BRUNNIAN LINKS, CLASPERS AND GOUSSAROV-VASSILIEV FINITE TYPE INVARIANTS

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Dedicated to the memory of Mikhail Goussarov

Abstract. We will prove that if $n \geq 1$, then an $(n+1)$-component Brunnian link L in a connected, oriented 3-manifold is C_n-equivalent to an unlink. We will also prove that if $n \geq 2$, then L can not be distinguished from an unlink by any Goussarov-Vassiliev finite type invariant of degree $< 2n$.

1. Introduction

Goussarov [5, 6] and Vassiliev [19] independently introduced the notion of finite type invariants of knots, which provides a beautiful, unifying view over the quantum link invariants [2, 3, 12, 1]. For each oriented, connected 3-manifold M, there is a filtration

$$\mathbb{Z}\mathcal{L} = J_0 \supset J_1 \supset \dots$$

of the free abelian group $\mathbb{Z}\mathcal{L}$ generated by the set $\mathcal{L} = \mathcal{L}(M)$ of ambient isotopy classes of oriented, ordered links in M, where for $n \geq 0$, the subgroup J_n is generated by all the n-fold alternating sums of links defined by 'singular links' with n double points. An abelian-group-valued link invariant is said to be of degree$\leq n$ if it vanishes on J_{n+1}.

Goussarov [8, 9] and the author [10] independently introduced theories of surgery along embedded graphs in 3-manifolds, which are called *Y-graphs* or *variation axes* by Goussarov, and *claspers* by the author. For links, one has the notion of *n-variation equivalence* (simply called *n-equivalence* in [9]) or *C_n-equivalence*, which is generated by *n-variation* [9] or *C_n-moves* [10], respectively. As proved by Goussarov [9, Theorem 9.3], for string links and knots in S^3, the n-variation (or C_n-) equivalence is the same as the Goussarov-Ohyama n-equivalence [6, 15]. The C_n-equivalence is generated by the local move depicted in Figure 1, i.e., band-summing Milnor's link of $(n+1)$-components [13, Figure 7], see Figure 2.

One of the main achievements of these theories is the following characterization of the topological information carried by Goussarov-Vassiliev finite type invariants.

Theorem 1 ([9, 10]). *Two knots K and K' in S^3 are n-variation (or C_n-)equivalent if and only if we have $K - K' \in J_n$ (i.e., K and K' are not distinguished by any Goussarov-Vassiliev invariants of degree$< n$.)*

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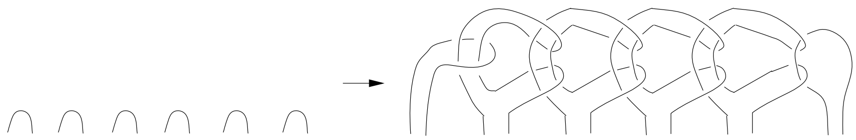

FIGURE 1. A special n-variation or a special C_n-move ($n=5$).

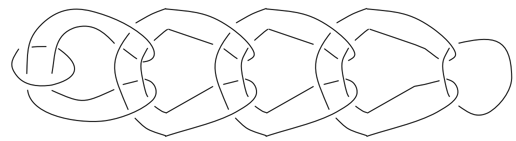

FIGURE 2. Milnor's link of 6-components.

The variant of Theorem 1, with n-variation equivalence replaced by Goussarov-Ohyama n-equivalence, is proved previously by Goussarov [7].

In [10, Proposition 7.4], we observed that for links in S^3 there is a certain difference between the notion of C_n-equivalence and the notion of the Goussarov-Vassiliev finite type invariants of degree $< n$, i.e., Theorem 1 does not extend to links in S^3. More specifically, we showed that if $n \geq 2$, then Milnor's link L_{n+1} of $(n+1)$-components is C_n-equivalent but *not* C_{n+1}-equivalent to the unlink U_{n+1}, but we have $L_{n+1} - U_{n+1} \in J_{2n}$. (For 2-component links, one can easily observe a similar facts for the Whitehead link W_2: W_2 is C_2- but not C_3-equivalent to the unlink U_2, but we have $W_2 - U_2 \in J_3, \notin J_4$.)

