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MATRIX FACTORIZATIONS AND REPRESENTATIONS OF QUIVERS II: TYPE ADE CASE

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ABSTRACT. We study a triangulated category of graded matrix factorizations for a polynomial of type ADE. We show that it is equivalent to the derived category of finitely generated modules over the path algebra of the corresponding Dynkin quiver. Also, we discuss a special stability condition for the triangulated category in the sense of T. Bridgeland, which is naturally defined by the grading.

CONTENTS

1. INTRODUCTION

The universal deformation and the simultaneous resolution of a simple singularity are described by the corresponding simple Lie algebra (Brieskorn [Bs]). Inspired by that theory, the second named author associated in [Sa2],[Sa4] a generalization of root systems, consisting of vanishing cycles of the singularity, to any regular weight systems [Sa1], and asked to construct a suitable Lie theory in order to reconstruct the primitive forms for the singularities. In fact, the simple singularities correspond exactly to the weight systems having only positive exponents, and, in this case, this approach gives the classical finite root systems as in [Bs].

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As the next case, the approach is worked out for simple elliptic singularities corresponding to weight systems having only non-negative exponents, from where the theory of elliptic Lie algebras is emerging [Sa2]. However, the root system in this approach in general is hard to manipulate because of the transcendental nature of vanishing cycles. Hence, he asked ([Sa4], Problem in p.124 in English version) *an algebraic and/or a combinatorial construction of the root system starting from a regular weight system.*

In [T2], based on the mirror symmetry for the Landau-Ginzburg orbifolds and also based on the duality theory of the weight systems [Sa3],[T1], the third named author proposed a new approach to the root systems, answering to the above problem. He introduced a triangulated category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ of graded matrix factorizations for a weighted homogeneous polynomial f attached to a regular weight system and showed that the category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ for a polynomial of type A_l is equivalent to the bounded derived category of modules over the path algebra of the Dynkin quiver of type A_l. He conjectured ([T2], Conjecture 1.3) further that the same type of equivalences hold for all simple polynomials of type ADE. The main goal of the present paper is to answer affirmatively to the conjecture.

One side of this conjecture: the properties of the category of modules over a path algebra of a Dynkin quiver are already well-understood by the Gabriel's theorem [Ga], which states that the number of the indecomposable objects in the category for a Dynkin quiver coincides with the number of the positive roots of the root system corresponding to the Dynkin diagram. The other side of the conjecture: the triangulated categories of (ungraded) matrix factorizations were introduced and developed by Eisenbud [E] and Knörrer [K] in the study of the maximal Cohen-Macaulay modules. Recently, the categories of matrix factorizations are rediscovered in string theory as the categories of topological D-branes of type B in Landau-Ginzburg models (see [KL1],[KL2]). The category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ of graded matrix factorizations is then motivated by the work on the categories of topological D-branes of type B in Landau-Ginzburg *orbifolds* $(f, \mathbb{Z}/h\mathbb{Z})$ by Hori-Walcher [HW], where the orbifolding corresponds to introducing the $\mathbb{Q}$-grading. In fact, in [T2], the triangulated category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ is constructed from a special A_∞-category with $\mathbb{Q}$-grading via the twisted complexes in the sense of Bondal-Kapranov [BK]. Independently, D. Orlov defines a triangulated category, called the category of graded D-branes of type B, which is in fact equivalent to $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ (see the end of subsection 2.2). Though some notions, for instance the central charge of the stability condition (see section 4), can be understood more naturally in $D^b_{\mathbb{Z}}(\mathcal{A}_f)$, the Orlov's construction of categories requires less terminologies and is easier to understand in a traditional way in algebraic geometry. Therefore, in this paper we shall use the Orlov's construction with a slight modification of the scaling of degrees and denote the modified category by $HMF^{gr}_R(f)$.

Let us explain details of the contents of the present paper. In section 2, we recall the construction of triangulated categories of matrix factorizations. Since we compare the category $HMF^{gr}_R(f)$ with the ungraded version $HMF_{\mathcal{O}}(f)$ in the proof of our main theorem (Theorem 3.1), we first introduce the ungraded version $HMF_{\mathcal{O}}(f)$ corresponding to that given in [O1] in subsection 2.1, and then we define the graded version $HMF^{gr}_R(f)$ based on [O2] in subsection 2.2, where we also explain the relation of the category $HMF^{gr}_R(f)$ with the category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ introduced in [T2]. Section 3 is the main part of the present paper. In subsection 3.1, we state the main theorem (Theorem 3.1): for a polynomial f of type ADE, $HMF^{gr}_R(f)$ is equivalent as a triangulated category to the bounded derived category of modules over the path algebra of the Dynkin quiver of type of f. Subsection 3.2 is devoted to the proof of Theorem 3.1. The proof is based on various explicit data on the matrix factorizations; the complete list of the matrix factorizations (Table 1), their gradings (Table 2) and the complete list of the morphisms in $HMF^{gr}_R(f)$ (Table 3). The tables are arranged in the final section (section 5). In section 4, we construct a stability condition, the notion of which is introduced by Bridgeland [Bd], for the triangulated category $HMF^{gr}_R(f)$. One can see that, as in the A_l case [T2], the phase of objects (see Theorem 3.6 or Table 2) and the central charge $\mathcal{Z}$ (Definition 4.1) can be naturally given by the grading of matrix factorizations in Table 2 (c.f. [W]). They in fact define a stability condition on $HMF^{gr}_R(f)$ (Theorem 4.2), from which an abelian category is obtained as a full subcategory of $HMF^{gr}_R(f)$. In Proposition 4.3, we show that this abelian category is equivalent to an abelian category of modules over the path algebra $\mathbb{C}\vec{\Delta}_{principal}$, where $\vec{\Delta}_{principal}$ is the Dynkin quiver with the orientation being taken to be the principal orientation introduced in [Sa5].

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2. Triangulated categories of matrix factorizations

In this section, we set up several definitions which are used in the present paper. The goal of this section is the introduction of the categories $HMF_A(f)$ and $HMF^{gr}_A(f)$ attached to a weighted homogeneous polynomial $f \in A$ following [O1],[O2] with slight modifications.

2.1. The triangulated category $HMF_A(f)$ of matrix factorizations.

Let A be either the polynomial ring $R := \mathbb{C}[x, y, z]$, the convergent power series ring $\mathcal{O} := \mathbb{C}\{x, y, z\}$ or the formal power series ring $\hat{\mathcal{O}} := \mathbb{C}[[x, y, z]]$ in three variables x, y and z.

Definition 2.1 (Matrix factorization)**.** For a polynomial $f \in A$, a *matrix factorization* M of f is defined by

$$M := \Big(P_0 \overset{p_0}{\underset{p_1}{\rightleftarrows}} P_1 \Big) ,$$

where P_0, P_1 are right *free* A-modules of finite rank, and $p_0 : P_0 \to P_1$, $p_1 : P_1 \to P_0$ are A-homomorphisms such that $p_1 p_0 = f \cdot \mathbf{1}_{P_0}$ and $p_0 p_1 = f \cdot \mathbf{1}_{P_1}$. The set of all matrix factorizations of f is denoted by $MF_A(f)$.

Since $p_0 p_1$ and $p_1 p_0$ are f times the identities, where f is nonzero element of A, the rank of P_0 coincides with that of P_1. We call the rank the *size* of the matrix factorization M.

Definition 2.2 (Homomorphism)**.** Given two matrix factorizations $M := (P_0 \overset{p_0}{\underset{p_1}{\rightleftarrows}} P_1)$ and $M' := (P'_0 \overset{p'_0}{\underset{p'_1}{\rightleftarrows}} P'_1)$, a *homomorphism* $\Phi : M \to M'$ is a pair of A-homomorphisms $\Phi = (\phi_0, \phi_1)$

$$\phi_0 : P_0 \to P'_0 , \qquad \phi_1 : P_1 \to P'_1 ,$$

such that the following diagram commutes:

$$\begin{array}{ccccc} P_0 & \xrightarrow{p_0} & P_1 & \xrightarrow{p_1} & P_0 \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} \\ P'_0 & \xrightarrow{p'_0} & P'_1 & \xrightarrow{p'_1} & P'_0 \end{array} .$$

The set of all homomorphisms from M to M', denoted by $\mathrm{Hom}_{MF_A(f)}(M, M')$, is naturally an A-module and is finitely generated, since the sizes of the matrix factorizations are finite. For three matrix factorizations M, M', M'' and homomorphisms $\Phi : M \to M'$ and $\Phi' : M' \to M''$, the composition $\Phi'\Phi$ is defined by

$$\Phi'\Phi = (\phi'_0 \phi_0, \phi'_1 \phi_1) .$$

This composition is associative: $\Phi''(\Phi'\Phi) = (\Phi''\Phi')\Phi$ for any three homomorphisms.

Definition 2.3 ($HMF_A(f)$)**.** An additive category $HMF_A(f)$ is defined by the following data. The set of objects is given by the set of all matrix factorizations:

$$\mathrm{Ob}(HMF_A(f)) := MF_A(f) .$$

For any two objects $M, M' \in MF_A(f)$, the set of morphisms is given by the quotient module:

$$\mathrm{Hom}_{HMF_A(f)}(M, M') = \mathrm{Hom}_{MF_A(f)}(M, M')/\sim ,$$

where two elements Φ, Φ' in $\mathrm{Hom}_{MF_A(f)}(M, M')$ are equivalent (homotopic) $\Phi \sim \Phi'$ if there exists a *homotopy* (h_0, h_1), *i.e.*, a pair $(h_0, h_1) : (P_0 \to P'_1, P_1 \to P'_0)$ of A-homomorphisms such that $\Phi' - \Phi = (p_1' h_0 + h_1 p_0, p_0' h_1 + h_0 p_1)$. The composition of morphisms on $\mathrm{Hom}_{HMF_A(f)}$ is induced from that on $\mathrm{Hom}_{MF_A(f)}$ since $\Phi \sim \Phi'$ and $\Psi \sim \Psi'$ imply $\Psi\Phi \sim \Psi'\Phi'$.

Note that the matrix factorization $M = (P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1) \in MF_A(f)$ of size one with $(p_0, p_1) = (1, f)$ or $(p_0, p_1) = (f, 1)$ defines the zero object in $HMF_A(f)$, that is: one has $\mathrm{Hom}_{HMF_A(f)}(M, M') = \mathrm{Hom}_{HMF_A(f)}(M', M) = 0$ for any matrix factorization $M' \in MF_A(f)$ or, equivalently, $\mathbf{1}_M \in \mathrm{Hom}_{MF_A(f)}(M, M)$ is homotopic to zero.

Lemma 2.4. *For any two matrix factorizations $M, M' \in HMF_A(f)$, the space of morphisms $\mathrm{Hom}_{HMF_A(f)}(M, M')$ is a finitely generated $A/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$-module.*

Proof. Since $\mathrm{Hom}_{MF_A(f)}(M, M')$ is a finitely generated A-module and the equivalence relation $\sim$ is given by quotienting out by an A-submodule, $\mathrm{Hom}_{HMF_A(f)}(M, M')$ is also a finitely generated A-module. On the other hand, the Jacobi ideal $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ annihilates $\mathrm{Hom}_{HMF_A(f)}(M, M')$: that is, $\frac{\partial f}{\partial x}(\phi_0, \phi_1) \sim \frac{\partial f}{\partial y}(\phi_0, \phi_1) \sim \frac{\partial f}{\partial z}(\phi_0, \phi_1) \sim 0$ for any morphism $\Phi = (\phi_0, \phi_1)$. This can be shown for instance by differentiating $p_1 p_0 = f \cdot \mathbf{1}_{P_0}$ and $p_0 p_1 = f \cdot \mathbf{1}_{P_1}$ by $\frac{\partial}{\partial x}$. Then we have two identities $\frac{\partial p_1}{\partial x} p_0 + p_1 \frac{\partial p_0}{\partial x} = \frac{\partial f}{\partial x} \cdot \mathbf{1}_{P_0}$ and $\frac{\partial p_0}{\partial x} p_1 + p_0 \frac{\partial p_1}{\partial x} = \frac{\partial f}{\partial x} \cdot \mathbf{1}_{P_1}$. Multiplying ϕ_0 and ϕ_1 by these two identities, respectively, leads to $\frac{\partial f}{\partial x}(\phi_0, \phi_1) \sim 0$, where $(\phi_1 \frac{\partial p_0}{\partial x}, \phi_0 \frac{\partial p_1}{\partial x})$ is the corresponding homotopy. In a similar way one can obtain $\frac{\partial f}{\partial y}(\phi_0, \phi_1) \sim \frac{\partial f}{\partial z}(\phi_0, \phi_1) \sim 0$. □

Definition 2.5 (Shift functor)**.** The *shift functor* $T : HMF_A(f) \to HMF_A(f)$ is defined as follows. The action of T on $M = (P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1) \in HMF_A(f)$ is given by

$$T\Big(P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1 \Big) := \Big(P_1 \underset{-p_0}{\overset{-p_1}{\rightleftarrows}} P_0 \Big) .$$

For any $M, M' \in HMF_A(f)$, the action of T on $\Phi = (\phi_0, \phi_1) \in \mathrm{Hom}_{HMF_A(f)}(M, M')$ is given by

$$T(\phi_0, \phi_1) := (\phi_1, \phi_0) .$$

Note that the square T^2 of the shift functor is isomorphic to the identity functor on $HMF_A(f)$.

Definition 2.6 (Mapping cone)**.** For an element $\Phi = (\phi_0, \phi_1) \in \mathrm{Hom}_{MF_A(f)}(M, M')$, the *mapping cone* $C(\Phi) \in MF_A(f)$ is defined by

$$C(\Phi) := \left(C_0 \underset{c_1}{\overset{c_0}{\rightleftarrows}} C_1 \right), \quad \text{where}$$

$$C_0 := P_1 \oplus P_0', \qquad C_1 := P_0 \oplus P_1', \qquad c_0 = \begin{pmatrix} -p_1 & 0 \\ \phi_1 & p_0' \end{pmatrix}, \qquad c_1 = \begin{pmatrix} -p_0 & 0 \\ \phi_0 & p_1' \end{pmatrix}.$$

The following is stated in [O1] Proposition 3.3.

Proposition 2.7. *The additive category $HMF_A(f)$ endowed with the shift functor T and the distinguished triangles forms a triangulated category, where a distinguished triangle is a sequence of morphisms which is isomorphic to the sequence*

$$M \xrightarrow{\Phi} M' \to C(\Phi) \to T(M)$$

for some $M, M' \in MF_A(f)$ and $\Phi \in \mathrm{Hom}_{MF_A(f)}(M, M')$.

Proof. The proof is the same as the proof of the analogous result for a usual homotopic category (see e.g. [GM], [KS]). □

2.2. The triangulated category $HMF^{gr}_R(f)$ of graded matrix factorizations.

In this subsection, we study graded matrix factorizations for a weighted homogeneous polynomial f and construct the corresponding triangulated category, denoted by $HMF^{gr}_R(f)$.

A quadruple $W := (a, b, c; h)$ of positive integers with g.c.d$(a, b, c) = 1$ is called a *weight system*. For a weight system W, we define the *Euler vector field* $E = E_W$ by

$$E := \frac{a}{h} x \frac{\partial}{\partial x} + \frac{b}{h} y \frac{\partial}{\partial y} + \frac{c}{h} z \frac{\partial}{\partial z}.$$

For a given weight system W, R becomes a graded ring by putting $\deg(x) = \frac{2a}{h}$, $\deg(y) = \frac{2b}{h}$ and $\deg(z) = \frac{2c}{h}$. Let $R = \oplus_{d \in \frac{2}{h}\mathbb{Z}_{\geq 0}} R_d$ be the graded piece decomposition, where $R_d := \{f \in R \mid 2Ef = df\}$. A weight system W is called *regular* ([Sa1]) if the following equivalent conditions are satisfied:

(a) $\chi_W(T) := T^{-h} \frac{(T^h - T^a)(T^h - T^b)(T^h - T^c)}{(T^a - 1)(T^b - 1)(T^c - 1)}$ has no poles except at $T = 0$.

(b) A generic element of the eigenspace $R_2 = \{f \in R \mid Ef = f\}$ has an isolated critical point at the origin, *i.e.*, the Jacobi ring $R/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is finite dimensional over $\mathbb{C}$.

Such an element f of R_2 as in (b) shall be called a *polynomial of type* W.

