

INTEGRABLE HAMILTONIAN SYSTEM ON THE JACOBIAN OF A SPECTRAL CURVE — AFTER BEAUVILLE

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ABSTRACT. Beauville [1] introduced an integrable Hamiltonian system whose general level set is isomorphic to the complement of the theta divisor in the Jacobian of the spectral curve. This can be regarded as a generalization of the Mumford system [7]. In this article, we construct a variant of Beauville's system whose general level set is isomorphic to the complement of the *intersection* of the translations of the theta divisor in the Jacobian. A suitable subsystem of our system can be regarded as a generalization of the even Mumford system [11, 4].

1. INTRODUCTION

The Mumford system [7] is an integrable Hamiltonian system with the Lax matrix

$$(1.1) \quad A(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix} \in M_2(\mathbb{C}[x]).$$

Here $u(x)$ and $w(x)$ are monic of degree $d-1$ and d , and $v(x)$ is of degree $\leq d-2$ where d is a fixed positive integer. The space of Lax matrices $A(x)$ is endowed with $d-1$ independent Hamiltonian vector fields, defining an algebraically completely integrable dynamical system. Its general level set is isomorphic to the complement of the theta divisor in the Jacobian of the spectral curve of the Lax matrix, which is a hyperelliptic curve of genus $d-1$.

A variant called the even Mumford system was introduced by Fernandes and Vanhaecke [11, 4], whose Lax matrix has the same form as (1.1) but the polynomial $w(x)$ is monic of degree $d+1$. This small difference gives rise to another type of general level set, which is isomorphic to the complement of the *union* of two translates of the theta divisor in the Jacobian of a hyperelliptic curve.

On the other hand, Beauville [1] introduced a generalization of the Mumford system. The Lax matrix is given by $A(x) \in M_r(\mathbb{C}[x])$ with a certain condition on the degree of each entry, where $r \geq 2$ can be an arbitrary integer. He constructed a completely integrable Hamiltonian system on the space of (the gauge equivalence classes of) the Lax matrix

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$A(x)$. Its general level set is isomorphic to the complement of the theta divisor in the Jacobian of the spectral curve of the Lax matrix, which is not hyperelliptic in general. The Mumford system can be recovered as the case $r = 2$ of Beauville's system.

In this paper, we employ Beauville's method to construct a system which generalize the even Mumford system. The Lax matrix is again given by $A(x) \in M_r(\mathbb{C}[x])$ with arbitrary $r \geq 2$, but we impose a condition, different from Beauville's, on the degree of each entry. (Hence the spectral curve is not hyperelliptic in general.) We construct a completely integrable Hamiltonian system on the space of (the gauge equivalence classes of) the Lax matrix $A(x)$. An interesting feature of this system is that the general level set is isomorphic to the complement of the *intersection* of r translates of the theta divisor (Theorem 2.8 and 3.11), which is not an affine variety. In addition, we construct a family of subsystems, which provides an open (finite) covering of our system. The level set of each subsystem is isomorphic to the complement of the *union* of r translates of the theta divisor in the Jacobian (Theorem 4.5). We also construct the spaces of representatives of the subsystems, and explicitly describe the Hamiltonian vector fields (Proposition 4.11), and the correspondence between the Lax matrix and the divisor (Proposition 4.9). The even Mumford system can be recovered as the case $r = 2$ of a subsystem.

This paper is organized as follows: in §2 we study the Jacobian of the spectral curves for the Lax matrix. §3 is devoted to the construction of Hamiltonian vector fields, and to the proof of the integrability. In §4 we introduce a family of subsystems and show that each of them is algebraically completely integrable. Further we construct the spaces of representatives of the subsystems, and study the integrable structure.

The proofs of many results in §2 and §3 are given by a modification of the argument of Beauville [1], nevertheless we included a rather whole proof in the present paper for the sake of completeness, and for the importance of Beauville's argument.

2. JACOBIAN OF THE SPECTRAL CURVE

2.1. Intersection of translations of the theta divisor. Let C be a smooth projective irreducible curve of genus g (over \mathbb{C}). For each integer k , we write J^k for the space of invertible sheaves of degree k , which we regard as a principal homogeneous space under the Jacobian J^0 of C . We define the *theta divisor* $\Theta \subset J^{g-1}$ by

$$\begin{aligned}\Theta &= \{L \in J^{g-1} \mid H^0(C, L) \neq 0\} \\ &= \{\mathcal{O}_C(E) \mid E \text{ is an effective divisor of degree } g-1\}.\end{aligned}$$

For each point $q \in C$, we write Θ_q for the translation $\Theta + q = \{L(q) = L \otimes \mathcal{O}_C(q) \mid L \in \Theta\}$ of Θ . This is a divisor on J^g . Let $\pi : C \rightarrow \mathbb{P}^1$ be a finite morphism of degree r . We define a subvariety J' of J^g by

$$J' = \{L \in J^g \mid \pi_* L \cong \mathcal{O} \oplus \mathcal{O}(-1)^{\oplus r-1}\},$$

where we abbreviate $\mathcal{O}_{\mathbb{P}^1}$ to \mathcal{O} . (In [1], J' is denoted by $J(0, -1, \dots, -1)$.) In this subsection, we prove the following.

Proposition 2.1. *For any point $a \in \mathbb{P}^1$ unramified with respect to π , we have*

$$J' = J^g \setminus \left(\bigcap_{q \in C} \Theta_q \right) = J^g \setminus \left(\bigcap_{q \in \pi^{-1}(a)} \Theta_q \right).$$

It is enough to show the following two lemmas:

Lemma 2.2. *For any point $q \in C$, we have $J^g \setminus \Theta_q \subset J'$.*

Lemma 2.3. *For any point $a \in \mathbb{P}^1$ unramified with respect to π , we have*

$$J' \subset J^g \setminus \left(\bigcap_{q \in \pi^{-1}(a)} \Theta_q \right).$$

We need some preliminaries to prove them. Let L be an arbitrary invertible sheaf on C . We can write $\pi_* L \cong \bigoplus_{i=1}^r \mathcal{O}(d_i)$ for some integers $d_1 \leq d_2 \leq \dots \leq d_r$ such that $\deg L = g - 1 + r + \sum d_i$. We have

$$(2.1) \quad h^0(C, L) = h^0(\mathbb{P}^1, \pi_* L) = \sum_i h^0(\mathbb{P}^1, \mathcal{O}(d_i)) = \sum_{i \in \{j \mid d_j \geq 0\}} (d_i + 1),$$

$$(2.2) \quad h^1(C, L) = h^1(\mathbb{P}^1, \pi_* L) = \sum_i h^0(\mathbb{P}^1, \mathcal{O}(-2 - d_i)) = - \sum_{i \in \{j \mid d_j \leq -2\}} (d_i + 1),$$

where we used the notation $h^*(X, F) = \dim H^*(X, F)$. This computation, together with the Riemann-Roch theorem, implies the following two lemmas:

Lemma 2.4. (cf. [1] 1.8) *For $L \in J^{g-1}$, the following conditions are equivalent:*

$$(1) L \in J^{g-1} \setminus \Theta, \quad (2) h^0(C, L) = 0, \quad (3) h^1(C, L) = 0, \quad (4) \pi_* L \cong \mathcal{O}(-1)^{\oplus r}.$$

Lemma 2.5. *For $L \in J^g$, the following conditions are equivalent:*

$$(1) L \in J' \text{ (i.e. } \pi_* L \cong \mathcal{O} \oplus \mathcal{O}(-1)^{\oplus r-1}), \quad (2) h^0(C, L) = 1, \quad (3) h^1(C, L) = 0.$$

Proof of Lemma 2.2. For an invertible sheaf L on C , we have an exact sequence

$$(2.3) \quad 0 \rightarrow H^0(C, L(-q)) \rightarrow H^0(C, L) \xrightarrow{s_q} \mathbb{C} \rightarrow H^1(C, L(-q)) \rightarrow H^1(C, L) \rightarrow 0$$

deduced from the short exact sequence $0 \rightarrow L(-q) \rightarrow L \rightarrow \mathbb{C}_q \rightarrow 0$. Now we assume $L \in J^g \setminus \Theta_q$. This amounts to assuming $L(-q) \in J^{g-1} \setminus \Theta$, and Lemma 2.4 shows $h^0(C, L(-q)) = h^1(C, L(-q)) = 0$. Then the exact sequence (2.3) implies $h^0(C, L) = 1$, which means $L \in J'$ by Lemma 2.5. This completes the proof. \square

Proof of Lemma 2.3. We take $L \in J'$. By lemma 2.5, we have $h^0(C, L) = 1$. For $q \in C$, we regard $H^0(C, L(-q))$ as a subspace of $H^0(C, L)$ by the injection appeared in eq. (2.3).

