Virtual Turning Points — A gift of microlocal analysis to the exact WKB analysis *

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Summary. Several aspects of the notion of virtual turning points are discussed; its background, its relevance to the bifurcation phenomena of a Stokes curve, its importance in the analysis of the Noumi-Yamada system (a particular higher order Painlevé equation) and a concrete recipe for locating them. Examples given here make it manifest that virtual turning points are indispensable in WKB analysis of higher order linear ordinary differential equations with a large parameter.

0 Introduction

Microlocal analysis and the exact WKB analysis are intimately related and they are often complementary. A typical example is the exact steepest descent method ([AKT3], [AKT4], [T]), where a global version of the quantized Legendre transformation is given in terms of exact steepest descent paths. Here in

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this report we discuss another important example of their interactions, namely the notion of virtual turning points. Since this notion does not find any precedents in traditional asymptotic analysis, we first explain why it is needed. As pointed out by Silverstone ([S]), notorious ambiguities in the connection problems in WKB analysis are resolved if we make use of the Borel resummation method; in a word we have to first specify the region (the so-called Stokes region) where the Borel sum of a WKB solution is well-defined. Parenthetically we note that the importance of the Borel resummation in WKB analysis is also shown by Bender-Wu ([BW]), Voros ([V]), Zinn-Justin ([Z]), Pham ([P]) and others from several points of view and that the exact WKB analysis means the WKB analysis based on the Borel resummation. (See [DDP], where the wording "exact semi-classical expansion" is also used.) In the description of Stokes regions for a second order linear ordinary differential operator with a large parameter η that is of the form $P = P(x, \eta^{-1}d/dx) = P(x, \eta^{-1}\xi)$, we need to consider only Stokes curves emanating from (ordinary, or traditional) turning points; it suffices to consider the union of integral curves of the direction field

$$\mathrm{Im}(\xi_j(x) - \xi_k(x))dx = 0 \tag{1}$$

that emanate from some traditional turning point a of type (j,k), i.e., a point a satisfying

$$\xi_j(a) = \xi_k(a), \tag{2}$$

where $\xi_j(x)$ and $\xi_k(x)$ are characteristic roots of the operator P. For higher order operators, however, the totality of Stokes curves emanating from turning points (i.e., points satisfying (2)) does not suffice to describe the Stokes region as Berk-Nevins-Roberts ([BNR]) first pointed out; we need a "new Stokes curve" that does not emanate from a traditional turning point if we want to find correct Stokes regions. As we discuss in Section 1, the needed "new Stokes curves" emanate from "new turning points", which were first detected in [AKT1] through microlocal analysis. The wording "a new turning point" has been superseded by "a virtual turning point" in recent literature, and here, and in what follows, we use this new wording. As the effect of a virtual turning point is inert in its immediate vicinity, we discuss in Section 2 its relevance to the bifurcation of a Stokes curve so that its effect may become impressively visible. In Section 3 we discuss how this relevance gives a neat interpretation of the strange and intriguing phenomenon that one of us (S. Sasaki; [Sa1], [Sa2]) has found in analyzing the Noumi-Yamada system ([NY]) with the help of a computer. In Appendix we briefly describe how to find a virtual turning point with the help of a computer.

As we discuss only some illuminating examples in this report, for the detailed argument we refer the reader to [AKT1], [AKT2], [AKT5], [AKSST], [Ho], [Sa1], and [Sa2]. We use the same notions and notations used in [KT].

1 The background of the introduction of a virtual turning point

Let P denote the following third order operator with a large parameter $\eta (>> 1)$:

$$\eta^{-3}\frac{d^3}{dx^3} + 3\eta^{-1}\frac{d}{dx} + 2ix. \tag{3}$$

In what follows we call this operator the BNR operator after Berk-Nevins-Roberts ([BNR]), who first observed the importance of this operator in WKB analysis. The turning points of (3) are $x = \pm 1$, and the traditional Stokes geometry is given in Fig. 1.

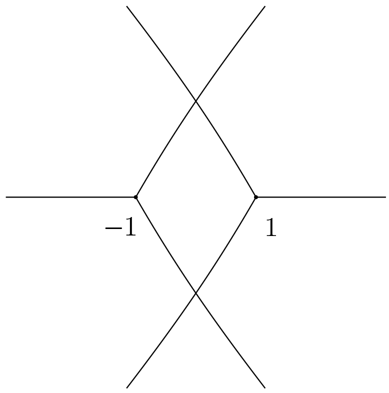

Figure 1

As Berk et al. ([BNR]) observed, we have to add two Stokes rays γ_1 and γ_2 to obtain the correct Stokes regions. See Fig. 2.

