A trace theorem for Dirichlet forms on fractals

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Dedicated to Professor S. Watanabe on the occasion of his 70th birthday.

19 October, 2005

Abstract

We consider a trace theorem for self-similar Dirichlet forms on self-similar sets to self-similar subsets. In particular, we characterize the trace of the domains of Dirichlet forms on the Sierpinski gaskets and the Sierpinski carpets to their boundaries, where boundaries mean the triangles and rectangles which confine gaskets and carpets. As an application, we construct diffusion processes on a collection of fractals called fractal fields, which behave as the appropriate fractal diffusion within each fractal component of the field.

MSC: 46E35; 28A80; 31C25; 60J60
Keywords: Trace theorem; Self-similar sets; Dirichlet forms; Diffusions on fractals; Lipschitz spaces; Besov spaces; Sierpinski carpets

*Research partially supported by Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for Encouragement of Young Scientists, 15740089.
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1 Introduction

The trace of Sobolev spaces on $\mathbb{R}^n$ to linear subspaces have been studied in various directions as generalizations of the Sobolev imbedding theorem. There has also been extensive study how to extend Sobolev, Besov and Lipschitz spaces from subdomains of $\mathbb{R}^n$ to the whole spaces (see for example, [1, 26] and the references therein). Since 80's, there are generalizations of these problems for Besov-type spaces on more complicated spaces, namely on the so-called Alfors d-regular sets ([16, 29]).

On the other hand, recent developments of analysis on fractals give new lights to these problems. On many fractals such as Sierpinski gaskets and Sierpinski carpets, diffusion processes and the "Laplace" operators are constructed. It turns out that the domains of the corresponding Dirichlet forms are Besov-Lipschitz spaces.

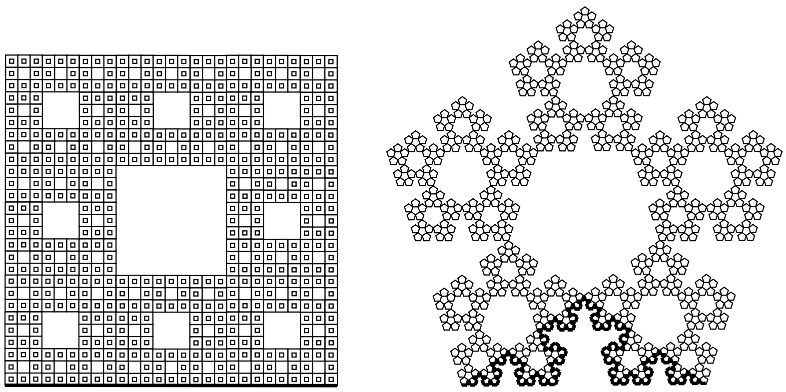

Figure 1: The Sierpinski carpet and the Pentakun

In this paper, we consider the following natural question: given a Besov-type space on a self-similar fractal K, what is the trace of the space to a self-similar subspace L? We would indicate two examples in Figure 1. The left figure is when K is the so-called 2-dimensional Sierpinski carpet (see Section 5 3) for the definition) and L is the line on the bottom (drawn by the thick line). The right figure is when K is the Pentakun (a self-similar fractal determined by five contraction maps; see Section 5 2) for the definition) and L is a Koch-like curve (drawn by the thick curve). In each case, the domain of the Dirichlet form on K is the Besov-Lipschitz space, but one cannot obtain the trace using the general theory given by Jonsson-Wallin ([16]) and Triebel ([29]).

This problem was quite recently solved by Jonsson ([15]) for one typical case, i.e. when K is the 2-dimensional Sierpinski gasket and L is the bottom line. But his methods rely strongly on the structure of the Sierpinski gasket and its Dirichlet form, and they cannot be applied to the so-called infinitely ramified fractals such as Sierpinski carpets. Instead, we use the self-similarity of the form and some kind of uniform property of harmonic functions which can be guaranteed by the Harnack inequalities. Our methods can be applied to the Sierpinski carpets (even to the high dimensional ones) and we can state the trace theorem under some abstract framework. In fact, we would need various assumptions for K and for the Dirichlet form on K, which are stated in Section 2. Unless these conditions are satisfied, there may be various possibility of the trace, because of the "complexity" of the space (see Section 5 4) for an example).

In order to prove our trace theorem, we give a discrete approximation of our Besov-Lipschitz space in Section 3.1. This approximation result is also new and is regarded as a generalization of the main result in [17]. The restriction theorem is given in Section 3.2; the key estimate (Proposition 3.8) is based on the idea used by one of the author in [13]. The extension theorem is given in Section 3.3, where the classical construction of the Whitney decomposition and the extension map is modified and generalized to this framework.

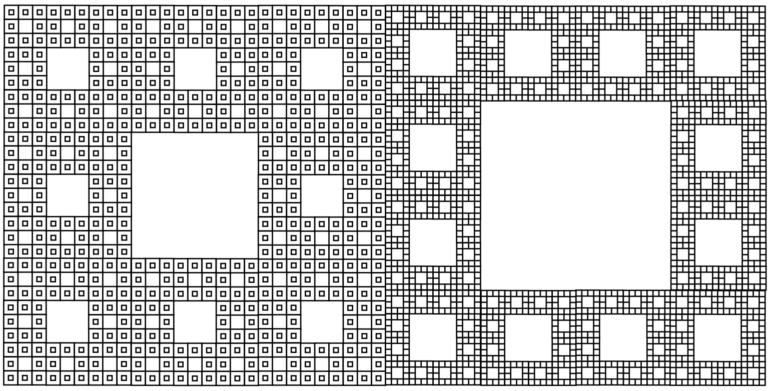

Figure 2: An example of fractal fields

Such a trace theorem has an important application to the penetrating process, which is discussed in Section 6. Let us indicate one concrete example. Given two types of Sierpinski carpets as in Figure 2 (the left carpet is determined by contraction maps with the contraction rate 1/3 and there is one hole in the middle, while the right carpet is determined by contraction maps with the contraction rate 1/4 and there is one bigger hole in the middle). On each carpet, one can construct a self-similar diffusion; the question is whether one can construct a diffusion which behaves as the appropriate fractal diffusions within each carpet and which penetrates each fractal. In order to construct such a diffusion by the superposition of Dirichlet forms on each carpet, the key problem is whether there is enough functions whose restriction to each carpet is in the domain of each Dirichlet form. To answer this question, it is crucial to get the information of the trace of the Dirichlet form on each carpet to the line, which is the intersection of the two carpets. Indeed, when one of the author studied this problem on fractals in [20, 12], he needed a very strong assumption on each fractal because of the lack of the information of the trace. Our trace theorem can be applied here and we can construct penetrating processes on much wider class of fractals.

Throughout this article, if f and g depend on a variable x ranging in a set A, $f \asymp g$ means that there exists $C > 0$ such that $C^{-1}f(x) \le g(x) \le C\,f(x)$ for all $x \in A$. We will use c, with or without subscripts, to denote strictly positive constants whose values are insignificant.

2 Framework and the main theorem

Let (X, d) be a complete separable metric space. For $\alpha > 1$ and a finite index set W, let $\{F_i\}_{i\in W}$ be a family of α-*similitudes* on X, i.e. $\mathsf{d}(F_i(x), F_i(y)) = \alpha^{-1}\mathsf{d}(x, y)$ for all $x, y \in X$. Let S be a subset of W and let N denote the cardinality of S. Since $\{F_i\}_{i\in S}$ is a family of contraction maps, there exists a unique non-void compact set K such that $K = \bigcup_{i\in S} F_i(K)$. We assume that K is connected. Note that W will be needed in general when we define a self-similar subset L below. In various important examples such as 1), 3) in Section 5, we can take $W = S$.

We will make the relation to the shift space. The one-sided shift space Σ is defined by $\Sigma = W^{\mathbb{N}}$. For $w \in \Sigma$, we denote the i-th element in the sequence by w_i and write $w = w_1w_2w_3\cdots$. When $w \in W^n$, $|w|$ denotes n. For $v \in W^m$ and $w \in W^n$, we define $v\cdot w \in W^{m+n}$ by $v\cdot w = v_1v_2\cdots v_m w_1w_2\cdots w_n$. For $A \subset W^m$ and $B \subset W^n$, $A\cdot B$ denotes $\{v\cdot w : v \in A,\ w \in B\}$. The set $w\cdot A$ is defined as $\{w\}\cdot A$. By definition, $W^0 = \{\emptyset\}$ and $\emptyset\cdot A = A$.

Let $\mathfrak{G}$ be a group consisting of isometries on K. We assume the following.

- For each $i \in W$, there exist $j = j(i) \in S$ and $\Psi_i \in \mathfrak{G}$ such that $F_i = F_j \circ \Psi_i$.
- For each $(\Psi, \alpha) \in \mathfrak{G} \times S$, there exists $(\hat\Psi, \hat\alpha) \in \mathfrak{G}\times S$ such that $\Psi \circ F_\alpha = F_{\hat\alpha} \circ \hat\Psi$.

Note that, when $W = S$, we can always take as $\mathfrak{G}$ the trivial group consisting of one element. We write $F_{w_1\cdots w_n} = F_{w_1}\circ F_{w_2}\circ\cdots\circ F_{w_n}$ for $w = w_1w_2\cdots w_n$. We regard $F_\emptyset$ as an identity map. For $w \in W^n$ and $A \subset W^n$ for some $n \in \mathbb{Z}_+$, define $K_w = F_w(K)$ and $K_A = \bigcup_{v\in A} K_v$.

Lemma 2.1. *There exist maps $\Phi : \bigcup_{n\in\mathbb{Z}_+} W^n \to \bigcup_{n\in\mathbb{Z}_+} S^n$ and $\Psi : \bigcup_{n\in\mathbb{Z}_+} W^n \to \mathfrak{G}$ such that $F_w = F_{\Phi(w)} \circ \Psi(w)$ for each $w \in \bigcup_{n\in\mathbb{Z}_+} W_n$. In particular, $K_w = K_{\Phi(w)}$.*

Proof. Set $\Phi(\emptyset) = \emptyset$ and $\Psi(\emptyset) =$ the unit element of $\mathfrak{G}$. When $i \in W^1$, it suffices to set $\Phi(i) = j(i)$ and $\Psi(i) = \Psi_i$. Suppose that $\Phi(w)$ is defined for $w \in W^n$. Then, for $w' = w\cdot i$ with $i \in W$, $F_{w'} = F_w \circ F_i = F_{\Phi(w)} \circ \Psi(w) \circ F_{j(i)} \circ \Psi_i$. This is equal to $F_{\Phi(w)} \circ F_{\hat i} \circ \hat\Psi \circ \Psi_i$ for some $(\hat\Psi, \hat i) \in \mathfrak{G}\times S$. Therefore, it is enough to define $\Phi(w') = \Phi(w)\cdot \hat i$ and $\Psi(w') = \hat\Psi \circ \Psi_i$. $\square$

Define $\pi : \Sigma \to K$ by the relation $\{\pi(w)\} = \bigcap_m K_{w_1\cdots w_m}$ for $w = w_1w_2\cdots \in \Sigma$. Define

$$C_K := \pi^{-1}\left(\bigcup_{i,j\in S, i\neq j} (K_i \cap K_j)\right), \qquad P_K := \bigcup_{n\geq 1} \sigma^n(C_K), \tag{2.1}$$

where $\sigma : \Sigma \to \Sigma$ is the left shift map, i.e. $\sigma w = w_2w_3\cdots$ if $w = w_1w_2w_3\cdots$.

For $v, w \in W^n$, we write $v \overset{n,K}{\sim} w$ if $K_v \cap K_w \neq \emptyset$. For $w \in W^n$ and $A \subset W^n$, $w \overset{n,K}{\sim} A$ means that $w \overset{n,K}{\sim} v$ for some $v \in A$. For $A \subset W^n$, define $\mathcal{N}_0(A) = A$ and $\mathcal{N}_k(A) = \{v \in W^n \mid v \overset{n,K}{\sim} \mathcal{N}_{k-1}(A)\}$ for $k \in \mathbb{N}$ inductively. We set $\mathcal{N}_k(w) = \mathcal{N}_k(\{w\})$ for $w \in W^n$.

Let I be a subset of W. We assume that the cardinality N_I of I is less than N. Let L be a unique non-void compact set such that $L = \bigcup_{i\in I} F_i(L)$. Clearly, L is a subset of K. Denote $F_w(L)$ by L_w for $w \in \bigcup_{n\in\mathbb{Z}_+} I^n$. Let $M \in \mathbb{N}$. For $v,\ w \in I^n$, we write $v \underset{M}{\overset{n,L}{\longleftrightarrow}} w$ if $v \in \mathcal{N}_M(w)$. We fix M so that for each $i,\ j \in I$, there exist $i_1, i_2, \ldots \in I$ satisfying $i \underset{M}{\overset{1,L}{\longleftrightarrow}} i_1 \underset{M}{\overset{1,L}{\longleftrightarrow}} i_2 \underset{M}{\overset{1,L}{\longleftrightarrow}} \cdots \underset{M}{\overset{1,L}{\longleftrightarrow}} j$. In what follows, we omit M from the notation $\underset{M}{\overset{n,L}{\longleftrightarrow}}$. We assume the following.

(A1) $\sup_{n\in\mathbb{Z}_+} \max_{w\in S^n} \#(\mathcal{N}_1(w)\cap S^n) < \infty$ and $C_0 := \sup_{n\in\mathbb{Z}_+} \max_{w\in I^n} \#(\mathcal{N}_M(w)\cap I^n) < \infty$.

(A2) There exist $k_1, k_2 > 0$ such that, for $x, y \in L$, $n \in \mathbb{Z}_+$ and $v, w \in I^n$ with $x \in K_v$ and $y \in K_w$, $\mathsf{d}(x,y) < k_1\alpha^{-n}$ implies $v \overset{n,L}{\leftrightarrow} w$ and $v \overset{n,L}{\leftrightarrow} w$ implies $\mathsf{d}(x,y) < k_2\alpha^{-n}$.

(A3) There exist $k_1, k_2 > 0$ such that, for $x, y \in K$, $n \in \mathbb{Z}_+$ and $v, w \in S^n$ with $x \in K_v$ and $y \in K_w$, $\mathsf{d}(x,y) < k_1\alpha^{-n}$ implies $v \overset{n,K}{\sim} w$ and $v \overset{n,K}{\sim} w$ implies $\mathsf{d}(x,y) < k_2\alpha^{-n}$.

Let $\hat{\mu}$ and $\hat{\nu}$ be the canonical Bernoulli measures on $S^{\mathbb{N}}$ and $I^{\mathbb{N}}$, respectively. That is, they are infinite product measures of S (resp. I) with uniformly distributed measure. Denote by μ the image measures of $\hat{\mu}$ by the map $\pi|_{S^{\mathbb{N}}} : S^{\mathbb{N}} \to K$. In the same way, the probability measure ν on L is defined. By conditions (A1), (A2), and (A3) and [18, Theorem 1.5.7], the Hausdorff dimensions of K and L are equal to $d_f := \log N/\log\alpha$ and $d := \log N_I/\log\alpha$, respectively, and μ and ν are equivalent to the Hausdorff measures on K and L, respectively.

We will further assume the following.

(A4) $\mu(\{x \in K : \#(\pi^{-1}(x) \cap S^{\mathbb{N}}) = \infty\}) = 0$ and $\nu(\{x \in L : \#(\pi^{-1}(x) \cap I^{\mathbb{N}}) = \infty\}) = 0$.

Then, by Theorem 1.4.5 in [18], $\mu(K_w) = N^{-|w|}$ for every $w \in \bigcup_{n\in\mathbb{Z}_+} S^n$ and $\nu(L_w) = N_I^{-|w|}$ for every $w \in \bigcup_{n\in\mathbb{Z}_+} I^n$. It also holds that $\mu(L) = 0$.

Suppose that we are given a strong local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K,\mu)$. $\mathcal{F}$ is equipped with a norm $\|f\|_{\mathcal{F}} = (\mathcal{E}(f) + \|f\|^2_{L^2(\mu)})^{1/2}$. Here and throughout the paper, for each quadratic form $E(\cdot,\cdot)$, we abbreviate $E(f,f)$ as $E(f)$. We assume the following.

(A5) (Self-similarity) For each $f \in \mathcal{F}$ and $i \in S$, $F_i^* f \in \mathcal{F}$ where $F_i^* f = f \circ F_i$. Further, there exists $\rho > 0$ such that
$$\mathcal{E}(f) = \rho \sum_{i\in S} \mathcal{E}(F_i^* f), \quad f \in \mathcal{F}.$$

(A6) For every $\Psi \in \mathfrak{G}$, $\Psi^*\mathcal{F} = \mathcal{F}$, that is, $\{f \circ \Psi : f \in \mathcal{F}\} = \mathcal{F}$. Further, $\mathcal{E}(\Psi^* f) = \mathcal{E}(f)$ for all $f \in \mathcal{F}$.

(A7) Let $d_w = (\log \rho N)/(\log\alpha)$. Then $d_w > d_f - d$.

(B1) The space $\mathcal{F}$ is compactly imbedded in $L^2(K,\mu)$, and $\mathcal{E}(f) = 0$ if and only if f is a constant function.