Now we assume $L \in \bigcap_{q \in \pi^{-1}(a)} \Theta_q$. This amounts to assuming $L(-q) \in \Theta$ for any $q \in \pi^{-1}(a)$. Then Lemma 2.4 shows that the inclusion $H^0(C, L(-q)) \rightarrow H^0(C, L)$ is bijective

for any $q \in \pi^{-1}(a)$. In other words, any non-zero global section of L must have a zero at q for any $q \in \pi^{-1}(a)$. Therefore $H^0(C, L(-\pi^*a)) = \cap_{q \in \pi^{-1}(a)} H^0(C, L(-q))$ is isomorphic to $H^0(C, L)$, and we have $h^0(C, L(-\pi^*a)) = h^0(C, L) = 1$. However, by the projection formula (and the assumption $L \in J'$), we have

$$h^0(C, L(-\pi^*a)) = h^0(\mathbb{P}^1, \pi_*L \otimes \mathcal{O}(-1)) = h^0(\mathbb{P}^1, \mathcal{O}(-1) \oplus \mathcal{O}(-2)^{\oplus r-1}) = 0.$$

This is a contradiction, and the proof is done. \square

2.2. Jacobian of the spectral curve. We fix natural numbers r and d . Let us consider a polynomial of the form

$$P(x, y) = y^r + s_1(x)y^{r-1} + \cdots + s_r(x)$$

with $s_i(x) \in \mathbb{C}[x]$ is of degree $\leq di$. We regard x as a fixed coordinate function on \mathbb{P}^1 , so that the equation $P(x, y) = 0$ defines a finite map $\pi : C_P \rightarrow \mathbb{P}^1$ of degree r , where C_P is the *spectral curve* of P . One can define C_P to be the closure of the affine curve defined by $P(x, y) = 0$ in the Hirzebruch surface of degree d . More explicitly, C_P can be described by gluing two plane affine curves defined by the polynomials $P(x, y)$ and $z^{dr}P(z^{-1}, z^{-d}w) \in \mathbb{C}[z, w]$ by the relation $x = z^{-1}$, $y = z^{-d}w$. The aim of this subsection is to give an explicit representation (the matrix realization) of the variety J' considered in §2.1 assuming $C = C_P$ is smooth (hence irreducible). We remark that, under this assumption, the genus of C_P is $g = \frac{1}{2}(r-1)(rd-2)$.

We introduce some notations:

$$S_k(x) = \{s(x) \in \mathbb{C}[x] \mid \deg s(x) \leq k\},$$

$$M(r, d) = \left\{ A(x) \in M_r(\mathbb{C}[x]) \left| \begin{array}{l} A(x)_{11} \in S_d(x), \quad A(x)_{1j} \in S_{d+1}(x), \\ A(x)_{i1} \in S_{d-1}(x), \quad A(x)_{ij} \in S_d(x), \end{array} \right. \quad (2 \leq i, j \leq r) \right\},$$

$$V(r, d) = \{P(x, y) = y^r + s_1(x)y^{r-1} + \cdots + s_r(x) \in \mathbb{C}[x, y] \mid s_i(x) \in S_{di}(x)\},$$

$$G_r = \left\{ g(x) = \begin{pmatrix} 1 & {}^t\vec{b}_1x + {}^t\vec{b}_0 \\ 0 & B \end{pmatrix} \left| B \in GL_{r-1}(\mathbb{C}), \quad \vec{b}_1, \vec{b}_0 \in \mathbb{C}^{r-1} \right. \right\}.$$

In this article we denote column vectors using a notation such as \vec{b} . We write the adjoint action of G_r on $M(r, d)$ as

$$(2.4) \quad g(A(x)) = g(x)^{-1}A(x)g(x) \text{ for } g(x) \in G_r, \quad A(x) \in M(r, d).$$

Further we introduce a map:

$$\psi : M(r, d) \rightarrow V(r, d); \quad A(x) \mapsto \det(y\mathbb{I}_r - A(x)),$$

and define subsets of $V(r, d)$ or $M(r, d)$ as follows:

$$M_P = \psi^{-1}(P(x, y)),$$

$$V_{ir}(r, d) = \{P(x, y) \in V(r, d) \mid C_P \text{ is irreducible}\},$$

$$V_{sm}(r, d) = \{P(x, y) \in V_{ir}(r, d) \mid C_P \text{ is smooth}\},$$

$$M_{ir}(r, d) = \psi^{-1}(V_{ir}(r, d)),$$

$$M_{sm}(r, d) = \psi^{-1}(V_{sm}(r, d)).$$

Then we have $V(r, d) \supset V_{ir}(r, d) \supset V_{sm}(r, d)$ and $M(r, d) \supset M_{ir}(r, d) \supset M_{sm}(r, d)$. Note that each $M_P, M_{ir}(r, d)$ and $M_{sm}(r, d)$ is stable with respect to the action of G_r (2.4). For the later use we introduce a lemma:

Lemma 2.6. *The action (2.4) of G_r on $M_{ir}(r, d)$ is free.*

Proof. We have to show that the stabilizer is trivial for all $A(x) \in M_{ir}(r, d)$. Since any element of G_r has an eigenvalue 1, this follows from the following lemma on elementary linear algebra: \square

Lemma 2.7. *Let $K = \mathbb{C}(x)$ be the field of rational functions over \mathbb{C} . Let $r \in \mathbb{N}$, and suppose $A, B \in M_r(K)$ satisfies the following conditions: (1) $AB = BA$, (2) B is not a scalar matrix, (3) B has an eigenvalue b in K . Then $\det(y\mathbb{I}_r - A) \in K[y]$ is a reducible polynomial in y .*

Proof. This follows at once by noting that the eigenspace of B with respect to b is a non-trivial, proper subspace of $K^{\oplus r}$ stable under A . \square

We define a projection map η :

$$(2.5) \quad \eta : M_{ir}(r, d) \rightarrow M_{ir}(r, d)/G_r.$$

In the following, we respectively write J_P and J'_P for the variety J and J' defined in §2.1 associated to (C_P, π) . For $k \in \mathbb{Z}$ and an invertible sheaf L on C_P , we use a notation $L(k) = L \otimes \pi^*O(k)$. The main result in this subsection is the following:

Theorem 2.8. *(cf. [1] 1.4) Let $P(x, y) \in V_{sm}(r, d)$, and let $\pi : C_P \rightarrow \mathbb{P}^1$ be the finite map defined by x . Then, M_P is a principal fiber bundle under G_r , and the base space M_P/G_r is isomorphic to J'_P .*

Proof. The first part follows from Lemma 2.6. We construct a surjective map $M_P \rightarrow J'_P$ and show that each fiber is a principal homogeneous space under G_r . We remark that a matrix $A(x) \in M(r, d)$ defines an O -linear map $O \oplus O(-1)^{\oplus r-1} \rightarrow O(d) \oplus O(d-1)^{\oplus r-1}$. (Here we consider $O(d) = O(d \cdot \infty)$.) Due to [2] (see also [1] 1.4), the set

$$(2.6) \quad \{(L, v) \mid L \in J'_P, v : O \oplus O(-1)^{\oplus r-1} \cong \pi_*L\}$$

is in one-to-one correspondence with M_P in such a way that the diagram

$$(2.7) \quad \begin{array}{ccc} O \oplus O(-1)^{\oplus r-1} & \xrightarrow{A(x)} & O(d) \oplus O(d-1)^{\oplus r-1} \\ v \downarrow & & v(d) \downarrow \\ \pi_*L & \xrightarrow{\pi_*y} & \pi_*L(d) \end{array}$$

commutes whenever (L, v) corresponds to $A(x) \in M_P$. (Note that $A(x)$ must be in M_P because of the relation $P(x, y) = 0$ in \mathcal{O}_C .) By composing this correspondence with the ‘forgetful’ map $(L, v) \mapsto L$, we obtain the desired surjection $M_P \rightarrow J'_P$. The fiber of this map over $L \in J'_P$ is the set of isomorphisms $O \oplus O(-1)^{\oplus r-1} \cong \pi_* L$ which is a principal homogeneous space under G_r where the action of $g(x) \in G_r$ is given by $v \mapsto g(x)^{-1} \circ v \circ g(x)$. (Here we regard $g(x)$ as an automorphism on $O \oplus O(-1)^{\oplus r-1}$ as well as $O(d) \oplus O(d-1)^{\oplus r-1}$.) On the set M_P , this action corresponds to the conjugation. This completes the proof. \square

Remark 2.9. Given an invertible sheaf $L \in J'_P$, a corresponding matrix $A(x) \in M_P$ is constructed in the following way. We have to choose an isomorphism $v : O \oplus O(-1)^{\oplus r-1} \rightarrow \pi_* L$. This amounts to a choice of a basis of $H^0(C_P, L(1))$ of the form $(f_0, f_1, \dots, f_{r-1}, x f_0)$ with $f_0 \in H^0(C_P, L)$. The multiplication by y defines elements $y f_0 \in H^0(C_P, L(d)) = (f_0 S_d(x)) \oplus (\oplus_{j=1}^{r-1} f_j S_{d-1}(x))$ and $y f_1, \dots, y f_{r-1} \in H^0(C, L(d+1)) = (f_0 S_{d+1}(x)) \oplus (\oplus_{j=1}^{r-1} f_j S_d(x))$. Now the matrix $A(x)$ is characterized by

$$y(f_0, f_1, \dots, f_{r-1}) = (f_0, f_1, \dots, f_{r-1})A(x).$$

In other words, the set M_P is in one-to-one correspondence with the set of pairs (L, v) where $L \in J'_P$ and $v : S_1(x) \oplus \mathbb{C}^{\oplus r-1} \xrightarrow{\cong} H^0(C_P, L(1))$. A matrix $A(x) \in M_P$ corresponds to (L, v) iff

$$(2.8) \quad \begin{array}{ccc} S_1(x) \oplus \mathbb{C}^{\oplus r-1} & \xrightarrow[\cong]{v} & H^0(C_P, L(1)) \\ \downarrow A(x) & & \downarrow y \\ S_{d+1}(x) \oplus S_d(x)^{\oplus r-1} & \xrightarrow[\cong]{v(d)} & H^0(C_P, L(d+1)) \end{array}$$

commutes.

2.3. Characterization of a translation of the theta divisor. We fix $P \in V_{sm}(r, d)$. Let $A(x) \in M_P$, and let $L \in J'_P$ be the corresponding invertible sheaf. We take $a \in \mathbb{P}^1 \setminus \{\infty\}$ unramified with respect to π , so that $\pi^{-1}(a) = \{q_1, \dots, q_r\}$ consists of r distinct points. Then $y(q_1), \dots, y(q_r)$ are the distinct eigenvalues of the matrix $A(a)$. Let $\rho_{q_i} : \mathbb{C}^r \rightarrow \mathbb{C}$ be the projection to the eigenspace associated with the eigenvalue $y(q_i)$. For each $i = 1, \dots, r$, we write $s_{q_i} : H^0(C_P, L) \rightarrow \mathbb{C}$ for the map in the exact sequence (2.3) applied to $q = q_i$. In this subsection, we show the following.

