THE PRINCIPAL NUMBERS OF K. SAITO FOR THE TYPES A_l, D_l AND E_l

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Abstract

The principal number $\sigma(\Gamma)$ for a tree Γ is introduced by K. Saito [S2] as the maximal number of chambers contained in components of the complement of the graphic arrangement attached to Γ (see §1).

The purpose of the present paper is to determine the principal numbers for the Coxeter-Dynkin graphs [1] of types A_l, D_l and E_l. We show that the generating series $\sum_{l=1}^{\infty} \frac{\sigma(A_l)}{l!} x^l$, $\sum_{l=3}^{\infty} \frac{\sigma(D_l)}{l!} x^l$ and $\sum_{l=4}^{\infty} \frac{\sigma(E_l)}{l!} x^l$ satisfy certain differential equations of the first order. By solving the equations, we obtain the following results:

$$\begin{aligned} \sum_{l=1}^{\infty} \frac{\sigma(A_l)}{l!} x^l &= \tan(\tfrac{x}{2} + \tfrac{\pi}{4}) - 1. \\ \sum_{l=3}^{\infty} \frac{\sigma(D_l)}{l!} x^l &= 2(x-1)\tan(\tfrac{x}{2} + \tfrac{\pi}{4}) - x^2 + 2. \\ \sum_{l=4}^{\infty} \frac{\sigma(E_l)}{l!} x^l &= (\tfrac{1}{2}x^2 - 2x + 3)\tan(\tfrac{x}{2} + \tfrac{\pi}{4}) - 3x^3 - x - 3. \end{aligned}$$

1. Introduction

First, we recall the principal numbers of K. Saito for trees.

Let Γ be a tree (i.e. a connected graph without cycles) with l vertices, and let Π be the vertex set of Γ. Put $|\Gamma| := |\Pi| = l$. Denote the edge set of Γ by $Edge(\Gamma)$.

Any orientation o on Γ defines a partially order on Π by transitive closure. Let $\Sigma(o)$ be the set of all linear orderings of Π which agree with the ordering of o. The enumeration of $\sigma(o) := |\Sigma(o)|$, i.e. the enumeration of linear extensions of a partially ordered set, is one of basic problems in combinatorics (see, for instance, [St1]).

Here, let us consider the problem to maximize $\sigma(o)$ such that o is an orientation on the tree Γ. Put:

$$\sigma(\Gamma) := \max\{\sigma(o) \mid o : \text{all orientations on } \Gamma\} \tag{1}$$

[1] In this paper, we denote the Coxeter-Dynkin graphs of types A_l, D_l and E_l also by the same symbols A_l, D_l and E_l, respectively.

K. Saito [S2] has proved that this maximal number $\sigma(\Gamma)$ is attained by the two principal orientations o_{Π_1,Π_2}, o_{Π_2,Π_1}, where the principal orientations are defined as follows. Since Γ is a tree, the vertex set Π decomposes uniquely as $\Pi = \Pi_1 \sqcup \Pi_2$ (disjoint union), where Π_1 and Π_2 are totally disconnected in Γ. The ordered decomposition $\{\Pi_1, \Pi_2\}$ is called a *principal decomposition* of the tree Γ. Choosing a principal decomposition, we define a partial order on Π: for $\alpha, \beta \in \Pi$,

$$\alpha < \beta \Longleftrightarrow \overline{\alpha\beta} \in Edge(\Gamma) \text{ with } \alpha \in \Pi_1, \beta \in \Pi_2.$$

This partial order on Π is called a *principal orientation* on Γ and is denoted by o_{Π_1,Π_2}. The number $\sigma(\Gamma) = \sigma(o_{\Pi_1,\Pi_2})$ is called the *principal number* of Γ. In other words, the principal number $\sigma(\Gamma)$ of the tree Γ is the number of linear extensions of the partially ordered set $(\Pi, <_{o_{\Pi_1,\Pi_2}})$.

K. Saito has shown further that $\sigma(\Gamma)$ is expressed by a finite sum of hook length formulae for some rooted trees. The purpose of the present paper is to determine the principal numbers for the three series of the Coxeter-Dynkin graphs of types A_l, D_l and E_l by a different approach, i.e. by a use of generating series. We determine the generating series as described in Abstract.

The construction of this paper is as follows.

In the section 2, we see the inductive relation among the principal numbers, and we get an induction formula for the principal numbers.

