FINITE TYPE INVARIANTS AND MILNOR INVARIANTS FOR BRUNNIAN LINKS

KAZUO HABIRO AND JEAN-BAPTISTE MEILHAN

Abstract. A link L in the 3-sphere is called *Brunnian* if every proper sublink of L is trivial. In a previous paper, the first author proved that the restriction to Brunnian links of any Goussarov-Vassiliev finite type invariant of $(n+1)$-component links of degree $< 2n$ is trivial. The purpose of this paper is to study the first nontrivial case. We will show that the restriction of an invariant of degree $2n$ to $(n+1)$-component Brunnian links can be expressed as a quadratic form on the Milnor link-homotopy invariants of length $n+1$.

1. Introduction

The notion of *Goussarov-Vassiliev finite type link invariants* [8, 9, 34] enables us to understand the various quantum invariants from a unifying viewpoint, see e.g. [1, 32]. The theory involves a descending filtration

$$\mathbb{Z}\mathcal{L}(m) = J_0(m) \supset J_1(m) \supset \dots$$

of the free abelian group $\mathbb{Z}\mathcal{L}(m)$ generated by the set $\mathcal{L}(m)$ of the ambient isotopy classes of m-component, oriented, ordered links in S^3. Here each $J_n(m)$ is generated by alternating sums of links over n independent crossing changes. A homomorphism from $\mathbb{Z}\mathcal{L}(m)$ to an abelian group A is said to be a Goussarov-Vassiliev invariant of degree n if it vanishes on $J_{n+1}(m)$. Thus, for $L, L' \in \mathcal{L}(m)$, we have $L - L' \in J_{n+1}(m)$ if and only if L and L' have the same values of Goussarov-Vassiliev invariants of degree $\leq n$ with values in any abelian group. A fundamental result in the theory of Goussarov-Vassiliev invariants [20, 1] is that the graded quotient $\bar{J}_n(m) = J_n(m)/J_{n+1}(m)$ is isomorphic, after being tensored by $\mathbb{Q}$, to the space of certain *unitrivalent diagrams* (also called *Feynman diagrams* or *Jacobi diagrams*) 'of degree n'. This gives a complete classification of rational-valued Goussarov-Vassiliev invariants via the Kontsevich integral [20].

It is natural to ask what kind of informations a Goussarov-Vassiliev link invariants can contain and what is the topological meaning of the unitrivalent diagrams. *Calculus of claspers*, introduced by Goussarov and the first author [10, 11, 17], answers these questions. (We will recall the definition of claspers in Section 2.) A special type of claspers, called *graph claspers*, can be regarded as topological realizations of unitrivalent diagrams. For *knots*, claspers enables us to give a complete topological characterization of the informations that can be contained by Goussarov-Vassiliev invariants of degree $< n$ [11, 17]: The difference of two knots is

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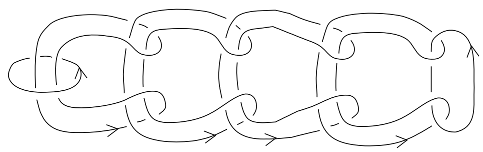

FIGURE 1.1. Milnor's link L_6 of 6 components

in J_n if and only if these two knots are C_n-*equivalent*. Here C_n-equivalence is generated by a certain type of local moves, called C_n-*moves* (called $(n-1)$-*variations* by Goussarov), which is defined as surgeries along certain tree claspers.

For links with more than 1 components, the above-mentioned properties of Goussarov-Vassiliev invariants does not hold. It is true that if $L, L' \in \mathcal{L}(m)$ are C_n-equivalent, then we have $L - L' \in J_n(m)$, but the converse does not hold in general. A counterexample is Milnor's link L_{n+1} of $n+1$ components depicted in Figure 1.1: If $n \geq 2$, L_n is (C_n-equivalent but) not C_{n+1}-equivalent to the $(n+1)$-component unlink U, while we have $L_{n+1} - U \in J_{2n}(n+1)$ (but $L_{n+1} - U \notin J_{2n+1}(n+1)$), see [17, Proposition 7.4]. (This fact is contrasting to the case of *string links*: Conjecturally [17, Conjecture 6.13], two string links L, L' of the same number of components are C_n-equivalent if and only if $L - L' \in J_n$.)

Milnor's links are typical examples of *Brunnian links*. Recall that a link in an oriented, connected 3-manifold is said to be Brunnian if every proper sublink of it is an unlink. In some sense, an n-component Brunnian link is a 'pure n-component linking'. Thus studying the behavior of Goussarov-Vassiliev invariants on Brunnian links would be a first step in understanding the Goussarov-Vassiliev invariants for links.

The first author generalized a part of the above-mentioned properties of Milnor's links to Brunnian links:

Theorem 1.1 ([19])**.** *Let L be an $(n+1)$-component Brunnian link in a connected, oriented 3-manifold M $(n \geq 1)$, and let U be an $(n+1)$-component unlink in M. Then we have the following.*

(1) *L and U are C_n-equivalent.*
(2) *If $n \geq 2$, then we have $L - U \in J_{2n}(n+1)$. Hence L and U are not distinguished by any Goussarov-Vassiliev invariants of degree $< 2n$.*

The case $M = S^3$ of Theorem 1.1 was announced in [17], and was later proved also by Miyazawa and Yasuhara [29], independently to [19].

The purpose of the present paper is to study the restrictions of Goussarov-Vassiliev invariants of degree $2n$ to $(n+1)$-component Brunnian links in S^3, which is the first nontrivial case according to Theorem 1.1. The main result in the present paper expresses any such restriction as a *quadratic* form of Milnor link-homotopy invariants of length $n+1$:

Theorem 1.2. *Let f be any $\mathbb{Z}$-valued Goussarov-Vassiliev link invariant of degree $2n$. Then there are (non-unique) integers $f_{\sigma,\sigma'}$ for elements σ, σ' of the symmetric group S_{n-1} on the set $\{1, \ldots, n-1\}$ such that, for any $(n+1)$-component Brunnian link L, we have*

$$f(L) - f(U) = \sum_{\sigma,\sigma' \in S_{n-1}} f_{\sigma,\sigma'} \bar{\mu}_\sigma(L) \bar{\mu}_{\sigma'}(L). \tag{1.1}$$

Here, U is an $(n+1)$-component unlink, and we set

$$\bar{\mu}_\sigma(L) = \bar{\mu}_{\sigma(1),\sigma(2),\ldots,\sigma(n-1),n,n+1}(L) \in \mathbb{Z}$$

for $\sigma \in S_{n-1}$.

We give two proofs of Theorem 1.2, one involving a rather heavy use of claspers, and the other involving the Kontsevich integral.

Recall that Milnor invariants of length $n+1$ for string links are Goussarov-Vassiliev invariants of degree $\leq n$ [2, 24] (see also [15]). As is well-known, Milnor's invariants is not well-defined for all links, and hence it does not make sense to ask whether Milnor invariants of length $n+1$ is of degree $\leq n$ or not. However, as Theorem 1.2 indicates, a quadratic expression in such Milnor invariants, which is well-defined at least for $(n+1)$-component Brunnian links, may extend to a link invariant of degree $\leq 2n$. In fact, Theorem 8.3, in theory, gives a necessary and sufficient condition for a quadratic form of Milnor invariants to be extendable to a link invariant of degree $\leq 2n$.

In the study of Milnor's invariants, tree claspers seem at least as useful as Cochran's construction [3]. For the use of claspers in the study of the Milnor invariants, see also [6, 12, 27]. For other relationships between finite type invariants and the Milnor invariants, see [2, 24, 15, 14, 25].

We organize the rest of the paper as follows.

In Section 2, we recall some definitions from clasper calculus.

In Section 3, we provide a description of the graded quotients of the Goussarov-Vassiliev filtration for links. We establish a surjective homomorphism

$$\xi_n\colon \mathcal{A}_n(m) \to \bar{J}_n(m)$$

from an abelian group $\mathcal{A}_n(m)$ of unitrivalent diagrams of degree n on m circles to $\bar{J}_n(m)$, which is announced in [17, Section 8.2].

In Section 4, we recall the notion of C_k^a-equivalence for links, studied in [19]. If a link L is C_k^a-equivalent (for any k) to a Brunnian link, then L also is a Brunnian link.

In Section 5, we study the group $\overline{BSL}_{n+1}$ of C_{n+1}^a-equivalence classes of $(n+1)$-component string links. We establish an isomorphism

$$\theta_n\colon \mathcal{T}_{n+1} \xrightarrow{\simeq} \overline{BSL}_{n+1}$$

from an abelian group $\mathcal{T}_{n+1}$ of certain tree diagrams. This map is essentially the inverse to the Milnor link-homotopy invariants of length $n+1$.

In Section 6, we apply the results in Section 5 to Brunnian links. The operation of closing string links induces a bijection

$$\bar{c}_{n+1}\colon \overline{BSL}_{n+1} \xrightarrow{\simeq} \overline{B}_{n+1},$$

where $\overline{B}_{n+1}$ is the set of C_{n+1}^a-equivalence classes of $(n+1)$-component Brunnian links. As a byproduct, we obtain another proof of a result of Miyazawa and Yasuhara [29].

In Section 7, we study the behavior of Goussarov-Vassiliev invariants of degree $2n$ for $(n+1)$-component Brunnian links. We first show that two C_{n+1}^a-equivalent, $(n+1)$-component Brunnian links cannot be distinguished by Goussarov-Vassiliev invariants of degree $2n$. We have a quadratic map

$$\kappa_{n+1} : \overline{B}_{n+1} \longrightarrow \bar{J}_{2n}(n+1)$$

defined by $\kappa_{n+1}([L]_{C_{n+1}^a}) = [L-U]_{J_{2n+1}}$. We prove Theorem 1.2, using κ_{n+1}.

In Section 8, we study the *Brunnian part* $\mathrm{Br}(\bar{J}_{2n}(n+1))$ of $\bar{J}_{2n}(n+1)$, which is defined as the subgroup generated by the elements $[L-U]_{J_{2n+1}}$, where L is Brunnian. We construct a homomorphism

$$h_n\colon \mathcal{A}_{n-1}^c(\emptyset) \to \bar{J}_{2n}(n+1),$$

where $\mathcal{A}_{n-1}^c(\emptyset)$ is a $\mathbb{Z}$-module of connected trivalent diagrams with $2n-2$ vertices. We show that h_n is surjective for $n \geq 3$, and is an isomorphism over $\mathbb{Q}$ for $n \geq 2$.

