How to Observe Quantum Fields and Recover Them from Observational Data? – Takesaki Duality as a Micro-Macro Duality –

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Abstract

After the mathematical notion of "Micro-Macro Duality" for understanding mutual relations between microsopic quantum systems (Micro) and their macroscopic manifestations (Macro) is explained on the basis of the notion of sectors and order parameters, a general mathematical scheme is proposed for detecting the state-structure inside of a sector through measurement processes of a maximal abelian subalgebra of the algebra of observables. For this purpose, the Kac-Takesaki operators controlling group duality play essential roles, which naturally leads to the composite system of the observed system and the measuring system formulated by a crossed product. This construction of composite systems will be shown to make it possible for the Micro to be reconstructed from its observational data as Macro in the light of the Takesaki duality for crossed products.

1 Quantum-Classical Correspondence and Micro-Macro Duality

The essence of "quantum-classical correspondence" (q-c correspondence, for short) is usually understood in such an intuitive way that *macroscopic classical* objects arise from a *microscopic quantum* system as *condensates of infinite quanta* in the latter. Aiming at a satisfactory understanding of mutual relations among different hierarchical levels in the physical world, we try here to provide this heuristic idea with a mathematically sound formulation on the basis of what we call "Micro-Macro duality" [1]; this is the mathematical notion of *duality* (or, *categorical adjunction* in more general situations), which allows us to connect microscopic and macroscopic levels in the physical nature in *bi-directional* ways from Micro to Macro and vice versa. To be precise, the contrasts of [Micro vs. Macro] (according to length scales) and of [Quantum vs. Classical] (due to the essential differences in their structures) are to certain extent independent of each other, as exemplified by the presence of such interesting phenomena as "macroscopic quantum effects" owing to the absence of an intrinsic length scale to separate quantum and classical domains. Since this kind of mixtures can usually be taken as 'exceptional', we restrict, for simplicity, our consideration here to such generic situations that processes taking place at microscopic levels are of quantum nature to be described by non-commutative quantities and that the macroscopic levels are described in the standard framework of classical physics in terms of commutative variables, unless the considerations on the above point become crucial.

First we note that, in formulating a physical theory, we need the following four basic ingredients, algebra of physical variables, its states & representations, their dynamical changes and a classifying space to classify, describe and interpret the obtained theoretical and experimental results, among which the algebra and its representations are mutually dual. When we try to provide collected experimental results with a physical interpretation, the most relevant points of the discussion starts from the problem to identify the states responsible for the phenomena under consideration. On the premise of the parallelisms among micro / quantum / non-commutative and macro / classical / commutative, respectively, the essential contents of q-c correspondence can be examined in the following steps and forms:

1) Superselection *sectors* and *intersectorial structures* described by order parameters: the first major gap between the microscopic levels described by non-commutative algebras of physical variables and the macroscopic ones by commutative algebras can be concisely formulated and understood in terms of the notion of a (*superselection*) *sector structure* consisting of a family of *sectors* (or *pure phases*) described mathematically by factor states and representations: the algebra of observables is represented within a sector by isomorphic von Neumann algebras with trivial centres and representations corresponding to different sectors are mutually disjoint. The totality of sectors (relevant to a given specific physical situation) constitutes physically a *mixed phase* involving both classical and quantum aspects. Sectors or pure phases are faithfully parametrized by the spectrum of the centre of a relevant representation of the C*-algebra of microscopic quantum observables describing a physical system under consideration. Physically speaking, operators belonging to the centre are mutually commutative classical observables which can be interpreted as *macroscopic order parameters*. In this way, the intersectorial structure describes the coexistence of and the gap between quantum(=intrasectorial) and classical(=intersectorial) aspects.

2) *Intrasectorial* quantum structures and measurement processes: it is evident, however, that we cannot attain a satisfactory description of a given quantum system unless we succeed in analyzing and describing the intrinsic quantum structures *within* a given sector, not only theoretically but also operationally (up to the resolution limits imposed by quantum theory itself). The detection of these invisible microscopic quantum aspects necessarily

involves the problem of quantum measurements. In the usual discussions in quantum mechanics, a maximal abelian subalgebra (MASA, for short) plays canonical roles in specifying a quantum state according to measured data, in place of the centre trivialized by Stone-von Neumann theorem of the uniqueness (up to unitary equivalence) of irreducible representations of CCR algebras with finite degrees of freedom. As seen below, the notion of MASA $\mathcal{A}$ plays central roles also in our context, whereas it need be reformulated, in such a quantum system as quantum fields with infinite degrees of freedom, whose algebra $\mathcal{M}$ of observables may have representations of non-type I. The present formulation will be seen also to determine the precise form of the coupling between the object system and the apparatus required for implementing a measurement process, on the basis of which the central notion of *instrument* can be concisely formulated.

3) Inverse problem to *reconstruct the algebra* of Micro system from the observational Macro data: the roles played by the above MASA and by its measured data suggest the possibility for us to reconstruct a microscopic non-commutative algebra $\mathcal{M}$ from the macroscopic information $Spec(\mathcal{A})$ obtained by measuring the MASA $\mathcal{A}$, in parallel with the structure theory of semisimple Lie algebras based on their root systems corresponding to chosen Cartan subalgebras. In fact, we show in Sec.5 that this analogy precisely works by means of the Takesaki duality applied to the crossed product $\mathcal{M} \rtimes_\alpha \mathcal{U}$ arising from the above coupling between the object system and the apparatus, where $\mathcal{U}$ is a locally compact group acting on $\mathcal{A}$ and generating it: $\mathcal{A} = \mathcal{U}''$ (in combination with a modest technical assumption of "semi-duality"). This observation is conceptually very important as the supporting evidence for the above-mentioned *bi-directionality* expected naturally in the notion of q-c correspondence. Here, the notion of *co-action* $\hat{\alpha}$ of $\hat{\mathcal{U}}$ on $\mathcal{M} \rtimes_\alpha \mathcal{U}$ plays a crucial role to reproduce $\mathcal{M} = (\mathcal{M} \rtimes_\alpha \mathcal{U}) \rtimes_{\hat{\alpha}} \hat{\mathcal{U}}$ from $\mathcal{M} \rtimes_\alpha \mathcal{U}$, according to which the von Neumann type of $\mathcal{M}$ is determined by the abelian dynamical system $[\mathfrak{Z}(\mathcal{M} \rtimes_\alpha \mathcal{U}) = \mathcal{A}] \underset{\hat{\alpha}}{\curvearrowleft} \hat{\mathcal{U}}$ on the classifying space $Spec(\mathcal{A})$ of the intrasectorial structure. On this last point, the presence of certain subtle points is exhibited in relation with the essential features of quantum systems with infinite degrees of freedom in the following form:

4) Roles of *intrinsic dynamics* responsible for the recovery of *non-type I* algebras: when the algebra of observables is represented in a Hilbert space as a von Neumann algebra of non-type I (like the typical case with local subalgebras of *type III* in relativistic quantum field theory), a state vector within a sector cannot uniquely be specified by means of quantum observables for *lack of minimal projections*. In view of the above von Neumann-type classification, the compatibility between the above reconstruction of $\mathcal{M}$ in 3) and its non-trivial type forces the action α of $\mathcal{U}$ on $\mathcal{M}$ to deviate from the adjoint form $Ad_u(X) = uXu^{-1}$, which invalidates the usual approximation adopted in most discussions of measurements to neglect the *intrinsic dy-*

namics of the object system closing up the coupling between the system and the apparatus. In this case, therefore, we need the information not only about states of the system but also its *intrinsic dynamics*, which can operationally be provided by the measurement of energy-momentum tensor. In this way, all the basic ingredients, $\mathcal{M}$, states on $\mathcal{M}$, the dynamics α and the classifying space $Spec(\mathcal{A})$ of the intrasectorial structure exhibit themselves in the discussion and to be determined operationally.

1.1 Q-C correspondence (I): Sectors & centre = order parameters

At the level of *sectors*, quantum and classical aspects can be separated in a clear-cut way by means of order parameters to specify a sector. To see this, we first recall the standard notion of *quasi-equivalence* $\pi_1 \approx \pi_2$ [2] of representations π_1, π_2 of an abstract C*-algebra $\mathfrak{A}$ describing the observables of a given microscopic quantum system: taken as unitary equivalence *up to multiplicity*, this notion can be reformulated into many equivalent forms such as the isomorphism of von Neumann algebras associated with representations:

$$\pi_1 \approx \pi_2 \Longleftrightarrow \pi_1(\mathfrak{A})'' \simeq \pi_2(\mathfrak{A})'' \Longleftrightarrow c(\pi_1) = c(\pi_2),$$

where $c(\pi)$ denotes the *central support* of a representation π. In the universal representation [2] of $\mathfrak{A}$, $(\pi_u := \underset{\omega\in E_{\mathfrak{A}}}{\oplus} \pi_\omega, \mathfrak{H}_u := \underset{\omega\in E_{\mathfrak{A}}}{\oplus} \mathfrak{H}_\omega)$, $\pi_u(\mathfrak{A})'' \simeq \mathfrak{A}^{**} =: \mathfrak{A}''$, consisting of all the GNS representations $(\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ for states $\omega \in E_{\mathfrak{A}}$(: state space of $\mathfrak{A}$), the central support $c(\pi)$ of $(\pi, \mathfrak{H}_\pi = P_\pi \mathfrak{H}_u)$ with support projection $P_\pi \in \pi_u(\mathfrak{A})'$ can be defined as the smallest projection in the centre $\mathfrak{Z}(\mathfrak{A}'') := \mathfrak{A}'' \cap \pi_u(\mathfrak{A})'$ to pick up all the representations quasi-equivalent to π: $c(\pi) =$ projection onto $\overline{\pi(\mathfrak{A})'\mathfrak{H}_\pi} \subset \mathfrak{H}_u$. On this basis, we introduce a basic scheme for q-c correspondence in terms of sectors and order parameters: the Gel'fand spectrum $Spec(\mathfrak{Z}(\mathfrak{A}''))$ of $\mathfrak{Z}(\mathfrak{A}'')$ arising from the "simultaneous diagonalization" of the commutative algebra $\mathfrak{Z}(\mathfrak{A}'')$ can be identified with the factor spectrum $\widehat{\mathfrak{A}}$ [2] of $\mathfrak{A}$:

$$Spec(\mathfrak{Z}(\mathfrak{A}'')) \simeq \widehat{\mathfrak{A}} := F_{\mathfrak{A}}/\approx: \textbf{factor spectrum},$$

defined by all the quasi-equivalence classes of *factor* states $\omega \in F_{\mathfrak{A}}$(: set of all factor states of $\mathfrak{A}$) with trivial centres $\mathfrak{Z}(\pi_\omega(\mathfrak{A})'') = \pi_\omega(\mathfrak{A})'' \cap \pi_\omega(\mathfrak{A})' = \mathbb{C}\mathbf{1}_{\mathfrak{H}_\omega}$ in the GNS representations $(\pi_\omega, \mathfrak{H}_\omega)$.

Definition 1 *A* ***sector*** *(or, physically speaking,* ***pure phase****) of observable algebra $\mathfrak{A}$ is defined by a* ***quasi-equivalence class of factor states*** *of $\mathfrak{A}$.*

In view of the commutativity of $\mathfrak{Z}(\mathfrak{A}'')$ and of the role of its spectrum, we can regard [3]

- $\mathfrak{Z}(\mathfrak{A}'')$ as the algebra of **macroscopic order parameters** to specify sectors, and
- $Spec(\mathfrak{Z}(\mathfrak{A}'')) \simeq \widehat{\mathfrak{A}}$ as the **classifying space of sectors** to distinguish among different sectors.

Then the dual map

$$\textbf{Micro:} \quad \mathfrak{A}^* \supset E_{\mathfrak{A}} \twoheadrightarrow Prob(\widehat{\mathfrak{A}}) \subset L^\infty(\widehat{\mathfrak{A}})^* \ \ : \textbf{Macro},$$

of the embedding $\mathfrak{Z}(\mathfrak{A}'') \simeq L^\infty(\widehat{\mathfrak{A}}) \hookrightarrow \mathfrak{A}''$ can be interpreted as a *universal q(uantum)→c(lassical) channel* which transforms a microscopic quantum state $\phi \in E_{\mathfrak{A}}$ into a macroscopic classical state $\mu_\phi \in Prob(\widehat{\mathfrak{A}})$ [3]:

$$E_{\mathfrak{A}} \ni \phi \longmapsto \mu_\phi = \phi'' \restriction_{\mathfrak{Z}(\mathfrak{A}'')} \in E_{\mathfrak{Z}(\mathfrak{A}'')} = M^1(Spec(\mathfrak{Z}(\mathfrak{A}''))) = Prob(\widehat{\mathfrak{A}})\,.$$

μ_ϕ is the probability distribution of sectors contained in a mixed-phase state ϕ of $\mathfrak{A}$ in a quantum-classical composite system,

$$\widehat{\mathfrak{A}} \supset \Delta \longmapsto \phi''(\chi_\Delta) = \mu_\phi(\Delta) = Prob(\text{sector} \in \Delta \mid \phi),$$

wherer ϕ'' denotes the normal extension of $\phi \in E_{\mathfrak{A}}$ to $\mathfrak{A}''$. While it tells us as to which sectors appear in ϕ, it cannot specify as to which representative factor state appears within each sector component of ϕ. In other words, our vocabulary at this level of resolution consists of words to indicate a representation of $\mathfrak{A}$ as a whole which cannot pinpoint a specific state belonging to it.

1.2 Q-C correspondence: (II) Inside of sectors and maximal abelian subalgebra

To detect operationally the *intrasectorial structures* inside of a sector ω described by a factor representation $(\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$, we need to choose a *maximal abelian subalgebra* (MASA) $\mathcal{A}$ of a factor algebra $\mathcal{M} := \pi_\omega(\mathfrak{A})''$, characterized by the condition $\mathcal{A}' \cap \mathcal{M} = \mathcal{A} \cong L^\infty(Spec(\mathcal{A}))$ [4]. Note that, if we adopt the usual definition of MASA, $\mathcal{A}' = \mathcal{A}$, found in many discussions on quantum-mechanical systems with finite degrees of freedom, the relation $\mathcal{A}' = \mathcal{A} \subset \mathcal{M}$ implies $\mathcal{M}' \subset \mathcal{A}' = \mathcal{A} \subset \mathcal{M}$, and hence, $\mathcal{M}' = \mathcal{M}' \cap \mathcal{M} = \mathfrak{Z}(\mathcal{M})$ is of type I, which does not fit to the general context of infinite systems involving algebras of non-type I. Since a tensor product $\mathcal{M} \otimes \mathcal{A}$ (acting on the Hilbert-space tensor product $\mathfrak{H}_\omega \otimes L^2(Spec(\mathcal{A}))$) has a centre given by

$$\mathfrak{Z}(\mathcal{M} \otimes \mathcal{A}) = \mathfrak{Z}(\mathcal{M}) \otimes \mathcal{A} = \mathbf{1} \otimes L^\infty(Spec(\mathcal{A})),$$

we see that the spectrum $Spec(\mathcal{A})$ of a MASA $\mathcal{A}$ to be measured can be understood as parametrizing a *conditional sector structure* of the coupled system of the object system $\mathcal{M}$ and $\mathcal{A}$, the latter of which can be identified with the measuring apparatus $\mathcal{A}$ in the simplified version [3] of Ozawa's measurement scheme [5]. This picture of conditional sector structure is consistent with the physical essence of a measurement process as "classicalization" of some restricted aspects $\mathcal{A}(\subset \mathcal{M})$ of a quantum system, conditional on the coupling $\mathcal{M} \otimes \mathcal{A}$ of $\mathcal{M}$ with the apparatus identified with $\mathcal{A}$.

In addition to the choice of relevant algebras of observables, the essential point in the mathematical description of a measurement process is to find a *coupling term* between algebras $\mathcal{M}$ and $\mathcal{A}$ of observables of the object system and of the measuring apparatus in such a way that a microscopic quantum state of $\mathcal{M}$ can be determined by knowing the macroscopic data of the pointer positions on $Spec(\mathcal{A})$ of the measuring apparatus. To solve this problem we note that the algebra $\mathcal{A}$ is generated by its unitary elements which constitute an abelian unitary group $\mathcal{U}(\mathcal{A})$. As an infinite-dimensional group, $\mathcal{U}(\mathcal{A})$ is, in general, not ensured to have an invariant Haar measure. In the physically meaningful situations where observables are represented in *separable* Hilbert spaces, however, $\mathcal{A}$ as a commutative von Neumann algebra can be shown to be generated by a single element $A_0 = A_0^*$: $\mathcal{A} = \{A_0\}''$ [6]. This allows us to focus upon a one-parameter subgroup $\{\exp(itA_0); t \in \mathbb{R}\}$ of $\mathcal{U}(\mathcal{A})$ generating $\mathcal{A}$ and equipped with an invariant Haar measure. In concrete situations (where what is most relevant is as to which quantities are actually measured), the existence of a *single* generator valid at the level of von Neumann algebras may sound too idealistic, but this point can easily be remedied by relaxing it to a finite number of mutually commuting generators consistently with the existence of a Haar measure. Thus, we treat in what follows an abelian (Lie) group $\mathcal{U}$ equipped with a Haar measure du which generates the MASA $\mathcal{A}$:

$$\mathcal{U} \subset \mathcal{U}(\mathcal{A}), \mathcal{A} = \mathcal{U}''.$$

Rewriting the condition $\mathcal{A} = \mathcal{A}' \cap \mathcal{M}$ for $\mathcal{A}$ to be a MASA of $\mathcal{M}$ into such a form as

$$\mathcal{A} = \mathcal{M} \cap \mathcal{A}' = \mathcal{M} \cap \mathcal{U}' = \mathcal{M}^{\alpha(\mathcal{U})},$$

we see that $\mathcal{A}$ is the fixed-point subalgebra of the adjoint action $\alpha_u := Ad(u) : \mathcal{M} \ni X \longmapsto uXu^*$ of $\mathcal{U}$ on $\mathcal{M}$ [1]. From this viewpoint, the relevance of the group duality and of the Galois extension can naturally be expected. On the basis of a formulation with a *Kac-Takesaki operator* [7, 8] (*K-T operator*, for short) or a *multiplicative unitary* [9], the universal essence of the problem can be understood in the following form.

