Matching Procedure for the Sixth Painlevé Equation

Davide Guzzetti

Abstract

In the context of the isomonodromy deformation method, we present a constructive procedure (a matching procedure) to obtain the critical behavior of Painlevé VI transcendents and solve the connection problem. This procedure yields two and one parameter families of solutions, including logarithmic behaviors, and three classes of solutions with Taylor expansion at a critical point.

1   Introduction

The sixth Painlevé equation is:

$$
\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]\frac{dy}{dx}
$$

$$
+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right], \qquad \text{(PVI)}.
$$

The generic solution has essential singularities and/or branch points in $0,1,\infty$. It's behavior at these points will be called *critical*. The other singularities, which depend on the initial conditions, are poles. A solution of PVI can be analytically continued to a meromorphic function on the universal covering of $\mathbf{P}^1\backslash\{0,1,\infty\}$. For generic values of the integration constants and of the parameters $\alpha,\beta,\gamma,\delta$, it can not be expressed via elementary or classical transcendental functions. For this reason, it is called a *Painlevé transcendent*.

Solving (PVI) means: i) Determine the critical behavior of the transcendents at the *critical points* $x = 0, 1, \infty$. Such a behavior must depend on two integration constants. ii) Solve the *connection problem*, namely: find the relation between couples of integration constants at $x = 0, 1, \infty$.

In this paper we present a constructive procedure, called *matching procedure*, to compute the first leading terms of the critical behavior at $x = 0$. No a priori assumption is necessary. The procedure is based on the isomonodromy deformation theory. It allows to compute the *monodromy data* associated to a solution, and thus solve the connection problem. [1] The matching procedure was developed by A.Kitaev for other Painlevé equations.

As a result of the matching procedure, we obtain: 1) A two-parameter family of solutions, of the type found by Jimbo [15]. 2) One-parameter families of solutions, including a class of logarithmic solutions. 3) The solutions which admit a Taylor expansion at $x = 0$. 4) Then, we compute the corresponding monodromy data. In virtue of the symmetries of (PVI) (birational transformations of $(x, y(x))$), it can be shown that the solutions with Taylor expansion at $x = 0$, obtained by the matching procedure, are the representatives of three equivalent classes which include all the solutions admitting a Taylor expansion at a critical point.

The purpose of this paper is to present a constructive procedure to solve (PVI), and to show its effectiveness by both reproducing known solutions and by finding new ones. It is not our purpose

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[1]For reasons of space, we limit ourselves to the computation of monodromy data, without explaining how the connection problem is practically solved once the monodromy data are computed, and how the analytic continuation is done. We refer the reader to [15], [7], [8], [10], [4]. The behaviors at $x = 1$ and $x = \infty$, and the dependence of them on the monodromy data are deduced from the behavior at $x = 0$ by symmetry transformations. PVI is invariant for the change of variables $y(x) = 1 - \tilde{y}(t)$, $x = 1 - t$ and simultaneous permutation of θ_0, θ_1. This means that $y(x)$ solves PVI if and only if $\tilde{y}(t)$ solves PVI with permuted parameters and independent variable t. Similarly, PVI is invariant for $y(x) = 1/\tilde{y}(t)$, $x = 1/t$ and simultaneous permutation of θ_∞, θ_0. It is invariant for $y(x) = (\tilde{y}(t) - t)/(1 - t)$, $x = t/(t - 1)$ and simultaneous permutation of θ_0, θ_x. By composing the third, first and again third symmetries, we get $y(x) = \tilde{y}(t)/t$, $t = 1/x$ with the permutation of θ_1, θ_x. Therefore, the critical points $0, 1, \infty$ are equivalent.

here to discuss the problem of characterization and classification of all the solutions of (PVI) in terms of the monodromy data of the associated linear system. Nevertheless, we show that the matching procedure is effective to produce new solutions associated to monodromy data for which the connection problem has not been studied so far. Therefore, it is a tool to study the classification problem. This will be done in another paper.

PART I: Matching Procedure and Results

2   Matching Procedure

PVI is the isomonodromy deformation equation of a Fuchsian system of differential equations [16]:

$$\frac{d\Psi}{d\lambda} = A(\lambda, x, \theta) \ \Psi, \qquad A(\lambda, x, \theta) := \left[\frac{A_0(x, \theta)}{\lambda} + \frac{A_x(x, \theta)}{\lambda - x} + \frac{A_1(x, \theta)}{\lambda - 1} \right], \quad \lambda \in \mathbf{C}. \tag{1}$$

The 2×2 matrices $A_i(x, \theta)$ depend on x in such a way that the monodromy of a fundamental solution $\Psi(\lambda, x)$ does not change for small deformations of x. They also depend on the parameters $\alpha, \beta, \gamma, \delta$ of PVI through more elementary parameters $\theta = (\theta_0, \theta_x, \theta_1, \theta_\infty)$ according to the following relations:

$$-A_\infty := A_0 + A_1 + A_x = -\frac{\theta_\infty}{2}\sigma_3. \qquad \text{Eigenvalues } (A_i) = \pm\frac{1}{2}\theta_i, \quad i = 0, 1, x;$$

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad -\beta = \frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \left(\frac{1}{2} - \delta\right) = \frac{1}{2}\theta_x^2 \tag{2}$$

Here σ_3 is the Pauli matrix. The equations of monodromy-preserving deformation (Schlesinger equations), can be written in Hamiltonian form and reduce to PVI, being the transcendent $y(x)$ solution of $A(y(x), x, \theta)_{1,2} = 0$. Namely:

$$y(x) = \frac{x \ (A_0)_{12}}{x \ [(A_0)_{12} + (A_1)_{12}] - (A_1)_{12}}, \tag{3}$$

The matrices $A_i(x, \theta)$, $i = 0, x, 1$, depend on $y(x)$, $\frac{dy(x)}{dx}$ and $\int y(x)$ through rational functions, which are given in [16]. In short, we will write $A_i = A_i(x)$.

The product of the monodromy matrices M_0, M_x, M_1 of a fundamental matrix solution Ψ at $\lambda = 0, x, 1$ respectively, is equal to the monodromy at $\lambda = \infty$. The order of the producs depends on the choice of a basis of loops. As a consequence, the following relation must hold:

$$\cos(\pi\theta_0)\mathrm{tr}(M_1 M_x) + \cos(\pi\theta_1)\mathrm{tr}(M_0 M_x) + \cos(\pi\theta_x)\mathrm{tr}(M_1 M_0)$$

$$= 2\cos(\pi\theta_\infty) + 4\cos(\pi\theta_1)\cos(\pi\theta_0)\cos(\pi\theta_x).$$

2.1   Leading Terms of $y(x)$ as a result of Matching

We present the constructive procedure to obtain the leading terms of a solution $y(x)$ when $x \to 0$. This procedure has been used for the fifth Painlevé equation by F.V. Andeev and A.V. Kitaev in [1].

Since we are considering $x \to 0$, we divide the λ-plane into two domains. The "outside" domain is defined for λ sufficiently big:

$$|\lambda| \geq |x|^{\delta_{OUT}}, \qquad \delta_{OUT} > 0. \tag{4}$$

Therefore, (1) can be written as:

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_x}{\lambda} \sum_{n=1}^{\infty} \left(\frac{x}{\lambda}\right)^n + \frac{A_1}{\lambda - 1} \right] \ \Psi. \tag{5}$$

The "inside" domain is defined for λ comparable with x, namely:

$$|\lambda| \leq |x|^{\delta_{IN}}, \qquad \delta_{IN} > 0. \tag{6}$$

Therefore, $\lambda \to 0$ as $x \to 0$, and we rewrite (1) as:

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} - A_1 \sum_{n=0}^{\infty} \lambda^n \right] \Psi. \tag{7}$$

If the behavior of $A_0(x)$, $A_1(x)$ and $A_x(x)$ is sufficiently good, we expect that the higher order terms in the series of (5) and (7) are small corrections which can be neglected when $x \to 0$. If this is the case, (5) and (7) reduce respectively to:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_x}{\lambda} \sum_{n=1}^{N_{OUT}} \left(\frac{x}{\lambda} \right)^n + \frac{A_1}{\lambda - 1} \right] \Psi_{OUT}, \tag{8}$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} - A_1 \sum_{n=0}^{N_{IN}} \lambda^n \right] \Psi_{IN}, \tag{9}$$

where N_{IN}, N_{OUT} are suitable integers. The simplest reduction is to Fuchsian systems:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_1}{\lambda - 1} \right] \Psi_{OUT}, \tag{10}$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} \right] \Psi_{IN}. \tag{11}$$

Generally speaking, we can parameterize the elements of $A_0 + A_x$ and A_1 of (10) in terms of θ_1, the eigenvalues of $A_0 + A_x$ and the eigenvalues θ_∞ of $A_0 + A_x + A_1$. We also need an additional unknown function of x. In the same way, we can explicitly parameterize the elements of A_0 and A_x in (11) in terms of θ_0, θ_x, the eigenvalues of $A_0 + A_x$ and another additional unknown function of x. When the reductions (8) and (9) are non-fuchsian, particular care must be payed. This will be explained case by case in the paper. Our purpose is to find the leading term of the unknown functions when $x \to 0$, in order to determine the critical behavior of $A_0(x)$, $A_1(x)$, $A_x(x)$ and (3).

The leading term can be obtained as a result of two facts:

i) Systems (8) and (9) are isomonodromic. This imposes constraints on the form of the unknown functions. Typically, one of them must be constant.

ii) Two fundamental matrix solutions $\Psi_{OUT}(\lambda, x)$, $\Psi_{IN}(\lambda, x)$ must match in the region of overlap, provided this is not empty:

$$\Psi_{OUT}(\lambda, x) \sim \Psi_{IN}(\lambda, x), \qquad |x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}, \quad x \to 0 \tag{12}$$

This relation is to be intended in the sense that the leading terms of the local behavior of Ψ_{OUT} and Ψ_{IN} for $x \to 0$ must be equal. This determines a simple relation between the two functions of x appearing in A_0, A_x, A_1, $A_0 + A_x$. (12) also implies that $\delta_{IN} \leq \delta_{OUT}$.

Practically, to fulfill point ii), we will match a fundamental solution of (8) for $\lambda \to 0$, with a fundamental solution of (9) when $\mu := \lambda/x \to \infty$, namely with a solution of:

$$\frac{d\Psi_{IN}}{d\mu} = \left[\frac{A_0}{\mu} + \frac{A_x}{\mu - 1} - x A_1 \sum_{n=0}^{N_{IN}} x^n \mu^n \right] \Psi_{IN}, \qquad \mu := \frac{\lambda}{x}. \tag{13}$$

To summarize, *matching* two fundamental solutions of the reduced *isomonodromic* systems (8) and (9), we obtain the leading term(s), for $x \to 0$, of the entries of the matrices of the original system (1). The procedure is algorithmic, no *a priori* assumption about the behavior being necessary.

This method is sometimes called *coalescence of singularities*, because the singularity $\lambda = 0$ and $\lambda = x$ coalesce to produce system (8), while the singularity $\mu = \frac{1}{x}$ and $\mu = \infty$ coalesce to produce system (13). Coalescence of singularities was first used by M. Jimbo in [15] to compute the monodromy matrices of (1) for a class of solutions of (PVI) with leading term $y(x) \sim a\, x^{1-\sigma}$, $0 < \Re\sigma < 1$.

2.2 Computation of the Monodromy Data

Let Ψ be a fundamental matrix solution of (1), and let M_0, M_x, M_1, M_∞ be its monodromy matrices at $\lambda = 0, x, 1, \infty$ respectively (M_∞ is the product of M_0, M_x, M_1, the order depending on the choice of a basis of loops). As a consequence of isomonodromicity, there exists a fundamental solution Ψ_{OUT} of (8) such that

$$M_1^{OUT} = M_1, \qquad M_\infty^{OUT} = M_\infty,$$

where M_1^{OUT} and M_∞^{OUT} are the monodromy matrices of Ψ_{OUT} at $\lambda = 1, \infty$. Moreover, $M_0^{OUT} = M_0 M_x$ or $M_x M_0$, depending on the order of loops. A detailed proof of these facts can be found in [8]. There also exists a fundamental solution Ψ_{IN} of (9) such that:

$$M_0^{IN} = M_0, \qquad M_x^{IN} = M_x,$$

where M_0^{IN} and M_x^{IN} are the monodromy matrices of Ψ_{IN} at $\lambda = 0, x$.

The method of coalescence of singularities is useful when the monodromy of the reduced systems (8), (9) can be explicitly computed. This is the case when the reduction is fuchsian (namely (10), (11)), because fuchsian systems with three singular points are equivalent to a Gauss hyper-geometric equation (see Appendix 1). For the non-fuchsian reduction, in general we can compute the monodromy when (8), (9) are solvable in terms of special or elementary functions. This will be discussed case by case in the paper.

In order for this procedure to work, not only Ψ_{OUT} and Ψ_{IN} must match with each other, as in subsection 2.1, but also Ψ_{OUT} must match with a fundamental matrix solution Ψ of (1) in a domain of the λ plane, and Ψ_{IN} must match with *the same* Ψ in another domain of the λ plane.

The standard choice of Ψ is as follows:

$$\Psi(\lambda) = \begin{cases} \left[I + O\left(\frac{1}{\lambda}\right)\right] \ \lambda^{-\frac{\theta_\infty}{2}\sigma_3}\lambda^{R_\infty}, & \lambda \to \infty; \\[2mm] \psi_0(x)\left[I + O(\lambda)\right] \ \lambda^{\frac{\theta_0}{2}\sigma_3}\lambda^{R_0}C_0, & \lambda \to 0; \\[2mm] \psi_x(x)\left[I + O(\lambda - x)\right] \ (\lambda - x)^{\frac{\theta_x}{2}\sigma_3}(\lambda - x)^{R_x}C_x, & \lambda \to x; \\[2mm] \psi_1(x)\left[I + O(\lambda - 1)\right] \ (\lambda - 1)^{\frac{\theta_1}{2}\sigma_3}(\lambda - 1)^{R_1}C_1, & \lambda \to 1; \end{cases} \qquad (14)$$

Here $\psi_0(x)$, $\psi_x(x)$, $\psi_1(x)$ are the diagonalizing matrices of $A_0(x)$, $A_1(x)$, $A_x(x)$ respectively. They are defined by multiplication to the right by arbitrary diagonal matrices, possibly depending on x. C_κ, $\kappa = \infty, 0, x, 1$, are invertible *connection matrices*, independent of x [16]. Each R_κ, $\kappa = \infty, 0, x, 1$, is also independent of x, and:

$$R_\kappa = 0 \text{ if } \theta_\kappa \notin \mathbf{Z}, \qquad R_\kappa = \begin{cases} \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, & \text{if } \theta_\kappa > 0 \text{ integer} \\[3mm] \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, & \text{if } \theta_\kappa < 0 \text{ integer} \end{cases}$$

If $\theta_i = 0$, $i = 0, x, 1$, then R_i is to be considered the Jordan form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of A_i. If $\theta_\infty = 0$, $R_\infty = 0$.

Note that for the loop $\lambda \mapsto \lambda e^{2\pi i}$, $|\lambda| > \max\{1, |x|\}$, we immediately compute the monodromy at infinity:

$$M_\infty = \exp\{-i\pi\theta_\infty\} \ \exp\{2\pi i R_\infty\}.$$

Let Ψ_{OUT} and Ψ_{IN} be the solutions of (8) and (9) matching as in (12). We explain how they are matched with (14).

(*) Matching $\Psi \leftrightarrow \Psi_{OUT}$:
$\lambda = \infty$ is a fuchsian singularity of (8), with residue $-A_\infty/\lambda$. Therefore, we can always find a fundamental matrix solution with behavior:

$$\Psi_{OUT}^{Match} = \left[I + O\left(\frac{1}{\lambda}\right)\right] \lambda^{-\frac{\theta_\infty}{2}\sigma_3}\lambda^{R_\infty}, \quad \lambda \to \infty.$$

This solution matches with Ψ. Also $\lambda = 1$ is a fuchsian singularity of (8). Therefore, we have:

$$\Psi_{OUT}^{Match} = \psi_1^{OUT}(x)\left[I + O(\lambda - 1)\right] (\lambda - 1)^{\frac{\theta_1}{2}\sigma_3}(\lambda - 1)^{R_1}C_1^{OUT}, \quad \lambda \to 1;$$

Here C_1^{OUT} is a suitable connection matrix. $\psi_1^{OUT}(x)$ is the matrix that diagonalizes the leading terms of $A_1(x)$. Therefore, $\psi_1(x) \sim \psi_1^{OUT}(x)$ for $x \to 0$. As a consequence of isomonodromicity, R_1 is the same of Ψ.

As a consequence of the matching $\Psi \leftrightarrow \Psi_{OUT}^{Match}$, the monodromy of Ψ at $\lambda = 1$ is:

$$M_1 = C_1^{-1}\exp\{i\pi\theta_1\sigma_3\}\exp\{2\pi i R_1\}C_1, \quad \text{with } C_1 \equiv C_1^{OUT}.$$

We finally need an invertible connection matrix C_{OUT} to connect Ψ_{OUT}^{Match} with the solution Ψ_{OUT} appearing in (12). Namely, $\Psi_{OUT}^{Match} = \Psi_{OUT}C_{OUT}$.

(*) Matching $\Psi \leftrightarrow \Psi_{IN}$:

As a consequence of the matching $\Psi \leftrightarrow \Psi_{OUT}^{Match}$, we have to choose the IN-solution which matches with Ψ_{OUT}^{Match}. This is $\Psi_{IN}^{Match} := \Psi_{IN}C_{OUT}$.