In the present paper, *by a graded module, we mean a graded right module with degrees only in* $\frac{2}{h}\mathbb{Z}$. Namely, a graded R-module $\widetilde{P}$ decomposes into the direct sum:

$$\widetilde{P} = \oplus_{d \in \frac{2}{h}\mathbb{Z}} \widetilde{P}_d\,. \tag{2.1}$$

For two graded R-modules $\widetilde{P}$ and $\widetilde{P}'$, a graded R-homomorphism ϕ of degree $s \in \frac{2}{h}\mathbb{Z}$ is an R-homomorphism $\phi : \widetilde{P} \to \widetilde{P}'$ such that $\phi(\widetilde{P}_d) \subset \widetilde{P}'_{d+s}$ for any d. The category of graded R-modules has a degree shifting automorphism τ defined by [1]

$$(\tau(\widetilde{P}))_d = \widetilde{P}_{d+\frac{2}{h}} \ .$$

For any two graded R-modules $\widetilde{P}$, $\widetilde{P}'$ and a graded R-homomorphism $\phi : \widetilde{P} \to \widetilde{P}'$, we denote the induced graded R-homomorphism by $\tau(\phi) : \tau(\widetilde{P}) \to \tau(\widetilde{P}')$. On the other hand, an R-homomorphism $\phi : \widetilde{P} \to \widetilde{P}'$ of degree $\frac{2m}{h}$ induces a degree zero R-homomorphism from $\widetilde{P}$ to $\tau^m(\widetilde{P}')$, which we denote again by $\phi : \widetilde{P} \to \tau^m(\widetilde{P}')$.

Definition 2.8 (Graded matrix factorization)**.** For a polynomial f of type W, a *graded matrix factorization* $\widetilde{M}$ of $f \in R$ is defined by

$$\widetilde{M} := \left(\widetilde{P}_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} \widetilde{P}_1 \right) ,$$

where $\widetilde{P}_0$, $\widetilde{P}_1$ are free graded right R-modules of finite rank, $p_0 : \widetilde{P}_0 \to \widetilde{P}_1$ is a graded R-homomorphism of degree zero, $p_1 : \widetilde{P}_1 \to \widetilde{P}_0$ is a graded R-homomorphism of degree two such that $p_1 p_0 = f \cdot \mathbf{1}_{\widetilde{P}_0}$ and $p_0 p_1 = f \cdot \mathbf{1}_{\widetilde{P}_1}$. The set of all graded matrix factorizations of f is denoted by $MF_R^{gr}(f)$.

Definition 2.9 (Homomorphism)**.** Given two graded matrix factorizations $\widetilde{M}, \widetilde{M}' \in MF_R^{gr}(f)$, a *homomorphism* $\Phi = (\phi_0, \phi_1) : \widetilde{M} \to \widetilde{M}'$ is a homomorphism in the sense of Definition 2.2 such that ϕ_0 and ϕ_1 are graded R-homomorphisms of *degree zero*. The vector space of all graded R-homomorphisms from $\widetilde{M}$ to $\widetilde{M}'$ is denoted by $\mathrm{Hom}_{MF_R^{gr}(f)}(\widetilde{M}, \widetilde{M}')$.

For three graded matrix factorizations $\widetilde{M}, \widetilde{M}', \widetilde{M}'' \in MF_R^{gr}(f)$ and morphisms $\Phi : \widetilde{M} \to \widetilde{M}'$, $\Phi' : \widetilde{M}' \to \widetilde{M}''$, the composition is again a graded R-homomorphism: $\Phi'\Phi \in \mathrm{Hom}_{MF_R^{gr}(f)}(\widetilde{M}, \widetilde{M}'')$.

Definition 2.10 ($HMF_R^{gr}(f)$)**.** An additive category $HMF_R^{gr}(f)$ of graded matrix factorizations is defined by the following data. The set of objects is given by the set of all graded matrix factorizations:

$$\mathrm{Ob}(HMF_R^{gr}(f)) := MF_R^{gr}(f) \ .$$

For any two objects $\widetilde{M}, \widetilde{M}' \in MF_R^{gr}(f)$, the set of morphisms is given by

$$\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}, \widetilde{M}') := \mathrm{Hom}_{MF_R^{gr}(f)}(\widetilde{M}, \widetilde{M}')/\sim \ ,$$

where two elements Φ, Φ' in $\mathrm{Hom}_{MF_R^{gr}(f)}(\widetilde{M}, \widetilde{M}')$ are equivalent $\Phi \sim \Phi'$ if there exists a *homotopy*, *i.e.*, a pair $(h_0, h_1) : (\widetilde{P}_0 \to \widetilde{P}'_1, \widetilde{P}_1 \to \widetilde{P}'_0)$ of graded R-homomorphisms such that

[1] This τ is what is often denoted (for instance [Y],[O2]) by (1), *i.e.* , $\tau(\widetilde{P}) = \widetilde{P}(1)$.

h_0 is of degree minus two, h_1 is of degree zero and $\Phi'-\Phi = (\tau^{-h}(p_1')h_0+h_1p_0, p_0'h_1+\tau^h(h_0)p_1)$. The composition of morphisms is induced from that on $\mathrm{Hom}_{MF_R^{gr}(f)}(\widetilde{M},\widetilde{M}')$.

A graded matrix factorization is the zero object in $HMF_R^{gr}(f)$ if and only if it is a direct sum of the graded matrix factorizations of the forms $(\ \tau^n(R) \underset{f}{\overset{1}{\rightleftarrows}} \tau^n(R)\ ) \in MF_R^{gr}(f)$ and $(\ \tau^{n'}(R) \underset{1}{\overset{f}{\rightleftarrows}} \tau^{n'+h}(R)\ ) \in MF_R^{gr}(f)$ for some $n, n' \in \mathbb{Z}$.

Lemma 2.11. *The category $HMF_R^{gr}(f)$ is Krull-Schmidt, that is,*
(a) for any two objects $\widetilde{M},\widetilde{M}' \in HMF_R^{gr}(f)$, $\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M},\widetilde{M}')$ is finite dimensional;
(b) for any object $\widetilde{M} \in HMF_R^{gr}(f)$ and any idempotent $e \in \mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M},\widetilde{M})$, there exists a matrix factorization $\widetilde{M}' \in HMF_R^{gr}(f)$ and a pair of morphisms $\Phi \in \mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M},\widetilde{M}')$, $\Phi' \in \mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}',\widetilde{M})$ such that $e = \Phi'\Phi$ and $\Phi\Phi' = \mathrm{Id}_{\widetilde{M}'}$.

Proof. (a) Due to Lemma 2.4, $\oplus_{n\in\mathbb{Z}}\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M},\tau^n(\widetilde{M}'))$ is a finitely generated graded $R/\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$-module. Since the Jacobi ring $R/\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$ is finite dimensional, the space $\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M},\widetilde{M}')$ is finite dimensional over $\mathbb{C}$.

(b) Let R_+ be the maximal ideal of R of all positive degree elements. Note that any graded matrix factorization is isomorphic in $HMF_R^{gr}(f)$ to a graded matrix factorization whose entries belong to $\tau^n(R_+)$ for some $n \in \mathbb{Z}$. Thus, we may assume that $\widetilde{M} := (\ \widetilde{P}_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} \widetilde{P}_1\ ) \in HMF_R^{gr}(f)$ is such a graded matrix factorization. Suppose that $\widetilde{M}$ has an idempotent $e \in \mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M},\widetilde{M})$, $e^2 = e$. This implies that there exists $\hat{e} \in \mathrm{Hom}_{MF_R^{gr}(f)}(\widetilde{M},\widetilde{M})$ such that

$$\hat{e}^2 - \hat{e} = (\tau^{-h}(p_1)h_0 + h_1p_0, p_0h_1 + \tau^h(h_0)p_1) \tag{2.2}$$

for some homotopy (h_0, h_1) on $\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M},\widetilde{M})$. However, since each entry of p_0 and p_1 belongs to $\tau^n(R_+)$, each entry in the right hand side also belongs to $\tau^n(R_+)$. Let π : $\mathrm{Hom}_{MF_R^{gr}(f)}(\widetilde{M},\widetilde{M}) \to \mathrm{Hom}_{MF_R^{gr}(f)}(\widetilde{M},\widetilde{M})$ be the canonical projection given by restricting each entry on $R/R_+ = \mathbb{C}$. Then, eq.(2.2) in fact implies that $\pi(\hat{e})^2 - \pi(\hat{e}) = 0$. Thus, for $\pi(\hat{e}) =: (\hat{e}_0, \hat{e}_1)$, defining a matrix factorization $\widetilde{M}' \in HMF_R^{gr}(f)$ by

$$\widetilde{M}' := \left(\ \hat{e}_0\widetilde{P}_0 \underset{\hat{e}_0p_1\hat{e}_1}{\overset{\hat{e}_1p_0\hat{e}_0}{\rightleftarrows}} \hat{e}_1\widetilde{P}_1\ \right),$$

one obtains a pair of morphisms $\Phi \in \mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M},\widetilde{M}')$ and $\Phi' \in \mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}',\widetilde{M})$ such that $e = \Phi'\Phi$ and $\Phi\Phi' = \mathrm{Id}_{\widetilde{M}'}$. □

One can see that τ induces an automorphism on $HMF_R^{gr}(f)$, which we shall denote by the same notation $\tau : HMF_R^{gr}(f) \to HMF_R^{gr}(f)$. Explicitly, the action of τ on $\widetilde{M} =$

$(\widetilde{P}_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} \widetilde{P}_1) \in HMF^{gr}_R(f)$ is given by

$$\tau\Big(\widetilde{P}_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} \widetilde{P}_1 \Big) := \Big(\tau(\widetilde{P}_0) \underset{\tau(p_1)}{\overset{\tau(p_0)}{\rightleftarrows}} \tau(\widetilde{P}_1) \Big) .$$

The action of τ on morphisms are naturally induced from that on graded R-homomorphisms between two graded right R-modules.

Also, we have the shift functor T on $HMF_R(f)$, the graded version of that in Definition 2.5. [2]

Definition 2.12 (Shift functor on $HMF^{gr}_R(f)$)**.** The *shift functor* $T : HMF^{gr}_R(f) \to HMF^{gr}_R(f)$ is defined as follows. The action of T on $\widetilde{M} \in HMF^{gr}_R(f)$ is given by

$$T\Big(\widetilde{P}_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} \widetilde{P}_1 \Big) = \Big(\widetilde{P}_1 \underset{-\tau^h(p_0)}{\overset{-p_1}{\rightleftarrows}} \tau^h(\widetilde{P}_0) \Big) .$$

For any $\widetilde{M}, \widetilde{M}' \in HMF^{gr}_R(f)$, the action of T on $\Phi = (\phi_0, \phi_1) \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}, \widetilde{M}')$ is given by

$$T(\phi_0, \phi_1) = (\phi_1, \tau^h(\phi_0)) .$$

We remark that the square T^2 of the shift functor is not isomorphic to the identity functor on $HMF^{gr}_R(f)$ but $T^2 = \tau^h$.

Definition 2.13 (Mapping cone)**.** For an element $\Phi = (\phi_0, \phi_1) \in \mathrm{Hom}_{MF^{gr}_R(f)}(\widetilde{M}, \widetilde{M}')$, the *mapping cone* $C(\Phi) \in MF^{gr}_R(f)$ is defined by

$$C(\Phi) := \Big(C_0 \underset{c_1}{\overset{c_0}{\rightleftarrows}} C_1 \Big) , \text{where}$$

$$C_0 := \widetilde{P}_1 \oplus \widetilde{P}'_0 , \qquad C_1 := \tau^h(\widetilde{P}_0) \oplus \widetilde{P}'_1 , \qquad c_0 = \begin{pmatrix} -p_1 & 0 \\ \phi_1 & p'_0 \end{pmatrix} , \qquad c_1 = \begin{pmatrix} -\tau^h(p_0) & 0 \\ \tau^h(\phi_0) & p'_1 \end{pmatrix} .$$

This mapping cone is well-defined. In fact, one can see that the degree of c_0 and c_1 are zero and two, since the graded R-homomorphisms $p_1 : \widetilde{P}_1 \to \widetilde{P}_0$ of degree two and $p_0 : \widetilde{P}_0 \to \widetilde{P}_1$ of degree zero induce graded R-homomorphisms $-p_1 : \widetilde{P}_1 \to \tau^h(\widetilde{P}_0)$ of degree zero and $-\tau^h(p_0) : \tau^h(\widetilde{P}_0) \to \widetilde{P}_1$ of degree two, respectively.

The following is stated in [O2] Proposition 3.4.

[2]The shift functor T is often denoted by [1].

Theorem 2.14. *The additive category $HMF^{gr}_R(f)$ endowed with the shift functor T and the distinguished triangles forms a triangulated category, where a distinguished triangle is defined by a sequence isomorphic to the sequence*

$$\widetilde{M} \xrightarrow{\Phi} \widetilde{M}' \to C(\Phi) \to T(\widetilde{M})$$

for some $\widetilde{M}, \widetilde{M}' \in MF^{gr}_R(f)$ and $\Phi \in \mathrm{Hom}_{MF^{gr}_R(f)}(\widetilde{M}, \widetilde{M}')$.

Proof. As in the case for $HMF_A(f)$, the proof is the same as the proof of the analogous result for a usual homotopic category (see e.g. [GM], [KS]). □

Let $\widetilde{M} = (\widetilde{P}_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} \widetilde{P}_1) \in HMF^{gr}_R(f)$ be a graded matrix factorization of size r. One can choose homogeneous free basis $(b_1, \cdots, b_r; \bar{b}_1, \cdots, \bar{b}_r)$ such that $\widetilde{P}_0 = b_1 R \oplus \cdots \oplus b_r R$ and $\widetilde{P}_1 = \bar{b}_1 R \oplus \cdots \oplus \bar{b}_r R$. Then, the graded matrix factorization $\widetilde{M}$ is expressed as a $2r$ by $2r$ matrix

$$Q = \begin{pmatrix} & \varphi \\ \psi & \end{pmatrix} , \qquad \varphi, \psi \in \mathrm{Mat}_r(R) \tag{2.3}$$

satisfying

$$Q^2 = f \cdot \mathbf{1}_{2r} , \qquad -SQ + QS + 2EQ = Q , \tag{2.4}$$

where S is the diagonal matrix of the form $S := \mathrm{diag}(s_1, \cdots, s_r; \bar{s}_1, \cdots, \bar{s}_r)$ such that $s_i = \deg(b_i)$ and $\bar{s}_i = \deg(\bar{b}_i) - 1$ for $i = 1, 2, \cdots, r$. We call this S a *grading matrix* of Q.

This procedure $\widetilde{M} \mapsto (Q, S)$ gives the equivalence between the triangulated category $HMF^{gr}_R(f)$ and the triangulated category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ in [T2]. This implies that $HMF^{gr}_R(f)$ is an enhanced triangulated category in the sense of Bondal-Kapranov [BK].

We shall represent the matrix factorization $\widetilde{M} = (\widetilde{P}_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} \widetilde{P}_1)$ by (Q, S).

3. $HMF^{gr}_R(f)$ for type ADE and representations of Dynkin quivers

In this section, we formulate the main theorem (Theorem 3.1) of the present paper in subsection 3.1. The proof of the theorem is given in subsection 3.2.

3.1. Statement of the main theorem (Theorem 3.1).

The main theorem states an equivalence between the triangulated category $HMF^{gr}_R(f)$ for a polynomial $f \in R$ of type ADE with the derived category of modules over a *path algebra* of a *Dynkin quiver*. In order to formulate the results, we recall (i) the weighted homogeneous polynomials of type ADE and (ii) the notion of the path algebras of the Dynkin quivers.

(i) **ADE polynomials.** For a regular weight system W, we have the following facts [Sa1].

(a) There exist integers $m_1, \cdots, m_l$, called the *exponents* of W, such that $\chi_W(T) = T^{m_1} + \cdots + T^{m_l}$, where the smallest exponent is given by $\epsilon := a+b+c-h$.

(b) The regular weight systems with $\epsilon > 0$ are listed as follows.

$$\begin{aligned} &A_l : (1, b, l+1-b; l+1)\ , 1 \le b \le l\ , \quad D_l : (l-2, 2, l-1; 2(l-1))\ , \\ &E_6 : (4,3,6;12)\ , \quad E_7 : (6,4,9;18)\ , \quad E_8 : (10,6,15;30)\ . \end{aligned} \tag{3.1}$$

Here, the naming in the left hand side is given according to the identifications of the exponents of the weight systems with those of the simple Lie algebras. As a consequence, one obtains $\epsilon = 1$ for all regular weight system with $\epsilon > 0$. For the polynomials of type ADE, without loss of generality we may choose the followings:

$$f(x,y,z) = \begin{cases} x^{l+1} + yz, & h = l+1, & A_l\ (l \ge 1), \\ x^2y + y^{l-1} + z^2, & h = 2(l-1), & D_l\ (l \ge 4), \\ x^3 + y^4 + z^2, & h = 12, & E_6, \\ x^3 + xy^3 + z^2, & h = 18, & E_7, \\ x^3 + y^5 + z^2, & h = 30, & E_8\ . \end{cases}$$

We denote by $X(f)$ the type of f.