Note that Milnor's links are examples of *Brunnian links*. Here, a link L is *Brunnian* if any proper sublink of L is an unlink. The purpose of this paper is to prove the following results, which are generalizations of the above-mentioned facts about Milnor's links to Brunnian links.

Let M be a connected, oriented 3-manifold.

Theorem 2 (Announced in [10, Remark 7.5] for $M = S^3$)**.** *For $n \geq 1$, every $(n+1)$-component Brunnian link in M is C_n-equivalent to an unlink.*

Theorem 3 (Announced in [10, Remark 7.5] for $M = S^3$)**.** *Let $n \geq 2$, and let U denote $(n+1)$-component unlink in M. For every $(n+1)$-component Brunnian link L in M, we have $L - U \in J_{2n}$. (Consequently, L and U can not be distinguished by any Goussarov-Vassiliev invariant of degree $< 2n$ with values in any abelian group.)*

We remark that Theorem 2 follows from a stronger, but more technically stated, result (see Theorem 6 below), which is proved also by Miyazawa and Yasuhara [14] for $M = S^3$, independently to the present paper.

2. PRELIMINARIES

2.1. **Preliminaries.** In the rest of this paper, we will freely use the definitions, notations and conventions in [10].

Throughout the paper, let M denote a connected, oriented 3-manifold (possibly noncompact, possibly with boundary).

By a *tangle* γ in M, we mean a 'link' in the sense of [10, §1.1], i.e., a proper embedding $f\colon \alpha \to M$ of a compact, oriented 1-manifold α into M. We will systematically confuse γ and the image $\gamma(\alpha) \subset M$. It is often convenient to think of a tangle in M as a proper, compact, oriented 1-submanifold of M together with an ordering of the circle components. In this paper, by a *link*, we mean a tangle consisting only of circle components.

Two tangles γ and γ' in M are *equivalent*, denoted by $\gamma \cong \gamma'$, if γ and γ' are ambient isotopic fixing the endpoints.

2.2. **Claspers and tree claspers.** Here we recall some definition of claspers and tree claspers. See [10, §2, §3] for the details.

A *clasper* C for a tangle γ in a 3-manifold M is a (possibly unorientable) compact surface C in $\int M$ with some structure. G is decomposed into finitely many subsurfaces called *edges*, *leaves*, *disk-leaves*, *nodes* and *boxes*. We do not repeat here all the rules that should be satisfied by the subsurfaces. For the details, see [10, Definition 2.5]. We follow the drawing convention for claspers [10, Convention 2.6], in which we draw an edge as a line instead of a band.

Given a clasper C, there is defined a way to associate a framed link L_C, see [10, §2.2]. *Surgery along* C is defined to be surgery along L_C. A clasper C is called *tame* if surgery along C preserves the homeomorphism type of a regular neighborhood of C relative to the boundary. All the clasper which appear in the present paper are tame, and thus surgery along a clasper can be regarded as a move of tangle in a fixed 3-manifold. The result from a tangle γ of surgery along a clasper C is denoted by γ^C.

A *strict tree clasper* T is a simply-connected clasper T consisting only of disk-leaves, nodes and edges. The degree of T is defined to be the number of nodes plus 1, which is equal to the number of disk-leaves minus 1. For $n \geq 1$, a *C_n-tree* will mean a strict tree clasper of degree n. A *C_n-move* is surgery along a C_n-tree, which may be regarded as a local move of tangle since the regular neighborhood of T is a 3-ball. The *C_n-equivalence* of tangles is the equivalence relation generated by C_n-moves and equivalence of tangles.

A disk-leaf in a clasper is said to be *simple* if it intersects the tangle by one point. A strict tree clasper is *simple* if all its leaves are simple.