2 Measurement Coupling and Instrument

In the context of a *Hopf-von Neumann algebra* $M(\subset B(\mathfrak{H}))$ [10] equipped with a Haar weight, a K-T operator $V \in \mathcal{U}((M \bar{\otimes} M_*)^-) \subset \mathcal{U}(\mathfrak{H} \bar{\otimes} \mathfrak{H})$ is defined as the unitary implementer of its coproduct $\Gamma : M \to M \otimes M$ in the sense of $\Gamma(x) = V^*(\mathbf{1} \otimes x)V$. Corresponding to the co-associativity of Γ, the K-T operator V is characterized by the pentagonal relation, $V_{12}V_{13}V_{23} = V_{23}V_{12}$, on $\mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{H}$, where subscripts i, j of V_{ij} indicate the places in $\mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{H}$ on which the operator V acts. It plays most fundamental roles as an intertwiner, $V(\lambda \otimes \iota) = (\lambda \otimes \lambda)V$, due to the quasi-equivalence among tensor powers of the regular representation $\lambda : M_* \ni \omega \longmapsto \lambda(\omega) := (i \otimes \omega)(V) \in \hat{M}$ given through a generalized Fourier transform, $\lambda(\omega_1 * \omega_2) = \lambda(\omega_1)\lambda(\omega_2)$, of the convolution algebra M_* with $\omega_1 * \omega_2 := \omega_1 \otimes \omega_2 \circ \Gamma$. On these bases, a generalization of *group duality* can be formulated for Kac algebras [10]. In the case of $M = L^\infty(G, dg)$ with a locally compact group G equipped with a (left-invariant) Haar measure dg, the K-T operator V is given on $L^2(G \times G)$ by

$$(V\xi)(s,t) := \xi(s, s^{-1}t) \quad \text{for } \xi \in L^2(G \times G), s, t \in G,$$

or symbolically, $V|s,t\rangle = |s, st\rangle$, in the Dirac-type notation.

To apply this machinery to our discussion involving the MASA $\mathcal{A}$, we first recall the notion of the group dual $\hat{G}$ of a group G defined by the set of equivalence classes of irreducible unitary representations of G. For our abelian group $\mathcal{U}$, its group dual $\widehat{\mathcal{U}}$ consists of the characters γ of $\mathcal{U}$: $\gamma(u_1u_2) = \gamma(u_1)\gamma(u_2)$, $\gamma(e) = 1$ (u_1, $u_2 \in \mathcal{U}$). Identifying the above M with $L^\infty(\widehat{\mathcal{U}}) = \lambda(\mathcal{U})''$, we consider the K-T operator $V \in L^\infty(\widehat{\mathcal{U}}) \otimes \lambda(\widehat{\mathcal{U}})'' = L^\infty(\widehat{\mathcal{U}} \times \mathcal{U})$ associated with $\widehat{\mathcal{U}}$ taken as the above G:

$$(V\xi)(\gamma_1, \gamma_2) := \xi(\gamma_1, \gamma_1^{-1}\gamma_2) \quad \text{for } \xi \in L^2(\widehat{\mathcal{U}} \times \widehat{\mathcal{U}}), \gamma_1, \gamma_2 \in \widehat{\mathcal{U}},$$

which satisfies the pentagonal relation, $V_{12}V_{13}V_{23} = V_{23}V_{12}$, and the intertwining relation $V(\lambda_\gamma \otimes I) = (\lambda_\gamma \otimes \lambda_\gamma)V$ ($\gamma \in \widehat{\mathcal{U}}$) for the regular representation $\widehat{\mathcal{U}} \ni \gamma \longmapsto \lambda_\gamma \in \mathcal{U}(L^2(\widehat{\mathcal{U}}))$. We note here the following implications of the inclusion relations $\mathcal{U} \subset \mathcal{A} \subset \mathcal{M}$:

i) Any character $\chi \in Spec(\mathcal{A})$ of the MASA $\mathcal{A}$ defined as an *algebraic homomorphism* $\chi : \mathcal{A} \to \mathbb{C}$ is also a character $\chi \restriction_{\mathcal{U}} \in \widehat{\mathcal{U}}$ of the abelian unitary group $\mathcal{U}$ as a *group homomorphism* $\chi \restriction_{\mathcal{U}}: \mathcal{U} \to \mathbb{T}$ by the restriction to $\mathcal{U}$. This implies the inclusion $Spec(\mathcal{A}) \hookrightarrow \widehat{\mathcal{U}}$, by which we identify $\chi \in Spec(\mathcal{A})$ and $\chi \restriction_{\mathcal{U}} \in \widehat{\mathcal{U}}$. While physically measured quantities would be points χ in $Spec(\mathcal{A})$ which, in general, has no intrinsic base point, the identity character $\iota \in \widehat{\mathcal{U}}$, $\iota(u) \equiv 1$ ($\forall u \in \mathcal{U}$) present in $\widehat{\mathcal{U}}$ can be physically distinguished by its important role as the **neutral position** of measuring pointer. To be precise, when $\mathcal{U}$ is not compact, there is no vector $|\iota\rangle$ corresponding to $\iota \in \widehat{\mathcal{U}}$ in $L^2(\mathcal{U})$, which can, however, be remedied by replacing $\langle\iota|\cdots|\iota\rangle$ with the *invariant mean* $m_{\mathcal{U}}$ meaningful for all such amenable groups as the abelian group $\mathcal{U}$. The importance of this neutral position remarked earlier by Ozawa has been overlooked in the usual approaches for lack of the suitable place to accommodate it in an intrinsic way.

ii) The inclusion map $E : \mathcal{A} = L^\infty(Spec(\mathcal{A})) \hookrightarrow \mathcal{M}$ defines an $\mathcal{M}$-valued spectral measure dE on $Spec(\mathcal{A})$ by $E(\Delta) = E(\chi_\Delta)$ for Borel sets $\Delta \subset Spec(\mathcal{A})$, and its restriction to $\mathcal{U}$ induces a spectral decomposition of $\mathcal{U}$ (as an application of the SNAG theorem):

$$E(u) = \int_{\chi \in Spec(\mathcal{A}) \subset \widehat{\mathcal{U}}} \overline{\chi(u)} dE(\chi) \qquad (u \in \mathcal{U}).$$

Then the group homomorphism $\mathcal{U} \ni u \longmapsto E(u) \in \mathcal{M}$ gives an $\mathcal{M}$-valued unitary representation E of the group $\mathcal{U}$ in a Hilbret space $\mathfrak{H}_\mathcal{M}$ of $\mathcal{M}$ with spectral support given by $Spec(\mathcal{A})$:

$$supp(E) = supp(dE) = Spec(\mathcal{A})(\subset \widehat{\mathcal{U}}),$$

where we can take $\mathfrak{H}_\mathcal{M}$ as $L^2(\mathcal{M})$ (a non-commutative L^2-space of $\mathcal{M}$), the Hilbert space where $\mathcal{M}$ is represented in its standard form so that any normal state ω of $\mathcal{M}$ is expressed in a vectorial form: $\omega(A) = \langle \xi_\omega | A \xi_\omega \rangle$. Corresponding to this representation E of $\mathcal{U}$, a representation $E_*(V) = \int_{\chi \in Spec(\mathcal{A})} dE(\chi) \otimes \lambda_\chi$ of the K-T operator V on $L^2(\mathcal{M}) \otimes L^2(\widehat{\mathcal{U}})$ is defined by

$$E_*(V)(\xi \otimes |\gamma\rangle) = \int_{\chi \in Spec(\mathcal{A})} dE(\chi)\xi \otimes |\chi\gamma\rangle, \qquad \text{for } \gamma \in \widehat{\mathcal{U}}, \qquad \xi \in L^2(\mathcal{M}), \tag{1}$$

satisfying the modified pentagonal relation $E_*(V)_{12}E_*(V)_{13}V_{23} = V_{23}E_*(V)_{12}$.

iii) In view of the inclusion relations $E(u) = u \in \mathcal{U} \subset \mathcal{A} \subset \mathcal{M}$, it may appear strange or pedantic to introduce the map E and to talk about it as a unitary *representation* $(E, L^2(\mathcal{M}))$ of $\mathcal{U}$ in $\mathfrak{H}_\mathcal{M}$. As will be shown later, however, this is not the case, since it turns out to be crucial to distinguish $\mathcal{U}$ itself as an "*abstract*" group from the *represented* unitary group $\mathcal{U} \subset \mathcal{M}$ embedded in $\mathcal{M}$. First, we note that the group $\mathcal{U}$ has the *regular representation* $(\lambda, L^2(\mathcal{U}, du))$ as its canonically defined representation in the Hilbert space $L^2(\mathcal{U}, du)$ with the Haar measure du of $\mathcal{U}$, which is isomorphic to the Hilbert space $L^2(\widehat{\mathcal{U}}, d\gamma)$ through the *Fourier transform* $\mathcal{F}$ from $\mathcal{U}$ to $\widehat{\mathcal{U}}$ as a unitary transformation given by

$$(\mathcal{F}\xi)(\gamma) = \int_\mathcal{U} \overline{\gamma(u)}\xi(u)du, \qquad (\mathcal{F}^{-1}\eta)(u) = \int_{\widehat{\mathcal{U}}} \gamma(u)\eta(\gamma)d\gamma. \tag{2}$$

While $\mathcal{U}(\subset \mathcal{M})$ and $\lambda(\mathcal{U})(\subset B(L^2(\mathcal{U})))$ are isomorphic *as groups*, the corresponding von Neumann algebras given by their weak closures are, in general, different, $\mathcal{U}'' = \mathcal{A} \hookrightarrow L^\infty(\widehat{\mathcal{U}})$, $Spec(\mathcal{A}) \subset \widehat{\mathcal{U}}$, owing to the difference in their representation Hilbert spaces: the former is represented through the action of $\mathcal{U}$ on $\mathcal{M}$ in the representation space $\mathfrak{H}_{\mathcal{M}}$ of $\mathcal{M}$, and the latter in $L^2(\mathcal{U}) \cong L^2(\widehat{\mathcal{U}})$. As will be shown in Sec.4, the differences between $\mathcal{U}'' = \mathcal{A}$ and $L^\infty(\widehat{\mathcal{U}}) = \lambda(\mathcal{U})''$, or between $Spec(\mathcal{A})$ and $\widehat{\mathcal{U}}$, determine the von Neumann type of $\mathcal{M}$, according to which $\mathcal{A} \cong L^\infty(\widehat{\mathcal{U}})$, or equivalently, $Spec(\mathcal{A}) = \widehat{\mathcal{U}}$, holds if and only if $\mathcal{M}$ is of type I.