(ii) **Path algebras.** (a) The *path algebra* $\mathbb{C}\vec{\Delta}$ of a *quiver* is defined as follows (see [Ga], [R] and [Ha] Chapter 1, 5.1). A *quiver* $\vec{\Delta}$ is a pair (Δ_0, Δ_1) of the set Δ_0 of vertices and the set Δ_1 of arrows (oriented edges). Any arrow in Δ_1 has a starting point and end point in Δ_0. A path of *length* $r \ge 1$ from a vertex v to a vertex v' in a quiver $\vec{\Delta}$ is of the form $(v|\alpha_1, \cdots, \alpha_r|v')$ with arrows $\alpha_i \in \Delta_1$ satisfying the starting point of α_1 is v, the end point of α_i is equal to the starting point of α_{i+1} for all $1 \le i \le r-1$, and the end point of α_r is v'. In addition, we also define a path of length zero $(v|v)$ for any vertex v in $\vec{\Delta}$. The *path algebra* $\mathbb{C}\vec{\Delta}$ of a quiver $\vec{\Delta}$ is then the $\mathbb{C}$-vector space with basis the set of all paths in $\vec{\Delta}$. The product structure is defined by the composition of paths, where the product of two non-composable paths is set to be zero.

The category of finitely generated right modules over the path algebra $\mathbb{C}\vec{\Delta}$ is denoted by mod-$\mathbb{C}\vec{\Delta}$. It is an abelian category, and its derived category is denoted by $D^b(\text{mod-}\mathbb{C}\vec{\Delta})$.

If Δ_0 and Δ_1 are finite sets and Δ does not have any oriented cycle, then $D^b(\text{mod-}\mathbb{C}\vec{\Delta})$ is a Krull-Schmidt category.

(b) A *Dynkin quiver* $\vec{\Delta}_X$ of type X of ADE is a Dynkin diagram Δ_X of type X listed in Figure 1 together with an orientation for each edge of the Dynkin diagram. It is known [Ga] that the number of all the isomorphism classes of the indecomposable objects of the abelian category $\text{mod-}\mathbb{C}\vec{\Delta}$ of a quiver $\vec{\Delta}$ is finite if and only if the quiver $\vec{\Delta}$ is a Dynkin quiver (of type ADE).

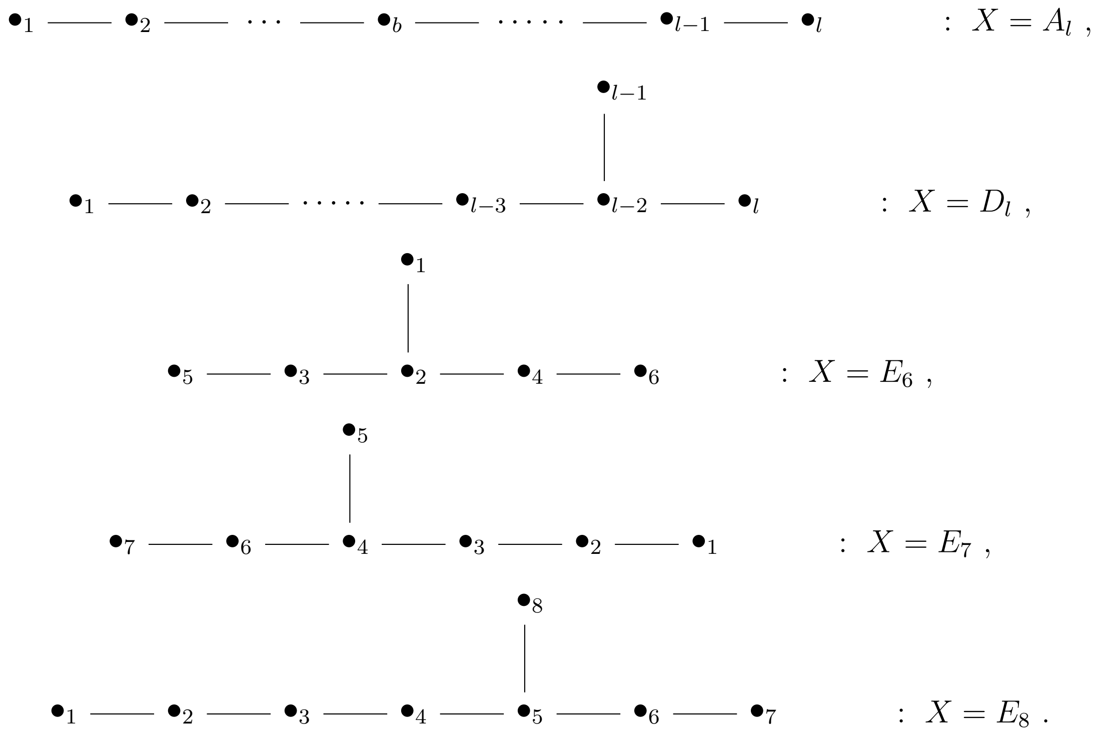

FIGURE 1. ADE Dynkin diagram

For a Dynkin diagram Δ_X, we shall denote by Π_X the set of vertices. For later convenience, the elements of Π_X are labeled by the integers $\{1, \cdots, l\}$ as in the above figures, and we shall sometimes confuse vertices in Π_X with labels in $\{1, \cdots, l\}$.

The following is the main theorem of the present paper.

Theorem 3.1. *Let $f(x, y, z) \in \mathbb{C}[x, y, z]$ be a polynomial of type $X(f)$, and let $\vec{\Delta}_{X(f)}$ be a Dynkin quiver of type $X(f)$ with a fixed orientation. Then, we have the following equivalence of the triangulated categories*

$$HMF^{gr}_R(f) \simeq D^b(\text{mod-}\mathbb{C}\vec{\Delta}_{X(f)}) \ .$$

3.2. **The proof of Theorem 3.1.**

The construction of the proof of Theorem 3.1 is as follows.

Step 1. We describe the Auslander-Reiten (AR-)quiver for the triangulated category $HMF_{\hat{\mathcal{O}}}(f)$ of matrix factorizations due to [E], [AR2] and [A] and give the matrix factorizations explicitly.

Step 2. We determine the structure of the triangulated category $HMF^{gr}_R(f)$ of graded matrix factorizations (Theorem 3.6).

Step 3. By comparing the AR-quiver of the category $HMF_{\hat{\mathcal{O}}}(f)$ with the category $HMF^{gr}_R(f)$ we find the exceptional collections corresponding to $\vec{\Delta}_{X(f)}$ in $HMF^{gr}_R(f)$ and complete the proof of the main theorem (Theorem 3.1).

Step 1. The Auslander-Reiten quiver for $HMF_{\hat{\mathcal{O}}}(f)$.

We recall the known results on the equivalence of the McKay quiver for Kleinean group and the AR-quiver for the simple singularities [AR2], [A] and [E].

For a Krull-Schmidt category $\mathcal{C}$ over $\mathbb{C}$, an object $X \in \mathrm{Ob}(\mathcal{C})$ is called *indecomposable* if any idempotent $e \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ is zero or the identity Id_X. For two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, denote by $\mathrm{rad}_{\mathcal{C}}(X,Y)$ the subspace of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of non-invertible morphisms from X to Y. We denote by $\mathrm{rad}^2_{\mathcal{C}}(X,Y) \subset \mathrm{rad}_{\mathcal{C}}(X,Y)$ the space of morphisms each of which is described as a composition $\Phi'\Phi$ with $\Phi \in \mathrm{rad}_{\mathcal{C}}(X,Z)$, $\Phi' \in \mathrm{rad}_{\mathcal{C}}(Z,Y)$ for some object $Z \in \mathrm{Ob}(\mathcal{C})$. For two indecomposable objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, an element in $\mathrm{rad}_{\mathcal{C}}(X,Y)\backslash\mathrm{rad}^2_{\mathcal{C}}(X,Y)$ is called an *irreducible morphism*. The space $\mathrm{Irr}_{\mathcal{C}}(X,Y) := \mathrm{rad}_{\mathcal{C}}(X,Y)/\mathrm{rad}^2_{\mathcal{C}}(X,Y)$ in fact forms a subvector space of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. We call by the *AR-quiver* $\Gamma(\mathcal{C})$ of a Krull-Schmidt category $\mathcal{C}$ the quiver $\Gamma(\mathcal{C}) := (\Gamma_0, \Gamma_1)$ whose vertex set Γ_0 consists of the isomorphism classes $[X]$ of the indecomposable objects $X \in \mathrm{Ob}(\mathcal{C})$ and whose arrow set Γ_1 consists of $\dim_{\mathbb{C}}(\mathrm{Irr}_{\mathcal{C}}(X,Y))$ arrows from $[X] \in \Gamma_0$ to $[Y] \in \Gamma_0$ for any $[X], [Y] \in \Gamma_0$ (see [R],[Ha],[Y]).

On the other hand, for a Dynkin diagram Δ_X of type X listed in Figure 1, we define a quiver consisting of the vertex set Π_X and arrows in both directions $k \rightleftarrows k'$ for each edge of Δ_X between vertices $k, k' \in \Pi_X$. The resulting quiver is denoted by $\overset{\leftrightarrow}{\Delta}_X$.

Note that the category $HMF_{\hat{\mathcal{O}}}(f)$ is Krull-Schmidt (see [Y], Proposition 1.18).

Theorem 3.2. *Let f be a polynomial f of type ADE, which we regard as an element of $\hat{\mathcal{O}}$.*

(i) ([AR2],[A],[E]) *The AR-quiver of the category $HMF_{\hat{\mathcal{O}}}(f)$ is isomorphic to the quiver $\overset{\leftrightarrow}{\Delta}_X$ of type $X(f)$ corresponding to f :*

$$\Gamma(HMF_{\hat{\mathcal{O}}}(f)) \simeq \overset{\leftrightarrow}{\Delta}_{X(f)} \ .$$

(ii) *According to (i), fix an identification $\Pi_{X(f)} \simeq \Gamma_0(HMF_{\hat{\mathcal{O}}}(f))$, $k \mapsto [M^k]$. A representative M^k of the isomorphism classes $[M^k]$ of the indecomposable matrix factorizations of* *minimum size is given explicitly in Table 1. The size of* M^k *is* $2\nu_k$*, where* ν_k *is the coefficient of the highest root for* $k \in \Pi_{X(f)}$.

Proof. (i) This statement follows from the combination of results of [AR2] and [E], where the Auslander-Reiten quivers of the categories of the maximal Cohen-Macaulay modules over $\hat{\mathcal{O}}/(f)$ for type ADE are determined in [AR2], and the equivalence of the category of Maximal Cohen-Macaulay modules with the category $HMF_{\hat{\mathcal{O}}}(f)$ of the matrix factorizations is given in [E].

(ii) For each type $X(f)$, since we have $\sharp(\Pi_{X(f)})$ non-isomorphic matrix factorizations (Table 1), these actually complete all the vertices of the AR-quiver $\Gamma(HMF_{\hat{\mathcal{O}}}(f))$. □

Remark 3.3. In [Y], matrix factorizations for a polynomial of type ADE in two variables x, y are listed up completely. On the other hand, for type A_l and D_l in both two and three variables, all the matrix factorizations and the AR-quivers are presented in [Sc], where the relation of the results in two variables and those in three variables is given. This gives a method of finding the matrix factorizations of a polynomial of type E_l, $l = 6, 7, 8$, in three variables from the ones in two variables case [Y]. For a recent paper in physics, see also [KL3].

Hereafter we fix an identification of $\Gamma(HMF_{\hat{\mathcal{O}}}(f))$ with $\overset{\leftrightarrow}{\Delta}_{X(f)}$ by $k \leftrightarrow [M^k]$.

Step 2. Indecomposable objects in $HMF^{gr}_R(f)$

Recall that one has the inclusion $R \hookrightarrow \mathcal{O} \subset \hat{\mathcal{O}}$. We prepare some definitions for any fixed weighted homogeneous polynomial $f \in R$.

Definition 3.4 (Forgetful functor from $HMF^{gr}_R(f)$ to $HMF_{\hat{\mathcal{O}}}(f)$)**.** For a fixed weighted homogeneous polynomial $f \in R$, there exists a functor $F : HMF^{gr}_R(f) \to HMF_{\hat{\mathcal{O}}}(f)$ given by $F(\widetilde{M}) := \widetilde{M} \otimes_R \hat{\mathcal{O}}$ for $\widetilde{M} \in HMF^{gr}_R(f)$ and the naturally induced homomorphism $F : \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}, \widetilde{M'}) \to \mathrm{Hom}_{HMF_{\hat{\mathcal{O}}}(f)}(F(\widetilde{M}), F(\widetilde{M'}))$ for any two objects $\widetilde{M}, \widetilde{M'} \in HMF^{gr}_R(f)$. We call this F the *forgetful functor*.

It is known that $F : HMF^{gr}_R(f) \to HMF_{\hat{\mathcal{O}}}(f)$ brings an indecomposable object to an indecomposable object ([Y], Lemma 15.2.1).

Let us introduce the notion of *distance* between two indecomposable objects in $HMF_{\hat{\mathcal{O}}}(f)$ and in $HMF^{gr}_R(f)$ as follows.

Definition 3.5 (Distance)**.** For any two indecomposable objects $M, M' \in HMF_{\hat{\mathcal{O}}}(f)$, define the *distance* $d(M, M') \in \mathbb{Z}_{\geq 0}$ from M to M' by the minimal length of the paths from $[M]$ to $[M']$ in the AR-quiver $\Gamma(HMF_{\hat{\mathcal{O}}}(f))$ of $HMF_{\hat{\mathcal{O}}}(f)$. In particular, we have $d(M, M') = 0$ if and only if $M \simeq M'$ in $HMF_{\hat{\mathcal{O}}}(f)$.

For any two indecomposable objects $\widetilde{M}, \widetilde{M}' \in HMF_R^{gr}(f)$, the *distance* $d(\widetilde{M}, \widetilde{M}') \in \mathbb{Z}_{\geq 0}$ from $\widetilde{M}$ to $\widetilde{M}'$ is defined by

$$d(\widetilde{M}, \widetilde{M}') := d(F(\widetilde{M}), F(\widetilde{M}')) \ .$$

By definition, an irreducible morphism exists in $\mathrm{Hom}_{HMF_{\hat{\mathcal{O}}}(f)}(M, M')$ if and only if $d(M, M') = 1$.

Let us return to the case that f is of type ADE. In this case, the distance is in fact symmetric: $d(M^k, M^{k'}) = d(M^{k'}, M^k)$ for any $[M^k], [M^{k'}] \in \Pi$, since, due to Theorem 3.2, there exists an arrow from $[M^k]$ to $[M^{k'}]$ if and only if there exists an arrow from $[M^{k'}]$ to $[M^k]$. For two indecomposable objects $\widetilde{M}^k, \widetilde{M}^{k'} \in HMF_R^{gr}(f)$ such that $F(\widetilde{M}^k) = M^k$, $F(\widetilde{M}^{k'}) = M^{k'}$, we denote $d(\widetilde{M}^k, \widetilde{M}^{k'}) = d(M^k, M^{k'}) := d(k, k')$. Since we fix a polynomial $f \in R$ of type ADE and then the corresponding type $X(f)$ of the Dynkin quiver $\vec{\Delta}_{X(f)}$, hereafter we drop the subscript $X(f)$.

In order to state the following Theorem 3.6, it is convenient to introduce the ordered decomposition $\Pi = \{\Pi_1, \Pi_2\}$ of Π, called a *principal decomposition* of Π [Sa5], as follows. We first define the base vertex $[M_o] \in \Pi$. For a diagram of D_l or E_l, $l = 6, 7, 8$, we choose $[M_o]$ as the vertex on which three edges join. Explicitly, it is $[M^{k=l-2}]$ for type D_l, $[M^{k=2}]$ for type E_6, $[M^{k=4}]$ for type E_7 and $[M^{k=5}]$ for type E_8 (Figure 1). For type A_l, we set $[M_o] := [M^{k=b}]$ (see Figure 1) depending on the index b, $1 \leq b \leq l$ (see eq.(3.1)). Then, we define the decomposition of Π by

$$\Pi_1 := \{k \in \Pi \mid d(M_o, M^k) \in 2\mathbb{Z}_{\geq 0} + 1\} \ , \qquad \Pi_2 := \{k \in \Pi \mid d(M_o, M^k) \in 2\mathbb{Z}_{\geq 0}\} \ .$$

Recall that we express a graded matrix factorization $\widetilde{M} \in HMF_R^{gr}(f)$ by the pair $\widetilde{M} = (Q, S)$, where Q denotes a matrix factorization in eq.(2.3) and S is the grading matrix defined in eq.(2.4).