A *forest* F will mean 'strict forest clasper' in the sense of [10, Definition 3.2], i.e., a clasper consisting of finitely many disjoint strict tree claspers. F is said to be simple if all the components of F are simple. A *C_n-forest* is a forest consisting only of C_n-trees.

3. Brunnian links and C_n^a-moves

3.1. Definition of C_n^a-moves.

Definition 4. For $k \geq 1$, a *C_k^a-tree* for a tangle γ in M is a C_k-tree T for γ in M, such that

(1) for each disk-leaf A of T, all the strands intersecting A are contained in one component of γ, and
(2) each component of γ intersects at least one disk-leaf of T. (In other words, T intersects *all* the components of γ; this explains 'a' in 'C_k^a'.)

Note that such a tree exists only when $k \geq l-1$, where l is the number of components in γ. Note also that the condition (1) is vacuous if T is simple.

A C^a_k-*move* on a link is surgery along a C^a_k-tree. The C^a_k-*equivalence* is the equivalence relation on tangles generated by C^a_k-moves. A C^a_k-*forest* is a forest consisting only of C^a_k-trees.

What makes the notion of C^a_k-move useful in the study of Brunnian links is the following.

Proposition 5. *A C^a_k-move on a tangle preserves the types of the proper subtangles. In particular, if a link L' is C^a_k-equivalent to a Brunnian link L, then L' also is a Brunnian link.*

Proof. Let T be a C^a_k-tree for a tangle γ. For any proper subtangle γ', T viewed as a clasper for γ' has at least one disk-leaf which intersects no components of γ'. Hence, by [10, Proposition 3.4], we have $\gamma'_T \cong \gamma'$. □

Obviously, C^a_k-equivalence implies C_k-equivalence. But the converse does not hold in general, since a C_k-move can transform an unlink into a non-Brunnian link (e.g., a link with a knotted component).

The following result gives a characterization of Brunnian links in terms of clasper moves.

Theorem 2 follows from Theorem 6 below.

Theorem 6. *An $(n+1)$-component link L in M $(n \geq 1)$ is Brunnian if and only if L is C^a_n-equivalent to an n-component unlink U in M.*

As mentioned in the introduction, Theorem 6 is proved independently by Miyazawa and Yasuhara [14] for $M = S^3$.

The rest of this subsection is devoted to proving Theorem 6.

The following two lemmas easily follow from the proof of the corresponding results in [10].

Lemma 7 (C^a-version of [10, Theorem 3.17])**.** *For two tangles γ and γ' in M, and an integer $k \geq 1$, the following conditions are equivalent.*

(1) *γ and γ' are C^a_k-equivalent.*
(2) *There is a simple C^a_k-forest F for γ in M such that $\gamma^F \cong \gamma'$.*

Lemma 8 (C^a-version of [10, Proposition 4.5])**.** *Let γ be a tangle in M, and let γ_0 be a component of γ. Let T_1 and T_2 be C_k-trees for a tangle γ in M, differing from each other by a crossing change of an edge with the component γ_0. Suppose that T_1 and T_2 are C^a_k-trees for either γ or $\gamma \setminus \gamma_0$. Then γ^{T_1} and γ^{T_2} are related by one C^a_{k+1}-move.*

Now we prove Theorem 6.

Proof of Theorem 6. Let $L = L_0 \cup L_1 \cup \cdots \cup L_n$.

The 'if' part follows since a C^a_n-move for an $(n+1)$-component link L preserves each proper sublinks of L up to isotopy.

The proof of the 'only if' part is by induction on n.

Suppose $n = 1$. Since $L = L_0 \cup L_1$ is Brunnian, it follows that both L_0 and L_1 are unknotted in M. In M we can homotop L_1 into an unknot U_1, such that $L_0 \cup U_1$ is an unlink. This homotopy can be done by ambient isotopy and crossing changes between distinct components, i.e., (simple) C^a_1-moves. This shows the assertion.