Now, $\lambda = 0, x$ are fuchsian singularities of (9). Therefore:

$$\Psi_{IN}^{Match} = \begin{cases} \psi_0^{IN}(x)\left[I + O(\lambda)\right]\lambda^{\frac{\theta_0}{2}\sigma_3}\lambda^{R_0}C_0^{IN}, & \lambda \to 0; \\[2ex] \psi_x^{IN}(x)\left[I + O(\lambda - x)\right](\lambda - x)^{\frac{\theta_x}{2}\sigma_3}(\lambda - x)^{R_x}C_x^{IN}, & \lambda \to x; \end{cases}$$

The above hold for fixed small $x \neq 0$. Here C_0^{IN} and C_x^{IN} are suitable connection matrices. $\psi_0^{IN}(x)$ and $\psi_x(x)^{IN}$ are diagonalizing matrices of the leading terms of $A_0(x)$ and $A_x(x)$. For $x \to 0$ they match with $\psi_0(x)$ and $\psi_x(x)$ of Ψ in (14). On the other hand, as a consequence of isomonodromicity, the matrices R_0 and R_x are the same of Ψ.

By virtue of the matching $\Psi \leftrightarrow \Psi_{IN}^{Match}$, the connection matrices C_0 and C_x coincide with the x-independent connection matrices C_0^{IN}, C_x^{IN} respectively. As a result, we obtain the monodromy matrices for Ψ:

$$M_0 = C_0^{-1}\exp\{i\pi\theta_0\sigma_3\}\exp\{2\pi i R_0\}C_0, \quad C_0 \equiv C_0^{IN},$$
$$M_x = C_x^{-1}\exp\{i\pi\theta_x\sigma_3\}\exp\{2\pi i R_x\}C_x, \quad C_x \equiv C_x^{IN}.$$

Our reduction is useful if the connection matrices C_1^{OUT}, C_0^{IN}, C_x^{IN} can be computed explicitly. This is possible for the fuchsian reduced systems (10), (11). For non-fuchsian reduced systems, we will discuss the computability case by case.

3 Results

In the following, it is understood that $x \to 0$ inside a sector. Namely, $\arg(x)$ is bounded.

3.1 Critical Behaviors: Result I

The novelty of this paper is that the matching procedure is applied to non-fuchsian systems (8) and (9). As a result, we obtain all the solutions that admit a Taylor expansion

$$y(x) = b_0 + b_1 x + b_2 x^2 + \ldots = \sum_{n=0}^{\infty} b_n x^n, \quad x \to 0.$$

Precisely, we obtain the representative solutions of three equivalence classes, the equivalence relation being the birational transformations [22] of Appendix 3 and formula (19). Our result is the following.

Theorem 1 *The solutions of (PVI) with Taylor expansion at $x = 0$ are divided into four equivalent classes (one being that of singular solutions $y = 0, 1, x$). The representatives can be chosen as follows:*

1) Singular solution $y = 1$.

2) $\theta_1 \neq 1$, $\theta_1 - \theta_\infty \notin \mathbf{Z}$:

$$y(x) = \frac{\theta_1 - \theta_\infty + 1}{1 - \theta_\infty} + \frac{\theta_1[(\theta_1 - \theta_\infty)(\theta_1 - \theta_\infty + 2) + \theta_x^2 - \theta_0^2]}{2(\theta_\infty - 1)(\theta_\infty - \theta_1)(\theta_\infty - \theta_1 - 2)} x + \sum_{n=3}^{\infty} b_n(\theta_1, \theta_\infty, \theta_0, \theta_x) x^n. \quad (15)$$

The coefficients are certain rational functions of $\theta_0, \theta_\infty, \theta_0, \theta_x$.

3) $\theta_1 = \theta_\infty \neq 1$, $\theta_0 = \pm\theta_x$:

$$y(x) = \frac{1}{1 - \theta_\infty} + ax + \sum_{n=2}^{\infty} b_n(a; \theta_0, \theta_\infty) x^n. \quad (16)$$

The coefficients are certain rational functions of θ_0, θ_∞ and a parameter $a \in \mathbf{C}$.

4) $\theta_\infty = 1$, $\theta_1 = 0$:

$$y(x) = a + \frac{1-a}{2}(1 + \theta_0^2 - \theta_x^2) x + \sum_{n=2}^{\infty} b_n(a; \theta_0; \theta_x) x^n. \quad (17)$$

The coefficients are certain rational functions of θ_0, θ_x and a parameter $a \in \mathbf{C}$.

The monodromy data associated to the above solutions is given in theorem 3.

The symmetry $\theta_1 \mapsto -\theta_1$, which leaves (PVI) invariant, transforms (15) into:

$$y(x) = \frac{\theta_1 + \theta_\infty - 1}{\theta_\infty - 1} + \frac{\theta_1[(\theta_1 + \theta_\infty)(\theta_1 + \theta_\infty - 2) + \theta_x^2 - \theta_0^2]}{2(1 - \theta_\infty)(\theta_\infty + \theta_1)(\theta_\infty + \theta_1 - 2)} x + \sum_{n=3}^{\infty} b_n(-\theta_1, \theta_\infty, \theta_0, \theta_x) x^n. \quad (18)$$

Here $\theta_1 \neq 1$, $\theta_1 + \theta_\infty \notin \mathbf{Z}$. The coefficients b_n are the same of (15).

The convergence of the Taylor series can be proved by a Briot-Bouquet like argument. This will not be done here, for reasons of space. The reader can find the general procedure in [13] and an application to the fifth Painlevé equation in [18]

Comments:

1) Characterization of solutions $y(x) = \sum_{n=0}^{\infty} b_n x^n$, $b_0 \neq 0$.

(a) There always exists one solution (15) when $\theta_1 - \theta_\infty \notin \mathbf{Z}$; there always exists one solution (18) when $\theta_1 + \theta_\infty \notin \mathbf{Z}$. The coefficients b_n depend rationally on θ_κ, $\kappa = 0, x, 1, \infty$. **(b)** There is a one-parameter family of solutions equivalent to (16), when $\theta_1 \pm \theta_\infty \in \mathbf{Z}$ and $\theta_0 \pm \theta_x$ has a particular integer value. The coefficients b_n depend rationally on a complex parameter a and θ_∞, θ_0. **(c)** Finally, there is a one-parameter family of solutions equivalent to (17), when $\theta_1 \pm \theta_\infty \in \mathbf{Z}$, and θ_∞ has a particular integer value; the coefficients b_n depend rationally on a complex parameter a and θ_0, θ_x. The singular solutions $y = 0, 1, x$ are possibly obtained by birational transformations of (15), (16), (17).

The coefficients b_n can always be computed recursively by direct substitution into (PVI). We will clarify these facts by some examples in Appendix 4.

2) Characterization of solutions $y(x) = \sum_{n=1}^{\infty} b_n x^n$, $b_1 \neq 0$.

These solutions are obtained from those of theorem 1 by the symmetry.

$$\theta_x \mapsto \theta_1, \quad \theta_0 \mapsto \theta_\infty - 1, \quad \theta_1 \mapsto \theta_x, \quad \theta_\infty \mapsto \theta_0 + 1; \quad y(x) \mapsto \frac{x}{y(x)}. \quad (19)$$

The solutions obtained from the singular solution $y = 1$ and (15), (16), (17) are respectively:

1) Singular solution $y(x) = x$.

2) $\theta_0 \neq 0$, $\theta_0 \pm \theta_x \notin \mathbf{Z}$:

$$y(x) = \frac{\theta_0}{\theta_0 \pm \theta_x} x \pm \frac{\theta_0 \theta_x \left[(\theta_0 \pm \theta_x)^2 + \theta_1^2 - \theta_\infty^2 + 2\theta_\infty - 2\right]}{2(\theta_0 \pm \theta_x)^2 \left[(\theta_0 \pm \theta_x)^2 - 1\right]} x^2 + \sum_{n=3}^{\infty} b_n(\theta_0, \theta_x, \theta_1, \theta_\infty) x^n. \quad (20)$$

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3) $\theta_0 + \theta_x = 1$, $\theta_0 \neq 0$, $\theta_1 = \pm(\theta_\infty - 1)$:

$$y(x) = \theta_0 x \ + a \ x^2 \ + \sum_{n=3}^{\infty} b_n(a; \theta_0, \theta_\infty) x^n. \tag{21}$$

4) $\theta_x = \theta_0 = 0$.

$$y(x) = ax \ + \frac{a(a-1)}{2}(\theta_1^2 - (\theta_\infty - 1)^2 - 1)x^2 \ + \sum_{n=3}^{\infty} b_n(a; \theta_1, \theta_\infty) x^n. \tag{22}$$

(a) (PVI) has always one or both solutions (20) when $\theta_0 \pm \theta_x \notin \mathbf{Z}$. Also when $\theta_0 + \theta_x$ (or $\theta_0 - \theta_x$) is integer, (PVI) has a solution (20) corresponding to $\theta_0 - \theta_x$ not integer (or $\theta_0 + \theta_x$ not integer). **(b)** When $\theta_0 + \theta_x$ or $\theta_0 - \theta_x$ is integer, (PVI) has a 1-parameter family of solutions equivalent (by birational transformations) to (21); this family exists provided that $\theta_1 \pm \theta_\infty$ has a particular integer value. **(c)** When $\theta_0 + \theta_x$ or $\theta_0 - \theta_x$ is integer and θ_0 has a particular integer value, there is a one parameter family of solutions equivalent to (22).

3) (PVI) has a one-parameter family of solutions of the type:

$$y(x) = y_0(x) + y_1(x) \ ax^\omega \ + y_2(x) \left(ax^\omega\right)^2 \ + ... \ = \sum_{N=0}^{\infty} y_N(x) \left(ax^\omega\right)^N, \quad x \to 0; \tag{23}$$

where the parameter is $a \in \mathbf{C}$, and the $y_N(x)$'s are Taylor series:

$$y_N(x) = \sum_{k=0}^{\infty} b_{k,N}(\theta_1, \theta_\infty, \theta_0, \theta_x) \ x^k, \qquad x \to 0.$$

Either $y_0(x)$ is (18) and $\omega = \pm(\theta_1 + \theta_\infty - 1)$, or $y_0(x)$ is (15) and $\omega = \pm(\theta_\infty - \theta_1 - 1)$. The conditions $|\Re\omega| < 1$, $\omega \neq 0$ hold. The coefficients $b_{k,N}(\theta_1, \theta_\infty, \theta_0, \theta_x)$ are certain rational functions that can be recursively determined by direct substitution into (PVI). These solutions are the immages of solutions (25) and (26) respectively, through the symmetry (19). Solutions (25) and (26) are a sub-case of theorem 2, obtained by the matching procedure.

Taylor solutions (15), (18) are a special case of (23), when the parameter is zero. Solutions (16) and (17) – and their images by symmetry – are one parameters families of type (23), in non generic cases when $\sigma \in \mathbf{Z}$.

Further study of one-parameter solutions, including non-generic cases when θ_ν and/or some sum of two θ_ν's are integer (including logarithmic one parameter families), will be presented in another paper devoted to the general classification problem.

4) Solutions (15) and the equivalent solutions (18), (20) were also derived in [17] by substitution of a Taylor expansion in (PVI). The corresponding monodromy was computed by coalescence of singularities of a Heun's type (scalar) equation.

3.2 Critical Behaviors: Result II

We now consider cases when (1) can be reduced to the fuchsian systems (10) and (11). Let σ be a complex number defined, up to sign, by:

$$\mathrm{tr} \ (M_0 M_x) = 2\cos(\pi\sigma), \qquad |\Re\sigma| \leq 1.$$

Actually, $\pm\sigma/2$ are the eigenvalues of $\lim_{x\to 0}(A_0 + A_x)$. The matching procedure yields the following result.

Theorem 2 *Let $r \in \mathbf{C}$ and σ be as above, with the restriction $|\Re\sigma| < 1$. (PVI) has a family of solutions depending on the two parameters r, σ. The leading terms of the critical behavior for $x \to 0$ may be parametrized as follows:*

For $\sigma \neq 0$:

$$y(x) \sim \begin{cases} \frac{1}{r} \frac{[\sigma^2 - (\theta_0 + \theta_x)^2][(\theta_0 - \theta_x)^2 - \sigma^2]}{16\sigma^3} x^{1-\sigma}, & \text{if } \Re\sigma > 0; \\[3mm] -\frac{r}{\sigma} x^{1+\sigma}, & \text{if } \Re\sigma < 0; \\[3mm] \frac{1}{r} \frac{[\sigma^2 - (\theta_0 + \theta_x)^2][(\theta_0 - \theta_x)^2 - \sigma^2]}{16\sigma^3} x^{1-\sigma} + \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{2\sigma^2} x - \frac{r}{\sigma} x^{1+\sigma}, & \text{if } \Re\sigma = 0. \end{cases} \quad (24)$$

In the above formulae, $r \neq 0$.

For special values of $\sigma \neq 0$:

$$y(x) = \frac{\theta_0}{\theta_0 + \theta_x} x \mp \frac{r}{\theta_0 + \theta_x} x^{1+\sigma}, \quad \sigma = \pm(\theta_0 + \theta_x) \neq 0, \quad (25)$$

$$y(x) = \frac{\theta_0}{\theta_0 - \theta_x} x \mp \frac{r}{\theta_0 - \theta_x} x^{1+\sigma}, \quad \sigma = \pm(\theta_0 - \theta_x) \neq 0. \quad (26)$$

For $\sigma = 0$:

$$y(x) \sim \begin{cases} x \left(\frac{\theta_x^2 - \theta_0^2}{4} \log^2 x - 2 \left(r + \frac{\theta_0}{2}\right) \log x + \frac{4 \; r(r+\theta_0)}{\theta_x^2 - \theta_0^2} \right), & \theta_0 \neq \pm\theta_x, \\[3mm] x \left(r \pm \theta_0 \ln x \right), & \theta_0 = \pm\theta_x. \end{cases} \quad (27)$$

Comments:

1) r can be computed as a function of the monodromy data. See (30) and comments there.

2) Sub-cases of theorem 2.

i) When $\sigma \neq 0$, the result of the theorem includes the sub-cases (25) and (26). If $r = 0$, $\theta_0 \neq 0$, $\theta_0 \pm \theta_x \notin \mathbf{Z}$, direct substitution into (PVI) gives the two Taylor expansions (20).

If $r \neq 0$, (25) and (26) are a 1-parameter family, with the restriction $|\Re\sigma| < 1$. The symmetry (19) transforms them into the solutions (23), the leading terms being respectively:

$$y(x) \sim \frac{\theta_\infty + \theta_1 - 1}{\theta_\infty - 1} \left(1 \pm \frac{r}{\theta_\infty - 1} x^\omega\right), \quad \omega = \pm(\theta_\infty + \theta_1 - 1) \neq 0,$$

$$y(x) \sim \frac{\theta_\infty - \theta_1 - 1}{\theta_\infty - 1} \left(1 \pm \frac{r}{\theta_\infty - 1} x^\omega\right), \quad \omega = \pm(\theta_\infty - \theta_1 - 1) \neq 0,$$

with the restriction $|\Re\omega| < 1$.

ii) The case $\sigma = 0$ includes the sub-case $y(x) \sim rx$, which occurs for $\theta_0 = \theta_x$, $\theta_0 = 0$. By direct substitution in (PVI) we obtain a series:

$$y(x) = r \; x + \sum_{n=3}^{\infty} b_n(r, \theta_1, \theta_\infty) x^n, \quad \theta_0 = \theta_x = 0, \quad r \neq 0, 1.$$

This is again solution (22). Note that for $r = 0, 1$ we have the singular solutions $y = 0$, $y = 1$. Note also that the special sub-sub-case $\theta_0 = \theta_x = \theta_1 = 0$ has applications in the theory of *semi-simple Frobenius manifolds* of dimension three [6] [9].

3) Solutions (24) were studied in [15]. Their existence was proved by assuming that the matrices A_0, A_x, A_1 have a certain critical behavior for $x \to 0$, and proving that such matrices solve the Schlesinger equations. Then, the monodromy data were computed by coalescence of singularity. These solutions where further studied in [7], [8], [10], [4]. We show that these solutions can be obtained without any assumption by the matching procedure, together with the solutions (27), which do not appear in [15].

4) The class of solutions (24) was enlarged in [23] and [10], to the values $\sigma \in \mathbf{C}$, $\sigma \notin (-\infty, 0] \cup [1, +\infty)$. When $\Re\sigma \geq 1$ or $\Re\sigma \leq 0$, the critical behavior is like the first of (24), and it holds for $x \to 0$ in

a spiral-shaped domain in the universal covering of a punctured neighborhood of $x = 0$, along a paths joining a point x_0 to $x = 0$. Along special paths which approach the movable poles, these solution may have behavior $y(x) \sim \sin^{-2}\left(\frac{i\sigma}{2}\ln x + \varphi(x, r)\right)$, where $\varphi(x, r)$ is a phase depending on the parameter r. The transformation $\sigma \mapsto \pm\sigma + 2N$, $N \in \mathbf{Z}$, leaves the identity $\operatorname{tr}(M_0 M_x) = 2\cos(\pi\sigma)$ invariant. Its effect on the solutions is studied in [10]. As a result, one can reduce to the values $0 \leq \Re\sigma \leq 1$, $\sigma \neq 0, 1$. We cannot enter into more details here. The reader may find a synthetic description of these results in the review paper [11].

5) Solutions with expansion:

$$y(x) = x(A_1 + B_1 \ln x + C_1 \ln^2 x + D_1 \ln^3 x + ...) + x^2(A_2 + B_2 \ln x + ...) + ..., \qquad x \to 0.$$

are all included it theorems 1 and 2. Actually, only the following cases are possible:

$$y(x) = \begin{cases} \frac{\theta_0}{\theta_0 \pm \theta_x}x + O(x^2) \quad \text{[Taylor expansion]}, \\[2mm] x\left(\frac{\theta_0^2 - B_1^2}{\theta_0^2 - \theta_x^2} + B_1 \ln x + \frac{\theta_x^2 - \theta_0^2}{4}\ln^2 x\right) + x^2(...) + ..., \\[2mm] x\left(A_1 \pm \theta_0 \ln x\right) + x^2(...) + ..., \quad \text{and } \theta_0 = \pm\theta_x. \end{cases} \tag{28}$$

A_1 and B_1 are parameters.

