 & \dots & l_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ l_{bn} & l_{2n} & \dots & s_n \end{pmatrix}$$ where

- $\cdot\ l_{ab} = lk(L_a, L_b)$,
- $\cdot\ l_{ai} = lk(L_a, L_i)$ and $l_{bi} = lk(L_b, L_i)$; $2 \leq i \leq n$,
- $\cdot\ l_{ij} = lk(L_i, L_j)$; $2 \leq i \neq j \leq n$,
- $\cdot\ s_i$ is the framing of L_i ; $i \in \{1, ..., n\}$.

Then Johannes formula gives:

$$\lambda(S^3_{L_+}) - \lambda(S^3_{L_-}) = \frac{det\,(S(L))}{det\,(M)}. \tag{2.1}$$

Note that the formulas given in [J] are for the Casson-Walker invariant, which is equal to 2λ. Note also that we changed the formula in [J] by a sign so that (2.1) agrees with Thm.2.1.[2]

Remark 2.2. So far, we refered to two types of moves relating all integral homology spheres, namely surgery along a ±1-framed knot and crossing change within a component of a surgery link. Casson and Johannes respectively give a formula for the variation of the Casson invariant under these moves. In this paper, we give such a formula for a third type of move which relates all integral homology spheres, namely Borromean surgery.

2.2. **Clasper theory for 3-manifolds.** Let us briefly recall from [H, GGP, G] the basic notions of clasper theory for 3-manifolds.

Definition 2.3. Let M be a compact connected oriented 3-manifold. A *clasper* G in M is an embedding

$$G : F \longrightarrow M$$

of a surface F which is a thickening of a (non-necessarily connected) uni-trivalent graph having a copy of S^1 attached to each of its univalent vertices (here we do not allow connected components without trivalent vertex).
The (thickened) circles are called the *leaves* of G, the trivalent vertices are called the *nodes* of G and we still call the thickened edges of the graph the *edges* of G.

In particular, a *tree clasper* is a connected clasper obtained from the thickening of a simply connected unitrivalent graph (with circles attached).
The *degree* of a clasper G is the minimal number of nodes of its connected components.

Given a clasper G in M, there is a precise procedure to construct, in a regular neighbourhood of the clasper, an associated framed link L_G: *surgery along the clasper* G simply means surgery along L_G. Though the procedure for the construction of L_G will not be explained here, it is well illustrated by the two examples of Figure 2.2. We respectively call these two particular types of claspers $\mathbf{Y}$-*graphs* and $\mathbf{H}$-*graphs*.

[2]For example, consider the right-handed trefoil knot T in S^3: (2.1) gives us $\lambda(S^3_{T_{+1}}) = 1$, which can be easily verified by Thm.2.1(2). It is actually well known that $S^3_{T_{+1}}$ is the Poincaré sphere P and that $\lambda(P) = 1$ (see for example [L3]).

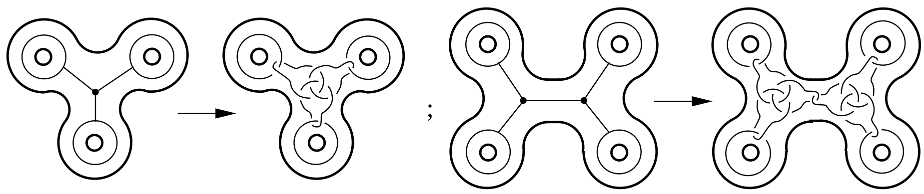


FIGURE 2.2. A degree 1 and a degree 2 clasper and the associated framed links in their regular neighbourhoods.

In [H, Prop. 2.7], K. Habiro gives a list of 12 moves on claspers which gives *equivalent* claspers, that is claspers with diffeomorphic surgery effect. We will freely use *Habiro's moves* (which are essentially derived from Kirby calculus) by refering to their numbering in Habiro's paper.

Definition 2.4. Let $k \geq 1$ be an integer. A surgery move on M along a connected clasper G of degree k is called a *Y_k-move*.
The *Y_k-equivalence*, denoted by $\sim_{Y_k}$, is the equivalence relation on 3-manifolds generated by the Y_k-moves and orientation-preserving diffeomorphisms (with respect to the boundary).

Y_1-moves were originally introduced by S. Matveev under the name of *Borromean surgery* (as Fig.2.2 suggests). It is shown [Mt, Thm. 2] that two closed oriented 3-manifolds are Y_1-equivalent if and only if they have the same first homology groups and isomorphic linking forms. In particular, any two integral homology spheres are Y_1-equivalent.

Finally, an important property of this Y_k-equivalence relation is that it becomes finer as k increases: if $k \leq l$ and if $M \sim_{Y_l} N$, then we also have $M \sim_{Y_k} N$.

2.3. **The Goussarov-Habiro finite type invariants theory.** Borromean surgery is the elementary move defining a finite type invariant theory for closed compact oriented 3-manifolds known as the *Goussarov-Habiro theory* [G, GGP],[H, §8.4.2]. For integral homology spheres, this theory essentially coincides with Ohtsuki's theory, based on surgery along (±1)-framed algebraically split links [O1].