Suppose $n > 1$. Since L is Brunnian in M, it follows that $L' = L \setminus L_0$ is an n-component Brunnian link in $M \setminus L_0$. By induction hypothesis, it follows that L'

is C^a_{n-1}-equivalent in $M \setminus L_0$ to an n-component unlink U' in $M \setminus L_0$. By Lemma 7, there is a C^a_{n-1}-forest F for U' in $M \setminus L_0$ satisfying $(U')^F \cong L'$ in $M \setminus L_0$. Since $L_0 \cup U'$ is an unlink, there is a disk D_0 in M disjoint from U'. We may assume that D_0 intersects F only by finitely many transverse intersections with the edges of F. By crossing changes between L_0 and edges of F intersecting D, we obtain from L_0 an unknot U_0 in M which bounds a disk disjoint from L' and F. By Lemma 8, it follows that these crossing changes do not change the C^a_n-equivalence class of the result of surgery. Hence we have

$$L = L_0 \cup L' \cong L_0 \cup (U')^F \cong (L_0 \cup U')^F \underset{C^a_n}{\sim} (U_0 \cup U')^F \cong U_0 \cup (U')^F \cong U_0 \cup L'.$$

Since $U_0 \cup L'$ is an unlink, the assertion follows. □

3.2. **Generalization to tangles.** One can generalize Theorem 6 to tangles as follows.

Let $c_0, \dots, c_n \subset \partial M$ be disjoint arcs, and set $c = c_0 \cup \cdots \cup c_n$. A $(n+1)$-component *tangle in* M *with arc basing* c is a tangle γ consisting of $n+1$ properly embedded arcs $\gamma_0, \dots, \gamma_n$ in M such that $\partial\gamma_i = \partial c_i$ for $i = 0, \dots, n$. A tangle γ with arc basing c is called *trivial* (with respect to c) if simple closed curves $\gamma_i \cup c_i$ for $i = 0, \dots, n$ bounds disjoint disks in M. A tangle γ with arc basing c is *Brunnian* if every proper subtangle of γ is trivial with respect to the corresponding 1-submanifold of c.

Theorem 9. *If $\gamma = \gamma_0 \cup \cdots \cup \gamma_n$ $(n \geq 1)$ is an $(n+1)$-component tangle in M with arc basing $c = c_1 \cup \cdots \cup c_n$. Then γ is Brunnian if and only if γ is C^a_n-equivalent to an $(n+1)$-component trivial tangle with respect to c.*

Proof. Similar to the proof of Theorem 6. □

Remark 10. The case $M = B^3$ of Theorem 9 is independently proved by Miyazawa and Yasuhara [14, Proposition 4.1].

Remark 11. Taniyama [18] (see also Stanford [17]) proved that an $(n+1)$-component Brunnian link is *n-trivial*, or *n-equivalent* to an unlink. Here, by 'n-triviality' and 'n-equivalence' we mean the notion introduced independently by Goussarov [6] and Ohyama [15] (see also [18, 9]). It is well known that C_n-equivalence implies n-equivalence, but the converse seems open for links with at least 2-components. However, Goussarov [9] proved that C_n-equivalence (or n-variation equivalence) and n-equivalence are the same for string links in $D^2 \times [0,1]$, and hence the case $M = B^3$ of Theorem 9 follows from the fact (which seems to be well known) that $(n+1)$-component Brunnian tangle of arcs in B^3 is n-trivial.

Using Theorems 6 and 9, we can prove the following fact, which means that *a Brunnian link in S^3 is the closure of a Brunnian tangle in B^3*. (It is clear that, conversely, the closure of a Brunnian tangle is Brunnian.)