Theorem 3.6. *Let $f \in R$ be a polynomial of type ADE. The triangulated category $HMF_R^{gr}(f)$ is described as follows.*
(i) **(Objects)**: *The set of isomorphism classes of all indecomposable objects of $HMF_R^{gr}(f)$ is given by*

$$[\widetilde{M}_n^k := (Q^k, S_n^k)], \quad k \in \Pi \ , \quad n \in \mathbb{Z} \ .$$

Here, • *Q^k is the matrix factorizations of size $2\nu_k$ given in Table 1,*

• *the grading matrix S_n^k for $k \in \Pi_\sigma$, $\sigma = 1, 2$, and $n \in \mathbb{Z}$ is given by:*

$$S_n^k := \mathrm{diag}\left(q_1^k, -q_1^k, \cdots, q_{\nu_k}^k, -q_{\nu_k}^k; \bar{q}_1^k, -\bar{q}_1^k, \cdots, \bar{q}_{\nu_k}^k, -\bar{q}_{\nu_k}^k\right) + \phi_n^k \cdot \mathbf{1}_{4\nu_k} \ , \qquad (3.2)$$

where the data of the first term, called the traceless part: $q_j^k \in \frac{2}{h}\mathbb{Z} - \frac{\sigma}{h}$ *and* $\bar{q}_j^k \in \frac{2}{h}\mathbb{Z} - \frac{\sigma}{h} - 1$ *for* $1 \le j \le \nu_k$ *are given in* Table 2, *and the coefficient of the second term, called the phase, is given by* $\phi_n^k := \phi(\widetilde{M}_n^k) = \frac{2n+\sigma}{h}$.

(ii) **(Morphisms)**: (ii-a) *An irreducible morphism exists in* $\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_n^k, \widetilde{M}_{n'}^{k'})$ *if and only if* $d(k,k') = 1$ *and* $\phi_{n'}^{k'} = \phi_n^k + \frac{1}{h}$.

(ii-b) $\dim_{\mathbb{C}}(\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_n^k, \widetilde{M}_{n'}^{k'})) = 1$ *for* $\phi_{n'}^{k'} - \phi_n^k = \frac{1}{h} d(k,k')$.

(iii) **(The Serre duality)**: *The automorphism* $\mathcal{S} := T\tau^{-1} : HMF_R^{gr}(f) \to HMF_R^{gr}(f)$ *satisfies the following properties.*

(iii-a) $\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_n^k, \mathcal{S}(\widetilde{M}_n^k)) \simeq \mathbb{C}$ *for any indecomposable object* $\widetilde{M}_n^k \in HMF_R^{gr}(f)$.

(iii-b) *This induces the following nondegenerate bilinear map for any* $k, k' \in \Pi$ *and* $n, n' \in \mathbb{Z}$:

$$\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_n^k, \widetilde{M}_{n'}^{k'}) \otimes \mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_{n'}^{k'}, \mathcal{S}(\widetilde{M}_n^k)) \to \mathbb{C} \ .$$

Proof. (i) For each M^k, by direct calculations based on the explicit form of M^k in Table 1, we can attach grading matrices S satisfying eq.(2.4). This, together with the fact that $F : HMF_R^{gr}(f) \to HMF_{\hat{\mathcal{O}}}(f)$ brings an indecomposable object to an indecomposable object, implies that the union $\coprod_{k=1}^l F^{-1}(M^k)$ gives the set of all indecomposable objects in $HMF_R^{gr}(f)$ and then $F : HMF_R^{gr}(f) \to HMF_{\hat{\mathcal{O}}}(f)$ is a surjection (see also [AR3] and [Y], Theorem 15.14). Actually, S is unique up to an addition of a constant multiple of the identity. Therefore, we decompose S into the traceless part and the phase part as in eq.(3.2). Due to the restriction of degrees (2.1), one has $\pm q_j^k + \phi_n^k \in \frac{2}{h}\mathbb{Z}$ and $\pm \bar{q}_j^k + \phi_n^k \in \frac{2}{h}\mathbb{Z} - 1$ for any $k \in \Pi$ and $1 \le j \le \nu_k$ (see below eq.(2.4)), and by solving these conditions, we obtain Statement (i), *i.e.*, Table 2. In particular, one has $F^{-1}(M^k) = \{\widetilde{M}_n^k \mid n \in \mathbb{Z}\}$ and $\tau(\widetilde{M}_n^k) = \widetilde{M}_{n+1}^k$.

(ii-a) For any two indecomposable objects $\widetilde{M}_n^k, \widetilde{M}_{n'}^{k'} \in HMF_R^{gr}(f)$, Statement (i) implies that $F : \mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_n^k, \widetilde{M}_{n'}^{k'}) \to \mathrm{Hom}_{HMF_{\hat{\mathcal{O}}}(f)}(M^k, M^{k'})$ is injective and then

$$F : \oplus_{n''\in\mathbb{Z}} \mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_n^k, \tau^{n''}(\widetilde{M}_{n'}^{k'})) \simeq \mathrm{Hom}_{HMF_{\hat{\mathcal{O}}}(f)}(M^k, M^{k'})$$

is an isomorphism of vector spaces. This induces the following isomorphisms:

$$\begin{array}{llcl}
F: & \oplus_{n''\in\mathbb{Z}} \mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_n^k, \tau^{n''}(\widetilde{M}_{n'}^{k'})) & \xrightarrow{\sim} & \mathrm{Hom}_{HMF_{\hat{\mathcal{O}}}(f)}(M^k, M^{k'}) \ , \\
 & \cup & & \cup \\
F: & \oplus_{n''\in\mathbb{Z}} \mathrm{rad}_{HMF_R^{gr}(f)}(\widetilde{M}_n^k, \tau^{n''}(\widetilde{M}_{n'}^{k'})) & \xrightarrow{\sim} & \mathrm{rad}_{HMF_{\hat{\mathcal{O}}}(f)}(M^k, M^{k'}) \ , \\
 & \cup & & \cup \\
F: & \oplus_{n''\in\mathbb{Z}} \mathrm{rad}^2_{HMF_R^{gr}(f)}(\widetilde{M}_n^k, \tau^{n''}(\widetilde{M}_{n'}^{k'})) & \xrightarrow{\sim} & \mathrm{rad}^2_{HMF_{\hat{\mathcal{O}}}(f)}(M^k, M^{k'}) \ ,
\end{array}$$

and hence, the isomorphism $F : \oplus_{n''\in\mathbb{Z}} \mathrm{Irr}_{HMF_R^{gr}(f)}(\widetilde{M}_n^k, \tau^{n''}(\widetilde{M}_{n'}^{k'})) \xrightarrow{\sim} \mathrm{Irr}_{HMF_{\hat{\mathcal{O}}}(f)}(M^k, M^{k'})$.

For $k, k' \in \Pi$, define a multi-set $\mathfrak{C}(k,k')$ of non-negative integers by

$$\mathfrak{C}(k,k') := \{h(\phi(\tau^{n''}(\widetilde{M}^{k'}_{n'})) - \phi(\widetilde{M}^k_n)) \mid \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \tau^{n''}(\widetilde{M}^{k'}_{n'})) \neq 0 \ , \quad n'' \in \mathbb{Z}\} \ ,$$

where the integer $h(\phi(\tau^{n''}(\widetilde{M}^{k'}_{n'})) - \phi(\widetilde{M}^k_n)) = h(\phi^{k'}_{n'} - \phi^k_n) + 2n''$ appears with multiplicity $\dim_{\mathbb{C}}(\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \tau^{n''}(\widetilde{M}^{k'}_{n'})))$. The multi-set $\mathfrak{C}(k,k')$ in fact depends only on k and k', and is independent of n and n'.

For $k, k' \in \Pi$ such that $d(k,k') = 1$, by calculating $\mathrm{Hom}_{HMF_{\hat{\mathcal{O}}}(f)}(M^k, M^{k'})$ using the explicit forms of the matrix factorizations $M^k, M^{k'}$ in Table 1, one can see that $\mathfrak{C}(k,k')$ consists of positive odd integers including 1 of multiplicity one. This implies that, for two indecomposable objects $\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'} \in HMF^{gr}_R(f)$, a morphism in $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'})$ is an irreducible morphism if and only if $h(\phi^{k'}_{n'} - \phi^k_n) = 1$ (Statement (ii-a)). (A morphism in $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'})$ which corresponds to $c \in \mathfrak{C}(k,k')$ with $c \geq 3$ can be obtained by composing irreducible morphisms.)

(ii-b) By direct calculations of $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \widetilde{M}^k_n)$ and $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \mathcal{S}(\widetilde{M}^k_n))$ using the explicit forms of the matrix factorizations in Table 1 again, one gets:

Lemma 3.7. 1) *For an indecomposable object $\widetilde{M}^k_n \in HMF^{gr}_R(f)$, one has*

1-a) $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \widetilde{M}^k_n) \simeq \mathbb{C}$, 1-b) $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \mathcal{S}(\widetilde{M}^k_n)) \simeq \mathbb{C}$.

2) *For an indecomposable object $\widetilde{M}^k_n \in HMF^{gr}_R(f)$, a morphism $\Psi \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \mathcal{S}(\widetilde{M}^k_n))$ and an indecomposable object $\widetilde{M}^{k'}_{n'} \in HMF^{gr}_R(f)$ such that $d(k,k') = 1$,*

2-a) $\Psi\Phi \sim 0$ *holds for an irreducible morphism* $\Phi \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \widetilde{M}^k_n))$,

2-b) $\mathcal{S}(\Phi)\Psi \sim 0$ *holds for an irreducible morphism* $\Phi \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'})$. □

Note that Lemma 3.7.1-a also follows from the AR-quiver $\Gamma(HMF_{\hat{\mathcal{O}}}(f))$ of $HMF_{\hat{\mathcal{O}}}(f)$ in Theorem 3.2 and Statement (ii-a).

Due to Lemma 3.7.1-b, for any indecomposable object $\widetilde{M}^k_n \in HMF^{gr}_R(f)$, one can consider the cone $C(\Psi) \in HMF^{gr}_R(f)$ of a nonzero morphism $\Psi \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\mathcal{S}^{-1}(\widetilde{M}^k_n), \widetilde{M}^k_n)$, that is, the cone of a lift of the morphism $\Psi \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\mathcal{S}^{-1}(\widetilde{M}^k_n), \widetilde{M}^k_n)$ to a homomorphism in $\mathrm{Hom}_{MF^{gr}_R(f)}(\mathcal{S}^{-1}(\widetilde{M}^k_n), \widetilde{M}^k_n)$. We denote it simply by $C(\Psi)$, since two different lifts of the morphism $\Psi \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\mathcal{S}^{-1}(\widetilde{M}^k_n), \widetilde{M}^k_n)$ lead to two cones which are isomorphic to each other in $HMF^{gr}_R(f)$.

Recall that an *Auslander-Reiten (AR-)triangle* (see [Ha],[Y], and also an Auslander-Reiten sequence or equivalently an almost split sequence [AR1]) in a Krull-Schmidt triangulated category $\mathcal{C}$ is a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \tag{3.3}$$

satisfying the following conditions:

(AR1) X, Z are indecomposable objects in $\mathrm{Ob}(\mathcal{C})$.
(AR2) $w \neq 0$
(AR3) If $\Phi : W \to Z$ is not a split epimorphism, then there exists $\Phi' : W \to Y$ such that $v\Phi' = \Phi$.

Then, it is known (see [Ha], Proposition in 4.3) that u and v are irreducible morphisms.

Lemma 3.8. *For an indecomposable object $\widetilde{M}^k_n \in HMF^{gr}_R(f)$ and the cone $C(\Psi) \in HMF^{gr}_R(f)$ of a nonzero morphism $\Psi \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\mathcal{S}^{-1}(\widetilde{M}^k_n), \widetilde{M}^k_n)$, the distinguished triangle*

$$\widetilde{M}^k_n \to C(\Psi) \to \tau(\widetilde{M}^k_n) \overset{T(\Psi)}{\to} T(\widetilde{M}^k_n) \tag{3.4}$$

is an AR-triangle.

Proof. Since by definition the condition (AR1) and (AR2) are already satisfied, it is enough to show that the condition (AR3) holds. Consider the functor $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, -)$ on the distinguished triangle (3.4) for an indecomposable object $\widetilde{M}^{k'}_{n'} \in HMF^{gr}_R(f)$. One obtains the long exact sequence as follows:

$$\begin{aligned} \cdots \to \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \mathcal{S}^{-1}(\widetilde{M}^k_n)) &\overset{\Psi}{\to} \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \widetilde{M}^k_n) \\ &\to \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, C(\Psi)) \to \\ \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \tau(\widetilde{M}^k_n)) &\overset{T(\Psi)}{\to} \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, T(\widetilde{M}^k_n)) \to \cdots . \end{aligned} \tag{3.5}$$

A nonzero morphism in $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \tau(\widetilde{M}^k_n))$ is a split epimorphism if and only if $\widetilde{M}^{k'}_{n'} = \tau(\widetilde{M}^k_n)$, in which case $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \tau(\widetilde{M}^k_n)) \simeq \mathbb{C}$ is spanned by the identity $\mathbf{1}_{\widetilde{M}^{k'}_{n'}} \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \widetilde{M}^{k'}_{n'})$. For $\widetilde{M}^{k'}_{n'} \neq \tau(\widetilde{M}^k_n)$, due to Lemma 3.7.2-a, the map $T(\Psi)$ in the long exact sequence (3.5) turns out to be the zero map, which implies that we have the surjection $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, C(\Psi)) \to \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \tau(\widetilde{M}^k_n))$ and then the condition (AR3). □

The existence of the AR-triangle together with the corresponding long exact sequence (3.5) leads to the two key lemmas as follows.

Lemma 3.9. *Let $\widetilde{M}^k_n \in HMF^{gr}_R(f)$ be an indecomposable object. The cone $C(\Psi)$ of a nonzero morphism $\Psi \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\mathcal{S}^{-1}(\widetilde{M}^k_n), \widetilde{M}^k_n)$ is isomorphic to the direct sum of indecomposable objects $\widetilde{M}^{k_1}_{n_1} \oplus \cdots \oplus \widetilde{M}^{k_m}_{n_m}$ for some $m \in \mathbb{Z}_{>0}$ such that $\{k_1, \cdots, k_m\} = \{k' \in \Pi \mid d(k, k') = 1\}$ and $\phi(\widetilde{M}^{k_i}_{n_i}) = \phi(\widetilde{M}^k_n) + \frac{1}{h}$ for any $i = 1, \cdots, m$.*

Proof. For the AR-triangle in eq.(3.4), the morphisms $\widetilde{M}^k_n \to C(\Psi)$ and $C(\Psi) \to \tau(\widetilde{M}^k_n)$ are irreducible and hence one has $\phi(\widetilde{M}^{k_i}_{n_i}) = \phi(\widetilde{M}^k_n) + \frac{1}{h}$ and $d(k', k_i) = 1$, for any $i = 1, \cdots, m$, with the direct sum decomposition of indecomposable objects $C(\Psi) \simeq \oplus_{i=1}^m \widetilde{M}^{k_i}_{n_i}$ above. Also, the fact that the distinguished triangle (3.4) is an AR-triangle further guarantees that, for an indecomposable object $\widetilde{M}^{k'}_{n'} \in HMF^{gr}_R(f)$, there exists an irreducible morphism in $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'})$ if and only if $(k', n') = (k_i, n_i)$ for some $i = 1, \cdots, m$ (see Lemma in p.40 of [Ha]). Thus, the rest of the proof is to show $k_i \neq k_j$ if $i \neq j$, for which it is enough to check this lemma at the level of the grading matrices (see eq.(5.1) below Table 2). □

Lemma 3.10. *For an indecomposable object $\widetilde{M}^k_n \in HMF^{gr}_R(f)$, let $C(\Psi)$ be the cone of a nonzero morphism $\Psi \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\mathcal{S}^{-1}(\widetilde{M}^k_n), \widetilde{M}^k_n)$, and let $C(\Psi) \simeq \oplus_{i=1}^m \widetilde{M}^{k_i}_{n_i}$, $m \in \mathbb{Z}_{>0}$, be the direct sum decomposition of indecomposable objects as above. Then, for an indecomposable object $\widetilde{M}^{k'}_{n'} \in HMF^{gr}_R(f)$ and the given multi-set $\mathfrak{C}(k', k)$, one obtains*

$$\coprod_{i=1}^m \mathfrak{C}(k', k_i) = \{c - 1 \mid c \in \mathfrak{C}(k', k), c \neq 0\} \coprod \{c + 1 \mid c \in \mathfrak{C}(k', k), c \neq h - 2 \ \mathit{if}\ k = {k'}^{\mathcal{S}}\ \},$$

where, for $k \in \Pi$, $k^{\mathcal{S}} \in \Pi$ denotes the vertex such that $[F(\mathcal{S}(\widetilde{M}^k_n))] = [M^{k^{\mathcal{S}}}] \in \Pi$ for any $n \in \mathbb{Z}$.