For each subset A of W^m for some $m \in \mathbb{Z}_+$, let $\mathcal{F}_A$ be a function space on K_A such that $\{f|_{K_A} : f \in \mathcal{F}\} \subset \mathcal{F}_A \subset \{f \in L^2(K_A) : F_w^* f \in \mathcal{F} \text{ for all } w \in A\}$. The space $\mathcal{F}_A$ will be specified later for some class of Dirichlet forms in Section 4. Define, for $f, g \in \mathcal{F}_A$,
$$\mathcal{E}_A(f,g) = \rho^m \sum_{w\in A} \mathcal{E}(F_w^* f, F_w^* g). \tag{2.2}$$

We assume that $\mathcal{F}_A = \mathcal{F}_{A\cdot S^n}$ for all $n \in \mathbb{N}$ and $(\mathcal{E}_A, \mathcal{F}_A)$ is a closed form on $L^2(K_A, \mu|_{K_A})$. In what follows, we always consider $\mathcal{F}_A$ as a normed space with norm $\|f\|_{\mathcal{F}_A} = (\mathcal{E}_A(f) + \|f\|^2_{L^2(K_A)})^{1/2}$. Due to (A5), $\mathcal{E}_A(f) = \mathcal{E}_{A\cdot S^n}(f)$ holds for any $f \in \mathcal{F}_A$, and $\mathcal{E}_{\Phi(A)}(f) = \mathcal{E}_A(f)$ if $\#\Phi(A) = \#A$ by (A6). When $A = \{w\}$, we use the notation $\mathcal{E}_w$ in place of $\mathcal{E}_{\{w\}}$. Functions in $\mathcal{F}$ can be naturally considered as elements in $\mathcal{F}_A$ by the restriction of the domain. We often write simply f in place of $f|_{K_A}$ when we regard $f \in \mathcal{F}$ as an element of $\mathcal{F}_A$, for notational conveniences.

Definition 2.2. Let A be a nonempty subset of W^m for some $m \in \mathbb{Z}_+$. We say that A is $\mathcal{E}_A$-connected if, for $f \in \mathcal{F}_A$, $\mathcal{E}_A(f) = 0$ implies that f is constant on K_A.

Definition 2.3. Let $A \subset W^m$ and $B \subset W^n$ for some m and n. We say that A and B are of the same type if there exist a homeomorphism $F : K_A \to K_B$ and a bijection $\chi : A \to B$ such that $F \circ F_u = F_{\chi(u)}$ for all $u \in A$ and $F^*(\mathcal{F}_B) = \mathcal{F}_A$.

We assume the following.

(B2) There exists $\hat{I} \subset W$ such that the following hold.

(1) $\hat{I} \supset I$ and $\#\hat{I} < N$.

(2) For each $w \in I^n$, $\mathcal{N}_M(w) \cap \hat{I}^n$ is an $\mathcal{E}_{\mathcal{N}_M(w)\cap\hat{I}^n}$-connected set.

(3) There exist finite elements $u_1, \dots, u_k \in \bigcup_{n\in\mathbb{Z}_+} I^n$ such that, for any $w \in \bigcup_{n\in\mathbb{Z}_+} I^n$, there exists $j \in \{1, \dots, k\}$ such that $\mathcal{N}_M(w) \cap \hat{I}^{|w|}$ and $\mathcal{N}_M(u_j) \cap \hat{I}^{|u_j|}$ are of the same type, and moreover, $F(L_{\mathcal{N}_M(w)\cap\hat{I}^{|w|}}) = L_{\mathcal{N}_M(u_j)\cap\hat{I}^{|u_j|}}$ where F is provided in Definition 2.3.

(4) $C_1 := \sup_{n\in\mathbb{Z}_+} \max_{w\in\hat{I}^n} \#(\mathcal{N}_M(w)\cap I^n) < \infty$ and $C_2 := \sup_{n\in\mathbb{Z}_+} \max_{w\in S^n} \#\{v \in \hat{I}^n : \Phi(v) = w\} < \infty$.

For an open set $U \subset K$, define the capacity of U by

$$\mathrm{Cap}(U) = \inf\{\|u\|_{\mathcal{F}}^2 : u \in \mathcal{F}, u \geq 1 \ \mu\text{-a.e. on } U\}.$$

The capacity of any set $D \subset K$ is defined as the infimum of the capacity of open sets that contain D. We denote a quasi-continuous modification of $f \in \mathcal{F}$ by $\tilde{f}$. We assume the following.

(A8) There exists some $c > 0$ such that $\nu(D) \leq c\,\mathrm{Cap}(D)$ for every compact set $D \subset K$.

By Theorem 3.1 of [7], (A8) is equivalent to the following.

(A8)' The measure ν charges no set of zero capacity and $f \mapsto \tilde{f}|_L$ is a continuous map from $\mathcal{F}$ to $L^2(L, \nu)$.

We will provide sufficient conditions for (A8) in Section 4.

For each $n \in \mathbb{Z}_+$, define $Q_n : L^1(L, \nu) \to \mathbb{R}^{I^n}$ as

$$Q_n f(w) = \fint_{L_w} f(y)\,d\nu(y), \quad w \in I^n,$$

where in general $\fint_A \cdots d\lambda(y) := \lambda(A)^{-1} \int_A \cdots d\lambda(y)$ denotes the normalized integral on A. Then, one can easily check

$$N_I^{-1} \sum_{j\in I} Q_{m+1} f(w \cdot j) = Q_m f(w), \quad w \in I^m. \tag{2.3}$$

Let $m \in \mathbb{N}$, $A \subset S^m$, and $J \subset I^m$. Define $\mathcal{F}(J, A) = \{f \in \mathcal{F} : f = 0 \text{ on } K_{S^m\setminus A},\ Q_m(\tilde{f}|_L) = 0 \text{ on } J\}$, and define a closed subspace $\mathcal{H}(J, A)$ of $\mathcal{F}$ by

$$\mathcal{H}(J, A) = \{h \in \mathcal{F} : \mathcal{E}(h, f) = 0 \ \text{ for all } f \in \mathcal{F}(J, A)\}.$$

When J is an empty set, we omit it from the notation. We assume the following.

(B3) There exist some $l_0, m_0 \in \mathbb{Z}_+$, $C > 0$, a proper subset $D'(w)$ of $S^{|w|}$ with $w \in D'(w)$ for each $w \in \bigcup_{n\in\mathbb{Z}_+} \Phi(\hat{I}^{n+m_0})$, a finite subset $\Xi \subset \bigcup_{n\in\mathbb{Z}_+} \Phi(\hat{I}^{n+m_0})$ and subsets $D^\sharp(v)$ of $D'(v)$ with $v \in D^\sharp(v)$ for each $v \in \Xi$ such that the following hold.

(1) For each $w \in \bigcup_{n\in\mathbb{Z}_+} \Phi(\hat{I}^{n+m_0})$,

(a) $w \in D'(w)$ and $D'(w) \subset \mathcal{N}_{l_0}(w) \cap (\Phi(\hat{I}^n) \cdot S^{m_0})$,

(b) there exists $v \in \Xi$ such that

$$F_w^*(\{h \in \mathcal{H}(I^{|w|}, D'(w)) : \int_{K_{D'(w)}} h\, d\mu = 0,\ \rho^{-|w|}\mathcal{E}_{D'(w)}(h) \le 1\}) \subset F_v^*(\{h \in \mathcal{H}(I^{|v|}, D^\sharp(v)) : \|h\|_{\mathcal{F}_{D^\sharp(v)}} \le C\}).$$

(2) For each $v \in \Xi$, the operator $F_v^* : \mathcal{H}(D^\sharp(v))|_{K_{D^\sharp(v)}} \to \mathcal{F}$ is a compact operator, where $\mathcal{H}(D^\sharp(v))|_{K_{D^\sharp(v)}}$ is regarded as a subspace of $\mathcal{F}_{D^\sharp(v)}$.

We set $D(w) = D'(\Phi(w))$ for $w \in \bigcup_{n\in\mathbb{Z}_+} \hat{I}^{n+m_0}$. We have a sufficient condition concerning (B3); see Section 4.

The following assumption (B4) will be used in the restriction theorem.

(B4) For $f \in \mathcal{F}$, if $\mathcal{E}_{S^m\setminus\Phi(\hat{I}^m)}(f) = 0$ for every $m \in \mathbb{Z}_+$, then f is a constant function.

We next introduce Besov spaces.

Definition 2.4. For $1 \le p < \infty,\ 1 \le q \le \infty,\ \beta \ge 0$ and $m \in \mathbb{Z}_+$, set

$$a_m(\beta, f) := \gamma^{m\beta}\left(\gamma^{md_f} \iint_{\{(x,y)\in K\times K: \mathsf{d}(x,y) < c\gamma^{-m}\}} |f(x) - f(y)|^p\, d\mu(x)d\mu(y)\right)^{1/p}$$

for $f \in L^p(K, \mu)$, where $1 < \gamma < \infty,\ 0 < c < \infty$. Define a *Besov space* $\Lambda^\beta_{p,q}(K)$ as a set of all $f \in L^p(K,\mu)$ such that $\bar{a}(\beta, f) := \{a_m(\beta, f)\}_{m=0}^\infty \in l^q$. $\Lambda^\beta_{p,q}(K)$ is a Banach space with the norm $\|f\|_{\Lambda^\beta_{p,q}(K)} := \|f\|_{L^p(K)} + \|\bar{a}(\beta, f)\|_{l^q}$. Let $\hat{\Lambda}^\beta_{p,q}(K)$ denote the closure of $\Lambda^\beta_{p,q}(K) \cap C(K)$ in $\Lambda^\beta_{p,q}(K)$. $\Lambda^\beta_{p,q}(L)$ and $\hat{\Lambda}^\beta_{p,q}(L)$ are defined in the same way by replacing (K, μ) by (L, ν).

We remark that this definition is valid for general Alfors regular compact sets K with normalized Hausdorff measure μ. We use the notation $\Lambda^\beta_{p,q}(K)$ following [11]. $\Lambda^\beta_{p,q}(K)$ was denoted by Lip $(\beta, p, q)(K)$ in [14, 20] and by $\Lambda^{p,q}_\beta(K)$ in [28]. Note that different choices of $c > 0$ and $\gamma > 1$ provide the same space $\Lambda^\beta_{p,q}(K)$ with equivalent norms. In what follows, we will take $\gamma = \alpha$.

We are now ready to state our main theorems. Let $\beta = d_w/2 - (d_f - d)/2$.

Theorem 2.5. *Suppose that* (A1)–(A8) *and* (B1)–(B4) *hold. Then, for every $f \in \mathcal{F}$, $\tilde{f}|_L$ belongs to $\hat{\Lambda}^\beta_{2,2}(L)$. Moreover, there exists $c > 0$ such that $\|\tilde{f}|_L\|_{\Lambda^\beta_{2,2}(L)} \le c\|f\|_{\mathcal{F}}$ for every $f \in \mathcal{F}$.*

Theorem 2.6. *Suppose that* (A1)–(A8) *and* (C1)–(C2) *hold. (The conditions* (C1) *and* (C2) *will be defined in Section 3.3). Then, there exists a bounded linear map* ξ *from* $\hat{\Lambda}^{\beta}_{2,2}(L)$ *to* $\mathcal{F}$ *such that* $\xi(\Lambda^{\beta}_{2,2}(L) \cap C(L)) \subset \mathcal{F} \cap C(K)$ *and* $\widetilde{\xi f}|_L = f$ ν*-a.e. for all* $f \in \hat{\Lambda}^{\beta}_{2,2}(L)$*.*

In what follows, we often write $\mathcal{F}|_L = \hat{\Lambda}^{\beta}_{2,2}(L)$ to denote the assertions of two theorems above.

Remark 2.7. In the following two cases, we can prove $\hat{\Lambda}^{\beta}_{2,2}(L) = \Lambda^{\beta}_{2,2}(L)$.
1) $L \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$ and $\beta < 1$. In this case, the following trace theorem holds due to [16]; $B^{2,2}_{\beta+(n-d)/2}(\mathbb{R}^n)|_L = \Lambda^{\beta}_{2,2}(L)$ where $B^{2,2}_{\gamma}(\mathbb{R}^n)$ is the classical Besov space with smoothness order γ. Since $C^{\infty}_0(\mathbb{R}^n)$ is dense in $B^{2,2}_{\gamma}(\mathbb{R}^n)$ for $\gamma > 0$, it follows that functions from $C^{\infty}_0(\mathbb{R}^n)$ restricted to L are dense in $\Lambda^{\beta}_{2,2}(L)$.
2) $\beta > d/2$. In this case, the following holds due to [11] Theorem 8.1; $\Lambda^{\beta}_{2,\infty}(L) \subset \mathcal{C}^{\beta-d/2}(L)$, where $\mathcal{C}^{\lambda}(L)$ is a Hölder space defined as follows. $u \in \mathcal{C}^{\lambda}(L)$ if

$$\|u\|_{\mathcal{C}^{\lambda}(L)} := \|u\|_{L^{\infty}(L)} + \nu - \operatorname*{esssup}_{x,y\in L,\ x\neq y} \frac{|u(x)-u(y)|}{\mathsf{d}(x,y)^{\lambda}} < \infty. \tag{2.4}$$

Since $\Lambda^{\beta}_{2,2}(L) \subset \Lambda^{\beta}_{2,\infty}(L)$, we see that any element in $\Lambda^{\beta}_{2,2}(L)$ is continuous in this case.

Remark 2.8. Since ν is smooth with respect to $(\mathcal{E},\mathcal{F})$, we can consider the time changed Markov process with respect to the positive continuous additive functional associated with ν via the Revuz correspondence. By the general theory of Dirichlet forms, this has an associated regular Dirichlet form $(\check{\mathcal{E}},\check{\mathcal{F}})$ on $L^2(L,\nu)$ with $\check{\mathcal{F}} = \{f \in L^2(L,\nu) : f = \tilde{u}\ \nu\text{-a.e. on } L \text{ for some } u \in \mathcal{F}_e\}$, where $\mathcal{F}_e$ is the family of μ-measurable functions u on K such that $|u| < \infty$ μ-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}_{n\in\mathbb{N}}$ of functions in $\mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ μ-a.e. As is seen in the proposition below, $\mathcal{F}_e = \mathcal{F}$ in our framework. So, our main theorems determine the function space $\check{\mathcal{F}}$.

Proposition 2.9. *Under the condition (B1),* $\mathcal{F}_e = \mathcal{F}$*.*

Proof. By (B1), there exists some $c > 0$ such that

$$\left\| f - \int_K f\, d\mu \right\|^2_{L^2(K)} \le c\mathcal{E}(f), \quad f \in \mathcal{F}. \tag{2.5}$$

Let $u \in \mathcal{F}_e$. Take $\{u_n\}_{n\in\mathbb{N}}$ from $\mathcal{F}$ as in the definition of $\mathcal{F}_e$ in Remark 2.8. Define $g_n = u_n - \int_K u_n\, d\mu$ for each n. Then, $\{g_n\}_{n\in\mathbb{N}}$ is $\mathcal{E}$-Cauchy. Since $\int_K g_n\, d\mu = 0$, (2.5) implies that $\{g_n\}$ is also $L^2(K)$-Cauchy. Therefore, g_n converges to some g in $\mathcal{F}$. By taking a subsequence, we may assume that $g_n \to g$ μ-a.e. Thus, $\int_K u_n\, d\mu\, (= u_n - g_n)$ converges to some $C \in \mathbb{R}$. In particular, $\int_K u_n\, d\mu$ converges to C in $\mathcal{F}$ as a sequence of constant functions. Therefore, u_n converges to $g + C$ in $\mathcal{F}$. This implies that $u = g + C$ belongs to $\mathcal{F}$. □

3 Proof of main theorems

3.1 Discrete approximation

In this section, we assume (A1)–(A8). For $n \in \mathbb{Z}_+$, define a bilinear form on I^n as

$$E_{(n)}(g,g) = \sum_{v,w\in I^n,\ v \overset{n,L}{\leftrightarrow} w} (g(v)-g(w))^2 \quad \text{for} \quad g \in \mathbb{R}^{I^n}.$$

We then have the following discrete characterization of $\Lambda^{\beta}_{2,q}(L)$ (for related results, see [17]).

Lemma 3.1. *Let $\beta > 0$ and $q \in [1,\infty]$. Then, there exists $c_1 > 0$ such that for each $f \in L^2(L,\nu)$,*

$$\begin{aligned} c_1 &\left\| \left\{ \alpha^{n\beta} \left(\alpha^{nd} \iint_{\{(x,y)\in L\times L:\mathsf{d}(x,y)<k_1\alpha^{-n}\}} |f(x)-f(y)|^2\,d\nu(x)\,d\nu(y) \right)^{1/2} \right\}_{n=0}^{\infty} \right\|_{l^q} \\ \le &\left\| \left\{ \alpha^{n\beta} \left(\alpha^{-nd} E_{(n)}(Q_n f) \right)^{1/2} \right\}_{n=0}^{\infty} \right\|_{l^q} \\ \le &\left\| \left\{ \alpha^{n\beta} \left(\alpha^{nd} \iint_{\{(x,y)\in L\times L:\mathsf{d}(x,y)<k_2\alpha^{-n}\}} |f(x)-f(y)|^2\,d\nu(x)\,d\nu(y) \right)^{1/2} \right\}_{n=0}^{\infty} \right\|_{l^q}. \end{aligned} \tag{3.1}$$

Here, k_1 and k_2 are provided in (A2).