Proposition 12. *Let $n \geq 2$. Given an n-component Brunnian link $L = L_1 \cup \cdots \cup L_n$ in S^3, there is an n-component Brunnian tangle $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$ in a 3-ball B^3 with respect to a basing $c = c_1 \cup \cdots \cup c_n \subset \partial B^3$ such that the union $\bigcup_{i=1}^n \gamma_i \cup c_i \subset B^3 \subset S^3$ viewed as a link in S^3 is equivalent to L.*

Proof. By Theorem 6 and Lemma 7, there is a simple C^a_{n-1}-forest F for an n-component unlink $U = U_1 \cup \cdots \cup U_n$ such that $U^F \cong L$. Let $D_1, \dots, D_n$ be disjoint discs in S^3 bounded by $U_1, \dots, U_n$, and set $D = D_1 \cup \dots \cup D_n$. Choose a point $p_0 \in S^3$ disjoint from $F \cup D$. For each $i = 1, \dots, n$, let $p_i \in U_i \setminus F$ and let g_i be a simple arc in $M \setminus F$ from p_0 to p_i such that $g_i \cap D = p_i$. Here we may assume that $g_i \cap g_j = p_0$ if $i \neq j$. Let N be a small regular neighborhood of $g_1 \cup \dots \cup g_n$, which is a 3-ball. Set $B^3 = \overline{S^3 \setminus N}$. For $i = 1, \dots, n$, set $c_i = \partial B^3 \cap D_i$, and set $\gamma_i^0 = U_i \cap B^3$. Then, by Theorem 9 the result of surgery $\gamma = (\gamma_1^0 \cup \dots \cup \gamma_n^0)^F$ is Brunnian with respect to $c_1 \cup \dots \cup c_n$, and satisfies the assertion. □

4. Brunnian links and the Goussarov-Vassiliev filtration

4.1. **Definition of the Goussarov-Vassiliev filtration.** Here we recall the definition of the Goussarov-Vassiliev filtration for links using strict tree claspers. For the details, see [10, §6].

Let $\mathcal{L}(M)$ denote the set of equivalence classes of tangles in M. For $n \geq 0$, define $J_n = J_n(M) \subset \mathbb{Z}\mathcal{L}(M)$ as follows.

By a *forest scheme* for a tangle γ in M, we mean a 'strict forest scheme' in the sense of [10, Definition 6.6], i.e., a set $S = \{T_1, \dots, T_p\}$ of disjoint, strict tree claspers $T_1, \dots, T_p$ for a tangle γ in M. The *degree* of S is defined to be the sum of the degrees of $T_1, \dots, T_p$. Set

$$[\gamma, S] = [\gamma; T_1, \dots, T_p] = \sum_{S' \subset S} (-1)^{p-|S'|} \gamma^{\bigcup S'} \in \mathbb{Z}\mathcal{L}(M),$$

where the sum is over all subsets S' of S, $|S'|$ denotes the cardinality of S', and $\bigcup S'$ denote the clasper consisting of the elements of S'.

For $n \geq 0$, let $J_n = J_n(M)$ denote the $\mathbb{Z}$-submodule of $\mathbb{Z}\mathcal{L}(M)$ spanned by the elements $[\gamma, S]$ for any pair (γ, S) of a link γ in M and a forest scheme S for γ in M of degree n. This defines a descending filtration of $\mathbb{Z}\mathcal{L}(M)$:

$$\mathbb{Z}\mathcal{L}(M) = J_0(M) \supset J_1(M) \supset \dots,$$

which is the same as the Goussarov-Vassiliev filtration in the usual sense, defined using singular tangles.

4.2. **Proof of Theorem 3.** We need some lemmas before proving Theorem 3.

Lemma 13 (A variant of [10, Lemma 3.20])**.** *Let γ be a tangle in M, and let T be a strict tree clasper for γ in M. Let N be a small regular neighborhood of T in M. Then the pair $(N, (\gamma \cap N)^T)$ is homeomorphic to $(D^2, (p \text{ points})) \times [0, 1]$, where p is the number of points in $T \cap \gamma$.*

Proof. The case where T is simple is a part of [10, Lemma 3.20]. The general case immediately follows from this case. □

Lemma 14. *Let $1 \leq n \leq r$, and let L be an $(n+1)$-component Brunnian link in M. Then there is a forest F for an $(n+1)$-component unlink U in M satisfying the following properties.*

1. *F consists of C_l^a-trees with $n \leq l < r$.*
2. *U bounds $n+1$ disjoint disks $D_1 \cup \dots \cup D_{n+1}$ in M which are disjoint from edges and trivalent vertices of F.*
3. *L is C_r^a-equivalent to U^F.*

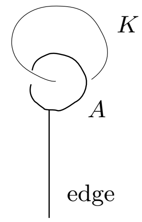

Figure 3. A monopoly.