Now the important operational meaning of the equality (1) and the role of the neutral position ι can clearly be seen, especially if $\widehat{\mathcal{U}}$ is a *discrete* group which is equivalent to the *compactness* of the group $\mathcal{U}$: choosing $\chi = \iota$, we have the equality, $E_*(V)(\xi_\gamma \otimes |\iota\rangle) = \xi_\gamma \otimes |\gamma\rangle$ $(\forall\gamma \in \widehat{\mathcal{U}},\ \xi_\gamma \in E(\gamma)\mathfrak{H}_{\mathcal{M}})$. With a generic state $\xi = \sum_{\gamma\in\widehat{\mathcal{U}}} c_\gamma \xi_\gamma$ of $\mathcal{M}$, an initial *uncorrelated* state $\xi \otimes |\iota\rangle$ is transformed by $E_*(V)$ to a *correlated* one:

$$E_*(V)(\xi \otimes |\iota\rangle) = \sum_{\gamma\in\widehat{\mathcal{U}}} c_\gamma \xi_\gamma \otimes |\gamma\rangle.$$

If $\mathcal{M}$ is not of type III equipped with a normal faithful semi-finite (n.f.s., for short) trace, this establishes a *one-to-one* correspondence ("perfect correlation" due to Ozawa [11]) between a state $|\gamma\rangle$ of the measuring probe system $\mathcal{A}$ specified by an observed value $\gamma \in Spec(\mathcal{A})$ on the pointer and the corresponding unique state $\xi_\gamma \in \mathcal{M}_\gamma$ of the microscopic system $\mathcal{M}$. If $\mathcal{M}$ is of type III, $\dim(\mathcal{M}_\gamma) \leq 1$ is not guaranteed for lack of a trace, and hence, the notion of perfect correlation may fail to hold in such cases. Moreover, if we find some evidence for such a kind of uncertainty as violating $\dim(\mathcal{M}_\gamma) \leq 1$, then it implies that $\mathcal{M}$ should be of type III.

On these bases, we can define the notion of an **instrument** $\mathfrak{I}$ as a (completely) positive operation-valued measure to unify all the ingredients relevant to a measurement as follows:

$$\begin{aligned}\mathfrak{I}(\Delta|\omega_\xi)(B) &:= (\omega_\xi \otimes m_{\mathcal{U}})(E_*(V)^*(B \otimes \chi_\Delta)E_*(V)) \\ &= (\langle\ \xi| \otimes \langle\iota|)E_*(V)^*(B \otimes \chi_\Delta)E_*(V)(|\xi\rangle \otimes |\iota\rangle).\end{aligned}$$

In the situation with a state $\omega_\xi = \langle\ \xi|\ (-)\xi\rangle$ of $\mathcal{M}$ as an initial state of the system, the instrument describes simultaneously the probability $p(\Delta|\omega_\xi) = \mathfrak{I}(\Delta|\omega_\xi)(\mathbf{1})$ for measured values of observables in $\mathcal{A}$ to be found in a Borel set Δ and the final state $\mathfrak{I}(\Delta|\omega_\xi)/p(\Delta|\omega_\xi)$ realized through the detection of measured values [5]. The merits of the present formulation of instrument consist in such points that it is free from the restriction on the types of von Neumann algebras and that it can be applied to *any* measurement, irrespective of whether repeatable or not, since any drastic changes between initial and final states can be easily absorbed in the system with *infinite degrees of freedom.*

3 Crossed Product $\mathcal{M} \rtimes_\alpha \mathcal{U}$ as Composite System of System & Apparatus

Here we clarify the important meaning of the coupling $E_*(V)$ between the system $\mathcal{M}$ to be observed and the measuring apparatus corresponding to a MASA $\mathcal{A}$ of $\mathcal{M}$: its essential roles in the whole measurement processes are closely related with the crossed product $\mathcal{M} \rtimes_\alpha \mathcal{U}$ to describe the composite system of $\mathcal{M}$ and $\mathcal{A}$ to be put in the context of ***Fourier-Galois duality*** and with the amplification processes for the measured data $\chi \in Spec(\mathcal{A}) \subset \widehat{\mathcal{U}}$ to take macroscopically visible forms emerging from the small changes at the microscopic tip of the measuring apparatus caused by this coupling. For this purpose, we consider the *Fourier transform* of the K-T operator $V \in L^\infty(\widehat{\mathcal{U}}) \otimes \lambda(\widehat{\mathcal{U}})''$ on $\widehat{\mathcal{U}}$, $(V\xi)(\gamma_1, \gamma_2) = \xi(\gamma_1, \gamma_1^{-1}\gamma_2)$ $(\xi \in L^2(\widehat{\mathcal{U}} \times \widehat{\mathcal{U}}))$, given by

$$W := (\mathcal{F} \otimes \mathcal{F})^{-1} V (\mathcal{F} \otimes \mathcal{F}),$$
$$(W\xi)(u_1, u_2) := \xi(u_2 u_1, u_2) \quad \text{for } \xi \in L^2(\mathcal{U} \times \mathcal{U}), u_1, u_2 \in \mathcal{U}.$$

This $W \in \lambda(\mathcal{U})'' \otimes L^\infty(\mathcal{U})$ is seen also to be a K-T operator on $\mathcal{U}$ belonging to $\lambda(\mathcal{U})'' \otimes L^\infty(\mathcal{U})$ characterized by the pentagonal and the intertwining relations:

$$W_{12}W_{13}W_{23} = W_{23}W_{12},$$
$$W(\lambda_u \otimes \lambda_u) = (I \otimes \lambda_u)W, \quad (u \in \mathcal{U}),$$

for the regular representation $\lambda = \lambda^{\mathcal{U}}$ of $\mathcal{U}$ on $L^2(\mathcal{U})$. Through the embedding map $E : \mathcal{U} \hookrightarrow \mathcal{A} \hookrightarrow \mathcal{M}$, this K-T operator W is represented in $\mathcal{M}$ by $EW := (E \otimes id)(W) \in \mathcal{A} \otimes L^\infty(\mathcal{U}) \subset \mathcal{M} \otimes L^\infty(\mathcal{U}))$, which satisfies the modified version of pentagonal and intertwining relations:

$$(EW)_{12}(EW)_{13}W_{23} = W_{23}(EW)_{12},$$
$$EW(u \otimes \lambda_u) = (I \otimes \lambda_u)EW.$$

In view of the relation

$$[(EW)\hat{X}(EW^*)](u) = u^{-1}\hat{X}(u)u = \alpha_u^{-1}(\hat{X}(u)) \tag{3}$$

valid for $\hat{X} \in \mathcal{M} \otimes L^\infty(\mathcal{U})$, we define an injective *-homomorphism $\pi_\alpha : \mathcal{M} \to L^\infty(\mathcal{U}, \mathcal{M}) = \mathcal{M} \otimes L^\infty(\mathcal{U})$ by

$$(\pi_\alpha(X)\xi)(u) := \alpha_u^{-1}(X)(\xi(u)) = (u^{-1}Xu)(\xi(u)) \tag{4}$$
$$\text{for } \xi \in L^2(\mathcal{M}) \otimes L^2(\mathcal{U}), u \in \mathcal{U},$$

which is implemented by EW:

$$\pi_\alpha(X) = (EW)(X \otimes I)(EW)^* \quad \text{for } X \in \mathcal{M}.$$

According to [8], the von Neumann algebra generated by $\pi_\alpha(\mathcal{M})$ and $\mathbb{C}I \otimes \lambda(\mathcal{U})''$ is just a crossed product $\mathcal{M} \rtimes_\alpha \mathcal{U}$:

$$\mathcal{M} \rtimes_\alpha \mathcal{U} := \pi_\alpha(\mathcal{M}) \vee (\mathbb{C} \otimes \lambda(\mathcal{U})'').$$

This can also be viewed as (the weak-operator closure of) the image of the convolution *-algebra $L^1(\mathcal{U}, \mathcal{M}) = \mathcal{M} \otimes L^1(\mathcal{U})$ with the product structure given for $X, Y \in L^1(\mathcal{U}, \mathcal{M})$ by

$$\begin{aligned} (X * Y)(u) &= \int_{\mathcal{U}} X(v)\alpha_v(Y(v^{-1}u))dv, \\ X^{\#}(u) &= \alpha_u(X(u^{-1}))^*, \end{aligned}$$

under the operator-valued Fourier transform $\mathfrak{F}$:

$$\begin{gathered} \mathfrak{F}(X) = (Xdu \otimes id)(\sigma(EW)^*\sigma) = \int_{\mathcal{U}} X(u)udu \\ \text{for } X \in L^1(\mathcal{U}, \mathcal{M}) = \mathcal{M} \otimes L^1(\mathcal{U}); \\ \mathfrak{F}(X * Y) = \mathfrak{F}(X)\mathfrak{F}(Y) \ \text{ and } \mathfrak{F}(X^{\#}) = \mathfrak{F}(X)^*, \end{gathered}$$

where σ is the flip operator interchanging tensor factors: $\sigma(\xi \otimes \eta) := \eta \otimes \xi$. In this way, the crucial roles played by the coupling EW between the observed system $\mathcal{M}$ and the probe system $\mathcal{A}$ can be seen in the the formation of their composite system in the form of a crossed product $\mathcal{M} \rtimes_\alpha \mathcal{U}$. In the process $\iota \to \alpha \to \iota$ of switching-on and -off the coupling α starting from $\alpha = \iota$, the structure of $\mathcal{M} \rtimes_\alpha \mathcal{U}$ will be seen to change as $\mathcal{M} \otimes L^\infty(\widehat{\mathcal{U}}) \to \mathcal{M} \rtimes_\alpha \mathcal{U} \to \mathcal{M} \otimes L^\infty(\widehat{\mathcal{U}})$.