6) The symmetry (19) applied to solutions (27) gives:

$$y(x) = \frac{4}{\left[\theta_1^2 - (\theta_\infty - 1)^2\right] \ln^2 x}\left[1 + \frac{8r + 4(\theta_\infty - 1)}{\theta_1^2 - (\theta_\infty - 1)^2}\frac{1}{\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \tag{29}$$

and

$$y(x) = \frac{\pm 1}{(\theta_\infty - 1)\ \ln x}\left[1 \mp \frac{r}{(\theta_\infty - 1)\ \ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \qquad \theta_\infty \mp \theta_1 = 1.$$

The higer orders $O(1/\ln^2 x)$ include powers $x^n(\ln x)^{\pm m}$. The so called *Chazy solutions*, studied in [19] for the special case $\theta_0 = \theta_x = \theta_1 = 0$, $\theta_\infty = -1$, have the behavior (29).

7) When this paper was completed, I received a communication by the first author of [5]. In [5] it is proved that (PVI) has solutions with expansion at $x = \infty$, or $x = 0$, of the form $y = c_r x^r + \sum_s c_s x^s$, $c_r \in \mathbf{C}$. The c_s's are either complex constants or polynomials in $\ln x$. r and s are integer or complex. If r is complex, the restriction $\Re r \in (0, 1)$ holds. The method used in [5] is a power geometry technique. The connection problem and the characterization of the associated monodromy data are not studied.

3.3 Monodromy: Result III

In this paper, we computed the monodromy for the Taylor-expanded solutions, which correspond to non-fuchsian reductions of system (1). Because of the symmetries of (PVI), we can limit ourselves to the monodromy data for the representative solutions (15), (16) and (17).

Theorem 3 *a) Let $\theta_\kappa \notin \mathbf{Z}$, $\kappa = 0, 1, x, \infty$. A representation for the monodromy matrices of the solution (15) is:*

$$M_0 = C_{0\infty}\ \exp\{i\pi\theta_0\sigma_3\}\ C_{0\infty}^{-1},$$

$$M_x = C_{0\infty}\ C_{01}^{-1}\ \exp\{i\pi\theta_x\sigma_3\}\ C_{01}\ C_{0\infty}^{-1}.$$

$$M_1 = \exp\{-i\pi\theta_1\sigma_3\}, \quad M_\infty = \exp\{-i\pi\theta_\infty\sigma_3\}.$$

The matrices $C_{0\infty}$ and C_{01} are (91) and (90). The subgroup generated by $M_0 M_x$ and M_1 is reducible. As for the solution (18), we just need to change $\theta_1 \mapsto -\theta_1$.

b) It is convenient to re-parameterize the solution (16) by introducing a parameter s through the equality:

$$a = \frac{\theta_\infty(2s + \theta_x + 1)}{2(\theta_\infty - 1)}.$$

Let θ_x, $\theta_\infty \notin \mathbf{Z}$. Then, a representation for the monodromy group is:

$$M_0 = G \ \exp\{i\pi\theta_x\sigma_3\} \ G^{-1}, \qquad M_1 = \exp\{-i\pi\theta_\infty\sigma_3\}$$

$$M_x = G \ \exp\{-i\pi\theta_x\sigma_3\} \ G^{-1}, \qquad M_\infty = \exp\{-i\pi\theta_\infty\sigma_3\}$$

In particular, $M_1 = M_\infty$, $M_0 M_x = I$. We can choose G as follows:

$$G = \begin{pmatrix} 1 & 1 \\ \frac{s+\theta_x}{r} & \frac{s}{r} \end{pmatrix}.$$

Conversely, we may express s as a funiction of the monodromy data:

$$s = \frac{\theta_x\big[2\cos(\pi(\theta_\infty + \theta_x)) - \operatorname{tr}(M_1 M_0)\big]}{2\big[\cos(\pi(\theta_\infty - \theta_x)) - \cos(\pi(\theta_\infty + \theta_x))\big]}.$$

c) We re-parameterize solution (17) introducing a new parameter s defined by $a =: (1-s)^{-1}$. Let θ_0, $\theta_x \notin \mathbf{Z}$. Then, a monodromy representation for the solutions (17) is:

$$M_0 = \left(C_{\infty 0}\right)^{-1} \ \exp\{i\pi\theta_0\sigma_3\} \ C_{\infty 0}, \qquad M_\infty = \begin{pmatrix} -1 & 0 \\ 2\pi i\,(1-s) & -1 \end{pmatrix}$$

$$M_x = \left(C_{\infty 0}\right)^{-1}\left(C_{01}\right)^{-1} \ \exp\{i\pi\theta_x\sigma_3\} \ C_{01}C_{\infty 0}, \qquad M_1 = \begin{pmatrix} 1 & 0 \\ 2\pi i s & 1 \end{pmatrix}.$$

where $C_{\infty 0}$ and C_{01} are (94) and (93).
Conversely, we may express s as a function of the monodromy data:

$$s = \frac{\operatorname{tr}(M_1 M_0) - 2\cos(\pi\theta_0)}{4\pi\sin(\pi\theta_0)} \frac{(C_{\infty 0})_{21}}{(C_{\infty 0})_{22}}.$$

The conditions $\theta_\kappa \notin \mathbf{Z}$ can be eliminated, and the computations can be repeated without conceptual changes, but with different results.

In the above theorem, the subgroups generated by $M_0 M_x$ and M_1 are *reducible*. This characterizes the monodromy associated to solutions which have a Taylor series at $x = 0$. The same characterization at $x = 1$ involves the subgroup generated by $M_1 M_x$ and M_0. At $x = \infty$, it involves the subgroup generated by $M_0 M_1$ and M_x. [2] In another paper, we will consider again this characterization, together with the general problem of classification.

Let us define again σ by $\operatorname{tr}(M_0 M_x) = 2\cos\pi\sigma$. Then, in case a), $\sigma = \pm(\theta_1 - \theta_\infty)$ [and $\pm(\theta_1 + \theta_\infty)$ for the change $\theta_1 \mapsto -\theta_1$]. In case b), $\operatorname{tr}(M_0 M_x) = 2$ and $\sigma = 0$. In case c), $\operatorname{tr}(M_0 M_x) = -2$, $\sigma = \pm 1$. The matching procedure is effective to produce solutions corresponding to monodromy data for which the connection problem is *so far* not well studied, such as the case $\operatorname{tr}(M_i M_j) = -2$ (see [11]).[3]

Note: Also the 1-parameter solutions (25) (26) and the second solution in (27) are characterized by a reducible subgroup generated by M_0, M_x.

Comments.

1) The monodromy group for the solutions (20) was derived also in [17], by confluence of singularities of scalar equations (including a Heun's type equation). The result is equivalent to that in point a) of the above theorem.

2) The computation of the monodromy group of the fuchsian systems (10) and (11) is quite clear [15] [7] [10] [4]. It allows to express the parameter r of (24), (25), (26) and (27) as a function of the

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[2] In the appendix of [10], the reader may find explanantions about how to obtain results at $x = 1, \infty$ from the results at $x = 0$

[3] Here I remark that the formula (1.30), page 1293, of my paper [10] is wrong. The correct one is $\operatorname{tr}(M_i M_j) \notin (-\infty, -2]$. In [10] the connection problem is solved for $\operatorname{tr}(M_i M_j) \neq \pm 2$. The case $\operatorname{tr}(M_i M_j) = 2$ yields (27). For the special choice of the parameters $\theta_0 = \theta_x = \theta_1 = 0$, it was studied in [7] and [8] (no logarithmic terms appear in such a special case). The result (27) for the general (PVI), corresponding to $\operatorname{tr}(M_0 M_x) = 2$, appears in the present paper for the first time.

monodromy data. Though the computation for (25), (26) and (27) does not appear in the literature, the procedure is clear (see section 4.8), so we do not repeat it. We just report the result for (24), which can be found in [15] [10] [4]:

$$r = \frac{(\theta_0 - \theta_x + \sigma)(\theta_0 + \theta_x - \sigma)(\theta_\infty + \theta_1 - \sigma)}{4\sigma(\theta_\infty + \theta_1 + \sigma)} \frac{1}{\mathbf{F}}, \tag{30}$$

where

$$\mathbf{F} := \frac{\Gamma(1+\sigma)^2 \Gamma\left(\frac{1}{2}(\theta_0 + \theta_x - \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_x - \theta_0 - \sigma) + 1\right)}{\Gamma(1-\sigma)^2 \Gamma\left(\frac{1}{2}(\theta_0 + \theta_x + \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_x - \theta_0 + \sigma) + 1\right)} \times$$

$$\times \frac{\Gamma\left(\frac{1}{2}(\theta_\infty + \theta_1 - \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_1 - \theta_\infty - \sigma) + 1\right)}{\Gamma\left(\frac{1}{2}(\theta_\infty + \theta_1 + \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_1 - \theta_\infty + \sigma) + 1\right)} \frac{V}{U},$$

and:

$$U := \left[\frac{i}{2}\sin(\pi\sigma)\mathrm{tr}(M_1 M_x) - \cos(\pi\theta_x)\cos(\pi\theta_\infty) - \cos(\pi\theta_0)\cos(\pi\theta_1)\right] e^{i\pi\sigma} +$$

$$+\frac{i}{2}\sin(\pi\sigma)\mathrm{tr}(M_0 M_1) + \cos(\pi\theta_x)\cos(\pi\theta_1) + \cos(\pi\theta_\infty)\cos(\pi\theta_0)$$

$$V := 4\sin\frac{\pi}{2}(\theta_0 + \theta_x - \sigma)\sin\frac{\pi}{2}(\theta_0 - \theta_x + \sigma) \ \sin\frac{\pi}{2}(\theta_\infty + \theta_1 - \sigma)\sin\frac{\pi}{2}(\theta_\infty - \theta_1 + \sigma).$$

The above formula was computed with the assumption that $\sigma \pm (\theta_0 + \theta_x)$, $\sigma \pm (\theta_0 - \theta_x)$, $\sigma \pm (\theta_1 + \theta_\infty)$, $\sigma \pm (\theta_1 - \theta_\infty)$ are not even integers. [4]

3) Reducible Monodromy. The monodromy groups in theorem 3 are not reducible, but they have a reducible subgroup. If the entire group itself is completely reducible, all the Painleveé transcendents are known. Solutions of (PVI) corresponding to a reducible monodromy were found in [12]). We summarize the results:

Proposition 1 *All the solutions of (PVI) corresponding to a reducible monodromy group are equivalent by birational canonical transformations to the following one-parameter family of solutions, with* $\theta_\infty + \theta_1 + \theta_0 + \theta_x = 0$:

$$y(x) = \frac{\theta_1 + \theta_\infty - 1 + x(1 + \theta_x)}{\theta_\infty - 1} - \frac{1}{\theta_\infty - 1}\frac{x(1-x)}{u(x;a)}\frac{du(x;a)}{dx}, \tag{31}$$

where $u(x;a) = u_1(x) + au_2(x)$; $a \in \mathbf{C}$, $u_1(x)$ *and* $u_2(x)$ *are linear independent solutions of the hyper-geometric equation:*

$$x(1-x)\frac{d^2u}{dx^2} + \{[2 - (\theta_\infty + \theta_1)] - (4 - \theta_\infty + \theta_x)x\}\frac{du}{dx} - (2 - \theta_\infty)(1 + \theta_x)u = 0$$

The monodromy matrices are

$$M_0 = \begin{pmatrix} \frac{\theta_0}{2} & * \\ 0 & -\frac{\theta_0}{2} \end{pmatrix}, \quad M_x = \begin{pmatrix} \frac{\theta_x}{2} & * \\ 0 & -\frac{\theta_x}{2} \end{pmatrix}, \quad M_1 = \begin{pmatrix} \frac{\theta_1}{2} & * \\ 0 & -\frac{\theta_1}{2} \end{pmatrix}.$$

The parameter a does not appear in the monodromy.

Remark: The *rational solutions* of (PVI) are a special case of the above proposition. They were studied in [20]. Up to canonical birational transformations, they are realized for $\theta_\infty + \theta_1 + \theta_0 + \theta_x = 0$ and:

$$\theta_0 = 1: \quad y(x) = \frac{\theta_\infty + \theta_1}{\theta_\infty} \frac{x-1}{x(1+\theta_1) - (\theta_1 + \theta_\infty)};$$

$$\theta_0 = -2: \quad y(x) = \frac{\left(2 - (\theta_\infty + \theta_1) + \theta_1 x\right)^2 - 2 + \theta_\infty + \theta_1 - \theta_1 x^2}{(1 - \theta_\infty)\left(2 - (\theta_\infty + \theta_1) + \theta_1 x\right)}.$$

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[4]In [10] there is a missprint in formula (A.30), which must be re-calculated. In [15], in formula (1.8) at the bottom of page 1141, the last sign is $\pm\sigma$ instead of $\mp\sigma$.

The computation of the expansion at $x = 0$ of (31) is just a consequence of the expansions of $u_1(x)$ and $u_2(x)$. The reader can find by himself a behavior $y \sim x(r(a) \pm \theta_x \ln(x))$ for $\theta_1 + \theta_\infty = \theta_0 + \theta_x = 0$, namely a sub-case of the second solution in (27). For $\theta_1 + \theta_\infty \notin \mathbf{Z}$, we find behaviors of the type (23) (and (15), (20) for $a = 0$).

PART II – Derivation Results: Fuchsian Reduction

4 Fuchsian Case

Let $x \to 0$. The reduction to the fuchsian systems (10) is possible if in the domain (4) we have:

$$|(A_0 + A_x)_{ij}| \gg \left|(A_x)_{ij} \frac{x}{\lambda}\right|, \quad \text{namely:} \quad |(A_0 + A_x)_{ij}| \gg \left|(A_x)_{ij} \, x^{1-\delta_{OUT}}\right|. \tag{32}$$

Let us denote with $\hat{A}_i$ the leading term of the matrix A_i, $i = 0, x, 1$. We can substitute (10) with:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{\hat{A}_0 + \hat{A}_x}{\lambda} + \frac{\hat{A}_1}{\lambda - 1} \right] \Psi_{OUT} \tag{33}$$

We suppose that $\theta_\infty \neq 0$. This is not a loss in generality, because $\theta_\infty = 0$ is equivalent to $\theta_\infty = 2$.

Lemma 1 *If the approximation (10) is possible, then $\hat{A}_0 + \hat{A}_x$ has eigenvalues $\pm \frac{\sigma}{2} \in \mathbf{C}$ independent of x, defined (up to sign and addition of an integer) by $\mathrm{tr}(M_x M_0) = 2\cos(\pi\sigma)$. Let $r_1 \in \mathbf{C}$, $r_1 \neq 0$. For $\theta_\infty \neq 0$, the leading terms are:*

$$\hat{A}_1 = \begin{pmatrix} \frac{\sigma^2 - \theta_\infty^2 - \theta_1^2}{4\theta_\infty} & -r_1 \\ \frac{[\sigma^2 - (\theta_1 - \theta_\infty)^2][\sigma^2 - (\theta_1 + \theta_\infty)^2]}{16\theta_\infty^2} \frac{1}{r_1} & -\frac{\sigma^2 - \theta_\infty^2 - \theta_1^2}{4\theta_\infty} \end{pmatrix}, \tag{34}$$

and

$$\hat{A}_0 + \hat{A}_x = \begin{pmatrix} \frac{\theta_1^2 - \sigma^2 - \theta_\infty^2}{4\theta_\infty} & r_1 \\ -\frac{[\sigma^2 - (\theta_1 - \theta_\infty)^2][\sigma^2 - (\theta_1 + \theta_\infty)^2]}{16\theta_\infty^2} \frac{1}{r_1} & -\frac{\theta_1^2 - \sigma^2 - \theta_\infty^2}{4\theta_\infty} \end{pmatrix}. \tag{35}$$

Proof: Observe that $\mathrm{tr}(\hat{A}_0 + \hat{A}_x) = \mathrm{tr}(A_0 + A_x) = 0$, thus, for any x, $\hat{A}_0 + \hat{A}_x$ has eigenvalues of opposite sign, that we denote $\pm\tilde{\sigma}(x)/2$. Then, we recall that x is a monodromy preserving deformation, therefore the monodromy matrices of (33) are independent of x. At $\lambda = 0, 1, \infty$ they are:

$$M_0^{OUT} = \begin{cases} M_x M_0 \\ M_0 M_x \end{cases}, \quad M_1^{OUT} = M_1, \quad M_\infty^{OUT} = M_\infty.$$

Thus, $\det(M_0^{OUT}) = 1$, because $\det(M_x) = \det(M_0) = 1$. Therefore, there exists a constant matrix D and a complex constant number σ such that:

$$D^{-1} M_0^{OUT} D = \begin{cases} \mathrm{diag}(\exp\{-i\pi\sigma\}, \exp\{i\pi\sigma\}), \\ \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}, \text{ or } \begin{pmatrix} \pm 1 & 0 \\ * & \pm 1 \end{pmatrix}, \quad \sigma \in \mathbf{Z} \end{cases}$$

We conclude that $\tilde{\sigma}(x) \equiv \sigma$. We also have $\mathrm{tr}(M_0^{OUT}) = 2\cos(\pi\sigma)$.

Now consider the gauge:

$$\Phi_1 := \lambda^{-\frac{\sigma}{2}}(\lambda - 1)^{-\frac{\theta_1}{2}} \Psi_{OUT}. \qquad \frac{d\Phi_1}{d\lambda} = \left[\frac{\hat{A}_0 + \hat{A}_x - \frac{\sigma}{2}}{\lambda} + \frac{\hat{A}_1 - \frac{\theta_1}{2}}{\lambda - 1} \right] \Phi_1 \tag{36}$$

We can identify $\hat{A}_0 + \hat{A}_x - \frac{\sigma}{2}$ and $\hat{A}_1 - \frac{\theta_1}{2}$ with B_0 and B_1 of Proposition 2 in Appendix 1, case (95), with $a = \frac{\theta_\infty}{2} + \frac{\theta_1}{2} + \frac{\sigma}{2}$, $b = -\frac{\theta_\infty}{2} + \frac{\theta_1}{2} + \frac{\sigma}{2}$, $c = \sigma$. $\square$

Remark: r_1 may be a function of x. If the monodromy of system (33) depends on r_1, then r_1 is a constant independent of x. This is the case here, but we do not prove it for reasons of space. See the references in Part I.