Let $\mathcal{S}$ denote the free $\mathbf{Z}$-module generated by orientation-preserving diffeomorphism classes of integral homology spheres. In $\mathcal{S}$, put

$$[M;F] = \sum_{F'\subset F} (-1)^{|F'|} S^3_{F'},$$

where the sum ranges over all possible subset of F (starting from the empty set). Denote by $\mathcal{S}_k$ the submodule generated by the elements $[M;F]$ for all claspers $F = F_1 \cup ... \cup F_p$ in an integral homology sphere M such that $\sum_i degF_i = k$. We call *finite type invariant of degree k* any map $f : \mathcal{S} \longrightarrow A$, where A is an Abelian group, which vanishes on $\mathcal{S}_{k+1}$. It is well known that the Casson invariant is of finite type.

Proposition 2.5. [O1, GGP] *The Casson invariant is a finite type invariant of degree* 2 *(in the Goussarov-Habiro sense).*

This proposition can also be proved as a consequence of C. Lescop's sum formula for the Casson-Walker invariant [L2].

The Y_k-equivalence relation plays an important role in the understanding of the Goussarov-Habiro invariants. Indeed, two homology spheres within the same Y_{k+1}-equivalence class are not distinguished by any invariant of degree k (it is an easy consequence of the definition of a finite type invariant and elementary calculus of claspers). As a consequence, $\lambda(M_G) = \lambda(M)$ for all clasper G of degree at least 3 in an integral homology sphere M.

3. Variation of the Casson invariant under Y_2-surgery

We first compute the Casson invariant of the homology sphere obtained from S^3 by surgery along a Θ-clasper. A Θ-*clasper* is a connected degree 2 clasper without leaves, and whose edges form a (trivialy embedded) θ graph as in Fig.3.1.

Lemma 3.1. *If G is a Θ-clasper, then* $\lambda(S^3_G) = -2$.

Proof. Consider the surgery link L_G associated to G. L_G is Kirby-equivalent to the two-component 0-framed link L shown in the right part of Fig.3.1 (this can be seen by applying Habiro's moves 2 and 9 to G). By successively applying formula (2.1)

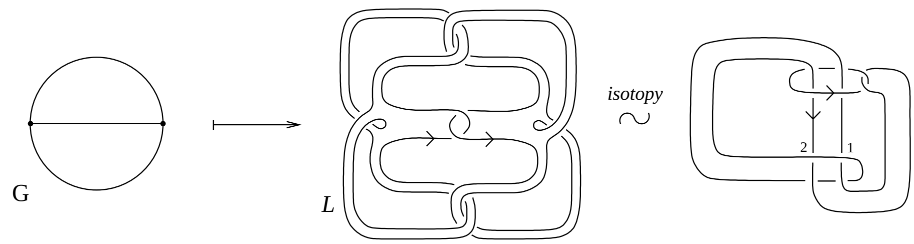
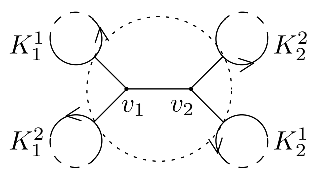


Figure 3.1. A Θ-clasper and the associated surgery link L.

at crossings 1 and 2, we obtain $\lambda(S^3_G) = \lambda(S^3_H) - 2$, where H denotes the positive 0-framed Hopf link. It follows that $\lambda(S^3_G) = -2$. □

This lemma will be used to prove the general surgery formula for the degree 2 case, which we now state. For that purpose we need to fix a little more notations and conventions.

Notations 3.2. Let G be a connected degree 2 clasper in M, whose vertices are denoted by v_1 and v_2. By Habiro's move 2, we can freely assume that G is a **H**-graph, i.e. G is a tree clasper. Up to isotopy, we can always suppose that there exists a 3-ball in M which intersects the edges and nodes of G as shown below.



G is an oriented surfaces, so at each vertex the two adjacent leaves are naturally ordered: we denote by K_i^1 and K_i^2 the two leaves adjacent to v_i ($i = 1, 2$). The leaves are oriented as depicted.

Theorem 3.3. *Let M be an integral homology sphere, and G be a connected degree 2 clasper in M. Then*

$$\lambda(M_G) - \lambda(M) = -2\big(lk(K_1^1, K_2^2).lk(K_1^2, K_2^1) - lk(K_1^1, K_2^1).lk(K_1^2, K_2^2)\big).$$

Remark 3.4. As said in the introduction, this formula is not really new. Indeed, it comes as a consequence of [GR, Prop. 3.4], which expresses the degree $2n$ of the Aarhus integral of $S^3_G - S^3$, where G is a degree $2n$ clasper, in terms of the so-called *complete contraction* of G. It can also be seen as a direct corollary of the sum formula of C. Lescop [L2]. However, our proof only uses elementary properties of the Casson invariant and standard calculus of claspers.

Proof. First, observe that M is obtained from S^3 by surgery along a (non-necessarily connected) clasper G_0. We can homotop G in a small ball of S^3 disjoint from G_0: by [O2, Lem. E.16 and E.17], this does not change the Y_3-equivalence class of $M_G \cong S^3_{G_0 \cup G}$. By Prop.2.5 we thus have $\lambda(M_G) = \lambda(M) + \lambda(S^3_G)$. So we can simply suppose that $M = S^3$ in the rest of the proof.
Now, we can suppose that the leaves are unknotted and unframed, that is, we can *simplify* the leaves of G. By using [GGP, Cor. 4.3], $\lambda(S^3_G)$ is equal to a sum $\sum_i \lambda(S^3_{G_i})$, where each leaf of G_i satisfies one of the following conditions:

(i) the leaf bounds a genus 1 surface disjoint from G_i and with respect to which the leaf is 0-framed;
(ii) the leaf bounds a disk disjoint from G and with respect to which the leaf is (± 1)-framed;
(iii) the leaf links another leaf of G_i as a 0-framed Hopf link.