3.1 Physical meaning of crossed product and Takesaki duality

The importance of the crossed product $\mathcal{M} \rtimes_\alpha \mathcal{U}$ can be seen in the relation with the Takesaki duality [12]:

$$(\mathcal{M} \rtimes_\alpha \mathcal{U}) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}} \simeq \mathcal{M} \otimes B(L^2(\mathcal{U})) \simeq \mathcal{M},$$

where the last isomorphism $\mathcal{M} \simeq \mathcal{M} \otimes B(L^2(G))$ holds for any *properly infinite* von Neumann algebras $\mathcal{M}$ as applies to the present situation discussing a quantum system with infinite degreees of freedom. Here $\hat{\alpha}$ is the dual co-action [8] of $\mathcal{U}$ on $\mathcal{M} \rtimes_\alpha \mathcal{U}$ defined for $Y \in \mathcal{M} \rtimes_\alpha \mathcal{U}$ by

$$\pi_{\hat{\alpha}}(Y) := Ad(1 \otimes \sigma W^*\sigma)(Y \otimes 1),$$

which reduces just to the action of the group dual $\widehat{\mathcal{U}}$ on $\mathcal{M} \rtimes_\alpha \mathcal{U}$ in the case of abelian group $\mathcal{U}$. In this context, a crossed product $\mathcal{M} \rtimes_\alpha \mathcal{U}$ can be viewed as a kind of the non-commutative Fourier dual of $\mathcal{M}$ whose precise knowledge enables us to **recover** the original **algebra** $\mathcal{M}$ of the microscopic quantum system by forming the *second crossed product* with the dual action $\hat{\alpha}$ by $\widehat{\mathcal{U}}$. Our original purpose of considering the composite system $\mathcal{M} \rtimes_\alpha \mathcal{U}$ was to prepare a measurement process just for analyzing the structure of *states* within a sector starting from the postulated knowledge of the algebra $\mathcal{M}$ of a Micro-system on the basis of the coupling term to yield experimental data in $Spec(\mathcal{A}) \subset \widehat{\mathcal{U}}$. The fullfilment of this step, however, drives us into the next step in the opposite direction of *reconstructing* the *original algebra* $\mathcal{M}$ from the observational data on states. As a result, the essential idea of *Micro-Macro duality* [1] is implemented mathematically by the duality of crossed products as an operator-algebraic extension of Fourier-Galois duality: if the algebra $\mathcal{M}$ of the Micro-system is known beforehand for one reason or another, this scheme can be used for checking whether $\mathcal{M}$ is correctly chosen or not through the comparison of the theoretical predicitions encoded in $\mathcal{M} \rtimes_\alpha \mathcal{U}$ and the actually observed data. On the other hand, if $\mathcal{M}$ is *unknown* (as is in the usual situations), the latter data can serve for constructing $\mathcal{M}$ from which one should rederive the observational data to ensure the consistency.

To proceed further, we add here a mathematical postulate called *semi-duality* [8] of the action α on $\mathcal{M}$, which assumes the existence of such a unitary $v \in \mathcal{M} \otimes \lambda(\mathcal{U})''$ that the condition $\overline{\alpha}(v) = (v \otimes 1)(1 \otimes V')$ holds with a K-T operator V' given by $(V'\xi)(u_1, u_2) = \xi(u_1 u_2, u_2)$ and $\overline{\alpha} := (\iota \otimes \sigma) \circ (\alpha \otimes \iota)$. From this assumption follows the relation $(\mathcal{M} \otimes B(L^2(\mathcal{U})))^{\tilde{\alpha}(\mathcal{U})} = \mathcal{M}^{\alpha(\mathcal{U})} \otimes B(L^2(\mathcal{U}))$ (see [8]), which implies, in combination with the relation $\mathcal{M} \rtimes_\alpha \mathcal{U} \simeq (\mathcal{M} \otimes B(L^2(\mathcal{U})))^{\tilde{\alpha}(\mathcal{U})}$ with $\tilde{\alpha} = \alpha \otimes Ad \circ \lambda$ (valid for any crossed products with abelian group actions), the following interesting structure for the crossed product:

$$\mathcal{M} \rtimes_\alpha \mathcal{U} \simeq (\mathcal{M} \otimes B(L^2(\mathcal{U})))^{\tilde{\alpha}(\mathcal{U})} = \mathcal{M}^{\alpha(\mathcal{U})} \otimes B(L^2(\mathcal{U})) = \mathcal{A} \otimes B(L^2(\mathcal{U})).$$

We see that in this situation the Takesaki duality splits into two parts as follows:

Theorem 2 *Let $\mathcal{M}$ and $\mathcal{A} = \mathcal{A}' \cap \mathcal{M}$ be, respectively, a properly infinite von Neumann algebra and its MASA generated by a locally compact abelian unitary group $\mathcal{U} \subset \mathcal{A} = \mathcal{U}'' = \mathcal{M}^{\alpha(\mathcal{U})}$. If the semi-duality condition $\overline{\alpha}(v) = (v \otimes 1)(1 \otimes V')$ holds for the action α of $\mathcal{U}$, then the Takesaki duality [12] for $\mathcal{M}$ and $\mathcal{A}$, $(\mathcal{M} \rtimes_\alpha \mathcal{U}) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}} \simeq \mathcal{M} \otimes B(L^\infty(\mathcal{U})) \simeq \mathcal{M}$ and $(\mathcal{A} \otimes B(L^\infty(\mathcal{U}))) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}}) \rtimes_\mu \mathcal{U} \simeq \mathcal{A} \otimes B(L^\infty(\widehat{\mathcal{U}}))$, can be decomposed into the following two mutually equivalent isomorphisms:*

i) $\mathcal{M} \rtimes_\alpha \mathcal{U} \simeq \mathcal{A} \otimes B(L^\infty(\mathcal{U}))$[: amplification process],

ii) $(\mathcal{A} \otimes B(L^\infty(\mathcal{U})) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}} \simeq \mathcal{M}$[: reconstruction].

By means of this, we can attain the following clear-cut mathematical description of the physical situations relevant to measurement processes of quantum dynamical systems with infinite degrees of freedom, which explains both aspects at the same time, the amplification processes from invisible Micro to visible Macro data and the recovery of invisible Micro from visible Macro data.