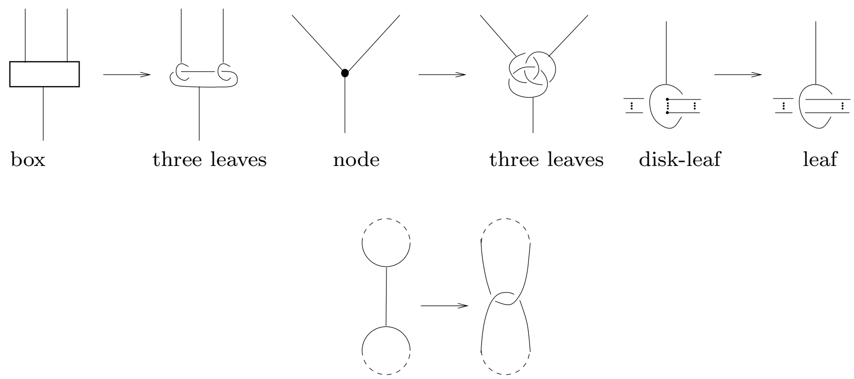

FIGURE 2.1. How to obtain the associated framed link L_G from G. First one replaces boxes, nodes and disk-leaves by leaves. Then replace each 'I-shaped' clasper by a 2-component framed link as depicted.

In Section 9, we give an alternative proof of Theorem 1.2 using the Kontsevich integral.

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2. Claspers

In this section, we recall some definitions from calculus of claspers. For the details, we refer the reader to [17].

A *clasper* in an oriented 3-manifold M is a compact, possibly unorientable, embedded surface G in $\operatorname{int} M$ equipped with a decomposition into connected subsurfaces called *leaves*, *disk-leaves*, *nodes*, *boxes*, and *edges*. Two distinct non-edge subsurfaces are disjoint. Edges are disjoint bands which connect two subsurfaces of the other types. A connected component of the intersection of one edge E and another subsurface F (of different type), which is an arc in $\partial E \cap \partial F$, is called an *attaching region* of F.

- A *leaf* is an annulus with one attaching region.
- A *disk-leaf* is a disk with one attaching region.
- A *node* is a disk with three attaching regions. (Usually, a node is incident to three edges, but it is allowed that the two ends of one edge are attached to a node.)
- A *box* is a disk with three attaching regions. (The same remark as that for node applies here, too.) Moreover, one attaching region is distinguished with the other two. (This distinction is done by drawing a box as a rectangle, see [17].)

A clasper G *for* a link L in M is a clasper in M such that the intersection $G \cap L$ consists of finitely many transverse double points and is contained in the interior of the union of disk-leaves.

We often use the drawing convention for claspers as described in [17].

Surgery along a clasper G is defined to be surgery along the *associated framed link* L_G to G. Here L_G is obtained from G by the rules described in Figure 2.1.

A *graph clasper* is a clasper without boxes. A graph clasper G is called *strict* if each component of G has no leaves and at least one disk-leaf. Surgery along a strict graph clasper G is *tame* in the sense of [17, Section 2.3], i.e., the result of surgery along G preserves the 3-manifold and the surgery may be regarded as a move on a link.

A *tree clasper* is a connected graph clasper T such that the union of edges and nodes of T is simply connected.

A graph clasper G for a link L is *simple* (with respect to L) if each disk-leaf of G has exactly one intersection point with L.

The *degree* (or *C-degree*) of a connected, strict graph clasper G is defined to be the number of nodes of T plus 1.

For $n \geq 1$, a *C_n-tree* (resp. *C_n-graph*) is a strict tree (resp. connected graph) clasper of degree n. A *(simple) C_n-move* is a local move on links defined as surgery along a (simple) C_n-tree. For example, a simple C_1-move is a crossing change, and a simple C_2-move is a delta move [26, 30] The *C_n-equivalence* is the equivalence relation on links generated by C_n-moves. This equivalence relation is also generated by simple C_n-moves, and also by surgeries along (simple) C_n-graphs, see [17, 18]. The C_n-equivalence becomes finer as n increases.

3. Graph claspers as topological realizations of unitrivalent diagrams

In this section, we recall the formulation of the Goussarov-Vassiliev filtrations for links using claspers. For each $k \geq 0$, we will establish a *surgery map* ξ_k from a $\mathbb{Z}$-module of 'unitrivalent diagrams of degree k' to the kth graded quotient of the Goussarov-Vassiliev filtration, see Theorem 3.4 below. This surgery map has been announced by the first author [17, Section 8.2].

3.1. **Goussarov-Vassiliev filtration.** We briefly recall the standard definition of the Goussarov-Vassiliev filtration for links, involving 'singular links'. See e.g. [1] for the details.

A *singular link* L in a 3-manifold M is an immersed 1-manifold in M whose only singularities are finitely many transverse double points.

Let $\mathcal{L}(M,n)$ denote the set of the equivalence classes of n-component links in M. Let $\mathbb{Z}\mathcal{L}(M,n)$ denote the free $\mathbb{Z}$-module generated by $\mathcal{L}(M,n)$.

For $n,k \geq 0$, let $J_k^{GV}(n)$ denote the $\mathbb{Z}$-submodule of $\mathbb{Z}\mathcal{L}(M,n)$ generated by the k-fold alternating sums determined by n-component singular links, each with k double points. We have a descending filtration

$$\mathbb{Z}\mathcal{L}(M,n) = J_0^{GV}(n) \supset J_1^{GV}(n) \supset J_2^{GV}(n) \supset \cdots ,$$

which is called the *Goussarov-Vassiliev filtration*.

3.2. **Graph schemes and Goussarov-Vassiliev filtration.** Here we give another definition of the Goussarov-Vassiliev filtration for links, which involves *graph schemes*. This definition is a slightly modified version of the one given in [17].

Definition 3.1. A *graph scheme* of degree k for a link L in a 3-manifold M will mean a collection $S = \{G_1, \ldots, G_l\}$ of disjoint connected (strict) graph claspers $G_1, \ldots, G_l$ for L such that $\sum_{i=1}^k \deg G_i = k$. A graph scheme S is said to be *simple* if every element of S is simple.

For a graph scheme $S = \{G_1, \ldots, G_l\}$ for a link L in M, we set

$$[L, S] = [L; G_1, \ldots, G_l] = \sum_{S' \subset S} (-1)^{|S'|} L_{\bigcup S'} \in \mathbb{Z}\mathcal{L}(M,n),$$

where the sum runs over all subsets S' of S, and $|S'|$ denote the number of elements of S'.

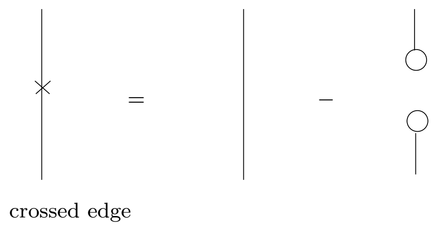


FIGURE 3.1. The crossed edge notation

For $k \geq 0$, let $J_k(M,n)$ (resp. $J^s_k(M,n)$) denote the $\mathbb{Z}$-submodule of $\mathbb{Z}\mathcal{L}(M,n)$ generated by the elements of the form $[L,S]$, where $L \in \mathcal{L}(M,n)$ and S is a graph scheme (resp. a simple graph scheme) for L of degree k.

Proposition 3.2. *We have $J_k(M,n) = J^s_k(M,n) = J^{GV}_k(M,n)$. Thus the $J_k(M,n)$ is the same as the Goussarov-Vassiliev filtration for n-component links.*

Proof. For $k \geq 0$, let $J^f_k(M,n)$ denote $\mathbb{Z}$-submodule of $\mathbb{Z}\mathcal{L}(M,n)$ generated by the elements of the form $[L;S]$, where $L \in \mathcal{L}(M,n)$ and S is a *simple forest scheme* for L of degree k. Here a 'forest scheme' is a graph scheme consisting only of (strict) tree claspers. It is known [17] that $J^f_k(M,n) = J^{GV}_k(M,n)$.

It is obvious that $J^f_k(M,n) \subset J^s_k(M,n) \subset J_k(M,n)$. Hence it suffices to prove that

$$J_k(M,n) \subset J^f_k(M,n) \tag{3.1}$$

Let $S = \{G_1,\ldots,G_l\}$ be a graph scheme for $L \in \mathcal{L}(M,n)$. By [18] each G_i can be replaced by a disjoint union of simple $C_{\deg G_i}$-trees $T_{i,1},\ldots,T_{i,m_i}$ ($m_i \geq 0$) in a small regular neighborhood of G_i. Hence we have

$$\begin{aligned}[L,S] &= [L;T_{i,1}\cup\cdots\cup T_{1,m_1},\ldots,T_{l,1}\cup\cdots\cup T_{l,m_l}]\\ &= \sum_{j_1=1}^{m_1}\cdots\sum_{j_l=1}^{m_l}[L_{\bigcup_{i=1}^{l}\bigcup_{j=1}^{j_i-1}T_{i,j}};T_{j_1},\ldots,T_{j_l}] \in J^f_k(M,n).\end{aligned}$$

This implies (3.1). □

A similar statement for forest schemes has been proved in [17, Section 6].

3.3. **Crossed edge notation.** It is useful to introduce a notation for depicting certain linear combinations of surgery along claspers, which we call *crossed edge notation*.

Let G be a clasper for a link L in a 3-manifold M. Let E be an edge of G. By putting a cross on the edge E in a figure, we mean the difference $L_G - L_{G_0}$, where G_0 is obtained from G by inserting two trivial leaves into E. See Figure 3.1. If we put several crosses on the edges of G, then we understand it in a multilinear way. I.e., a clasper with several crosses is an alternating sum of the result of surgery along claspers obtained from G by inserting pairs of trivial, unlinked leaves into the crossed edges. We will freely use the identities depicted in Figure 3.2, which can be easily verified. The second identity implies that if G' is a connected graph clasper contained in G and there are several crosses on G', then one can safely replace these crosses by just one cross on one edge in G'. This properties can be generalized to the case where G' is a connected subsurface of G consisting only of nodes, edges, leaves and disk-leaves. Note also that if $S = \{G_1,\ldots,G_l\}$, is a graph scheme for L, then $[L,S]$ can be expressed by the clasper $G_1\cup\cdots\cup G_l$ with one cross on each component G_i.

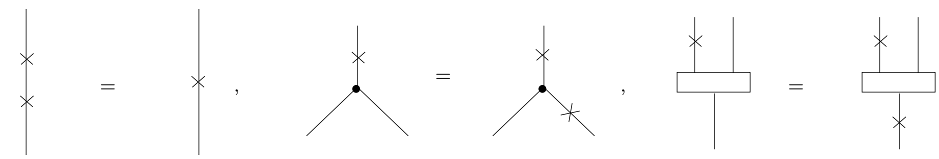


Figure 3.2. Identities for the crossed edge notation

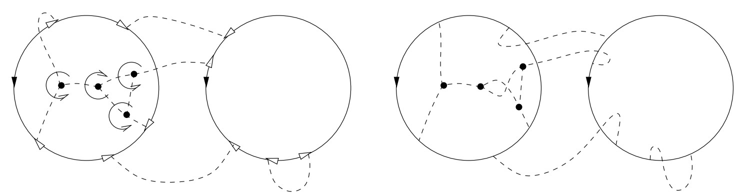


Figure 3.3

3.4. **The map $\chi_k\colon \mathcal{S}_k(M,n) \to \bar{J}_k(M,n)$.** We set

$$\bar{J}_k(M,n) = J_k(M,n)/J_{k+1}(M,n).$$

Let $\mathcal{S}_k = \mathcal{S}_k(M,n)$ denote the free $\mathbb{Z}$-module generated by ambient isotopy classes of pairs (L,S) of an n-component link L in M and a simple graph scheme S for L of degree k. Define a $\mathbb{Z}$-homomorphism

$$\chi_k\colon \mathcal{S}_k(M,n) \to \bar{J}_k(M,n)$$

by

$$\chi_k(L,S) = [L,S] \mod J_{k+1}.$$

Obviously, χ_k is surjective.