If G_i contains a leaf of type (i), it is equivalent to a degree 3 clasper (apply Habiro's move 9) and thus $\lambda(G_i) = 0$. By lemma [GGP, Lem. 4.9], the same holds if G_i has a leaf of type (ii). So the only terms contributing to the Casson invariant are those G_i with all leaves of type (iii), which are counted by the products of the linking numbers of the leaves of G. More precisely,

$$\begin{aligned} \lambda(S^3_G) &= lk(K_1^1, K_1^2).lk(K_2^1, K_2^2).\lambda(S^3_{G_1}) \\ &+ lk(K_1^1, K_2^2).lk(K_1^2, K_2^1).\lambda(S^3_{G_2}) + lk(K_1^1, K_2^1).lk(K_1^2, K_2^2).\lambda(S^3_{G_3}), \end{aligned}$$

where G_1, G_2 and G_3 are the 3 claspers depicted in Figure 3.2. Now, by applying

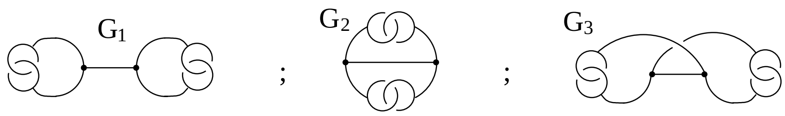
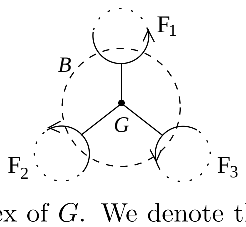


FIGURE 3.2. The claspers G_1, G_2 and G_3

Habiro's move 2, each pair of Hopf-linked leaves is equivalent to an edge. So G_1 is equivalent to a clasper with two looped edges, which has no surgery effect (see [GGP, Lem. 2.3]). G_2 being equivalent to a Θ-clasper, we have $\lambda(S^3_{G_2}) = -2$ by Lem.3.1. Likewise $\lambda(S^3_{G_3}) = 2$, as a consequence of Lem.3.1 and [GGP, Cor. 4.6]. □

4. VARIATION OF THE CASSON INVARIANT UNDER Y_1-SURGERY

4.1. **Statement of the main result.** Let G be a connected degree 1 clasper in an integral homology sphere M. First of all, note that up to isotopy one can always assume that there is a 3-ball B in M which intersects G as depicted below.



Fix an orientation of the vertex of G. We denote the three leaves of G by F_1, F_2 and F_3 according to this orientation. Denote by f_i the framing of F_i, and by l_{ij} the linking number $lk(F_i, F_j)$, $1 \leq i \neq j \leq 3$ (the leaves are oriented as in the figure above).

Theorem 4.1. *The difference* $\lambda(M_G) - \lambda(M)$ *is given by the formula*

$$-f_1.f_2.f_3 - 2.l_{12}.l_{13}.l_{23} - 2.\mu_{123}(G) - 2.l_{12}.l_{23} + \sum_{\circlearrowleft_{1,2,3}} l_{23}.(l_{23}+1).f_1,$$

where the sum is over all cyclic permutations of the indices $(1,2,3)$ *and where* $\mu_{123}(G)$ *denotes* Milnor's triple linking number of G.

Indeed, as explained in the next section, there is a 3-component string link σ_G naturally associated to G. Recall that a string link is a pure tangle, whithout closed components. $\mu_{123}(G)$ is thus defined as Milnor's triple linking number of this string link – see below for a definition.

4.2. **Milnor invariants for Y-graphs.** We first recall the classical definition of Milnor's invariants, then show how to extend it to **Y**-graphs, and finally recall some helpful formulas.

4.2.1. *Definition of Milnor invariants.* J. Milnor defined in [Mi1, Mi2] a family of invariants of oriented, ordered links in S^3, known as Milnor's $\overline{\mu}$-invariants.

Given an n-component link in S^3, we have a presentation of the fundamental group π of the link complement using the Wirtinger relations: for any diagram of L, each arc gives a generator and each crossing gives a relation. Now, when working modulo the q^{th} subgroup π_q of the lower central series of π, we have a presentation with only n generators, given by a meridian m_i of each component of L.[3] In particular, for $1 \leq i \leq n$, the longitude l_i of the i^{th} component of L is expressed modulo π_q as a word in the m_i's. The *Magnus expansion* of l_i is the formal power series in non-commuting variables $X_1, ..., X_n$ obtained by substituting $1+X_i$ for m_i and $1 - X_i + X_i^2 - X^3 + ...$ for m_i^{-1}. Denote by $\mu_{ijk}(L)$ the coefficient of X_iX_j in the Magnus expansion of λ_k. *Milnor's triple linking number* $\overline{\mu}_{ijk}(L)$ is the residue class of $\mu_{ijk}(L)$ modulo the greatest commun divisor of the linking numbers of components i, j, k (for any $q \geq 3$).
The indeterminacy comes from the choice of the meridians m_i. In other words, it comes from the indeterminacy of representing the link as the closure of a string link [HL]. Indeed, μ_{ijk} is a well-defined invariant for string links.