Proposition 2.10. *For each $i = 1, \dots, r$, the following conditions are equivalent:*

$$(1) \ \rho_{q_i}(1, 0, \dots, 0) \neq 0, \quad (2) \ \text{Im}(s_{q_i}) \neq 0, \quad (3) \ L \in J'_P \setminus \Theta_{q_i}.$$

Proof. The equivalence between (2) and (3) is a consequence of Lemma 2.4 and the exact sequence (2.3), as is shown in the same way as Lemma 2.2. We show the equivalence

between (1) and (2). We recall that the map s_{q_i} is induced by the map \tilde{s}_{q_i} in the following short exact sequence of sheaves on C_P

$$0 \rightarrow L(-q_i) \longrightarrow L \xrightarrow{\tilde{s}_{q_i}} \mathbb{C}_{q_i} \rightarrow 0.$$

We then have a commutative diagram

$$\begin{array}{ccc} \pi_* L & \xrightarrow{\oplus \tilde{s}_{q_i}} & \oplus_{i=1}^r \pi_* \mathbb{C}_{q_i} \\ \pi_* y \downarrow & & \downarrow \oplus_i y(q_i) \\ \pi_* L(d) & \xrightarrow{\oplus \tilde{s}_{q_i}(d)} & \oplus_{i=1}^r \pi_* \mathbb{C}_{q_i}, \end{array}$$

where the right vertical map is defined as the multiplication by $y(q_i)$ on the i -th component. Let $v : O \oplus O(-1)^{\oplus r-1} \cong \pi_* L$ be the isomorphism corresponding to $A(x)$. The pull-back of this diagram by v is written as

$$\begin{array}{ccc} O \oplus O(-1)^{\oplus r-1} & \xrightarrow{l_1} & \mathbb{C}_a^{\oplus r} \\ \downarrow A(x) & & \downarrow A(a) \\ O(d) \oplus O(d-1)^{\oplus r-1} & \xrightarrow{l_2} & \mathbb{C}_a^{\oplus r}, \end{array}$$

where l_1 and l_2 are defined simply by the direct sum of $O(k) \rightarrow \mathbb{C}_a$ for $k \in \{0, -1, d, d-1\}$. This means that $\pi_* \mathbb{C}_{q_i}$ maps to the eigenspace of $y(q_i)$ in \mathbb{C}^r under the isomorphism $v_a : \mathbb{C}_a^r \cong \oplus_{i=1}^r \pi_* \mathbb{C}_{q_i}$. The image of the map $H^0(\mathbb{P}^1, O \oplus O(-1)^{\oplus r-1}) \rightarrow \mathbb{C}^r$ induced by l_1 is generated by $(1, 0, \dots, 0)$. Therefore the image of s_{q_i} is non-trivial if and only if $\rho_{q_i}(1, 0, \dots, 0) \neq 0$. This shows the proposition. \square

Remark 2.11. Let us consider the case $a = \infty$ (still assuming that π is unramified at $a = \infty$). The statement of Proposition 2.10 remains true if we replace $A(a)$ by $A(\infty)$, where the (i, j) -component of $A(\infty)$ is the coefficient of the leading term of $A(x)_{ij}$. Note that, if we set $w = y/x^d$, then $w(q_1), \dots, w(q_r)$ are the distinct eigenvalues of $A(\infty)$.

3. INTEGRABLE SYSTEM

3.1. Vector Fields. We identify the tangent space $T_{A(x)}M(r, d)$ at $A(x) \in M(r, d)$ with the affine space $M(r, d)$ and write vector fields on $M(r, d)$ in the matrix form. For a positive integer p and $a \in \mathbb{C}$, we define a vector field $\Upsilon_a^{(p)}$ on $M(r, d)$ by the Lax form

$$(3.1) \quad \Upsilon_a^{(p)}(A(x)) := \frac{1}{x-a} [A(a)^p, A(x)].$$

If we let $a \in \mathbb{C}$ vary, $\Upsilon_a^{(p)}$ can be written as a polynomial in a of degree pd . For $j = 0, \dots, pd$, we define a vector field $Y_j^{(p)}$ to be the coefficient of a^j in this polynomial, viz.

$$(3.2) \quad \Upsilon_a^{(p)} = \sum_{j=0}^{pd} a^j Y_j^{(p)}.$$

Remark 3.1. For each $a \in \mathbb{C}$, the sets of the vector fields $\{\Upsilon_a^{(p)} | 1 \leq p \leq r-1\}$ and $\{\Upsilon_a^{(p)} | 1 \leq p\}$ generate the same vector space by Hamilton-Cayley's formula for $A(a)$. Further for each $p \geq 1$, the sets $\{\Upsilon_a^{(p)} | a \in \mathbb{C}\}$ and $\{Y_j^{(p)} | 0 \leq j \leq pd\}$ generate the same vector space by Vandermond's determinant formula.

Lemma 3.2. *The projection map η (2.5) induces the equality $\eta_* \Upsilon_a^{(p)}(A(x)) = \eta_* \Upsilon_a^{(p)}(g(A(x)))$ in $T_{\eta(A(x))}(M_{ir}(r, d)/G_r)$ for all $g(x) \in G_r$ and $A(x) \in M_{ir}(r, d)$.*

Proof. A vector field X on $M_{ir}(r, d)$ satisfies $\eta_* X(A(x)) = \eta_* X(g(A(x)))$ in $T_{\eta(A(x))}(M_{ir}(r, d)/G_r)$ if and only if $X(A(x)) - g_* X(A(x))$ is tangent to G_r -orbits for any $g(x) \in G_r$. A direct calculation shows that $\Upsilon_a^{(p)}(A(x)) - g_* \Upsilon_a^{(p)}(A(x))$ is a linear combination of the vector fields of Lie G_r :

$$(3.3) \quad X_E(A(x)) = [E, A(x)], \text{ for } E = E_{ij}, E_{1j}, E'_{1j} \ (2 \leq i, j \leq r).$$

Here E_{ij} is given by $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, and $E'_{1j} = xE_{1j}$. Thus the claim follows. \square

Corollary 3.3. *For each $a \in \mathbb{C}, 1 \leq p \leq r-1, 0 \leq j \leq pd$, we have well-defined vector fields $\tilde{\Upsilon}_a^{(p)}$ and $\tilde{Y}_j^{(p)}$ on $M_{ir}(r, d)/G_r$ which satisfies at $[A(x)] = \eta(A(x))$*

$$\tilde{\Upsilon}_a^{(p)}([A(x)]) = \eta_* \Upsilon_a^{(p)}(A(x)), \quad \tilde{Y}_j^{(p)}([A(x)]) = \eta_* Y_j^{(p)}(A(x)).$$

We collect some properties of $\tilde{Y}_j^{(p)}$.

- Lemma 3.4.** 1. *For each $P \in V_{ir}(r, d)$, the vector field $Y_j^{(p)}$ is tangent to M_P and $\tilde{Y}_j^{(p)}$ is tangent to M_P/G_r .*
2. *For any i and j , the vector fields $Y_i^{(p)}$ and $Y_j^{(q)}$ commute. So do $\tilde{Y}_i^{(p)}$ and $\tilde{Y}_j^{(q)}$.*
3. *We have $\tilde{Y}_{pd}^{(p)} = \tilde{Y}_{pd-1}^{(p)} = 0$. The dimension of the vector space generated by $\tilde{Y}_j^{(p)}$ with $1 \leq p \leq r-1, 0 \leq j \leq pd-2$ is at most g .*

Proof. 1: A vector field on $M(r, d)$ is equivalently given as a derivation on the affine ring of $M(r, d)$. We write $t_k(x) = \text{tr} A(x)^k$ and let $s_k(x)$ be the coefficients of y^{r-k} in $\det(y\mathbb{I}_r - A(x))$ for $1 \leq k \leq r$. By Newton's formula, each $s_k(x)$ is written as a function in $\mathbb{Q}[t_1(x), \dots, t_k(x)]$. Since $\Upsilon_a^{(p)}$ is given by the Lax form (3.1), the associated derivation satisfies $\Upsilon_a^{(p)}(t_k(x)) = 0$. Thus we see $\Upsilon_a^{(p)}(s_k(x)) = 0$, and the claim follows.

2: This is shown by a direct computation.

3: Since $Y_{pd}^{(p)}$ and $Y_{pd-1}^{(p)}$ are tangent to G_r -orbits, $\tilde{Y}_{pd}^{(p)}$ and $\tilde{Y}_{pd-1}^{(p)}$ vanish. Therefore the space in question is generated by $\tilde{Y}_j^{(p)}$ with $1 \leq p \leq r-1, 0 \leq j \leq pd-2$. The number of the members is $\sum_{p=1}^{r-1} (pd-1) = \frac{1}{2}(r-1)(dr-2) = g$. \square

3.2. Translation invariance. We have seen that M_P/G_r is isomorphic to an open subset J'_P of J_P^g for $P(x, y) \in V_{sm}(r, d)$ (Theorem 2.8). We regard the restriction of the vector fields $\tilde{\Upsilon}_a^{(p)}$ and $\tilde{Y}_j^{(p)}$ as vector fields on J'_P . In this subsection, we show that $\tilde{\Upsilon}_a^{(p)}|_{M_P/G_r}$ and $\tilde{Y}_j^{(p)}|_{M_P/G_r}$ are translation invariant under the action of the Jacobian J_P^0 on J_P^g .