Lemma 1 (and Lemma 3 which follows) includes all cases (95)—(99) for system (36). Cases (96)—(99) are obtained substituting $\sigma = -(\theta_\infty + \theta_1)$, $\theta_\infty - \theta_1$, $\theta_\infty + \theta_1$, $\theta_1 - \theta_\infty$ respectively. For all the computations which follow, involving system (33) or (36), we note that the hypothesis $\theta_\infty \neq 0$ excludes cases (100), (101) and the Jordan cases (102)–(104).

The reduction to the fuchsian system (11) is possible for $x \to 0$ in the domain (6) if:

$$\left| \frac{(A_0)_{ij}}{\lambda} + \frac{(A_x)_{ij}}{\lambda - x} \right| \gg |(A_1)_{ij}|, \quad \text{namely:} \quad \left| \frac{(A_0 + A_x)_{ij}}{x^{\delta_{IN}}} \right| \gg |(A_1)_{ij}|. \tag{37}$$

We can rewrite (11) using just the leading terms of the matrices:

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{\hat{A}_0}{\lambda} + \frac{\hat{A}_x}{\lambda - x} \right] \Psi_{IN}, \tag{38}$$

Then, we re-scale λ and consider the following system:

$$\frac{d\Psi_{IN}}{d\mu} = \left(\frac{\hat{A}_0}{\mu} + \frac{\hat{A}_x}{\mu - 1} \right) \Psi_{IN}, \quad \mu := \frac{\lambda}{x}$$

We know that there exists a matrix $K_0(x)$ such that:

$$K_0{}^{-1}(x) \, (\hat{A}_0 + \hat{A}_x) \, K_0(x) = \begin{pmatrix} \frac{\sigma}{2} & 0 \\ 0 & -\frac{\sigma}{2} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $\hat{\hat{A}}_i := K_0{}^{-1} \hat{A}_i K_0$, $i = 0, x$. By a gauge transformation, we get the system:

$$\Psi_{IN} =: K_0(x) \, \Psi_0, \qquad \frac{d\Psi_0}{d\mu} = \left[\frac{\hat{\hat{A}}_0}{\mu} + \frac{\hat{\hat{A}}_x}{\mu - 1} \right] \Psi_0, \tag{39}$$

Lemma 2 *Let $r \in \mathbf{C}$, $r \neq 0$. If $\sigma \neq 0$, we have:*

$$\hat{\hat{A}}_0 = \begin{pmatrix} \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{4\sigma} & r \\ -\frac{[\sigma^2 - (\theta_0 - \theta_x)^2][\sigma^2 - (\theta_0 + \theta_x)^2]}{16\sigma^2} \frac{1}{r_0} & -\frac{\theta_0^2 - \theta_x^2 + \sigma^2}{4\sigma} \end{pmatrix}, \tag{40}$$

$$\hat{\hat{A}}_x = \begin{pmatrix} \frac{\sigma^2 + \theta_x^2 - \theta_0^2}{4\sigma} & -r \\ \frac{[\sigma^2 - (\theta_0 - \theta_x)^2][\sigma^2 - (\theta_0 + \theta_x)^2]}{16\sigma^2} \frac{1}{r} & -\frac{\sigma^2 + \theta_x^2 - \theta_0^2}{4\sigma} \end{pmatrix}. \tag{41}$$

Proof: We do a gauge transformation:

$$\Phi_0 := \mu^{-\frac{\theta_0}{2}} (\mu - 1)^{-\frac{\theta_x}{2}} \, \Psi_0, \qquad \frac{d\Phi_0}{d\mu} = \left[\frac{\hat{\hat{A}}_0 - \frac{\theta_0}{2}}{\mu} + \frac{\hat{\hat{A}}_x - \frac{\theta_x}{2}}{\mu - 1} \right] \Phi_0. \tag{42}$$

We identify $\hat{\hat{A}}_0 - \frac{\theta_0}{2}$, $\hat{\hat{A}}_x - \frac{\theta_x}{2}$ with B_0 and B_1 in the Appendix 1, Proposition 2, case (95), with $a = \frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{\sigma}{2}$, $b = \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\sigma}{2}$, $c = \theta_0$. $\square$

Remark: If the monodromy of the system (39) depends on r, then r is a constant independent of x. This is the case here.

The Lemma 2 (and lemma 4 which follows) includes also the cases (96)–(99) for the system (42). These cases correspond respectively to the values $\sigma = \theta_0 + \theta_x$, $-\theta_0 - \theta_x$, $\theta_x - \theta_0$, $\theta_0 - \theta_x$, with $\theta_0 \neq \pm \theta_x$.

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4.1 Matching for $\sigma \notin \mathbf{Z}$ and proof of (24)

We match Ψ_{OUT} and Ψ_{IN} in the intersection of the "outside" and "inside" domains, namely the region $|x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}$, $x \to 0$. As a consequence, we obtain the leading term of $y(x)$.

Lemma 3 *If $\sigma \notin \mathbf{Z}$ and $\theta_\infty \neq 0$, system (33) has a fundamental matrix solution $\Psi_{OUT}(\lambda)$ with the following behavior at $\lambda = 0$:*

$$\Psi_{OUT} = \sum_{n=0}^\infty G_n \lambda^n \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix}, \qquad G_0 = \begin{pmatrix} 1 & 1 \\ \frac{(\theta_\infty+\sigma)^2-\theta_1^2}{4\theta_\infty r_1} & \frac{(\theta_\infty-\sigma)^2-\theta_1^2}{4\theta_\infty r_1} \end{pmatrix}.$$

G_n are matrices which depend rationally on θ_∞, θ_1, σ, r_1. The series is convergent for $|\lambda| < 1$.

Proof: It is an immediate consequence of the standard theory of linear systems of fuchsian differential equations. $\square$

Lemma 4 *If $\sigma \notin \mathbf{Z}$, system (39) has a fundamental matrix solution with the following behavior at $\mu = \infty$:*

$$\Psi_0(\mu) = \left[I + \sum_{n=1}^\infty K_n \mu^{-n} \right] \begin{pmatrix} \mu^{\frac{\sigma}{2}} & 0 \\ 0 & \mu^{-\frac{\sigma}{2}} \end{pmatrix},$$

where I is the identity matrix, K_n are matrices which depend rationally on θ_0, θ_x, σ, r. The series is convergent for $|\mu| > 1$.

Proof: It is a consequence of the standard theory of systems of fuchsian equations. $\square$

The matching relation $\Psi_1(\lambda) \sim K_0(x)\Psi_0(\lambda/x)$, $|x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}$, $x \to 0$, is:

$$G_0 \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix} \sim K_0(x) \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix} \begin{pmatrix} x^{-\frac{\sigma}{2}} & 0 \\ 0 & x^{\frac{\sigma}{2}} \end{pmatrix}.$$

This gives the result:

$$K_0(x) \sim \begin{pmatrix} 1 & 1 \\ \frac{(\theta_\infty+\sigma)^2-\theta_1^2}{4\theta_\infty r_1} & \frac{(\theta_\infty-\sigma)^2-\theta_1^2}{4\theta_\infty r_1} \end{pmatrix} \begin{pmatrix} x^{\frac{\sigma}{2}} & 0 \\ 0 & x^{-\frac{\sigma}{2}} \end{pmatrix}.$$

We compute the matrices $\hat{A}_0(x) = K_0(x)\hat{\hat{A}}_0 K_0(x)^{-1}$, $\hat{A}_x(x) = K_0(x)\hat{\hat{A}}_x K_0(x)^{-1}$ making use of Lemma 2. We obtain:

$$\hat{A}_0(x) = G_0 \begin{pmatrix} \frac{\theta_0^2-\theta_x^2+\sigma^2}{4\sigma} & r\,x^\sigma \\ -\frac{(\sigma+\theta_x-\theta_0)(\sigma+\theta_x+\theta_0)(\sigma-\theta_x+\theta_0)(\sigma-\theta_x-\theta_0)}{16\sigma^2 r} x^{-\sigma} & -\frac{\theta_0^2-\theta_x^2+\sigma^2}{4\sigma} \end{pmatrix} G_0^{-1}$$

$$\hat{A}_x(x) = G_0 \begin{pmatrix} \frac{\sigma^2+\theta_x^2-\theta_0^2}{4\sigma} & -r\,x^\sigma \\ \frac{(\sigma+\theta_x-\theta_0)(\sigma+\theta_x+\theta_0)(\sigma-\theta_x+\theta_0)(\sigma-\theta_x-\theta_0)}{16\sigma^2 r} x^{-\sigma} & -\frac{\sigma^2+\theta_x^2-\theta_0^2}{4\sigma} \end{pmatrix} G_0^{-1}.$$

This result shows that the matrix elements of $\hat{A}_0$ and $\hat{A}_x$ diverge as $|x|^{-|\Re\sigma|}$ when $x \to 0$ inside a sector (i.e. for $|\arg(x)|$ bounded). In particular, we find $(\hat{A}_1)_{12} = -r_1$ and

$$(\hat{A}_0)_{12} = \frac{r_1}{r} \frac{[\sigma^2-(\theta_0+\theta_x)^2][(\theta_0-\theta_x)^2-\sigma^2]}{16\sigma^3} x^{-\sigma} + r_1 \frac{\theta_0^2-\theta_x^2+\sigma^2}{2\sigma^2} - \frac{rr_1}{\sigma} x^\sigma.$$

The above are enough to compute the leading term(s) of $y(x)$ from the formula:

$$y(x) = \frac{x(A_0)_{12}}{x[(A_0)_{12} + (A_1)_{12}] - (A_1)_{12}} = -\frac{x(A_0)_{12}}{(A_1)_{12}} \left[1 - x \left(1 + \frac{(A_0)_{12}}{(A_1)_{12}} \right) \right]^{-1} \tag{43}$$

Thus:

$$y(x) \sim -x \frac{(\hat{A}_0)_{12}}{(\hat{A}_1)_{12}} = \left[\frac{1}{r} \frac{[\sigma^2-(\theta_0+\theta_x)^2][(\theta_0-\theta_x)^2-\sigma^2]}{16\sigma^3} x^{1-\sigma} + \frac{\theta_0^2-\theta_x^2+\sigma^2}{2\sigma^2} x - \frac{r}{\sigma} x^{1+\sigma} \right]. \tag{44}$$

We have ignored $1/[1 - x(1 + (\hat{A}_0)_{12}/(\hat{A}_1)_{12})]$ because condition (32) is equivalent to: $\left|x^{1-\delta_1\pm\sigma}\right| \to 0$ for $x \to 0$, which implies that $\left|x\,(\hat{A}_0)_{12}/(\hat{A}_1)_{12}\right| \sim \left|x^{1\pm\sigma}\right| \to 0$. Therefore, $1/[1 - x(1 + (\hat{A}_0)_{12}/(\hat{A}_1)_{12})] = 1/(1 + O(x)) = 1 + O(x)$.

If $\Re\sigma \neq 0$, the leading term of (44) is certainly correct, but some higher order corrections may be bigger than the next two terms of (44). If $\Re\sigma = 0$, the three terms of (44) are of the same order.

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4.2 Range of σ

Conditions (32) and (37) must be verified. Let C denote a non zero constant. We suppose that $x \to 0$ inside a sector with center on $x = 0$. Then:

$$\text{Condition (37) is } |x|^{-\delta_{IN}} \gg C \iff \delta_{IN} > 0.$$

$$\text{Condition (32) is } C \gg |x|^{-|\Re\sigma|+1-\delta_{OUT}} \iff |\Re\sigma| < 1 - \delta_{OUT}.$$

The last condition implies that $|\Re\sigma| < 1$. We also conclude that $0 < \delta_{IN} \leq \delta_{OUT} < 1$.

4.3 Leading term for $\sigma = \pm(\theta_0 + \theta_x), \pm(\theta_0 - \theta_x) \neq 0$. Proof of (25) and (26)

Formula (44) holds for any $\sigma \neq 0$ such that $|\Re\sigma| < 1$. However, we cannot naively substitute the value of $\sigma = \pm(\theta_0 + \theta_x), \pm(\theta_0 - \theta_x)$, for which the coefficient of $x^{1-\sigma}$ vanishes. This is because only the leading term is certainly correct, and it may be the term in $x^{1-\sigma}$. Therefore, here we briefly give the explicit derivation of (25) and (26), using cases (96)–(99) for system (42).

Case (96), $a = 0$: This is the case $\sigma = \theta_0 + \theta_x \neq 0$. The matching procedure does not change. From (96) we compute:

$$\hat{A}_0 = G_0 \begin{pmatrix} \frac{\theta_0}{2} & r\, x^\sigma \\ 0 & -\frac{\theta_0}{2} \end{pmatrix} G_0{}^{-1}, \quad \hat{A}_x = G_0 \begin{pmatrix} \frac{\theta_x}{2} & -r\, x^\sigma \\ 0 & -\frac{\theta_x}{2} \end{pmatrix} G_0{}^{-1}.$$

This implies that $(\hat{A}_0)_{12} = r_1 \left(\frac{\theta_0}{\theta_0+\theta_x} - \frac{r}{\theta_0+\theta_x} x^\sigma \right)$, while $(\hat{A}_1)_{12} = -r_1$ as in the generic case. Therefore,

$$y(x) \sim \frac{\theta_0}{\theta_0 + \theta_x}\, x \; - \; \frac{r}{\theta_0 + \theta_x}\, x^{\sigma+1}.$$

It is interesting to note that for $\Re\sigma > 0$ we have $\hat{y}(x) \sim \theta_0/(\theta_0 + \theta_x)\, x$. Such a behavior is what one would naively expect from the generic behavior (44) when $\sigma = 0$.

Case (97), $b = 0$, is $\sigma = -\theta_0 - \theta_x \neq 0$. Case (98), $a = c$, is $\sigma = \theta_x - \theta_0$. Case (99), $b = c$, is $\sigma = \theta_0 - \theta_x$. Proceeding as above, we find (25) and (26).

Remark: If we substitute $y = b_1 x + b_2 x^2 + b_3 x^3 + ...$ into (PVI) we find all the coefficients b_n by identifying equal powers of x. The result is (20). We need to assume that $\theta_0 \pm \theta_x$ is not integer or zero.

4.4 Matching for $\sigma = 0$. Proof of (27)

4.5 Case $\theta_0 \pm \theta_x \neq 0$

Lemma 5 *Let $r_1 \in \mathbf{C}$, $r_1 \neq 0$. The matrices of system (33) are:*

$$\hat{A}_1 = \begin{pmatrix} -\frac{\theta_\infty{}^2+\theta_1{}^2}{4\theta_\infty} & -r_1 \\ \frac{[\theta_1^2-\theta_\infty^2]^2}{16\theta_\infty^2 r_1} & \frac{\theta_\infty{}^2+\theta_1{}^2}{4\theta_\infty} \end{pmatrix}, \quad \hat{A}_0 + \hat{A}_x = \begin{pmatrix} \frac{\theta_1^2-\theta_\infty^2}{4\theta_\infty} & r_1 \\ -\frac{[\theta_\infty^2-\theta_1^2]^2}{16\theta_\infty^2 r_1} & \frac{\theta_\infty^2-\theta_1^2}{4\theta_\infty} \end{pmatrix}, \quad \forall r_1 \neq 0.$$

A fundamental matrix solution can be chosen with the following behavior at $\lambda = 0$:

$$\Psi_{OUT}(\lambda) = [G_0 + O(\lambda)] \begin{pmatrix} 1 & \log\lambda \\ 0 & 1 \end{pmatrix}, \qquad G_0 = \begin{pmatrix} 1 & 0 \\ \frac{\theta_\infty{}^2-\theta_1{}^2}{4\theta_\infty\, r_1} & \frac{1}{r_1} \end{pmatrix}.$$

Proof: The system (36) is:

$$\frac{d\Phi_1}{d\lambda} = \left[\frac{\hat{A}_0 + \hat{A}_x}{\lambda} + \frac{\hat{A}_1 - \frac{\theta_1}{2}}{\lambda - 1} \right] \Phi_1,$$

We identify $\hat{A}_0 + \hat{A}_x$ and $\hat{A}_1 - \frac{\theta_1}{2}$ with B_0 and B_1 of proposition 2 in Appendix 1, diagonalizable case (95)–(99) (we recall that (100)–(104) never occur when $\theta_\infty \neq 0$) with $a = \frac{\theta_\infty}{2} + \frac{\theta_1}{2}$, $b = -\frac{\theta_\infty}{2} + \frac{\theta_1}{2}$, $c = 0$.