4.2.2. *The string link representation of a* **Y***-graph.* Observe that the leaves of a **Y**-graph G in a homology sphere M form a 3-component link with a natural 'basing' (a base point on each component) given by the attaching region of the edges. There is thus a natural way of representing this link as the closure of a string link. Consider a 3-ball B as in §4.1. By isotopying the leaves of G, we can regard $G \cap (S^3 \setminus B)$ as a framed 3-component string link σ_G in the homology ball $M \setminus B$ – see Fig.4.1. We call σ_G the *string link representation of* G.
By *Milnor's triple linking number of* G $\mu_{ijk}(G)$, we mean the Milnor invariant μ_{ijk} of σ_G (recall that the definition of Milnor's invariant for string links extends to string links in any homology ball).

4.2.3. *Some formulas.* We now recall some formulas for Milnor's triple linking numbers of a **Y**-graph G in S^3 which are used in the proof of our main theorem.

First, we have the following formulas relating the various Milnor's triple linking numbers of G [Me1, §2.3]:

$$\mu_{123}(G) = -\mu_{213}(G) + l_{13}l_{23} = -\mu_{132}(G) + l_{12}l_{13} - l_{13} \quad (4.1)$$

[3]For any group G, its lower central series is defined inductively by $G_1 = G$ and $G_k = [G, G_{k-1}]$.

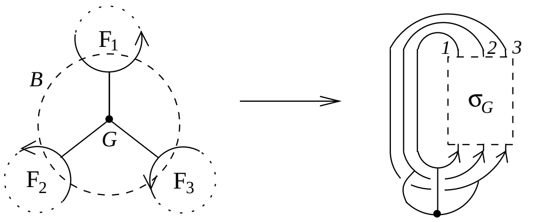


FIGURE 4.1. The string link representation of a **Y**-graph.

$$\mu_{123}(G) = \mu_{231}(G) - l_{12}l_{23} + l_{13}l_{23} = \mu_{312}(G) - l_{12}l_{23} + l_{12}l_{13} - l_{13} \tag{4.2}$$

Also, let us briefly recall Polyak's Skein relations for the computation of Milnor invariants [P].
Let σ_+ and σ_- be two 3-component string links in $D^2 \times I$ which are identical outside of a ball where their components σ_j and σ_k look as depicted in Fig.4.2. By using the two possible ways of smoothing this crossing, we consider the two tangles α'_j and α'_k as depicted in Fig.4.2. Note that α'_j respects the orientation of both σ_j and σ_k, whereas α'_k follows σ_j against the orientation.

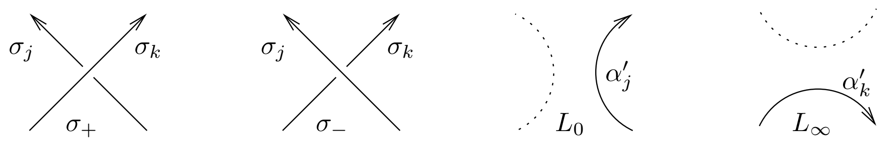


FIGURE 4.2

Let $\hat{\sigma}_i$ denote the (usual) closure of σ_i. We denote by L_0 (resp. L_∞) a 2-component link whose diagram is obtained by closing the tangle α'_j (resp. α'_k) with an arc which overpasses $\hat{\sigma}_i$. Then we have

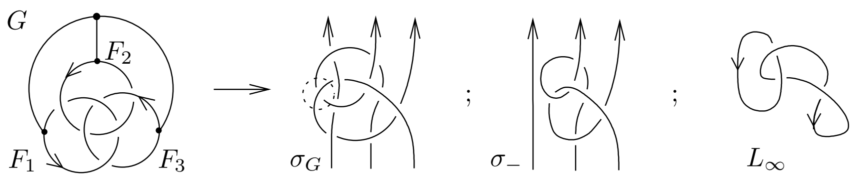
$$\mu_{jik}(\sigma_+) = \mu_{jik}(\sigma_-) + lk(L_0) \quad \text{and} \quad \mu_{ijk}(\sigma_+) = \mu_{ijk}(\sigma_-) + lk(L_\infty), \tag{4.3}$$

where $G_\pm$ is a **Y**-graph in S^3 with string link representation $\sigma_\pm$ and where lk denotes the linking number.

Example 4.2. Consider the **Y**-graph G whose three leaves form a Borromean link. Denote by σ_+ its string link representation. By suitably applying Polyak's formula, we obtain:

$$\mu_{123}(G) = \mu_{123}(\sigma_-) + lk(L_\infty) = 0 + 1$$

where σ_- and L_∞ are as depicted below.



4.3. **Some Lemmas.** Before proving Theorem 4.1, it is necessary to state some lemmas. First, we prove the surgery formula for a simple situation.