Proof. The proof is by induction on r. The case $r = n$ follows immediately from Theorem 6 by setting $F = \emptyset$. Suppose that the result is true for $r \geq n$ and let us verify the case for $r + 1$. Let F be as in the statement of the lemma. Let N be a small regular neighborhood of F in M. Then U^F is obtained from U by replacing the part $U \cap N$ by $(U \cap N)^F$. Since L is C^a_r-equivalent to U^F, it follows from Lemma 7 that there is a C^a_r-forest F' for U^F such that

$$(U^F)^{F'} \cong L. \tag{4.1}$$

Using Lemma 13, we may assume that F' is disjoint from N, and thus can be regarded as a forest for U disjoint from F. Hence we have

$$(U^F)^{F'} \cong U^{F \cup F'}. \tag{4.2}$$

Now F' may intersects $D = D_1 \cup \cdots \cup D_{n+1}$. We may assume that F' intersects D only by disk-leaves and finitely many transverse intersection of D and edges of F'. By Lemma 8, without changing the result of surgery up to C^a_{r+1}-equivalence, we can remove the intersection of D and the edges of F' by crossing changes between components of U and edges of F' intersecting D. Let F'' denote the forest obtained from this operation. Now D is disjoint from the edges and trivalent vertices of F'', and $U^{F \cup F''}$ and $U^{F \cup F'}$ are C^a_{r+1}-equivalent. From this, (4.1) and (4.2), it follows that $F \cup F''$ is a forest with the desired properties. □

Definition 15. Let C be a clasper for a tangle γ in M. We say that a simple disk-leaf A of C *monopolizes* a circle component K of γ in (C, γ) if there is a 3-ball $B \subset M$ such that $(\gamma \cup C) \cap B$ looks as depicted in Figure 3. We call the pair (A, K) a *monopoly* in (C, γ). The monopolized component K bounds a disk D in $\int M$ which intersect C by an arc $A \cap D$. We call D a *monopoly disk* for K.

Lemma 16 (Monopoly Lemma). *Suppose $l \geq 1$ and $0 \leq k \leq l + 1$ be integers. Let T be a C_l-tree for a tangle γ in M with k distinct monopolies in (T, γ). Then we have*

$$\gamma^T - \gamma \in J_{d(l,k)}(M), \tag{4.3}$$

where

$$d(l,k) = \begin{cases} 1 & \text{if } l = 1,\ 0 \leq k \leq 2, \\ l + k & \text{if } l \geq 2,\ 0 \leq k \leq l, \\ l + k - 1 & \text{if } l \geq 2,\ k = l + 1. \end{cases}$$

Proof. The case $l = 1$ is trivial. Also, the case $k = l + 1$ and $l \geq 2$ follows from the case $k = l \geq 2$ by ignoring one monopoly. Hence it suffices to prove the case $l \geq 2$, $0 \leq k \leq l$. Note that if $(l, k) = (1, 0)$, then we have $d = l + k$. We will prove

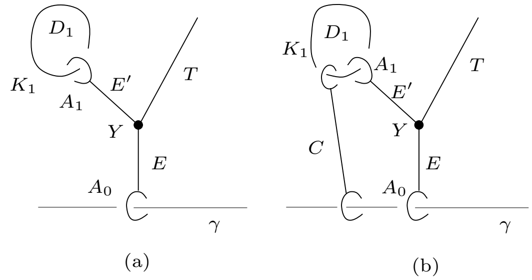

FIGURE 4. Here the lines labeled γ depicts a parallel family of strands of γ.