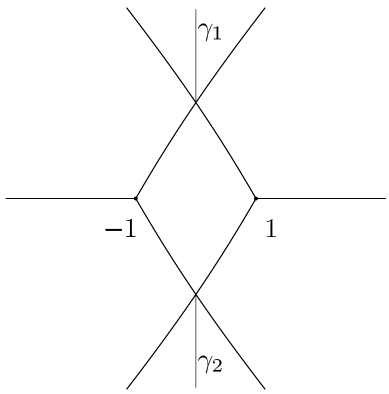

Figure 2

Then a natural question to be raised is: From what points do these rays emanate? Let us try to answer this question in terms of the singularity structure of the Borel transform $\psi_B(x, y)$ of a WKB solution $\psi(x, \eta)$ of the equation

$P\psi = 0$. In view of the definition of the Borel sum, we know ([V]) that the Stokes phenomenon of ψ is due to the interplay of singularities of $\psi_B(x, y)$: the Stokes phenomenon is observed when the path $C(x)$ of integration used to define the Borel sum $\int_{C(x)} \exp(-y\eta)\psi_B(x, y)dy$ is hit by some "related" singular point of $\psi_B(x, y)$. Hence it is reasonable to surmise that a starting point of a Stokes curve should be the x-component of a point where two "related", or "cognate", singularities of $\psi_B(x, y)$ coalesce. Actually a traditional turning point of a Schrödinger operator $\widetilde{P} = d^2/dx^2 - \eta^2 Q(x)$ is of this character: the Borel transform $\varphi_B(x, y)$ of a WKB solution φ of the equation $\widetilde{P}\varphi = 0$ has two singularities $s_\pm = \{(x, y); y = \pm \int_a^x \sqrt{Q}dx\}$ with $Q(a) = 0$, and they coalesce at $(x, y) = (a, 0)$. Let us now raise the following question: In what sense are s_+ and s_- cognate? To answer this question, we have to understand the structure of singularities of $\varphi_B(x, y)$. Fortunately enough, a clear-cut answer to this question has been given by microlocal analysis: Assuming that the point a is a simple turning point, i.e., a is a simple zero of the potential $Q(x)$, there exists a non-singular bicharacteristic strip of the Borel transform $\widetilde{P}_B$ of $\widetilde{P}$ whose projection to the base manifold $\mathbb{C}^2_{(x,y)}$ is $s_+ \cup s_-$ near $(x, y) = (a, 0)$. Here $\widetilde{P}_B$ is a partial differential operator given by $\partial^2/\partial x^2 - Q(x)\partial^2/\partial y^2$, and a bicharacteristic strip is, by definition, a solution curve of the Hamilton-Jacobi equation associated with $\widetilde{P}_B$. (See e.g. [CH, p.558].) Its projection to the base manifold is called a bicharacteristic curve. A fundamental result in microlocal analysis ([H], [SKK]) asserts that each bicharacteristic strip is the most "elementary" carrier of the singularities of solutions of a linear partial differential equation with simple characteristics. Hence a singular point in s_+ and that in s_- should be cognate, as they are both the projections of points in one and the same connected non-singular curve, a bicharacteristic strip. The next question is, then: Are there any other pairs of cognate singularities that coalesce? In a generic situation it is rather difficult to find such pairs. But the dimension of the base manifold is 2 in our case, and hence a bicharacteristic curve "generically" forms a self-intersection point.

Remark 1. We encounter self-intersection points of bicharcteristics "normally" even in higher dimensional case if we start with a subholonomic system with a large parameter instead of an ordinary differential equation with a large parameter. See [Sh] for such equations.