The space of translation invariant (holomorphic) vector fields on J_P is canonically dual to $H^0(C_P, \Omega_{C_P}^1)$. Let C_P^0 be the set of points $q \in C_P$ such that $\pi : C_P \rightarrow \mathbb{P}^1$ is unramified at q and $\pi(q) \neq \infty$. For $q \in C_P^0$, we write X_q for the vector field corresponding to the linear form $\omega \mapsto \frac{\omega}{d(x-x(q))}(q)$ on $H^0(C_P, \Omega_{C_P}^1)$. (Recall we have fixed a coordinate x on \mathbb{P}^1 .) Equivalently, X_q is characterized as follows: the short exact sequence $0 \rightarrow \mathcal{O}_{C_P} \rightarrow \mathcal{O}_{C_P}(q) \rightarrow T_q C_P \rightarrow 0$ induces the connecting homomorphism

$$T_q C_P \rightarrow H^1(C_P, \mathcal{O}_{C_P}).$$

The image of the vector $\frac{\partial}{\partial(x-x(q))} \in T_q C_P$ in $H^1(C_P, \mathcal{O}_{C_P})$ corresponds to X_q under the Serre duality.

Remark 3.5. If Q is an infinite subset of C_P^0 , the vectors X_q ($q \in Q$) generate the full space of translation invariant vector fields. Indeed, this is equivalent to the triviality of the cokernel of

$$\bigoplus_{q \in Q} T_q C_P \rightarrow H^1(C_P, \mathcal{O}_{C_P}),$$

which is dual to the kernel of

$$H^0(C_P, \Omega_{C_P}^1) \rightarrow \prod_{q \in Q} T_q^* C_P;$$

but this kernel is trivial by the simple fact that any non-zero differential form has only finitely many zeros.

The main result in this subsection is the following.

Theorem 3.6. (cf. [1] 2.2) *Let $a \in \mathbb{P}^1$ be a point such that $\pi : C_P \rightarrow \mathbb{P}^1$ is unramified over a , and let $\pi^{-1}(a) = \{q_1, \dots, q_r\}$. Then, for each $p \geq 1$, the vector field $\tilde{\Upsilon}_a^{(p)}|_{M_P/G_r}$ coincides with $y(q_1)^p X_{q_1} + \dots + y(q_r)^p X_{q_r}$.*

Proof. Let $A(x) \in M_P$. Then $A(a)$ has r distinct eigenvalues $y(q_1), \dots, y(q_r)$. For each $q \in \pi^{-1}(a)$, we write $\Pi_q \in M_r(\mathbb{C})$ for the projector to the eigenspace of $y(q)$, and we define a vector field \dot{A}_q on M_P by

$$\dot{A}_q(A(x)) = \frac{1}{x-a} [\Pi_q, A(x)].$$

Since $\Upsilon_a^{(p)}|_{M_P/G_r} = y(q_1)^p \dot{A}_{q_1} + \dots + y(q_r)^p \dot{A}_{q_r}$, the theorem is reduced to the following lemma. □

Lemma 3.7. *We have $\eta_*(\Upsilon_a^{(p)})(A(x)) = X_q(\eta(A(x)))$ for any $q \in \pi^{-1}(a), A(x) \in M_P$.*

Proof. In this proof, we omit to indicate P and write $C = C_P, J = J_P$ etc. Let C_ϵ be the scheme whose underlying topological space is C but with the structure sheaf $\mathcal{O}_C[\epsilon], \epsilon^2 = 0$. For $L \in J$, the tangent space $T_L J$ is in one-to-one correspondence with the set of invertible

sheaves on C_ϵ , which reduce to L modulo ϵ . If $q \in C^0$ and $L \in J'$, the vector $X_q(L)$ corresponds to the invertible sheaf L_q^ϵ is given by

$$H^0(U, L_q^\epsilon) = \left\{ s + \epsilon t \mid \begin{array}{l} s \in H^0(U, L), \ t \in H^0(U, L(q)), \\ s/(x-a) + t \text{ is holomorphic at } q \end{array} \right\}$$

for an open set U of C (cf. [1] 2.2).

Recall that the set M_P is in one-to-one correspondence with the set of pairs (L, v) where $L \in J'$ and v is an isomorphism $H^0(C, L(1)) \xrightarrow{\cong} S_1(x) \oplus \mathbb{C}^{\oplus r-1}$ (cf. Remark 2.9). If $A(x) \in M_P$ corresponds to (L, v) , the tangent space $T_{A(x)}M_P$ is in one-to-one correspondence with the pairs of (L^ϵ, v^ϵ) where L^ϵ is an invertible sheaf on C_ϵ which reduces to L modulo ϵ , and v^ϵ is an isomorphism $(S_1(x) \oplus \mathbb{C}^{\oplus r-1}) \otimes \mathbb{C}[\epsilon] \cong H^0(C_\epsilon, L^\epsilon(1))$ of $\mathbb{C}[\epsilon]$ -modules, which reduces to v modulo ϵ . A vector $\dot{A}(x) \in T_{A(x)}M_P \subset T_{A(x)}M_{sm}(r, d) \cong M(r, d)$ corresponds to a pair (L^ϵ, v_ϵ) iff

$$(3.4) \quad \begin{array}{ccc} (S_1^\epsilon \oplus \mathbb{C}[\epsilon]^{\oplus r-1}) & \xrightarrow[\cong]{v^\epsilon} & H^0(C_\epsilon, L^\epsilon(1)) \\ A(x) + \epsilon \dot{A}(x) \downarrow & & \downarrow y \\ (S_{d+1}^\epsilon \oplus S_d^{\epsilon \oplus r-1}) & \xrightarrow[\cong]{v^\epsilon(d)} & H^0(C_\epsilon, L^\epsilon(d+1)) \end{array}$$

commutes. Here we denote $S_k^\epsilon = S_k(x) \otimes \mathbb{C}[\epsilon]$.

Now let $q \in C^0$. Let $A(x) \in M_P$ and let (L, v) be the corresponding pair. Recall that L_q^ϵ is the invertible sheaf on C_ϵ corresponding to $X_q(L)$. In order to complete the proof, we are going to construct an isomorphism $v_q^\epsilon : S_1^\epsilon \oplus \mathbb{C}[\epsilon]^{\oplus r-1} \xrightarrow{\cong} H^0(C_\epsilon, L_q^\epsilon(1))$ such that v_q^ϵ reduces to v modulo ϵ , and that the diagram

$$(3.5) \quad \begin{array}{ccc} (S_1^\epsilon \oplus \mathbb{C}[\epsilon]^{\oplus r-1}) & \xrightarrow[\cong]{v_q^\epsilon} & H^0(C_\epsilon, L_q^\epsilon(1)) \\ A(x) + \epsilon \dot{A}_q(x) \downarrow & & \downarrow y \\ (S_{d+1}^\epsilon \oplus S_d^{\epsilon \oplus r-1}) & \xrightarrow[\cong]{v_q^\epsilon(d)} & H^0(C_\epsilon, L_q^\epsilon(d+1)) \end{array}$$

commutes.

Let $a = \pi(q)$ and write $\pi^{-1}(a) = \{q_1 = q, q_2, \dots, q_r\}$. There exists a section $s_i \in H^0(C, L(1))$ which does not vanish at q_i but vanishes at q_j for $j \neq i$. However, such an s_i is not unique. We specify a choice of s_i as follows. We write $f_0, f_1, \dots, f_{r-1} \in H^0(C, L(1))$ for the images of $(1, (0, \dots, 0)), (0, (1, 0, \dots, 0)), \dots, (0, (0, \dots, 1))$ under the isomorphism v . Then $((x-a)f_0, f_0, f_1, \dots, f_{r-1})$ is a \mathbb{C} -basis of $H^0(C, L(1))$ (and $(x-a)f_0$ is a \mathbb{C} -base of $H^0(C, L)$). On the other hand, $((x-a)f_0, s_1, \dots, s_r)$ is also a basis of $H^0(C, L(1))$.

Thus we can write

$$((x-a)f_0, s_1, \dots, s_r) = ((x-a)f_0, f_0, f_1, \dots, f_{r-1}) \cdot \tilde{\Lambda},$$

$$\tilde{\Lambda} = \begin{pmatrix} 1 & * \\ 0 & \Lambda \end{pmatrix}, \quad \Lambda = (\vec{\lambda}_1, \dots, \vec{\lambda}_r) \in GL_r(\mathbb{C}).$$

We can choose s_1, \dots, s_r so that $\tilde{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix}$. This condition determines s_i up to a multiplication by a non-zero scalar. By definition we have $((x-a)f_0/s_i)(q_i) = 0$ and $s_j/s_i(q_i) = \delta_{i,j}$. Hence, if we set $\mathbf{f} := ((f_j/s_i)(q_i))_{ij}$, then $\mathbf{f} \cdot \Lambda = \mathbb{I}_r$.

Now we define v_q^ϵ to be the composition of

$$(3.6) \quad \sigma : H^0(C, L(1)) \oplus H^0(C, L(1))\epsilon \xrightarrow{\cong} H^0(C_\epsilon, L_q^\epsilon(1))$$

$$(t_1, t_2\epsilon) \mapsto t_1 + \left(t_2 - \frac{t_1}{s_1}(q) \frac{s_1}{x-a}\right)\epsilon$$

with an isomorphism

$$v \otimes id_{\mathbb{C}[\epsilon]} : (S_1^\epsilon \oplus \mathbb{C}[\epsilon]^{r-1}) \xrightarrow{\cong} H^0(C, L(1)) \otimes \mathbb{C}[\epsilon] = H^0(C, L(1)) \oplus H^0(C, L(1))\epsilon.$$

The change of s_1 by a scalar multiplication does not affect the definition of this map.