The section 3 is the main part of this paper. Using the inductive relations among the principal numbers, we obtain ordinary differential equations of the first order. By solving these ordinary differential equations, we obtain the generating functions of the series $\{\sigma(A_l)\}$, $\{\sigma(D_l)\}$ and $\{\sigma(E_l)\}$, i.e. we give closed expressions to the generating series $\sum_{l=1}^{\infty} \frac{\sigma(A_l)}{l!} x^l$, $\sum_{l=3}^{\infty} \frac{\sigma(D_l)}{l!} x^l$ and $\sum_{l=4}^{\infty} \frac{\sigma(E_l)}{l!} x^l$. (It should be noted that the number $\sigma(A_l)$ is well known classically as the number of alternating permutations in the symmetric group of rank l.)

In the appendix, we present the principal numbers concretely for small l.

Note. It should be noted that the principal numbers of trees have the discrete geometric background related to Γ-*cones* (see [S2]). For the convenience of readers, we recall the definition of a Γ-cone.

Let Π be a finite set. Put $l := |\Pi|$. And let $V_\Pi := \oplus_{\alpha \in \Pi} \mathbb{R} v_\alpha / \mathbb{R} \cdot v_\Pi$ be a quotient vector space, where $v_\Pi := \sum_{\alpha \in \Pi} v_\alpha$. The permutation group $\mathfrak{S}(\Pi)$ acts on $\{v_\alpha\}_{\alpha \in \Pi}$ fixing v_Π, and, hence, the action extends linearly on V_Π. Let $\{\lambda_\alpha\}_{\alpha \in \Pi}$ be the dual basis of $\{v_\alpha\}_{\alpha \in \Pi}$, so that the differences $\lambda_{\alpha\beta} := \lambda_\alpha - \lambda_\beta$ for $\alpha, \beta \in \Pi$ are well defined linear forms on V_Π (forming the root system of type A_{l-1}). The zero locus $H_{\alpha\beta}$ of $\lambda_{\alpha\beta}$ $(\alpha \neq \beta)$ in

V_Π is the reflection hyperplane of the reflection action induced by the transposition (α,β). The complement of the union $\cup_{\alpha,\beta\in\Pi,\alpha\neq\beta}H_{\alpha\beta}$ in V_Π decomposes into $l!$ connected components, called *chambers*. Let Γ be a tree which has Π as its vertex set. The complement of the union $\cup_{(\alpha,\beta)\in Edge(\Gamma)}H_{\alpha\beta}$ in V_Π decomposes into 2^{l-1} connected components, called Γ-*cones*. Note that there exists one-to-one correspondence between Γ-cones and orientations on Γ.

What the above definitions make clear at once is that each Γ-cone is subdivided into chambers. This fact leads us to one question:
"How many chambers are contained in each Γ*-cone?"*

K. Saito have shown that two particular Γ-cones which correspond to the principal orientations, called *principal* Γ*-cones*, contain the most number of chambers. In terms of the Γ-cones, the principal number is the number of chambers contained in a principal Γ-cone. This geometric background is our original motivation for calculating the principal numbers.

And it should be also noted that the principal numbers for types A_l, D_l and E_l have geometric meanings as the number of topological types of Morsification of simple polynomials (see [S1]), and are of particular interest.

2. **An induction formula**

In this section, we calculate the principal number $\sigma(\Gamma)$ of a tree Γ inductively.

Suppose that we know the principal numbers for all the trees such that the number of vertices is less than $l := |\Gamma|$.

Consider one linear extension of o_{Π_1,Π_2}. Then, the minimum (maximum) element is in Π_1 (Π_2). Let $v \in \Pi_1$ be the minimum element. And consider a graph $\Gamma\setminus\{v\}$. This graph consists of some trees. So we denote the connected components of the graph $\Gamma\setminus\{v\}$ by $\Gamma_{v,(1)}$, $\Gamma_{v,(2)}$, ..., $\Gamma_{v,(t_v)}$, where t_v is the number of the connected components of the graph $\Gamma\setminus\{v\}$. It is obvious that the number of vertices of each tree is less than l.