For example, in the case of the operator P given by (3), the associated bicharacteristic strip passing through $(x, y; \xi, \eta) = (1, 0; -i, 1)$ is given by

$$\begin{cases} x(t) = -4(t + 1/2)(t^2 + t - 1/2) \\ y(t) = -6it^2(t + 1)^2 \\ \xi(t) = -2it - i \\ \eta(t) = 1. \end{cases} \tag{4}$$

Hence its projection to the base space forms a (unique) self-intersection point at $(x, y) = (0, -3i/2)$. The situation is schematically illustrated in Fig. 3 with an appropriate labelling of solutions of the following characteristic equation of P:

$$\xi^3 + 3\eta^2\xi + 2ix\eta^3 = 0 \text{ with } \eta = 1. \tag{5}$$

The label j attached to a curve in Fig. 3 indicates that the curve is determined by the factor $\xi - \xi_j(x)\eta$ of the characteristic polynomial written in the form of $\prod_{l=1}^{3}(\xi - \xi_l(x)\eta)$. Note that the point A(resp., B) corresponds to the traditional

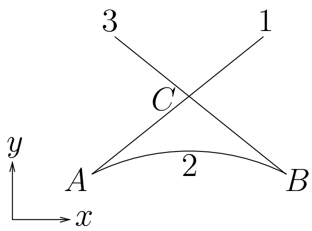

Figure 3

turning point $x = -1$(resp., 1). Thus the x-component of the point C, i.e., $x = 0$, is expected to play a role similar to a traditional turning point in the description of Stokes regions. Fortunately the actual situation is exactly as expected: the Stokes curve of type (1,3) that emanates from $x = 0$, that is,

$$\mathrm{Im} \int_0^x (\xi_1(x) - \xi_3(x))dx = 0 \tag{6}$$

contains Stokes rays γ_1 and γ_2 in Fig. 2. Furthermore, by using the reasoning of Voros ([V, p.244]) we find that the Stokes curve is inert near $x = 0$ (until it hits a crossing point of other Stokes curves) in the sense that no Stokes phenomena are observed there. We also note that the Voros argument ceases to work at the crossing points of Stokes curves as the singularity originating from the factor $\xi - \xi_2(x)\eta$ intervenes there. To emphasize the inert character of a portion of a Stokes curve, we usually use a dotted line to describe it. See Fig. 4.

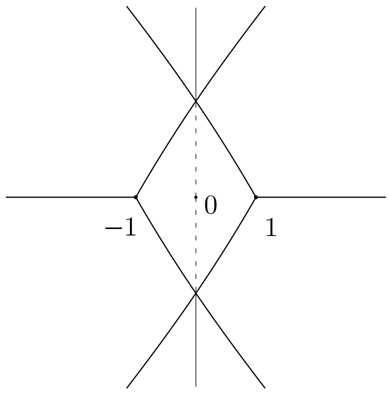

Figure 4

Thus we have found the correct Stokes regions shown in Fig. 2 by making use of an ordinary Stokes curve that emanates from the hitherto undetected point $x = 0$, and we are now entitled to call the point $x = 0$

a virtual turning point (of the BNR operator),

a turning point which cannot be detected with the naked eye but whose effect may resurge when it hits a crossing point of Stokes curves, due to the interplay of three singularities that occurs there. Although we have so far discussed the particular operator (3), the reasoning goes as well in the general case.

Definition 1.1 ([AKT1],[AKT2],[AKKT]) *Let $P = P(x, \eta^{-1}d/dx, \eta^{-1})$ be a linear differential operator with a large parameter η that is of the following form:*

$$P_0(x, \eta^{-1}d/dx) + \eta^{-1}P_1(x, \eta^{-1}d/dx) + \eta^{-2}P_2(x, \eta^{-1}d/dx) + \cdots . \tag{7}$$

Assume that its Borel transform $P_B = P(x, \partial_y^{-1}\partial_x, \partial_y^{-1})$ is a well-defined microdifferntial operator and that its traditional turning points are all simple (in the sense of [AKKT]). Then a virtual turning point of P is, by definition, the x-component of a self-intersection point of a bicharacteristic curve associated with the operator P_B. If the crossing bicharacteristic curves are respectively associated with the factor $(\eta^{-1}\xi - \xi_j(x))$ and $(\eta^{-1}\xi - \xi_k(x))$ of $P_0(x, \zeta) = \prod_l (\eta^{-1}\xi - \xi_l(x))$, we say the virtual turning point is of type (j, k).

Definition 1.2 *Let τ be a virtual turning point of type (j, k) of the operator P in Definition 1.1. Then an integral curve of the direction field*

$$\mathrm{Im}(\xi_j(x) - \xi_k(x))dx = 0 \tag{8}$$

that emanates from τ is called a new Stokes curve of type (j, k), or just a Stokes curve of type (j, k).