The behavior of a fundamental solution is a standard result in the theory of Fuchsian systems. The matrix G_0 is defined by $G_0{}^{-1} \left(\hat{A}_0 + \hat{A}_x \right) G_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $\square$

Lemma 6 *Let $r \in \mathbf{C}$. The matrices of system (39) are:*

$$\hat{\hat{A}}_0 = \begin{pmatrix} r + \frac{\theta_0}{2} & \frac{4\ r\ (r+\theta_0)}{\theta_x^2 - \theta_0^2} \\ \frac{\theta_0^2 - \theta_x^2}{4} & -r - \frac{\theta_0}{2} \end{pmatrix}, \qquad \hat{\hat{A}}_x = \begin{pmatrix} -r - \frac{\theta_0}{2} & 1 - \frac{4\ r\ (r+\theta_0)}{\theta_x^2 - \theta_0^2} \\ \frac{\theta_x^2 - \theta_0^2}{4} & r + \frac{\theta_0}{2}. \end{pmatrix}. \tag{45}$$

There exist a fundamental solution of (39) with the following behavior at $\mu = \infty$:

$$\Psi_0(\mu) = \left[I + O\left(\frac{1}{\mu}\right) \right] \begin{pmatrix} 1 & \log \mu \\ 0 & 1 \end{pmatrix}, \quad \mu \to \infty.$$

Proof: To compute $\hat{\hat{A}}_0$ and $\hat{\hat{A}}_x$ for the generic case, we consider the case (102) in Proposition 2, applied to the system (42). The parameters are $a = \frac{\theta_0}{2} + \frac{\theta_x}{2}$, $c = \theta_0$. In particular,

$$\hat{\hat{A}}_0 - \frac{\theta_0}{2} + \hat{\hat{A}}_x - \frac{\theta_x}{2} = \begin{pmatrix} -\frac{\theta_0 + \theta_x}{2} & 1 \\ 0 & -\frac{\theta_0 + \theta_x}{2} \end{pmatrix} \tag{46}$$

Here the values of the parameters satisfy the conditions $a \neq 0$ and $a \neq c$, namely $\theta_0 \pm \theta_x \neq 0$. From the matrices (102), we obtain $\hat{\hat{A}}_0 = B_0 + \theta_0/2$ and $\hat{\hat{A}}_x = B_1 + \theta_x/2$. Keeping into account (46), by the standard theory of fuchsian systems we have:

$$\Phi_0(\mu) = \left[I + O\left(\frac{1}{\mu}\right) \right] \mu^{-\frac{\theta_0 + \theta_x}{2}} \begin{pmatrix} 1 & \log \mu \\ 0 & 1 \end{pmatrix}, \quad \mu \to \infty.$$

This proves the behavior of $\Psi_0(\mu)$. $\square$

The matching condition $\Psi_{OUT}(\lambda) \sim K_0(x)\ \Psi_0(\lambda/x)$ becomes:

$$K_0(x) \begin{pmatrix} 1 & \log\left(\frac{\lambda}{x}\right) \\ 0 & 1 \end{pmatrix} \sim G_0 \begin{pmatrix} 1 & \log \lambda \\ 0 & 1 \end{pmatrix} \implies K_0(x) \sim \begin{pmatrix} \frac{1}{\theta_\infty^2 - \theta_1^2} & 0 \\ \frac{\theta_\infty^2 - \theta_1^2}{4\ \theta_\infty\ r_1} & \frac{1}{r_1} \end{pmatrix} \begin{pmatrix} 1 & \log x \\ 0 & 1 \end{pmatrix}.$$

From the above result, together with (45), we compute $\hat{A}_0 = K_0 \hat{\hat{A}}_0 K_0^{-1}$, $\hat{A}_1 = K_0 \hat{\hat{A}}_1 K_0^{-1}$. For example,

$$\hat{A}_0 = G_0 \begin{pmatrix} r + \frac{\theta_0}{2} + \frac{\theta_0^2 - \theta_x^2}{4} \log x & \frac{\theta_x^2 - \theta_0^2}{4} \log^2 x\ -2\left(r + \frac{\theta_0}{2}\right)\log x\ + \frac{4\ r(r+\theta_0)}{\theta_x^2 - \theta_0^2} \\ \frac{\theta_0^2 - \theta_x^2}{4} & \frac{\theta_x^2 - \theta_0^2}{4} \log x - \left(r + \frac{\theta_0}{2}\right) \end{pmatrix} G_0^{-1}.$$

A similar expression holds for $\hat{A}_x$. The reader can verify that the matching conditions (32), (37) are satisfied.

The leading terms of $y(x)$ are obtained from (43) with matrix entries $(\hat{A}_1)_{12} = -r_1$ and:

$$(\hat{A}_0)_{12} = r_1 \left[\frac{\theta_x^2 - \theta_0^2}{4} \log^2 x - 2\left(r + \frac{\theta_0}{2}\right) \log x + \frac{4\ r(r+\theta_0)}{\theta_x^2 - \theta_0^2} \right].$$

The result is:

$$y(x) \sim x \left[\frac{\theta_x^2 - \theta_0^2}{4} \log^2 x - 2\left(r + \frac{\theta_0}{2}\right) \log x + \frac{4\ r(r+\theta_0)}{\theta_x^2 - \theta_0^2} \right]. \tag{47}$$

4.6   Case $\theta_0 \pm \theta_x = 0$

We consider here the cases (103), (104) of Proposition 2 applied to the system (42).

Case (103) is the case $\sigma = 0$, $\theta_0 = -\theta_x$, with $a = 0$, $c = \theta_0$ in the system (42). From Proposition 2 we immediately have:

$$\hat{\hat{A}}_0 = \begin{pmatrix} \frac{\theta_0}{2} & r \\ 0 & -\frac{\theta_0}{2} \end{pmatrix}, \quad \hat{\hat{A}}_x = \begin{pmatrix} \frac{\theta_x}{2} & 1 - r \\ 0 & -\frac{\theta_x}{2} \end{pmatrix}.$$

The behavior of Ψ_0 and Ψ_{OUT}, and the matching are the same of subsection 4.5. We obtain the same $K_0(x)$. Therefore:

$$(\hat{A}_0)_{12} = r_1\ (r - \theta_0\ \ln x), \quad (\hat{A}_1)_{12} = -r_1.$$

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This gives the leading terms:

$$y(x) \sim x(r - \theta_0 \ln x) = x(r + \theta_x \ln x). \tag{48}$$

In the same way, we treat the other cases. *Case (103)* with $a = c$, is the case $\sigma = 0$, $\theta_0 = \theta_x$. As above, we find $y(x) \sim x(r - \theta_0 \ln x) = x(r - \theta_x \ln x)$. *Case (104)* with $a = 0$, is the case $\sigma = 0$, $\theta_0 = -\theta_x$. We find $y(x) \sim x(r + \theta_0 \ln x) = x(r - \theta_x \ln x)$. *Case (104)* with $a = c$, is the case $\sigma = 0$, $\theta_0 = \theta_x$. We find $y(x) \sim x(r + \theta_0 \ln x) = x(r + \theta_x \ln x)$.

Both (47) and (48) contain more than one term, and in principle only the leading one is certainly correct. To prove that they are all correct, we observe that (47) and (48) can be obtained also by direct substitution of $y(x) = x(A_1 + B_1 \ln x + C_1 \ln^2 x + D_1 \ln^3 x + ...) + x^2(A_2 + B_2 \ln x + ...) + ...$ into (PVI). We can recursively determine the coefficients by identifying the same powers of x and $\ln x$. As a result we obtain only the five cases (28), which include (47) and (48).

4.7 No Naive Matching for $\sigma = 1$

The condition $|\Re\sigma| < 1$ suggests that the matching above does not work in the case $\sigma = 1$ (and $\sigma = -1$, being equivalent). Let us convince ourselves of this fact by repeating the procedure above. A fundamental matrix solution for (33) at $\lambda = 0$ is non-generic:

$$\Psi_{OUT}(\lambda) = (G_0 + O(\lambda)) \begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \log\lambda \\ 0 & 1 \end{pmatrix}. \tag{49}$$

where:

$$G_0 = \begin{pmatrix} 1 & \frac{4}{\theta_1^2 - (\theta_\infty - 1)^2} \\ \frac{(\theta_\infty + 1)^2 - \theta_1^2}{4\theta_\infty r_1} & -\frac{1}{\theta_\infty r_1} \end{pmatrix}, \qquad \forall r_1 \neq 0.$$

A fundamental matrix solution of (39) at $\mu = \infty$ is non-generic:

$$\Psi_0(\mu) = \left(I + O\left(\frac{1}{\mu}\right)\right) \begin{pmatrix} \mu^{\frac{1}{2}} & 0 \\ 0 & \mu^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R \log\mu & 1 \end{pmatrix}, \tag{50}$$

where:

$$R := (\hat{\hat{A}}_x)_{21} = \frac{[(\theta_0 + \theta_x)^2 - 1][(\theta_0 - \theta_x)^2 - 1]}{16\,r}, \qquad r \neq 0.$$

The matching relation:

$$K_0(x) \begin{pmatrix} \left(\frac{\lambda}{x}\right)^{\frac{1}{2}} & 0 \\ R\left(\frac{\lambda}{x}\right)^{-\frac{1}{2}}\log\left(\frac{\lambda}{x}\right) & \left(\frac{\lambda}{x}\right)^{-\frac{1}{2}} \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{4}{\theta_1^2 - (\theta_\infty - 1)^2} \\ \frac{(\theta_\infty + 1)^2 - \theta_1^2}{4\theta_\infty r_1} & -\frac{1}{\theta_\infty r_1} \end{pmatrix} \begin{pmatrix} \lambda^{\frac{1}{2}} & \lambda^{\frac{1}{2}}\log\lambda \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix},$$

shows that we cannot eliminate λ to obtain $K_0(x)$.

One case $\sigma = 1$ is studied in Part III, making use of a non-fuchsian reductions of the system (1).

4.8 Monodromy Data

Systems (33) (39) are equivalent to Gauss hyper-geometric equations, as it is explained in Appendix 1 (make use of the systems (36) and (42) respectively). Therefore, the monodromy can be computed in a standard way, using the connection formulae for the hyper-geometric functions.

We obtain in this way the monodromy of Ψ_{OUT} and Ψ_{IN}. As it is explained in section 2.2, it may be necessary to do a transformation $\Psi_{OUT} \mapsto \Psi_{OUT}^{Match} := \Psi_{OUT}C_{OUT}$, in order to match the "out" and "in" solutions with a solution Ψ of (1). In this way, the monodromy matrices M_0, M_x, M_1 of Ψ can be obtained. They depend on r. We then compute the traces of M_iM_j and extract r, which is thus obtained as a function of the monodromy data.

We do not repeat the computations here. One example is the computation of (30) in [15] and [7] [8] [10] [4].

PART III – Derivation of Results for the Non-fuchsian Reduction

5 Case $\theta_1 = \pm\theta_\infty$, $\theta_x = \pm\theta_0$, $\lim_{x\to 0}(A_x + A_0) = 0$. Solution (16)

We begin by observing that for $\theta_x = \pm\theta_0$, system (42) may fall in cases (100) and (101). If it is so, then $\hat{A}_0 + \hat{A}_x = 0$, and therefore $\hat{A}_0 + \hat{A}_x = 0$. More precisely, we start from the following hypotheses:

$$\lim_{x\to 0}\big(A_0(x) + A_x(x)\big) = 0,$$

$$A := \lim_{x\to 0} A_x(x) = \text{ a constant matrix with eigenvalues } \pm\frac{\theta_x}{2}$$

The first hypothesis means that we can write (the trace is zero):

$$A_0 + A_x = \begin{pmatrix} a(x) & b(x)\, r \\ c(x)\,\frac{1}{r} & -a(x) \end{pmatrix}, \quad \lim_{x\to 0} a(x) = \lim_{x\to 0} b(x) = \lim_{x\to 0} c(x) = 0,$$

The second hypotheses implies that the general form of A is:

$$A = \begin{pmatrix} s + \frac{\theta_x}{2} & -r \\ \frac{(s+\theta_x)\,s}{r} & -s - \frac{\theta_x}{2} \end{pmatrix}, \quad r, s \in \mathbf{C}, \quad r \neq 0.$$

We also write:

$$A_x(x) - A =: \Delta_x(x), \quad A_0 + A =: \Delta_0(x), \qquad \Delta_0 + \Delta_x \equiv A_0 + A_x.$$

$\Delta_x(x)$ and $\Delta_0(x)$ are vanishing. We suppose that the slowest vanishing behavior be of order x^{σ_0}, for some $\sigma_0 > 0$. Namely:

$$a(x),\ b(x),\ c(x),\ (\Delta_x)_{ij}(x),\ (\Delta_0)_{ij}(x)\ = O(x^{\sigma_0}), \quad \sigma_0 > 0.$$

Finally, we have:

$$A_1(x) = -\frac{\theta_\infty}{2}\,\sigma_3\ - (A_0 + A_x)\ \longrightarrow\ -\frac{\theta_\infty}{2}\,\sigma_3, \quad x \to 0.$$

5.1 Coalescence of Singularities

1) THE SYSTEM for Ψ_{OUT}.

We consider system (8), in the domain $|\lambda| \geq |x|^{\delta_{OUT}}$. Let us determine the conditions to neglect a term $x^n A_x / \lambda^{n+1}$ – and all the terms following it – with respect to $(A_0 + A_x)/\lambda$, when $x \to 0$, $\lambda \sim x^\delta$, $\delta \leq \delta_{OUT}$.

$$\text{We can neglect } \frac{x^n A_x}{\lambda^{n+1}} \iff \left|\frac{x^n}{\lambda^n}A_x\right| \ll \left|(A_0 + A_x)_{ij}\right|, \quad \forall i, j \in \{1, 2\}.$$

Since $\lim_{x\to 0}(A_x)_{ij}$ are non-zero constants, the above condition is: $|x|^{n-n\delta} \ll |x|^{\sigma_0}$, namely: $\delta < 1 - \sigma_0/n$. We state this result as a lemma.

Lemma 7 *Let $N_{OUT} \geq 2$ be an integer. We can approximate (8) with:*

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{(A_0 + A_x)}{\lambda} + \frac{A_x}{\lambda}\sum_{n=1}^{N_{OUT}-1}\left(\frac{x}{\lambda}\right)^n + \frac{A_1}{\lambda - 1}\right]\Psi_{OUT}.$$

if and only if:

$$\delta_{OUT} < 1 - \frac{\sigma_0}{N_{OUT}}. \tag{51}$$

Suppose that all term $x^n A_x / \lambda^{n+1}$, with $n \geq N_{OUT}$, have been neglected. We can also make the substitution $A_1 \mapsto -\frac{\theta_\infty}{2}\sigma_3$ in $\frac{A_1}{\lambda-1}$, if and only if the error term $-\frac{A_0+A_x}{\lambda-1}$, is smaller than $\frac{x^{N_{OUT}-1}A_x}{\lambda^{N_{OUT}}}$. Namely, if and only if:

$$\left|(A_0 + A_x)_{ij}\right| \ll \left|\frac{x^{N_{OUT}-1}(A_x)_{ij}}{\lambda^{N_{OUT}}}\right|.$$

This is $|x|^{\sigma_0} \ll |x|^{N_{OUT}-1-N_{OUT}\delta_{OUT}}$, namely $\delta_{OUT} > 1 - \frac{1+\sigma_0}{N_{OUT}}$.

We can also do the substitution $A_x \mapsto A$, provided that $\delta_{OUT} < 1$. This is because we can neglecting terms $\frac{x^n \Delta_x}{\lambda^{n+1}}$ with respect to $\frac{A_0+A_x}{\lambda}$, where both Δ_x and $A_0 + A_x$ are $O(x^{\sigma_0})$, $\lambda \sim x^\delta$, $\delta < 1$. We summarize the result in the following lemma.

Lemma 8 *Let $N_{OUT} \geq 2$ be integer. We can approximate (8) with:*

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A}{\lambda} \sum_{n=1}^{N_{OUT}-1} \left(\frac{x}{\lambda}\right)^n - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda-1}\right] \Psi_{OUT}.$$

if and only if:

$$1 - \frac{1+\sigma_0}{N_{OUT}} < \delta_{OUT} < 1 - \frac{\sigma_0}{N_{OUT}}.$$

In particular, this means that $\delta_{OUT} < 1$.

Example:

If $\sigma_0 = 1$ and $N_{IN} = 2$ we have:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{x\,A}{\lambda^2} + \frac{A_0 + A_x}{\lambda} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda-1}\right] \Psi_{OUT}, \qquad 0 < \delta_{OUT} < \frac{1}{2}.$$

If $\sigma_0 = 1$ and $N_{IN} = 3$ we have:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{x^2 A}{\lambda^3} + \frac{x\,A}{\lambda^2} + \frac{A_0 + A_x}{\lambda} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda-1}\right] \Psi_{OUT}, \qquad \frac{1}{3} < \delta_{OUT} < \frac{2}{3}.$$

2) THE SYSTEM for Ψ_{IN}.