Proof. Consider the long exact sequence (3.5) for the AR-triangle (3.4). As discussed in the proof of Lemma 3.8, for $\widetilde{M}^{k'}_{n'} \neq \tau(\widetilde{M}^k_n)$, the map $T(\Psi)$ in eq.(3.5) defines the zero map due to Lemma 3.7.2-a. Similarly, for $\widetilde{M}^{k'}_{n'} \neq \mathcal{S}^{-1}(\widetilde{M}^k_n)$, the map Ψ in eq.(3.5) defines the zero map due to Lemma 3.7.2-b. On the other hand, if $\widetilde{M}^{k'}_{n'} = \tau(\widetilde{M}^k_n)$, the map $T(\Psi)$ in eq.(3.5) defines an isomorphism and hence $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, C(\Psi)) \to \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \tau(\widetilde{M}^k_n))$ turns out to be a zero map. Similarly, if $\widetilde{M}^{k'}_{n'} = \mathcal{S}^{-1}(\widetilde{M}^k_n)$, the map Ψ in eq.(3.5) defines an isomorphism and hence $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \widetilde{M}^k_n) \to \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, C(\Psi))$ turns out to be a zero map. Combining these facts, one has the exact sequences as follows:

- $0 \to \mathrm{Hom}(\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'}) \to \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, C(\Psi)) \to \mathrm{Hom}(\widetilde{M}^k_n, \tau(\widetilde{M}^{k'}_{n'})) \to 0$ if $\widetilde{M}^k_n \neq \tau(\widetilde{M}^{k'}_{n'})$ and $\widetilde{M}^k_n \neq \mathcal{S}^{-1}(\widetilde{M}^{k'}_{n'})$,
- $0 \to \mathrm{Hom}(\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'}) \to \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, C(\Psi)) \to 0$ if $\widetilde{M}^k_n = \tau(\widetilde{M}^{k'}_{n'})$, and
- $0 \to \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, C(\Psi)) \to \mathrm{Hom}(\widetilde{M}^k_n, \tau(\widetilde{M}^{k'}_{n'})) \to 0$ if $\widetilde{M}^k_n = \mathcal{S}^{-1}(\widetilde{M}^{k'}_{n'})$, though in this case $\mathrm{Hom}(\widetilde{M}^k_n, \tau(\widetilde{M}^{k'}_{n'})) = 0$.

From these results follows the statement of this lemma. □

Calculate $\mathrm{Hom}_{HMF^{gr}_R(f)}(M^k, \tau^n(M^k))$, $n \in \mathbb{Z}$, where we set $k = 1$ for type A_l, D_l and E_8, $k = 5$ or 6 for E_6, and $k = 7$ for E_7, by using the explicit forms of matrix factorizations M^k in Table 1. Then, one obtains $\mathfrak{C}(k, k)$. Using Lemma 3.9 and Lemma 3.10 repeatedly, one can actually obtain $\mathfrak{C}(k', k'')$ for all $k', k'' \in \Pi$ (Table 3). In particular, by a use of Table 3, one can get Statement (ii-b) and

Corollary 3.11. *The following equivalent statements hold: for two indecomposable objects $\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'} \in HMF^{gr}_R(f)$,*

(a) *If* $\phi(\mathcal{S}(\widetilde{M}^{k'}_{n'})) - \phi(\widetilde{M}^{k}_{n}) < \frac{1}{h}d(k, k'^{\mathcal{S}})$, *then* $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \widetilde{M}^{k}_{n}) = 0$.
(b) *If* $\phi(\widetilde{M}^{k}_{n}) - \phi(\widetilde{M}^{k'}_{n'}) < \frac{1}{h}d(k', k)$, *then* $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k}_{n}, \mathcal{S}(\widetilde{M}^{k'}_{n'})) = 0$. □

(iii-a) This is already proven in Lemma 3.7.1-b.

(iii-b) Suppose that there exists a nonzero morphism $\Phi \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \widetilde{M}^{k}_{n})$ for two indecomposable objects $\widetilde{M}^{k}_{n}, \widetilde{M}^{k'}_{n'} \in HMF^{gr}_R(f)$ and let us show that there exists a nonzero morphism $\Phi^{\mathcal{S}} \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k}_{n}, \mathcal{S}(\widetilde{M}^{k'}_{n'}))$ such that $\Phi^{\mathcal{S}}\Phi$ is nonzero in $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \mathcal{S}(\widetilde{M}^{k'}_{n'}))$. If $\widetilde{M}^{k}_{n} = \mathcal{S}(\widetilde{M}^{k'}_{n'})$, then one may take $\Phi^{\mathcal{S}} = \mathbf{1}_{\mathcal{S}(\widetilde{M}^{k'}_{n'})}$, so assume that $\widetilde{M}^{k}_{n} \neq \mathcal{S}(\widetilde{M}^{k'}_{n'})$, *i.e.*, $0 \le \phi^{k}_{n} - \phi^{k'}_{n'} < 1 - \frac{2}{h}$. As in the proof of Lemma 3.9, from the long exact sequence (3.5) of the AR-triangle (3.4) one obtains the exact sequence

$$0 \to \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \widetilde{M}^{k}_{n}) \to \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, C(\Psi))$$

for the cone $C(\Psi)$ of a nonzero morphism $\Psi : \mathcal{S}^{-1}(\widetilde{M}^{k}_{n}) \to \widetilde{M}^{k}_{n}$. Namely, there exists an indecomposable object $\widetilde{M}^{k''}_{n''} \in HMF^{gr}_R(f)$ such that $d(k, k'') = 1$, $\phi^{k''}_{n''} - \phi^{k}_{n} = \frac{1}{h}$ and $\Phi'\Phi$ is not zero in $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \widetilde{M}^{k''}_{n''})$ with an irreducible morphism $\Phi' \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k}_{n}, \widetilde{M}^{k''}_{n''})$. Repeating this procedure together with Corollary 3.11 (a) leads that the path corresponding to the morphism arrives at $\mathcal{S}(\widetilde{M}^{k'}_{n'})$. Namely, for a given morphism Φ in $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \widetilde{M}^{k}_{n})$, there exists a nonzero morphism $\Phi^{\mathcal{S}} \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k}_{n}, \mathcal{S}(\widetilde{M}^{k'}_{n'}))$ such that $\Phi^{\mathcal{S}}\Phi$ is nonzero in $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \mathcal{S}(\widetilde{M}^{k'}_{n'}))$.

Conversely, suppose that there exists a morphism $\Phi^{\mathcal{S}} \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k}_{n}, \mathcal{S}(\widetilde{M}^{k'}_{n'}))$ for two indecomposable objects $\widetilde{M}^{k}_{n}, \widetilde{M}^{k'}_{n'} \in HMF^{gr}_R(f)$. One can show that there exists a nonzero morphism $\Phi \in \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \widetilde{M}^{k}_{n})$ such that $\Phi^{\mathcal{S}}\Phi$ is nonzero in $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \mathcal{S}(\widetilde{M}^{k'}_{n'}))$ by considering the functor $\mathrm{Hom}_{HMF^{gr}_R(f)}(-, \mathcal{S}(\widetilde{M}^{k'}_{n'}))$ on the AR-triangle (3.4) together with Corollary 3.11 (b), instead of $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, -)$ with Corollary 3.11 (a). Thus, one gets Statement (iii-b) of Theorem 3.6. □

Step 3. Exceptional collections in $HMF^{gr}_R(f)$

In this step, for a fixed polynomial f of type ADE and a Dynkin quiver $\vec{\Delta}_{X(f)}$, we find exceptional collections in $HMF^{gr}_R(f)$ and then show the equivalence of the category $HMF^{gr}_R(f)$ with the derived category of modules over the path algebra of the Dynkin quiver $\vec{\Delta}_{X(f)}$ (which we again denote simply by $\vec{\Delta}$).

Lemma 3.12. *For any Dynkin quiver* $\vec{\Delta}$*, there exists* $\mathbf{n} := (n_1, \cdots, n_l) \in \mathbb{Z}^l$ *such that one has an isomorphism of* $\mathbb{C}$*-algebras*:

$$\mathbb{C}\vec{\Delta} \simeq \mathbb{C}\vec{\Gamma}(\mathbf{n}) := \bigoplus_{k,k' \in \Pi} \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k}_{n_k}, \widetilde{M}^{k'}_{n_{k'}}) \ ,$$

with the natural correspondence between the path from $k \in \Pi$ to $k' \in \Pi$ and the space $\mathrm{Hom}(\widetilde{M}^k_{n_k}, \widetilde{M}^{k'}_{n_{k'}})$ *for any $k, k' \in \Pi$. Such a tuple of integers is unique up to the $\mathbb{Z}$ shift $(n_1 + n, \cdots, n_l + n)$ for some $n \in \mathbb{Z}$.*

Proof. Let us fix $n_1 \in \mathbb{Z}$ of $\widetilde{M}^1_{n_1}$. For each $k \in \Pi$, $d(1,k) = 1$, take $\widetilde{M}^k_{n_k} \in HMF^{gr}_R(f)$ such that $\phi^k_{n_k} - \phi^1_{n_1} = \frac{1}{h}$ (resp. $-\frac{1}{h}$) if there exists an arrow in $\vec{\Delta}$ from $1 \in \Pi$ to $k \in \Pi$ (resp. from $k \in \Pi$ to $1 \in \Pi$). Repeating this process leads to the tuple $\mathbf{n} = (n_1, \cdots, n_l)$ such that $\mathbb{C}\vec{\Gamma}(\mathbf{n}) \simeq \mathbb{C}\vec{\Delta}$. On the other hand, Theorem 3.6 (ii-b) implies that there exists a morphism for any given path and there are no relations among them. Thus, one gets this lemma. □

Corollary 3.13. *Let $\mathfrak{S}_l$ be the permutation group of $\Pi = \{1, \cdots, l\}$. Given a Dynkin quiver $\vec{\Delta}$ and $\mathbf{n} \in \mathbb{Z}^l$ such that $\mathbb{C}\vec{\Gamma}(\mathbf{n}) \simeq \mathbb{C}\vec{\Delta}$, one can take an element $\sigma \in \mathfrak{S}_l$ such that $\phi^{\sigma(k')}_{n_{\sigma(k')}} \geq \phi^{\sigma(k)}_{n_{\sigma(k)}}$ if $k' > k$. Thus, for $\{E^1, \cdots, E^l\}$, $E^k := \widetilde{M}^{\sigma(k)}_{n_{\sigma(k)}}$, $\mathrm{Hom}_{HMF^{gr}_R(f)}(E^k, E^{k'}) \neq 0$ only if $k' > k$.* □

Lemma 3.14. *Given a Dynkin quiver $\vec{\Delta}$ and $\mathbf{n} \in \mathbb{Z}^l$ such that $\mathbb{C}\vec{\Gamma}(\mathbf{n}) \simeq \mathbb{C}\vec{\Delta}$, one has* $\mathrm{Hom}_{HMF^{gr}_R(f)}(\tau^n(\widetilde{M}^k_{n_k}), \widetilde{M}^{k'}_{n_{k'}}) = 0$ *for any $k, k' \in \Pi$ if $n \geq 1$.* □

Proof. Let us first concentrate on irreducible morphisms. For any $k, k' \in \Pi$ such that $d(k, k') = 1$, one obtains $\phi^{k'}_{n_{k'}} - \phi^k_{n_k+n} = \frac{\pm 1 - 2n}{h}$ since $\phi^{k'}_{n_{k'}} - \phi^k_{n_k} = \pm\frac{1}{h}$. On the other hand, Theorem 3.6 (ii-a) implies that there exists an irreducible morphism in $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_{n_k+n}, \widetilde{M}^{k'}_{n_{k'}})$ only if $d(k, k') = 1$ and $n = 0$ or $n = -1$, since $\frac{\pm 1 - 2n}{h}$ can be $\frac{1}{h}$ only if $n = 0$ or $n = -1$.

Since, by definition, all morphisms except the identities can be described by the composition of the irreducible morphisms, one gets this lemma. □

By Lemma 3.14 and the Serre duality (Theorem 3.6 (iii)), we obtain the following lemma.

Lemma 3.15. *Given a Dynkin quiver $\vec{\Delta}$ and $\mathbf{n} \in \mathbb{Z}^l$ such that $\mathbb{C}\vec{\Gamma}(\mathbf{n}) \simeq \mathbb{C}\vec{\Delta}$, $\{E^1, \ldots, E^l\}$ given in Corollary 3.13 is a strongly exceptional collection, that is,*

$$\begin{cases} \mathrm{Hom}_{HMF^{gr}_R(f)}(E^i, E^j) = 0 & \text{for } i > j \ , \\ \mathrm{Hom}_{HMF^{gr}_R(f)}(E^i, T^k(E^j)) = 0 & \text{for } k \neq 0 \text{ and any } i, j \ . \end{cases}$$

Proof. Due to Corollary 3.13, it is sufficient to show that $\mathrm{Hom}_{HMF^{gr}_R(f)}(E^i, T^n(E^j)) = 0$ for $n \geq 1$. On the other hand, by the Serre duality,

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{HMF^{gr}_R(f)}(E^i, T^n(E^j))) = \dim_{\mathbb{C}}(\mathrm{Hom}_{HMF^{gr}_R(f)}(T^{n-1}\tau(E^j), E^i))$$

holds. If $n = 1$, $\mathrm{Hom}_{HMF^{gr}_R(f)}(T^{n-1}\tau(E^j), E^i) = \mathrm{Hom}_{HMF^{gr}_R(f)}(\tau(E^j), E^i) = 0$ follows from Lemma 3.14. If $n \geq 2$, first we have $\phi(T^{n-1}\tau(E^j)) = \phi(E^j) + (n - 1 + \frac{2}{h})$ and then

$$\phi(E^i) - \phi(T^{n-1}\tau(E^j)) = (\phi(E^i) - \phi(E^j)) - \left(n - 1 + \frac{2}{h}\right) \ .$$

Here, it is clear that

$$|\phi(E^i) - \phi(E^j)| \leq \frac{l-1}{h}$$

holds for $\{E^1, \cdots, E^l\}$. Thus, we have the following inequality:

$$\phi(E^i) - \phi(T^{n-1}\tau(E^j)) \leq \frac{l-1}{h} - \left(n - 1 + \frac{2}{h}\right) < 0 \quad \text{if } n \geq 2\ .$$

On the other hand, since all morphisms except the identities can be obtained by the composition of the irreducible morphisms, Theorem 3.6 (ii-a) implies that $\mathrm{Hom}_{HMF^{gr}_R(f)}(E^j, E^i) \neq 0$ only if $\phi(E^i) - \phi(E^j) \geq \frac{1}{h} d(E^j, E^i) \geq 0$. This implies
$\mathrm{Hom}_{HMF^{gr}_R(f)}(T^{n-1}\tau(E^j), E^i) = 0$. □

Corollary 3.16. $D^b(\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta})$ *is a full triangulated subcategory of* $HMF^{gr}_R(f)$.

Proof. Use the fact that $\{E^1, \ldots, E^l\}$ is a strongly exceptional collection and

$$\mathbb{C}\vec{\Delta} \simeq \bigoplus_{i,j=1}^{l} \mathrm{Hom}_{HMF^{gr}_R(f)}(E^i, E^j)\ .$$

Since $HMF^{gr}_R(f)$ is an enhanced triangulated category, we can apply the theorem by Bondal-Kapranov ([BK] Theorem 1) which implies that $D^b(\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta})$ is full as a triangulated subcategory. □

Recall that the number of indecomposable objects of $HMF^{gr}_R(f)$ up to the shift functor T is $\frac{l \cdot h}{2}$, which coincides with the number of the positive roots for the root system of type ADE. This number is the number of indecomposable objects of $D^b(\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta})$ up to the shift functor T by Gabriel's theorem [Ga]. Therefore, one obtains $D^b(\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta}) \simeq HMF^{gr}_R(f)$.

4. A stability condition on $HMF^{gr}_R(f)$

Let $K_0(HMF^{gr}_R(f))$ be the Grothendieck group of the category $HMF^{gr}_R(f)$ [Gr]. For a triangulated category $\mathcal{C}$, let F be a free abelian group generated by the isomorphism classes of objects $[X]$ for $X \in \mathrm{Ob}(\mathcal{C})$. Let F_0 be the subgroup generated by $[X] - [Y] + [Z]$ for all distinguished triangles $X \to Y \to Z \to T(M)$ in $\mathcal{C}$. The Grothendieck group $K_0(\mathcal{C})$ of the triangulated category $\mathcal{C}$ is, by definition, the factor group F/F_0.

Definition 4.1. For a graded matrix factorization $\widetilde{M} = (Q, S) \in HMF^{gr}_R(f)$, we associate a complex number as follows:

$$\mathcal{Z}(\widetilde{M}) := \mathrm{Tr}(e^{\pi\sqrt{-1}S})\ .$$

The map $\mathcal{Z}$ extends to a group homomorphism $\mathcal{Z} : K_0(HMF^{gr}_R(f)) \to \mathbb{C}$.

Theorem 4.2. *Let $f \in R$ be a polynomial of type ADE. Let $\mathcal{P}(\phi)$ be the full additive subcategory of $HMF^{gr}_R(f)$ additively generated by the indecomposable objects of phase ϕ. Then $\mathcal{P}(\phi)$ and $\mathcal{Z}$ define a stability condition on $HMF^{gr}_R(f)$ in the sense of Bridgeland* [Bd].