Remark 2. A bicharacteristic strip is a curve in the complex cotangent bundle. Hence a virtual turning point is a complex-analytic notion; unlike Stokes curves or their crossing points, real structure is irrelevant. To avoid the possible confusion of the reader, we note that the assertion contrary to this remark in [HLO, p.2292, l.3] originates from their erroneous quotation of the wording 'a new turning point' ([HLO, p.2291]). We also note that they make a misleading claim in p.2291, $l.8 \sim l.10$; actually $f_0(a)$ and $f_2(a)$ coalesce at a "new turning point". (Logically speaking, their argument in p.2291, l.2 resulted in counting a virtual turning point as a (traditional) turning point, contrary to their intention. At the same time Fig. 4 of [HLO] indicates that they overlooked the relation $f_0(0) = f_2(0)$. They could have avoided losing their way in the logical labyrinth if they had noticed this relation.) Parenthetically we also note that we can actually characterize (either traditional or virtual) turning points by the comparison of phase functions evaluated at different saddle points if the differential equation in question admits some "nice" integral representation of its solutions. See [Sh] for the concrete examples related to the quantized Hénon map.

Remark 3. If some turning points of the operator P in question are double, some care is needed in the definition of virtual turning points. See [AKT5] for the details. We note that the care is needed due to the complexity of microlocal structure of solutions of the equation $P_B\psi_B = 0$; the operator P_B is an operator with multiple characteristics in this case.

2 The relevance of a virtual turning point to the bifurcation phenomenon of a Stokes curve

The Stokes geometry given in Fig. 4 of the BNR operator is described with η being real and positive. Let us now study what happens when we change $\arg\eta$. This amounts to considering the operator

$$\eta^{-3}\frac{d^3}{dx^3} + 3a^2\eta^{-1}\frac{d}{dx} + 2ia^3x \tag{9}$$

with a parameter a satisfying $|a| = 1$, keeping η to be positive. Note that Stokes curves do depend on $\arg\eta$ by their definition but that (virtual and traditional) turning points remain fixed. (See Remark 2.) The resulting Stokes geometry for (i) $\arg\eta = (\frac{1}{2} - \frac{1}{12})\pi$, (ii) $\arg\eta = \frac{1}{2}\pi$ and (iii) $\arg\eta = (\frac{1}{2} + \frac{1}{12})\pi$ are respectively given in Fig. 5 (i), (ii) and (iii).

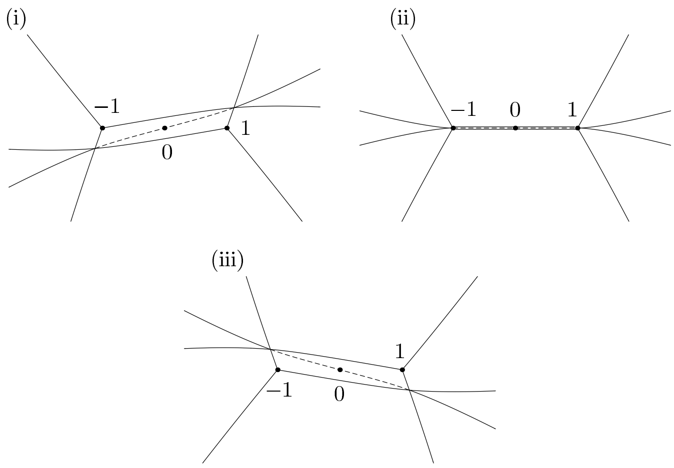

Figure 5: The Stokes geometry of the BNR operator for (i) $\arg\eta = (\frac{1}{2} - \frac{1}{12})\pi$, (ii) $\arg\eta = \frac{1}{2}\pi$ and (iii) $\arg\eta = (\frac{1}{2} + \frac{1}{12})\pi$.

The bifurcation of a Stokes curve observed in Fig. 5 (ii) is due to the singularity that the direction field (1) acquires at a simple turning point. Impressively enough, the smooth transition between Fig. 5 (i) and Fig. 5 (iii) via Fig. 5 (ii) is attained with the addition of Stokes curves emanating from the virtual turning point $x = 0$. One should observe some clumsy transition if they were not added. A subtle and interesting fact is that Fig. 5 (ii) switches the relative location of a Stokes curve emanating from a traditional turning point and that from a virtual turning point. As we will see in Section 3, this fact plays an important role in understanding the intriguing fact which Sasaki ([Sa1]) has found in the study of the Noumi-Yamada system.