Lemma 4.3. *Suppose the leaves F_i of G are three unlinked f_i-framed trivial knots in S^3 ($i = 1, 2, 3$). Then we have*

$$\lambda(S^3_G) = -f_1.f_2.f_3.$$

Proof. By applying three Rolfsen moves [R], one sees that the surgery link associated to G is equivalent to the Borromean rings B with framings $-1/f_1$, $-1/f_2$, $-1/f_3$. B being algebraically split, we know by a formula of T. Hoste [Ho] that

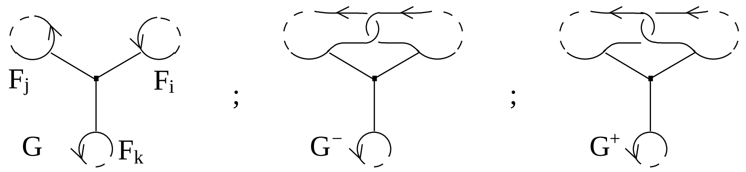
$$\lambda(S^3_G) = (-f_1).(-f_2).(-f_3).a_1(L'),$$

where a_1 denotes the coefficient of $z^{\sharp B+1}$ in the Conway polynomial. The result follows, as the Conway polynomial of the Borromean link is z^4. □

For example, if the three leaves of G are (-1)-framed trivial knots, then S^3_G is the Poincaré sphere, and Lem.4.3 gives $\lambda(S^3_G) = 1$ (see §2.1.2).

The next lemma considers the behaviour of the Casson invariant under the addition of a clasp between the leaves of a **Y**-graph.

Lemma 4.4. *Let G be a* **Y***-graph in S^3. Let G^- and G^+ be obtained by adding respectively a* negative *or* positive *clasp between leaves F_i and F_j of G as shown below.*



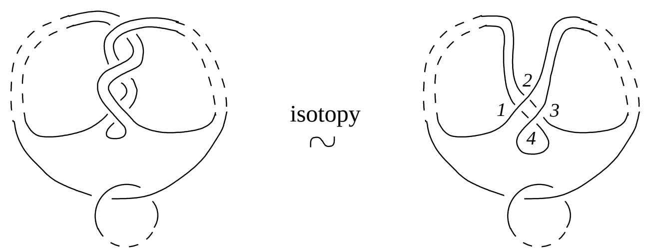
Then

$$\lambda(S^3_G) = \lambda(S^3_{G^-}) - 2.l_{ik}.l_{jk} + 2.f_k.l_{ij},$$

and

$$\lambda(S^3_G) = \lambda(S^3_{G^+}) + 2.l_{ik}.l_{jk} - 2.f_k.(l_{ij} + 1).$$

Proof. The proof is a straightforward application of the crossing change formula (2.1). For example, consider the 2-component link depicted below, which is Kirby-equivalent to the link $L_{G'}$ associated to G^-.



By successively changing the crossing numbered from 1 to 4, we obtain a link associated to G. The formula for G^+ is a consequence of the one for G^-. Indeed, adding a negative clasp to G^+ as in Lem.4.4 gives G. □

We now state a technical lemma which is used on several occasions in the proof of our main theorem.

Lemma 4.5. *Let G be a* **Y***-graph in S^3. Suppose T is a degree k clasper with a leaf f which is a small 0-framed meridian of a leaf F of G (see Fig.4.3 for an example). Denote by T' the degre $k+1$ clasper obtained by connecting the edges incident to f and F. We have*

(1). $\lambda(S^3_{G\cup T}) = \lambda(S^3_G)$, *if* $k \geq 2$.
(2). $\lambda(S^3_{G\cup T}) = \lambda(S^3_G) + \lambda(S^3_{T'})$, *if* $k = 1$.

Proof. Simple calculus of claspers displayed in Fig.4.3 (apply Habiro's move 7, then moves 2, 11 and 5) shows that $G \cup T$ is equivalent to the union $P \cup T'$, where the clasper P satisfies $S^3_P \cong S^3_G$ thanks to Habiro's move 3.

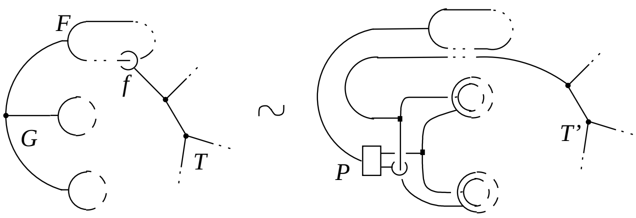


FIGURE 4.3

By [O2, Lem. E.16 and E.17], we can homotop the degree $k+1$ clasper T' in a ball disjoint from P without changing the Y_{k+2}-equivalence class of $S^3_{P\cup T'}$, that is we have $S^3_{G\cup T} \sim_{Y_{k+2}} S^3_G \sharp S^3_{T'}$. The result follows from Thm.2.1(4) and Prop.2.5. □

4.4. **Proof of the main result.** In this section we prove Theorem 4.1, which expresses the variation of the Casson invariant of a homology sphere M under surgery along a **Y**-graph G. The proof is organized as follows. In a first part, we prove the formula in the simplest case where $M = S^3$. In a second part, we show that the formula remains valid in the general case.

Let us consider the case where G is a **Y**-graph in $M = S^3$. The proof goes in three steps, where we simplify the clasper until we obtain an elementary situation where the Casson invariant is easily computed. At each step, we perform simple transformations on the clasper, so that we can compute how the various invariants involved in the statement vary.