We consider system (9) in the domain $|\lambda| \leq |x|^{\delta_{IN}}$. We investigate the condition necessary and sufficient to neglect a term $\lambda^n A_1$ (and all its next terms) with respect to $\frac{A_0}{\lambda} + \frac{A_x}{\lambda-x}$. It is convenient to write:

$$\frac{A_0}{\lambda} + \frac{A_x}{\lambda-x} = \frac{A_0 + A_x}{\lambda-x} - \frac{xA_0}{\lambda(\lambda-x)}.$$

Suppose that $\lambda \sim x^\delta$, $\delta \geq \delta_{IN}$.

$$\text{We neglect} \quad A_1 \lambda^n \quad \Longleftrightarrow \quad \begin{cases} \left|\frac{xA_0}{\lambda(\lambda-x)}\right| \gg \left|A_1 \lambda^n\right|, & \text{namely: } |x|^{1-2\delta} \gg |x|^{n\delta} \;\Leftrightarrow\; \delta > \frac{1}{n+2}; \\[2mm] \left|\frac{A_0+A_x}{\lambda-x}\right| \gg \left|A_1 \lambda^n\right|, & \text{namely: } |x|^{\sigma_0-\delta} \gg |x|^{n\delta} \;\Leftrightarrow\; \delta > \frac{\sigma_0}{n+1}. \end{cases}$$

Thus, we have the condition $\delta > \max\left\{\frac{\sigma_0}{n+1}, \; \frac{1}{n+2}\right\}$. We have proven the following:

Lemma 9 *Let $N_{IN} \geq 1$ be an integer. We approximate (9) with:*

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda-x} - A_1 \sum_{n=0}^{N_{IN}-1} \lambda^n\right] \Psi_{IN}.$$

if and only if:

$$\delta_{IN} > \max\left\{\frac{\sigma_0}{N_{IN}+1}, \; \frac{1}{N_{IN}+2}\right\}.$$

We further make the substitution $A_1 \mapsto -\frac{\theta_\infty}{2}\sigma_3$. This is possible if and only if two conditions are true: 1) $\left|\frac{xA_0}{\lambda(\lambda-x)}\right|$ and $\left|\frac{A_0+A_x}{\lambda-x}\right|$ are dominant w.r.t. the term $A_0 + A_x$ appearing in $A_1 = -\frac{\theta_\infty}{2}\sigma_3 - (A_0 + A_x)$. 2) $\left|\lambda^{N_{IN}-1} A_1\right|$, i.e. $\left|\lambda^{N_{IN}-1}\sigma_3\right|$, is dominant w.r.t. the term $A_0 + A_x$ in A_1. Esplicitely, the conditions are:

$$\left|\frac{A_0 + A_x}{\lambda-x}\right| \gg \left|A_0 + A_x\right| \quad \text{(this is always true)},$$

$$\left| \frac{xA_0}{\lambda(\lambda - x)} \right| \gg |A_0 + A_x| \quad \Longleftrightarrow \quad |x|^{1-2\delta} > x^{\sigma_0}, \text{ namely: } \delta > \frac{1 - \sigma_0}{2},$$

and

$$\left| \lambda^{N_{IN}-1}\sigma_3 \right| \gg |A_0 + A_x| \quad \Longleftrightarrow \quad |x|^{(N_{IN}-1)\delta} > |x|^{\sigma_0}, \text{ namely: } \delta < \frac{\sigma_0}{N_{IN} - 1}.$$

We have:

Lemma 10 *Let $N_{IN} \geq 1$ be an integer. We approximate (9) with:*

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2}\sigma_3 \sum_{n=0}^{N_{IN}-1} \lambda^n \right] \Psi_{IN}$$

$$= \left[\frac{A_0 + A_x}{\lambda - x} - \frac{xA_0}{\lambda(\lambda - x)} + \frac{\theta_\infty}{2}\sigma_3 \sum_{n=0}^{N_{IN}-1} \lambda^n \right] \Psi_{IN},$$

if and only if:

$$\max\left\{ \frac{1 - \sigma_0}{2}, \frac{\sigma_0}{N_{IN} + 1}, \frac{1}{N_{IN} + 2} \right\} < \delta_{IN} < \frac{\sigma_0}{N_{IN} - 1}.$$

As a final simplification, we substitute $A_0 = -A + \Delta_0 \mapsto -A$. This is possible if and only if:

$$\left| \frac{x\,\Delta_0}{\lambda(\lambda - x)} \right| \ll \left| \frac{A_0 + A_x}{\lambda - x} \right| \quad \Longleftrightarrow \quad |x|^{1+\sigma_0-2\delta} < |x|^{\sigma_0-\delta}, \text{ namely } \delta < 1,$$

and:

$$\left| \frac{x\,\Delta_0}{\lambda(\lambda - x)} \right| \ll \left| \lambda^{N_{IN}-1}\frac{\theta_\infty}{2}\sigma_3 \right| \quad \Longleftrightarrow \quad |x|^{1+\sigma_0-2\delta} \ll x^{(N_{IN}-1)\delta}, \text{ namely: } \delta < \frac{\sigma_0 + 1}{N_{IN} + 1}.$$

We have proven the following:

Lemma 11 *Let $N_{IN} \geq 1$ be an integer. We can approximate (9) with:*

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda - x} + \frac{xA}{\lambda(\lambda - x)} + \frac{\theta_\infty}{2}\sigma_3 \sum_{n=0}^{N_{IN}-1} \lambda^n \right] \Psi_{IN},$$

if and only if:

$$\max\left\{ \frac{1 - \sigma_0}{2}, \frac{\sigma_0}{N_{IN} + 1}, \frac{1}{N_{IN} + 2} \right\} < \delta_{IN} < \min\left\{ \frac{\sigma_0}{N_{IN} - 1}, \frac{\sigma_0 + 1}{N_{IN} + 1} \right\}.$$

Examples:

If $\sigma_0 = 1$ and $N_{IN} = 1$, we have:

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda - x} + \frac{xA}{\lambda(\lambda - x)} + \frac{\theta_\infty}{2}\sigma_3 \right] \Psi_{IN}, \qquad \frac{1}{2} < \delta_{IN} < 1.$$

If we keep $-A_0$ instead of A, with no change in the condition on δ_{IN}, we can also rewrite:

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2}\sigma_3 \right] \Psi_{IN},$$

If $\sigma_0 = 1$ and $N_{IN} = 2$, we have:

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda - x} + \frac{xA}{\lambda(\lambda - x)} + \frac{\theta_\infty}{2}\sigma_3(1 + \lambda) \right] \Psi_{IN}, \qquad \frac{1}{3} < \delta_{IN} < \frac{2}{3}.$$

Equivalently, we can write:

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2}\sigma_3(1 + \lambda) \right] \Psi_{IN}.$$

5.2 Matching

We do the matching in the overlapping region $|x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}$. This imposes: $\delta_{IN} \leq \delta_{OUT}$. In order for the overlapping region not to be empty, we must choose suitable reductions of (8) and (9). If we expect σ_0 to be close to 1, we try to match solutions Ψ_{OUT} and Ψ_{IN} satisfying one of the following sets of systems:

First choice:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{xA}{\lambda^2} + \frac{A_0 + A_x}{\lambda} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda - 1} \right] \Psi_{OUT},$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2}\sigma_3(1 + \lambda) \right] \Psi_{IN}.$$

The condition to be satisfied for $\sigma_0 \cong 1$ is: $\frac{\sigma_0}{3} < \delta_{IN} \leq \delta_{OUT} < 1 - \frac{\sigma_0}{2}$. For $\sigma_0 = 1$, this is:

$$\frac{1}{3} < \delta_{IN} \leq \delta_{OUT} < \frac{1}{2}.$$

Second choice:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{x^2 A}{\lambda^3} + \frac{xA}{\lambda^2} + \frac{A_0 + A_x}{\lambda} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda - 1} \right] \Psi_{OUT},$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2}\sigma_3 \right] \Psi_{IN}.$$

For $\sigma_0 \cong 1$, the condition to be satisfied is: $\frac{\sigma_0}{2} < \delta_{IN} \leq \delta_{OUT} < 1 - \frac{\sigma_0}{3}$. For $\sigma_0 = 1$, this is:

$$\frac{1}{2} < \delta_{IN} \leq \delta_{OUT} < \frac{2}{3}.$$

In both cases, the overlapping regions are not empty. The matching procedure will determine the leading terms (order x^{σ_0}) of the unknown matrix elements $a(x)$, $b(x)$, $c(x)$ of $A_0 + A_x$.

5.3 Matching for the First Choice: $\frac{1}{3} < \delta_{IN} \leq \delta_{OUT} < \frac{1}{2}$.

We rewrite the systems in a more convenient form:

$$\nu := \frac{1}{\lambda}, \quad \mu := \frac{\lambda}{x};$$

$$\frac{d\Psi_{OUT}}{d\nu} = \left[-xA - \frac{A_0 + A_x}{\nu} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\nu(\nu - 1)} \right] \Psi_{OUT} \tag{52}$$

$$\frac{d\Psi_{IN}}{d\mu} = \left[x^2 \frac{\theta_\infty}{2}\sigma_3 \, \mu \, + x \, \frac{\theta_\infty}{2}\sigma_3 + \frac{A_0}{\mu} + \frac{A_x}{\mu - 1} \right] \Psi_{IN}$$

$$= \left[x^2 \frac{\theta_\infty}{2}\sigma_3 \, \mu \, + x \, \frac{\theta_\infty}{2}\sigma_3 + \frac{A_0 + A_x}{\mu} - \frac{A_0}{\mu(\mu - 1)} \right] \Psi_{IN}.$$

Then we substitute $A_0 \mapsto -A$ in the last term.

In the matching region $|x|^{-\delta_{IN}} \leq |\nu| \leq |x|^{-\delta_{OUT}}$, $|x|^{\delta_{OUT}-1} \leq \mu \leq |x|^{\delta_{IN}-1}$, we have $\nu \to \infty$, $\mu \to \infty$. The point at infinity is a non-fuchsian singularity. [5] In order to find the local behavior at

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[5]System (52) can be also written with $\frac{\theta_\infty}{2}\sigma_3 \mapsto -A_1$:

$$\frac{d\Psi_{OUT}}{d\nu} = \left[-xA - \frac{A_0 + A_x}{\nu} + \frac{A_1}{\nu(\nu - 1)} \right] \Psi_{OUT}$$

$$\left[-xA - \frac{A_0 + A_x + A_1}{\nu} + \frac{A_1}{\nu - 1} \right] \Psi_{OUT}$$

After diagonalization, we get:

$$\frac{d\tilde{\Psi}_{OUT}}{d\nu} = \left[x\frac{\theta_x}{2}\sigma_3 + \frac{\theta_\infty}{2} \frac{G^{-1}\sigma_3 \, G}{\nu} + \frac{G^{-1}A_1 \, G}{\nu - 1} \right] \tilde{\Psi}_{OUT}.$$

This form is that of a system of isomonodromy deformation for the fifth Painlevé equation.

this point, it is convenient to put the leading term in diagonal form. Let G be the invertible matrix such that:

$$G^{-1} \, AG = -\frac{\theta_x}{2}\sigma_3, \quad \text{for example:} \quad G = \begin{pmatrix} 1 & \frac{r}{s} \\ \frac{s+\theta_x}{r} & 1 \end{pmatrix},$$

and put:

$$\Psi_{OUT} =: G \, \tilde{\Psi}_{OUT}.$$

Then:

$$\frac{d\tilde{\Psi}_{OUT}}{d\nu} = \left[x \, \frac{\theta_x}{2}\sigma_3 - \frac{G^{-1}(A_0 + A_x)G}{\nu} - \frac{\theta_\infty}{2} \, G^{-1}\sigma_3 G \, \left(\frac{1}{\nu^2} + \frac{1}{\nu^3} + ... \right) \right] \, \tilde{\Psi}_{OUT}, \quad \nu \to \infty; \quad (53)$$

$$\frac{d\Psi_{IN}}{d\mu} = \left[x^2 \, \frac{\theta_\infty}{2}\sigma_3 \, \mu \, + x \, \frac{\theta_\infty}{2}\sigma_3 + \frac{A_0 + A_x}{\mu} + A \, \left(\frac{1}{\mu^2} + \frac{1}{\mu^3} + ... \right) \right] \, \Psi_{IN}, \quad \mu \to \infty. \quad (54)$$

In order to write the local behavior of $\tilde{\Psi}_{OUT}$ and Ψ_{IN} at infinity, we observe that the systems (53) and (54) respectively have the following forms:

$$\frac{dY_1}{dz} = \left[\Omega + \frac{D_1}{z} + \frac{D_2}{z^2} + \frac{D_3}{z^3} + ... \right] \, Y_1, \tag{55}$$

$$\frac{dY_2}{dz} = \left[x^2\Lambda \, z + x\Lambda \, z + \frac{E_1}{z} + \frac{E_2}{z^2} + \frac{E_3}{z^3} + ... \right] \, Y_2, \tag{56}$$

where Ω and Λ are diagonal matrices with *distinct eigenvalues*. In our case:

$$\Omega = x \, \frac{\theta_x}{2}\sigma_3, \qquad \Lambda = \frac{\theta_\infty}{2}\sigma_3.$$

The eigenvalues are distinct iff $\theta_x \neq 0$, $\theta_\infty \neq 0$.

The theory for such systems is developed in [2] (see also [3]). For any sector of angular width $\pi + \epsilon$, $\epsilon > 0$ sufficiently small, there exists a unique solution of (55) with asymptotic expansion:

$$Y_1(z) \sim \left[I + \frac{G_1}{z} + \frac{G_2}{z^2} + ... \right] \, \exp\{\Omega \, z\} \, z^{\Omega_1}, \quad z \to \infty.$$

$$\Omega_1 = \text{ diagonal part of } D_1.$$

For any sector of angular width $\frac{\pi}{2} + \epsilon$, $\epsilon > 0$ sufficiently small, there exist a unique solution of (56) with asymptotic expansion:

$$Y_2(z) \sim \left[I + \frac{K_1}{z} + \frac{K_2}{z^2} + ... \right] \, \exp\left\{ \frac{x^2}{2}\Lambda \, z^2 \, + x\Lambda \, z \right\} \, z^{\Lambda_1}, \quad z \to \infty.$$

$$\Lambda_1 = \text{ diagonal part of } E_1.$$

We can always find two solutions $Y_1(z)$ and $Y_2(z)$ as above, such that the sectors where the asymptotic expansions hold are overlapping. We refer the reader to [2] for the general description of irregular system with a Stokes phenomenon, and to the Appendix 2 for the computation of the matrices G_i, K_i, $i = 1, 2, ...$

The systems (53), (54) are *isomonodromic*. This imposes that Ω_1 and Λ_1 must be independent of x. They are:

$$\Omega_1 = \text{ diagonal of } \left(-G^{-1}(A_0 + A_x) \, G \right),$$

$$\Lambda_1 = \text{ diagonal of } (A_0 + A_x) = \begin{pmatrix} a(x) & 0 \\ 0 & -a(x) \end{pmatrix}.$$

We compute:

$$G^{-1}(A_0 + A_x) \, G = \frac{1}{\theta_x} \begin{pmatrix} -(2s + \theta_x)a - s(s + \theta_x)b + c & \left(-2a - sb + \frac{c}{s} \right) \, r \\ \frac{s}{r} \left(2(s + \theta_x)a + (s + \theta_x)^2 b - c \right) & (2s + \theta_x)a + s(s + \theta_x)b - c \end{pmatrix}$$

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Since a, b, c vanish, the condition of isomonodromicity implies that:

$$\Omega_1 = 0, \qquad \Lambda_1 = 0$$

This means that *the leading terms of $a(x)$, $b(x)$, $c(x)$ satisfy the conditions*:

$$a(x) = 0, \qquad c(x) = s(s + \theta_x)b(x).$$

The above conditions mean that if $b(x)$, $c(x) = O(x^{\sigma_0})$, then $a(x)$ is of higher order, i.e. it vanishes faster than x^{σ_0}. Note that with this choice of a, b, c we get:

$$G^{-1}(A_0 + A_x)\, G = \begin{pmatrix} 0 & br \\ \frac{s(s+\theta_x)}{r}\, b & 0 \end{pmatrix}$$

We are ready to write the behavior of Ψ_{OUT}:

$$\Psi_{OUT} = G \left[I + \frac{G_1}{\nu} + \frac{G_2}{\nu^2} + ... \right] \exp\left\{ x\frac{\theta_x}{2}\sigma_3 \nu \right\}, \qquad \nu \to \infty$$

$$=$$

$$= G \left[I + G_1\lambda + G_2\lambda^2 + ... \right] \exp\left\{ \frac{\theta_x}{2}\sigma_3\, \frac{x}{\lambda} \right\}, \qquad \lambda \to 0.$$

We use the formulae of Appendix 2 to determine G_1:

$$(G_1)_{ij} = 2\, \frac{\left[G^{-1}(A_0 + A_x)\, G \right]_{ij}}{x\, \theta_x \big((\sigma_3)_{ii} - (\sigma_3)_{jj} \big)}, \qquad i \neq j.$$

$$(G_1)_{ii} = \frac{\theta_\infty}{2}\big(G^{-1}\sigma_3\, G \big)_{ij} + 2\, \frac{\left[G^{-1}(A_0 + A_x)\, G \right]_{ij} \left[G^{-1}(A_0 + A_x)\, G \right]_{ji}}{x\, \theta_x \big((\sigma_3)_{jj} - (\sigma_3)_{ii} \big)}.$$

In the second term of the last formula $j = 2$ if $i = 1$, $j = 1$ if $i = 2$. We compute:

$$G^{-1}\sigma_3\, G = \frac{1}{\theta_x} \begin{pmatrix} -(2s + \theta_x) & -2r \\ \frac{2s(s+\theta_x)}{r} & 2s + \theta_x \end{pmatrix}. \tag{57}$$

Therefore:

$$(G_1)_{12} = \frac{r}{x\, \theta_x}\, b, \qquad (G_1)_{21} = -\frac{s(s + \theta_x)}{x\, \theta_x\, r}\, b,$$

$$(G_1)_{11} = -\frac{\theta_\infty}{2\theta_x}(2s + \theta_x) + \frac{s(s + \theta_x)}{x\, \theta_x}\, b^2, \qquad (G_1)_{22} = -(G_1)_{11} \tag{58}$$

On the other hand, the local behavior of Ψ_{IN} is:

$$\Psi_{IN} = \left[I + \frac{K_1}{\mu} + \frac{K_2}{\mu^2} + ... \right] \exp\left\{ x^2\frac{\theta_\infty}{4}\sigma_3\, \mu^2 + x\frac{\theta_\infty}{2}\sigma_3\, \mu \right\}, \qquad \mu \to \infty$$

$$= \left[I + K_1\frac{x}{\mu} + K_2\frac{x^2}{\lambda^2} + ... \right] \exp\left\{ \frac{\theta_\infty}{4}\sigma_3\, \lambda^2 + \frac{\theta_\infty}{2}\sigma_3\, \lambda \right\}, \qquad \lambda \to 0.$$

We determine K_1 from the formulas of Appendix 2.

$$K_1 = \text{ diagonal part of } (-A) = \begin{pmatrix} -\big(s + \frac{\theta_x}{2} \big) & 0 \\ 0 & s + \frac{\theta_x}{2} \end{pmatrix}$$

The matching condition:

$$\Psi_{OUT}(\lambda, x) \sim \Psi_{IN}(\lambda, x), \qquad x \to 0, \quad |x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}.$$

is restricted to the overlapping sector where both expansions of Ψ_{OUT} and Ψ_{IN} hold. Noting that $\Psi_{OUT} \sim G$ and $\Psi_{IN} \sim I$, we choose the new solution $\Psi_{OUT} \mapsto \Psi_{OUT} \; G^{-1}$. Then, we expand the exponents:

$$\Psi_{OUT} = \left[I + GG_1G^{-1}\lambda + GG_2G^{-1}\lambda^2 + ... \right] \left[I + \frac{\theta_x}{2}G\sigma_3 G^{-1}\frac{x}{\lambda} + \frac{\theta_x^2}{8}\frac{x^2}{\lambda^2} + ... \right].$$

The point here is quite delicate. We consider the relation of dominance among terms – and write the leading terms of the expansion – as they are in case the $G_n(x)$'s are not divergent when $x \to 0$. Keeping into account that $\frac{\theta_x}{2}G\sigma_3 G^{-1} = -A$, the dominant terms are:

$$\Psi_{OUT}(\lambda, x) = I + GG_1G^{-1}\lambda \; - A\,\frac{x}{\lambda} \; + O\left(\lambda^2, \frac{x^2}{\lambda^2}, x\right)$$

It is important to note that λ is dominant w.r.t $\frac{x}{\lambda}$, because $\delta_{OUT} < \frac{1}{2}$; namely, $\lambda \sim x^\delta$ vanishes slower than $\frac{x}{\lambda} \sim x^{1-\delta}$, as $x \to 0$.