3 Deformation of the linear differential equations that underlie the Noumi-Yamada system

The Noumi-Yamada system ([NY]) is one of the Painlevé hierarchies, i.e., a family of higher order non-linear equations whose first member coincides with one of the second order Painlevé equations. The first member of the Noumi-Yamada hierarchy is the following $(NY)_2$, which is a symmetric form of the fourth Painlevé equation (P_{IV});

$$(NY)_2 : \eta^{-1}\frac{df_j}{dt} = f_j(f_{j+1} - f_{j+2}) + \alpha_j \quad (j = 0, 1, 2), \tag{10}$$

where $f_j = f_{j-3}$ $(j = 3, 4)$ and α_j $(j = 0, 1, 2)$ are constants that satisfy

$$\alpha_0 + \alpha_1 + \alpha_2 = \eta^{-1}. \tag{11}$$

Its underlying Lax pair, i.e., an overdetermined system of linear differential equations whose compatibility conditions are given by (10), is as follows:

$$-\eta^{-1}x\frac{\partial}{\partial x}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}(2\alpha_1+\alpha_2)/3 & f_1 & 1\\ x & (-\alpha_1+\alpha_2)/3 & f_2\\ xf_0 & x & -(\alpha_1+2\alpha_2)/3\end{pmatrix}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix}, \tag{12}$$

$$-\eta^{-1}\frac{\partial}{\partial t}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}f_2 - t/2 & -1 & 0\\ 0 & f_0 - t/2 & -1\\ -x & 0 & f_1 - t/2\end{pmatrix}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix}. \tag{13}$$

In what follows we regard (12) as the main equation containing a parameter t that is to be deformed obeying (13), and we study the Stokes geometry of (12) for each t. Our earlier study ([KT]) of the connection problems for the Painlevé transcendents indicates that, if t_0 lies on a Stokes curve γ properly defined for the Painlevé equation, the Stokes geometry of (12) should degenerate in the sense that some turning points are connected by a Stokes curve. A more accurate statement would be that the Stokes geometry topologically changes off the curve γ near $t = t_0$; actually the degeneration of the Stokes geometry is a symptom of such a change. In studying the Stokes geometry of (12), Sasaki ([Sa1]) found the following intriguing FACT 3.1:

FACT 3.1 *When t_0 moves along a Stoke curve γ of* (10)*, we observe the following phenomena in the Stokes geometry of* (12)*:*

(i) *If t_0 is close to the starting point τ (i.e., a turning point of* (10)*) of γ, a double turning point and a simple turning point are connected by a Stokes curve of* (12)*.*

(ii) *If t_0 is far away from τ, no (traditional) turning points are connected by a Stokes curve of* (12)*.*

The situation is illustrated in Fig.6.

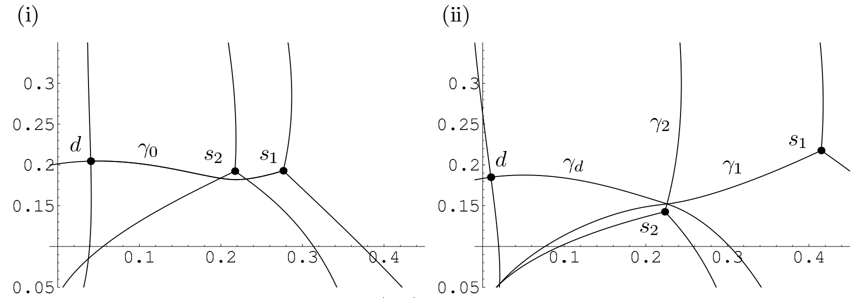

Figure 6: Stokes geometry of (12) with $\alpha_0 = 1 + 0.6i$ and $\alpha_1 = 0.2 - 0.1i$ for (i) $t_0 = -1.6104 - 0.2268i$ and (ii) $t_0 = -1.5783 - 0.4130i$.