According to i), the composite system $\mathcal{M} \rtimes_\alpha \mathcal{U}$ of a Micro-quantum system $\mathcal{M}$ (of local fields, for instance) and of a measuring apparatus coupled through an action α of the unitary group $\mathcal{U}$ generating a MASA $\mathcal{U}'' = \mathcal{A} = \mathcal{A}' \cap \mathcal{M}$ can be decomposed into a classical system with a commutative algebra $\mathcal{A}$ to be measured and a quantum-mechanical one $B(L^2(\mathcal{U}))$ of CCR with finite degrees of freedom. Arising from the Heisenberg group composed of two abelian groups $\mathcal{U}$ and $\widehat{\mathcal{U}}$ in Fourier-Pontryagin duality, this latter component will be seen to play physically interesting role as the "reservoir" in the relaxation processes of *amplification* to extract Macro from Micro; namely, the former half of the Takesaki duality, $\mathcal{M} \rtimes_\alpha \mathcal{U} \simeq \mathcal{A} \otimes B(L^2(\mathcal{U}))$, provides the mathematical basis for the process to amplify the measured quantities in $\mathcal{A}$ into macroscopically visible data at the expense of the dissipative damping effects to suppress other irrelevant quantities. This picture is based upon the following two points, one being the homotopical notion of strong Morita equivalence and the other the quasi-equivalence of arbitrary tensor powers $\lambda^{\otimes n}$ of regular representation λ of $\mathcal{U}$. The notion of strong Morita equivalence $\mathfrak{A}_1 \approx \mathfrak{A}_2$ of algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$ is defined by the isomorphism $Rep_{\mathfrak{A}_1} \simeq Rep_{\mathfrak{A}_1}$ of their representation categories which is equivalent to the stability $\mathfrak{A}_1 \otimes \mathcal{K} \simeq \mathfrak{A}_2 \otimes \mathcal{K}$ under tensoring the compact operator algebra $\mathcal{K} = \mathcal{K}(\mathfrak{H})$ [13]. Physically this notion fits to the purpose of ensuring the stability of the object system against noise perturbations coming from its neglected surroundings. In the present context of focusing on the internal structure of a sector ω of $\mathfrak{A}$, the *topological* form $\mathfrak{A} \approx \mathfrak{A} \otimes \mathcal{K}(L^2(\widehat{\mathcal{U}}))$ of Morita equivalence for the C*-algebra $\mathfrak{A}$ of observables is converted into the *measure-theoretical* one $\mathcal{M} = \pi_\omega(\mathfrak{A})'' \simeq \mathcal{M} \otimes B(L^2(\widehat{\mathcal{U}}))$ as the isomorphism of von Neumann algebras, which automatically holds for an arbitrary *properly infinite* von Neumann algebra $\mathcal{M}$ describing a quantum dynamical system with infinite degrees of freedom like quantum fields. This allows us to interchange $\mathcal{M}$ and $\mathcal{M} \otimes B(L^2(\widehat{\mathcal{U}}))$ freely without any changes. On the other hand, arbitrary tensor powers $(\lambda^{\otimes n}, L^2(\mathcal{U})^{\otimes n})$ $(n \in \mathbb{N})$ of regular representation $(\lambda, L^2(\mathcal{U}))$ of $\mathcal{U}$ are quasi-equivalent via the K-T operator W, and hence, the relation $\mathcal{M} \rtimes_\alpha \mathcal{U} \simeq \mathcal{A} \otimes B(L^\infty(\mathcal{U}))$ related with $\tilde{\alpha} = \alpha \otimes Ad \circ \lambda$ can be extended to the situation involving $\tilde{\alpha}^{(n)} = \alpha \otimes Ad \circ \lambda^{\otimes n}$. This provides the mathematical support for the *repeatability* hypothesis of the measurement processes, which can be formulated consistently in the framework of quantum stochastic processes. (Note that the distinction between repeatable and non-repeatable ones disappears in the system with infinite degrees of freedom.) In this context the notion of operator-valued weights associated with the "*integrability*" [8] of an action following from the assumption of semi-duality plays important roles. Along this line, the stochastic processes developed on the tensor algebra generated by $(\lambda, L^2(\mathcal{U}))$ are expected to provide a natural basis for the processes to *amplify* microscopic changes caused by the coupling between the microscopic end (called a probe system) of measuring apparatus and the observed microscopic system into the macroscopically visible motions of the measuring pointer. Note at the same time, however, the sharp contrast between the situations with $n = 0$ and $n > 1$, since the above isomorphism valid for a properly infinite algebra $\mathcal{M}$ does not apply to the MASA $\mathcal{A}$ which are commutative, and hence, *not* properly infinite. This will be seen also to be related with such complication that uniqueness of MASA up to unitary conjugacy valid in a von Neumann algebra of type I is not guaranteed in non-type I cases.

More interesting is the second isomorphism,

$$\mathcal{M} \simeq \mathcal{M} \otimes B(L^2(\mathcal{U})) \simeq (\mathcal{M} \rtimes_\alpha \mathcal{U}) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}} \simeq (\mathcal{A} \otimes B(L^2(\mathcal{U}))) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}},$$

by which the *invisible* microscopic algebra $\mathcal{M}$ of quantum observables is *recovered* from the information on the *macroscopically visible* MASA $\mathcal{A}$ with its measured valued in the spectrum $Spec(\mathcal{A})$ together with that of the dual group $\widehat{\mathcal{U}}$ of an abelian group $\mathcal{U}$ in $\mathcal{A}$ to generate $\mathcal{A} = \mathcal{U}''$, both constituting the quantum-mechanical CCR algebra $B(L^2(\mathcal{U}))$. As shown in the next section, this is not merely a matter of interpretation but it actually provides the crucial information on the von Neumann type classification of the quantum algebra $\mathcal{M}$ on the basis of which the claimed bi-directionality at the beginning is ensured.

4 Reconstruction of Micro-Algebra $\mathcal{M}$ & its Type-Classification

The main purpose here is to analyze the structure of the von Neumann factor $\mathcal{M}$ describing a fixed sector from the viewpoint of ii), $\mathcal{M} \simeq (\mathcal{A} \otimes B(L^2(\mathcal{U}))) \rtimes_{\hat{\alpha}} \widehat{\mathcal{U}}$, in the last section, in the systematic use of the observable data provided by the measurement processes described by i), $\mathcal{M} \rtimes_\alpha \mathcal{U} \simeq \mathcal{A} \otimes B(L^2(\mathcal{U})) =: \mathcal{N}$. To achieve it in an effective way, we need the description of the modular structure of $\mathcal{M}$ given as a crossed product $\mathcal{M} = \mathcal{N} \rtimes_\theta G$ of the W*-dynamical system $\mathcal{N} \underset{\theta}{\curvearrowleft} G$ in terms of its component algebra $\mathcal{N}$ and the (co-)action $\theta := \hat{\alpha}$ of an locally compact abelian group $G = \widehat{\mathcal{U}}$.

4.1 Dynamical systems and crossed products

We first need some basic notions related to the W*-dynamical system $\mathcal{N} \underset{\theta}{\curvearrowleft} G$, where the action $\theta = \hat{\alpha}$ of G on $\mathcal{N}$ is given in the form $\theta_g = \beta_g \otimes AdU_g$

with U_g a unitary representation of G on $\mathfrak{H}$. $\mathcal{N}$ can be identified with a subalgebra $\pi_\theta(\mathcal{N})$ in $\mathcal{N} \otimes L^\infty(G)$ through $(\pi_\theta(X)\xi)(g) := \theta_g^{-1}(X)(\xi(g))$ for $\xi \in L^2(G, \mathfrak{H})$. By restriction on the centre $\mathfrak{Z}(\mathcal{N}) = \mathcal{A}$ of $\mathcal{N}$ the action θ of G defines a W*-dynamical system $\mathfrak{Z}(\mathcal{N}) \underset{\beta}{\curvearrowleft} G$, which we call a central W*-dynamical system. The corresponding crossed product $\mathcal{Q} := \mathfrak{Z}(\mathcal{N}) \rtimes_\beta G$ can be regarded as a subalgebra of $\mathcal{N} \rtimes_\theta G$. We recall that a W*-dynamical system $\mathcal{N} \underset{\theta}{\curvearrowleft} G$ is *ergodic* if its θ-fixed point algebra is trivial: $\mathcal{N}^\theta = \mathbb{C}1$, and is called *free* if there exists $X \in \mathcal{N}$ for any non-zero $A \in \mathcal{N}$ and $s \in G$, $s \neq e$, such that $\beta_s(X)A \neq AX$. When applied to the abelian algebra $\mathcal{A}$, the latter condition is equivalent to the requirement that, for any compact subset $K \subset G \setminus \{e\}$ and a non-zero projection $P \in \mathcal{A}$, there exists a non-zero projection $E \in \mathcal{A}$ such that $E \leq P$ and $E\beta_s(E) = 0$ for any $s \in K$. For an action of an *abelian* group, its ergodicity automatically implies that it is free. The ergodicity and freeness of the action β in the W*-dynamical system $\mathcal{A} \underset{\beta}{\curvearrowleft} G$ are related with the algebraic properties of the corresponding crossed product $\mathcal{Q} = \mathcal{A} \rtimes_\beta G$ in the following way:

Proposition 3 *For an abelian W*-dynamical system $\mathcal{A} \underset{\beta}{\curvearrowleft} G$ and the corresponding von Neumann algebra $\mathcal{Q} = \mathcal{A} \rtimes_\beta G$,*

(i) the action β is free if and only if $\mathcal{A}$ is maximally abelian in $\mathcal{Q}$: $\mathcal{A} = \mathcal{Q} \cap \mathcal{A}'$;

(ii) when β is free, $\mathcal{Q}$ is a factor if and only if β is ergodic. In this case, the centre of $\mathcal{Q}$ is equal to $\mathcal{A}^\beta$: $\mathfrak{Z}(\mathcal{Q}) = \mathcal{A}^\beta$.

(The proofs of the above proposition and of all the following statements are omitted here, which will be given in a separate paper [14].)

In view of the close relations between the W*-dynamical systems $\mathcal{N} \underset{\theta}{\curvearrowleft} G$ and its central subsystem $\mathfrak{Z}(\mathcal{N}) \underset{\beta}{\curvearrowleft} G$, the action θ is said to be *centrally ergodic* if its restriction β is ergodic on $\mathfrak{Z}(\mathcal{N})$, and *centrally free* if β is free on $\mathfrak{Z}(\mathcal{N})$. We can verify some commutant relations between $\mathfrak{Z}(\mathcal{N})$ and $\mathcal{N}$ in $\mathcal{M}$ valid for a centrally free action θ, which plays essential roles for the analysis of $\mathcal{M}$:

Proposition 4 *The following relations hold:*

$$\pi_\beta(\mathfrak{Z}(\mathcal{N})) = \mathcal{M} \cap \pi_\theta(\mathcal{N})',$$
$$\pi_\theta(\mathcal{N}) = \mathcal{M} \cap \pi_\beta(\mathfrak{Z}(\mathcal{N}))',$$
$$\pi_\theta(\mathcal{N}^\theta) = \mathcal{M} \cap \mathcal{Q}'.$$

While $\mathfrak{Z}(\mathcal{N}) \cong \mathcal{A}$ is not maximal abelian in $\mathcal{M} = \mathcal{N} \rtimes_\theta G$ owing to the above relations, $\mathcal{A}$ can be easily extended to a MASA $\mathcal{R} = \mathcal{A} \otimes \mathcal{L}$ in $\mathcal{N}$ by tensoring a MASA $\mathcal{L} = \mathcal{L}'$ in $B(\mathfrak{H})$. If θ is centrally free, the subalgebra $\pi_\theta(\mathcal{R})$ is shown to be a MASA in $\mathcal{M}$.