It is immediate that $v_q^\epsilon \bmod \epsilon$ is v_q . We check the commutativity of (3.5). We write $\vec{f} = (f_0, \dots, f_{r-1})$ and $\vec{f}/s_i(q) = (f_0/s_i(q), \dots, f_{r-1}/s_i(q))$. Then the map (3.6) can be written in terms of matrices

$$\sigma(\vec{f}, \epsilon \vec{f}) = \vec{f} + \epsilon \left(\vec{f} - \frac{1}{x-a} \vec{f} \cdot \Pi \right), \quad \Pi = \vec{\lambda}_1 \cdot \vec{f}/s_1(q) \in M_r(\mathbb{C}).$$

Therefore, the commutativity of (3.5) means

$$\vec{f}A(x) \left(\mathbb{I} - \frac{\epsilon}{x-a} \Pi \right) = \vec{f} \left(\mathbb{I} - \frac{\epsilon}{x-a} \Pi \right) (A(x) + \epsilon \dot{A}_q(x)),$$

which follows if we have $\Pi = \Pi_{q_1}$. To show the last assertion, we note that the equation $ys_i = \vec{f}A(x)\vec{\lambda}_i$ holds in $H^0(C, L(d+1))$. Thus we have $\mathbf{f}A(a)\Lambda = \text{diag}(y(q_1), \dots, y(q_r))$. Since $\mathbf{f} = \Lambda^{-1}$, this means $\vec{\lambda}_i$ is an eigenvector of $A(a)$ belonging to the eigenvalue $y(q_i)$. In particular, $\Pi = \vec{\lambda}_1 \cdot \vec{f}/s_1(q_1)$ is the projector Π_{q_1} . This completes the proof. \square

By Lemma 3.4-3 and Remark 3.5, we obtain

Corollary 3.8. *The space of vector fields on $M_{ir}(r, d)/G_r$ generated by $\tilde{Y}_j^{(p)}$ ($1 \leq p \leq r-1, 0 \leq j \leq pd-2$) is g -dimensional.*

3.3. Hamiltonian structure. In this subsection, we show that the vector fields $\tilde{\Upsilon}_a^{(p)}$ on $M_{ir}(r, d)/G_r$ are Hamiltonian, following the method of [1] §5 (see also [6] §15, [8]).

Let a_1, \dots, a_{d+2} be distinct points in \mathbb{C} , and $\varphi : M(r, d) \rightarrow M_r(\mathbb{C})^{d+2}$ be a map defined by

$$(3.7) \quad \varphi(A(x)) = (c_1 A(a_1), \dots, c_{d+2} A(a_{d+2})).$$

Here $c_\alpha = P_\alpha(a_\alpha)^{-1}$ with $P_\alpha(x) = \prod_{\rho \neq \alpha} (x - a_\rho)$. This map is injective, and the preimage of $\mathbf{Y} = (Y_1, Y_2, \dots, Y_{d+2}) \in \varphi(M(r, d))$ is obtained as $\varphi^{-1}(\mathbf{Y}) = \sum_{\alpha=1}^{d+2} Y_\alpha P_\alpha(x)$ by Lagrange's interpolation formula.

We set the coordinate on $M_r(\mathbb{C})^{d+2}$ by using y_{ij}^α ($1 \leq \alpha \leq d+2, 1 \leq i, j \leq r$) as $Y_\alpha = (y_{ij}^\alpha)_{1 \leq i, j \leq r} \in M_r(\mathbb{C})$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_{d+2}) \in M_r(\mathbb{C})^{d+2}$. We define the G_r -action on $M_r(\mathbb{C})^{d+2}$ by

$$(3.8) \quad g(x) : (Y_\alpha)_{1 \leq \alpha \leq d+2} \mapsto (g(a_\alpha)^{-1} Y_\alpha g(a_\alpha))_{1 \leq \alpha \leq d+2},$$

which is compatible with the G_r -action on $M(r, d)$. We equip $M_r(\mathbb{C})^{d+2}$ with the Poisson bracket which comes from that of $gl_r(\mathbb{C}) \cong M_r(\mathbb{C})$:

$$(3.9) \quad \{y_{ij}^\alpha, y_{kl}^\beta\} = \delta_{\alpha, \beta} (\delta_{j, k} y_{il}^\alpha - \delta_{l, i} y_{kj}^\alpha).$$

The associated Casimir functions are $t_{k, \alpha} = \text{tr}(Y_\alpha^k)$ for $1 \leq \alpha \leq d+2, k \in \mathbb{Z}_{>0}$.

For $E \in \text{Lie } G_r$, we introduce the Hamiltonian functions H_E on $M_r(\mathbb{C})^{d+2}$:

$$H_{E_{1j}} = \sum_{\alpha} y_{j1}^\alpha, \quad H_{E'_{1j}} = \sum_{\alpha} a_\alpha y_{j1}^\alpha, \quad H_{E_{ij}} = \sum_{\alpha} y_{ji}^\alpha, \quad \text{for } 2 \leq i, j \leq r.$$

These satisfy $H_{[E, E']} = \{H_E, H_{E'}\}$ for any $E, E' \in \text{Lie } G_r$. Each H_E generates a vector field on $M_r(\mathbb{C})^{d+2}$ compatible with X_E (3.3) on $M(r, d)$ via the map φ . The associated moment map $\mu : M_r(\mathbb{C})^{d+2} \rightarrow (\text{Lie } G_r)^*$ is the unique map which satisfies $H_E(\mathbf{Y}) = \langle \mu(\mathbf{Y}), E \rangle$ for all $\mathbf{Y} \in M_r(\mathbb{C})^{d+2}$ and $E \in \text{Lie } G_r$. Here $\langle \cdot, \cdot \rangle$ is the pairing between $(\text{Lie } G_r)^*$ and $\text{Lie } G_r$.

Lemma 3.9. 1. *The image of φ is an affine subvariety of $M_r(\mathbb{C})^{d+2}$ determined as the intersection of $\mu^{-1}(0)$ and $t_1^{-1}(0)$, where $t_1 = \sum_{\alpha} t_{1, \alpha}$.*
 2. *The Poisson structure (3.9) induces the Poisson structure on $\varphi(M_{ir}(r, d))/G_r$, and hence on $M_{ir}(r, d)/G_r$ via φ .*

Proof. 1: The image $\varphi(M(r, d))$ of φ is a subvariety of $M_r(\mathbb{C})^{d+2}$ determined by the following conditions:

$$(3.10) \quad \begin{aligned} & \sum_{\alpha=1}^{d+2} y_{11}^\alpha = 0, \\ & \sum_{\alpha=1}^{d+2} y_{j1}^\alpha = 0, \quad \sum_{\alpha=1}^{d+2} a_\alpha y_{j1}^\alpha = 0, \quad \sum_{\alpha=1}^{d+2} y_{ji}^\alpha = 0, \quad \text{for } 2 \leq i, j \leq r. \end{aligned}$$

We see that the last three conditions are nothing but the defining equations for $\mu^{-1}(0)$ (i.e. the zero of the Hamiltonian functions H_E). Summing up the first one and the last one for $2 \leq i = j \leq r$, we obtain the defining equation for $t_1^{-1}(0)$.

2: Recall that the action of G_r on $\varphi(M_{ir}(r, d)) \subset M_r(\mathbb{C})_0^{d+2}$ is free, and that $\varphi(M_{ir}(r, d)) \subset \mu^{-1}(0) \cap t_1^{-1}(0)$. Then the Poisson structure (3.9) on $M_r(\mathbb{C})^{d+2}$ induces the Poisson structure on the quotient space $\varphi(M_{ir}(r, d))/G_r$. This is passed to the Poisson structure on $M_{ir}(r, d)/G_r$ by φ . \square

The following lemma is shown by a direct computation.

Lemma 3.10. *The vector fields $(p+1) \prod_{\alpha=1}^{d+2} (a - a_\alpha) \tilde{\Upsilon}_a^{(p)}$ on $M_{ir}(r, d)/G_r$ is Hamiltonian. They are generated by the G_r -invariant function $\text{tr}A(a)^{p+1}$ on $M_{ir}(r, d)$ with respect to the Poisson bracket of Lemma 3.9-2.*

Summarizing Theorem 2.8, 3.6 and Lemma 3.10, we conclude that

Theorem 3.11. *(cf. [1] 5.3) The Hamiltonian system $\psi|_{M_{ir}(r, d)/G_r} : M_{ir}(r, d)/G_r \rightarrow V(r, d)$ is completely integrable. In particular, the general level set is isomorphic to an open subvariety of a Jacobian. More precisely, we have $M_P/G_r \cong J'_P$ if $P \in V_{sm}(r, d)$.*

4. GENERALIZATION OF EVEN MUMFORD SYSTEM

4.1. Matrix realization of the affine Jacobian. In this section, we construct a family of subsystems of $M_{ir}(r, d)/G_r$ whose general level set is isomorphic to the complement of the union of r translates of the theta divisor in the Jacobian.

In the following, we write $A(x) \in M(r, d)$ as

$$(4.1) \quad A(x) = \begin{pmatrix} v(x) & {}^t \vec{w}(x) \\ \vec{u}(x) & T(x) \end{pmatrix},$$

where $v(x) \in S_d(x)$, $\vec{u}(x) \in S_{d-1}(x)^{\oplus r-1}$, $\vec{w}(x) \in S_{d+1}(x)^{\oplus r-1}$ and $T(x) \in M_{r-1}(S_d(x))$. The coefficients of x^k ($k \geq 0$) in $v(x), \vec{w}(x), \vec{u}(x)$ and $T(x)$ will be denoted by $v_k, \vec{w}_k, \vec{u}_k$ and T_k . For $A(x) \in M(r, d)$, we define

$$(4.2) \quad D(A(x); x) = (\vec{u}(x), T(x)\vec{u}(x), \dots, T(x)^{r-2}\vec{u}(x)) \in M_{r-1}(\mathbb{C}[x]),$$

$$(4.3) \quad D(A(x); \infty) = (\vec{u}_{d-1}, T_d \vec{u}_{d-1}, \dots, T_d^{r-2} \vec{u}_{d-1}) \in M_{r-1}(\mathbb{C}).$$

Note that $\det D(A(x); x)$ is a polynomial in x of degree at most g , and that the coefficients of x^g is $\det D(A(x); \infty)$.