Proof. By definition, what we should check is that $\mathcal{P}(\phi)$ and $\mathcal{Z}$ satisfy the following properties:

(i) if $\widetilde{M} \in \mathcal{P}(\phi)$, then $\mathcal{Z}(\widetilde{M}) = m(\widetilde{M})\exp(\pi\sqrt{-1}\phi)$ for some $m(\widetilde{M}) \in \mathbb{R}_{>0}$,

(ii) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi+1) = T(\mathcal{P}(\phi))$,

(iii) if $\phi_1 > \phi_2$ and $\widetilde{M}_i \in \mathcal{P}(\phi_i)$, $i = 1,2$, then $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}_1, \widetilde{M}_2) = 0$,

(iv) for each nonzero object $\widetilde{M} \in HMF^{gr}_R(f)$, there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of distinguished triangles

$$\begin{array}{ccccccccccc}
0 = \widetilde{M}_0 & \longrightarrow & \widetilde{M}_1 & \longrightarrow & \widetilde{M}_2 & \longrightarrow & \cdots\cdots & \longrightarrow & \widetilde{M}_{n-1} & \longrightarrow & \widetilde{M}_n = \widetilde{M} \\
 & \nwarrow \;\; \swarrow & & \nwarrow \;\; \swarrow & & & & & & \nwarrow \;\; \swarrow & \\
 & \widetilde{N}_1 & & \widetilde{N}_2 & & & & & & \widetilde{N}_n &
\end{array}$$

with $\widetilde{N}_j \in \mathcal{P}(\phi_j)$ for all $j = 1, \cdots, n$.

Here, for any indecomposable object $\widetilde{M}^k_n \in HMF^{gr}_R(f)$, $\mathcal{Z}(\widetilde{M}^k_n)$ is of the form:

$$\mathcal{Z}(\widetilde{M}^k_n) = m(\widetilde{M}^k_n) e^{\pi\sqrt{-1}\phi^k_n} , \qquad m(\widetilde{M}^k_n) \in \mathbb{R}_{>0} .$$

In fact, by a straightforward calculation, one obtains

$$\begin{aligned}
\mathcal{Z}(\widetilde{M}^k_n) &= \sum_{j=1}^{\nu_k} \left(e^{\pi\sqrt{-1}(q^k_j+\phi^k_n)} + e^{\pi\sqrt{-1}(-q^k_j+\phi^k_n)} + e^{\pi\sqrt{-1}(\bar{q}^k_j+\phi^k_n)} + e^{\pi\sqrt{-1}(-\bar{q}^k_j+\phi^k_n)} \right) \\
&= 2e^{\pi\sqrt{-1}\phi^k_n} \sum_{j=1}^{\nu_k} \left(\cos(\pi q^k_j) + \cos(\pi \bar{q}^k_j) \right) \\
&= m(\widetilde{M}^k_n) e^{\pi\sqrt{-1}\phi^k_n} , \qquad m(\widetilde{M}^k_n) := 2\sum_{j=1}^{\nu_k} \left(\cos(\pi q^k_j) + \cos(\pi \bar{q}^k_j) \right) .
\end{aligned}$$

This shows Statement (i) in Theorem 4.2. It is clear that Statement (ii),(iii) and (iv) follow from Theorem 3.6. □

Proposition 5.3 in [Bd] states that a stability condition $(\mathcal{Z}, \mathcal{P})$ on a triangulated category defines an abelian category. In our case, for the triangulate category $HMF^{gr}_R(f)$, we obtain an abelian category $\mathcal{P}((0,1])$ which consists of objects $\widetilde{M} \in HMF^{gr}_R(f)$ having the decomposition $\widetilde{N}_1, \cdots, \widetilde{N}_n$ given by Theorem 4.2 (iv) such that $1 \geq \phi_1 > \cdots > \phi_n > 0$.

Proposition 4.3. *Given a polynomial f of type ADE, the following equivalence of abelian categories holds:*

$$\mathcal{P}((0,1]) \simeq \mathrm{mod}\text{-}\mathbb{C}\vec{\Delta}_{principal}\ ,$$

where $\vec{\Delta}_{principal}$ is the Dynkin quiver of the corresponding ADE Dynkin diagram Δ with the principal orientation [Sa5]*, i.e., all arrows start from Π_1 and end on Π_2.* □

Proof. Let us take the following collection $\{\widetilde{M}_0^1, \cdots, \widetilde{M}_0^k\}$. The corresponding $\mathbb{C}$-algebra is denoted by $\mathbb{C}\vec{\Gamma}(\mathbf{n}=0)$ (see Lemma 3.12). First, we show that

$$\mathbb{C}\vec{\Gamma}(\mathbf{n}=0) \simeq \mathbb{C}\vec{\Delta}_{principal}\ , \tag{4.1}$$

for which it is enough to say that $\widetilde{M}_0^k \in \mathcal{P}((0,1])$ is projective for each $k \in \Pi$. For any $\widetilde{N} \in \mathcal{P}(\phi(\widetilde{N}))$ such that $0 < \phi(\widetilde{N}) \leq 1$, the Serre duality implies

$$\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_0^k, T(\widetilde{N})) \simeq \mathrm{Hom}_{HMF_R^{gr}(f)}(\tau(\widetilde{N}), \widetilde{M}_0^k)\ .$$

Here, one has $\mathrm{Hom}_{HMF_R^{gr}(f)}(\tau(\widetilde{N}), \widetilde{M}_0^k) \neq 0$ only if $\phi(\widetilde{N}) + \frac{2}{h} \leq \phi(\widetilde{M}_0^k)$. However, such $\widetilde{N}$ does not exist since $\phi(\widetilde{M}_0^k) = \frac{\sigma}{h}$ for $k \in \Pi_\sigma$, $\sigma = 1, 2$. Thus, one has $\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_0^k, T(\widetilde{N})) = 0$ and hence eq.(4.1).

On the other hand, the collection $\langle \widetilde{M}_0^1, \cdots, \widetilde{M}_0^k \rangle$ is a full sub-abelian category of $\mathcal{P}((0,1])$ and is equivalent to the abelian category $\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta}_{principal}$. Also, the Gabriel's theorem [Ga] asserts that the number of the indecomposable objects in $\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta}_{principal}$ is equal to the number of the positive roots of Δ, which coincides with the number of the indecomposable objects in $\mathcal{P}((0,1])$. Thus, one obtains the equivalence $\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta}_{principal} \simeq \mathcal{P}((0,1])$. □

Remark 4.4 (Principal orientation)**.** In the triangulated category $HMF_R^{gr}(f)$, the principal oriented quiver $\vec{\Delta}_{principal}$ is realized by a strongly exceptional collection $\{\widetilde{M}_{n_1}^1, \cdots, \widetilde{M}_{n_l}^l\}$ with $n_1 = \cdots = n_l = n \in \mathbb{Z}$ for some $n \in \mathbb{Z}$ as above. It is interesting that a strongly exceptional collection of this type has minimum range of the phase: $\frac{2n+1}{h} \leq \phi(\widetilde{M}_{n_k}^k) \leq \frac{2n+2}{h}$ for any $k \in \Pi$.

5. Tables of data for matrix factorizations of type ADE

Table 1. We list up the Auslander-Reiten (AR)-quiver of the category $HMF_{\hat{\mathcal{O}}}(f)$ for a polynomial f of type ADE. We label each isomorphism class of the indecomposable objects (matrix factorizations) in $MF_{\hat{\mathcal{O}}}(f)$ by upperscript $\{1, \cdot\cdot, k, \cdot\cdot, l\}$ such as $[M^k]$. We also list up a representative M^k of $[M^k]$ by giving the pair (φ^k, ψ^k) for each matrix factorization $Q^k := \begin{pmatrix} & \psi^k \\ \varphi^k & \end{pmatrix}$ (see eq.(2.3)). In addition, for type D_l and E_l ($l = 6, 7, 8$) cases, we attach

the indices $\begin{pmatrix} q_1^k \\ \vdots \\ q_{\nu_k}^k \end{pmatrix}$ defined in Theorem 3.6 (eq.(3.2)) to each of those pairs as

$$\varphi^k_{\begin{pmatrix} q_1^k \\ \vdots \\ q_{\nu_k}^k \end{pmatrix}}, \psi^k_{\begin{pmatrix} q_1^k \\ \vdots \\ q_{\nu_k}^k \end{pmatrix}} .$$

Those indices are for later convenience and irrelevant for the statement of Theorem 3.2. They will be listed up again in the next table (Table 2).

Type A_l. The AR-quiver for the polynomial $f = x^{l+1} + yz$ is of the form:

$$[M^1] \rightleftarrows [M^2] \rightleftarrows \cdots\cdots \rightleftarrows [M^{l-1}] \rightleftarrows [M^l] \quad ,$$

where, for each $k \in \Pi$, (φ^k, ψ^k) is given by

$$\varphi^k := \begin{pmatrix} y & x^{l+1-k} \\ x^k & -z \end{pmatrix}, \quad \psi^k := \begin{pmatrix} z & x^{l+1-k} \\ x^k & -y \end{pmatrix} .$$

Type D_l. The AR-quiver for the polynomial $f = x^2 y + y^{l-1} + z^2$ is given by:

$$\begin{array}{ccccccccccc} & & & & & & & & & & [M^{l-1}] \\ & & & & & & & & & \nearrow\!\!\swarrow & \\ [M^1] & \rightleftarrows & [M^2] & \rightleftarrows & \cdots & \cdots \rightleftarrows & [M^{l-3}] & \rightleftarrows & [M^{l-2}] & & \\ & & & & & & & & & \searrow\!\!\nwarrow & \\ & & & & & & & & & & [M^l] \quad , \end{array}$$

where, for each $k \in \Pi$, (φ^k, ψ^k) is given by

$$\varphi^1_{(l-3)} = \psi^1_{(l-3)} = \begin{pmatrix} z & x^2 + y^{l-2} \\ y & -z \end{pmatrix} ,$$

$$\varphi^k_{\substack{l-2-k \\ l-k}} = \psi^k_{\substack{l-2-k \\ l-k}} = \begin{pmatrix} -z & 0 & xy & y^{\frac{k}{2}} \\ 0 & -z & y^{l-1-\frac{k}{2}} & -x \\ x & y^{\frac{k}{2}} & z & 0 \\ y^{l-1-\frac{k}{2}} & -xy & 0 & z \end{pmatrix} , \quad k\text{: even} \quad (2 \le k \le l-2)$$

$$\varphi^k_{\substack{l-2-k \\ l-k}} = \psi^k_{\substack{l-2-k \\ l-k}} = \begin{pmatrix} -z & y^{\frac{k-1}{2}} & xy & 0 \\ y^{l-\frac{k+1}{2}} & z & 0 & -x \\ x & 0 & z & y^{\frac{k-3}{2}} \\ 0 & -xy & y^{l-\frac{k-1}{2}} & -z \end{pmatrix} , \quad k\text{: odd} \quad (3 \le k \le l-2)$$

The forms of $(\varphi^{l-1}, \psi^{l-1})$ and (φ^l, ψ^l) depend on whether l is even or odd.

• If l is even, one obtains

$$\varphi^{l-1}_{(1)} = \psi^{l-1}_{(1)} = \begin{pmatrix} z & y(x+\sqrt{-1}y^{\frac{l-2}{2}}) \\ x-\sqrt{-1}y^{\frac{l-2}{2}} & -z \end{pmatrix} ,$$

$$\varphi^{l}_{(1)} = \psi^{l}_{(1)} = \begin{pmatrix} z & y(x-\sqrt{-1}y^{\frac{l-2}{2}}) \\ x+\sqrt{-1}y^{\frac{l-2}{2}} & -z \end{pmatrix} .$$

• If l is odd, one obtains

$$\varphi^{l-1}_{(1)} = \psi^{l}_{(1)} = \begin{pmatrix} z+\sqrt{-1}y^{\frac{l-1}{2}} & xy \\ x & -(z-\sqrt{-1}y^{\frac{l-1}{2}}) \end{pmatrix} ,$$

$$\varphi^{l}_{(1)} = \psi^{l-1}_{(1)} = \begin{pmatrix} z-\sqrt{-1}y^{\frac{l-1}{2}} & xy \\ x & -(z+\sqrt{-1}y^{\frac{l-1}{2}}) \end{pmatrix} .$$

Type E_6. The AR-quiver for the polynomial $f = x^3 + y^4 + z^2$ is given by:

$$\begin{array}{ccccccccc} & & & & [M^1] & & & & \\ & & & & \uparrow\downarrow & & & & \\ [M^5] & \rightleftarrows & [M^3] & \rightleftarrows & [M^2] & \rightleftarrows & [M^4] & \rightleftarrows & [M^6] \end{array} .$$

For $Y^{\pm} := y^2 \pm \sqrt{-1}z$, each pair (φ^k, ψ^k) is given by

$$\varphi^{1}_{\binom{1}{5}} = \varphi^{1}_{\binom{1}{5}} = \begin{pmatrix} -z & 0 & x^2 & y^3 \\ 0 & -z & y & -x \\ x & y^3 & z & 0 \\ y & -x^2 & 0 & z \end{pmatrix} ,$$

$$\varphi^{2}_{\begin{smallmatrix}0\\2\\4\end{smallmatrix}} = \begin{pmatrix} -\sqrt{-1}z & -y^2 & xy & 0 & x^2 & 0 \\ -y^2 & -\sqrt{-1}z & 0 & 0 & 0 & x \\ 0 & 0 & -\sqrt{-1}z & -x & 0 & y \\ 0 & xy & -x^2 & -\sqrt{-1}z & y^3 & 0 \\ x & 0 & 0 & y & -\sqrt{-1}z & 0 \\ 0 & x^2 & y^3 & 0 & xy^2 & -\sqrt{-1}z \end{pmatrix} ,$$

$$\psi^{2}_{\begin{smallmatrix}0\\2\\4\end{smallmatrix}} = \begin{pmatrix} \sqrt{-1}z & -y^2 & xy & 0 & x^2 & 0 \\ -y^2 & \sqrt{-1}z & 0 & 0 & 0 & x \\ 0 & 0 & \sqrt{-1}z & -x & 0 & y \\ 0 & xy & -x^2 & \sqrt{-1}z & y^3 & 0 \\ x & 0 & 0 & y & \sqrt{-1}z & 0 \\ 0 & x^2 & y^3 & 0 & xy^2 & \sqrt{-1}z \end{pmatrix} ,$$

$$\varphi^{3}_{\binom{1}{3}} = \psi^{4}_{\binom{1}{3}} = \begin{pmatrix} -Y^- & 0 & xy & x \\ -xy & Y^+ & x^2 & 0 \\ 0 & x & \sqrt{-1}z & y \\ x^2 & -xy & y^3 & \sqrt{-1}z \end{pmatrix} ,$$

$$\varphi^4_{\binom{1}{3}} = \psi^3_{\binom{1}{3}} = \begin{pmatrix} -Y^+ & 0 & xy & x \\ -xy & Y^- & x^2 & 0 \\ 0 & x & -\sqrt{-1}z & y \\ x^2 & -xy & y^3 & -\sqrt{-1}z \end{pmatrix},$$

$$\varphi^5_{(2)} = \psi^6_{(2)} = \begin{pmatrix} -Y^- & x \\ x^2 & Y^+ \end{pmatrix}, \qquad \varphi^6_{(2)} = \psi^5_{(2)} = \begin{pmatrix} -Y^+ & x \\ x^2 & Y^- \end{pmatrix}.$$

Type E_7. The AR-quiver for the polynomial $f = x^3 + xy^3 + z^2$ is given by:

$$\begin{array}{ccccccccccc} & & & & & & [M^4] & & & & \\ & & & & & & \uparrow\downarrow & & & & \\ [M^7] & \rightleftarrows & [M^6] & \rightleftarrows & [M^5] & \rightleftarrows & [M^3] & \rightleftarrows & [M^2] & \rightleftarrows & [M^1]\,. \end{array}$$