The central ergodicity of θ is related with the factoriality of the crossed product:

Corollary 5 *If θ is a centrally free action, we have*

$$\mathfrak{Z}(\mathcal{M}) = \mathfrak{Z}(\mathcal{Q}) = \pi_\beta(\mathfrak{Z}(\mathcal{N})^\beta).$$

Therefore the following conditions are equivalent:

(i) the action θ is centrally ergodic;

(ii) $\mathcal{M}$ is a factor;

(iii) $\mathcal{Q}$ is a factor.

Note that the action θ is free because G is abelian, and is ergodic when $\mathcal{M}$ is a factor. $\mathcal{Q} = \mathcal{A} \rtimes_\beta G$ is a factor of finite type only when G is discrete and is properly infinite, otherwiseD

4.2 Modular structure and von Neumann types of $\mathcal{M}$ in terms of observable data

To extract the modular data from the dynamical system $\mathcal{N} \underset{\theta}{\curvearrowleft} G$ necessary for the classification of the Micro-algebra $\mathcal{M} = \mathcal{N} \rtimes_\theta G$, we consider a n.f.s. weight φ of the von Neumann algebra $\mathcal{N}$. Let $(\pi_\varphi, U_\varphi, \mathfrak{H}_\varphi, J_\varphi, \mathcal{P}_\varphi)$ be the corresponding standard representation of $\mathcal{N} \underset{\theta}{\curvearrowleft} G$ which consists of the semi-cyclic representation $(\pi_\varphi, \mathfrak{H}_\varphi)$ of $\mathcal{N}$ associated to φ, a modular conjugation operator J_φ, a natural positive cone $\mathcal{P}_\varphi$ and the covariant representation U_φ of θ in $\mathfrak{H}_\varphi$, $\pi_\varphi(\theta_g(X)) = U_\varphi(g)\pi_\varphi(X)U_\varphi(g)^*$. As $\mathcal{N}$ is considered in $\mathfrak{H}_\varphi$, we omit here the symbol π_φ identifying $\mathcal{N}$ with $\pi_\varphi(\mathcal{N})$.

Using the covariant representation U_φ, we construct a left Hilbert algebra in the representation space $\mathfrak{H}_\varphi \otimes L^2(G) = L^2(G, \mathfrak{H}_\varphi)$ of $\mathcal{N} \rtimes_\theta G$ [15] as follows. First in the space $\mathcal{C}_c(G, \mathcal{N})$ of σ^*-strongly continuous $\mathcal{N}$-valued functions on G with compact supports, a convolution and an involution are defined for $X, Y \in \mathcal{C}_c(G, \mathcal{N})$, $s, t \in G$ by

$$(X * Y)(s) = \int_G X(t)\theta_t\left(Y(t^{-1}s)\right) ds,$$
$$X^\sharp(s) = \theta_s\left(X(s^{-1})^*\right).$$

With the left and right actions of $A \in \mathcal{N}$ on $X \in C_c(G, \mathcal{N})$ defined by

$$(A \cdot X)(s) := AX(s), \qquad (X \cdot A)(s) := X(s)\theta_s(A), \qquad \text{for } s \in G,$$

$C_c(G,\mathcal{N})$ is a bimodule over $\mathcal{N}$ satisfying the compatibility conditions:

$$A\cdot(X*Y)=(A\cdot X)*Y, \qquad (X*Y)\cdot A = X*(Y\cdot A),$$
$$(X\cdot A)^\sharp = A^*\cdot X^\sharp, \qquad (A\cdot X)^\sharp = X^\sharp\cdot A^*,$$

for $A\in\mathcal{N}$ and $X,Y\in C_c(G,\mathcal{N})$.

Next we denote by $\mathcal{K}_\varphi := Lin\{X\cdot A : X\in C_c(G,\mathcal{N}), A\in\mathcal{N}\}$, the linear hull of $C_c(G,\mathcal{N})\cdot\mathcal{N}$. Since $\mathfrak{n}_\varphi := \{A\in\mathcal{N}; \varphi(A^*A)\}$ is a left ideal, we have $Y(s)A\in\mathfrak{n}_\varphi$ for $Y\in\mathcal{C}_c(G,\mathcal{N})$, $A\in\mathfrak{n}_\varphi$, $s\in G$, and hence, $\eta_\varphi(Y(s)A) = Y(s)\eta_\varphi(A)$ is meaningful. Accordingly, $\eta_\varphi(X(s))$ makes sense for $X\in\mathcal{K}_\varphi$, and we see that a function $G\ni s\mapsto \eta_\varphi(X(s))\in\mathfrak{H}_\varphi$ belongs to $\mathcal{C}_c(G,\mathcal{N})$. With a map $\widetilde{\eta}_\varphi$ from $\mathcal{K}_\varphi$ to $L^2(G,\mathfrak{H}_\varphi)$ is defined by

$$\widetilde{\eta}_\varphi(X)(s) = \eta_\varphi(X(s)), \qquad X\in\mathcal{K}_\varphi, \quad s\in G,$$

$\widetilde{\mathfrak{A}}_\varphi := \widetilde{\eta}_\varphi(\mathcal{K}_\varphi\cap\mathcal{K}_\varphi^\#)$ is a left Hilbert algebra equipped with the following product and involution:

$$\widetilde{\eta}_\varphi(X)\widetilde{\eta}_\varphi(Y) = \widetilde{\eta}_\varphi(X*Y),$$
$$\widetilde{\eta}_\varphi(X)^\sharp = \widetilde{\eta}_\varphi(X^\#), \qquad X,Y\in\mathcal{K}_\varphi\cap\mathcal{K}_\varphi^\#.$$

With the definition,

$$\widetilde{\pi}_\theta(X) := \int_G X(s)(U_\varphi(s)\otimes\lambda_s)ds,$$

we see the following relation

$$\widetilde{\eta}_\varphi(X*Y) = \widetilde{\pi}_\theta(X)\widetilde{\eta}_\varphi(Y), \qquad X,Y\in\mathcal{K}_\varphi\cap\mathcal{K}_\varphi^*,$$

which shows the equality $\pi_l(\widetilde{\eta}_\varphi(X)) = \widetilde{\pi}_\theta(X)$ for the left multiplication π_l on $\widetilde{\mathfrak{A}}_\varphi$. Therefore $\widetilde{\pi}_\theta$ is a *-representation of $\mathcal{K}_\varphi\cap\mathcal{K}_\varphi^*$, and $\widetilde{\pi}_\theta(\mathcal{K}_\varphi\cap\mathcal{K}_\varphi^*)$ generates the crossed product $\mathcal{N}\rtimes_\theta G$ which is isomorphic with the left von Neumann algebra $\mathcal{R}_l(\widetilde{\mathfrak{A}}_\varphi)$ of $\widetilde{\mathfrak{A}}_\varphi$. Therefore, the modular structure of the crossed product $\mathcal{N}\rtimes_\theta G$ is determined by the standard form $(\pi_\varphi,\mathfrak{H}_\varphi,J_\varphi,\mathcal{P}_\varphi)$ of $\mathcal{N}$. The modular operator $\widetilde{\Delta}$ and modular conjugation $\widetilde{J}$ are given by

$$\left(\widetilde{\Delta}^{it}\xi\right)(s) = \Delta^{it}_{\varphi\circ\theta_s,\varphi}\xi(s),$$
$$\left(\widetilde{J}\xi\right)(s) = U_\varphi(s)J_\varphi\xi(s^{-1}), \qquad \xi\in L^2(G,\mathfrak{H}_\varphi), \quad s\in G,$$

where $\Delta_{\varphi\circ\theta_s,\varphi}$ is the relative modular operator from φ to $\varphi\circ\theta_s$ with which Connes cocycle derivative $V_t = (D(\varphi\circ\theta_s):D\varphi)_t$ is related through $\Delta^{it}_{\varphi\circ\theta_s,\varphi} = V_t\Delta^{it}_\varphi$ (see [16]). The dual weight $\widehat{\varphi}$ of $\mathcal{R}_l(\widetilde{\mathfrak{A}}_\varphi) = \mathcal{N}\rtimes_\theta G$ is defined by such a n.f.s. weight induced from the left Hilbert algebra $\widetilde{\mathfrak{A}}_{\varphi}$ as given for $X \in \mathcal{N}_{+}$ by

$$\widehat{\varphi}(X) = \begin{cases} \|\xi\|^2, & X = \pi_l(\xi)^* \pi_l(\xi), \quad \xi \in \mathfrak{B}, \\ +\infty, & \end{cases}$$

where $\mathfrak{B}$ is the set of left bounded vector in $\widetilde{\mathfrak{A}}_{\varphi}$. The modular automorphism group $\sigma^{\widehat{\varphi}}$ of $\widehat{\varphi}$ is given by $\sigma_t^{\widehat{\varphi}}(X) = \widetilde{\Delta}^{it} X \widetilde{\Delta}^{-it}$ for $X \in \mathcal{N} \rtimes_{\theta} G$, whose action on the generators $\pi_{\theta}(\mathcal{N})\mathrm{C}\lambda(G)$ of $\mathcal{N} \rtimes_{\theta} G$ can be specified explicitly by:

$$\begin{aligned} \sigma_t^{\widehat{\varphi}}(\pi_{\theta}(X)) &= \pi_{\theta}(\sigma_t^{\varphi}(X)), \qquad X \in \mathcal{N}, \quad t \in \mathbb{R}, \\ \sigma_t^{\widehat{\varphi}}(\lambda(s)) &= \lambda(s)\pi_{\theta}((D\varphi \circ \theta_s : D\varphi)_t), \qquad s \in G. \end{aligned}$$