For each $c \in \mathbb{P}^1$, we define the subspaces $\mathcal{M}_c, \mathcal{M}_c^{ir}$ and $\mathcal{M}_{c,P}$ of $M(r, d)$:

$$\mathcal{M}_c = \{A(x) \in M(r, d) \mid \det D(A(x); c) \neq 0\},$$

$$\mathcal{M}_c^{ir} = \mathcal{M}_c \cap M_{ir}(r, d),$$

$$\mathcal{M}_{c,P} = \mathcal{M}_c \cap M_P.$$

Lemma 4.1. 1. *The subset \mathcal{M}_c is invariant under the action of G_r on $M(r, d)$.*

2. *The action of G_r on \mathcal{M}_c is free.*

3. *Let c_1, \dots, c_{g+1} be distinct points on \mathbb{P}^1 . Then we have*

$$M_{ir}(r, d) \subset \bigcup_{i=1}^{g+1} \mathcal{M}_{c_i} = \bigcup_{c \in \mathbb{P}^1} \mathcal{M}_c \subset M(r, d).$$

Proof. Let $A(x) \in \mathcal{M}_c$ and $g(x) = \begin{pmatrix} 1 & {}^t\vec{b}(x) \\ \vec{0} & B \end{pmatrix} \in G_r$.

1: This follows from the relation $\det D(g(A(x)); x) = \det B^{-1} \cdot \det D(A(x); x)$.

2: A computation

$$(4.4) \quad g(A(x)) = \begin{pmatrix} v - {}^t\vec{b} \cdot B^{-1}\vec{u} & {}^t\vec{w} \cdot B + v{}^t\vec{b} - {}^t\vec{b}B^{-1}\vec{u}{}^t\vec{b} - {}^t\vec{b}B^{-1}TB \\ B^{-1}\vec{u} & B^{-1}\vec{u}{}^t\vec{b} + B^{-1}TB \end{pmatrix}$$

shows that the condition $g(A(x)) = A(x)$ implies $BD(A(x); x) = D(A(x); x)$ and ${}^t\vec{b}D(A(x); x) = 0$. If we further assume $A(x) \in \mathcal{M}_c$, then we obtain $B = \mathbb{I}_{r-1}$ and $\vec{b} = 0$.

3: The equality in the middle holds since $\deg_x D(A(x); x) \leq g$. We show the left inclusion.

Assume $A(x) \notin \mathcal{M}_c$ for all $c \in \mathbb{P}^1$. Then $D(A(x); x)$ is identically zero. Hence we have

$$\det \left(\begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}, A(x) \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}, \dots, A(x)^{r-1} \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \right) = 0,$$

which implies that the column vectors span a proper subspace in $\mathbb{C}(x)^{\oplus r}$ invariant under $A(x)$. Therefore the characteristic polynomial of $A(x)$ is reducible if $A(x) \notin \mathcal{M}_c$. \square

This lemma implies that \mathcal{M}_c^{ir}/G_r is a subsystem of the completely integrable system $M_{ir}(r, d)/G_r$. The general level set is described in the following:

Proposition 4.2. *Let $c \in \mathbb{P}^1$ and $P \in V_{sm}(r, d)$ such that $\pi : C_P \rightarrow \mathbb{P}^1$ is unramified over c . Then the level set $\mathcal{M}_{c,P}/G_r$ of \mathcal{M}_c^{ir}/G_r is isomorphic to $J_P \setminus (\bigcup_{q \in \pi^{-1}(c)} \Theta_q)$.*

Proof. Let $A(x) \in M_P$ and let $L \in J'_P$ be the image of $A(x)$ under the map $M_P \rightarrow M_P/G_r \cong J'_P$. According to Proposition 2.10 and Theorem 2.8, L is in $\bigcup_{q \in \pi^{-1}(c)} \Theta_q$ if and only if the first entry of any eigenvector of ${}^tA(c)$ is nonzero. Thus the following lemma on linear algebra completes the proof. \square

Lemma 4.3. *Let $C \in M_r(\mathbb{C})$ be a semi-simple matrix. Writing ${}^tC = \begin{pmatrix} * & * \\ \vec{c} & C_0 \end{pmatrix}$ with $C_0 \in M_{r-1}(\mathbb{C})$ and $\vec{c} \in \mathbb{C}^{r-1}$, we set $D = {}^t(\vec{c}, C_0\vec{c}, \dots, C_0^{r-2}\vec{c}) \in M_{r-1}(\mathbb{C})$. We write W for the subspace of \mathbb{C}^r generated by all eigenvectors of C whose first entries are zero. Then we have $\dim W = r - 1 - \text{rank } D$.*

Proof. Define $i : \mathbb{C}^{r-1} \rightarrow \mathbb{C}^r$ by setting the first entry to be zero, and let $V_0 = i(\mathbb{C}^{r-1})$. Let $W_0 = \{i(\vec{w}) \in V_0 \mid \vec{w} \in \mathbb{C}^{r-1}, D\vec{w} = 0\}$. Since $\dim W_0 = r - 1 - \text{rank } D$, it is enough to show $W = W_0$. The lemma below shows that W is the maximal subspace of V_0 which satisfies the condition $CW \subset W$. Since $CW_0 \subset W_0$, we have $W_0 \subset W$. To show the converse, we take $\vec{w} \in W$. Since $CW \subset W$, we have $C^k\vec{w} \in W (\subset V_0)$ for all $k \geq 0$. By writing down the condition $C^k\vec{w} \in V_0$ for $k = 0, 1, \dots$, we see $\vec{w} \in W_0$. This shows $W \subset W_0$ and we have done. \square

Lemma 4.4. *Let $f : V \rightarrow V$ be a semi-simple endomorphism of a finite dimensional \mathbb{C} -vector space. For a subspace V' of V , we write $Ev(V')$ for the set of eigenvectors of f in V' . Let W be a subspace of V . Let W_{st} be the maximal subspace in W which satisfies $f(W_{st}) \subset W_{st}$, and let W_{eig} be the subspace of V generated by $Ev(W)$. Then we have $W_{st} = W_{eig}$.*

Proof. We have $W_{eig} \subset W_{st}$ because $f(W_{eig}) \subset W_{eig}$. It holds that

$$W_{st} \stackrel{(1)}{=} \langle Ev(W_{st}) \rangle \stackrel{(2)}{\subset} \langle Ev(W) \rangle \stackrel{(3)}{=} W_{eig}.$$

Here (1), (2) and (3) follows by the semi-simplicity of f , by $Ev(W_{st}) \subset Ev(W)$ and by definition, respectively. \square

We summarize our main result.

Theorem 4.5. *The Hamiltonian system $\psi|_{\mathcal{M}_c^{ir}/G_r} : \mathcal{M}_c^{ir}/G_r \rightarrow V(r, d)$ is algebraically completely integrable. In particular the general level set is isomorphic to an affine subvariety of a Jacobian. More precisely, if $P \in V_{sm}(r, d)$ and if $\pi : C_P \rightarrow \mathbb{P}^1$ is unramified over c , we have $\mathcal{M}_{c,P}/G_r \cong J_P^g \setminus (\bigcup_{q \in \pi^{-1}(c)} \Theta_q)$.*

Remark 4.6. The Hamiltonian vector fields $\tilde{\Upsilon}_a^{(p)}$ are defined on \mathcal{M}_c/G_r (not only on \mathcal{M}_c^{ir}/G_r) because of Lemma 4.1-2.

4.2. Space of Representatives. We introduce a space of representatives of \mathcal{M}_c/G_r . For Beauville's system, Donagi and Markman [3] constructed such a space of representatives.

We define subspaces \mathcal{S}_c of $M(r, d)$ for $c \in \mathbb{P}^1$ as follows:

$$\begin{aligned} \mathcal{S}_c = & \left\{ A(x) \in M(r, d) \left| A(x) = \begin{pmatrix} v^{(0)} & {}^t \vec{w}^{(0)} \\ \vec{\nu} & \tau \end{pmatrix} + (x - c) \begin{pmatrix} v^{(1)} & {}^t \vec{w}^{(1)} \\ \vec{u}^{(1)} & T^{(1)} \end{pmatrix} \right. \right. \\ & \left. \left. + \text{higher terms in } (x - c), T^{(1)} \in \mathcal{T} \right\}, \quad \text{for } c \in \mathbb{C}, \\ \mathcal{S}_\infty = & \left\{ A(x) \in M(r, d) \left| A(x) = \begin{pmatrix} 0 & {}^t \vec{w}_{d+1} \\ \vec{0} & O \end{pmatrix} x^{d+1} + \begin{pmatrix} v_d & {}^t \vec{w}_d \\ \vec{0} & \tau \end{pmatrix} x^d + \begin{pmatrix} v_{d-1} & {}^t \vec{w}_{d-1} \\ \vec{\nu} & T_{d-1} \end{pmatrix} x^{d-1} \right. \right. \\ & \left. \left. + \text{lower terms in } x, T_{d-1} \in \mathcal{T} \right\}, \quad \text{for } c = \infty. \end{aligned}$$

Here $\tau, \vec{\nu}$ and the set \mathcal{T} is as follows:

$$(4.5) \quad \tau = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \in M_{r-1}(\mathbb{C}), \quad \vec{\nu} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{r-1},$$

$$\mathcal{T} = \{ \rho \in M_{r-1}(\mathbb{C}) \mid \rho_{1j} = 0 \text{ for } j = 1, \dots, r-1 \}.$$

By definition, $\mathcal{S}_c \subset \mathcal{M}_c$ since $\det D(A(x); x) = 1$ for all $A(x) \in \mathcal{S}_c$.