The corresponding matrix factorizations are:

$$\varphi^1_{\binom{2}{8}} = \psi^1_{\binom{2}{8}} = \begin{pmatrix} z & 0 & -x^2 & y \\ 0 & z & xy^2 & x \\ -x & y & -z & 0 \\ xy^2 & x^2 & 0 & -z \end{pmatrix}, \quad \varphi^4_{\binom{1}{5}} = \psi^4_{\binom{1}{5}} = \begin{pmatrix} -z & y^2 & 0 & x \\ xy & z & -x^2 & 0 \\ 0 & -x & -z & y \\ x^2 & 0 & xy^2 & z \end{pmatrix}$$

$$\varphi^2_{\substack{1\\3\\7}} = \psi^2_{\substack{1\\3\\7}} = \begin{pmatrix} -z & y^2 & xy & 0 & x^2 & 0 \\ xy & z & 0 & 0 & 0 & -x \\ 0 & 0 & z & -x & 0 & y \\ 0 & -xy & -x^2 & -z & xy^2 & 0 \\ x & 0 & 0 & y & z & 0 \\ 0 & -x^2 & xy^2 & 0 & x^2y & -z \end{pmatrix},$$

$$\varphi^5_{\substack{1\\3\\5}} = \psi^5_{\substack{1\\3\\5}} = \begin{pmatrix} -z & 0 & xy & 0 & 0 & x \\ -xy & z & 0 & -y^2 & -x^2 & 0 \\ y^2 & 0 & z & -x & xy & 0 \\ 0 & -xy & -x^2 & -z & 0 & 0 \\ 0 & -x & 0 & 0 & -z & -y \\ x^2 & 0 & 0 & xy & -xy^2 & z \end{pmatrix},$$

$$\varphi^3_{\substack{0\\2\\4\\6}\big)} = \psi^3_{\substack{0\\2\\4\\6}\big)} = \begin{pmatrix} -z & 0 & xy & -y^2 & 0 & 0 & x^2 & 0 \\ 0 & -z & 0 & y^2 & 0 & 0 & 0 & x \\ y^2 & y^2 & z & 0 & 0 & -x & 0 & 0 \\ 0 & xy & 0 & z & -x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & -z & 0 & 0 & y \\ 0 & 0 & -x^2 & 0 & 0 & -z & xy^2 & y^2 \\ x & 0 & 0 & 0 & -y^2 & y^2 & z & 0 \\ 0 & x^2 & 0 & 0 & xy^2 & 0 & 0 & z \end{pmatrix},$$

$$\varphi^6_{\binom{2}{4}} = \psi^6_{\binom{2}{4}} = \begin{pmatrix} z & 0 & -xy & x \\ 0 & z & x^2 & y^2 \\ -y^2 & x & -z & 0 \\ x^2 & xy & 0 & -z \end{pmatrix}, \quad \varphi^7_{(3)} = \psi^7_{(3)} = \begin{pmatrix} z & x \\ x^2+y^3 & -z \end{pmatrix}.$$

Type E_8. The AR-quiver for the polynomial $f = x^3 + y^5 + z^2$ is given by:

$$\begin{array}{ccccccccccccc} & & & & & & & & [M^6] & & & & \\ & & & & & & & & \uparrow\downarrow & & & & \\ [M^1] & \rightleftarrows & [M^2] & \rightleftarrows & [M^3] & \rightleftarrows & [M^4] & \rightleftarrows & [M^5] & \rightleftarrows & [M^7] & \rightleftarrows & [M^8]\,. \end{array}$$

The corresponding matrix factorizations are :

$$\varphi^1_{\binom{4}{14}} = \psi^1_{\binom{4}{14}} = \begin{pmatrix} z & 0 & x & y \\ 0 & z & y^4 & -x^2 \\ x^2 & y & -z & 0 \\ y^4 & -x & 0 & -z \end{pmatrix}, \quad \varphi^8_{\binom{2}{8}} = \psi^8_{\binom{2}{8}} = \begin{pmatrix} z & 0 & x & y^2 \\ 0 & z & y^3 & -x^2 \\ x^2 & y^2 & -z & 0 \\ y^3 & -x & 0 & -z \end{pmatrix},$$

$$\varphi^2_{\substack{3\\5\\13}} = \psi^2_{\substack{3\\5\\13}} = \begin{pmatrix} z & -y^2 & xy & 0 & -x^2 & 0 \\ -y^3 & -z & 0 & 0 & 0 & x \\ 0 & 0 & -z & x & 0 & y \\ 0 & -xy & x^2 & z & y^4 & 0 \\ -x & 0 & 0 & y & -z & 0 \\ 0 & x^2 & y^4 & 0 & -xy^3 & z \end{pmatrix}, \quad \varphi^6_{\substack{1\\5\\9}} = \psi^6_{\substack{1\\5\\9}} = \begin{pmatrix} -z & 0 & 0 & y^2 & 0 & x \\ xy & z & -y^3 & 0 & -x^2 & 0 \\ 0 & -y^2 & -z & x & 0 & 0 \\ y^3 & 0 & x^2 & z & -xy^2 & 0 \\ 0 & -x & 0 & 0 & -z & y \\ x^2 & 0 & -xy^2 & 0 & y^4 & z \end{pmatrix},$$

$$\varphi^3_{\begin{pmatrix}\substack{2\\4\\6\\12}\end{pmatrix}} = \psi^3_{\begin{pmatrix}\substack{2\\4\\6\\12}\end{pmatrix}} = \begin{pmatrix} -z & 0 & -xy & y^2 & 0 & 0 & x^2 & 0 \\ 0 & -z & y^3 & 0 & 0 & 0 & 0 & x \\ 0 & y^2 & z & 0 & 0 & -x & 0 & 0 \\ y^3 & xy & 0 & z & -x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & -z & 0 & y^3 & y \\ 0 & 0 & -x^2 & 0 & 0 & -z & 0 & y^2 \\ x & 0 & 0 & 0 & y^2 & -y & z & 0 \\ 0 & x^2 & 0 & 0 & 0 & y^3 & 0 & z \end{pmatrix},$$

$$\varphi^4_{\begin{pmatrix}\substack{1\\3\\5\\7\\11}\end{pmatrix}} = \psi^4_{\begin{pmatrix}\substack{1\\3\\5\\7\\11}\end{pmatrix}} = \begin{pmatrix} z & 0 & xy & 0 & 0 & -y^2 & y^3 & 0 & -x^2 & 0 \\ 0 & -z & 0 & 0 & 0 & 0 & 0 & -y^2 & 0 & x \\ 0 & 0 & -z & y^2 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & xy & y^3 & z & 0 & 0 & -x^2 & 0 & 0 & 0 \\ 0 & y^2 & 0 & 0 & z & -x & 0 & 0 & y^3 & 0 \\ -y^3 & 0 & 0 & 0 & -x^2 & -z & 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & -x & 0 & 0 & -z & 0 & 0 & y \\ 0 & -y^3 & x^2 & 0 & 0 & 0 & xy^2 & z & 0 & 0 \\ -x & 0 & 0 & 0 & y^2 & 0 & 0 & y & -z & 0 \\ 0 & x^2 & xy^2 & 0 & 0 & 0 & y^4 & 0 & 0 & z \end{pmatrix},$$

$$\varphi^5_{\begin{pmatrix} 0\\2\\4\\6\\8\\10 \end{pmatrix}} = \psi^5_{\begin{pmatrix} 0\\2\\4\\6\\8\\10 \end{pmatrix}} = \begin{pmatrix} -z & 0 & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 & 0 & x \\ 0 & -z & -xy & 0 & 0 & 0 & y^3 & -y^2 & 0 & 0 & x^2 & 0 \\ 0 & 0 & z & 0 & 0 & -y^2 & 0 & 0 & y^3 & -x & 0 & 0 \\ xy & 0 & 0 & z & -y^3 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y^2 & -z & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & -y^3 & 0 & 0 & -z & -x^2 & 0 & 0 & 0 & xy^2 & y^2 \\ y^2 & y^2 & 0 & 0 & 0 & -x & z & 0 & 0 & 0 & 0 & 0 \\ y^3 & 0 & 0 & 0 & x^2 & 0 & 0 & z & -xy^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & -z & 0 & 0 & y \\ 0 & 0 & -x^2 & -y^3 & 0 & 0 & xy^2 & 0 & 0 & -z & -y^4 & 0 \\ 0 & x & 0 & 0 & y^2 & 0 & 0 & 0 & 0 & -y & z & 0 \\ x^2 & 0 & 0 & 0 & -xy^2 & 0 & 0 & 0 & y^4 & 0 & 0 & z \end{pmatrix} .$$

$$\varphi^7_{\begin{pmatrix} 1\\3\\7\\9 \end{pmatrix}} = \psi^7_{\begin{pmatrix} 1\\3\\7\\9 \end{pmatrix}} = \begin{pmatrix} z & 0 & 0 & 0 & -y^3 & 0 & 0 & -x \\ xy & -z & 0 & 0 & 0 & y^2 & x^2 & 0 \\ 0 & 0 & -z & y^2 & 0 & x & -y^3 & 0 \\ 0 & 0 & 0 & z & -x^2 & 0 & 0 & y^2 \\ -y^2 & 0 & 0 & -x & -z & 0 & 0 & 0 \\ 0 & y^3 & x^2 & 0 & xy^2 & z & 0 & 0 \\ 0 & x & -y^2 & 0 & 0 & 0 & z & y \\ -x^2 & 0 & 0 & y^3 & 0 & 0 & 0 & -z \end{pmatrix} .$$

The shift functor T acts on these matrix factorizations as follows.

- For type A_l, $T(M^k) \simeq M^{l+2-k}$ for any $k \in \Pi$.
- For type D_l, $T(M^k) \simeq M^k$ for all $k \in \Pi$, except that $T(M^{l-1}) \simeq M^l$, $T(M^l) \simeq M^{l-1}$ if l is odd.
- For type E_6, $T(M^k) \simeq M^k$ for $k = 1, 2$ but $T(M^3) \simeq M^4$, $T(M^4) \simeq M^3$ and $T(M^5) \simeq M^6$, $T(M^6) \simeq M^5$.
- For type E_7 or E_8, $T(M^k) \simeq M^k$ for any $k \in \Pi$.

Remember that the Serre functor (in Theorem 3.6 (iii)) is defined by $\mathcal{S} := T\tau^{-1}$ when we shall introduce the grading.

Table 2. The list of all the isomorphism classes of the indecomposable objects in $HMF^{gr}_R(f)$ for a polynomial $f \in R$ of type ADE is given.

The set of the isomorphism classes of the indecomposable objects is given by

$$\left\{ [\widetilde{M}^k_n := (Q^k, S^k_n)], \quad k \in \Pi, \quad n \in \mathbb{Z} \right\} .$$

Here, for each $k \in \Pi$, Q^k is the matrix factorization of size $2\nu_k$ given in Table 1, and the grading matrix S^k_n is a diagonal matrix as follows:

$$S^k_n := \operatorname{diag}\left(q^k_1, -q^k_1, \cdots, q^k_{\nu_k}, -q^k_{\nu_k}; \bar{q}^k_1, -\bar{q}^k_1, \cdots, \bar{q}^k_{\nu_k}, -\bar{q}^k_{\nu_k} \right) + \phi^k_n \cdot \mathbf{1}_{4\nu_k} ,$$

where the phase is given by $\phi_n^k = \frac{2n+\sigma}{h}$ for $k \in \Pi_\sigma$, $\sigma = 1, 2$, and the data $q_j^k, \bar{q}_j^k$ are given below.

Type A_l $(h = l + 1)$: In this case, $\nu_k = 1$ for all $k \in \Pi$ and the grading is given by

$$(q_1^k; \bar{q}_1^k) = \frac{1}{l+1}(b - k; (l + 1 - b) - k) \ .$$

For type D_l and E_l, for any $k \in \Pi$ there exists a representative of the isomorphism classes of the indecomposable objects such that $q_j^k = \bar{q}_j^k$, $j = 1, \cdots, \nu_k$. The matrix factorizations Q^k's listed in Table 1 are just such ones. Therefore, we present only q_j^k and omit $\bar{q}_j^k$.

Type D_l $(h = 2(l - 1))$:

k	ν_k	$(q_1^k, \cdots, q_{\nu_k}^k)$
1	1	$\frac{1}{2(l-1)}(l-3)$
$2, \cdots, (l-2)$	2	$\frac{1}{2(l-1)}(l-k-2, l-k)$
$(l-1), l$	1	$\frac{1}{2(l-1)}(1)$.

Type E_6 $(h = 12)$:

k	ν_k	$(q_1^k, \cdots, q_{\nu_k}^k)$
1	2	$\frac{1}{12}(1,5)$
2	3	$\frac{1}{12}(0,2,4)$
$3,4$	2	$\frac{1}{12}(1,3)$
$5,6$	1	$\frac{1}{12}(2)$.

Type E_7 $(h = 18)$:

k	ν_k	$(q_1^k, \cdots, q_{\nu_k}^k)$
1	2	$\frac{1}{18}(2,8)$
2	3	$\frac{1}{18}(1,3,7)$
3	4	$\frac{1}{18}(0,2,4,6)$
4	2	$\frac{1}{18}(1,5)$
5	3	$\frac{1}{18}(1,3,5)$
6	2	$\frac{1}{18}(2,4)$
7	1	$\frac{1}{18}(3)$.

Type E_8 $(h = 30)$:

k	ν_k	$(q_1^k, \cdots, q_{\nu_k}^k)$
1	2	$\frac{1}{30}(4,14)$
2	3	$\frac{1}{30}(3,5,13)$
3	4	$\frac{1}{30}(2,4,6,12)$
4	5	$\frac{1}{30}(1,3,5,7,11)$
5	6	$\frac{1}{30}(0,2,4,6,8,10)$
6	3	$\frac{1}{30}(1,5,9)$
7	4	$\frac{1}{30}(1,3,7,9)$
8	2	$\frac{1}{30}(2,8)$.

(i) If we set $n = 0$ for all $k \in \Pi$, then $\{\widetilde{M}_0^1, \cdots, \widetilde{M}_0^l\}$ corresponds to the Dynkin quiver with a principal orientation (see Proposition 4.3).

(ii) For the grading matrix of $\widetilde{M}_n^k$, one has $q_j^k \neq 0$ for $j > 1$, and $q_1^k = 0$ if and only if $F(\widetilde{M}_n^k) = M_o$.

(iii) Lemma 3.9 can be checked at the level of the grading matrices S. Namely, for an indecomposable object $\widetilde{M}_n^k \in HMF_R^{gr}(f)$, the cone $C(\Psi)$ of a nonzero morphism $\Psi \in \mathrm{Hom}_{HMF_R^{gr}(f)}(\mathcal{S}^{-1}(\widetilde{M}_n^k), \widetilde{M}_n^k)$ is isomorphic to the direct sum of indecomposable objects $\widetilde{M}_{n_1}^{k_1} \oplus \cdots \oplus \widetilde{M}_{n_m}^{k_m}$ for some $m \in \mathbb{Z}_{>0}$ such that $\{k_1, \cdots, k_m\} = \{k' \in \Pi \mid d(k, k') = 1\}$ and $\phi(\widetilde{M}_{n_i}^{k_i}) = \phi(\widetilde{M}_n^k) + \frac{1}{h}$ for any $i = 1, \cdots, m$. Correspondingly, one can check that

$$\left\{q_1^k-\frac{1}{h},-q_1^k-\frac{1}{h},\cdots,q_{\nu_k}^k-\frac{1}{h},-q_{\nu_k}^k-\frac{1}{h}\right\}\coprod\left\{q_1^k+\frac{1}{h},-q_1^k+\frac{1}{h},\cdots,q_{\nu_k}^k+\frac{1}{h},-q_{\nu_k}^k+\frac{1}{h}\right\}$$
$$=\coprod_{i=1}^{m}\left\{q_1^{k_i},-q_1^{k_i},\cdots,q_{\nu_{k_i}}^{k_i},-q_{\nu_{k_i}}^{k_i}\right\} \tag{5.1}$$

and the same identities for $\bar{q}_j^k$, $k\in\Pi$, $j=1,\cdots,\nu_k$.

Table 3. The following tables give all the morphisms between all the isomorphism classes of the indecomposable objects in $HMF_R^{gr}(f)$ for a polynomial f of type ADE.

Recall that, for two indecomposable objects $\widetilde{M}_n^k,\widetilde{M}_{n'}^{k'}\in HMF_R^{gr}(f)$, $\mathfrak{C}(k,k')$ is the multi-set of non-negative integers such that

$$\mathfrak{C}(k,k'):=\{c:=h(\phi_{n'}^{k'}-\phi_n^k)+2n''\mid \mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_n^k,\tau^{n''}(\widetilde{M}_{n'}^{k'}))\neq 0\ ,\quad n\in\mathbb{Z}\}\ ,$$

where the integer c appears with multiplicity $d:=\dim_{\mathbb{C}}(\mathrm{Hom}_{HMF_R^{gr}(f)}(\widetilde{M}_n^k,\tau^{n''}(\widetilde{M}_{n'}^{k'})))$. We sometimes write c^d instead of d copies of c.

For each type X of ADE, $\mathfrak{C}(k,k')$ is listed up for any $k,k'\in\Pi$.