We expand the exponent in Ψ_{IN}, and keep only the first dominant terms (in the spirit of the observation on the dominance relations made above):

$$\Psi_{IN}(\lambda, x) = I + \frac{\theta_\infty}{2}\sigma_3 \; \lambda \; + K_1 \, \frac{x}{\lambda} + O\left(\lambda^2, \frac{x^2}{\lambda^2}, x\right)$$

Ψ_{OUT} and Ψ_{IN} match in the first term I. We impose the matching of the second term, namely the term in λ:

$$G \; G_1(x) \; G^{-1} \; \sim \; \frac{\theta_\infty}{2}\sigma_3, \qquad x \to 0.$$

Namely:

$$G_1(x) \; \sim \; \frac{\theta_\infty}{2} \; G^{-1}\sigma_3 \; G, \qquad x \to 0 \tag{59}$$

From the explicit form of G_1 and $G^{-1}\sigma_3 \; G$ given above, we conclude that the matching is satisfied if and only if:

$$b(x) \sim -x \; \theta_\infty, \qquad \text{and } \sigma_0 = 1.$$

The error in $b(x)$ is of higher order w.r.t. x. The determination of the leading behavior of $A_0 + A_x$ is complete, because $c(x) \sim s(s + \theta_x) \; b(x)$, $x \to 0$. Namely:

$$c(x) \sim -x \left\{ s(s + \theta_x) \; \theta_\infty \right\}, \qquad a(x) = o(x).$$

With such a choice of $b(x)$, one can verify that the terms in Ψ_{OUT} and Ψ_{IN} which follow the second (i.e. which follow the term in λ) are actually of higher order in x. Nevertheless, Ψ_{OUT} and Ψ_{IN} match only in the first and second term, being already the off-diagonal entries of the third term not matching (i.e. $-A$ and $K_1 = $ diagonal part of $-A$ respectively.)

5.4 Matching for the Second Choice: $\frac{1}{2} < \delta_{IN} \leq \delta_{OUT} < \frac{2}{3}$.

We rewrite the systems in the convenient form:

$$\nu := \frac{1}{\lambda}, \quad \mu := \frac{\lambda}{x};$$

$$\frac{d\Psi_{OUT}}{d\nu} = \left[-x^2 A \; \nu \; - xA - \frac{A_0 + A_x}{\nu} - \frac{\theta_\infty}{2}\frac{\sigma_3}{\nu(\nu - 1)} \right] \; \Psi_{OUT},$$

$$\frac{d\Psi_{IN}}{d\mu} = \left[x\,\frac{\theta_\infty}{2}\sigma_3 + \frac{A_0}{\mu} + \frac{A_x}{\mu - 1} \right] \; \Psi_{IN} = \left[x\,\frac{\theta_\infty}{2}\sigma_3 + \frac{A_0 + A_x}{\mu} - \frac{A_0}{\mu(\mu - 1)} \right] \; \Psi_{IN}. \tag{60}$$

Then, substitute $A_0 \mapsto -A$ in the last term

We rewrite the systems at infinity [6] :

$$\frac{d\tilde{\Psi}_{OUT}}{d\nu} = \left[x^2 \frac{\theta_x}{2}\sigma_3 \nu + x \frac{\theta_x}{2}\sigma_3 - \frac{G^{-1}(A_0 + A_x)G}{\nu} - \frac{\theta_\infty}{2} G^{-1}\sigma_3 G \left(\frac{1}{\nu^2} + \frac{1}{\nu^3} + ...\right)\right] \tilde{\Psi}_{OUT},$$

$$\frac{d\Psi_{IN}}{d\mu} = \left[x \frac{\theta_\infty}{2}\sigma_3 + \frac{A_0 + A_x}{\mu} + A \left(\frac{1}{\mu^2} + \frac{1}{\mu^3} + ...\right)\right] \Psi_{IN}, \qquad \nu, \mu \to \infty. \qquad (61)$$

This time the system of $\tilde{\Psi}_{OUT}$ is in the form (56), while the system of Ψ_{IN} is in the form (55), where:

$$\Omega = x \frac{\theta_\infty}{2}\sigma_3, \qquad \Omega_1 = \text{ diagonal part of } (A_0 + A_x),$$

$$\Lambda = \frac{\theta_x}{2}\sigma_3, \qquad \Lambda_1 = \text{ diagonal part of } \left(-G^{-1}(A_0 + A_x) G\right).$$

We impose that Ω_1 and Λ_1 do not depend on x, and we get the conditions $a = 0$, $c = s(s + \theta_x) b$. Then, we choose the following solutions:

$$\Psi_{OUT} = G \left[I + \frac{K_1}{\nu} + ...\right] \exp\left\{x^2 \frac{\theta_x}{4}\sigma_3 \nu^2 + x \frac{\theta_x}{2}\sigma_3 \nu\right\} G^{-1}$$

$$= I + \frac{\theta_x}{2}G\sigma_3 G^{-1} \frac{x}{\lambda} + GK_1 G^{-1}\lambda + O\left(\frac{x^2}{\lambda^2}, x, \lambda^2\right), \quad \nu \to \infty.$$

$$\Psi_{IN} = \left[I + \frac{G_1}{\mu} + ...\right] \exp\left\{x \frac{\theta_\infty}{2}\sigma_3\mu\right\} = I + G_1 \frac{x}{\lambda} + \frac{\theta_\infty}{2}\sigma_3\lambda + O\left(\frac{x^2}{\lambda^2}, x, \lambda^2\right), \quad \mu \to \infty.$$

The relation of dominance among terms are considered as if the G_n's and K_n's do not diverge as $x \to 0$. The matching condition:

$$\Psi_{OUT}(\lambda, x) \sim \Psi_{IN}(\lambda, x), \qquad x \to 0, \quad |x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}.$$

is restricted to the overlapping sector where both expansions of Ψ_{OUT} and Ψ_{IN} hold. We note that $\frac{x}{\lambda}$ vanishes slower than λ, because $\delta_{IN} > \frac{1}{2}$ (namely, $x/\lambda \sim x^{1-\delta}$, $\lambda \sim x^\delta$, $\delta > 1/2$). Ψ_{IN} and Ψ_{OUT} automatically match in the first term I. We impose the matching of the second leading term, i.e. the term in $\frac{x}{\lambda}$:

$$G_1(x) \sim \frac{\theta_x}{2} G \sigma_3 G^{-1} \equiv -A, \qquad x \to 0. \qquad (62)$$

As for G_1, the formulae in Appendix 2 give:

$$(G_1)_{ij} = -2 \frac{(A_0 + A_x)_{ij}}{x \theta_\infty \left[(\sigma_3)_{ii} - (\sigma_3)_{jj}\right]}, \qquad i \neq j.$$

$$(G_1)_{ii} = -(A)_{ii} + 2 \frac{(A_0 + A_x)_{ij} (A_0 + A_x)_{ji}}{x \theta_\infty \left[(\sigma_3)_{jj} - (\sigma_3)_{ii}\right]}.$$

In the last formula, $j = 2$ if $i = 1$, $j = 1$ if $i = 2$. Explicitly:

$$(G_1)_{12} = -\frac{r}{x \theta_\infty} b, \qquad (G_1)_{21} = \frac{s(s + \theta_x)}{x \theta_\infty r} b,$$

$$(G_1)_{11} = -(A)_{11} - \frac{s(s + \theta_x)}{x \theta_\infty} b^2, \quad (G_1)_{22} = -(A)_{22} + \frac{s(s + \theta_x)}{x \theta_\infty} b^2. \qquad (63)$$

Therefore (62) $\iff$ $b(x) \sim -x \theta_\infty$, and $\sigma_0 = 1$. As it must be, we get the same result of the matching for the first choice.

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[6]The system (60) is in the form of a system of isomonodromy deformation for the fifth Painlevé equation.

5.5 Critical Matching: $\frac{1}{2} - \epsilon < \delta_{IN} \le \delta_{OUT} < \frac{1}{2} + \epsilon.$

In between the first and the second choice – which hold respectively for $\frac{1}{3} < \delta_{IN} \le \delta_{OUT} < \frac{1}{2}$ and $\frac{1}{2} < \delta_{IN} \le \delta_{OUT} < \frac{2}{3}$ – we can also consider the following approximations of system (1):

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{xA}{\lambda^2} + \frac{A_0 + A_x}{\lambda} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda - 1} \right] \Psi_{OUT},$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2}\sigma_3 \right] \Psi_{IN}.$$

Rigorously speaking, the two systems cannot be considered simultaneously when $\sigma_0 = 1$. But we can consider $\sigma_0 = 1$ as a "limit" value – or "critical" value – for the matching of the two above systems in the region specified by $\frac{1}{2} - \epsilon < \delta_{IN} \le \delta_{OUT} < \frac{1}{2} + \epsilon$, where $\epsilon > 0$ is sufficiently small. We write again $\Psi_{OUT} =: G \ \tilde{\Psi}_{OUT}$. Then:

$$\frac{d\tilde{\Psi}_{OUT}}{d\nu} = \left[x \frac{\theta_x}{2}\sigma_3 - \frac{G^{-1}(A_0 + A_x)G}{\nu} - \frac{\theta_\infty}{2} G^{-1}\sigma_3 G \left(\frac{1}{\nu^2} + \frac{1}{\nu^3} + ... \right) \right] \tilde{\Psi}_{OUT}, \quad \nu \to \infty;$$

$$\frac{d\Psi_{IN}}{d\mu} = \left[x \frac{\theta_\infty}{2}\sigma_3 + \frac{A_0 + A_x}{\mu} + A \left(\frac{1}{\mu^2} + \frac{1}{\mu^3} + ... \right) \right] \Psi_{IN}, \quad \mu \to \infty.$$

When we impose isomonodromicity conditions, the diagonal parts of $A_0 + A_x$ and $G^{-1}(A_0 + A_x)G$ must be independent of x. This gives again $a = 0$, $c = s(s + \theta_x)\,b$. Then, we choose the fundamental solutions:

$$\Psi_{OUT} = G \left[I + \frac{G_1^{OUT}}{\nu} + ... \right] \ \exp\left\{ x \frac{\theta_x}{2}\sigma_3 \nu \right\} \ G =$$

$$= I + GG_1 G^{-1}\lambda + \frac{\theta_x}{2}G\sigma_3 G^{-1} \frac{x}{\lambda} + O\left(\lambda^2, x, \frac{x^2}{\lambda^2} \right)$$

$$\Psi_{IN} = \left[I + G_1^{IN} \frac{x}{\lambda} + ... \right] \ \exp\left\{ x \frac{\theta_\infty}{2}\mu \right\} = I + \frac{\theta_\infty}{2}\sigma_3 \lambda + G_1^{IN} \frac{x}{\lambda} + O\left(\lambda^2, x, \frac{x^2}{\lambda^2} \right)$$

We match them for $\lambda \sim x^\delta$ and $\delta \in (1/2 - \epsilon, 1/2 + \epsilon)$, in the overlapping sector where the above expansions hold. Here both $\lambda \sim x^\delta$ and $x/\lambda \sim x^{1-\delta}$ are the dominant terms. The matching conditions are $GG_1^{OUT}G^{-1} \sim \frac{\theta_\infty}{2}\sigma_3$ and $G_1^{IN} \sim \frac{\theta_x}{2}G\sigma_3 G^{-1}$. Namely:

$$G_1^{OUT}(x) \sim \frac{\theta_\infty}{2}G^{-1}\sigma_3 \ G, \qquad G_1^{IN}(x) \sim -A, \qquad x \to 0 \tag{64}$$

The matrix $\frac{\theta_\infty}{2}G^{-1}\sigma_3 G$ can be derived from (57). The matrix G_1^{OUT} is (58), the matrix G_1^{IN} is (63). Condition (64) is inclusive of both (59) and (62). Therefore, (64) $\iff$ $b(x) \sim -x\,\theta_\infty$, $x \to 0$. This is again the expected result.

5.6 Higher Order Terms

The final result obtained above is:

$$A_0 + A_x = \begin{pmatrix} 0 & -r\theta_\infty \ x \\ -\frac{(s+\theta_x)s\theta_\infty}{r} \ x & 0 \end{pmatrix} + o(x), \qquad A_1 = -\frac{\theta_\infty}{2}\sigma_3 - (A_0 + A_x), \tag{65}$$

$$A_x = \begin{pmatrix} s + \frac{\theta_x}{2} & -r \\ \frac{(s+\theta_x)\ s}{r} & -s - \frac{\theta_0}{2} \end{pmatrix} + o(1), \quad A_0 = -\begin{pmatrix} s + \frac{\theta_x}{2} & -r \\ \frac{(s+\theta_x)\ s}{r} & -s - \frac{\theta_0}{2} \end{pmatrix} + o(1) \tag{66}$$

Let us substitute the above results into (3). We obtain the first term with no error:

$$y(x) \sim \frac{1}{1 - \theta_\infty}, \qquad x \to 0.$$

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Here, r and s do not appear. Nevertheless, if we substitute in (PVI) the series $y = \frac{1}{1-\theta_\infty} + \sum_{n=1}^{\infty} b_n x^n$, we can compute recursively all the terms, for $\theta_0 = \pm\theta_x$ and $\theta_1 = \pm\theta_\infty$. We find a series:

$$y(x) = \frac{1}{1-\theta_\infty} + a\,x + \sum_{n=0}^{\infty} b_n(a;\theta_\infty,\theta_0)\,x^n, \quad x \to 0, \tag{67}$$

where a is an arbitrary parameter. This parameter is actually a function of s, as we prove now. The convergence of the Taylor expansion can be proved by a Briot-Bouquet like argument. This will not be done here. The reader can find a similar proof in [17] and the general procedure in [13].

5.6.1 Determination of $a = a(s)$

The system (1) is isomonodromic. This determines the structure A_x, A_0 A_1 as can be found in [16], Appendix C, formulae (C.47), (C.49), (C.51), (C.52), (C.55). If we substitute (67) in the formulae, we get a Taylor expansion for the matrix elements, in terms of the parameter a. The leading terms have exactly the structure of (65) and (66). We can identify the leading terms to express a as a function of s and r. The computations are quite long, so we give the result. When we write the leading terms as a function of a and impose that they coincide with (65) and (66), we find:

$$a = \frac{\theta_\infty(2s + \theta_x + 1)}{2(\theta_\infty - 1)} \in \mathbf{C}$$

The higher order terms are Taylor expansions. Explicitly, the first terms are:

$$A_1 = -\frac{\theta_\infty}{2}\sigma_3 - (A_0 + A_x) = \tag{68}$$

$$= \begin{pmatrix} -\frac{\theta_\infty}{2} + (s + \theta_x)s\theta_\infty\,x^2 & r\theta_\infty\left\{x - \frac{(\theta_\infty+1)(2s+\theta_x-1)}{2}\,x^2\right\} \\ \frac{s\theta_\infty}{r}\left\{(\theta_x + s)\,x - \frac{(\theta_\infty-1)(\theta_x+s)(2s+\theta_x+1)s}{2}\,x^2\right\} & \frac{\theta_\infty}{2} - (\theta_x + s)s\theta_\infty\,x^2 \end{pmatrix} + O(x^3)$$

$$A_x = \begin{pmatrix} \left(s + \frac{\theta_x}{2}\right) - 2(s + \theta_x)s\theta_\infty\,x & -r\left\{1 - (2s + \theta_x - 1)\theta_\infty\,x\right\} \\ \frac{s(s+\theta_x)}{r}\left\{1 - (2s + \theta_x + 1)\theta_\infty\,x\right\} & -\left(s + \frac{\theta_x}{2}\right) + 2(s + \theta_x)s\theta_\infty\,x \end{pmatrix} + O(x^2), \tag{69}$$

$$A_0 = \begin{pmatrix} -\left(s + \frac{\theta_x}{2}\right) + 2(s + \theta_x)s\theta_\infty\,x & r\left\{1 - (2s + \theta_x)\theta_\infty\,x\right\} \\ -\frac{s(s+\theta_x)}{r}\left\{1 - (2s + \theta_x)\theta_\infty\,x\right\} & \left(s + \frac{\theta_x}{2}\right) - 2(s + \theta_x)s\theta_\infty\,x \end{pmatrix} + O(x^2). \tag{70}$$

The above expansions are enough to obtain first two leading terms of (3):

$$y(x) = \frac{1}{1-\theta_\infty} + \frac{\theta_\infty(2s + \theta_x + 1)}{2(\theta_\infty - 1)}\,x + O(x^2),$$

Note that r simplifies. This is the solution (16).