3.5. **Unitrivalent diagrams.**

Definition 3.3. By a *unitrivalent diagram* on an oriented 1-manifold X, we mean a finite graph Γ with univalent and trivalent vertices with the following data and properties.

- Each trivalent vertex is equipped with a cyclic order of the three incident edges.
- Each univalent vertex is associated with a point in X and a local orientation of X near the point. (This local orientation may or may not be consistent within each component of X.) Any two distinct univalent vertices of Γ are associated with distinct points in X.

The *degree* of Γ is defined to be half the number of vertices in Γ.

A diagram Γ is called *strict* if every component of Γ has at least one univalent vertex. *In the rest of this section, by a 'diagram' we mean a 'strict diagram'.*

For example, Figure 3.3 (a) depicts a diagram on the disjoint union of two oriented circles. As usual, the diagram is drawn with dashed lines. We sometimes drop the cyclic orders and local orientations from figures by adopting the convention depicted in Figure 3.4. Thus the previous diagram can alternatively be drawn as in Figure 3.3 (b).

Two diagrams Γ and Γ' on X are said to be *equivalent* if there is an isomorphism g of graphs from Γ to Γ' such that g preserves the cyclic order at each trivalent

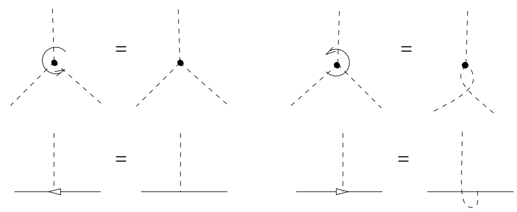


FIGURE 3.4

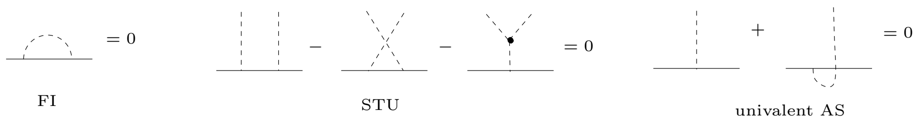


FIGURE 3.5

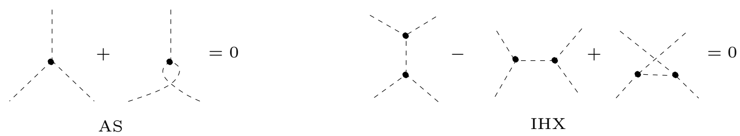


FIGURE 3.6

vertex and such that the associated points at the univalent vertices are the same up to isotopy of X.

For a compact 1-manifold X, let $\mathcal{A}_n = \mathcal{A}_n(X)$ denote the $\mathbb{Z}$-module generated by diagrams on X of degree n, subject to the *FI (framing independence) relations*, the *STU relations* and the *univalent AS relations*, see Figure 3.5. It is easy to see that the definition of $\mathcal{A}_n(X)$ is equivalent to the usual one for any oriented 1-manifold X. It follows that the *AS relations* and the *IHX relations* depicted in Figure 3.6 are valid in $\mathcal{A}_n(X)$. If X is the disjoint union of m copies of S^1, then $\mathcal{A}_n(X)$ is denoted by $\mathcal{A}(m)$.

3.6. **The map $\Gamma_n\colon \mathcal{S}_n \to \mathcal{A}_n$.** In this subsection, we set $M = S^3$ for simplicity.

Define a homomorphism

$$\Gamma_n\colon \mathcal{S}_n \to \mathcal{A}_n$$

as follows. Let $(L, S) \in \mathcal{S}_n$. The link L is regarded as an embedding $L\colon X \to M$. We choose orientations for the nodes and the disk-leaves of S, which may or may not extend to an orientation of the components of S. Let Γ be a diagram on X defined as follows. The trivalent vertices of Γ corresponds to the nodes of S and the univalent vertices of Γ corresponds to the intersections of disk-leaves of S and $L(X)$ via the map L. The orientation of each node induces a cyclic order at the corresponding trivalent vertex of Γ. At the intersection of each disk-leaf and X, the local orientation is such that the sign of the intersection of the locally-oriented strand of X and the disk-leaf is positive. Let s be the number of edges E of S such that the two nodes or disk-leaves incident to E have inconsistent orientations along E. Then, set

$$\Gamma_n(L, S) = (-1)^s \Gamma. \tag{3.2}$$

For example, see Figure 3.7. Using the AS and the univalent AS relations, we see easily that $\Gamma_n(L, S)$ does not depend on the choice of the orientations of the nodes and the disk-leaves, and hence is well-defined.

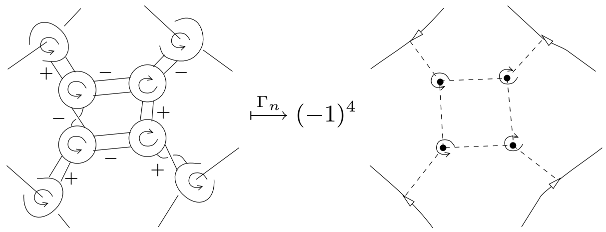


FIGURE 3.7. Here, each edge is labeled '+' if the incident nodes/disk-leaves have consistent orientations, and '−' otherwise. There are four inconsistent edges.

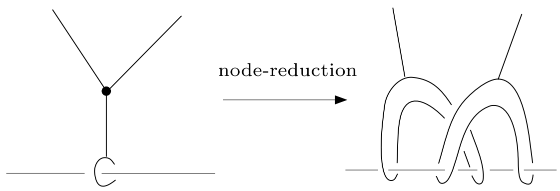


FIGURE 3.8. Here the strands may be replaced by a parallel family of strands.

Obviously, Γ_n is surjective.

3.7. **Statement of the result.**

Theorem 3.4. *There is a surjective homomorphism $\xi_n\colon \mathcal{A}_n(m) \to \bar{J}_n(S^3, m)$ such that the diagram*

(3.3)
$$\begin{tikzcd} \mathcal{S}_n(S^3,m) \arrow[d, "\Gamma_n"'] \arrow[dr, "\chi_n"] & \\ \mathcal{A}_n(m) \arrow[r, "\xi_n"'] & \bar{J}_n(S^3,m). \end{tikzcd}$$

commutes. Moreover, $\xi_n \otimes \mathbb{Q}\colon \mathcal{A}_n(m)\otimes\mathbb{Q} \to \bar{J}_n(S^3,m)\otimes\mathbb{Q}$ is an isomorphism.

3.8. **Lemmas.** In this subsection, we list some results which are necessary in the proof of Theorem 3.4.

We will refer to the move on a clasper depicted in Figure 3.8 as a *node-reduction*. It reduces the number of nodes by one, and preserves the result of surgery (cf. move 9 of [17, Proposition 2.7]). For a graph scheme, a node-reduction preserves the degree, and increases the number of elements by 0 or 1.

Lemma 3.5. *If two graph schemes are related by a node-reduction, then we have $[L, S] = [L, S']$.*

Proof. If $|S| = |S'|$, where $|S|$ denotes the cardinality of S, then the result clearly follows. Suppose $|S'| = |S| + 1$. Let G be the element of S on which a node-reduction is performed, and let $G_1, G_2 \in S'$ be the elements newly created by the move. By [17, Proposition 3.4], surgery along only G_1 and surgery along only G_2 both preserve the result of surgery from the regular neighborhood N of G. Hence we have

$$\begin{aligned}[L\cap N; G] &= (L\cap N)_{G_1\cup G_2} - L\cap N\\ &= (L\cap N)_{G_1\cup G_2} - (L\cap N)_{G_1} - (L\cap N)_{G_2} + L\cap N\\ &= [L\cap N; G_1, G_2].\end{aligned}$$

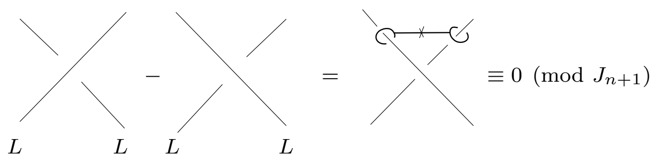


FIGURE 3.9. There is a graph scheme S of degree n invisible in the figure.

Hence we have the assertion. $\square$

Proposition 3.6 (Homotopy)**.** *Let S be a graph scheme for a link in M of degree n, and let S' be a graph scheme for a link in M' of degree n. Let N be a regular neighborhood of $L \cup \bigcup S$ in M. Suppose that there is an orientation-preserving embedding $f\colon N \to M$ such that $f(L) = L'$, $f(S) = S'$, where f respects the orientation and the orderings of links and also the structures of graph schemes. Assume also that f is homotopic to $i\colon N \subset M$. (If M is simply connected, then this assumption is vacuous.) Then we have*

$$[L, S] \equiv [L', S'] \pmod{J_{n+1}}$$

Proof. There is a sequence from (L, S) to (L', S') of ambient isotopies and the following moves:

(1) crossing change of two strands,
(2) crossing change of a strand of the link and an edge of a graph scheme,
(3) crossing change of two edges of a graph scheme,
(4) full twisting of an edge of a graph scheme.

Hence we may assume that (L, S) and (L', S') are related by one of the above moves.

We use induction on the number v of nodes in S. Suppose $v = 0$. In the cases (2), (3), (4) above, the move can be achieved by a sequence of crossing changes of strands, and hence reduces to the case (1). The case (1) follows from Figure 3.9.