Now we can calculate the principal number $\sigma(\Gamma)$ of the tree Γ. The number of linear extensions of each $\Gamma_{v,(k)}$ ($k = 1, ..., t_v$) is $\sigma(\Gamma_{v,(k)})$ which we know. And among the elements in different $\Gamma_{v,(k)}$, there are no order relations in o_{Π_1,Π_2}. So the number of the linear extensions of o_{Π_1,Π_2} such that v is the minimum element is:

$$\binom{(l-1)}{|\Gamma_{v,(1)}|}\cdot\binom{(l-1)-|\Gamma_{v,(1)}|}{|\Gamma_{v,(2)}|}\cdot\binom{(l-1)-|\Gamma_{v,(1)}|-|\Gamma_{v,(2)}|}{|\Gamma_{v,(3)}|}$$

$$\cdot \ldots \cdot \binom{(l-1)-\sum_{k=1}^{t_v-1}|\Gamma_{v,(k)}|}{|\Gamma_{v,(t_v)}|} \cdot \sigma(\Gamma_{v,(1)}) \cdot \sigma(\Gamma_{v,(2)}) \cdot \ldots \cdot \sigma(\Gamma_{v,(t_v)})$$

$$= \prod_{k=1}^{t_v} \binom{(l-1)-\sum_{i=1}^{k-1}|\Gamma_{v,(i)}|}{|\Gamma_{v,(k)}|} \cdot \prod_{k=1}^{t_v} \sigma(\Gamma_{v,(k)})$$

$$= \frac{(l-1)!}{|\Gamma_{v,(1)}|! \cdot |\Gamma_{v,(2)}|! \cdot \ldots \cdot |\Gamma_{v,(t_v)}|!} \cdot \prod_{k=1}^{t_v} \sigma(\Gamma_{v,(k)}).$$

At last, we can get the following induction formula by taking a summation for all $v \in \Pi_1$.

$$\sigma(\Gamma) = \sum_{v \in \Pi_1} \Big\{ (l-1)! \cdot \prod_{k=1}^{t_v} \frac{\sigma(\Gamma_{v,(k)})}{|\Gamma_{v,(k)}|!} \Big\}. \tag{2}$$

We also get the following formula in the same way by considering the maximum element.

$$\sigma(\Gamma) = \sum_{v \in \Pi_2} \Big\{ (l-1)! \cdot \prod_{k=1}^{t_v} \frac{\sigma(\Gamma_{v,(k)})}{|\Gamma_{v,(k)}|!} \Big\}. \tag{3}$$

From that $\Pi = \Pi_1 \sqcup \Pi_2$ (disjoint union) and the above equations (2) and (3), we obtain the next result (c.f. [S2] Assertion 2.1).

Proposition 2.1 (An induction formula for the principal number)**.** *Let Γ be a tree. The principal number $\sigma(\Gamma)$ of the tree Γ is given by the next induction formula.*

$$\sigma(\Gamma) = \frac{1}{2} \cdot (|\Gamma|-1)! \cdot \sum_{v \in \Pi} \Big\{ \prod_{k=1}^{t_v} \frac{\sigma(\Gamma_{v,(k)})}{|\Gamma_{v,(k)}|!} \Big\}, \tag{4}$$

where $\Gamma_{v,(k)}$ $(k = 1, ..., t_v)$ are connected components of a graph $\Gamma \setminus \{v\}$.

For the use in the next section, we reformulate this formula (4) as follows.

Corollary 2.2.

$$\sigma(\Gamma) = \frac{1}{2} \cdot \sum_{v \in \Pi} \Big\{ \prod_{k=1}^{t_v} \binom{(|\Gamma|-1)-\sum_{i=1}^{k-1}|\Gamma_{v,(i)}|}{|\Gamma_{v,(k)}|} \cdot \prod_{k=1}^{t_v} \sigma(\Gamma_{v,(k)}) \Big\}, \tag{5}$$

3. Generating functions for the series of types A_l, D_l and E_l

In this section, we give the generating functions for the series $\{\sigma(A_l)\}$, $\{\sigma(D_l)\}$ and $\{\sigma(E_l)\}$ of the principal numbers of types A_l $(l \geq 1)$, D_l $(l \geq 3)$ and E_l $(l \geq 4)$.

1. A_l-type.

The Coxeter-Dynkin graph of type A_l $(l \geq 1)$ is a tree given as follows:

$$\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\cdots-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\!\circ$$

where l is the number of vertices of the graph. We denote the principal number of type A_l by $\sigma(A_l)$.