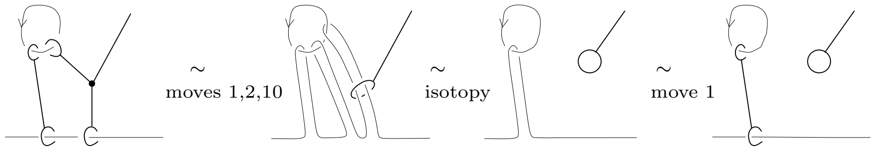

FIGURE 5. Here we use moves 1, 2, 10 of [10, Proposition 2.7]. The orientations given to the circle components are 'temporary' and may possibly be the opposite to the actual orientations simultaneously for all the four figures.

by induction on $l+k$ that the assertion is true if either $(l,k)=(1,0)$ or $l\geq 2$ and $0\leq k\leq l$.

As we have seen, the case $(l,k)=(1,0)$ is trivial. Assume $l+k\geq 2$. Let $(A_1,K_1),\dots,(A_k,K_k)$ be the k monopolies in (T,γ) with monopoly disks $D_1,\dots,D_k$, respectively. Since $k\leq l$, we can choose one disk-leaf A_0 of T distinct from $A_1,\dots,A_k$. Since $l\geq 2$, A_0 is adjacent to a node Y. Let E denote the edge between A_0 and Y. Let P' and P'' be the two components of $T\setminus(Y\cup E\cup A_0)$, which are two subtrees in T.

Let $l',l''\geq 1$ denote the number of disk-leaves in P' and P'', respectively. Let $k'\leq l'$ and $k''\leq l''$ denote the numbers of the monopolizing disk-leaves from $A_1,\dots,A_k$ contained in P', and P'', respectively. We have $l'+l''=l$ and $k'+k''=k$.

The proof is divided into two cases.

Case 1. Either (l',k') or (l'',k'') is $(1,1)$. We assume that $(l',k')=(1,1)$; the other case is proved by the same argument. Then P' consists of a monopolizing disk-leaf A_i, $i\in\{1,\dots,k\}$, and the incident edge E'. Without loss of generality, we may assume that $i=1$. See Figure 4 (a). Let C be a C_1-tree for γ disjoint from T, as depicted in Figure 4 (b). Figure 5 and [10, Proposition 3.4] imply that $\gamma^{T\cup C}\cong\gamma^C$. (This fact is implicit in the proof of [10, Proposition 7.4].) Hence we have

$$\gamma^T-\gamma=-(\gamma^{T\cup C}-\gamma^T-\gamma^C+\gamma)=-[\gamma;T,C]. \tag{4.4}$$

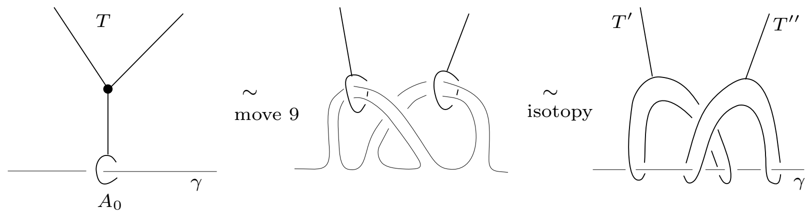

FIGURE 6

Let N be a small regular neighborhood of $T \cup D_2 \cup D_3 \cup \cdots \cup D_k$, which is a 3-ball. Then, by the induction hypothesis, we have

$$(\gamma \cap N)^T - \gamma \cap N \in J_{l+k-1}(N). \tag{4.5}$$

Since C is a C_1-tree, (4.4) and (4.5) implies (4.3).