Step 1. *Killing the linking numbers.* First, we can apply Lem.4.4 to kill all the linking numbers of the leaves. By applying $|l_{23}|$ times Lem.4.4 to leaves 2 and 3 of G, we obtain a clasper G' whose leaves 2 and 3 have linking number 0, and satisfying

$$\lambda(S^3_G) = \lambda(S^3_{G'}) - 2.l_{12}.l_{13}.l_{23} + f_1.l_{23}.(l_{23}+1).$$

Clearly, the framing of the leaves remains unchanged, but Milnor's triple linking number μ_{123} may change. The difference $\mu_{123}(G) - \mu_{123}(G')$ is easily computed using Polyak's Skein formula (4.3). We obtain $\mu_{123}(G) = \mu_{123}(G') - l_{12}.l_{23}$.
Likewise, we apply $|l_{13}|$ times Lem.4.4 to leaves 1 and 3, then $|l_{12}|$ times to leaves 1 and 2. We obtain a **Y**-graph G_0 whose leaves have vanishing linking numbers, same framing as those of G and such that

$$\lambda(S^3_G) = \lambda(S^3_{G_0}) - 2.l_{12}.l_{13}.l_{23} + \sum_{\circlearrowleft_{1,2,3}} f_1.l_{23}.(l_{23}+1). \tag{4.4}$$

On the other hand, (4.3) and (4.1) give us $\mu_{123}(G') = \mu_{123}(G_0)$, so we have

$$\mu_{123}(G) - \mu_{123}(G_0) = -l_{12}.l_{23}. \tag{4.5}$$

Step 2. *Unknotting and pairwise unlinking the leaves.* We now show that the leaves of G_0 can be supposed to be unknotted and pairwise unlinked.
Consider the leaves 1 and 2 of G_0, regarded as a 2-component link in S^3. As it has zero linking number, this link is Δ-equivalent to the 2-component unlink by [MN]. Each Δ-move is achieved by surgery along a **Y**-graph whose leaves are small meridians of leaves 1 and/or 2 (see [H, §7.1]). Likewise, we apply the theorem of Murakami and Nakanishi to the algebraically split 2-component link formed by the

leaves 1 and 3, and by the leaves 2 and 3. There is thus a sequence of n Δ-moves (of the type described above)

$$G_0 = G^1 \mapsto G^2 \mapsto ... \mapsto G^n = G_0^u,$$

such that the leaves of G_0^u form pairwise a 2-component unlink.
Now, such a surgery move doesn't change the Casson invariant of the corresponding manifold. Indeed, by Lem.4.5(2) we have $\lambda(S^3_{G^{k+1}}) = \lambda(S^3_{G^k})+\lambda(S^3_{H^k})$, where H^k is a degree 2 clasper which satisfies $\lambda(S^3_{H^k}) = 0$ by Thm.3.3. An example is depicted in Fig.4.4. We thus have $\lambda(S^3_{G_0}) = \lambda(S^3_{G_0^u})$.

On the other hand, note that these moves do not change the framings, linking

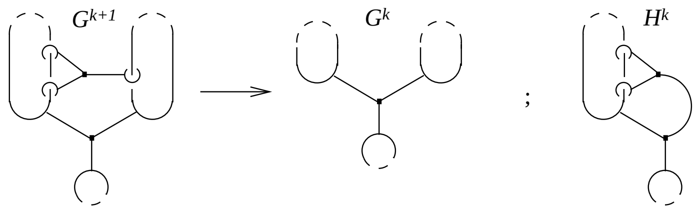


FIGURE 4.4. A Δ-move which doesn't affect the Casson invariant.

numbers and triple linking numbers of the leaves.

Step 3. *Killing the triple linking number and the framings.* Let $\varepsilon \in \{+1; -1\}$ be the sign of $\mu_{123}(G_0^u) = \mu_{123}(G_0)$, and let $m := |\mu_{123}(G_0)|$.
Denote by $G_1^{\pm}$ the **Y**-graph whose leaves have the same framing as those of G and form a Borromean link as shown in Fig.4.5. Denote respectively by B_+ and B_- these two Borromean links. As the notation suggests, $G_1^{\pm}$ satisfies $\mu_{123}(G_1^{\pm}) = \mu_{123}(B_{\pm}) = \pm 1$ (see [Mi1, Fig. 5] and Ex.4.2).
Likewise, denote by G_m^{ε} the **Y**-graph whose leaves have the same framing as those of G and form an algebraically split based link which is the band-sum of m copies of the Borromean link B_ε as shown in the right part of Fig.4.5.

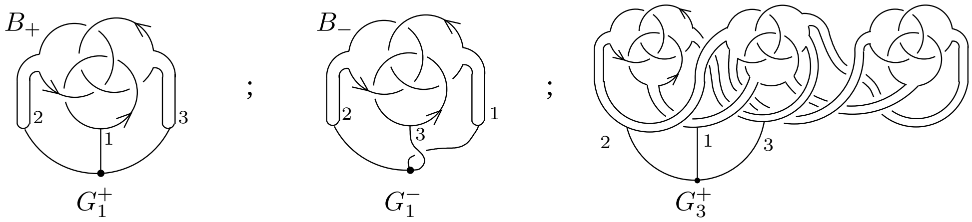


FIGURE 4.5. The **Y**-graphs G_1^+, G_1^- and G_3^+.

Recall from [TY] that two algebraically split links are C_3-equivalent (or clasp-pass equivalent) if and only if they have same Milnor's triple linking numbers, Arf invariants and Casson knot invariants.[4] Clearly, $\mu_{123}(G_0^u) = \mu_{123}(G_m^{\varepsilon})$. Moreover by Step 2 the leaves of G_0^u have zero Arf and Casson knot invariants, and so are the leaves of G_m^{ε} (by construction). So there is a sequence of *C_3-moves*

$$G_0^u = G_1 \mapsto G_2 \mapsto ... \mapsto G_p = G_m^{\varepsilon},$$

[4]Note that both this theorem of [TY] and the theorem of [MN] used in the previous step can be proven using the theory of claspers – see [Me1].

where each C_3-move is a surgery move along a degree 2 clasper whose leaves are all copies of a small meridians of a leaf of G_k $(1 \leq k \leq p-1)$. So by Lem.4.5(1) we have

$$\lambda(S^3_{G_0^u}) = \lambda(S^3_{G_0}) = \lambda(S^3_{G_m^\varepsilon}).$$

Now we have to compute $\lambda(G_m^\varepsilon)$.