Now suppose that $v > 0$. There is at least one disk-leaf A of S such that the edge E incident to A is not involved in the move, and such that A is adjacent to a node V. Let $\tilde{S}$ be the result from S of applying a node-reduction at A, E, V, and let $\tilde{S}'$ be the result from S' of the corresponding application of node-reduction. By Lemma 3.5, we have $[L, S] = [L, \tilde{S}]$ and $[L, S'] = [L, \tilde{S}']$. Since the number of nodes in $\tilde{S}$ (and in $\tilde{S}'$) is less than the number of nodes in S (and in S') by 1, the assertion follows from the induction hypothesis. $\square$

Proposition 3.7 (Half twist)**.** *Let S and S' be two graph schemes for a link L in M of degree n. Suppose that S and S' are related by a half twist of an edge. Then we have*

$$[L, S] + [L, S'] \equiv 0 \pmod{J_{n+1}}.$$

Proof. The proof is by induction on the number v of nodes in S (and in S'). Suppose $v = 0$. Let $T \in S$ be the C_1-tree on which half twist is performed. Let B be a regular neighborhood of T in M. We may assume without loss of generality that $S' = (S \setminus \{T\}) \cup \{T'\}$, where $T' \subset B$ is a C_1-tree obtained from T by half twisting the edge and sliding along the strands of S intersecting T as depicted in the left hand side of Figure 3.10. As depicted in the figure we have $(L \cap N)_{T \cup T'} \cong L \cap N$. Hence we have

$$0 = (L \cap N)_{T \cup T'} - L \cap N = [L \cap N; T, T'] + [L \cap N; T] + [L \cap N; T'].$$

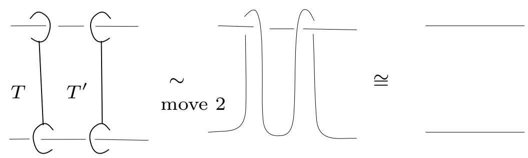


FIGURE 3.10. Here the strands may be replaced by parallel schemes of strands.

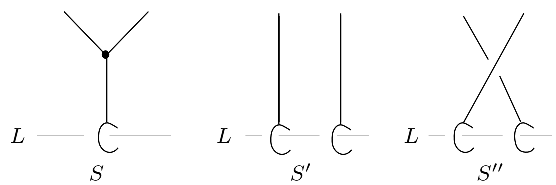


FIGURE 3.11

and hence

$$[L \cap N; T] + [L \cap N; T'] = -[L \cap N; T, T'] \equiv 0 \pmod{J_2}.$$

Hence we have $[L; S] + [L; S'] \equiv 0 \pmod{J_n}$.

The case $v > 0$ is similar to the case $v > 0$ in the proof of Proposition 3.6. Here we apply a node-reduction to a disk-leaf which is incident to an edge not involved in the half twist, and which is adjacent to a node. □

Proposition 3.8 (STU)**.** *Let S, S', S'' be three graph schemes for a link L in M of degree n which differ only in a small ball as depicted in Figure 3.11. Note that in each of the figures of S' and S'' the two leaves are contained in distinct tree claspers. Then we have*

$$[L, S] + [L, S'] - [L, S''] \equiv 0 \pmod{J_{n+1}}. \tag{3.4}$$

Proof. The proof is by induction on the number v of nodes in S' (and in S''). Consider the case $v = 0$. It suffices to consider the case $S = \{T\}$, where T is a C_2-tree as depicted in the left hand side of Figure 3.12. There are four C_1-claspers G_1, G_1', G_2, G_2' for L in M as depicted in the right hand side of the figure, such that $L_T \cong L_{G_1 \cup G_1' \cup G_2 \cup G_2'}$. It is not difficult to check that

$$[L; T] = [L_{G_1 \cup G_2}; G_1', G_2'] - [L; G_1, G_2].$$

We have

$$[L_{G_1 \cup G_2}; G_1', G_2'] \equiv [L; G_1', G_2'] \pmod{J_3}.$$

Hence, by Proposition 3.7, we have

$$[L; G_1', G_2'] \equiv [L; G_1'', G_2''] \pmod{J_3},$$

where G_1'' (resp. G_2'') is obtained from G_1' (resp. G_2') by inserting a negative half twist. Hence we have the assertion.

The case $v > 0$ can be proved by using Lemma 3.5, similarly to the case $v > 0$ in the proof of Proposition 3.6. □

3.9. **Proof of Theorem 3.4.** To prove the existence of ξ_n, it suffices to prove that $\ker \Gamma_n \subset \ker \chi_n$.

$\ker \Gamma_n$ is generated by the following elements.

1. (homotopy) $(L, S) - (L', S')$, where (L, S) and (L', S') are homotopic to each other in the sense of Proposition 3.6.

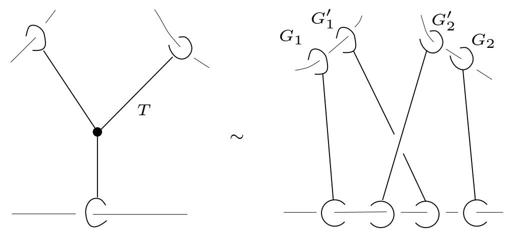


FIGURE 3.12. Here the strands may be replaced by parallel families of strands.

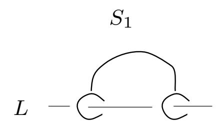


FIGURE 3.13

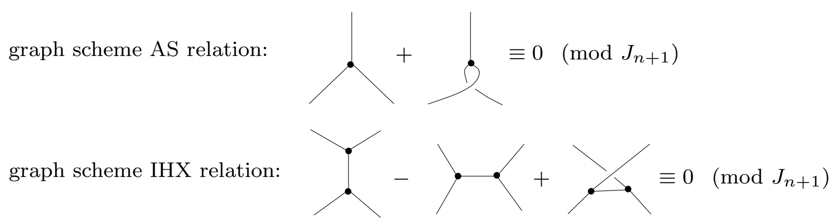


FIGURE 3.14

(2) (edge AS) $(L, S) + (L, S')$, where S and S' are related by a half twist of an edge.
(3) (FI) (L, S), where S contains a tree clasper S_1 as depicted in Figure 3.13.
(4) (STU) $(L, S) + (L, S') - (L, S'')$ as described in Proposition 3.8.

By Propositions 3.6, 3.7, 3.8, the elements in (1), (2) and (4) are contained in $\ker \chi_n$. The element in (3) is obviously 0.

Since χ_n is surjective, so is ξ_n.

$\xi_n \otimes \mathbb{Q}$ is an isomorphism since it can be naturally identified with the well-known isomorphism

$$\mathcal{A}_n^{\mathrm{ch}}(m) \otimes \mathbb{Q} \simeq \bar{J}_n \otimes \mathbb{Q},$$

where $\mathcal{A}_n^{\mathrm{ch}}(m) \otimes \mathbb{Q}$ is the $\mathbb{Q}$-vector space of chord diagrams for m circles of degree n, modulo the 4T and the FI relations.

This completes the proof of Theorem 3.4.

Remark 3.9 (AS and IHX relations). As we have mentioned, the AS and the IHX relations are valid in $\mathcal{A}_n$. Therefore, it follows from Theorem 3.4 that graph scheme versions of these relations are also valid, see Figure 3.14. (One can also prove these relations by induction directly using Proposition 3.8.) The graph scheme AS relation can also be directly proved by applying Proposition 3.7 three times. The graph scheme IHX relation can also be proved directly using the IHX relation for tree claspers which has already appeared in the literature [11, 5, 4] (see also [16] for an earlier version of topological IHX relation).

4. C_k^a-equivalence

We recall from [19] the definition of the C_k^a-equivalence.

Definition 4.1. Let L be an m-component link in a 3-manifold M. For $k \geq m-1$, a C_k^a-*tree* for L in M is a C_k-tree T for L in M, such that

(1) for each disk-leaf A of T, all the strands intersecting A are contained in one component of L, and
(2) each component of L intersects at least one disk-leaf of T, i.e., T intersects all the components of L.

Note that the condition (1) is vacuous if T is simple.

A C_k^a-*move* on a link is surgery along a C_k^a-tree. The C_k^a-*equivalence* is the equivalence relation on links generated by C_k^a-moves. A C_k^a-*forest* is a clasper consisting only of C_k^a-trees.

Clearly, the above notions are defined also for tangles, particularly for string links.

What makes the notion of C_k^a-equivalence useful in the study of Brunnian links is the fact that a link which is C_k^a-equivalent (for any k) to a Brunnian link is again a Brunnian link ([19, Proposition 5]).

Note that the C_k^a-equivalence is generated by *simple* C_k^a-moves, i.e., surgeries along simple C_k^a-trees [19]. In the following, we use technical lemmas from [19].

Lemma 4.2 ([19, Lemma 7], C^a-version of [17, Theorem 3.17])**.** *For two tangles β and β' in a 3-manifold M, and an integer $k \geq 1$, the following conditions are equivalent.*

(1) *β and β' are C_k^a-equivalent.*
(2) *There is a simple C_k^a-forest F for β in M such that $\beta_F \cong \beta'$.*

Lemma 4.3 ([19, Lemma 8], C^a-version of [17, Proposition 4.5])**.** *Let β be a tangle in a 3-manifold M, and let β_0 be a component of β. Let T_1 and T_2 be C_k-trees for a tangle β in M, differing from each other by a crossing change of an edge with the component β_0. Suppose that T_1 and T_2 are C_k^a-trees for either β or $\beta \setminus \beta_0$. Then β_{T_1} and β_{T_2} are related by one C_{k+1}^a-move.*

5. The group $\overline{BSL}_{n+1}$

5.1. **The monoids BSL_{n+1} and $\overline{BSL}_{n+1}$.** Let us recall the definition of string links. (For the details, see e.g. [13, 17]). Let $x_1, \ldots, x_{n+1} \in \operatorname{int} D^2$ be distinct points. An $(n+1)$-component *string link* $\beta = \beta_1 \cup \cdots \cup \beta_{n+1}$ is a tangle in the cylinder $D^2 \times [0,1]$, consisting of arc components $\beta_1, \ldots, \beta_{n+1}$ such that $\partial\beta_i = \{x_i\} \times \{0,1\}$ for each i. Let SL_{n+1} denote the set of $(n+1)$-component string links up to ambient isotopy fixing endpoints. There is a natural, well-known monoid structure for SL_{n+1} with multiplication given by 'stacking' of string links. The identity string link is denoted by $\mathbf{1} = \mathbf{1}_{n+1}$.

Let BSL_{n+1} denote the submonoid of SL_{n+1} consisting of *Brunnian string links.* Here a string link β is said to be Brunnian if every proper subtangle of β is the identity string link.

We have the following characterization of Brunnian string links.

Theorem 5.1 ([19, Theorem 9], [29, Proposition 4.1])**.** *An $(n+1)$-component link (resp. string link) is Brunnian if and only if it is C_n^a-trivial, i.e., it is C_n^a-equivalent to the unlink (resp. the identity string link).*

Set

$$\overline{BSL}_{n+1} = BSL_{n+1}/(C_{n+1}^a\text{-equivalence}).$$

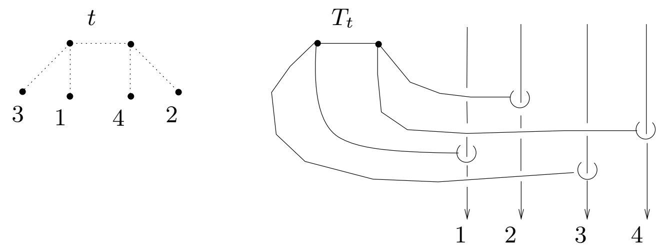


FIGURE 5.1

By Theorem 5.1, $\overline{BSL}_{n+1}$ can be regarded as the monoid of C^a_{n+1}-equivalence classes of C^a_n-trivial, $(n+1)$-component string links (in $D^2 \times [0,1]$).