Proof. Due to the choice of M, the exists some $c_2 > 0$ such that

$$\sum_{i\in I}\left(g(i) - N_I^{-1}\sum_{j\in I} g(j)\right)^2 \le c_2 E_{(1)}(g), \quad g \in \mathbb{R}^I.$$

For $f \in L^2(L,\nu)$ and $n \in \mathbb{Z}_+$, we have

$$\begin{aligned} &\iint_{\{(x,y)\in L\times L:\mathsf{d}(x,y)<k_1\alpha^{-n}\}} |f(x)-f(y)|^2\,d\nu(x)\,d\nu(y) \\ &\le \sum_{(v,w)\in I^n\times I^n,\, v \overset{n,L}{\leftrightarrow} w} \iint_{L_v\times L_w} |f(x)-f(y)|^2\,d\nu(x)\,d\nu(y) \qquad \text{(by (A2))} \\ &\le \sum_{(v,w)\in I^n\times I^n,\, v \overset{n,L}{\leftrightarrow} w} \iint_{L_v\times L_w} 3\{|f(x)-Q_nf(v)|^2 + |Q_nf(v)-Q_nf(w)|^2 \\ &\qquad + |Q_nf(w)-f(y)|^2\}\,d\nu(x)\,d\nu(y) \\ &\le 6C_0 N_I^{-n}\sum_{v\in I^n}\int_{L_v}(f(x)-Q_nf(v))^2\,d\nu(x) + 3N_I^{-2n}E_{(n)}(Q_nf), \end{aligned}$$

where C_0 is what appeared in (A1). Concerning the first term, we have

$$\begin{aligned}
&\sum_{v\in I^n}\int_{L_v}(f(x)-Q_nf(v))^2\,d\nu(x)\\
&= \int_L f(x)^2\,d\nu(x) - N_I^{-n}\sum_{v\in I^n}Q_nf(v)^2\\
&= \sum_{m=n}^{\infty}\left(N_I^{-(m+1)}\sum_{v\in I^{m+1}}Q_{m+1}f(v)^2 - N_I^{-m}\sum_{w\in I^m}Q_mf(w)^2\right)\\
&= \sum_{m=n}^{\infty}N_I^{-(m+1)}\sum_{w\in I^m}\sum_{i\in I}\left(Q_{m+1}f(w\cdot i) - N_I^{-1}\sum_{j\in I}Q_{m+1}f(w\cdot j)\right)^2\\
&\le c_2\sum_{m=n}^{\infty}N_I^{-(m+1)}\sum_{w\in I^m}E_{(1)}(Q_{m+1}f(w\cdot *))\\
&\le c_2\sum_{m=n}^{\infty}N_I^{-(m+1)}E_{(m+1)}(Q_{m+1}f),
\end{aligned}$$

where the martingale convergence theorem was used in the second equality and (2.3) was used in the third equality. Note that $\alpha^d = N_I$. Suppose $q\in[1,\infty)$. Then,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\alpha^{n(\beta+d/2)q}\left(\iint_{\{(x,y)\in L\times L:\mathsf{d}(x,y)<k_1\alpha^{-n}\}}|f(x)-f(y)|^2\,d\nu(x)\,d\nu(y)\right)^{q/2}\\
&\le \sum_{n=0}^{\infty}\alpha^{n(\beta+d/2)q}\left(6c_2C_0N_I^{-n}\sum_{m=n}^{\infty}N_I^{-(m+1)}E_{(m+1)}(Q_{m+1}f) + 3N_I^{-2n}E_{(n)}(Q_nf)\right)^{q/2}\\
&\le c_3\sum_{n=0}^{\infty}\alpha^{n\beta q}\left(\sum_{m=n}^{\infty}\alpha^{-md}E_{(m)}(Q_mf)\right)^{q/2}\\
&\le c_4\sum_{m=0}^{\infty}\alpha^{m(\beta-d/2)q}E_{(m)}(Q_mf)^{q/2}\\
&= c_4\left\|\left\{\alpha^{n\beta}\left(\alpha^{-nd}E_{(n)}(Q_nf)\right)^{1/2}\right\}_{n=0}^{\infty}\right\|_{l^q}^q,
\end{aligned}$$

where in the third inequality, we used (A7) and the following inequality for $a>0$:

$$\sum_{i=0}^{\infty}2^{ai}\left(\sum_{j\in\Lambda_i}a_j\right)^p \le c\sum_{j=0}^{\infty}2^{aj}a_j^p \quad \text{for } a\ne 0,\ p>0,\ a_j\ge 0, \tag{3.2}$$

where $\Lambda_i=\{i,i+1,\dots\}$ when $a>0$ and $\Lambda_i=\{0,1,\dots,i\}$ when $a<0$. When $0<p\le 1$, this is obvious since $(x+y)^p\le x^p+y^p$ for $x,\ y\ge 0$. When $p>1$, this is proved by applications of Hölder's inequality; see e.g. [22].

When $q=\infty$, letting $\gamma = \left\| \left\{ \alpha^{n\beta} \left(\alpha^{-nd} E_{(n)}(Q_n f) \right)^{1/2} \right\}_{n=0}^{\infty} \right\|_{l^\infty}$, we have for every $n \in \mathbb{Z}_+$,

$$\begin{aligned}
&\left| \alpha^{n(\beta+d/2)} \left(\iint_{\{(x,y)\in L\times L:\mathsf{d}(x,y)<k_1\alpha^{-n}\}} |f(x)-f(y)|^2\,d\nu(x)\,d\nu(y) \right)^{1/2} \right|^2 \\
&\quad \le c_5 \alpha^{n(2\beta+d)} \left(N_I^{-n} \sum_{m=n}^{\infty} N_I^{-(m+1)} E_{(m+1)}(Q_{m+1}f) + N_I^{-2n} E_{(n)}(Q_n f) \right) \\
&\quad \le c_5 \alpha^{2n\beta} \sum_{m=n}^{\infty} \alpha^{-2m\beta} \gamma^2 \\
&\quad = \frac{c_5}{1-\alpha^{-2\beta}} \gamma^2 .
\end{aligned}$$

Thus, the first inequality in (3.1) is proved.

Next, we have

$$\begin{aligned}
E_{(n)}(Q_n f) &= \sum_{(v,w)\in I^n\times I^n,\, v \overset{n,L}{\leftrightarrow} w} \left| N_I^{2n} \iint_{L_v\times L_w} \{f(x)-f(y)\}\,d\nu(x)\,d\nu(y) \right|^2 \\
&\le \sum_{(v,w)\in I^n\times I^n,\, v \overset{n,L}{\leftrightarrow} w} N_I^{2n} \iint_{L_v\times L_w} |f(x)-f(y)|^2\,d\nu(x)\,d\nu(y) \\
&\le \alpha^{2nd} \iint_{\{(x,y)\in L\times L:\mathsf{d}(x,y)<k_2\alpha^{-n}\}} |f(x)-f(y)|^2\,d\nu(x)\,d\nu(y),
\end{aligned}$$

which deduces the second inequality of (3.1). □

Remark 3.2. Quite recently, M. Bodin ([8]) gives a discrete characterization of $\Lambda_{p,q}^{\beta}(K)$ for the Alfors d-regular set K if it has a regular triangular system with some property (property (B) in the thesis).

3.2 Proof of the restriction theorem

In this section, we assume (A1)–(A8) and (B1)–(B4), and prove Theorem 2.5. The following lemma is immediately proved by equation (2.2).

Lemma 3.3. *Let $A \subset W^m$, $B \subset W^n$, $f \in \mathcal{F}_A$, and $g \in \mathcal{F}_B$. Suppose that there exists a bijection ι from A to B and $F_v^* f = F_{\iota(v)}^* g$ for every $v \in A$. Then, $\rho^{-m}\mathcal{E}_A(f) = \rho^{-n}\mathcal{E}_B(g)$.*

Let $n \in \mathbb{Z}_+$ and $w \in I^n$. Let $A = \mathcal{N}_M(w) \cap \hat{I}^n$. Define $\mathcal{G}_w = \{f \in \mathcal{F}_A : Q_n(\tilde{f}|_{L_A}) = 0 \text{ on } \mathcal{N}_M(w)\cap I^n\}$ and $\mathcal{K}_w = \{h \in \mathcal{F}_A : \mathcal{E}_A(h,f) = 0 \text{ for all } f \in \mathcal{G}_w\}$. Here, we used (and will use) notations $Q_n(\tilde{f}|_{L_A})$ (on A) and $\mathcal{E}_A(f)$ for $f \in \mathcal{F}_A$ in the obvious sense.

Lemma 3.4. (1) *There exists some $c > 0$ such that $\|f\|_{L^2(K_A)}^2 \le c\mathcal{E}_A(f)$ for all $f \in \mathcal{G}_w$.*

(2) *For each $g \in \mathcal{F}_A$, there exists $h_g \in \mathcal{K}_w$ such that $Q_n(\tilde{h}_g|_{L_A}) = Q_n(\tilde{g}|_{L_A})$ on $\mathcal{N}_M(w)\cap I^n$ and $\mathcal{E}_A(h_g) \le \mathcal{E}_A(g)$.*

Proof. (1) Suppose that the claim does not hold. Then, there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{G}_w$ such that $\|f_k\|_{L^2(K_A)} = 1$ and $\lim_{k\to\infty} \mathcal{E}_A(f_k) = 0$. We may assume that f_k converges weakly to some f in $\mathcal{F}_A$ and $F_w^* f_k$ converges to $F_w^* f$ weakly in $\mathcal{F}$ for every $w \in A$. By (B1), $F_w^* f_k$ converges to $F_w^* f$ in $L^2(K)$ for each $w \in A$. Thus, f_k converges to f in $L^2(K_A)$. We also have $\mathcal{E}_A(f) \le \liminf_{k\to\infty} \mathcal{E}_A(f_k) = 0$. Therefore, $\mathcal{E}_A(f) = 0$. In view of (B2)(2), f is constant on K_A. Since f belongs to $\mathcal{G}_w$ by (A8)', we conclude $f = 0$ on K_A, which is a contradiction to the fact that $\|f\|_{L^2(K_A)} = \lim_{k\to\infty} \|f_k\|_{L^2(K_A)} = 1$.

(2) Let $\mathcal{F}_g = \{f \in \mathcal{F}_A : Q_n(\tilde{f}|_{L_A}) = Q_n(\tilde{g}|_{L_A})$ on $\mathcal{N}_M(w) \cap I^n\}$. Take a sequence $\{h_k\}_{k\in\mathbb{N}} \subset \mathcal{F}_g$ such that $\mathcal{E}_A(h_k)$ converges to the infimum of $\{\mathcal{E}_A(f) : f \in \mathcal{F}_g\}$. Since

$$\|h_k\|_{L^2(K_A)} \le \|h_k - g\|_{L^2(K_A)} + \|g\|_{L^2(K_A)} \le c^{1/2} \mathcal{E}_A(h_k - g)^{1/2} + \|g\|_{L^2(K_A)}, \tag{3.3}$$

we have $\sup_k \|h_k\|_{L^2(K_A)} < \infty$. There exists a weak limit $h \in \mathcal{F}_A$ of a subsequence of $\{h_k\}_{k\in\mathbb{N}}$ in $\mathcal{F}_A$. Then $h \in \mathcal{F}_g$ and h attains the infimum of $\{\mathcal{E}_A(f) : f \in \mathcal{F}_g\}$. Dividing by ϵ both sides of the inequality $\mathcal{E}_A(h + \epsilon f) - \mathcal{E}_A(h) \ge 0$ for $f \in \mathcal{G}_w$ and letting $\epsilon \to 0$, we obtain $h \in \mathcal{K}_w$. □

Lemma 3.5. *There exists some $c_1 > 0$ such that*

$$c_1 \rho^{-n} \mathcal{E}_{\Phi(\hat{I}^n)}(f) \ge E_{(n)}(Q_n(\tilde{f}|_L)) \quad \text{for all } f \in \mathcal{F} \text{ and } n \in \mathbb{Z}_+. \tag{3.4}$$

Proof. First, we prove that $\mathcal{K}_w$ is a finite dimensional vector space. For each $i \in \mathcal{N}_M(w) \cap I^n$, take a function $g_i \in \mathcal{F}$ such that $Q_n(\tilde{g}_i|_L)(j) = \begin{cases} 1 & j = i \\ 0 & j \ne i \end{cases}$ for all $j \in \mathcal{N}_M(w) \cap I^n$. Existence of such functions is assured by the regularity of the Dirichlet form $(\mathcal{E}, \mathcal{F})$. Define a linear map $\Theta\colon \mathcal{K}_w \to \mathbb{R}^{\mathcal{N}_M(w)\cap I^n}$ by $\Theta(f) = \{\mathcal{E}_A(f, g_i)\}_{i\in\mathcal{N}_M(w)\cap I^n}$. Suppose f belongs to the kernel of Θ. Then $\mathcal{E}_A(f, g) = 0$ for every $g \in \mathcal{F}_A$, which implies that f is constant on K_A by (B2) (2). Therefore, $\mathcal{K}_w$ is finite dimensional.

Since $\mathcal{E}_A(h) = 0$ implies $\sum_{v\in A} (Q_n(\tilde{h}|_{L_A})(v) - Q_n(\tilde{h}|_{L_A})(w))^2 = 0$ for $h \in \mathcal{K}_w$, there exists $c_2 > 0$ such that $\sum_{v\in A} (Q_n(\tilde{h}|_{L_A})(v) - Q_n(\tilde{h}|_{L_A})(w))^2 \le c_2 \rho^{-n} \mathcal{E}_A(h)$ for every $h \in \mathcal{K}_w$. By (B2) (3) and Lemma 3.3, we can take c_2 independently with respect to $w \in \bigcup_{n\in\mathbb{Z}_+} I^n$. Therefore, for any $f \in \mathcal{F}$ and $n \in \mathbb{Z}_+$, by taking $h_f \in \mathcal{K}_w$ as in Lemma 3.4 (2),

$$\begin{aligned}
\sum_{v\in\mathcal{N}_M(w)\cap I^n} (Q_n(\tilde{f}|_L)(v) - Q_n(\tilde{f}|_L)(w))^2 &= \sum_{v\in\mathcal{N}_M(w)\cap I^n} (Q_n(\tilde{h}_f|_{L_A})(v) - Q_n(\tilde{h}_f|_{L_A})(w))^2 \\
&\le c_2 \rho^{-n} \mathcal{E}_A(h_f) \\
&\le c_2 \rho^{-n} \mathcal{E}_A(f).
\end{aligned}$$

This implies that

$$\begin{aligned}
E_{(n)}(Q_n(\tilde{f}|_L)) &= \sum_{w\in I^n} \sum_{v\in\mathcal{N}_M(w)\cap I^n} (Q_n(\tilde{f}|_L)(v) - Q_n(\tilde{f}|_L)(w))^2 \\
&\le c_2 \rho^{-n} \sum_{w\in I^n} \mathcal{E}_{\mathcal{N}_M(w)\cap \hat{I}^n}(f) \\
&\le c_2 C_1 \rho^{-n} \mathcal{E}_{\hat{I}^n}(f) \le c_2 C_1 C_2 \rho^{-n} \mathcal{E}_{\Phi(\hat{I}^n)}(f),
\end{aligned}$$

where C_1 and C_2 are provided in (B2) (4). □

Recall finite sets Ξ and $D^\sharp(v)$ for $v \in \Xi$ introduced in (B3).

Lemma 3.6. *For each $v \in \Xi$, the operator $F_v^*\colon \mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}} \to \mathcal{F}$ is a compact operator. Here, $\mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}}$ is regarded as a subspace of $\mathcal{F}_{K_{D^\sharp(v)}}$.*

Proof. Define $I(v) = \{w \in I^{|v|} : L_w \not\subset K_{S^{|v|}\setminus D^\sharp(v)}\}$. Note that $\mathcal{H}(I^{|v|}, D^\sharp(v)) = \mathcal{H}(I(v), D^\sharp(v))$. For each $i \in I(v)$, take a function g_i in $\mathcal{F}(D^\sharp(v))$ such that $Q_{|v|}(\tilde{g}_i|_L)(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$ for all $j \in I(v)$. Define a linear map $\Theta : \mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}} \to \mathbb{R}^{I(v)}$ by $\Theta(f) = \{\mathcal{E}(f, g_i)\}_{i \in I(v)}$. Then, the kernel of Θ is equal to $\mathcal{H}(D^\sharp(v))|_{K_{D^\sharp(v)}}$. The homomorphism theorem implies that $\mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}} \Big/ \mathcal{H}(D^\sharp(v))|_{K_{D^\sharp(v)}} \simeq \Theta(\mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}})$ as a vector space. Therefore, there exists a finite dimensional vector space Z of $\mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}}$ such that $\mathcal{H}(I^{|v|}, D^\sharp(v))|_{K_{D^\sharp(v)}}$ is a direct sum of $\mathcal{H}(D^\sharp(v))|_{K_{D^\sharp(v)}}$ and Z. Condition (B3) (2) concludes the assertion. □

Lemma 3.7. *Let $m \in \mathbb{N}$, A a proper subset of S^m, and J a subset of I^m. For $g \in \mathcal{F}$, there exists a unique function g' in $\mathcal{H}(J, A)$ such that $g' = g$ on $K_{S^m\setminus A}$ and $Q_m(\widetilde{g'}|_L) = Q_m(\tilde{g}|_L)$ on J. Moreover, there exists $c > 0$ such that*

$$\|g'\|_{\mathcal{F}_A} \le c\|g\|_{\mathcal{F}_A}, \quad \mathcal{E}(g') \le \mathcal{E}(g) \tag{3.5}$$

for all $g \in \mathcal{F}$. Further, if $g \ge 0$ μ-a.e., then $g' \ge 0$ μ-a.e.