Theorem 3.1.

$$\sum_{l=1}^{\infty} \frac{\sigma(A_l)}{l!} x^l = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) - 1. \tag{6}$$

Proof. We put $\sigma(A_0) := 1$. From the formula (5), we get an equation:

$$\sigma(A_{l+1}) = \frac{1}{2} \sum_{i=0}^{l} \binom{l}{i} \sigma(A_i)\sigma(A_{l-i}) \qquad (l \geq 1)$$

We transform this equation as follows.

$$2 \cdot \frac{\sigma(A_{l+1})}{l!} = \sum_{i=0}^{l} \frac{\sigma(A_i)}{i!} \cdot \frac{\sigma(A_{l-i})}{(l-i)!}$$

$$2 \cdot \sum_{l=1}^{\infty} \frac{\sigma(A_{l+1})}{l!} x^l = \sum_{l=1}^{\infty} \Big(\sum_{i=0}^{l} \frac{\sigma(A_i)}{i!} \cdot \frac{\sigma(A_{l-i})}{(l-i)!} \Big) x^l$$

$$-2 + 2 \cdot \sum_{l=0}^{\infty} \frac{\sigma(A_{l+1})}{l!} x^l = -1 + \sum_{l=0}^{\infty} \Big(\sum_{i=0}^{l} \frac{\sigma(A_i)}{i!} \cdot \frac{\sigma(A_{l-i})}{(l-i)!} \Big) x^l$$

Here, put $f_A(x) := \sum_{l=0}^{\infty} \frac{\sigma(A_l)}{l!} x^l$. Then, above equation implies the differential equation:

$$2f_A'(x) = 1 + (f_A(x))^2.$$

We can solve this differential equation by using the separation of variables. With the initial condition: $f_A(0) = \sigma(A_0) = 1$, the solution is:

$$f_A(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

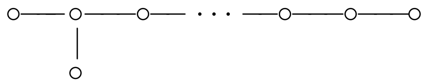
So we obtain the formula (6). $\square$

Remark. The formula (6) agrees with the classical formula.

$$f'_A(x) = \sum_{l=0}^{\infty} \frac{\sigma(A_{l+1})}{l!} x^l = \sec(x) + \tan(x).$$

2. D_l**-type.**

The Coxeter-Dynkin graph of type D_l ($l \geq 3$) is a tree given as follows:



where l is the number of vertices of the graph. We denote the principal number of type D_l by $\sigma(D_l)$.

Theorem 3.2.

$$\sum_{l=3}^{\infty} \frac{\sigma(D_l)}{l!} x^l = 2(x-1)\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) - x^2 + 2. \tag{7}$$

Proof. We note that $\sigma(D_3) = \sigma(A_3) = 2$. And from the formula (5), we get an equation:

$$2 \cdot \sigma(D_{l+1}) = 2 \cdot \sigma(A_l) + \binom{l}{2}\binom{2}{1}\sigma(A_1)\sigma(A_{l-1})$$

$$+ \sum_{i=3}^{l} \binom{l}{i} \sigma(D_i)\sigma(A_{l-i}) \qquad (l \geq 3).$$

Here, we put $\sigma(D_2) := 2$, $\sigma(D_1) := 0$ and $\sigma(D_0) := 2$ formally. Then this equation becomes a simpler form:

$$\sigma(D_{l+1}) = \frac{1}{2} \sum_{i=0}^{l} \binom{l}{i} \sigma(D_i)\sigma(A_{l-i}) \qquad (l \geq 3).$$

We transform this equation as follows.

$$2 \cdot \frac{\sigma(D_{l+1})}{l!} = \sum_{i=0}^{l} \frac{\sigma(D_i)}{i!} \cdot \frac{\sigma(A_{l-i})}{(l-i)!}.$$

$$2 \cdot \sum_{l=3}^{\infty} \frac{\sigma(D_{l+1})}{l!} x^l = \sum_{l=3}^{\infty} \Big(\sum_{i=0}^{l} \frac{\sigma(D_i)}{i!} \cdot \frac{\sigma(A_{l-i})}{(l-i)!} \Big) x^l.$$

$$2\Big(\sum_{l=0}^{\infty} \frac{\sigma(D_{l+1})}{l!} x^l - 2x - x^2 \Big) = \sum_{l=0}^{\infty} \Big(\sum_{i=0}^{l} \frac{\sigma(D_i)}{i!} \cdot \frac{\sigma(A_{l-i})}{(l-i)!} \Big) x^l - 2 - 2x - 2x^2.$$

Put $f_D(x) := \sum_{l=0}^{\infty} \frac{\sigma(D_l)}{l!} x^l$. Then, above equation implies the differential equation:

$$f_D'(x) = \frac{1}{2} f_D(x) f_A(x) + x - 1.$$

We can solve this differential equation by using variations of parameters. With the initial condition: $f_D(0) = \sigma(D_0) = 2$, the solution is:

$$f_D(x) = 2(x-1)\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) + 4.$$

So we obtain the formula (7). □

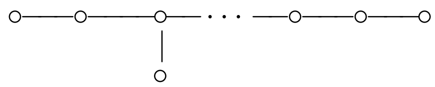
Using the relation: $f_D(x) = 2(x-1)f_A(x) + 4$, one obtains:

Corollary 3.3.

$$\sigma(D_l) = 2(l \cdot \sigma(A_{l-1}) - \sigma(A_l)) \qquad (l \geq 1). \tag{8}$$

3. E_l-type.

The Coxeter-Dynkin graph of type E_l ($l \geq 4$) is a tree given as follows:



where l is the number of vertices of the graph. We denote the principal number of type E_l by $\sigma(E_l)$.