To understand FACT 3.1 properly, we next include relevant virtual turning points in Fig. 6. For this purpose we study the Stokes geometry of (12) for t_0 slightly away from γ: near $t = t_0$ that realizes Fig. 6 (i), the resulting configuration is either one of the following:

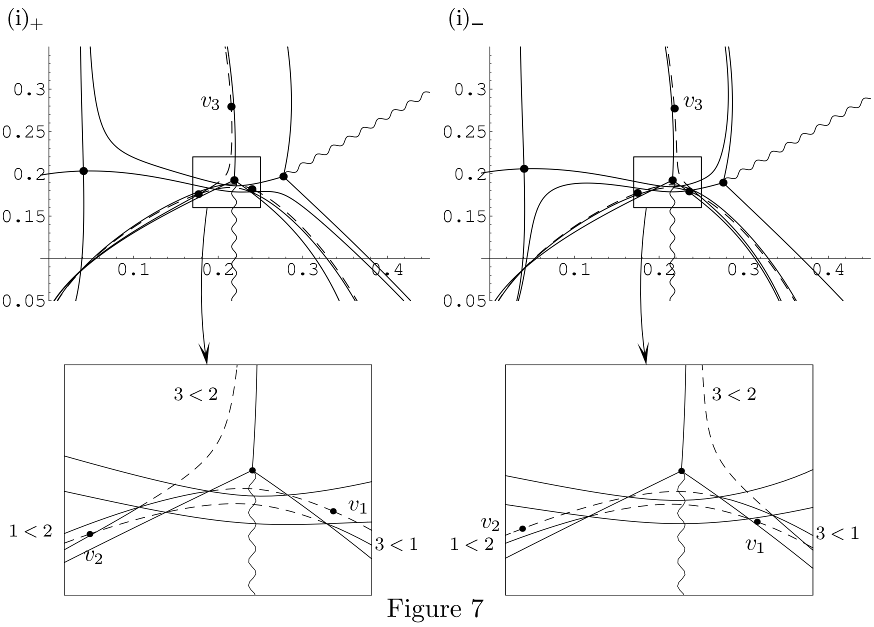

Figure 7

Similarly, near $t = t_0$ that realizes Fig. 6(ii), we find

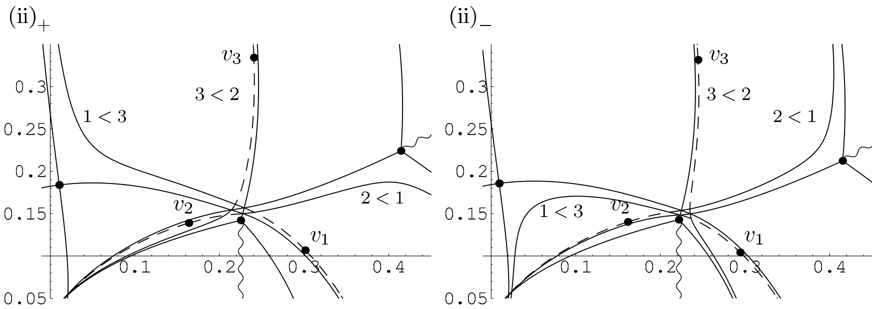

Figure 8

Here, and in what follows, a wiggly line designates a cut to fix the branch of a characteristic root, and the symbol $j > k$ attached to a Stokes curve indicates the dominance relation along the Stokes curve. (See, e.g., [AKT1], [AKSST] for the details.)

By letting the parameter t sit on the curve we then obtain the following Fig. 9 through the limiting procedure. (Cf. [AKSST, Fig. 2])

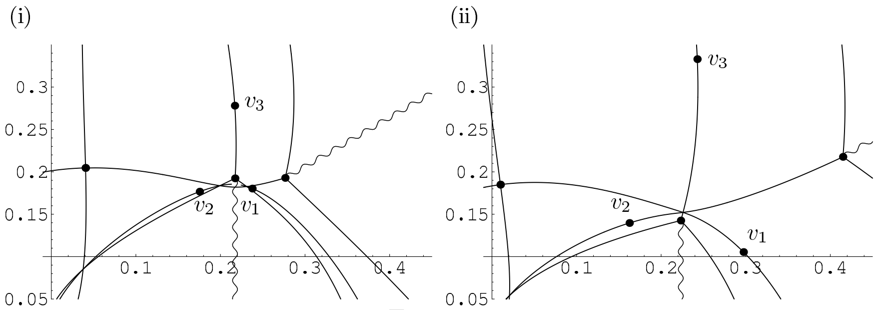

Figure 9

We now find the mechanism at the back of FACT 3.1: virtual turning points v_1, v_2 and v_3 should have been taken into account in Fig. 6 (ii). The degeneration of the Stokes geometry observed in Fig. 6 (i), that is, the existence of a pair of a double turning point d and a simple turning point s_1 which are connected by a Stokes curve, is superseded by another degeneration in Fig. 9 (ii), which is caused by the existence of a Stokes curve connecting the turning point d with a virtual turning point v_1 and that connecting s_1 with another virtual turning point v_2. We also note that Fig. 9 (i) and (ii) are switched by the following Fig. 10, which is observed when another simple turning point s_2 hits the Stokes curves in question and causes their bifurcation as we explained in Section 2.