Proposition 4.7. *For $c \in \mathbb{P}^1$, the map given by $\mathcal{S}_c \times G_r \rightarrow \mathcal{M}_c$; $(S(x), g(x)) \mapsto g(S(x))$ is an isomorphism. Thus the space \mathcal{S}_c is a set of representatives of \mathcal{M}_c/G_r .*

This is a consequence of the following lemma:

Lemma 4.8. *Let $c \in \mathbb{P}^1$.*

1. *If $A(x) \in \mathcal{M}_c$, then there exists $g(x) \in G_r$ such that $g(A(x)) \in \mathcal{S}_c$.*
2. *If $g(S(x)) = \tilde{S}(x)$ with $S(x), \tilde{S}(x) \in \mathcal{S}_c$ and $g(x) \in G_r$, then we have $g(x) = \mathbb{I}_r$.*

Proof. 1: We give a proof for $c \neq \infty$. (The case of $c = \infty$ can be shown in a similar way.) Define $B \in M_{r-1}(\mathbb{C})$ by

$$B = \left(\vec{u}(c), \zeta_1 \vec{u}(c), \dots, \zeta_{r-2} \vec{u}(c) \right).$$

Here ζ_i ($1 \leq i \leq r-2$) $\in M_{r-1}(\mathbb{C})$ are defined by

$$\zeta_i = T(c)^i + \beta_1 T(c)^{i-1} + \beta_2 T(c)^{i-2} + \dots + \beta_i \mathbb{I}_{r-1},$$

where β_i ($1 \leq i \leq r-1$) are the coefficients of y^i in the characteristic polynomial of $T(c)$: $\det(y\mathbb{I}_{r-1} - T(c)) = y^{r-1} + \beta_1 y^{r-2} + \dots + \beta_{r-1}$. Since we have assumed $A(x) \in \mathcal{M}_c$, B is invertible. Then we obtain

$$\begin{pmatrix} 1 & 0 \\ 0 & B^{-1} \end{pmatrix} A(x) \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} * & * \\ \vec{\nu} & \tau' \end{pmatrix} + (x-c) \begin{pmatrix} * & * \\ * & T \end{pmatrix} + \text{higher terms in } (x-c),$$

where

$$\tau' = \begin{pmatrix} -\beta_1 & -\beta_2 & \dots & -\beta_{r-1} \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}, \quad T \in M_{r-1}(\mathbb{C}).$$

We define \vec{b}_1 and \vec{b}_0 by

$$\vec{b}_1 c + \vec{b}_0 = {}^t(\beta_1, \dots, \beta_{r-1}), \quad \vec{b}_1 = -{}^t(T_{11}, T_{12}, \dots, T_{1r-1}).$$

Consequently we obtain the matrix

$$g(x) = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & {}^t\vec{b}_1 x + {}^t\vec{b}_0 \\ 0 & 1 \end{pmatrix},$$

which satisfies $g(A(x)) \in \mathcal{S}_c$.

2: By expanding the relation $g(S(x)) = \tilde{S}(x)$ in $(x-c)$ and comparing the coefficient matrices of $(x-c)^0$ and $(x-c)^1$, we see $g(x) = \mathbb{I}_r$. \square

4.3. Integrable structure of \mathcal{S}_∞ . Now we set $c = \infty$. We study an explicit relation between $\mathcal{S}_{\infty,P}$ and $\text{Div}_{eff}^g(C_P)$, then give a description of the vector field on \mathcal{S}_∞ . These two results may be regarded as the counterparts of the studies on Beauville's system by Smirnov and Zeitlin [10] §4.1-2, and by Fu [5] respectively.

Let $P \in V_{sm}(r, d)$ be such that $\infty \in \mathbb{P}^1$ is not a ramification point of π , and set $\mathcal{S}_{\infty,P} = \mathcal{S}_\infty \cap M_P$. We study the relation between $\mathcal{S}_{\infty,P}$ and $\text{Div}_{eff}^g(C_P)$ by applying the method of Sklyanin [9] (the separation of variables). Let $\tau : \text{Div}_{eff}^g(C_P) \rightarrow J_P^g$ be the Abel-Jacobi map. Its restriction $\tau|_{\tau^{-1}(J_P^g)}$ is injective, because the complete linear system of $L \in J_P^g$ is of dimension zero (cf. Lemma 2.5). By abuse of notation, we write τ^{-1} for the composition of

$$J_P^g \xrightarrow{\cong} \tau^{-1}(J_P^g) \hookrightarrow \text{Div}_{eff}^g(C_P).$$

Our aim is to give an explicit description of the composition κ of

$$\mathcal{S}_{\infty,P} \xrightarrow{\cong} \mathcal{M}_{\infty,P}/G_r \xrightarrow{\cong} J_P^g \setminus \left(\bigcup_{q \in \pi^{-1}(\infty)} \Theta_q \right) \subset J_P^g \xrightarrow{\tau^{-1}} \text{Div}_{eff}^g(C_P).$$

Unfortunately, our result is limited to a subset of $\mathcal{S}_{\infty,P}$ due to technical difficulties. Define

$$\mathcal{S}'_{\infty,P} = \{A(x) \in \mathcal{S}_{\infty,P} \mid \text{all roots of } \det D(A(x); x) \text{ are simple and belong to } \pi(C_P^0)\}.$$

Note that $\det D(A(x); x)$ of $A(x) \in \mathcal{S}_{\infty,P}$ is of degree g by the definition of $\mathcal{S}_{\infty,P}$.

Proposition 4.9. *Let $A(x) \in \mathcal{S}'_{\infty,P}$. Denote by x_1, \dots, x_g the simple roots of $\det D(A(x); x) = 0$. Let $\vec{v} \in \mathbb{C}^{r-1}$ be any vector satisfying*

$$(4.6) \quad \det(\vec{v}, \vec{u}(x), \dots, T(x)^{r-3}\vec{u}(x)) \neq 0.$$

With this \vec{v} , define

$$(4.7) \quad y_i := \frac{\det(T(x)\vec{v}, \vec{u}(x), \dots, T(x)^{r-3}\vec{u}(x))}{\det(\vec{v}, \vec{u}(x), \dots, T(x)^{r-3}\vec{u}(x))} \Big|_{x=x_i}.$$

(This is independent of the choice of \vec{v} .) Then we have $\kappa(A(x)) = \sum_{i=1}^g (x_i, y_i)$.

Proof. The assumption that x_i ($1 \leq i \leq g$) is a simple root of $\det D(A(x); x)$ implies that the rank of $D(A(x); x_i)$ is $r-2$. By Lemma 4.3, there exists a unique eigenvector of ${}^t A(x_i)$ whose first component is zero. Denote the eigenvalue by α_i . Then by Proposition 2.10, the invertible sheaf L corresponding to $A(x)$ satisfies $L \in \bigcap_{i=1}^g \Theta_{(x_i, \alpha_i)}$. Because of the injectivity of $\tau|_{\tau^{-1}(J_P^g)}$ mentioned above, we see $\kappa(A(x)) = \tau^{-1}(L) = \sum_{i=1}^g (x_i, \alpha_i)$. Thus what we have to show is that $y_i = \alpha_i$.

For simplicity, we show the case of $i = 1$. Since the eigenvalue α_1 of $A(x_1)$ is also an eigenvalue of $T(x_1)$, there exists an eigenvector $\vec{\mu}'$ of $T(x_1)$ of the eigenvalue α_1 . It is easy to show $\det(\vec{\mu}', \vec{u}(x_1), \dots, T(x_1)^{r-3}\vec{u}(x_1)) \neq 0$, and we obtain

$$\alpha_1 = \frac{\det(T(x_1)\vec{\mu}', \vec{u}(x_1), \dots, T(x_1)^{r-3}\vec{u}(x_1))}{\det(\vec{\mu}', \vec{u}(x_1), \dots, T(x_1)^{r-3}\vec{u}(x_1))}.$$

Let $\vec{\nu} \in \mathbb{C}^{r-1}$ be a vector satisfying (4.6). Then there exist rational functions $\beta(x)$, $\beta_0(x), \dots, \beta_{r-3}(x) \in \mathbb{C}(x)$ such that

$$\vec{\nu} = \beta(x)\vec{\mu}' + \sum_{k=0}^{r-3} \beta_k(x)T(x)^k\vec{u}(x).$$

Here $\beta(x) \neq 0$ by the assumption on $\vec{\nu}$. Now it is immediate to check that $y_1 = \alpha_1$. \square

Next we describe the vector field on \mathcal{S}_∞ induced from (3.1), using the following lemma:

Lemma 4.10. *Let X be a vector field on $\mathcal{M}_\infty \simeq \mathcal{S}_\infty \times G_r$. The isomorphism $\Phi : \mathcal{M}_\infty \xrightarrow{\sim} \mathcal{S}_\infty \times G_r$; $A(x) \mapsto (S(x), g(x))$ induces the decomposition of X as $\Phi_*X = F + G$, where $F \in H^0(\mathcal{S}_\infty \times G_r, T\mathcal{S}_\infty)$ and $G \in H^0(\mathcal{S}_\infty \times G_r, TG_r)$. Then*

$$(4.8) \quad X(A(x)) = g(F(S(x), g(x))) - [g(x)^{-1}G(S(x), g(x)), A(x)].$$

Here we identify $T_{g(x)}G_r$ with $\text{Lie}G_r$, and $T_{S(x)}\mathcal{S}_\infty$ with the subspace of $M(r, d)$ via the inclusion $\mathcal{S}_\infty \hookrightarrow M(r, d)$.