· Suppose $\mu_{123}(G) > 0$. Then by Habiro's move 10, each copy of B_+ in the leaves of G_m^+ can be replaced by a **Y**-graph whose leaves are meridians of each leaf of G_m^+ – see [H, Fig.34 (b)]. Using again Lem.4.5, we obtain $\lambda(S^3_{G_m^+}) = \lambda(S^3_{G_{m-1}^+}) + \lambda(S^3_T)$, where T is a Θ-clasper. By repeating this process m times, we obtain

$$\lambda(S^3_{G_m^+}) = \lambda(S^3_{G_f}) - 2.\mu_{123}(G_0^u),$$

where G_f is the **Y**-graph whose leaves are the f_i-framed trivial knots $(1 \leq i \leq 3)$. It follows by Lem.4.3 that

$$\lambda(S^3_{G_m^+}) = -f_1.f_2.f_3 - 2.\mu_{123}(G_0^u).$$

· If $\mu_{123}(G) < 0$, each copy of B_- in G_m^- is replaced by a **Y**-graph as above, such that Lem.4.5 gives $\lambda(S^3_{G_m^+}) = \lambda(S^3_{G_{m-1}^+}) + \lambda(S^3_{T_s})$, where T_s is a Θ-clasper having the attaching regions of two edges exchanged at some node. By Lem.3.1 and [GGP, Cor. 4.6] we have $\lambda(S^3_{T_s}) = 2$. We thus obtain

$$\lambda(S^3_{G_m^-}) = \lambda(S^3_{G_m^+}) = -f_1.f_2.f_3 - 2.\mu_{123}(G_0^u). \tag{4.6}$$

Theorem 4.1 for $M = S^3$ follows from (4.4), (4.5) and (4.6).

Now let us consider the general case. We aim to compute the difference $\lambda(M_G) - \lambda(M)$ where G is a **Y**-graph in an integral homology sphere M.
The pair (M, G) can be regarded as obtained by surgery along a clasper C from (S^3, G'), where G' is a **Y**-graph in S^3. In other words, $M_G \cong S^3_{G' \cup C}$.
Consider in S^3 the claspers G' and C. We can homotop C in a ball disjoint from G' in S^3. This homotopy is achieved by a sequence of crossing changes involving either edges or leaves of G' and C. Actually, up to isotopy of the claspers, and by eventually using Habiro's move 2, we can suppose that our crossing changes only involve the leaves of the claspers. Each such crossing change produces some new term, as illustrated by Fig.4.6 below.

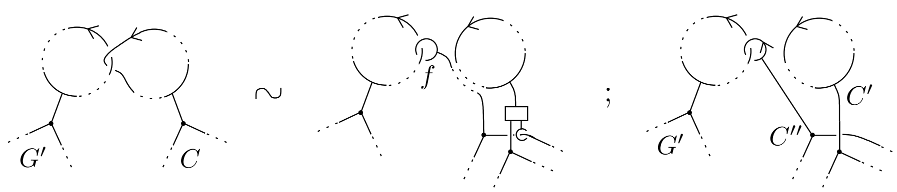


FIGURE 4.6. Here we use Habiro's moves 7, 2, 12 and 5 as in the proof of Lem.4.5.

If the connected component of C involved in the crossing change is of degree 1, [GGP, Lem. 4.10] and Thm.3.3 imply that $\lambda(S^3_{G' \cup C}) = \lambda(S^3_{G' \cup C' \cup C''})$, where C' and C'' are obtained by splitting the leaf of C as depicted in Fig.4.6. Otherwise, we simply have $\lambda(S^3_{G' \cup C}) = \lambda(S^3_{G' \cup C''})$ by Lem.4.5.
So after performing all the necessary crossing changes , we obtain

$$\lambda(M_G) = \lambda(S^3_{G' \cup C}) = \lambda(M) + \lambda(S^3_{G' \cup \overline{C}}),$$

where $\overline{C}$ is a union of **Y**-graphs whose three leaves are copies of a small meridian of each leaf of G' (as leaf f in Fig.4.6). Note that G' has same framings and linking numbers as G, as surgery along claspers preserves these invariants (see for example [Me2, §3.2.2]). We use the same arguments as in the first part of the proof (Step 3) to show that

$$\lambda(S^3_{G'\cup\overline{C}}) - \lambda(S^3_{G'}) = \mu_{123}(G) - \mu_{123}(G').$$

Namely, each component of $\overline{C}$ contributes on one hand to the Casson invariant by ± 2, as it produces a copy of the Θ-graph, and it contributes on the other hand to Milnor's triple linking number by ∓ 1, as it adds a copy of the Borromean rings to the leaves of G'. Details are left to the reader.

4.5. **Some remarks on Thm.4.1.** Here we list some remarks on our main theorem and its proof.