Comparison of Fig. 7, 8 and 9 shows the following:

FACT 3.2 *Resolution of the degeneration in Fig. 9 (ii) by a tiny change of* t *induces the change of topological configurations of Stokes curves of* (12)*, as*

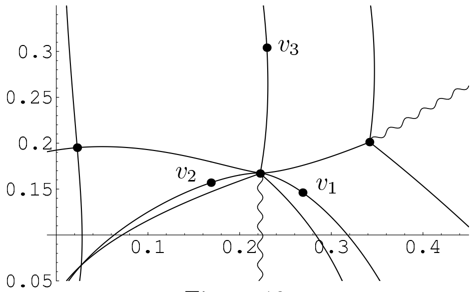

Figure 10

is observed in Fig. 8 $(\mathrm{ii})_+$ *and* $(\mathrm{ii})_-$ *; it is exactly in the same manner as the result of the resolution of the degeneration observed in Fig. 9 (i).*

Thus we are forced to conclude that the role of virtual turning points is commensurate with that of traditional turning points in describing the Stokes geometry.

The same comparison manifests the smooth transition from Fig. 7 $(\mathrm{i})_+$ (resp. Fig. 7 $(\mathrm{i})_-$) to Fig. 8 $(\mathrm{ii})_+$ (resp. Fig. 8 $(\mathrm{ii})_-$) outside a small neighborhood of the simple turning point s_2; this is what the reasoning in Section 2 predicts, but such a smooth transition can never be observed without virtual turning points.

Remark 4. The Noumi-Yamada system $(NY)_l$ ($l \geq 4$) is of higher order (higher than the second order), and the so-called Nishikawa phenomena ([KKNT]) are observed in its Stokes geometry. Its investigation requires some subtler study of Stokes geometry of the underlying linear differential equation (a counterpart of (12)). Such a study was initiated by Sasaki ([Sa2]) and its systematization is undertaken by Honda ([Ho]).

Appendix A practical recipe for locating a virtual turning point

If one wants to locate a virtual turning point following Definition 1.1, one needs to solve the Hamilton-Jacobi equation globally. In general it is a formidably difficult task. However, there is a practically satisfactory way to locate a virtual turning point with the help of a computer. For the reader's convenience we briefly describe the recipe. See also [AKKSST], [AKSST] and [Ho]; in particular [Ho] presents a systematic algorithm for describing a complete Stokes geometry for the underlying linear differential equation of $(NY)_4$. Probably the method of Honda ([Ho]) is applicable to general equations, beyond the framework of the Noumi-Yamada system.

To describe the recipe, let us first fix the situation to be considered: Suppose that a Stokes curve γ_1 of type (1,2) that emanates from a turning point τ_1 intersects at a point ι with another Stokes curve γ_2 of type (2,3) emanating from another turning point τ_2. In what follows τ_1 and τ_2 may be either virtual or traditional. (In case τ_1 or τ_2 is a simple turning point we need some care about the cut structure to fix the branches of solutions $\xi_j(x)$ of the characteristic equation.) Having the labeling of Fig. 3 in mind, a point x_* that satisfies

$$\int_{\tau_1}^{x_*} \xi_1 dx = \int_{\tau_1}^{\tau_2} \xi_2 dx + \int_{\tau_2}^{x_*} \xi_3 dx \tag{14}$$

is a virtual turning point. Next let us try to relate x_* with the point ι. Supposing that the cut structure is appropriately introduced if necessary, we use (14) to find the following:

$$\begin{aligned}\int_{\tau_1}^{x_*} (\xi_1 - \xi_2) dx &= \int_{\tau_1}^{\tau_2} \xi_2 dx + \int_{\tau_2}^{x_*} \xi_3 dx - \int_{\tau_1}^{x_*} \xi_2 dx \\ &= \int_{\tau_2}^{x_*} (\xi_3 - \xi_2) dx.\end{aligned} \tag{15}$$