In the rest of this section, we will describe the structure of $\overline{BSL}_{n+1}$.

5.2. **The group $\overline{BSL}_{n+1}$ and the surgery map $\theta_n \colon \mathcal{T}_{n+1} \to \overline{BSL}_{n+1}$.**

Proposition 5.2. *$\overline{BSL}_{n+1}$ is a finitely generated abelian group.*

Proof. The assertion is obtained by adapting the proof of [17, Lemma 5.5, Corollary 5.6] into the C^a setting. □

Let $n \geq 1$. By a *(labeled) unitrivalent tree* of degree n we mean a vertex-oriented, unitrivalent graph t such that the $n+1$ univalent vertices of t are labeled by distinct elements from $\{1, 2, \dots, n+1\}$. We use the usual drawing convention for the vertex-orientations as in Section 3.5.

Let $\mathcal{T}_{n+1}$ denote the free abelian group generated by unitrivalent trees of degree n, modulo the IHX and the AS relations.

For a unitrivalent tree t, let T_t denote a C^a_n-tree for $\mathbf{1}$ such that the tree shape and the labeling of T_t is induced by those of t, and such that after choosing an orientation of T_t, for each $i = 1, \dots, n+1$, the sign of the intersection of the ith string of $\mathbf{1}$ and the disk-leaf of T_t corresponding to the univalent vertex of t colored i is positive. See for example Figure 5.1.

Proposition 5.3. *There is a unique isomorphism*

$$\theta_{n+1} \colon \mathcal{T}_{n+1} \xrightarrow{\simeq} \overline{BSL}_{n+1}.$$

such that $\theta_{n+1}(t) = [\mathbf{1}_{T_t}]_{C^a_{n+1}}$ for each unitrivalent tree t, where T_t is as above.

Proof. Let $\mathcal{T}'_{n+1}$ be the free abelian group generated by unitrivalent trees of degree n, modulo the AS relations. By adapting the proof of [17, Theorem 4.7] into the C^a setting, we see that there is a unique surjective homomorphism

$$\theta'_{n+1} \colon \mathcal{T}'_{n+1} \to \overline{BSL}_{n+1}.$$

To see that θ'_{n+1} factors through the projection $\mathcal{T}'_{n+1} \to \mathcal{T}_n$, it suffices to see that the IHX relation is valid in $\overline{BSL}_{n+1}$, i.e., $t_I - t_H + t_X \in \mathcal{T}'_{n+1}$ is mapped to 0, where t_I, t_H, t_X locally differs as in the definition of the IHX relation (see the second line of Figure 3.14). This can be checked by adapting the IHX relation for tree claspers (see e.g. [11, 5, 4]) into the C^a setting.

Let

$$\theta_{n+1} \colon \mathcal{T}_{n+1} \to \overline{BSL}_{n+1}$$

be the surjective homomorphism induced by θ'_{n+1}. As in the statement of Theorem 1.2, for $\sigma \in S_{n-1}$ and $L \in B(n+1)$, we set

$$\mu_\sigma(T) = \mu_{\sigma(1),\sigma(2),\dots,\sigma(n-1),n,n+1}(T),$$

$t_\sigma =$

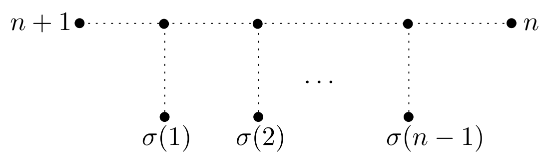


FIGURE 5.2

where $\mu_{\sigma(1),\sigma(2),\dots,\sigma(n-1),n,n+1}(T) \in \mathbb{Z}$ is the Milnor string link invariant of T. Let t_σ denote the unitrivalent tree as depicted in Figure 5.2. The t_σ for $\sigma \in S_{n-1}$ form a basis of $\mathcal{T}_{n+1}$. Define a homomorphism

$$\mu_{n+1} \colon \overline{BSL}_{n+1} \longrightarrow \mathcal{T}_{n+1}$$

by

$$\mu_{n+1}(L) = \sum_{\sigma \in S_{n-1}} \mu_\sigma(L) t_\sigma,$$

By [17, Theorem 7.2], μ_{n+1} is well defined.

To show that μ_{n+1} is left inverse to θ_{n+1}, it suffices to prove that $\mu_{n+1}\theta_{n+1}(t_\sigma) = t_\sigma$ for $\sigma \in S_{n-1}$. Let L_σ denote the closure of T_σ, which is Milnor's link as depicted in Figure 1.1. Milnor [28] proved that for $\tau \in S_{n-1}$

$$\mu_\tau(L_\sigma) = \begin{cases} 1 & \text{if } \tau = \sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Hence we have

$$\mu_{n+1}\theta_{n+1}(t_\sigma) = \sum_{\tau \in S_{n-1}} \bar{\mu}_\tau(L_\sigma) t_\tau = t_\sigma.$$

This completes the proof. □

Corollary 5.4. *For two Brunnian $(n+1)$-component string links $T, T' \in BSL_{n+1}$, the following conditions are equivalent.*

(1) *T and T' are C^a_{n+1}-equivalent.*
(2) *T and T' have the same Milnor invariants of length $n+1$.*
(3) *T and T' are link-homotopic.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) is due to Milnor [28]. The equivalence (1) $\Leftrightarrow$ (2) follows from the proof of Proposition 5.3. □

Remark 5.5. Miyazawa and Yasuhara [29] prove a similar result for Brunnian links. It seems that their proof can be applied to the case of string links. See also the Remark 6.4 below.

6. THE GROUP $\overline{B}_{n+1}$

6.1. **The set** B_{n+1}**.** Let B_{n+1} denote the set of the ambient isotopy classes of $(n+1)$-component Brunnian links. Let

$$c_{n+1} \colon BSL_{n+1} \to B_{n+1} \tag{6.1}$$

denote the map such that $c_{n+1}(\beta)$ is obtained from $\beta \in BSL_{n+1}$ by closing each component in the well-known manner.

Proposition 6.1. *The map c_{n+1} is onto.*

Proof. This is an immediate consequence of [19, Proposition 12]. □

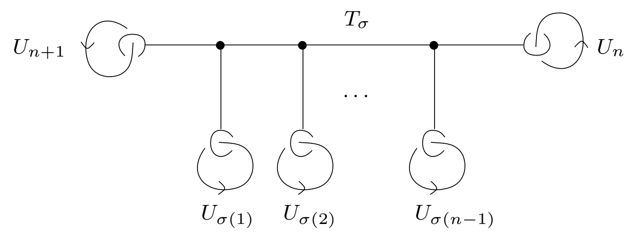


Figure 6.1

6.2. **The isomorphism $\bar{c}_{n+1}\colon \overline{BSL}_{n+1} \to \overline{B}_{n+1}$.** Set

$$\overline{B}_{n+1} = B_{n+1}/(C^a_{n+1}\text{-equivalence}),$$

and let

$$\bar{c}_{n+1}\colon \overline{BSL}_{n+1} \to \overline{B}_{n+1}$$

denote the map induced by c_{n+1}, which is onto by Proposition 6.1.

Proposition 6.2. *$\bar{c}_{n+1}$ is one-to-one.*

Proof. It suffices to prove that there is a map $\overline{B}_{n+1} \to \mathcal{T}_{n+1}$ which is inverse to $\bar{c}_{n+1}\theta_n\colon \mathcal{T}_{n+1} \to \overline{B}_{n+1}$. This is proved similarly as in the proof of Proposition 5.3. □

Proposition 6.2 provides the set $\overline{B}_{n+1}$ the well-known abelian group structure, with multiplication induced by band sums of Brunnian links.

As a corollary, we obtain another proof of a result of Miyazawa and Yasuhara [29].

Corollary 6.3 ([29, Theorem 1.2])**.** *Let L and L' be two $(n+1)$-component Brunnian links in S^3. Then the following conditions are equivalent.*

(1) *L and L' are C^a_{n+1}-equivalent.*
(2) *L and L' are C_{n+1}-equivalent.*
(3) *L and L' are link-homotopic.*

Proof. The result follows immediately from Propositions 5.4 and 6.2. □

Remark 6.4. Miyazawa and Yasuhara [29] do not explicitly state the equivalence of (1) and others, but this equivalence follows from their proof.

Note that, unlike the C^a_{n+1}-equivalence, neither the C_{n+1}-equivalence nor the link-homotopy are closed for Brunnian links.

Remark 6.5. It is possible to show directly that $\mathcal{T}_{n+1}$ is isomorphic to $\overline{B}_{n+1}$, without using string links and the closure map $\bar{c}_{n+1}$. The proof uses Milnor's $\overline{\mu}$-invariants and the above result of Miyazawa and Yasuhara. Our approach provides an alternative proof of the latter (instead of using it).

6.3. **Trees and the Milnor invariants.** In this subsection, we fix some notations which are used in later sections. (Some has appeared in the proof of Proposition 5.3.)

For $\sigma \in S_{n-1}$, let t_σ denote the unitrivalent tree as depicted in Figure 5.2. The t_σ for $\sigma \in S_{n-1}$ form a basis of $\mathcal{T}_{n+1}$. Let T_σ denote the corresponding C^a_n-tree for the $(n+1)$-component unlink $U = U_1 \cup \cdots \cup U_{n+1}$, see Figure 6.1.

For $i_1, \dots, i_{n+1}$ with $\{i_1, \dots, i_{n+1}\} = \{1, \dots, n+1\}$, let

$$\bar{\mu}_{i_1,\dots,i_{n+1}}\colon B_{n+1} \to \mathbb{Z}$$

denote the Milnor invariant, which is additive under connected sum [28] (see also [3, 33, 21]). For $\sigma \in S_{n-1}$, we set

$$\bar{\mu}_\sigma = \bar{\mu}_{\sigma(1),\sigma(2),\dots,\sigma(n-1),n,n+1} \colon B_{n+1} \to \mathbb{Z}.$$

It is well known [28] that for $\rho \in S_{n-1}$

$$\bar{\mu}_\rho(U_{T_\sigma}) = \begin{cases} 1 & \text{if } \rho = \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

7. Goussarov-Vassiliev invariants of Brunnian links

Throughout the rest of the paper, let $U = U_1 \cup U_2 \cup \cdots \cup U_{n+1}$ be the $(n+1)$-component unlink in the 3-sphere S^3.