Proof. First, we prove that there exists some $c' > 0$ such that $\|f\|^2_{L^2(K_A)} \le c'\mathcal{E}_A(f)$ for every $f \in \mathcal{F}(A)$. Suppose this does not hold. Then, there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(A)$ such that $\|f_n\|_{L^2(K_A)} = 1$ for every n and $\mathcal{E}_A(f_n)$ converges to 0 as $n \to \infty$. We may assume that f_n converges weakly to some f in $\mathcal{F}$. Then, f_n converges to $f \in \mathcal{F}$ in $L^2(K)$ by (B1), and $\mathcal{E}(f) \le \liminf_{n\to\infty} \mathcal{E}(f_n) = 0$. Therefore, $\mathcal{E}(f) = 0$ and f is constant on K. Since $f \in \mathcal{F}(A)$ and $A \neq S^m$, f is identically 0, which is contradictory to the fact $\|f\|_{L^2(K)} = 1$.

Now, given $g \in \mathcal{F}$, let $\mathcal{F}_g = \{f \in \mathcal{F} : f = g \text{ on } K_{S^m\setminus A} \text{ and } Q_m(\tilde{f}|_L) = Q_m(\tilde{g}|_L) \text{ on } J\}$. Then, in exactly the same way as the proof of Lemma 3.4 (2), there exists $h \in \mathcal{F}_g$ attaining the infimum of $\{\mathcal{E}(f) : f \in \mathcal{F}_g\}$ and $h \in \mathcal{H}(J, A)$. Such functions exist uniquely; indeed, if both h and h' attain the infimum above, we have

$$\mathcal{E}\left(\frac{h - h'}{2}\right) = \frac{1}{2}\left(\mathcal{E}(h) + \mathcal{E}(h')\right) - \mathcal{E}\left(\frac{h + h'}{2}\right) \le 0,$$

which implies that $h - h'$ is a constant. Since $h - h' = 0$ on $K_{S^m\setminus A}$, we conclude that $h = h'$. On the other hand, it is easy to see that g' should attain the infimum above. Therefore, g' is uniquely determined. By the inequality similar to (3.3), we conclude (3.5). The last assertion follows from the characterization of g' above and the Markov property of the Dirichlet form. □

The following is the key proposition. Condition (B4) will be used (only) here.

Proposition 3.8. *There exist $0 < c_0 < 1$ and $b_0 \in \mathbb{N}$ such that the following holds for all $n \in \mathbb{Z}_+$ and $h \in \mathcal{H}(I^n, \Phi(\hat{I}^n))$:*

$$\mathcal{E}_{\Phi(\hat{I}^{n+b_0})}(h) \le c_0 \mathcal{E}_{\Phi(\hat{I}^n)}(h).$$

Moreover, for all $i \ge j \ge 1$, $b = 0, 1, \dots, b_0 - 1$, and $h \in \mathcal{H}(I^{b_0 j}, \Phi(\hat{I}^{b_0 j}))$,

$$\mathcal{E}_{\Phi(\hat{I}^{b_0 i+b})}(h) \le c_0^{i-j} \mathcal{E}_{\Phi(\hat{I}^{b_0 j+b})}(h).$$

Proof. It is enough to prove the first claim. Recall l_0 and m_0 in condition (B3). By (B3), $C := \sup_{n\in\mathbb{Z}_+} \max_{w\in\hat{I}^{n+m_0}} \#D(w)$ is finite. Let $n \in \mathbb{Z}_+$ and $w \in \hat{I}^{n+m_0}$. Define

$$\begin{aligned} \mathcal{C}_w &= \{F_w^* f : f \in \mathcal{H}(I^{n+m_0}, D(w)),\ \int_{K_{D(w)}} f\, d\mu = 0,\ \rho^{-(n+m_0)}\mathcal{E}_{D(w)}(f) \le 1\}, \\ \mathcal{C} &= \text{the closure of } \bigcup_{w\in\bigcup_{n\in\mathbb{Z}_+} \hat{I}^{n+m_0}} \mathcal{C}_w \text{ in } \mathcal{F}. \end{aligned}$$

Then, $\mathcal{C}$ is a compact subset in $\mathcal{F}$ by Lemma 3.6 and (B3). Let $\delta = 1/(4C^2)$ and define $\mathcal{C}(\delta) = \{f \in \mathcal{C} : \mathcal{E}(f) \ge \delta\}$. Since (B4) holds, for each $f \in \mathcal{C}(\delta)$, there exist $m(f) \in \mathbb{N}$ and $a(f) \in (0,1)$ such that $\mathcal{E}_{\Phi(\hat{I}^m)}(f) < a(f)\mathcal{E}(f)$ for all $m \ge m(f)$. By continuity, $\mathcal{E}_{\Phi(\hat{I}^m)}(g) < a(f)\mathcal{E}(g)$ for all $m \ge m(f)$ for any g in some neighborhood of f in $\mathcal{F}$. Since $\mathcal{C}(\delta)$ is compact in $\mathcal{F}$, there exist $m_1 \in \mathbb{N}$ and $a_1 \in (0,1)$ such that $\mathcal{E}_{\Phi(\hat{I}^{m_1})}(f) < a_1\mathcal{E}(f)$ for every $f \in \mathcal{C}(\delta)$. In particular,

$$\mathcal{E}(f) \le a_2 \mathcal{E}_{S^{m_1}\setminus\Phi(\hat{I}^{m_1})}(f), \quad f \in \mathcal{C}(\delta) \tag{3.6}$$

with $a_2 = (1-a_1)^{-1} > 1$.

Now, take h as in the claim of the proposition. We construct an oriented graph such that the set of vertices is $\Phi(\hat{I}^{n+m_0})$ and a set of oriented edges is $E = \{(v,w) \in \Phi(\hat{I}^{n+m_0}) \times \Phi(\hat{I}^{n+m_0}) : v \in D'(w),\ \mathcal{E}_w(h) > 0 \text{ and } \mathcal{E}_w(h) \ge 2C\mathcal{E}_v(h)\}$. This graph does not allow any loops. Let Y be the set of all elements w in $\Phi(\hat{I}^{n+m_0})$ such that $\mathcal{E}_w(h) > 0$ and w is not a source of any edges. For $w \in Y$, define $N_0(w) = \{w\}$, $N_k(w) = \{v \in \Phi(\hat{I}^{n+m_0}) \setminus \bigcup_{l=0}^{k-1} N_l(w) : (v,u) \in E \text{ for some } u \in N_{k-1}(w)\}$ for $k \in \mathbb{N}$ inductively, and $N(w) = \bigcup_{k\ge0} N_k(w)$. It is clear that $\#N_k(w) \le C^k$ and $\mathcal{E}_v(h) \le (2C)^{-k}\mathcal{E}_w(h)$ for all $k \ge 0$ and $v \in N_k(w)$. Then, for each $w \in Y$,

$$\mathcal{E}_{N(w)}(h) = \sum_{k=0}^{\infty} \sum_{v\in N_k(w)} \mathcal{E}_v(h) \le \sum_{k=0}^{\infty} C^k (2C)^{-k}\mathcal{E}_w(h) = 2\mathcal{E}_w(h). \tag{3.7}$$

Suppose $w \in Y$ and $\mathcal{E}_w(h) \ge \delta\mathcal{E}_{D'(w)}(h)$. Then, since

$$F_w^*\left(\left(h - \fint_{K_{D'(w)}} h\, d\mu\right) \times \rho^{(n+m_0)/2}\mathcal{E}_{D'(w)}(h)^{-1/2}\right) \in \mathcal{C}(\delta),$$

(3.6) implies that $\mathcal{E}(F_w^* h) \le a_2 \mathcal{E}_{S^{m_1}\setminus\Phi(\hat{I}^{m_1})}(F_w^* h)$, namely,

$$\mathcal{E}_w(h) \le a_2 \mathcal{E}_{w\cdot(S^{m_1}\setminus\Phi(\hat{I}^{m_1}))}(h).$$

Next, suppose $w \in Y$ and $\mathcal{E}_w(h) < \delta \mathcal{E}_{D'(w)}(h)$. Since w is not a source of any edges, $\mathcal{E}_v(h) < 2C\mathcal{E}_w(h)$ for every $v \in D'(w) \cap \Phi(\hat{I}^{n+m_0})$. Then,

$$\mathcal{E}_{D'(w)\cap\Phi(\hat{I}^{n+m_0})}(h) < C \cdot 2C\mathcal{E}_w(h) < 2C^2\delta\mathcal{E}_{D'(w)}(h) = \frac{1}{2}\mathcal{E}_{D'(w)}(h),$$

which implies $\mathcal{E}_{D'(w)\cap\Phi(\hat{I}^{n+m_0})}(h) < \mathcal{E}_{D'(w)\cap((\Phi(\hat{I}^n)\cdot S^{m_0})\setminus\Phi(\hat{I}^{n+m_0}))}(h)$ by (B3) (1) (a). In particular,

$$\mathcal{E}_w(h) < \mathcal{E}_{D'(w)\cap((\Phi(\hat{I}^n)\cdot S^{m_0})\setminus\Phi(\hat{I}^{n+m_0}))}(h).$$

Therefore, in any cases, we have for $w \in Y$,

$$\begin{aligned} \mathcal{E}_w(h) &\leq a_2\mathcal{E}_{w\cdot(S^{m_1}\setminus\Phi(\hat{I}^{m_1}))\cup((D'(w)\cap((\Phi(\hat{I}^n)\cdot S^{m_0})\setminus\Phi(\hat{I}^{n+m_0})))\cdot S^{m_1})}(h) \\ &\leq a_2\mathcal{E}_{(D'(w)\cdot S^{m_1})\cap((\Phi(\hat{I}^n)\cdot S^{b_0})\setminus\Phi(\hat{I}^{n+b_0}))}(h), \end{aligned} \tag{3.8}$$

where $b_0 = m_0 + m_1$ and note that $\Phi(\hat{I}^{n+b_0}) \subset \Phi(\hat{I}^{n+m_0}) \cdot S^{m_1}$. Then we have

$$\begin{aligned} \mathcal{E}_{\Phi(\hat{I}^{n+b_0})}(h) &\leq \mathcal{E}_{\Phi(\hat{I}^{n+m_0})}(h) \\ &\leq \sum_{w\in Y}\mathcal{E}_{N(w)}(h) \\ &\leq 2a_2\sum_{w\in Y}\mathcal{E}_{(D'(w)\cdot S^{m_1})\cap((\Phi(\hat{I}^n)\cdot S^{b_0})\setminus\Phi(\hat{I}^{n+b_0}))}(h) \\ &\leq 2a_2C_3\mathcal{E}_{(\Phi(\hat{I}^n)\cdot S^{b_0})\setminus\Phi(\hat{I}^{n+b_0})}(h). \end{aligned}$$

Here, we used (3.7) and (3.8) in the third inequality and $C_3 := \sup_{n\in\mathbb{Z}_+}\max_{v\in S^{n+m_0}}\#(\mathcal{N}_{l_0}(v)\cap S^{n+m_0})$ is finite by (A1). Hence, the claim of the proposition holds with $c_0 = 2a_2C_3/(1+2a_2C_3)$. $\square$

Proof of Theorem 2.5. Fix $b \in \{0, 1, \ldots, b_0 - 1\}$. For each $f \in \mathcal{F}$, we can take $g_m \in \mathcal{H}(I^{b_0m+b}, \Phi(\hat{I}^{b_0m+b}))$ such that $g_m = f$ on $K_{S^{b_0m+b}\setminus\Phi(\hat{I}^{b_0m+b})}$ and $Q_{b_0m+b}(\tilde{g}_m|_L) = Q_{b_0m+b}(\tilde{f}|_L)$ by Lemma 3.7. By using the relations $\|g_m\|_{\mathcal{F}} \leq c\|f\|_{\mathcal{F}}$, $\mathcal{E}(g_m) \leq \mathcal{E}(f)$ (by Lemma 3.7), and $g_m \to f$ μ-a.e., we will prove $g_m \to f$ in $\mathcal{F}$ as $m \to \infty$. Here, note that the constant c is taken independently of m, which derives from the fact that c depends only on c' in the proof of Lemma 3.7. We first obtain that g_m converges weakly to f in $\mathcal{F}$ and $\limsup_{m\to\infty}\mathcal{E}(g_m - f) = \limsup_{m\to\infty}\mathcal{E}(g_m) - \mathcal{E}(f) \leq 0$. Therefore, $\mathcal{E}(g_m - f) \to 0$ as $m \to \infty$. By (B1), $g_m - f - \int_K(g_m - f)\,d\mu$ converges to 0 in $L^2(K)$. Since $\|g_m - f\|_{L^2(K)} \leq c\|f\|_{\mathcal{F}} + \|f\|_{L^2(K)}$, we have $\int_K(g_m - f)\,d\mu \to 0$ as $m \to \infty$, which implies that $\|g_m - f\|_{L^2(K)} \to 0$ as $m \to \infty$. Thus, $g_m \to f$ in $\mathcal{F}$ as $m \to \infty$.

Let $f_m = g_m - g_{m-1}$ where we set $g_{-1} \equiv 0$. Then, $f = \sum_{m=0}^{\infty} f_m$. Since $f_i \in \mathcal{F}(I^{b_0j+b}, \Phi(\hat{I}^{b_0j+b}))$ for $i > j$ and $f_j \in \mathcal{H}(I^{b_0j+b}, \Phi(\hat{I}^{B_0j+b}))$, we have $\mathcal{E}(f_i, f_j) = 0$ for $i \neq j$, so that

$$\mathcal{E}(f) = \sum_{m=0}^{\infty}\mathcal{E}(f_m). \tag{3.9}$$

Now, for each $f \in \mathcal{F}$,

$$
\begin{aligned}
(E_{(b_0 i+b)}(Q_{b_0 i+b}(\tilde{f}|_L)))^{1/2} &= (E_{(b_0 i+b)}(Q_{b_0 i+b}(\tilde{g}_i|_L)))^{1/2} = \left(E_{(b_0 i+b)}\left(\sum_{j=0}^{i} Q_{b_0 i+b}(\tilde{f}_j|_L)\right)\right)^{1/2} \\
&\le \sum_{j=0}^{i}(E_{(b_0 i+b)}(Q_{b_0 i+b}(\tilde{f}_j|_L)))^{1/2} \le \sum_{j=0}^{i}(c_1\rho^{-b_0 i-b}\mathcal{E}_{\Phi(\hat{I}^{b_0 i+b})}(f_j))^{1/2} \\
&\le \sum_{j=0}^{i}(c_1\rho^{-b_0 i-b}c_0^{i-j}\mathcal{E}_{\Phi(\hat{I}^{b_0 j+b})}(f_j))^{1/2} \\
&\le \sum_{j=0}^{i}(c_1\rho^{-b_0 i-b}c_0^{i-j}\mathcal{E}(f_j))^{1/2}, \qquad (3.10)
\end{aligned}
$$

where we apply Minkowski's inequality in the first inequality, (3.4) in the second inequality, and Proposition 3.8 in the third inequality.

Applying (3.10) and noting that $\alpha^{d_w-d_f}=\rho$, we have

$$
\begin{aligned}
\sum_{i=0}^{\infty}\alpha^{(d_w-d_f)(b_0 i+b)}E_{(b_0 i+b)}(Q_{b_0 i+b}(\tilde{f}|_L)) &\le \sum_{i=0}^{\infty}\rho^{b_0 i+b}\left(\sum_{j=0}^{i}(c_1\rho^{-b_0 i-b}c_0^{i-j}\mathcal{E}(f_j))^{1/2}\right)^2 \\
&= c_1\sum_{i=0}^{\infty}c_0^{i}\left(\sum_{j=0}^{i}(c_0^{-j}\mathcal{E}(f_j))^{1/2}\right)^2 \\
&\le c_2\sum_{j=0}^{\infty}c_0^{j}c_0^{-j}\mathcal{E}(f_j) = c_2\sum_{j=0}^{\infty}\mathcal{E}(f_j) = c_2\mathcal{E}(f).
\end{aligned}
$$

Here we used (3.2) in the second inequality and (3.9) in the last equality. Thus, we have

$$
\sum_{n=0}^{\infty}\alpha^{(d_w-d_f)n}E_{(n)}(Q_n(\tilde{f}|_L)) \le b_0 c_2\mathcal{E}(f).
$$

Combining this with Lemma 3.1 and (A8)', we have $\|\tilde{f}|_L\|_{\Lambda^{\beta}_{2,2}(L)} \le c_3\|f\|_{\mathcal{F}}$, so that $\mathcal{F}|_L \subset \Lambda^{\beta}_{2,2}(L)$ and $(\mathcal{F}\cap C(K))|_L \subset \Lambda^{\beta}_{2,2}(L)\cap C(L)$. Noting that $\mathcal{F}\cap C(K)$ is dense in $\mathcal{F}$ due to the regularity of $(\mathcal{E},\mathcal{F})$, the claim follows by a simple limiting procedure. □

Remark 3.9. Even if (B4) does not hold, $\mathcal{F}|_L \subset \hat{\Lambda}^{\beta}_{2,\infty}(L)$ hold. Indeed, for each $f \in \mathcal{F}$ and $n \in \mathbb{Z}_+$, we have by Lemma 3.5,

$$
c_1\mathcal{E}(f) \ge c_1\mathcal{E}_{\Phi(\hat{I}_n)}(f) \ge \rho^n E_{(n)}(Q_n(\tilde{f}|_L)) = \alpha^{(2\beta-d)n}E_{(n)}(Q_n(\tilde{f}|_L)).
$$

Therefore, we have

$$
\left\|\left\{\alpha^{n\beta}\left(\alpha^{-nd}E_{(n)}(Q_n(\tilde{f}|_L))\right)^{1/2}\right\}_{n=0}^{\infty}\right\|_{l^\infty} \le c_1^{1/2}\mathcal{E}(f)^{1/2},
$$

so the same argument as above gives the result.