Case 2. Otherwise. Apply move 9 in [10, Proposition 2.9] at the disk-leaf A_0, see Figure 6. The result is a union $T' \cup T''$ of a $C_{l'}$-tree T' and a $C_{l''}$-tree T'' for γ such that $\gamma^T \cong \gamma^{T' \cup T''}$. Let N' be a small regular neighborhood of the union of T' and the monopoly disks intersecting T'. Similarly, let N'' be a small regular neighborhood of the union of T'' and the monopoly disks intersecting T''. Since $(l', k'), (l'', k'') \neq (1, 1)$ and $l' + k', l'' + k'' < l + k$, it follows by induction hypothesis that we have

$$(\gamma \cap N')^{T'} - \gamma \cap N' \in J_{l'+k'}(N'),$$
$$(\gamma \cap N'')^{T''} - \gamma \cap N'' \in J_{l''+k''}(N'').$$

Using [10, Proposition 3.4], we see that $\gamma^{T'} \cong \gamma^{T''} \cong \gamma$. Hence it follows that

$$\gamma^T - \gamma = \gamma^{T' \cup T''} - \gamma^{T'} - \gamma^{T''} + \gamma \in J_{l+k}(M)$$

□

Remark 17. In [4, Lemma 7.1], a result similar to Lemma 16 is proved, but it is not strong enough for our purpose.

Now we prove Theorem 3.

Proof of Theorem 3. By Theorem 6 and Lemma 14 for $r = 2n$, there is a forest F for U in M consisting of simple C^a_l-trees with $n \leq l < 2n$ such that

(a) U bounds $n + 1$ disjoint disks $D_1, \ldots, D_{n+1}$ in M, disjoint from edges and trivalent vertices of F, and
(b) L is C^a_{2n}-equivalent to U^F.

By the condition (b), we have

$$L - U^F \in J_{2n}. \tag{4.6}$$

Let $S = \{T_1, \ldots, T_p\}$, $p \geq 0$, be a forest scheme for U in M consisting of the tree claspers $T_1, \ldots, T_p$ contained in F. By an easy calculation, we have

$$U^F = \sum_{S' \subset S} [U, S'], \tag{4.7}$$

where S' runs over all subsets of S. Since $\deg T_i \geq n$ for all i, we have $\deg S' \geq n|S'|$, where $|S'|$ denotes the number of elements in S'. Since $|S'| \geq 2$ implies $[U, S'] \in J_{2n}$, it follows from (4.7) that

$$U^F - U \equiv \sum_{i=1}^{p} [U; T_i] \pmod{J_{2n}}. \tag{4.8}$$

Hence, by (4.6) and (4.8), it suffices to prove the case $F = T$ is a C_l^a-tree with $n \leq l < 2n$. By assumption, there are at least $k = 2n + 1 - l$ monopolies in (T, U). Hence by Lemma 16, we have $U^T - U \in J_{d(l,k)}$, where $d(l, k)$ is defined in Lemma 16. Since $l \geq n \geq 2$, we have $d(l, k) \geq l + k - 1 \geq 2n$. Hence we have $U^T - U \in J_{2n}$. This completes the proof. □

4.3. **Remarks.**

Remark 18. Przytycki and Taniyama [16] proved a conjecture by Kanenobu and Miyazawa [11] about the homfly polynomial of Brunnian links, and also announced a similar result for the Kauffman polynomial. These results follow from Theorem 3.

Remark 19. Yasuhara pointed out to the author that Theorem 3 implies the following generalization.

Let $n \geq 2$, $m \geq 1$, and let M be a connected, oriented 3-manifold. Let L and L' be two $(n+1)$-component links in M such that

(1) *both L and L' are C_m-equivalent to an $(n+1)$-component unlink U,*
(2) *L and L' are C_n^a-equivalent to each other.*

Then we have $L' - L \in J_l$, where $l = \min(2n, n + m)$.

The proof is as follows. We may assume that $L = U^F$, where F is a C_m-forest for U. We may assume also that $L' = U^{F \cup F'}$, where F' is a C_n^a-forest for U, disjoint from F. Then we have

$$L' - L = U^{F \cup F'} - U^F = (U^{F \cup F'} - U^F - U^{F'} + U) + (U^{F'} - U).$$

Here we have $U^{F \cup F'} - U^F - U^{F'} + U \in J_{n+m}$. We also have $U^{F'} - U \in J_{2n}$ by Theorem 3. Hence the assertion.

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