(See [T, §3.3] for some diagrammatic interpretation of this relation.) By rewriting (15), we obtain

$$\begin{aligned}0 = &\int_{\tau_1}^{\iota} (\xi_1 - \xi_2) dx + \int_{\iota}^{x_*} (\xi_1 - \xi_2) dx + \int_{\tau_2}^{\iota} (\xi_2 - \xi_3) dx \\ &+ \int_{\iota}^{x_*} (\xi_2 - \xi_3) dx \\ = &\int_{\iota}^{x_*} (\xi_1 - \xi_3) dx + \int_{\tau_1}^{\iota} (\xi_1 - \xi_2) dx + \int_{\tau_2}^{\iota} (\xi_2 - \xi_3) dx.\end{aligned} \tag{16}$$

Since ι is an intersection point of Stokes curves γ_1 and γ_2, (16) entails

$$\operatorname{Im} \int_{\iota}^{x_*} (\xi_1 - \xi_3) dx = 0. \tag{17}$$

Hence ι is most likely to lie in the Stokes curve of type (1,3) that emanates from x_*. Here we say "most likely" just because we have not confirmed that x_* and ι belong to the same connected component of the real one-dimensional curve defined by (17). This point is, however, almost automatically checked in the computer-assisted study. This reasoning can be converted to find out a virtual turning point relevant to ι: We first consider a curve γ defined by

$$\operatorname{Im} \int_{\iota}^{x} (\xi_1 - \xi_3) dx = 0. \tag{18}$$

Defining a function $\rho(x)$ by

$$\operatorname{Re} \int_{\iota}^{x} (\xi_1 - \xi_3) dx, \tag{19}$$

we seek for a point x_0 in γ at which the following relation holds:

$$\rho(x_0) = \int_{\tau_1}^{\iota} (\xi_2 - \xi_1)dx + \int_{\tau_2}^{\iota} (\xi_3 - \xi_2)dx. \tag{20}$$

Since $\rho(x)$ is monotonically decreasing or increasing on the real one-dimensional curve γ, we can normally (i.e., except for the case where ρ is bounded on γ) locate such a point x_0 in γ. (Note that the right-hand side of (20) is a real number, as ι is an intersection point of Stokes curves.) Then we have

$$\int_{\iota}^{x_0} (\xi_1 - \xi_3)dx = \int_{\tau_1}^{\iota} (\xi_2 - \xi_1)dx + \int_{\tau_2}^{\iota} (\xi_3 - \xi_2)dx. \tag{21}$$

Hence the comparison of (16) and (21) entails that x_0 is coincident with a virtual turning point x_*. Actually all the virtual turning points studied by Sasaki ([Sa1], [Sa2]) and Honda ([Ho]) have been detected by this method.

Furthermore virtual turning points thus detected play important roles in describing the phase function $\phi(t)$ used in the instanton expansion of a solution of the Noumi-Yamada system: for example the function $\phi(t)$ is given by

$$\int_{d(t)}^{v_1(t)} (\xi_1 - \xi_3)dx \tag{22}$$

in the situation of Fig. 9 (ii) ([Sa1], [AKSST]). Using the terminology of [Ho], we can go further to give another interpretation of the point v_1 in the following manner: We can associate a function $\phi_T(t)$ to each effective bi-directional binary tree T that corresponds to degeneration of the Stokes geometry, and we can locate the required virtual turning point $v(t)$ in an appropriate Stokes curve of type, say (j, k), which emanates from a turning point τ so that the following relation holds:

$$\phi_T(t) = \int_{\tau(t)}^{v(t)} (\xi_j - \xi_k)dx \quad \text{(up to sign)}. \tag{23}$$

Let us note that $\phi_T(t)$ is given by

$$\int_{d}^{C} (\xi_1 - \xi_3)dx + \int_{s_1}^{C} (\xi_2 - \xi_1)dx + \int_{s_2}^{C} (\xi_3 - \xi_2)dx \tag{24}$$

with an appropriate indexing ξ_j in the situation of Fig. 6 (ii), where C designates the crossing point of three Stokes curves observed there. We note that (21) and (24) immediately entail (23) in this case. This interpretation of a virtual turning point plays an important role in grasping the behavior of napping virtual turning points ([Sa2]) in the framework of Honda ([Ho]).

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