3.3 Proof of the extension theorem

In this section, we assume (A1)–(A8) and (C1)–(C2), and prove Theorem 2.6. The conditions (C1) and (C2) will be defined below.

In order to construct an extension map ξ, we first define a Whitney-type decomposition and an associated partition of unity. Let $\Omega^{(n)} = \bigcup_{m=0}^{n} I^m$ for $n \in \mathbb{Z}_+$. For $w \in I^0 = \{\emptyset\}$, set $A_w = W \setminus \mathcal{N}_2(I)$ and $B_w = W \setminus \mathcal{N}_1(I)$. For $w \in I^n$ with $n \in \mathbb{N}$, set $A_w = (\mathcal{N}_2(w) \cdot W) \setminus \mathcal{N}_2(I^{n+1})$, $\hat{A}_w = \mathcal{N}_2(w) \cdot W$, $B_w = (\mathcal{N}_3(w) \cdot W) \setminus \mathcal{N}_1(I^{n+1})$, and $\hat{B}_w = \mathcal{N}_3(w) \cdot W$. Clearly $K_{A_w} \subset K_{B_w}$, $K_{\hat{A}_w} \subset K_{\hat{B}_w}$, $K_{A_w} \cap K_{W^{|w|+1} \setminus B_w} = \emptyset$, and $K_{\hat{A}_w} \cap K_{W^{|w|+1} \setminus \hat{B}_w} = \emptyset$.

By (A3), we see the following for $w, w' \in \bigcup_{n \in \mathbb{Z}_+} I^n$:

$$c_1 \alpha^{-|w|} \le \mathsf{d}(L, K_{B_w}) \le c_2 \alpha^{-|w|} \text{ if } B_w \neq \emptyset, \tag{3.11}$$

$$\text{there exists } l > 0 \text{ such that if } |w'| \ge |w| + l \text{ then } K_{B_w} \cap K_{\hat{B}_{w'}} = \emptyset. \tag{3.12}$$

For $n \in \mathbb{N}$ and $w \in \Omega^{(n)}$, we set

$$A_w^{(n)} = \begin{cases} A_w & \text{if } |w| < n \\ \hat{A}_w & \text{if } |w| = n \end{cases}, \quad B_w^{(n)} = \begin{cases} B_w & \text{if } |w| < n \\ \hat{B}_w & \text{if } |w| = n \end{cases},$$

and $R_w^{(n)} = \{w' \in \Omega^{(n)} : K_{B_w^{(n)}} \cap K_{B_{w'}^{(n)}} \neq \emptyset\}$. We assume the following.

(C1) There exists a finite subset Γ of $\bigcup_{n \in \mathbb{N}} (\{n\} \times \Omega^{(n)})$ such that, for any $n \in \mathbb{N}$ and $w \in \Omega^{(n)}$, there exist $(m, v) \in \Gamma$, a bijection $\iota \colon R_w^{(n)} \to R_v^{(m)}$, and a homeomorphism $F \colon K_{\bigcup_{u \in R_w^{(n)}} B_u^{(n)}} \to K_{\bigcup_{u \in R_v^{(m)}} B_u^{(m)}}$ satisfying that for every $u \in R_w^{(n)}$, $A_u^{(n)}$ and $A_{\iota(u)}^{(m)}$ are of the same type and so are $B_u^{(n)}$ and $B_{\iota(u)}^{(m)}$, for the homeomorphism F.

For each $(m, v) \in \Gamma$, take a function $\bar{\varphi}_v^{(m)} \in \mathcal{F} \cap C(K)$ such that $0 \le \bar{\varphi}_v^{(m)} \le 1$, $\bar{\varphi}_v^{(m)}(x) = 1$ on $K_{A_v^{(m)}}$, and $\bar{\varphi}_v^{(m)}(x) = 0$ on $K_{W^{|v|+1} \setminus B_v^{(m)}}$. Such a function exists since $(\mathcal{E}, \mathcal{F})$ is regular. For $n \in \mathbb{N}$ and $w \in \Omega^{(n)}$, define $\varphi_w^{(n)}(x) = \begin{cases} \bar{\varphi}_v^{(m)}(F(x)) & \text{if } x \in B_w^{(n)} \\ 0 & \text{otherwise} \end{cases}$, where m, v and F are given in (C1). We assume

(C2) $\varphi_w^{(n)} \in \mathcal{F} \cap C(K)$ for every $n \in \mathbb{N}$ and $w \in \Omega^{(n)}$.

For $n \in \mathbb{N}$ and $w \in \Omega^{(n)}$, define

$$\psi_w^{(n)}(x) = \frac{\varphi_w^{(n)}(x)}{\sum_{w' \in \Omega^{(n)}} \varphi_{w'}^{(n)}(x)}, \quad x \in K.$$

This is well-defined since the sum in the denominator is not less than 1. $\psi_w^{(n)}$ is continuous and takes values between 0 and 1. Since $\varphi_w^{(n)} \in \mathcal{F}$ and vanishes outside of $K_{B_w^{(n)}}$, so does $\psi_w^{(n)}$. For each $f \in \Lambda_{2,2}^{\beta}(L) \cap C(L)$, define

$$\xi^{(n)} f(x) = \sum_{w \in \Omega^{(n)}} \psi_w^{(n)}(x) Q_{|w|} f(w) = \sum_{w \in \Omega^{(n)}} \psi_w^{(n)}(x) ⨍_{L_w} f(s) d\nu(s).$$

$\xi^{(n)}$ is a linear map from $\Lambda^{\beta}_{2,2}(L)\cap C(L)$ to $\mathcal{F}\cap C(K)$. For $x\in K\setminus L$, $\xi^{(n)}f(x)$ is independent of n if n is sufficiently large because of (3.12). Therefore, for $f\in\Lambda^{\beta}_{2,2}(L)\cap C(L)$,

$$\xi f(x):=\begin{cases}\lim\limits_{n\to\infty}\xi^{(n)}f(x), & x\in K\setminus L\\ f(x), & x\in L\end{cases} \tag{3.13}$$

is well-defined and $\xi^{(n)}f$ converges to ξf μ-a.e.

Proof of Theorem 2.6. We first prove that ξf is continuous on K. Since ξf is continuous on $K\setminus L$ by the construction, it is enough to show for each $x_0\in L$ that

$$\lim_{\substack{x\to x_0\\ x\in K\setminus L}}\xi f(x)=f(x_0). \tag{3.14}$$

Since f is uniformly continuous on L, if we set $\omega_a(f)=\sup\{|f(s)-f(t)|: s,t\in L,\ \mathsf{d}(s,t)\le a\}$ for $a>0$, then $\lim_{a\to 0}\omega_a(f)=0$. Let $x_0\in L$, $x\in K\setminus L$, and $\delta=\mathsf{d}(x,x_0)$. Suppose that $w\in\bigcup_{n\in\mathbb{N}}I^n$ satisfies $x\in K_{B_w}$. Then, $c_1\alpha^{-|w|}\le\mathsf{d}(L,K_{B_w})\le\mathsf{d}(x_0,x)=\delta$ by (3.11). Next, take $y\in L_w$ and choose $z\in K_{B_w}$ that satisfies $\mathsf{d}(y,z)=\mathsf{d}(y,K_{B_w})\le c_2\alpha^{-|w|}$. Then, since $\operatorname{diam}(K_{B_w})\asymp\alpha^{-|w|}$, we have

$$\mathsf{d}(y,x_0)\le\mathsf{d}(y,z)+\mathsf{d}(z,x)+\mathsf{d}(x,x_0)\le c_2\alpha^{-|w|}+c_3\alpha^{-|w|}+\delta\le c_4\delta.$$

Therefore, $\fint_{L_w}|f(y)-f(x_0)|\,d\nu(y)\le\omega_{c_4\delta}(f)$. Now, take n sufficiently large so that $x\notin\bigcup_{w\in I^n}K_{\hat{B}_w}$. Then, $\xi^{(n)}f(x)=\xi f(x)$ and

$$\begin{aligned}|\xi f(x)-f(x_0)| &= \left|\sum_{w\in\Omega^{(n)}}\psi^{(n)}_w(x)\fint_{L_w}(f(y)-f(x_0))\,d\nu(y)\right|\\ &\le \sum_{w\in\Omega^{(n-1)},\ x\in K_{B_w}}\psi^{(n)}_w(x)\fint_{L_w}|f(y)-f(x_0)|\,d\nu(y)\\ &\le \omega_{c_4\delta}(f).\end{aligned}$$

Thus (3.14) is proved.

Next, we will prove $\{\xi^{(n)}f\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{F}$. Noting that $\int_K\psi^{(n)}_w(x)\,d\mu(x)\le c_5\alpha^{-d_f|w|}$

for all $n \in \mathbb{N}$ and $w \in \Omega^{(n)}$ for some $c_5 > 0$, we have

$$
\begin{aligned}
\|\xi^{(n)} f\|_{L^2(K,\mu)}^2 &= \int_K \left(\sum_{w\in\Omega^{(n)}} \psi_w^{(n)}(x) \fint_{L_w} f(s)\, d\nu(s) \right)^2 d\mu(x) \\
&\le \int_K \left(\sum_{w\in\Omega^{(n)}} \psi_w^{(n)}(x) \fint_{L_w} f(s)^2\, d\nu(s) \right) d\mu(x) \\
&\le \sum_{w\in\Omega^{(n)}} c_5 \alpha^{-d_f|w|} \alpha^{d|w|} \int_{L_w} f(s)^2 d\nu(s) \\
&= c_5 \sum_{k=0}^{n} \alpha^{(d-d_f)k} \|f\|_{L^2(L,\nu)}^2 \\
&\le \frac{c_5}{1-\alpha^{d-d_f}} \|f\|_{L^2(L,\nu)}^2. \qquad (3.15)
\end{aligned}
$$

For $n \in \mathbb{N}$, $w \in \Omega^{(n)}$ with $m = |w|$, let $\bar{R}_w^{(n)} = \bigcup_{v\in R_w^{(n)}} v \cdot I^{m+l-|v|} \subset I^{m+l}$, where l is provided in (3.12). For $g \in L^2(L,\nu)$, we define

$$
E_w^{(n)}(g) = \sum_{\substack{u,v\in\bar{R}_w^{(n)} \\ u \overset{m+l,L}{\longleftrightarrow} v}} (Q_{m+l}g(u) - Q_{m+l}g(v))^2, \quad \bar{E}_w^{(n)}(g) = \mathcal{E}_{\Phi(B_w^{(n)})}\left(\sum_{v\in R_w^{(n)}} Q_{|v|}g(v)\psi_v^{(n)} \right).
$$

Both $E_w^{(n)}(g)$ and $\bar{E}_w^{(n)}(g)$ are determined only by the values $\{Q_{m+l}g(u)\}_{u\in\bar{R}_w^{(n)}}$. If $E_w^{(n)}(g) = 0$, then $Q_{m+l}g$ is constant on $\bar{R}_w^{(n)}$, which implies that $\bar{E}_w^{(n)}(g) = 0$. Therefore, there exists $c_w^{(n)} > 0$ such that $\bar{E}_w^{(n)}(g) \le c_w^{(n)} E_w^{(n)}(g)$ for every $g \in \mathcal{F}$. Due to (C1) and Lemma 3.3, there exists some $c_6 > 0$ such that $\bar{E}_w^{(n)}(g) \le c_6 \rho^{|w|} E_w^{(n)}(g)$ for all $n \in \mathbb{N}$, $w \in \Omega^{(n)}$ and $g \in \mathcal{F}$. It also holds that there exists $c_7 > 0$ independent of m such that $\sum_{w\in I^m} E_w^{(n)}(g) \le c_7 E_{(m+l)}(Q_{m+l}g)$ for all n and $g \in L^2(L,\nu)$. Then, we have

$$
\begin{aligned}
\mathcal{E}(\xi^{(n)} f) &\le \sum_{w\in\Omega^{(n)}} \mathcal{E}_{\Phi(B_w^{(n)})}(\xi^{(n)} f) = \sum_{w\in\Omega^{(n)}} \bar{E}_w^{(n)}(f) \\
&\le c_6 \sum_{m=0}^{n} \sum_{w\in I^m} \rho^m E_w^{(n)}(f) \le c_6 c_7 \sum_{m=0}^{n} \rho^m E_{(m+l)}(Q_{m+l} f) \\
&\le c_8 \sum_{m=0}^{\infty} \rho^m E_{(m)}(Q_m f).
\end{aligned}
$$

Since $\alpha^{2\beta-d} = \alpha^{d_w-d_f} = \rho$, we obtain $\mathcal{E}(\xi^{(n)} f) \le c_8 \|f\|_{\Lambda_{2,2}^{\beta}(L)}^2$ by Lemma 3.1.

By combining this with (3.15), $\{\xi^{(n)} f\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{F}$ and we conclude that $\xi f \in \mathcal{F}$ and $\|\xi f\|_{\mathcal{F}} \le c_9 \|f\|_{\Lambda_{2,2}^{\beta}(L)}$ for some $c_9 > 0$.

Next, take any $\Lambda_{2,2}^{\beta}(L)$-Cauchy sequence $\{f_n\}_{n\in\mathbb{N}} \subset \Lambda_{2,2}^{\beta}(L) \cap C(L)$ and let $f \in \Lambda_{2,2}^{\beta}(L)$ be the limit point. By the above result, $\{\xi f_n\}_{n\in\mathbb{N}} \subset \mathcal{F} \cap C(K)$ is a $\mathcal{E}_1$-Cauchy sequence. Let

$g \in \mathcal{F}$ be the limit point. Since $\xi f_n|_L = f_n$ and a subsequence ξf_{n_k} converges to $\tilde{g}$ q.e., $\tilde{g}|_L = f$ ν-a.e. Thus, ξ can extend to a continuous map from $\hat{\Lambda}^\beta_{2,2}(L)$ to $\mathcal{F}$ such that $\widetilde{\xi f}|_L = f$ ν-a.e. for $f \in \hat{\Lambda}^\beta_{2,2}(L)$. $\square$

Remark 3.10. Let $\{L_i\}_{i=1}^m$ be a finite number of self-similar subsets of K where each L_i is constructed by the same number of contraction maps and satisfies (A2), the second identity of (A4), (A7), and (A8) in Section 2. Let $L = \bigcup_{i=1}^m L_i$. With suitable changes for A_w, B_w etc., we can consider conditions (C1)*–(C2)* as the corresponding (C1)–(C2). Define $\Lambda^\beta_{2,2}(L)$ as in Definition 2.4. Then, under such conditions, Theorem 2.6 is still valid, i.e. there is a linear map ξ from $\Lambda^\beta_{2,2}(L)$ to $\mathcal{F}$ such that $\xi\bigl(\Lambda^\beta_{2,2}(L) \cap C(L)\bigr) \subset \mathcal{F} \cap C(K)$, $\widetilde{\xi f}|_L = f$ and

$$\|\xi f\|_{\mathcal{F}} \le c_1 \sum_{i=1}^m \|f|_{L_i}\|_{\Lambda^\beta_{2,2}(L_i)}, \quad f \in \Lambda^\beta_{2,2}(L).$$

4 Complementary results

In this section, we give sufficient conditions concerning (A8) and (B3), and discuss a suitable choice of $\mathcal{F}_A$ for $A \subset W^m$. We first define fractional diffusions in the sense of [2] Definition 3.5.

Definition 4.1. Let (X, d) be a complete metric space where d has the midpoint property; for each $x, y \in X$, there exists $z \in X$ such that $\mathsf{d}(x,y) = \mathsf{d}(x,z)/2 = \mathsf{d}(z,y)/2$. For simplicity, we assume $\operatorname{diam} X = 1$. Let μ be a Borel measure on X such that there exists $d_f > 0$ with $\mu(B(x,r)) \asymp r^{d_f}$ for all $0 < r \le 1$. A Markov process $\{Y_t\}_{t\ge 0}$ is a fractional diffusion on X if
1) Y is a μ-symmetric conservative Feller diffusion,
2) Y has a symmetric jointly continuous transition density $p_t(x,y)$ ($t > 0, x, y \in X$) which satisfies the Chapman-Kolmogorov equations and has the following estimate,

$$\begin{aligned} & c_1 t^{-d_f/d_w} \exp(-c_2(\mathsf{d}(x,y)^{d_w} t^{-1})^{1/(d_w-1)}) \le p_t(x,y) \\ \le\ & c_3 t^{-d_f/d_w} \exp(-c_4(\mathsf{d}(x,y)^{d_w} t^{-1})^{1/(d_w-1)}) \qquad \text{for all } \ 0 < t < 1,\ x, y \in X, \end{aligned}$$

with some constant $d_w \ge 2$.

Proposition 4.2. (A8) *holds for the following three cases.*
1) *There exists $c > 0$ such that $\|f\|_{L^\infty(K)} \le c\|f\|_{\mathcal{F}}$ for all $f \in \mathcal{F}$.*
2) *The diffusion process corresponding to $(\mathcal{E}, \mathcal{F})$ is the fractional diffusion and* (A7) *holds.*
3) *$K \subset \mathbb{R}^n$,* (A7) *holds, and $\mathcal{F} = \Lambda^{d_w/2}_{2,\infty}(K)$.*

Proof. Suppose that 1) holds. Then, for any nonempty set D of K, $\operatorname{Cap}(D) \ge c^{-2}$. Therefore, $\nu(D) \le 1 \le c^2 \operatorname{Cap}(D)$.