Type A_l $(h-2=l-1)$: One obtains (essentially the same as those given in [HW])

$$\mathfrak{C}(k,k')=\left\{\begin{array}{l}|k'-k|,\ |k'-k|+2,\ |k'-k|+4,\ \cdots\cdots,\\ \qquad\cdots\cdots,l-3-|(l-1)-(k+k'-2)|,\ l-1-|(l-1)-(k+k'-2)|\end{array}\right\}.$$

For any $k,k'\in\Pi$, $\mathfrak{C}(k,k')$ does not contain multiple copies of the same integer. Pictorially, the table of $\mathfrak{C}(k,k')$ is displayed as

$k\backslash k'$	1	2	3	·········	$l-2$	$l-1$	l
1	0	1	2	·········	$l-3$	$l-2$	$l-1$
2	1	0 2	1 3	·········	$l-4$ $l-2$	$l-3$ $l-1$	$l-2$
3	2	1 3	0 2 4	·········	$l-5$ $l-3$ $l-1$	$l-4$ $l-2$	$l-3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$l-2$	$l-3$	$l-4$ $l-2$	$l-5$ $l-3$ $l-1$	·········	0 2 4	1 3	2
$l-1$	$l-2$	$l-3$ $l-1$	$l-4$ $l-2$	·········	1 3	0 2	1
l	$l-1$	$l-2$	$l-3$	·········	2	1	0

Type D_l $(h-2=2l-4)$:

$k\backslash k'$	1	k' $\left(2\leq k'\leq l-2\right)$	$l-1$	l
1	$0\ 2l{-}4$	$k'{-}1\ 2l{-}3{-}k'$	$l{-}2$	$l{-}2$
k $(2\leq k\leq l-2)$	$k{-}1$ $2l{-}3{-}k$	$\lvert k'{-}k\rvert\ \ \lvert k'{-}k\rvert{+}2\ \cdots\cdots$ $\cdots\cdots\ k{+}k'{-}4\ \ k{+}k'{-}2$, $2l{-}2{-}(k{+}k')\ 2l{-}(k{+}k')\ \cdots\cdots$ $\cdots\cdots\ 2l{-}6{-}\lvert k'{-}k\rvert\ \ 2l{-}4{-}\lvert k'{-}k\rvert$	$l{-}1{-}k\ l{+}1{+}k\ \cdots$ $\cdots\ l{-}5{+}k\ l{-}3{+}k$	$l{-}1{-}k\ l{+}1{+}k\ \cdots$ $\cdots\ l{-}5{+}k\ l{-}3{+}k$
$l-1$	$l{-}2$	$l{-}1{-}k\ l{+}1{+}k\ \cdots$ $\cdots\ l{-}5{+}k\ l{-}3{+}k$	$0\ 4\ 8\ \cdots\ 2l{-}4$ (l: even) $0\ 4\ 8\ \cdots\ 2l{-}6$ (l: odd)	$2\ 6\ 10\ \cdots\ 2l{-}6$ (l: even) $2\ 6\ 10\ \cdots\ 2l{-}4$ (l: odd)
l	$l{-}2$	$l{-}1{-}k\ l{+}1{+}k\ \cdots$ $\cdots\ l{-}5{+}k\ l{-}3{+}k$	$2\ 6\ 10\ \cdots\ 2l{-}6$ (l: even) $2\ 6\ 10\ \cdots\ 2l{-}4$ (l: odd)	$0\ 4\ 8\ \cdots\ 2l{-}4$ (l: even) $0\ 4\ 8\ \cdots\ 2l{-}6$ (l: odd)

Type E_6 $(h-2=10)$:

$k\backslash k'$	1	2	3	4	5	6
1	0 4 6 10	$1\ 3\ 5^2\ 7\ 9$	2 4 6 8	2 4 6 8	3 7	3 7
2	$1\ 3\ 5^2\ 7\ 9$	$0\ 2^2\ 4^3\ 6^3\ 8^2\ 10$	$1\ 3^2\ 5^2\ 7^2\ 9$	$1\ 3^2\ 5^2\ 7^2\ 9$	2 4 6 8	2 4 6 8
3	2 4 6 8	$1\ 3^2\ 5^2\ 7^2\ 9$	$0\ 2\ 4\ 6^2\ 8$	$2\ 4^2\ 6\ 8\ 10$	1 5 7	3 5 9
4	2 4 6 8	$1\ 3^2\ 5^2\ 7^2\ 9$	$2\ 4^2\ 6\ 8\ 10$	$0\ 2\ 4\ 6^2\ 8$	3 5 9	1 5 7
5	3 7	2 4 6 8	1 5 7	3 5 9	0 6	4 10
6	3 7	2 4 6 8	3 5 9	1 5 7	4 10	0 6

Type E_7 $(h-2=16)$:

$k\backslash k'$	1	2	3	4	5	6	7
1	0 6 10 16	1 5 7 9 11 15	2 4 6 8^2 10 12 14	3 7 9 13	3 5 7 9 11 13	4 6 10 12	5 11
2	1 5 7 9 11 15	0 2 4 6^2 8^2 10^2 12 14 16	1 3^2 5^2 7^3 9^3 11^2 13^2 15	2 4 6 8^2 10 12 14	2 4^2 6^2 8^2 10^2 12^2 14	3 5^2 7 9 11^2 13	4 6 10 12
3	2 4 6 8^2 10 12 14	1 3^2 5^2 7^3 9^3 11^2 13^2 15	0 2^2 4^3 6^4 8^4 10^4 12^3 14^2 16	1 3 5^2 7^2 9^2 11^2 13 15	1 3^2 5^3 7^3 9^3 11^3 13^2 15	2 4^2 6^2 8^2 10^2 12^2 14	3 5 7 9 11 13
4	3 7 9 13	2 4 6 8^2 10 12 14	1 3 5^2 7^2 9^2 11^2 13 15	0 4 6 8 10 12 16	2 4 6^2 8 10^2 12 14	3 5 7 9 11 13	4 8 12
5	3 5 7 9 11 13	2 4^2 6^2 8^2 10^2 12^2 14	1 3^2 5^3 7^3 9^3 11^3 13^2 15	2 4 6^2 8 10^2 12 14	0 2 4^2 6^2 8^3 10^2 12^2 14 16	1 3 5 7^2 9^2 11 13 15	2 6 8 10 14
6	4 6 10 12	3 5^2 7 9 11^2 13	2 4^2 6^2 8^2 10^2 12^2 14	3 5 7 9 11 13	1 3 5 7^2 9^2 11 13 15	0 2 6 8^2 10 14 16	1 7 9 15
7	5 11	4 6 10 12	3 5 7 9 11 13	4 8 12	2 6 8 10 14	1 7 9 15	0 8 16

Type E_8 $(h-2=28)$:

$k\backslash k'$	1	2	3	4	5	6	7	8
1	$0\ 10\ 18\ 28$	$1\ 9\ 11\ 17\ 19\ 27$	$2\ 8\ 10\ 12\ 16\ 18\ 20\ 26$	$3\ 7\ 9\ 11\ 13\ 15\ 17\ 19\ 21\ 25$	$4\ 6\ 8\ 10\ 12\ 14^2\ 16\ 18\ 20\ 22\ 24$	$5\ 9\ 13\ 15\ 19\ 23$	$5\ 7\ 11\ 13\ 15\ 17\ 21\ 23$	$6\ 12\ 16\ 22$
2	$1\ 9\ 11\ 17\ 19\ 27$	$0\ 2\ 8\ 10^2\ 12\ 16\ 18^2\ 20\ 26\ 28$	$1\ 3\ 7\ 9^2\ 11^2\ 13\ 15\ 17^2\ 19^2\ 21\ 25\ 27$	$2\ 4\ 6\ 8^2\ 10^2\ 12^2\ 14^2\ 16^2\ 18^2\ 20^2\ 22\ 24\ 26$	$3\ 5^2\ 7^2\ 9^2\ 11^2\ 13^3\ 15^3\ 17^2\ 19^2\ 21^2\ 23^2\ 25$	$4\ 6\ 8\ 10\ 12\ 14^2\ 16\ 18\ 20\ 22\ 24$	$4\ 6^2\ 8\ 10\ 12^2\ 14^2\ 16^2\ 18\ 20\ 22^2\ 24$	$5\ 7\ 11\ 13\ 15\ 17\ 21\ 23$
3	$2\ 8\ 10\ 12\ 16\ 18\ 20\ 26$	$1\ 3\ 7\ 9^2\ 11^2\ 13\ 15\ 17^2\ 19^2\ 21\ 25\ 27$	$0\ 2\ 4\ 6\ 8^2\ 10^3\ 12^2\ 14^2\ 16^2\ 18^3\ 20^2\ 22\ 24\ 26\ 28$	$1\ 3\ 5^2\ 7^2\ 9^2\ 11^3\ 13^3\ 15^3\ 17^3\ 19^3\ 21^2\ 23^2\ 25\ 27$	$2\ 4^2\ 6^3\ 8^3\ 10^3\ 12^4\ 16^4\ 18^3\ 20^3\ 22^3\ 24^2\ 26$	$3\ 5\ 7^2\ 9\ 11^2\ 13^2\ 15^2\ 17^2\ 19\ 21^2\ 23\ 25$	$3\ 5^2\ 7^2\ 9^2\ 11^2\ 13^3\ 15^3\ 17^2\ 19^2\ 21^2\ 23^2\ 25$	$4\ 6\ 8\ 10\ 12\ 14^2\ 16\ 18\ 20\ 22\ 24$
4	$3\ 7\ 9\ 11\ 13\ 15\ 17\ 19\ 21\ 25$	$2\ 4\ 6\ 8^2\ 10^2\ 12^2\ 14^2\ 16^2\ 18^2\ 20^2\ 22\ 24\ 26$	$1\ 3\ 5^2\ 7^2\ 9^2\ 11^3\ 13^3\ 15^3\ 17^3\ 19^3\ 21^2\ 23^2\ 25\ 27$	$0\ 2\ 4^2\ 6^3\ 8^3\ 10^4\ 12^4\ 14^4\ 16^4\ 18^4\ 20^3\ 22^3\ 24^2\ 26\ 28$	$1\ 3^2\ 5^3\ 7^4\ 9^4\ 11^5\ 13^5\ 15^5\ 17^5\ 19^4\ 21^4\ 23^3\ 25^2\ 27$	$2\ 4\ 6^2\ 8^2\ 10^2\ 12^3\ 14^2\ 16^3\ 18^2\ 20^2\ 22^2\ 24\ 26$	$2\ 4^2\ 6^2\ 8^3\ 10^3\ 12^3\ 14^4\ 16^3\ 18^3\ 20^3\ 22^2\ 24^2\ 26$	$3\ 5\ 7\ 9^2\ 11\ 13^2\ 15^2\ 17\ 19^2\ 21\ 23\ 25$
5	$4\ 6\ 8\ 10\ 12\ 14^2\ 16\ 18\ 20\ 22\ 24$	$3\ 5^2\ 7^2\ 9^2\ 11^2\ 13^3\ 15^3\ 17^2\ 19^2\ 21^2\ 23^2\ 25$	$2\ 4^2\ 6^3\ 8^3\ 10^3\ 12^4\ 16^4\ 18^3\ 20^3\ 22^3\ 24^2\ 26$	$1\ 3^2\ 5^3\ 7^4\ 9^4\ 11^5\ 13^5\ 15^5\ 17^5\ 19^4\ 21^4\ 23^3\ 25^2\ 27$	$0\ 2^2\ 4^3\ 6^4\ 8^5\ 10^6\ 12^6\ 14^6\ 16^6\ 18^6\ 20^5\ 22^4\ 24^3\ 26^2\ 28$	$1\ 3\ 5^2\ 7^2\ 9^3\ 11^3\ 13^3\ 15^3\ 17^3\ 19^3\ 21^2\ 23^2\ 25\ 27$	$1\ 3^2\ 5^2\ 7^3\ 9^4\ 11^4\ 13^4\ 15^4\ 17^4\ 19^4\ 21^3\ 23^2\ 25^2\ 27$	$2\ 4\ 6\ 8^2\ 10^2\ 12^2\ 14^2\ 16^2\ 18^2\ 20^2\ 22\ 24\ 26$
6	$5\ 9\ 13\ 15\ 19\ 23$	$4\ 6\ 8\ 10\ 12\ 14^2\ 16\ 18\ 20\ 22\ 24$	$3\ 5\ 7^2\ 9\ 11^2\ 13^2\ 15^2\ 17^2\ 19\ 21^2\ 23\ 25$	$2\ 4\ 6^2\ 8^2\ 10^2\ 12^3\ 14^2\ 16^3\ 18^2\ 20^2\ 22^2\ 24\ 26$	$1\ 3\ 5^2\ 7^2\ 9^3\ 11^3\ 13^3\ 15^3\ 17^3\ 19^3\ 21^2\ 23^2\ 25\ 27$	$0\ 4\ 6\ 8\ 10^2\ 12\ 14^2\ 16\ 18^2\ 20\ 22\ 24\ 28$	$2\ 4\ 6\ 8^2\ 10^2\ 12^2\ 14^2\ 16^2\ 18^2\ 20^2\ 22\ 24\ 26$	$3\ 7\ 9\ 11\ 13\ 15\ 17\ 19\ 21\ 25$
7	$5\ 7\ 11\ 13\ 15\ 17\ 21\ 23$	$4\ 6^2\ 8\ 10\ 12^2\ 14^2\ 16^2\ 18\ 20\ 22^2\ 24$	$3\ 5^2\ 7^2\ 9^2\ 11^2\ 13^3\ 15^3\ 17^2\ 19^2\ 21^2\ 23^2\ 25$	$2\ 4^2\ 6^2\ 8^3\ 10^3\ 12^3\ 14^4\ 16^3\ 18^3\ 20^3\ 22^2\ 24^2\ 26$	$1\ 3^2\ 5^2\ 7^3\ 9^4\ 11^4\ 13^4\ 15^4\ 17^4\ 19^4\ 21^3\ 23^2\ 25^2\ 27$	$2\ 4\ 6\ 8^2\ 10^2\ 12^2\ 14^2\ 16^2\ 18^2\ 20^2\ 22\ 24\ 26$	$0\ 2\ 4\ 6^2\ 8^2\ 10^3\ 12^3\ 14^2\ 16^3\ 18^3\ 20^2\ 22^2\ 24\ 26\ 28$	$1\ 5\ 7\ 9\ 11^2\ 13\ 15\ 17^2\ 19\ 21\ 23\ 27$
8	$6\ 12\ 16\ 22$	$5\ 7\ 11\ 13\ 15\ 17\ 21\ 23$	$4\ 6\ 8\ 10\ 12\ 14^2\ 16\ 18\ 20\ 22\ 24$	$3\ 5\ 7\ 9^2\ 11\ 13^2\ 15^2\ 17\ 19^2\ 21\ 23\ 25$	$2\ 4\ 6\ 8^2\ 10^2\ 12^2\ 14^2\ 16^2\ 18^2\ 20^2\ 22\ 24\ 26$	$3\ 7\ 9\ 11\ 13\ 15\ 17\ 19\ 21\ 25$	$1\ 5\ 7\ 9\ 11^2\ 13\ 15\ 17^2\ 19\ 21\ 23\ 27$	$0\ 6\ 10\ 12\ 16\ 18\ 22\ 28$

For each ADE case, one can easily check the followings.

(i) One has $\mathfrak{C}(k,k') = \mathfrak{C}(k',k)$ for any $k,k' \in \Pi$. This implies, for any $k,k' \in \Pi$, $\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'}) \simeq \mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \widetilde{M}^k_{n''})$ holds for some $n, n, n'' \in \mathbb{Z}$ such that $\phi^k_{n''} - \phi^{k'}_{n'} = \phi^{k'}_{n'} - \phi^k_n$.

(ii) A consequence of the Serre duality (Theorem 3.6 (iii)),

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'})) = \dim_{\mathbb{C}}(\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^{k'}_{n'}, \mathcal{S}(\widetilde{M}^k_n))) \ ,$$

can be checked as follows. For an indecomposable object $\widetilde{M}^k_n \in HMF^{gr}_R(f)$, $k^{\mathcal{S}} \in \Pi$ denotes the vertex such that $[F(\mathcal{S}(\widetilde{M}^k_n))] = [M^{k^{\mathcal{S}}}] \in \Pi$. Then, for the given multi-set $\mathfrak{C}(k,k')$, one has $\mathfrak{C}(k',k^{\mathcal{S}}) = \{h-2-c \mid c \in \mathfrak{C}(k,k')\}$.

(iii) One can check Theorem 3.6 (ii-b): for any two indecomposable objects $\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'} \in HMF^{gr}_R(f)$, $\dim_{\mathbb{C}}(\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'})) = 1$ if $h|\phi^{k'}_{n'} - \phi^k_n| = d(k,k')$. Correspondingly, one has $\sharp\{c \in \mathfrak{C}(k,k') \mid c = d(\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'})\} = 1$. In addition, the Serre duality implies that $\dim_{\mathbb{C}}(\mathrm{Hom}_{HMF^{gr}_R(f)}(\widetilde{M}^k_n, \widetilde{M}^{k'}_{n'})) = 1$ if $h|\phi^{k'}_{n'} - \phi^k_n| = h-2-d(k^{\mathcal{S}},k')$. Namely, one has $\sharp\{c \in \mathfrak{C}(k,k') \mid c = h-2-d(k^{\mathcal{S}},k')\} = 1$. These facts, together with Corollary 3.11, imply that $\mathfrak{C}(k,k') = \mathfrak{C}(k',k)$ is described in the following form:

$$\mathfrak{C}(k,k') = \{c_1, \cdots, c_s\}\ , \qquad d(k,k') = c_1 < c_2 \leq \cdots \leq c_{s-1} < c_s = h-2-d(k^{\mathcal{S}},k)$$

for some $s \in \mathbb{Z}_{>0}$.

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