As $\mathcal{N} = \mathcal{A} \otimes B(\mathfrak{H})$ is not finite, its crossed product $\mathcal{M}$ is not either and we have the following theorem:

Theorem 6 *For a centrally ergodic W*-dynamical system* $(\mathcal{N} \underset{\theta}{\curvearrowleft} G)$ *with its corresponding central W*-dynamical system* $(\mathfrak{Z}(\mathcal{N}) \underset{\beta}{\curvearrowleft} G)$*, the factor type of* $\mathcal{M} = \mathcal{N} \rtimes_{\theta} G$ *coincides with that of* $\mathcal{Q} = \mathfrak{Z}(\mathcal{N}) \rtimes_{\beta} G$ *and we have the following criteria:*

(i) $\mathcal{M}$ *is of type I if and only if* $(\mathfrak{Z}(\mathcal{N}) \underset{\beta}{\curvearrowleft} G)$ *is isomorphic to the flow on* $L^{\infty}(G)$*:* $(\mathfrak{Z}(\mathcal{N}) \underset{\beta}{\curvearrowleft} G) \cong (L^{\infty}(G) \underset{Ad\lambda_G}{\curvearrowleft} G)$*;*

(ii) $\mathcal{M}$ *is of type II if and only if* $(\mathfrak{Z}(\mathcal{N}) \underset{\beta}{\curvearrowleft} G)$ *is not isomorphic to* $(L^{\infty}(G) \underset{Ad\lambda_G}{\curvearrowleft} G)$ *and* $\mathfrak{Z}(\mathcal{N})$ *admits a* β*-invariant semifinite measure supported by* $\mathfrak{Z}(\mathcal{N})$*;*

(iii) $\mathcal{M}$ *is of type III if and only if* $\mathfrak{Z}(\mathcal{N})$ *admits no* β*-invariant semifinite measure with support* $\mathfrak{Z}(\mathcal{N})$*.*

It is remarkable that the modular structure of $\mathcal{M}$ is completely determined by the properties of the abelian dynamical system $\mathcal{A} \underset{\beta}{\curvearrowleft} G$. In more details in the above type classification, the spectrum $Spec(\mathcal{A})$ of the centre $\mathcal{A} = \mathfrak{Z}(\mathcal{N}) = \mathfrak{Z}(\mathcal{M} \rtimes_{\alpha} \mathcal{U})$ of the composite system $\mathcal{M} \rtimes_{\alpha} \mathcal{U}$ plays the crucial role as the classifying space of intrasectorial structure, in sharp contrast to the quantum-mechanical part $B(L^2(\mathcal{U}))$ playing no role. The former is in harmony with the general strategy adopted in Sec.2 in the sense that intrasectorial analysis reduces to the sector analysis of the composite system $\mathcal{M} \rtimes_{\alpha} \mathcal{U}$ and the latter is consistent with the interpretation of it in Sec. 3 as a (non-commutative) homotopy term. If $\mathcal{M}$ is of type III, we need more detailed characterization of the modular spectrum $S(\mathcal{M})$ which is also determined by $\mathcal{A}$ and the action of $\widehat{\mathcal{U}}$ on it:

$$S(\mathcal{M}) = \bigcap \{Spec(\Delta_{\varphi \circ \theta_{\gamma}, \varphi}) : \varphi \in \mathcal{W}_{\mathcal{A}}\},$$

where $\mathcal{W}_{\mathcal{A}}$ is the set of all normal semi-finite faithful weights on $\mathcal{A}$ and $\Delta_{\varphi\circ\theta_\gamma,\varphi} = (D(\omega\circ\theta_\gamma) : D\omega)_t \Delta_\varphi$. We recall here the Connes classification of type III von Neumann algebras [16]: (1) $\mathcal{M}$ is type III$_\lambda$, $(0<\lambda<1)$, if and only if $S(\mathcal{M}) = \{\lambda^n : n\in\mathbb{Z}\}\cup\{0\}$, (2) $\mathcal{M}$ is type III$_0$ if and only if $S(\mathcal{M}) = \{0,1\}$, (3) $\mathcal{M}$ is type III$_1$ if and only if $S(\mathcal{M}) = \mathbb{R}_+$.

While the type classification does not provide the whole data necessary for the complete recovery of Micro-algebra without such a uniqueness result as ensured for the AFD factor of type III$_1$, we can draw immediately some important lessons from it: the above (i) tells us that our starting assumption on the $\mathcal{U}$-action $\alpha = Ad$ on $\mathcal{M}$ was too restrictive to recover $\mathcal{M}$ of non-type I, since it implies that the corresponding coaction $\theta = \hat{\alpha}$ becomes isomorphic to the flow on $L^\infty(\widehat{\mathcal{U}})$. Recalling the presence of a non-trivial dynamics inherent to the system $\mathcal{M}$ to be observed, however, we can easily see that the measurement process described by the coupling $\alpha_u = Ad(u)$ is simply a convenient approximation commonly adopted in most discussions and that α should *not* be inner in general. To reconstruct the non-trivial algebra $\mathcal{M}$ of the observed system we need the data of the *intrinsic dynamics* of the system, which can be attained by measuring locally the energy-momentum tensor $T_{\mu\nu}$. For instance, we can approximate the dynamics locally on a subalgebra $\mathcal{M} = \pi(\mathfrak{A}(\mathcal{O}))''$ of local observables by the modular automorphism group corresponding to a local KMS state constructed by the Buchholz-Junglas method of heating-up [17], according to which the above fixed-point algebra $\mathcal{M}^{\alpha(\mathcal{U})}$ becomes of type II when $\mathcal{M}$ is of type III. Thus, starting from $\mathcal{M}\rtimes_\alpha\mathcal{U}\simeq\mathcal{A}\otimes B(\mathfrak{H})$ with $\mathcal{A}$ of type II$_1$, we can repeat the similar analysis to the one in Sec.3. According to the Takesaki duality [12], the crossed product $\mathcal{N} := \mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ with respect to the modular automorphism group of a n.f.s. weight φ of $\mathcal{M}$ is a von Neumann algebra of type II$_\infty$ with an n.f.s. trace τ such that $\tau\circ\theta_s = e^{-s}\tau, s\in\mathbb{R}$ with θ the action of $\mathbb{R}$ on $\mathcal{N}$ dual to σ^φ, and conversely, $\mathcal{M} = \mathcal{N}\rtimes_\theta\mathbb{R}$ holds. This decomposition of $\mathcal{M}$ is unique. Furthermore if $\mathcal{M}$ is a factor of type III$_1$, then $\mathcal{N}$ is a factor of type II$_\infty$. In such situations, we need some definitions in relation with a MASA $\mathcal{A}$ of $\mathcal{M}$. First, the normalizer of $\mathcal{A}$ in $\mathcal{U}(\mathcal{M})$ is defined by

$$N_{\mathcal{M}}(\mathcal{A}) := \{u\in\mathcal{U}(\mathcal{M}) : u\mathcal{A}u^* = \mathcal{A}\}.$$

A MASA $\mathcal{A}$ in a factor $\mathcal{M}$ is called *regular* if $N_{\mathcal{M}}(\mathcal{A})$ generates $\mathcal{M}$, and *semi-regular* if $N_{\mathcal{M}}(\mathcal{A})$ generates a subfactor of $\mathcal{M}$. The subalgebra $\mathcal{A}\cap\mathcal{N}$ is also a MASA of $\mathcal{N}$, for which we can derive the following result from [18]:

Proposition 7 *Let $\mathcal{N} = \mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ be a type II$_\infty$ factor von Neumann algebra defined above as a crossed product of a type III$_1$ factor von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space $\mathfrak{H}$ with an n.f.s. weight φ. Then $\mathcal{N}$ contains a maximal abelian subalgebra $\mathcal{A}$ which is also maximal abelian in $\mathcal{M}$ and semi-regular in $\mathcal{N}$. Moreover, if $\mathcal{N}$ is approximately finite dimensional, then $\mathcal{A}$ can be chosen to be regular in $\mathcal{N}$.*

In this way, several important steps for the formulation of a measurement scheme have been achieved in a form applicable to general quantum systems with *infinite degrees of freedom* as QFT, by removing the restriction on the choice of MASA inherent to the finite quantum systems and by specifying the coupling term necessary for constructing measurement processes. To be fair, however, we note that there remain some unsettled problems, such as the non-uniqueness of MASA $\mathcal{A} = \mathcal{A}' \cap \mathcal{M}$, which is one of the difficulties caused by the infinite dimensional non-commutativity. For lack of the uniqueness of MASA the uniqueness of the above reconstruction of $\mathcal{M}$ is not guaranteed either. In relation to this, the consistency problem should be taken serious between the mathematically relevant structures of type III and the *finite discrete* spectra inevitable at the operational level, which is closely related to such type of criteria as the nuclearity condition in algebraic QFT to select the most relevant states and observables. In this connection, it would be important to re-examine the general meaning of the so-called "ambiguity of interpolating fields" closely related to the notion of Borchers classes, relative locality and PCT invariance. At the end, we remark that the focal point in our consideration has shifted from states to algebra, from algebra to dynamics, through which all the basic ingredients constituting a mathematical framework for describing a physical qunatum system can and/or should be re-examined and re-constructed in close relations with observational and operational contexts.

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