The proof when 2) holds is similar to Lemma 2.5 of [5], but we will give it for completeness. Let $g_1(\cdot,\cdot)$ be the 1-order Green density given by

$$E^x\left[\int_0^\infty e^{-t} f(X_t)dt\right] = \int_K g_1(x,y) f(y) d\mu,$$

for any Borel measurable function f, where $\{X_t\}$ is the diffusion corresponding to $(\mathcal{E},\mathcal{F})$. Then, since $\{X_t\}$ is a fractional diffusion, we have

$$g_1(x,y) \asymp \begin{cases} c_1\mathsf{d}(x,y)^{d_w-d_f} & \text{if } d_f>d_w,\\ -c_2\log\mathsf{d}(x,y)+c_3 & \text{if } d_f=d_w,\\ c_4 & \text{if } d_f<d_w. \end{cases} \tag{4.1}$$

See [2] Proposition 3.28 for the proof. If $d_f<d_w$ then points have strictly positive capacity, and the result is immediate. We prove the result for $d_f>d_w$: the proof for $d_f=d_w$ is similar. It is well-known that for each compact set $M\subset K$,

$$\mathrm{Cap}(M)=\sup\left\{m(M):\begin{array}{l} m \text{ is a positive Radon measure, } \operatorname{supp} m\subset M,\\ G_1m(x)\equiv\int_M g_1(x,y)m(dy)\le 1 \text{ for every } x\in K\end{array}\right\}. \tag{4.2}$$

Using the above estimates of $g_1(\cdot,\cdot)$,

$$\begin{aligned}\int_M g_1(x,y)\nu(dy) &\le \int_K g_1(x,y)\nu(dy)\le\sum_{n=0}^\infty\int_{\alpha^{-n-1}\le\mathsf{d}(x,y)<\alpha^{-n}} g_1(x,y)\nu(dy)\\ &\le c_5\sum_n\alpha^{n(d_f-d_w)}\nu(\alpha^{-n-1}\le\mathsf{d}(x,y)<\alpha^{-n})\le c_6\sum_n\alpha^{n(d_f-d_w-d)}\equiv c_7<\infty,\end{aligned}$$

because of the assumption $d_f-d<d_w$. Thus, setting $\nu_M(\cdot)\equiv\nu(\cdot\cap M)$, we have $G_1\nu_M\le c_7$. Using (4.2), $\mathrm{Cap}(M)\ge\nu(M)/c_7$ for each compact set M.

For 3), we will use the results by Jonsson-Wallin in [16] and by Triebel in [29]. Denote the Lipschitz and the Besov spaces in the sense of Jonsson-Wallin by $\mathrm{Lip}_{JW}(\alpha,p,q,K)$ and $B^{p,q}_{\alpha,JW}(K)$ (see page 122–123 in [16] for definition). Note that $\mathrm{Lip}_{JW}(\alpha,p,q,K)\subset B^{p,q}_{\alpha,JW}(K)$ and they are equal when $\alpha\notin\mathbb{N}$ (page 125 in [16]). For each $f\in\Lambda^{d_w/2}_{2,\infty}(K)$, $(f,0,\dots,0)\in\mathrm{Lip}_{JW}(d_w/2,2,\infty,K)$. Thus, using the extension theorem in page 155 of [16], we have

$$\Lambda^{d_w/2}_{2,\infty}(K)\subset\mathrm{Lip}_{JW}(d_w/2,2,\infty,K)\subset B^{2,2}_{d_w/2,JW}(K)\subset\Lambda^{2,\infty}_\gamma(\mathbb{R}^n)|_K,$$

where $\gamma=(d_w+n-d_f)/2$ and $\Lambda^{p,q}_\gamma(\mathbb{R}^n)$ is a classical Besov space on $\mathbb{R}^n$. Now, since $d_w-d_f>-d$ (due to (A7)), $\Lambda^{2,\infty}_\gamma(\mathbb{R}^n)\subset\Lambda^{2,1}_{(n-d)/2}(\mathbb{R}^n)$. Finally, by Corollary 18.12 (i) in [29], we have $\mathrm{tr}_L\,\Lambda^{2,1}_{(n-d)/2}(\mathbb{R}^n)=L^2(L,\nu)$. (Note that this trace in the sense of Triebel is simply restriction and there is no corresponding extension.) Combining these facts, we have $\mathcal{F}|_L\subset L^2(L,\nu)$, which means $\|\tilde f|_L\|_{L^2(L,\nu)}\le c_9\|f\|_{\mathcal{F}}$ for all $f\in\mathcal{F}$. Therefore, (A8)' holds. □

We now make one concrete choice of $\mathcal{F}_A$ for $A\subset W^m$ and show that such a choice is suitable for Dirichlet forms whose corresponding processes are the fractional diffusions. By (A5), (A6), and the self-similarity of μ, for any $w\in\bigcup_{n\in\mathbb{Z}_+}W^n$, there exists $c>0$ such that $\mathrm{Cap}(D)\le c\,\mathrm{Cap}(F_w(D))$ for any $D\subset K$. We assume the converse as follows.

(A*) For any $w\in\bigcup_{n\in\mathbb{Z}_+}W^n$, there exists $c>0$ such that $\mathrm{Cap}(F_w(D))\le c\,\mathrm{Cap}(D)$ for any $D\subset K$.

For a subset A of W^m for some $m \in \mathbb{N}$, we say that a collection $\{f_w\}_{w\in A}$ of functions in $\mathcal{F}$ is compatible if $\tilde{f}_v(F_v^{-1}(x)) = \tilde{f}_w(F_w^{-1}(x))$ q.e. on $K_v \cap K_w$ for every $v, w \in A$. Note that this is well-defined by (A*). Define

$$\mathcal{F}_A = \{f \in L^2(K_A, \mu|_{K_A}) : F_w^* f \in \mathcal{F} \text{ for all } w \in A \text{ and } \{F_w^* f\}_{w\in A} \text{ is compatible}\}. \tag{4.3}$$

If we equip A with a graph structure so that $v \in A$ and $w \in A$ are connected if $\mathrm{Cap}(K_v \cap K_w) > 0$, then A is $\mathcal{E}$-connected when A is a connected graph. This is verified by using (B1).

Lemma 4.3. *For $A \subset W^m$ with $m \in \mathbb{Z}_+$, $(\mathcal{E}_A, \mathcal{F}_A)$ is a strong local Dirichlet form on $L^2(K_A, \mu|_{K_A})$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_A$ be a Cauchy sequence in $\mathcal{F}_A$. Let g be the limit in $L^2(K_A)$. Let $w \in A$. Since $\{F_w^* f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{F}$, $F_w^* f_n \to F_w^* g$ in $\mathcal{F}$. It is also easily deduced that $\{F_w^* g\}_{w\in A}$ is compatible. Therefore, $g \in \mathcal{F}_A$ and $f_n \to g$ in $\mathcal{F}_A$. This implies that $(\mathcal{E}_A, \mathcal{F}_A)$ is a closed form on $L^2(K_A)$. The Markov property and the strong locality are inherited from those of $(\mathcal{E}, \mathcal{F})$ via relation (2.2). □

Corollary 4.4. *Assume case* 1) *or* 2) *in Proposition 4.2. Then,* (A*) *holds.*

Proof. In case 1), non-empty sets have uniform positive capacities, which implies (A*). In case 2), (A*) is an easy consequence of (4.1) and (4.2). □

In the rest of this section, we will discuss sufficient conditions for (B3).

Lemma 4.5. *Let $A \subset W_m$, $m \in \mathbb{Z}_+$. Then, $\mathcal{F}_A$ is compactly imbedded in $L^2(K_A, \mu|_{K_A})$. Suppose that A is $\mathcal{E}_A$-connected. Then, when we set $\mathcal{A} = \{f \in \mathcal{F}_A : \int_{K_A} f\,d\mu = 0,\ \mathcal{E}_A(f) \le C\}$ for a constant $C > 0$, $\mathcal{A}$ is bounded in $\mathcal{F}_A$.*

Proof. Let $\mathcal{B}$ be a bounded subset of $\mathcal{F}_A$. For each $v \in A$, $\{F_v^* f : f \in \mathcal{B}\}$ is bounded in $\mathcal{F}$. By (B1), we can take a sequence $\{f_n\}_{n\in\mathbb{N}}$ from $\mathcal{B}$ such that $F_v^* f_n$ converges in $L^2(K)$. Therefore, we can take a sequence from $\mathcal{B}$ converging in $L^2(K_A)$. This implies the first assertion.

By combining this with the $\mathcal{E}_A$-connectedness of K_A, there exists $c > 0$ such that $\|f - ⨍_{K_A} f\,d\mu\|_{L^2(K_A)}^2 \le c\mathcal{E}_A(f)$ for every $f \in \mathcal{F}_A$. The latter assertion follows from this immediately. □

We now give a sufficient condition for (B3) (2).

Proposition 4.6. *The following condition* (EHI1) *implies* (B3) (2)*.*

(EHI1) *For any $v \in \Xi$, there exist some $c_1 > 0$, subsets $D''(v)$ and $D'''(v)$ of $D^\sharp(v)$ such that $D'''(v) \subset D''(v) \subset D^\sharp(v)$, $K_{D''(v)} \cap K_{S^{|v|}\setminus D^\sharp(v)} = \emptyset$, $K_v \cap K_{S^{|v|}\setminus D'''(v)} = \emptyset$ and $\operatorname{esssup}_{x\in K_{D'''(v)}} h(x) \le c_1 \operatorname{essinf}_{x\in K_{D'''(v)}} h(x)$ for every $h \in \mathcal{H}(D^\sharp(v))$ with $h \ge 0$ μ-a.e.*

Proof. First, we apply Lemma 3.7 to $g \in \mathcal{F}$ with $A = D^\sharp(v)$ and $J = \emptyset$, and denote g' there by Hg. We will follow the proof of Theorem 2.2 in [13]. For $h \in \mathcal{H}(D^\sharp(v))$ with $h \geq 0$ μ-a.e., we have, by (EHI1),

$$\operatorname*{esssup}_{x \in K_{D'''(v)}} h(x) \leq c_1 \operatorname*{essinf}_{x \in K_{D'''(v)}} h(x) \leq c_2 \|h\|_{L^2(K_{D'''(v)})}.$$

For $h \in \mathcal{H}(D^\sharp(v))$, let $h_+(x) = \max\{h(x), 0\}$ and $h_-(x) = \max\{-h(x), 0\}$. Since $h = Hh = Hh_+ - Hh_-$ and $Hh_\pm \geq 0$ μ-a.e., we have

$$\begin{aligned}
\operatorname*{esssup}_{x \in K_{D'''(v)}} |h(x)| &\leq \operatorname*{esssup}_{x \in K_{D'''(v)}} Hh_+(x) + \operatorname*{esssup}_{x \in K_{D'''(v)}} Hh_-(x) \\
&\leq c_2(\|Hh_+\|_{L^2(K_{D'''(v)})} + \|Hh_-\|_{L^2(K_{D'''(v)})}) \\
&\leq c_3(\|h_+\|_{\mathcal{F}_{D''(v)}} + \|h_-\|_{\mathcal{F}_{D''(v)}}) \\
&\leq 2c_3 \|h\|_{\mathcal{F}_{D''(v)}}.
\end{aligned} \tag{4.4}$$

In order to prove (B3) (2), it suffices to prove the following.

(∗) If a sequence $\{h_l\}$ in $\mathcal{H}(D^\sharp(v))$ converges weakly to 0 in $\mathcal{F}_{D^\sharp(v)}$, then there exists a subsequence $\{h_{l(k)}\}$ such that $F_v^* h_{l(k)}$ converges strongly to 0 in $\mathcal{F}$.

Indeed, suppose (∗) holds. Let $\{f_m\}$ be a sequence in $\mathcal{H}(D^\sharp(v))$ that is bounded in $\mathcal{F}_{D^\sharp(v)}$. We can take a subsequence $\{f_{m(l)}\}$ and $f \in \mathcal{F}_{D^\sharp(v)}$ such that $f_{m(l)}$ converges weakly to f in $\mathcal{F}_{D^\sharp(v)}$. Take $g_l \in \mathcal{H}(D^\sharp(v))$ such that $g_l \to f$ in $\mathcal{F}_{D^\sharp(v)}$. Applying (∗) to $h_l := f_{m(l)} - g_l$, we can take a sequence $\{l(k)\}$ diverging to ∞ such that $F_v^* f_{m(l(k))} \to F_v^* f$ in $\mathcal{F}$. This implies (B3) (2).

In order to prove (∗), recall the notion of the energy measure. For $f \in \mathcal{F} \cap L^\infty(K)$, the energy measure $\mu_{\langle f \rangle}$ is a unique positive Radon measure on K such that the following identity holds for every $g \in \mathcal{F} \cap C(K)$:

$$\int_K g \, d\mu_{\langle f \rangle} = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g).$$

Now, by (4.4), $C := \operatorname{esssup}_{x \in K_{D'''(v)}} |h_l(x)|$ is bounded in l. Define $\hat{h}_l = (-C) \vee h_l \wedge C$. Since $\mathcal{F}_{D^\sharp(v)}$ is compactly imbedded in $L^2(K_{D^\sharp(v)})$ by Lemma 4.5, $\{h_l\}$ converges to 0 in $L^2(K_{D^\sharp(v)})$. Take a subsequence $\{h_{l'}\}$ converging to 0 μ-a.e. on $K_{D^\sharp(v)}$. Since $(\mathcal{E}, \mathcal{F})$ is regular, we can take $\varphi \in \mathcal{F} \cap C(K)$ such that $0 \leq \varphi \leq 1$ on K, $\varphi = 1$ on K_w, and $\varphi = 0$ outside $K_{D'''(v)}$. We have

$$0 = 2\mathcal{E}(h_{l'}, \hat{h}_{l'}\varphi) = 2\mathcal{E}(\hat{h}_{l'}, \hat{h}_{l'}\varphi) = \mathcal{E}(\hat{h}_{l'}^2, \varphi) + \int_K \varphi \, d\mu_{\langle \hat{h}_{l'} \rangle},$$

because $\hat{h}_{l'}\varphi$ vanishes outside $K_{D'''(v)}$. Note that $\mathcal{E}(\hat{h}_{l'}^2) \leq 4C^2 \mathcal{E}(h_{l'})$, which is bounded in l'. A suitable subsequence $\hat{h}_{l''}$ can be taken so that $\{\hat{h}_{l''}^2\}$ converges weakly to some g in $\mathcal{F}$. Since $g = 0$ on $K_{D^\sharp(v)}$, $\mathcal{E}(\hat{h}_{l''}^2, \varphi) \to \mathcal{E}(g, \varphi) = 0$ as $l'' \to \infty$. On the other hand,

$$\begin{aligned}
\int_K \varphi \, d\mu_{\langle \hat{h}_{l''} \rangle} &= \sum_{z \in S^{|v|}} \rho^{|v|} \int_K F_z^* \varphi \, d\mu_{\langle F_z^* \hat{h}_{l''} \rangle} \\
&\geq \rho^{|v|} \int_K F_v^* \varphi \, d\mu_{\langle F_v^* \hat{h}_{l''} \rangle} \\
&= \rho^{|v|} \mu_{\langle F_v^* \hat{h}_{l''} \rangle}(K) = 2\rho^{|v|} \mathcal{E}(F_v^* \hat{h}_{l''}) = 2\rho^{|v|} \mathcal{E}(F_v^* h_{l''}).
\end{aligned}$$

Combining these estimates, we obtain $\overline{\lim}_{l''\to\infty}\mathcal{E}(F_v^* h_{l''}) \le 0$. Therefore, $F_v^* h_{l''}$ converges to 0 in $\mathcal{F}$. This proves $(*)$. $\square$

We next give sufficient conditions for (B3) (1) (b).

Proposition 4.7. *The following conditions imply* (B3) (1) (b).

(1) $\mathcal{F} = \Lambda^{\beta}_{2,\infty}(K)$ *for some* $\beta > 0$.

(2) *For each* $v \in \Xi$, $D'(v)$ *is* $\mathcal{E}_{D'(v)}$*-connected.*

(3) *For each* $w \in \bigcup_{n\in\mathbb{Z}_+} \hat{I}^{n+m_0}$, *there exist subsets* $D^\sharp(w)$, $D^{(1)}(w)$, $D^{(2)}(w)$ *of* $D'(w)$ *such that* $D^\sharp(w) \subset D^{(1)}(w) \subset D^{(2)}(w) \subset D'(w)$ *and the following hold.*

   (a) *There exists* $v \in \Xi$ *such that both* $D'(w)$ *and* $D'(v)$, *and* $D^\sharp(w)$ *and* $D^\sharp(v)$, *are of the same type by the same map* F.

   (b) $K_{D^\sharp(w)} \cap K_{S^{|w|}\setminus D^{(1)}(w)} = K_{D^{(2)}(w)} \cap K_{S^{|w|}\setminus D'(w)} = \emptyset$.