The proof is left to the reader. The Hamiltonian vector field on \mathcal{S}_∞ becomes as follows:

Proposition 4.11. *The projection of the vector field (3.1) onto \mathcal{S}_∞ is*

$$(4.9) \quad F_a^{(p)}(A(x)) = \frac{1}{x-a}[A(a)^p, A(x)] + \left[\begin{pmatrix} 0 & {}^t\vec{\gamma}_p x + {}^t\vec{\beta}_p \\ \vec{0} & C_p \end{pmatrix}, A(x) \right] \quad \text{at } A(x) \in \mathcal{S}_\infty.$$

Here $(\vec{\gamma}_p, \vec{\beta}_p, C_p) \in \mathbb{C}^{r-1} \oplus \mathbb{C}^{r-1} \oplus M_{r-1}(\mathbb{C})$ is a unique solution of

$$(4.10) \quad \begin{aligned} C_p \cdot \vec{\nu} &= (\tau - v_d \mathbb{I}_{r-1}) \cdot \vec{h}_p, \\ \vec{\nu} \cdot {}^t\vec{\gamma}_p - [C_p, \tau] &= \vec{h}_p \cdot {}^t\vec{w}_{d+1}, \\ (\vec{\nu} \cdot {}^t\vec{\beta}_p + \vec{u}_{d-2} \cdot {}^t\vec{\gamma}_p - C_p T_{d-1})_{1,i} &= (\vec{h}_p \cdot ({}^t\vec{w}_d + a {}^t\vec{w}_{d+1}) + J_p \tau)_{1,i}, \text{ for } 1 \leq i \leq r-1, \end{aligned}$$

where τ and $\vec{\nu}$ are defined in (4.5), and \vec{h}_p and J_p are

$$(4.11) \quad A(a)^p = \begin{pmatrix} * & * \\ \vec{h}_p & J_p \end{pmatrix}.$$

Proof. The equations (4.10) are obtained by solving (4.8) for $F(S(x), g(x))$ and $G(S(x), g(x))$ at $X = \Upsilon_a^{(p)}$ and $g = \mathbb{I}_r$. Eq. (4.8) becomes

$$(4.12) \quad \Upsilon_a^{(p)}(A(x)) = F_a^{(p)}(A(x)) - [G_a^{(p)}(A(x)), A(x)],$$

where $F_a^{(p)}(A(x))$ is of the form

$$F_a^{(p)}(A(x)) = x^{d+1} \begin{pmatrix} 0 & * \\ \vec{0} & O \end{pmatrix} + x^d \begin{pmatrix} * & * \\ \vec{0} & O \end{pmatrix} + x^{d-1} \begin{pmatrix} * & * \\ \vec{0} & \rho \end{pmatrix} + \text{lower terms in } x.$$

Here $\rho \in \mathcal{T}$ (4.5), and $G_a^{(p)}(A(x)) \in \text{Lie } G_r$ is of the form

$$G_a^{(p)}(A(x)) = \begin{pmatrix} 0 & {}^t\vec{\gamma}_p x + {}^t\vec{\beta}_p \\ \vec{0} & C_p \end{pmatrix} \quad (\vec{\gamma}_p, \vec{\beta}_p \in \mathbb{C}^{r-1}, C_p \in M_{r-1}(\mathbb{C})).$$

The matrix $G_a^{(p)}(A(x))$ is determined as follows. In the LHS of (4.12), the $(i, 1)$ -entries ($2 \leq i \leq r$) and (i, j) -entries ($2 \leq i, j \leq r$) are

$$\begin{aligned} & (v_d \mathbb{I}_{r-1} - \tau) \cdot \vec{h}_p x^{d-1} + \text{lower order in } x, \\ & \vec{h}_p \cdot {}^t\vec{w}_{d+1} x^d + (\vec{h}_p \cdot ({}^t\vec{w}_d + a {}^t\vec{w}_{d+1}) + [J_p, \tau]) x^{d-1} + \text{lower order in } x. \end{aligned}$$

In the RHS of (4.12), the $(i, 1)$ -entries ($2 \leq i \leq r$) and (i, j) -entries ($2 \leq i, j \leq r$) are

$$\begin{aligned} & -C_p \cdot \vec{v} x^{d-1} + \dots, \\ & (\vec{v} \cdot {}^t\vec{\gamma}_p - [C_p, \tau]) x^d + (\vec{v} \cdot {}^t\vec{\beta}_p + \vec{u}_{d-2} \cdot {}^t\vec{\gamma}_p - [C_p, T_{d-1}] + \rho) x^{d-1} + \dots. \end{aligned}$$

We obtain the equations (4.10) for $(\vec{\gamma}_p, \vec{\beta}_p, C_p)$ by comparing the LHS and the RHS.

The solution to eqs. (4.10) is unique since the first and second equations completely determine C_p and $\vec{\gamma}_p$ and then the third equation completely determines the value of $\vec{\beta}_p$. \square

4.4. Examples. The case of $r = 2$: we have the space of representatives as

$$\begin{aligned} \mathcal{S}_\infty &= \left\{ A(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & t(x) \end{pmatrix} \right. \\ &= \begin{pmatrix} 0 & w_{d+1} \\ 0 & 0 \end{pmatrix} x^{d+1} + \begin{pmatrix} v_d & w_d \\ 0 & 0 \end{pmatrix} x^d + \begin{pmatrix} v_{d-1} & w_{d-1} \\ 1 & 0 \end{pmatrix} x^{d-1} + \text{lower terms in } x \left. \right\}. \end{aligned}$$

For $P \in V_{sm}(r, d)$, the genus of the curve C_P is $d-1$. The isomorphism given in Proposition 4.9 becomes very simple: x_k ($k = 1, \dots, d-1$) are the zeros of $u(x)$ and $y_k = t(x_k)$. The vector field on \mathcal{S}_∞ (4.9) becomes

$$F_a^{(1)}(A(x)) = \left[\frac{1}{x-a} A(a) + u(a) \begin{pmatrix} 0 & (x+a-u_{d-2})w_{d+1}+w_d \\ 0 & -v_d \end{pmatrix}, A(x) \right].$$

Let $V = \{P(x, y) \in V(2, d) \mid s_1(x) \equiv 0\}$. The restriction $\psi^{-1}(V) \cap \mathcal{S}_\infty \rightarrow V$ of our system $\psi|_{\mathcal{S}_\infty} : \mathcal{S}_\infty \rightarrow V(2, d)$ coincides with the even Mumford system introduced by Fernandes and Vanhaeche [4].

The case of $r = 3$: this is a new system. \mathcal{S}_∞ is written as

$$\begin{aligned} \mathcal{S}_\infty = & \left\{ A(x) = \begin{pmatrix} v(x) & w^{(1)}(x) & w^{(2)}(x) \\ u^{(1)}(x) & T^{(1,1)}(z) & T^{(1,2)}(z) \\ u^{(2)}(x) & T^{(2,1)}(z) & T^{(2,2)}(z) \end{pmatrix} \right. \\ & = \begin{pmatrix} 0 & w_{d+1}^{(1)} & w_{d+1}^{(2)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x^{d+1} + \begin{pmatrix} v_d & w_d^{(1)} & w_d^{(2)} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x^d + \begin{pmatrix} v_{d-1} & w_{d-1}^{(1)} & w_{d-1}^{(2)} \\ 1 & 0 & 0 \\ 0 & T_{d-1}^{(2,1)} & T_{d-1}^{(2,2)} \end{pmatrix} x^{d-1} \\ & \left. + \text{lower terms in } x \right\}. \end{aligned}$$

For $P \in V_{sm}(r, d)$, the genus g of C_P is $3d - 2$. The isomorphism given in Proposition 4.9 becomes as follows: x_k are the zeros of $D(A(x))$ (4.2) and y_k (4.7) has two equivalent descriptions:

$$y_k = \frac{u^{(2)}(x)T^{(1,1)}(x) - u^{(1)}(x)T^{(2,1)}(x)}{u^{(2)}(x)} \Big|_{x=x_k} \quad \text{or} \quad \frac{u^{(1)}(x)T^{(2,2)}(x) - u^{(2)}(x)T^{(1,2)}(x)}{u^{(1)}(x)} \Big|_{x=x_k}.$$

The vector field on \mathcal{S}_∞ is written as

$$F_a^{(p)}(A(x)) = \left[\frac{1}{x-a} A(a) + \begin{pmatrix} 0 & {}^t\vec{\gamma}_p x + {}^t\vec{\beta}_p \\ 0 & C_p \end{pmatrix}, A(x) \right] \quad \text{for } p = 1, 2,$$

where

$$\begin{aligned} {}^t\vec{\gamma}_p x + {}^t\vec{\beta}_p &= h_p^{(1)}((x+a-u_{d-2}^{(1)}) {}^t\vec{w}_{d+1} + {}^t\vec{w}_d) \\ &\quad + {}^t(h_p^{(2)}w_{d+1}^{(2)}(x+T_{d-1}^{(2,1)}-u_{d-2}^{(1)}) + (J_p)_{1,2}), \quad h_p^{(2)}w_{d+1}^{(2)}T_{d-1}^{(2,2)} \\ C &= h_p^{(1)} \begin{pmatrix} -v_d & 0 \\ 1 & -v_d \end{pmatrix} + h_p^{(2)} \begin{pmatrix} 1 & w_{d+1}^{(2)} \\ -v_d & -w_{d+1}^{(1)} \end{pmatrix}. \end{aligned}$$

Here J_p and $\vec{h}_p = \begin{pmatrix} h_p^{(1)} \\ h_p^{(2)} \end{pmatrix}$ are given at (4.11).

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