7.1. The map $\kappa_{n+1} \colon \overline{B}_{n+1} \to \bar{J}_{2n}(n+1)$.

Proposition 7.1. *Let $n \geq 2$. Let L and L' be two $(n+1)$-component Brunnian links in an oriented, connected 3-manifold M. If L and L' are C^a_{n+1}-equivalent (or link-homotopic), then we have $L' - L \in J_{2n+1}$.*

Proposition 7.1 implies the following.

Corollary 7.2. *The restriction of any Goussarov-Vassiliev invariant of degree $2n$ to $(n+1)$-component Brunnian links is a link-homotopy invariant.*

Proof of Proposition 7.1. First, we consider the case $L = U$. By using the same arguments as in the proof of [19, Lemma 14], we see that there is a clasper G for U consisting of C^a_l-claspers with $n+1 \leq l < 2n+1$, such that U bounds $n+1$ disjoint disks which are disjoint from the edges and the nodes of G, and such that $U_G \sim_{C^a_{2n+1}} U_T$. The latter implies that $U_G - U_T \in J_{2n+1}$. We use the equality $U_G = \sum_{G' \subset G} [U, G']$. Clearly $[U, G'] \in J_{2n+1}$ for $|G'| > 1$, so we may safely assume that G has only one component. We then have $U_G - U \in J_{2n+1}$ as a direct application of [19, Lemma 16]. This completes the proof of the case $L = U$.

Now consider the general case. We may assume that L' is obtained from L by one simple C^a_n-move. Since L is an $(n+1)$-component Brunnian link, it follows from Theorem 5.1 and Lemma 4.2 that there exists a simple C^a_n-forest F for U such that $L = U_F$. Also, there exists a simple C^a_{n+1}-tree $\tilde{T}$ for $L = U$ such that $L' = L_{\tilde{T}}$. We may assume that $\tilde{T}$ is a simple C^a_{n+1}-tree for U disjoint from F such that $L' = U_{F \cup \tilde{T}}$. Let S be the forest scheme consisting of the trees $T_1, \dots, T_l$ of F. We have $L = \sum_{S' \subset S} [U, S']$ and $L' = \sum_{S' \subset S} [U_T, S']$. Hence we have

$$L' - L = \sum_{S' \subset S} [U, S' \cup \{T\}].$$

Since $\deg \tilde{T} = n+1$ and $\deg T_i = n$ for all i, the term in the above sum is contained in J_{2n+1} unless $S' = \emptyset$. Hence we have

$$L' - L \equiv [U, \tilde{T}] \equiv 0 \pmod{J_{2n+1}},$$

where the second congruence follows from the first case. □

By Proposition 7.1, we have a map

$$\kappa_{n+1} : \overline{B}_{n+1} \longrightarrow \bar{J}_{2n}(n+1)$$

defined by $\kappa_{n+1}(L) = [L - U]_{J_{2n+1}}$.

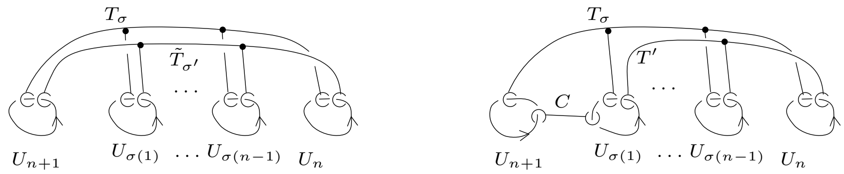


FIGURE 7.1

7.2. **Quadraticity of κ_{n+1}.** Let $n \geq 2$. In this subsection, we establish the following commutative diagram.

$$\begin{array}{ccc} \mathcal{T}_{n+1} & \xrightarrow[\simeq]{\psi_{n+1}} & \overline{B}_{n+1} \\ {\scriptstyle q_{n+1}}\downarrow & & \downarrow{\scriptstyle \kappa_{n+1}} \\ \tilde{\mathrm{Sym}}^2\mathcal{T}_{n+1} & \xrightarrow[\delta_{n+1}]{} & \bar{J}_{2n}(n+1) \end{array} \tag{7.1}$$

Definitions of ψ_{n+1}, $\tilde{\mathrm{Sym}}^2\mathcal{T}_{n+1}$, q_{n+1} and δ_{n+1} are in order.

- The isomorphism ψ_{n+1} is the composition of
$$\mathcal{T}_{n+1} \xrightarrow[\simeq]{\theta_n} \overline{BSL}_{n+1} \xrightarrow[\simeq]{\bar{c}_{n+1}} \overline{B}_{n+1}.$$
- Let $\mathrm{Sym}^2\,\mathcal{T}^{\mathbb{Q}}_{n+1}$ denote the symmetric product of two copies of $\mathcal{T}^{\mathbb{Q}}_{n+1} := \mathcal{T}_{n+1}\otimes\mathbb{Q}$, and let $\tilde{\mathrm{Sym}}^2\mathcal{T}_{n+1}$ denote the $\mathbb{Z}$-submodule of $\mathrm{Sym}^2\,\mathcal{T}^{\mathbb{Q}}_{n+1}$ generated by $\frac{1}{2}x^2$, $x \in \mathcal{T}_{n+1}$. One can easily verify that $\tilde{\mathrm{Sym}}^2\mathcal{T}_{n+1}$ is $\mathbb{Z}$-spanned by the elements $\frac{1}{2}t_\sigma^2$ for $\sigma \in S_{n-1}$ and $t_\sigma t_{\sigma'}$ for $\sigma, \sigma' \in S_{n-1}$. (Of course we have $t_\sigma t_{\sigma'} = t_{\sigma'} t_\sigma$. Thus $\tilde{\mathrm{Sym}}^2\mathcal{T}_{n+1}$ is a free abelian group of rank $\frac{1}{2}(n-1)!((n-1)!+1)$.)
- The arrow q_{n+1} is the quadratic map defined by $q_{n+1}(x) = \frac{1}{2}x^2$ for $x \in \mathcal{T}_{n+1}$.
- The arrow δ_{n+1} is the homomorphism defined as follows. For $\sigma, \sigma' \in S_{n-1}$, let T_σ and $T_{\sigma'}$ be the corresponding simple C^a_n-trees for U as in Section 6.3. Let $\tilde{T}_{\sigma'}$ denote a simple C^a_n-trees obtained from $T_{\sigma'}$ by a small isotopy if necessary so that $\tilde{T}_{\sigma'}$ is disjoint from T_σ. Set
$$\delta_{n+1}(t_\sigma t_{\sigma'}) = [U; T_\sigma, \tilde{T}_{\sigma'}]_{J_{2n+1}} \in \bar{J}_{2n}(n+1),$$
which does not depend on how we obtained $\tilde{T}_{\sigma'}$ from T_σ, since crossing changes between an edge of T_σ and an edge of $\tilde{T}_{\sigma'}$ preserves the right-hand side. (This can be verified by using a 'C^a-version' of [17, Proposition 4.6].) For the case of $\frac{1}{2}t_\sigma^2$, we modify the above definition with $\sigma' = \sigma$ as follows. Let T_σ and $\tilde{T}_\sigma$ be as above. See Figure 7.1. Let T' be the C_{n-1}-tree obtained from $\tilde{T}_\sigma$ by first removing the disk-leaf D intersecting U_{n+1}, the edge E incident to D, and the node N incident to E, and then gluing the ends of the two edges which were attached to N. Moreover, let C be a C_1-tree which intersects U_{n+1} and $U_{\sigma(1)}$ as depicted. Set
$$\delta_{n+1}(\frac{1}{2}t_\sigma^2) = [U; T_\sigma, T', C].$$

Lemma 7.3. *We have*

$$[U; T_\sigma, \tilde{T}_\sigma] \equiv 2[U; T_\sigma, T', C] \pmod{J_{2n+1}}. \tag{7.2}$$

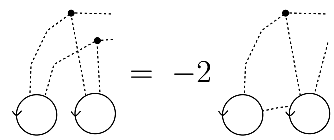


FIGURE 7.2

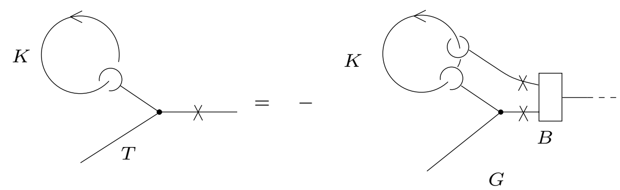


FIGURE 7.3

Proof. By Theorem 3.4, it suffices to prove the identity in $\mathcal{A}_{2n}(n+1)$ depicted in Figure 7.2, which can be easily verified using the STU relation several times. □

It follows from Lemma 7.3 that δ_{n+1} is a well-defined homomorphism. Set

$$\frac{1}{2}[U;T_\sigma,\tilde{T}_\sigma]_{J_{2n+1}} = [U;T_\sigma,T',C]_{J_{2n+1}}.$$

We have

$$\delta_{n+1}(\frac{1}{2}t_\sigma^2) = \frac{1}{2}[U;T_\sigma,\tilde{T}_\sigma]_{J_{2n+1}}.$$

Theorem 7.4. *The diagram* (7.1) *commutes. In particular, κ_{n+1} is a quadratic map.*

We need the following lemma before proving Theorem 7.4.

Lemma 7.5. *Let C be a clasper for a link L such that there is a disk-leaf D of T which 'monopolizes' a component K of L in the sense of* [19, Definition 15]*, and such that D is adjacent to a node. That is, T and L looks as depicted in the left hand side of Figure 7.3. Then we have the identity as depicted in the figure.*

Proof. The identity is easily verified and left to the reader. (Note that Lemma 7.5 is essentially the same as [19, (4.4)].) □

Proof of Theorem 7.4. Let $\sigma \in S_{n-1}$. We must show that

$$[U;T_\sigma]_{J_{2n+1}} = \frac{1}{2}[U;T_\sigma,\tilde{T}_\sigma]_{J_{2n+1}}.$$

For $i = 1,\ldots,n+1$, let D_i denote the disk-leaf of T_σ intersecting L_i, and let E_i denote the incident edge. For $i = 1,\ldots,n-1$, let N_i denote the node incident to E_i.

By applying Lemma 7.5 to the edge of T_σ which is incident to $N_{\sigma(1)}$ but not to D_{n+1} or $D_{\sigma(1)}$, we obtain the identity depicted in Figure 7.4. Let B be the box and E be the edge as depicted. Let G be the clasper in the right hand side. By zip construction [17, Section 3.3] at E, we obtain a crossed clasper depicted in Figure 7.5, which consists of two components T_σ and P. The component P has $n-2$ (non-disk) leaves.