Remark 4.6. The formula in Thm.4.1 may look surprising in view of its non-invariance under cyclic permutation of the indices. This lack of symmetry is due to the incidence of Milnor's triple linking number in the formula – see (4.1) and (4.2).

Remark 4.7. By reducing modulo 2 the formula and using Thm.2.1(5), we obtain a similar formula for the variation of Rochlin's μ-invariant

$$\mu(M_G) - \mu(M) = f_1.f_2.f_3 \pmod 2.$$

In the case where the leaves are three unlinked framed f_i-framed unknots (as in Lem.4.3), this formula was already established independently by G. Massuyeau in [Mas] and E. Auclair and C. Lescop [AL].

Remark 4.8. One can also use Lescop's general surgery formula to express the variation of the Casson invariant under surgery along a **Y**-graph G in terms of the multivariable Alexander polynomial Δ. Consider the 6-component link L associated to G (Fig. 2.2) and denote by F_1, F_2 and F_3 the three components of L corresponding to the three leaves of G. Denote by B the standard Borromean link. Then [L1, 1.4.8] gives that $\lambda(M_G) - \lambda(M)$ equals

$$-\begin{vmatrix} f_1 & l_{12} & l_{13} \\ l_{12} & f_2 & l_{23} \\ l_{13} & l_{23} & f_3 \end{vmatrix}.\zeta(B) - \sum_{\circlearrowleft_{1,2,3}} \begin{vmatrix} f_1 & l_{12} \\ l_{12} & f_2 \end{vmatrix}.\zeta(B\cup F_3) - \sum_{\circlearrowleft_{1,2,3}} f_1.\zeta(B\cup F_2\cup F_3) - \zeta(L),$$

where, for an n-component link K, $\zeta(K) = \frac{\partial^n}{\partial t_1 \ldots \partial t_n}\Delta_K(1,...,1)$. One can easily check that $\zeta(B) = 1$. Now, by the same arguments as in Step 2 of §4.4, we can suppose that each F_i is an unknot and that each pair $F_i \cup F_j$ is a band-sum of $|l_{ij}|$ copies of the Hopf link (with the appropriate sign). So we can compute, using these models, the two sums in the formula. We obtain

$$\zeta(B\cup F_i) = 0 \quad \text{and} \quad \zeta(B\cup F_i\cup F_j) = -l_{ij}, \qquad \text{for all } 1 \le i \ne j \ne k \le 3.$$

It follows in particular that $\frac{\partial^6}{\partial t_1 \ldots \partial t_6}\Delta_L(1,...,1) = 2.\mu_{123}(G) + 2.l_{12}.l_{23}$.

Remark 4.9. The reader familiar with [H] would notice that the difference $\lambda(M_G) - \lambda(M)$ only depends on the C_3-equivalence class of G. Indeed, both the framing and the linking number are invariants of C_2-equivalence of the **Y**-graph, and Milnor's triple linking number is an invariant of C_3-equivalence.

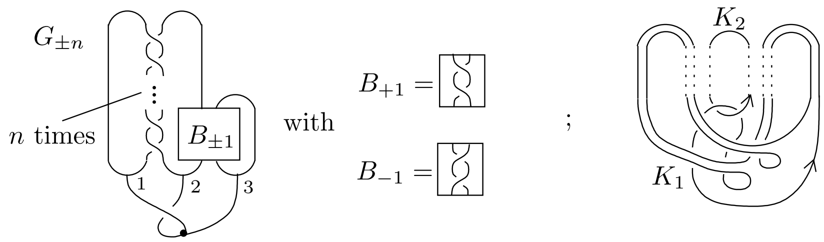
Remark 4.10. One can give a general formula for the variation of Casson invariant under surgery along a disjoint union G of **Y**-graphs by using strictly the same arguments as in the proof of our theorem. Such a formula involves a cubic expression in the linking numbers of the leaves of pairs of connected components of G.

4.6. **An application.** As a conclusion, we give an application of our main theorem. Namely we show how Thm.4.1 provides an alternative proof of a realization theorem for the Casson invariant of homology spheres of Mazur type.
Recall that a *homology sphere of Mazur type* is a 3-manifold given by surgery on S^3 along a 2-component link $K_1 \cup K_2$ such that $lk(K_1, K_2) = \pm 1$, K_1 is a trivial knot with framing 0 and K_2 has framing $r \in \mathbf{Z}$. Such an integral homology sphere bounds a contractible 4-manifold, and thus its Casson invariant is an even number. Y. Mizuma showed the following.

Theorem 4.11. [Mz] *For any even number k, there is a homology sphere of Mazur type M such that $\lambda(M) = k$.*

Mizuma explicitely describes this homology sphere in terms of double branched cover of S^3 along so-called knots of 1-*fusion*.

Thm.4.1 allows us to give explicit examples in terms of **Y**-graphs. Indeed, consider the **Y**-graph $G_{\pm n}$ depicted in the left part of Fig.4.6. Here, the three leaves have framing zero. Clearly, by Thm.4.1, surgery on S^3 along $G_{\pm n}$ produces an integral homology sphere with Casson invariant $\mp 2n$. On the other hand, the link



associated to $G_{\pm n}$ is Kirby-equivalent to a 2-component link $K_1 \cup K_2$ depicted on the right part (apply Habiro's move 9), which satisfies the above-mentioned conditions. Therefore $S^3_{G_{\pm n}} \cong S^3_{K_1 \cup K_2}$ is a homology sphere of Mazur type.

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