Theorem 3.4.

$$\sum_{l=4}^{\infty} \frac{\sigma(E_l)}{l!} x^l = \left(\frac{1}{2}x^2 - 2x + 3\right)\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) - 3x^3 - x - 3. \tag{9}$$

Proof. We note that $\sigma(E_5) = \sigma(D_5) = 18$, $\sigma(E_4) = \sigma(A_4) = 5$. And from the formula (5), we get an equation:

$$\begin{aligned} 2\sigma(E_{l+1}) = \sigma(A_l) + \sigma(D_l) + \binom{l}{1}\sigma(A_1)\sigma(A_{l-1}) \\ + \binom{l}{3}\binom{3}{1}\sigma(A_1)\sigma(A_2)\sigma(A_{l-3}) \\ + \sum_{i=4}^{l}\binom{l}{i}\sigma(E_i)\sigma(A_{l-i}) \qquad (l \geq 4). \end{aligned}$$

From Corollary 3.3, we have $\sigma(D_l) = 2(l \cdot \sigma(A_{l-1}) - \sigma(A_l))$ $(l \geq 1)$. Moreover we put $\sigma(E_3) := 3$, $\sigma(E_2) := 0$, $\sigma(E_1) := 3$ and $\sigma(E_0) := -1$ formally. Then this equation becomes a simpler form:

$$\sigma(E_{l+1}) = \frac{1}{2}\sum_{i=0}^{l}\binom{l}{i}\sigma(E_i)\sigma(A_{l-i}) \qquad (l \geq 4).$$

We transform this equation as follows.

$$2\Big(\sum_{l=0}^{\infty}\frac{\sigma(E_{l+1})}{l!}x^l-(3+\frac{3}{2}x^2+\frac{5}{6}x^3)\Big)$$

$$=\sum_{l=0}^{\infty}\Big(\sum_{i=0}^{l}\frac{\sigma(E_i)}{i!}\cdot\frac{\sigma(A_{l-i})}{(l-i)!}\Big)x^l-(-1+2x+\frac{5}{2}x^2+\frac{5}{3}x^3).$$

Put $f_E(x):=\sum_{l=0}^{\infty}\frac{\sigma(E_l)}{l!}x^l$. Then, above equation implies the differential equation:

$$f'_E(x)=\frac{1}{2}f_E(x)f_A(x)+\frac{1}{4}x^2-x+\frac{7}{2}.$$

We can solve this differential equation by using variations of parameters. With the initial condition: $f_E(0)=\sigma(E_0)=-1$, the solution is:

$$f_E(x)=\Big(\frac{1}{2}x^2-2x+3\Big)\tan\Big(\frac{x}{2}+\frac{\pi}{4}\Big)+2x-4.$$

So we obtain the formula (9). □

Using the relation: $f_E(x)=(\frac{1}{2}x^2-2x+3)f_A(x)+2x-4$, one obtains:

Corollary 3.5.

$$\sigma(E_l)=\frac{l(l-1)}{2}\sigma(A_{l-2})-2l\cdot\sigma(A_{l-1})+3\cdot\sigma(A_l)\qquad(l\geq 2). \tag{10}$$

4. Appendix: the concrete numbers for small l

the principal numbers			
l	$\sigma(A_l)$	$\sigma(D_l)$	$\sigma(E_l)$
(0)	(1)	(2)	(-1)
1	1	(0)	(3)
2	1	(2)	(0)
3	2	2	(3)
4	5	6	5
5	16	18	18
6	61	70	66
7	272	310	298
8	1385	1582	1511
9	7936	9058	8670
10	50521	57678	55168
11	353792	403878	386394
12	2702765	3085478	2951673
13	22368256	25535378	24428657
14	199360981	227589206	217723390
15	1903757312	2173314806	2079109386

The numbers in ( ) are just defined formally in the proofs in section 3, even though corresponding graphs do not exist.

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