We claim that we can unlink the leaves of P from T_σ without changing the class in $\bar{J}_{2n}(n+1)$. To see this, it suffices to show that

$$U_{T_\sigma\cup P} - U_{T_\sigma\cup P'} \in J_{2n+1}, \tag{7.3}$$

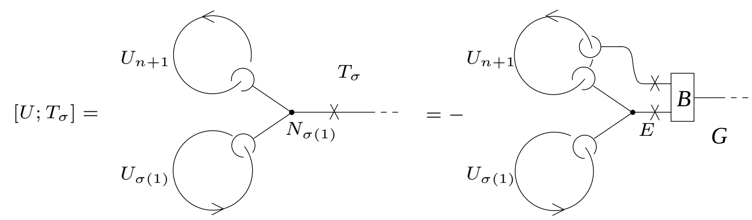


Figure 7.4

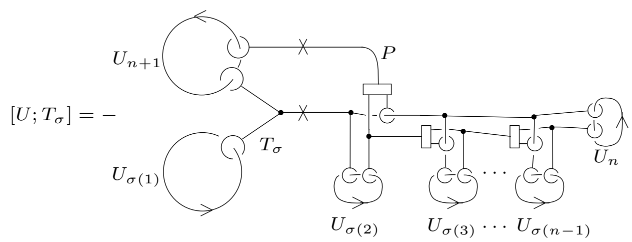


Figure 7.5

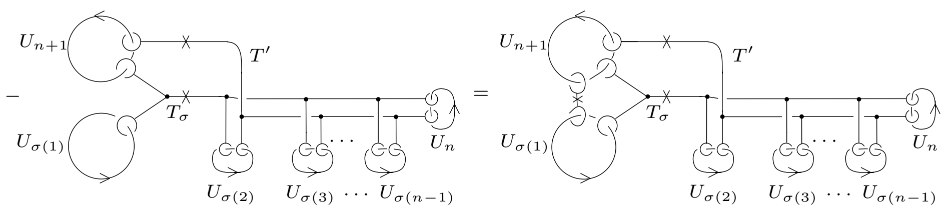


Figure 7.6

where P' is obtained from P by the unlinking operation. Note that each unlinking is performed by a sequence of crossing changes between an edge of the C^a_n-tree T_σ and a link component (after performing surgery along P in the regular neighborhood of P), and thus can be performed by C^a_{n+1}-moves. Since all the links appearing in this sequence is Brunnian, we have (7.3) by Proposition 7.1. This completes the proof of the claim.

By the above claim, it follows that

$$[U;T_\sigma,P] \equiv [U;T_\sigma,T'] \pmod{J_{2n+1}},$$

where T' is obtained from P by removing the leaves, the incident edges, and the boxes, and then smoothing the open edges, see the left hand side of Figure 7.6, which is equal to the right hand side by Lemma 7.5. The result is related to the desired clasper defining $\frac{1}{2}[U;T_\sigma,T',C]$ by half twists of two edges and homotopy with respect to U, and hence equivalent modulo J_{2n+1} to $\frac{1}{2}[U;T_\sigma,T',C]$. This completes the proof. □

7.3. **Proof of Theorem 1.2.** In this subsection we prove Theorem 1.2.

Let $L \in B_{n+1}$. We have

$$[L]_{C^a_{n+1}} = \sum_{\sigma\in S_{n-1}} \bar\mu_\sigma(L)[U_{T_\sigma}]_{C^a_{n+1}}$$

in $\overline{B}_{n+1}$. (Recall that the sum is induced by band-sum in $\overline{B}_{n+1}$.) Hence we have by the commutativity of (7.1)

$$\begin{aligned}
[L-U]_{J_{2n+1}} &= \kappa_{n+1}([L]_{C^a_{n+1}}) \\
&= \delta_{n+1}q_{n+1}\psi^{-1}_{n+1}(\sum_{\sigma\in S_{n-1}} \bar{\mu}_\sigma(L)[U_{T_\sigma}]_{C^a_{n+1}}) \\
&= \delta_{n+1}q_{n+1}(\sum_{\sigma\in S_{n-1}} \bar{\mu}_\sigma(L)t_\sigma) \\
&= \delta_{n+1}(\frac{1}{2}(\sum_{\sigma\in S_{n-1}} \bar{\mu}_\sigma(L)t_\sigma)^2) \\
&= \delta_{n+1}(\frac{1}{2}\sum_{\sigma,\sigma'\in S_{n-1}} \bar{\mu}_\sigma(L)\bar{\mu}_{\sigma'}(L)t_\sigma t_{\sigma'}) \\
&= \frac{1}{2}\sum_{\sigma,\sigma'\in S_{n-1}} \bar{\mu}_\sigma(L)\bar{\mu}_{\sigma'}(L)[U;T_\sigma,T_{\sigma'}]_{J_{2n+1}}.
\end{aligned}$$

Hence we have

$$f(L)-f(U) = \frac{1}{2}\sum_{\sigma,\sigma'\in S_{n-1}} \bar{\mu}_\sigma(L)\bar{\mu}_{\sigma'}(L)f([U;T_\sigma,T_{\sigma'}]). \tag{7.4}$$

We give any total order on the set S_{n-1}. Then we have

$$f(L) = f(U)+\sum_{\sigma\in S_{n-1}}(\frac{1}{2}f([U;T_\sigma,T_\sigma]))\bar{\mu}_\sigma(L)\bar{\mu}_\sigma(L)+\sum_{\sigma<\sigma'} f([U;T_\sigma,T_{\sigma'}])\bar{\mu}_\sigma(L)\bar{\mu}_{\sigma'}(L).$$

Note that $\frac{1}{2}f([U;T_\sigma,T_\sigma]) \in \mathbb{Z}$ and $f([U;T_\sigma,T_{\sigma'}]) \in \mathbb{Z}$. Hence we have (1.1) by setting

$$f(\sigma,\sigma') = \begin{cases} \frac{1}{2}f([U;T_\sigma,T_\sigma]) & \text{if } \sigma=\sigma', \\ f([U;T_\sigma,T_{\sigma'}]) & \text{if } \sigma<\sigma', \\ 0 & \text{if } \sigma>\sigma'. \end{cases}$$

This completes the proof of Theorem 1.2.

8. The Brunnian part of $\bar{J}_{2n}(n+1)$

If L is an $(n+1)$-component Brunnian link in S^3, then, by [19, Theorem 3], we have $L-U \in J_{2n}(n+1)$. Define $\mathrm{Br}(\bar{J}_{2n}(n+1))$ to be the $\mathbb{Z}$-submodule of $\bar{J}_{2n}(n+1)$ generated by the elements $[L-U]_{J_{2n+1}}$, where L is an $(n+1)$-component Brunnian link. We call $\mathrm{Br}(\bar{J}_{2n}(n+1))$ the *Brunnian part* of $\bar{J}_{2n}(n+1)$.

The purpose of this section is to give an almost complete description of the structure of the Brunnian part $\mathrm{Br}(\bar{J}_{2n}(n+1))$ of the $2n$th graded quotient $\bar{J}_{2n}(n+1)$ of Goussarov-Vassiliev filtration of $(n+1)$-component links, using the $\mathbb{Z}$-module $\mathcal{A}^c_{n-1}(\emptyset)$ of 'connected trivalent diagrams' (see below for the definition).

8.1. **Trivalent diagrams.** In Section 3.5, we have defined the notion of unitrivalent diagrams on a 1-manifold.

Definition 8.1. A *trivalent diagram* (also called *closed Jacobi diagram*) is defined to be a unitrivalent diagram on the empty 1-manifold $\emptyset$. In other words, a trivalent diagram is a vertex-oriented trivalent diagram. Note that the degree of a trivalent diagram is half the number of vertices.

No nonempty trivalent diagram is strict in the sense of Definition 3.3.

For $n \geq 1$, let $\mathcal{A}_n(\emptyset)$ denote the $\mathbb{Z}$-module generated by trivalent diagrams of degree n modulo the AS and IHX relations. This notation contradicts to the

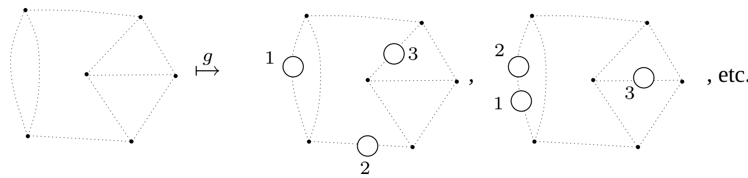


FIGURE 8.1

notation in Section 3.5, where $\mathcal{A}_n(\emptyset)$ means a strict unitrivalent diagram. Let us allow this abuse of notation for simplicity.

Let $\mathcal{A}^c_n(\emptyset)$ denote the $\mathbb{Z}$-submodule of $\mathcal{A}_n(\emptyset)$ generated by connected trivalent diagrams.

8.2. **The circle-insertion map g_n.** For $n \geq 2$, let

$$g_n\colon \mathcal{A}^c_{n-1}(\emptyset) \to \mathcal{A}_{2n}(n+1)$$

denote the homomorphism which maps each trivalent diagram Γ to the result of inserting $n+1$ ordered copies of S^1 in its edges, see Figure 8.1. This map is well-defined thanks to the STU relation, since Γ is connected.

We need the following result.

Proposition 8.2. *For $n \geq 2$, the map*

$$g_n \otimes \mathbb{Q}\colon \mathcal{A}^c_{n-1}(\emptyset) \otimes \mathbb{Q} \to \mathcal{A}_{2n}(n+1) \otimes \mathbb{Q}$$

is injective.

Proof. An *open unitrivalent diagram* (also called *open Jacobi diagrams*) Γ of degree $2n$ for $n+1$ colors $\{1, \ldots, n+1\}$ will mean a vertex-oriented unitrivalent diagram such that each univalent vertex is labeled by an element in $\{1, \ldots, n+1\}$. We assume that such a diagram Γ is strict, i.e., each component of Γ has at least one univalent vertex. Let $\mathcal{B}^l_{2n}(n+1)$ denote the $\mathbb{Q}$-vector space generated by open unitrivalent diagrams of degree $2n$ for $n+1$ colors $\{1, \ldots, n+1\}$, modulo the AS, the IHX, and the link relation. It is known [1] that there is a standard isomorphism

$$\chi_{2n}\colon \mathcal{B}^l_{2n}(n+1) \xrightarrow{\simeq} \mathcal{A}_{2n}(n+1) \otimes \mathbb{Q},$$

called the Poincaré-Birkhoff-Witt isomorphism.