(EHI2) *There exists* $c > 0$ *such that* $\operatorname{esssup}_{x\in K_{D^{(1)}(w)}} h(x) \le c \operatorname{essinf}_{x\in K_{D^{(1)}(w)}} h(x)$ *for* $h \in \mathcal{H}(D^{(2)}(w))$ *with* $h \ge 0$ μ*-a.e.*

Proof. Let g be a function in $\mathcal{F}$ such that $\int_{K_{D'(w)}} g\,d\mu = 0$ and $\rho^{|w|}\mathcal{E}_{D'(w)}(g) \le 1$. Let $f(x) = g(F^{-1}(x))$, $x \in K_{D'(v)}$. Then, $f \in \mathcal{F}_{D'(v)}$, $\int_{K_{D'(v)}} f\,d\mu = 0$ and $\rho^{|v|}\mathcal{E}_{D'(v)}(f) \le 1$. By Lemma 4.5, $\|f\|_{\mathcal{F}(D'(v))} \le C$, where C is a constant independent of w. Suppose moreover that $g \in \mathcal{H}(I^{|w|}, D'(w))$. Apply Lemma 3.7 to g with $A = D^{(2)}(w)$ and $J = \emptyset$ and denote g' there by g_1. Let $g_2 = g - g_1$. By (EHI2) and the same argument in the first part of the proof of Proposition 4.6, g_1 is bounded on $D^{(1)}(w)$. Take a function $\psi \in \mathcal{F}$ such that $0 \le \psi \le 1$, $\psi = 0$ on $K_{S^{|w|}\setminus D^{(1)}(w)}$ and $\psi = 1$ on $K_{D^\sharp(w)}$. Then, $g_1\psi \in \mathcal{F}$. Since both $g_1\psi$ and g_2 vanish on $K_{S^{|w|}\setminus D^{(2)}(w)}$, when we set $f'(x) = \begin{cases} (g_1\psi + g_2)(F^{-1}(x)), & x \in K_{D'(v)} \\ 0, & x \in K \setminus K_{D'(v)} \end{cases}$, f' belongs to $\mathcal{F}$ by using the fact $\mathcal{F} = \Lambda^{\beta}_{2,\infty}(K)$. Since $f' = f$ on $K_{D^\sharp(v)}$, we have $f' \in \mathcal{H}(I^{|v|}, D^\sharp(v))$ and $\|f'\|_{\mathcal{F}_v} = \|f\|_{\mathcal{F}_v} \le C$. These conclude the assertion. $\square$

5 Examples

In this section, we choose $\mathcal{F}_A$ as in (4.3) for $A \subset W^m$.

1) Sierpinski gaskets: Let $\{a_0, a_1, \dots, a_n\} \subset \mathbb{R}^n$ be the vertices of n-dimensional simplex. Let $W = S = \{0, 1, \dots, n\}$ and let $F_i(x) = (x - a_i)/2 + a_i$ for $x \in \mathbb{R}^n$ and $i = 0, 1, \dots, n$. Then the unique non-void compact set K which satisfies $K = \bigcup_{i=0}^n F_i(K)$ is the n-dimensional Sierpinski gasket. The map Φ in Lemma 2.1 is the identity map. It is well-known (see [2, 6, 18] etc.) that there is a self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ where the corresponding diffusion is the fractional diffusion. In particular, $\mathcal{F} = \Lambda^{d_w/2}_{2,\infty}(K) \subset C(K)$ where $d_w = (\log(n+3))/(\log 2)$. Note that $d_w - d_f > 0$ in this case. Let L be the $(n-1)$-dimensional gasket determined by $\{F_i\}_{i=0}^{n-1}$.

That is, $I = \{0, 1, \dots, n-1\}$. Let $\hat{I} = I$, $M = 1$. It is easy to see that $\mathcal{F}_A = \mathcal{F}|_{K_A}$ for each $A \subset W^m$. Then (A1)–(A7), (B2), and (C1)–(C2) are easy to check with $\rho = (n+3)/(n+1)$. (A8) holds by Proposition 4.2 and (B1) holds by [18] Lemma 3.4.5. For (B3), define $l_0 = 0$, $m_0 = 0$, $D'(w) = \{w\}$ for $w \in \bigcup_{n \in \mathbb{Z}_+} I^n$, $\Xi = \{\emptyset\}$, and $D^\sharp(\emptyset) = \{\emptyset\}$. It is easy to check (B3) (1) by using Lemma 3.3 and Lemma 4.5. Since $\mathcal{H}_{D'(w)}(D'(w))$ is a finite dimensional space, (B3) (2) is clearly true. We will prove (B4). Let $f \in \mathcal{F}$ and $\mathcal{E}_{S^m \setminus I^m}(f) = 0$ for some $m \in \mathbb{Z}_+$. Then, for each $w \in S^m \setminus I^m$, $\mathcal{E}(F_w^* f) = \rho^{-m} \mathcal{E}_w(f) = 0$. Therefore, f is constant on K_w for each $w \in S^m \setminus I^m$. We consider an unoriented graph with a vertex set $V = S^m \setminus I^m$ and an edge set $\{(v, w) \in V \times V : \mathrm{Cap}(K_v \cap K_w) > 0\}$. Then, V is a connected set. Note that (v, w) is an edge if and only if $K_v \cap K_w \neq \emptyset$. Therefore, f should be constant on $K_{S^m \setminus I^m}$, thus constant on $K \setminus L$. This concludes that (B4) holds. Therefore, we have by Theorem 2.5, Theorem 2.6 and Remark 2.7,

$$\mathcal{F}|_L = \Lambda_{2,2}^{\beta}(L) \qquad \text{where} \quad \beta = \frac{d_w}{2} - \frac{\log(1 + 1/n)}{2 \log 2}.$$

When $n = 2$, this relation was obtained in [15].

2) Pentakun: Let $a_k = e^{2k\sqrt{-1}\pi/5 + \sqrt{-1}\pi/2} \in \mathbb{C}$, $k = 0, 1, 2, 3, 4$. Let $W = \{0, 1, 2, 3, 4, 5, 6\}$, $S = \{0, 1, 2, 3, 4\}$, $I = \hat{I} = \{2, 3, 5, 6\}$ and $M = 1$. Let $\mathfrak{G} = \{G_k\}_{k=0}^4$ with $G_k : \mathbb{C} \to \mathbb{C}$ defined by $G_k(z) = e^{2k\sqrt{-1}\pi/5} z$. For $i = 0, 1, 2, 3, 4$, define a contraction map $F_i : \mathbb{C} \to \mathbb{C}$ by $F_i(z) = \alpha^{-1}(z - a_i) + a_i$, where $\alpha = \frac{3+\sqrt{5}}{2}$. We also define $F_5 = F_2 \circ G_1$ and $F_6 = F_3 \circ G_4$. Then, the resulted nested fractal K is called Pentakun and a subset L is a Koch curve (see Figure 1). The Hausdorff dimensions of K and L are $(\log 5)/(\log \alpha)$ and $(\log 4)/(\log \alpha)$, respectively. There exists a canonical Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ where the corresponding diffusion is the fractional diffusion (see [2, 19, 23] etc.), so $\mathcal{F} = \Lambda_{2,\infty}^{d_w/2}(K)$. It is known (see [19]) that $d_w = (\log \frac{\sqrt{161}+9}{2})/(\log \alpha)$ and we can check all the assumptions similarly to the case of the Sierpinski gasket. Note that C_2 given in (B2) (4) is equal to 4. Thus, by Theorem 2.5, Theorem 2.6 and Remark 2.7,

$$\mathcal{F}|_L = \Lambda_{2,2}^{\beta}(L) \qquad \text{where} \quad \beta = \frac{d_w}{2} - \frac{\log 5 - \log 4}{2 \log \alpha}.$$

For the Pentakun K, let $I' = \{2, 3\}$. Then the corresponding self-similar subset L' is a Cantor set with Hausdorff dimension $(\log 2)/(\log \alpha)$. In this case we should set $\hat{I} = \{2, 3, 5, 6\}$, so $I \neq \hat{I}$. Again we can check all the assumptions similarly, so by Theorem 2.5, Theorem 2.6 and Remark 2.7,

$$\mathcal{F}|_{L'} = \Lambda_{2,2}^{\beta'}(L') \qquad \text{where} \quad \beta' = \frac{d_w}{2} - \frac{\log 5 - \log 2}{2 \log \alpha}.$$

In general, if K is a nested fractal satisfying (A3), then there is a canonical Dirichlet form on $L^2(K, \mu)$ where the corresponding diffusion is the fractional diffusion (see [2, 19, 23] etc.). Let L be a self-similar subset of K given in the manner in the first part of Section 2, and satisfying (A2). In most cases, all the assumptions except (B4) can be checked similarly to the case of the Sierpinski gasket, so that we can use Theorem 2.5 and Theorem 2.6 to characterize the trace space if (B4) holds. However, there are cases where (B4) does not hold – see 4).

3) Sierpinski carpets: Let $H_0 = [0, 1]^n$, $n \geq 2$, and let $l \in \mathbb{N}$, $l \geq 2$ be fixed. Set $\mathcal{Q} = \{\Pi_{i=1}^n [(k_i - 1)/l, k_i/l] : 1 \leq k_i \leq l,\ k_i \in \mathbb{N}\ (1 \leq i \leq n)\}$, let $N \leq l^n$ and $W = S = \{1, \dots, N\}$.

Let F_i, $i \in S$ be orientation preserving affine maps of H_0 onto some element of $\mathcal{Q}$. We assume that the sets $F_i(H_0)$ are distinct. Set $H_1 = \bigcup_{i \in I} F_i(H_0)$. Then the unique non-void compact set K which satisfies $K = \bigcup_{i=1}^N F_i(K)$ is called the generalized Sierpinski carpet if the following holds:

(SC1) (Symmetry) H_1 is preserved by all the isometries of the unit cube H_0.

(SC2) (Connected) H_1 is connected.

(SC3) (Non-diagonality) Let B be a cube in H_0 which is the union of 2^n distinct elements of $\mathcal{Q}$. (So B has side length $2l^{-1}$.) Then if $\mathrm{Int}(H_1 \cap B)$ is non-empty, it is connected.

(SC4) (Borders included) H_1 contains the line segment $\{x : 0 \le x_1 \le 1, x_2 = \cdots = x_n = 0\}$.

Here (see [3]) (SC1) and (SC2) are essential, while (SC3) and (SC4) are included for technical convenience. The Sierpinski carpets are infinitely ramified: the critical set C_K in (2.1) is an infinite set, and K cannot be disconnected by removing a finite number of points.

It is known (see [3, 4, 21] etc.) that there is a self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ where the corresponding diffusion is the fractional diffusion. In particular, $\mathcal{F} = \Lambda_{2,\infty}^{d_w/2}(K)$ where $d_w = (\log \rho N)/(\log l)$, ρ given in (A5). Let $\mathfrak{G} = \{\text{the identity map}\}$ and $L = ([0,1]^{n-1} \times \{0\}) \cap K$ (cf. Figure 1). Let $I = \{i \in S : F_i(K) \cap L \ne \emptyset\}$, $N_I = \# I$, and assume

$$\rho N_I > 1. \tag{5.1}$$

For simplicity, we assume that the $(n-1)$-dimensional Sierpinski carpet L also satisfies the conditions corresponding to (SC1)–(SC4). Then, (A1)–(A6) and (C1)–(C2) are easy to check with $M = 1$. (A7) holds by (5.1), because $\beta = d_w/2 - (d_f - d)/2 = (\log \rho N_I)/(2 \log l)$. (A8) holds by Proposition 4.2. It is known that the corresponding self-adjoint operator has compact resolvents (see [3, 4, 21] etc.), so (B1) holds. Letting $\hat{I} = I$, we can check (B2). For $w \in I^m$, $m \in \mathbb{Z}_+$, let $x_0(w) \in [0,1]^n$ be the center of K_w and $\Lambda_k(w)$ the intersection of K and a cube in $\mathbb{R}^n$ with center $x_0(w)$ and length $(2k+1)l^{-m}$ for $k \in \mathbb{N}$. In order to assure (B3), assume for the moment that there exists some $k \ge 6$ such that $\Lambda_k(w)$ is connected for all $w \in \bigcup_{m \in \mathbb{Z}_+} I^m$. Let $l_0 = (2k+1)n$ and take $m_0 \in \mathbb{N}$ such that $l^{m_0} \ge 2k+1$. For each $w \in I^{m+m_0}$, $m \in \mathbb{Z}_+$, take $D'''(w) \subset D''(w) \subset D^\sharp(w) \subset D^{(1)}(w) \subset D^{(2)}(w) \subset D'(w)$ so that $K_{D'''(w)} = \Lambda_1(w)$, $K_{D''(w)} = \Lambda_2(w)$, $K_{D^\sharp(w)} = \Lambda_3(w)$, $K_{D^{(1)}(w)} = \Lambda_4(w)$, $K_{D^{(2)}(w)} = \Lambda_5(w)$, and $K_{D'(w)} = \Lambda_k(w)$. With the use of Proposition 4.6 and Proposition 4.7, (B3) can be checked. Here, the Harnack inequalities (EHI1) and (EHI2) are assured by [3, 4, 21]. To be more precise, let $\hat{K} = \bigcup_{x \in \{-1,0\}^n} (K + x)$, which is a subset of $[-1,1]^n$. Then, one can construct a Dirichlet form on $\hat{K}$ whose corresponding diffusion is the fractal diffusion in the same way as in [3, 4, 21]. Indeed, $\hat{K}$ has enough symmetry for the coupling arguments in [3] to work. In this way, the Harnack inequalities (EHI1) and (EHI2) are assured. If for each k, there exists $w \in \bigcup_{m \in \mathbb{Z}_+} I^m$ such that $\Lambda_k(w)$ is not connected, then take the connected component of $\Lambda_k(w)$ including K_w in place of $\Lambda_k(w)$ and discuss similarly as above. By the covering argument, we can check (B3). (B4) is confirmed by an argument similar to the case of Sierpinski gaskets. Thus, we have by Theorem 2.5 and Theorem 2.6,

$$\mathcal{F}|_L = \hat{\Lambda}_{2,2}^{\beta}(L) \qquad \text{where} \quad \beta = \frac{d_w}{2} - \frac{1}{2}\left(\frac{\log N}{\log l} - \dim_H L\right).$$

Note that when $\partial[0,1]^n \subset K$, then $0 < \beta < 1$, so (5.1) holds and $\mathcal{F}|_L = \Lambda^{\beta}_{2,2}(L)$ by Remark 2.7. Indeed, let $K_2 = [0,1]^n$ and K_1 be a generalized Sierpinski carpet in $\mathbb{R}^n$ with $\partial[0,1]^n \subset K_1$, which is determined by $\{F_i\}_i$ where $F_i([0,1]^n) \cap \partial[0,1]^n \neq \emptyset$ for all i. Clearly, $K_1 \subset K \subset K_2$. For each K_i, one can construct the self-similar Dirichlet form. Let ρ_i be the scaling factor given in (A5). By the shorting and cutting laws for electrical networks (see [9]), $\rho_2 \le \rho \le \rho_1$. Then, $\rho_2 = l^{2-n}$ and

$$\frac{2}{l^{n-1}} + \frac{l-2}{l^{n-1} - (l-2)^{n-1}} \le \rho_1 \le \frac{l}{l^{n-1} - (l-2)^{n-1}}, \tag{5.2}$$

due to (5.9) in [3]. Since $L = [0,1]^{n-1} \times \{0\}$ and $N_I = l^{n-1}$ in this case, we have $\rho N_I \ge \rho_2 N_I = l \ge 2$, so (5.1) holds and $\beta > 0$. Using (5.2),

$$\rho N_I \le \rho_1 N_I \le \frac{l^n}{l^{n-1} - (l-2)^{n-1}} < l^2,$$

where the last inequality is a simple computation. Thus $\beta < 1$.

4) Vicsek sets: Let $a_1 = (0,0), a_2 = (1,0), a_3 = (1,1), a_4 = (0,1), a_5 = (1/2,1/2)$ be points in $\mathbb{R}^2$ and define $F_i(x) = (x - a_i)/3 + a_i$ for $x \in \mathbb{R}^2$ and $i = 1, \ldots, 5$. The unique non-void compact set K which satisfies $K = \bigcup_{i=1}^5 F_i(K)$ is the Vicsek set. As in the case of 1), there is a self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K,\mu)$ with $\rho = 3$, where the corresponding diffusion is the fractional diffusion. In particular, $\mathcal{F} = \Lambda^{d_w/2}_{2,\infty}(K)$ where $d_w = (\log 15)/(\log 3)$. Let L be the line segment from $(0,0)$ to $(1,1)$. Then, (B4) does not hold. In this case, the trace of the Brownian motion on the Vicsek set is the Brownian motion on the line segment. Indeed, one can easily check condition $(H_1)-(H_3)$ in Section 8 of [6] on the 1-dimensional Sierpinski gasket which is the line. So, by [6] Theorem 8.1, one sees that the trace of the Brownian motion on the Vicsek set is a constant time change of the Brownian motion on the line. Thus,

$$\mathcal{F}|_L = \Lambda^1_{2,\infty}(L),$$

which is larger than $\Lambda^1_{2,2}(L)$. This shows that (B4) is necessary for Theorem 2.5.

6 Application: Brownian motion penetrating fractals

In [12], one of the authors constructed Brownian motions on *fractal fields*, a collection of fractals with (in general) different Hausdorff dimensions (see also [20]). They are diffusion processes which behave as the appropriate fractal diffusions within each fractal component of the field and they penetrate each fractal. In [12], a restricted assumption (Assumption 2.2 in [12]) was needed to construct such processes because we did not know the corresponding function spaces. Our result in this paper can be applied here and we can construct such penetrating diffusions without the restricted assumption.