Define a homomorphism

$$P\colon \mathcal{B}^l_{2n}(n+1) \longrightarrow \mathcal{A}^c_{n-1}(\emptyset) \otimes \mathbb{Q}$$

as follows. Let $\Gamma \in \mathcal{B}^l_{2n}(n+1)$ be a diagram. If there are exactly two univalent vertices colored by i for each $i = 1, \ldots, n+1$, then we set $P(\Gamma)$ to be the trivalent diagram obtained from Γ by joining each pair of univalent vertices of the same color. Otherwise, set $P(\Gamma) = 0$. Let

$$\pi\colon \mathcal{A}_{n-1}(\emptyset) \otimes \mathbb{Q} \longrightarrow \mathcal{A}^c_{n-1}(\emptyset) \otimes \mathbb{Q}$$

denote the projection, which maps each connected diagram into itself, and maps nonconnected diagrams to 0. One can easily check that the composition of

$$\mathcal{A}^c_{n-1}(\emptyset) \otimes \mathbb{Q} \xrightarrow{g_n \otimes \mathbb{Q}} \mathcal{A}_{2n}(n+1) \otimes \mathbb{Q} \xrightarrow{\chi^{-1}_{2n}} \mathcal{B}^l_{2n}(n+1) \xrightarrow{P} \mathcal{A}_{n-1}(\emptyset) \otimes \mathbb{Q} \xrightarrow{\pi} \mathcal{A}^c_{n-1}(\emptyset) \otimes \mathbb{Q}$$

is the identity. Hence $g_n \otimes \mathbb{Q}$ is injective. $\square$

8.3. **Structure of** $\mathrm{Br}(\bar{J}_{2n}(n+1))$**.** For $n \geq 2$, let h_n denote the composition

$$h_n\colon \mathcal{A}^c_{n-1}(\emptyset) \xrightarrow{g_n} \mathcal{A}_{2n}(n+1) \xrightarrow{\xi_{2n}} \bar{J}_{2n}(n+1).$$

(For the definition of ξ_{2n}, see Section 3.7.)

Theorem 8.3. *(1) For $n \geq 3$ we have $h_n(\mathcal{A}^c_{n-1}(\emptyset)) = \mathrm{Br}(\bar{J}_{2n}(n+1))$. For $n = 2$, $h_2(\mathcal{A}^c_1(\emptyset))$ is an index 2 subgroup of $\mathrm{Br}(\bar{J}_4(3))$.*

(2) For $n \geq 2$, the $\mathbb{Q}$-linear map

$$h_n \otimes \mathbb{Q}\colon \mathcal{A}^c_{n-1}(\emptyset) \otimes \mathbb{Q} \to \mathrm{Br}(\bar{J}_{2n}(n+1)) \otimes \mathbb{Q} \tag{8.1}$$

is an isomorphism.

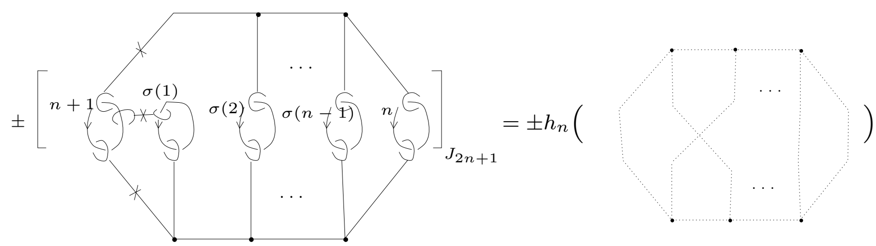
Proof. (1) Suppose $n \geq 3$. First we show that

$$\mathrm{Br}(\bar{J}_{2n}(n+1)) \subset h_n(\mathcal{A}^c_{n-1}(\emptyset)). \tag{8.2}$$

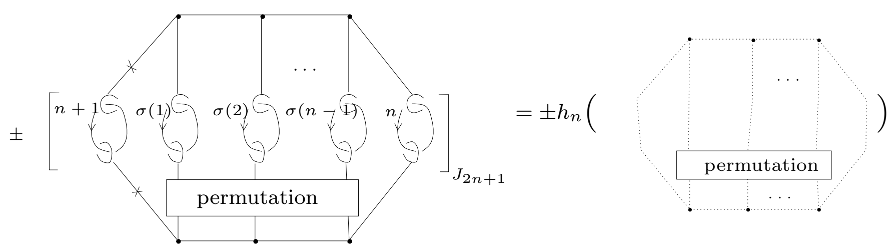
It follows from the proof of Theorem 1.2 that $\mathrm{Br}(\bar{J}_{2n}(n+1))$ is $\mathbb{Z}$-spanned by

$$\frac{1}{2}[U; T_\sigma, \tilde{T}_\sigma] \quad \text{for } \sigma \in S_{n-1},$$
$$[U; T_\sigma, \tilde{T}_{\sigma'}] \quad \text{for } \sigma, \sigma' \in S_{n-1}.$$

For $n \geq 3$, $\frac{1}{2}[U; T_\sigma, \tilde{T}_\sigma]_{J_{2n+1}}$ is equal to



For $n \geq 2$, $[U; T_\sigma, \tilde{T}_{\sigma'}]_{J_{2n+1}}$ is equal to



(8.3)

Hence we have (8.2) for $n \geq 3$.

Now we prove

$$h_n(\mathcal{A}^c_{n-1}(\emptyset)) \subset \mathrm{Br}(\bar{J}_{2n}(n+1)). \tag{8.4}$$

It suffices to prove that every element of $\mathcal{A}^c_{n-1}(\emptyset)$ is a $\mathbb{Z}$-linear combination of the elements of the form depicted in the right hand side of (8.3), which one can easily verify.

The case $n = 2$ follows, since $\mathrm{Br}(\bar{J}_4(3))$ is $\mathbb{Z}$-spanned by $[U; T_1]_{J_5} = \frac{1}{2}[U; T_1, \tilde{T}_1]_{J_5}$, and $h_2(\mathcal{A}^c_1(\emptyset))$ is $\mathbb{Z}$-spanned by $[U; T_1, \tilde{T}_1]_{J_5}$. (Here '1' in '$T_1$' denotes the unit in the (trivial) symmetric group S_1 of order 1.)

(2) The assertion follows immediately from (1) above, Proposition 8.2 and Theorem 3.4. $\square$

Corollary 8.4. *Let f be a $\mathbb{Z}$-valued invariant of degree $2n$ for $(n+1)$-component links in S^3. For any $(n+1)$-component Brunnian link L in S^3, we have the following.*

(1) *$f(L) = f(L_\sigma)$ for any $\sigma \in S_{n+1}$, where L_σ is obtained by reordering the components of L using σ.*
(2) *$f(L) = f(L_-)$, where L_- is obtained by reversing the orientation of any component of L.*

Remark 8.5. Recall that trivalent diagrams appear in the study of Ohtsuki finite type invariants of integral homology spheres [31, 7, 22]. The relation between Theorem 8.3 and Ohtsuki finite type invariants will be discussed in a future paper.

9. An alternative proof of Theorem 1.2

In this section, we give a sketch proof of Theorem 1.2 using the Kontsevich integral.

Suppose $L \in B_{n+1}$. By Proposition 6.1, there is $T \in BSL_{n+1}$ with $c_{n+1}(T) = L$. Consider the Kontsevich integral $Z(T) \in \mathcal{A}(\sqcup^{n+1} I)$ of T, where $\mathcal{A}(\sqcup^{n+1} I)$ is a completed $\mathbb{Q}$-vector space of unitrivalent diagrams on the disjoint union $\sqcup^{n+1} I$ of $n+1$ copies of the unit interval I, modulo the FI and the STU relations. Recall that $\mathcal{A}(\sqcup^{n+1} I)$ has an algebra structure with multiplication given by the 'stacking product' and with unit 1 given by the empty unitrivalent diagram.

Let $p\colon \mathcal{A}(\sqcup^{n+1} I) \simeq \mathcal{B}(n+1)$ denote the inverse of the Poincaré-Birkhoff-Witt isomorphism, where $\mathcal{B}(n+1)$ is the completed $\mathbb{Q}$-vector space of open unitrivalent diagrams modulo the AS and the IHX relations. Since $p(Z(T)) \in \mathcal{B}(n+1)$ is grouplike (see [23]), we have

$$p(Z(T)) = \exp_\sqcup(P), \quad P \in \mathcal{P}(n+1), \tag{9.1}$$

where $\mathcal{P}(n+1)$ denotes the primitive part of $\mathcal{B}(n+1)$, generated by connected diagrams, and where $\exp_\sqcup$ denotes exponential with respect to disjoint union of diagrams.

For each $i = 1, \dots, n+1$, consider the operation ϵ_i of omitting the ith string. At the level of string link we have $\epsilon_i(T) = \mathbf{1}_n$, since T is Brunnian. Hence we have $\epsilon_i(P) = 0$. It follows that P can be expressed as an infinite $\mathbb{Q}$-linear combination of connected diagrams, each having at least one univalent vertex of color i. Since we have this property for $i = 1, \dots, n+1$, we can deduce that P is an infinite $\mathbb{Q}$-linear combination of connected diagrams, each involving all the colors.

By an easy counting argument, we see that P can be expressed as

$$P = P_n + P_{n+1} + P_{n+2} + \dots,$$

where P_n is a linear combination of trees of degree n, and P_k for $k > n$ is a linear combination of diagrams of degree k. By Habegger and Masbaum's result [15], P_n is a $\mathbb{Z}$-linear combination of trees which corresponds to the Milnor invariants of length n without repeating indices.

By (9.1), we have

$$\begin{aligned} p(Z(T)) &= \exp_\sqcup(P_n + P_{n+1} + \dots + P_{2n} + \dots) \\ &= 1 + P_n + P_{n+1} + \dots + P_{2n-1} + (P_{2n} + \frac{1}{2}P_n^2) + O_{>2n}, \end{aligned}$$

where $O_{>2n}$ is a sum of terms of degree $> 2n$. Hence we have

$$Z(T) = 1 + P'_n + P'_{n+1} + \dots + P'_{2n-1} + P'_{2n} + p^{-1}(\frac{1}{2}P_n^2) + p^{-1}(O_{>2n}),$$

where $P'_k = p^{-1}(P_k)$ for $k = n, \dots, 2n$.

The Kontsevich integral $Z(L)$ of L can be obtained from $Z(T)$ by first multiplying by the Kontsevich integral of the unknot at each string, and then mapping into the space $\mathcal{A}(n+1)$ by closing the strings. For our purpose, it is more convenient to use a version $Z'(L)$ which is obtained from $Z(L)$ by multiplying the inverse of the

Kontsevich integral of the unknot to each string. Thus we have $Z'(L) = \pi(Z(T))$, where $\pi\colon \mathcal{A}(\sqcup^{n+1} I) \to \mathcal{A}(n+1)$ is the projection which closes the strings. The map π kills each diagram with at least one string having exactly one univalent vertex of the attached diagram. One can verify that each term in $P'_n, \ldots, P'_{2n}$ is killed in this way. Thus we have

$$Z'(L) = 1 + \pi p^{-1}(\frac{1}{2}P_n^2) + \pi p^{-1}(O_{>2n}).$$

It follows that the first nontrivial, $\mathbb{Z}$-valued, finite type invariants of $(n+1)$-component Brunnian links are of degree $2n$ and is a quadratic form of Milnor's link-homotopy invariant of length $n+1$. Further details are left to the reader.

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