Let A_0 be a countable set and let $\{K_i\}_{i\in A_0} \subset \mathbb{R}^n$ be a family of self-similar sets together with strong local, regular, and self-similar Dirichlet forms $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ on $L^2(K_i,\mu_i)$, where K_i and μ_i lie in the framework of Section 2. We also regard μ_i as a measure on $\mathbb{R}^n$ by letting $\mu_i(\mathbb{R}^n \setminus K_i) = 0$. We set $G = \bigcup_{i\in A_0} K_i$.

Let A_1 be another countable set and let $\{D_j\}_{j\in A_1} \subset \mathbb{R}^n$ be a family of disjoint domains in $\mathbb{R}^n \setminus G$. Denote the closure of D_j in $\mathbb{R}^n$ by K_j and the Lebesgue measure restricted on K_j by

μ_j. Define $\tilde{G} = G \cup \left(\bigcup_{j\in A_1} K_j\right)$. $\tilde{G}$ is called a *fractal field* generated by $\{K_i\}_{i\in A_0}$ and $\{D_j\}_{j\in A_1}$. (When G is connected as in the introduction, we also call G a *fractal field* or a *fractal tiling*.)

Denote by A the disjoint union of A_0 and A_1. For $i, j \in A$ with $i \neq j$, let $\Gamma_{ij} = K_i \cap K_j$. Define $\Gamma = \bigcup_{i,j\in A,\, i\neq j} \Gamma_{ij}$. For $x \in \Gamma$, let $J_x := \{i \in A : x \in K_i\}$ and define $N_x := \bigcup_{i,j\in J_x,\, i\neq j} \Gamma_{ij}$. Throughout this section, we impose the following assumption.

Assumption A (1) For each compact set $C \subset \mathbb{R}^n$, $\#\{i \in A : C \cap K_i \neq \emptyset\} < \infty$.
(2) For each $i \in A_1$, $K_i \setminus D_i$ is a null set with respect to the Lebesgue measure on $\mathbb{R}^n$.

For each $i \in A_1$, define $\mathcal{D}(\mathcal{E}_{K_i}) = \{u \in C_0(K_i) : u|_{D_i} \in W^{1,2}(D_i)\}$ and

$$\mathcal{E}_{K_i}(u, v) = \frac{1}{2}\int_{D_i} (\nabla u(x), \nabla v(x))_{\mathbb{R}^n}\, dx, \quad \text{for } \ u, v \in \mathcal{D}(\mathcal{E}_{K_i}).$$

Then, $(\mathcal{E}_{K_i}, \mathcal{D}(\mathcal{E}_{K_i}))$ is closable on $L^2(K_i, \mu_i)$. Its closure will be denoted by $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$. It is easy to see that $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ is a strong local regular Dirichlet form.

For $x \in \Gamma$ and $i \in J_x$, define $\beta_{x,i} = d_w(K_i)/2 - (d_f(K_i) - d_f(N_x \cap K_i))/2$. Here, $d_w(K_i)$ is defined in (A7) for $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ if $i \in A_0$ and $d_w(K_i)$ is defined as 2 if $i \in A_1$, and $d_f(K_i)$ and $d_f(N_x \cap K_i)$ are the Hausdorff dimensions of K_i and $N_x \cap K_i$, respectively.

We will also assume the following throughout this section.

Assumption B (1) For $i \in A_0$, $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ is a strong local regular Dirichlet form on $L^2(K_i, \mu_i)$ which satisfies (A1), (A3), the first identity of (A4), (A5), and (A6) in Section 2.
(2) For each $x \in \Gamma$ and $i \in J_x \cap A_0$, $N_x \cap K_i$ is a finite number of union of compact self-similar sets $\{L_j\}$ that are constructed by the same number of contraction maps and each of which satisfies (A2), the second identity of (A4), (A7), and (A8) in Section 2. Further, (C1)*–(C2)* in Remark 3.10 holds with $K = K_i$ and $L = N_x \cap K_i$.
(3) For each $x \in \Gamma$ and $i \in J_x \cap A_1$, $N_x \cap K_i$ is a closed Alfors $d_{x,i}$-regular set with some $d_{x,i}$.
(4) For every $x \in \Gamma$, $\beta_{x,i} > 0$ for all $i \in J_x$, and the set $\Lambda_x := \{f \in C_0(N_x) : f|_{N_x\cap K_i} \in \Lambda_{2,2}^{\beta_{x,i}}(N_x \cap K_i)$ for all $i \in J_x\}$ is dense in $C_0(N_x)$.

We will give several remarks. When $i \in A_1$, we have $d_f(K_i) = n$ and $d_f(K_i \cap N_x) = d_{x,i}$. The set Λ_x is closed under the operation of the normal contraction; $0 \vee f \wedge 1 \in \Lambda_x$ for $f \in \Lambda_x$. If N_x itself is an Alfors regular set and $\beta_{x,i} \in (0, 1)$ for all $i \in J_x$, then $\Lambda_{2,2}^{\max_{i\in J_x}\beta_{x,i}}(N_x) \cap C_0(N_x)$ (which is a subset of Λ_x) is dense in $C_0(N_x)$ by Chapter V, Proposition 1 in [16] and Theorem 3 in [27]. The condition $\beta_{x,i} \in (0, 1)$ holds, for example, if $i \in N_x \cap A_1$ and $d_{x,i} \in (n-2, n)$, because then $\beta_{x,i} = 1 - (n - d_{x,i})/2 \in (0, 1)$.

Define a measure $\tilde{\mu}$ on $\tilde{G}$ by $\tilde{\mu} = \sum_{i\in A} \mu_i$. We now define a bilinear form $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ on $L^2(\tilde{G}, \tilde{\mu})$ as follows:

$$\begin{aligned} \tilde{\mathcal{E}}(u, v) &= \sum_{i\in A} \mathcal{E}_{K_i}(u|_{K_i}, v|_{K_i}) \ \text{ for } u, v \in \mathcal{D}(\tilde{\mathcal{E}}), \\ \mathcal{D}(\tilde{\mathcal{E}}) &= \{u \in C_0(\tilde{G}) : u|_{K_i} \in \mathcal{F}_{K_i} \text{ for all } i \in A \text{ and } \tilde{\mathcal{E}}(u, u) < \infty\}. \end{aligned}$$

Then, the following is easy to check.

Lemma 6.1. (1) *$(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ is closable in $L^2(\tilde{G}, \tilde{\mu})$.*

(2) *$\mathcal{D}(\tilde{\mathcal{E}})$ is an algebra.*

(3) *For $i \in A$, $x \in K_i$, and for $U(x)$ which is a neighborhood of x in K_i, there exists $f \in \mathcal{F}_{K_i} \cap C_0(K_i)$ such that $f(x) > 0$ and $\operatorname{supp} f \subset U(x) \cap K_i$, where $\operatorname{supp} f$ denotes the support of f.*

Now, let $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ be the closure of $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$. We then have the following.

Theorem 6.2. *$(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is a strong local regular Dirichlet form on $L^2(\tilde{G}, \tilde{\mu})$.*

Note that the strong local property of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ can be easily deduced from those of the original forms on $\{K_i\}_{i \in A}$. Therefore, it is enough to prove the regularity of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. For the proof of it, the key part is to prove the following.

Proposition 6.3. (1) *For each $x \neq y \in \tilde{G}$, there exists $g \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $g(x) \neq g(y)$.*

(2) *For any compact set L in $\tilde{G}$, there exists $f \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $f = 1$ on L.*

Once this proposition is established, it is easy to prove the regularity of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ (see [12]), so we will only prove the proposition.

Proof of Proposition 6.3. Let $B(x, r)$ denote the open ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius r. When either x or y is in the compliment of Γ, then (1) is clear by Lemma 6.1 (3), so we will consider the case $x, y \in \Gamma$. By Assumption A (1), $\# J_x < \infty$. Since each K_j is closed, by Assumption A (1), there exists $r_x > 0$ such that $B(x, r_x) \cap K_j \neq \emptyset$ if and only if $j \in J_x$, and $y \notin B(x, r_x)$. Since Λ_x is dense in $C_0(N_x)$ by Assumption B (4), there exists $u \in \Lambda_x$ such that $u|_{B(x, r_x/2)} = 1$ and $u|_{B(x, 3r_x/4)^c} = 0$.

Now, by Assumption B (1), (2) and the extension theorem (Remark 3.10), for each $i \in J_x \cap A_0$, there exists $\hat{u}_i \in \mathcal{F}_{K_i} \cap C(K_i)$ such that $\hat{u}_i|_{N_x \cap K_i} = u$. For each $i \in J_x \cap A_1$, since $N_x \cap K_i$ is a closed Alfors $d_{x,i}$-regular set, we have

$$W^{1,2}(\mathbb{R}^n)|_{N_x \cap K_i} = \Lambda_{2,2}^{1-(n-d_{x,i})/2}(N_x \cap K_i) \tag{6.1}$$

(see [16]). By carefully tracing the proof of the extension theorem in (6.1), we see that there exists $\hat{u}_i \in W^{1,2}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ such that $\hat{u}_i|_{N_x \cap K_i} = u$ (see, for instance, pages 77–78 in [20]). For both cases, since $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ is regular, by multiplying a function in $\mathcal{F}_{K_i} \cap C_0(K_i)$ which is 1 in $B(x, 3r_x/4)$ and 0 outside $B(x, r_x)$, we may assume $\operatorname{supp} \hat{u}_i \subset B(x, r_x)$. Define $g \in C_0(\tilde{G})$ as $g|_{K_i} = \hat{u}_i$ for $i \in J_x$ and $g|_{K_i} \equiv 0$ otherwise. Then, $g \in \mathcal{D}(\tilde{\mathcal{E}})$, $g(x) = 1$ and $g(y) = 0$. We thus obtain the desired function.

The proof of (2) is quite similar, so we omit it (see Proposition 2.6 (2) in [12]). □

Denote the 1-capacity associated with $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ by Cap_{K_i} and $\mathrm{Cap}_{\tilde{G}}$, respectively. By definition, it is easy to see that $u|_{K_i} \in \mathcal{F}_{K_i}$ for any $i \in A$ and $u \in \tilde{\mathcal{F}}$. Further, $\mathrm{Cap}_{K_i}(H) \leq \mathrm{Cap}_{\tilde{G}}(H)$ for any $i \in A$ and $H \subset K_i$. For $i \in A$, let $\mathcal{F}_{K_i'} = \{f \in \mathcal{F}_{K_i} : \tilde{f} = 0 \text{ q.e. on } \Gamma\}$ and $\tilde{\mathcal{F}}_i = \{f \in \tilde{\mathcal{F}} : \tilde{f} = 0 \text{ q.e. on } \bigcup_{j \in A \setminus \{i\}} K_j\}$, where $\tilde{f}$ is a (corresponding) quasi-continuous modification of f.

We will denote by $(\{\tilde{X}_t\}_{t \geq 0}, \{\tilde{P}_x\}_{x \in \tilde{G}})$ the diffusion process corresponding to $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. The following proposition shows that $\{\tilde{X}_t\}$ behaves on K_i in the same way as the diffusion process associated with $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ until the process hits Γ.

Proposition 6.4. *$(\mathcal{E}_{K_i}, \mathcal{F}_{K_i'})$ and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}_i)$ give the same Dirichlet forms on $L^2(K_i, \mu_i|_{K_i\setminus\Gamma})$, by identifying the measure space $(\tilde{G}, \mu_i|_{K_i\setminus\Gamma})$ with $(K_i, \mu_i|_{K_i\setminus\Gamma})$. In particular, the corresponding parts of the processes on $K_i \setminus \Gamma$ are the same.*

Proof. It is easy to see that $f \in \tilde{\mathcal{F}}_i$ satisfies that $f|_{K_i} \in \mathcal{F}_{K_i'}$, so we will prove the converse. Let $f \in \mathcal{F}_{K_i'}$. By Theorem 4.4.3 of [10], we can take an approximation sequence of f from $\mathcal{F}_{K_i'} \cap C_0(K_i \setminus \Gamma)$. Therefore, the 0-extension of f outside K_i is an element of $\tilde{\mathcal{F}}_i$. □

For each distinct $i, j \in A$, we denote $K_i \sim K_j$ if $\mathrm{Cap}_{K_l}(\Gamma_{ij}) > 0$ for $l = i$ and j. We now assume the following in addition to Assumptions A and B.

Assumption C (1) For each $i \in A_0$, $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ is irreducible.
(2) For each distinct $i, j \in A$, there exist $k \in \mathbb{N}$ and a sequence $i_0, i_1, \ldots, i_k \in A$ such that $K_{i_0} = K_i$, $K_{i_k} = K_j$ and $K_{i_l} \sim K_{i_{l+1}}$ for $l = 0, 1, \ldots, k-1$.
(3) For each distinct $i, j \in A$ with $K_i \sim K_j$, there exists a positive Radon measure ν_{ij} on Γ_{ij} such that $\nu_{ij}(\Gamma_{ij}) > 0$ and ν_{ij} is smooth with respect to both $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ and $(\mathcal{E}_{K_j}, \mathcal{F}_{K_j})$.

Note that, when $i \in A_1$, $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ is irreducible since D_i is connected. (See, e.g. Theorem 4.5 in [24], for the proof.)

For each nearly Borel set $B \subset \mathbb{R}^n$, define $\sigma_B = \inf\{t > 0 : \tilde{X}_t \in B\}$. The next proposition shows that $\tilde{X}_t$ penetrates into each K_i.

Proposition 6.5. *The following holds for any nearly Borel set B with $\mathrm{Cap}_{\tilde{G}}(B) > 0$.*

$$\tilde{P}^x(\sigma_B < \infty) > 0 \text{ for } (\tilde{\mathcal{E}}, \tilde{\mathcal{F}})\text{-quasi every } x \in \tilde{G}. \tag{6.2}$$

Especially, if B is a subset of a certain K_i with $\mathrm{Cap}_{K_i}(B) > 0$, then (6.2) holds.

Proof. By virtue of Theorem 4.6.6 in [10], it is enough to prove that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is irreducible. We first recall the following fact. Let $(\mathcal{E}, \mathcal{F})$ be a local Dirichlet form. (Here, the locality means $\mathcal{E}(f, g) = 0$ if $fg = 0$ a.e. All Dirichlet forms appearing in this article are local in this sense; see [25].) Let Y be a measurable subset of the state space and $\mathcal{C}$ a dense set in $\mathcal{F}$. Then, Y is an invariant set if and only if $1_Y \cdot u \in \mathcal{F}$ for any $u \in \mathcal{C}$. This is verified by Theorem 1.6.1 in [10] and a usual approximation argument.

Now, let M be an invariant set for $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. Fix $i \in A$ and take $u \in \mathcal{F}_{K_i} \cap C_0(K_i)$. We can take $v \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $v = 1$ on $\operatorname{supp} u$ by Proposition 6.3 (2). Then, $1_M \cdot v \in \tilde{\mathcal{F}}$, which implies that $(1_M \cdot v)|_{K_i} \in \mathcal{F}_{K_i}$. Therefore, $u \cdot (1_M \cdot v)|_{K_i} = u \cdot 1_{M \cap K_i}$ also belongs to $\mathcal{F}_{K_i}$. Since $\mathcal{F}_{K_i} \cap C_0(K_i)$ is dense in $\mathcal{F}_{K_i}$, we obtain that $M \cap K_i$ is an invariant set for $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$. By the irreducibility of $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$, either $\mu_i(M \cap K_i) = 0$ or $\mu_i(K_i \setminus M) = 0$ holds.

By this argument, there exists a subset A' of A such that $M = \bigcup_{i \in A'} K_i$ $\tilde{\mu}$-a.e. Assume that M is a nontrivial invariance set. Then, $A' \neq \emptyset$, $A' \neq A$, and there exist $i \in A'$ and $j \in A \setminus A'$ such that $K_i \sim K_j$ by Assumption C (2). Take a compact set $H \subset \Gamma_{ij}$ such that $\nu_{ij}(H) > 0$, and a relatively compact open set H' including H. Take $v \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $v = 1$ on H' and let $u = 1_M \cdot v \in \tilde{\mathcal{F}}$. Denote by $\tilde{u}$ the quasi-continuous modification of u w.r.t. $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. Then, $\tilde{u}|_{K_l}$ is also quasi-continuous w.r.t. $(\mathcal{E}_{K_l}, \mathcal{F}_{K_l})$ for $l = i, j$. Since $\tilde{u} = 1$ μ-a.e. on $H' \cap K_i$, we have $\tilde{u} = 1$ $\mathcal{E}_{K_i}$-q.e. on $H \subset H' \cap K_i$. By Assumption C (3), $\tilde{u} = 1$ ν_{ij}-a.e. on H. On the other hand, since $\tilde{u} = 0$ μ-a.e. on $H' \cap K_j$, we have $\tilde{u} = 0$ $\mathcal{E}_{K_j}$-q.e. on H. Therefore, $\tilde{u} = 0$ ν_{ij}-a.e. on H. This is a contradiction, which deduces that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is irreducible. □

The fractal field in Figure 2 satisfies Assumptions A, B and C, so there is a penetrating diffusion on the field.

In [12], detailed properties of $\tilde{X}_t$ such as heat kernel bounds and large deviation estimates are established under strong assumptions such as Assumption 2.2 in [12]. Using the results given in this section, one can relax the assumption and obtain the same results by the same proof given in [12], when each Dirichlet form is the resistance form in the sense of [18].

ACKNOWLEDGEMENTS. The authors thank Professor R. S. Strichartz for fruitful discussions and valuable comments.

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