5.7 Monodromy Data

We assume that the matching has been completed as above, and in particular $\sigma_0 = 1$. Thus, the system (1) can be approximated by:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{xA_x}{\lambda^2} - \frac{\theta_\infty}{2}\frac{\sigma_3}{\lambda-1}\right]\Psi_{OUT}, \quad \text{for } |\lambda| \geq |x|^\delta, \quad \delta < \frac{1}{2}; \tag{71}$$

or

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda} + \frac{\theta_\infty}{2}\sigma_3\right]\Psi_{IN}, \quad \text{for } |\lambda| \leq |x|^\delta, \quad \delta > \frac{1}{2}; \tag{72}$$

The first two leading terms are :

$$\Psi_{OUT} = I + \frac{\theta_\infty}{2}\sigma_3 \, \lambda + O\left(\frac{x}{\lambda}\right), \quad \lambda \sim x^\delta \to 0, \quad \delta < \frac{1}{2}; \tag{73}$$

$$\Psi_{IN} = I - A \, \frac{x}{\lambda} + O(\lambda), \quad \lambda \sim x^\delta \to 0, \quad \delta > \frac{1}{2}. \tag{74}$$

The above solutions match in the first three terms in the "critical" region $\lambda \sim x^\delta$, $\delta \simeq \frac{1}{2}$ (the region is restricted to a sector). Namely:

$$\Psi_{OUT} \sim \Psi_{IN} \sim I + \frac{\theta_\infty}{2}\sigma_3 \, \lambda - A \, \frac{x}{\lambda} \, + O\left(x, \frac{x^2}{\lambda^2}, \lambda^2\right), \quad \lambda \sim x^\delta, \; \delta \simeq \frac{1}{2} \tag{75}$$

Now, for $\delta < \frac{1}{2}$, we have:

$$\frac{A_0 + A_x}{\lambda} \sim x^{1-\delta} \to 0, \qquad \frac{xA_x}{\lambda^2} \sim x^{1-2\delta} \to 0.$$

Thus, (71) can be further reduced to:

$$\frac{d\hat{\Psi}_{OUT}}{d\lambda} = -\frac{\theta_\infty}{2} \, \frac{\sigma_3}{\lambda - 1} \, \hat{\Psi}_{OUT} \tag{76}$$

In system (72) we rewrite:

$$\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} = \frac{A_0 + A_x}{\lambda - x} - \frac{xA_0}{\lambda(\lambda - x)}$$

Then, we observe that, for $\delta > \frac{1}{2}$, we have the behaviors:

$$\frac{A_0 + A_x}{\lambda - x} \sim x^{1-\delta} \to 0, \quad \frac{xA_0}{\lambda(\lambda - x)} \sim -\frac{xA}{\lambda(\lambda - x)} \sim x^{1-2\delta} \to \infty, \qquad \delta > \frac{1}{2}$$

Thus, as $x \to 0$, the system (72) can be further reduced to:

$$\frac{d\hat{\Psi}_{IN}}{d\lambda} = \frac{xA}{\lambda(\lambda - x)} \, \hat{\Psi}_{IN}. \tag{77}$$

System (76) has the same monodromy of (1) at $\lambda = 1, \infty$. System (77) has the same monodromy of (1) at $\lambda = 0, x$.

MATCHING $\Psi \leftrightarrow \hat{\Psi}_{OUT}$:

We choose $\hat{\Psi}_{OUT}$ such that it matches with Ψ at $\lambda = \infty$. The behavior of this last is:

$$\Psi(\lambda, x) = \left[I + O\left(\frac{1}{\lambda}\right)\right]\lambda^{-\frac{\theta_\infty}{2}\sigma_3}, \qquad \lambda \to \infty,$$

for $\theta_\infty \notin \mathbf{Z}$. The solution of (76) with the same behavior is:

$$\hat{\Psi}_{OUT}(\lambda) := (\lambda - 1)^{-\frac{\theta_\infty}{2}\sigma_3} \tag{78}$$

As a consequence, the monodromy of Ψ at $\lambda = 1, \infty$ coincides with that of $\hat{\Psi}_{OUT}$. To compute it, we consider the loops $\lambda - 1 \mapsto (\lambda - 1)e^{2\pi i}$ and $\lambda \mapsto \lambda e^{2\pi i}$. The corresponding monodromy is:

$$M_1 = M_\infty = \exp\{i\pi\theta_\infty\sigma_3\}.$$

MATCHING $\hat{\Psi}_{OUT} \leftrightarrow \Psi_{OUT}$:

We match $\hat{\Psi}_{OUT}$ with Ψ_{OUT} for $x/\lambda \to 0$. Let us choose the branch $(\lambda - 1) = (1 - \lambda)\, e^{i\pi}$, $(1 - \lambda) > 0$ for $0 < \lambda < 1$. Solution (78) has expansion:

$$\hat{\Psi}_{OUT}(\lambda) = e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}\left[I + \frac{\theta_\infty}{2}\sigma_3\lambda + O(\lambda^2)\right], \quad \lambda \to 0$$

Therefore, for $\lambda \to 0$, $\hat{\Psi}_{OUT}$ matches with $\Psi_{OUT}\, e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}$, where Ψ_{OUT} is (73).

MATCHING $\Psi_{OUT} \leftrightarrow \Psi_{IN}$: This is (75).

MATCHING $\Psi_{IN} \leftrightarrow \Psi$: The above matchings imply that $\Psi_{IN}\, e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}$ matches with Ψ (where Ψ_{IN} is (74)).

MATCHING $\Psi_{IN} \leftrightarrow \hat{\Psi}_{IN}$:

In order to determine the monodromy of system (1) at $\lambda = 0, x$, we need to find a fundamental solution $\hat{\Psi}_{IN}$ of (77), that matches with $\Psi_{IN}\, e^{-i\frac{\pi}{2}\sigma_3}$ for $x/\lambda \to 0$, where Ψ_{IN} is (74). A fundamental solution satisfying these requirements is:

$$\hat{\Psi}_{IN}(\lambda, x) := G\left[\left(1 - \frac{x}{\lambda}\right)^{-\frac{\theta_x}{2}\sigma_3}\right] G^{-1}\, e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}.$$

Actually, this has the behavior:

$$\hat{\Psi}_{IN}(\lambda, x) = \left[I + \frac{\theta_x}{2}G\sigma_3 G^{-1}\frac{x}{\lambda} + O\left(\frac{x^2}{\lambda^2}\right)\right] e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}, \qquad \frac{x}{\lambda} \to 0,$$

where $\frac{\theta_x}{2}G\sigma_3 G^{-1} \equiv -A$. The first two terms match with $\Psi_{IN}\, e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}$, as required.

As a consequence of the matching, the monodromy matrices of Ψ and $\hat{\Psi}_{IN}$ at $\lambda = 0, x$ coincide. To compute them, we write the local behaviors (for $x \neq 0$ fixed) of $\hat{\Psi}_{IN}$:

$$\hat{\Psi}_{IN}(\lambda, x) = \begin{cases} \left[Gx^{\frac{\theta_x}{2}\sigma_3}\right]\left[1 + O(\lambda - x)\right](\lambda - x)^{-\frac{\theta_x}{2}\sigma_3}\left[G^{-1}e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}\right], \quad \lambda \to x, \\[2mm] \left[G(-x)^{-\frac{\theta_x}{2}\sigma_3}\right]\left[1 + O(\lambda)\right]\lambda^{\frac{\theta_x}{2}\sigma_3}\left[G^{-1}e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}\right], \quad \lambda \to 0. \end{cases}$$

It is not necessary to specify the branch of $(\pm x)^{-\frac{\theta_x}{2}\sigma_3}$. We resctict to the case θ_0 and $\theta_x \notin \mathbf{Z}$, so that the matrices R_0, R_x in (14) are zero and the matching of Ψ with the above behaviors of $\hat{\Psi}_{IN}$ is realized.

We can compute the monodromy matrices for $\lambda \mapsto \lambda e^{2\pi i}$ and $(\lambda - x) \mapsto (\lambda - x)e^{2\pi i}$ respectively:

$$M_0 = e^{i\frac{\pi}{2}\theta_\infty\sigma_3}G\, \exp\{i\pi\theta_x\sigma_3\}\, G^{-1}e^{-i\frac{\pi}{2}\theta_\infty\sigma_3},$$

$$M_x = e^{i\frac{\pi}{2}\theta_\infty\sigma_3}G\, \exp\{-i\pi\theta_x\sigma_3\}\, G^{-1}e^{-i\frac{\pi}{2}\theta_\infty\sigma_3},$$

Note that $M_1 = M_\infty$ is invariant for $M_1 \mapsto e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}M_1 e^{i\frac{\pi}{2}\theta_\infty\sigma_3}$. With this fact in mind, we obtain the result of theorem 3, point b). In particular, computing the trace of $M_0 M_1$, we get:

$$s = \frac{\theta_x\left[2\cos(\pi(\theta_\infty + \theta_x)) - \mathrm{tr}(M_1 M_0)\right]}{2\left[\cos(\pi(\theta_\infty - \theta_x)) - \cos(\pi(\theta_\infty + \theta_x))\right]}.$$

6 Case $\sigma = \pm(\theta_1 - \theta_\infty),\ \pm(\theta_1 + \theta_\infty)$. Solution (15).

This case shows new features, namely r and r_1 may be functions of x. For $\sigma = \pm(\theta_1 - \theta_\infty),\ \pm(\theta_1 + \theta_\infty)$ the matrices (34) (35) become:

$$\sigma = \pm(\theta_1 - \theta_\infty): \quad \hat{A}_1 = \begin{pmatrix} -\frac{\theta_1}{2} & -r_1 \\ 0 & \frac{\theta_1}{2} \end{pmatrix}, \quad \hat{A}_0 + \hat{A}_x = \begin{pmatrix} \frac{\theta_1 - \theta_\infty}{2} & r_1 \\ 0 & -\frac{\theta_1 - \theta_\infty}{2} \end{pmatrix}. \tag{79}$$

$$\sigma = \pm(\theta_1 + \theta_\infty): \qquad \hat{A}_1 = \begin{pmatrix} \frac{\theta_1}{2} & -r_1 \\ 0 & -\frac{\theta_1}{2} \end{pmatrix}, \qquad \hat{A}_0 + \hat{A}_x = \begin{pmatrix} -\frac{\theta_1+\theta_\infty}{2} & r_1 \\ 0 & \frac{\theta_1+\theta_\infty}{2} \end{pmatrix}.$$

The transpose matrices may be considered (namely, re-define $r_1 \mapsto -\frac{[\sigma^2-(\theta_1-\theta_\infty)^2][\sigma^2-(\theta_1+\theta_\infty)^2]}{16\theta_\infty^2 r_1}$).

The matrices $\hat{A}_0$ and $\hat{A}_x$ are again (40) and (41). For definiteness, we will consider the upper triangular matrices $\hat{A}_1$ and $\hat{A}_0 + \hat{A}_x$, and the choice $\sigma = \theta_1 - \theta_\infty$. We distinguish three cases:

(I) r_1 is a non zero constant: In this case we just repeat the general matching procedure and find $y(x)$ as in (44). $r \neq 0$ is constant.

(II) $r_1 = 0$ and r constant. This is a very easy case, because $\hat{A}_1 = -\frac{\theta_1}{2}\sigma_3$, $\hat{A}_0 + \hat{A}_x = \frac{\theta_1-\theta_\infty}{2}\sigma_3$. Therefore a fundamental solution of (33) is:

$$\Psi_{OUT}(\lambda) = (\lambda-1)^{-\frac{\theta_1}{2}\sigma_3}\lambda^{\frac{\theta_1-\theta_\infty}{2}\sigma_3}(-1)^{-\frac{\theta_1}{2}\sigma_3} = \left[I + \frac{\theta_1}{2}\sigma_3 + O(\lambda^2)\right]\lambda^{\frac{\theta_1-\theta_\infty}{2}\sigma_3}, \quad \lambda \to 0.$$

The solution of (39), with r constant, is the same of the general case. The matching is possible as in the general case, with $K_0 = \begin{pmatrix} x^{\frac{\theta_1-\theta_\infty}{2}} & 0 \\ 0 & x^{\frac{\theta_\infty-\theta_1}{2}} \end{pmatrix}$. Therefore:

$$\hat{A}_0 = \begin{pmatrix} \frac{\theta_0^2-\theta_x^2+\sigma^2}{4\sigma} & r\, x^{\theta_1-\theta_\infty} \\ -\frac{[\sigma^2-(\theta_0-\theta_x)^2][\sigma^2-(\theta_0+\theta_x)^2]}{16\sigma^2 r}\, x^{\theta_\infty-\theta_1} & -\frac{\theta_0^2-\theta_x^2+\sigma^2}{4\sigma} \end{pmatrix}.$$

We obtain: $y(x) \sim \frac{x(\hat{A}_0)_{12}}{x[(\hat{A}_0)_{12}+(\hat{A}_1)_{12}]-(\hat{A}_1)_{12}} = \frac{x\, r\, x^{\theta_1-\theta_\infty}}{x[\,x^{\theta_1-\theta_\infty}+0]-0} = 1$. This is the *singular solution* $y(x) \equiv 1$.

The case $r_1 = r_1(x) \to 0$ may give a non-singular $y(x)$, provided that also $r = r(x) \to 0$. This will be proved below.

(III) Case $r_1 = r_1(x)$.

A priori, we don't know if it is possible to have an x-dependent r_1. This is not actually possible if we approximate (1) with the fuchsian system (33) (i.e. (10)). Actually, (33) is a system with reducible monodromy (upper triangular):

$$\frac{d\Psi_{OUT}}{d\lambda} = \left\{\begin{pmatrix} \frac{\theta_1-\theta_\infty}{2} & r_1 \\ 0 & -\frac{\theta_1-\theta_\infty}{2} \end{pmatrix}\frac{1}{\lambda} + \begin{pmatrix} -\frac{\theta_1}{2} & -r_1 \\ 0 & \frac{\theta_1}{2} \end{pmatrix}\frac{1}{\lambda-1}\right\}\Psi_{OUT}.$$

There is a fundamental solution (obtained by variation of parameters):

$$\Psi_{OUT} = \begin{bmatrix} (1-\lambda)^{-\frac{\theta_1}{2}} & 0 \\ 0 & (1-\lambda)^{\frac{\theta_1}{2}} \end{bmatrix} \begin{bmatrix} 1 & \frac{r_1 F(1-\theta_1,\theta_\infty-\theta_1,\theta_\infty-\theta_1+1;\lambda)}{\theta_\infty-\theta_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda^{\frac{\theta_1-\theta_\infty}{2}} & 0 \\ 0 & \lambda^{-\frac{\theta_1-\theta_\infty}{2}} \end{bmatrix}.$$

Here, $F(...)$ denotes a Gauss hyper-geometric function. This solution has a diagonal monondromy matrix at $\lambda = 0$ and an upper triangular monodromy matrix at $\lambda = 1$, with r_1 in the $(1,2)$ matrix element. Therefore, r_1 must be independent of the monodromy preserving deformation x. We are sent back to the case **(I)** of constant r_1.

The only possibility for r_1 to depend on x is that the matrices $\hat{A}_0$, $\hat{A}_x$ have a behavior, for $x \to 0$, such that the system (1) is approximated by a system (8) with singularity of the second kind at $\lambda = 0$. Namely, system (1) must be approximated *at least* by:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left\{\frac{xA_x}{\lambda^2} + \frac{A_0 + A_x}{\lambda} + \frac{A_1}{\lambda-1}\right\}\Psi_{OUT} \tag{80}$$

Hypothesis: *We consider the case:*

$$xA_x = \begin{pmatrix} 0 & \rho(x) \\ 0 & 0 \end{pmatrix} + \text{ higher orders} \quad x \to 0.$$

In the above hypothesis, (80) is:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left\{ \begin{pmatrix} 0 & \rho(x) \\ 0 & 0 \end{pmatrix} \frac{1}{\lambda^2} + \begin{pmatrix} \frac{\theta_1-\theta_\infty}{2} & r_1 \\ 0 & \frac{\theta_1-\theta_\infty}{2} \end{pmatrix} \frac{1}{\lambda} + \begin{pmatrix} -\frac{\theta_1}{2} & -r_1 \\ 0 & \frac{\theta_1}{2} \end{pmatrix} \frac{1}{\lambda-1} \right\} \Psi_{OUT} \tag{81}$$

Let us write a Ψ_{OUT} as a vector $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. The system becomes:

$$\frac{d\psi_1}{d\lambda} = \left(\frac{\theta_1-\theta_\infty}{2\lambda} - \frac{\theta_1}{2(\lambda-1)} \right) \psi_1 + \left(\frac{r_1}{\lambda} + \frac{\rho}{\lambda^2} - \frac{r_1}{\lambda-1} \right) \psi_2, \quad \frac{d\psi_2}{d\lambda} = \left(\frac{\theta_\infty-\theta_1}{2\lambda} + \frac{\theta_1}{2(\lambda-1)} \right) \psi_2.$$

This system is solvable by variation of parameters. Let C_1 and C_2 be integration constants. The general solution is:

$$\psi_2(\lambda) = C_2 \, \lambda^{\frac{\theta_\infty-\theta_1}{2}} (\lambda-1)^{\frac{\theta_1}{2}}, \tag{82}$$

$$\psi_1(\lambda) = C_1 \, \lambda^{\frac{\theta_1-\theta_\infty}{2}} (\lambda-1)^{-\frac{\theta_1}{2}} + C_2 \, e^{i\pi\theta_1} \left[\frac{\rho}{\theta_\infty-\theta_1-1} \frac{F(1-\theta_1, \theta_\infty-\theta_1-1, \theta_\infty-\theta_1:\lambda)}{\lambda} + \right.$$

$$\left. + \frac{r_1-\rho}{\theta_\infty-\theta_1} F(1-\theta_1, \theta_\infty-\theta_1, \theta_\infty-\theta_1+1;\lambda) \right] \lambda^{\frac{\theta_\infty-\theta_1}{2}} (\lambda-1)^{-\frac{\theta_1}{2}}. \tag{83}$$

Here $F(...)$ denotes a Gauss Hyper-geometric equation. The choice of the branch is such that for $0 < \lambda < 1$, we have $0 < 1-\lambda = e^{-i\pi}(\lambda-1)$.

In order to write the local behavior for $\lambda \to 0$, we expand the hypergemoetric functions and $(\lambda-1)^{\pm\theta_1/2} = \left(e^{i\pi}(1-\lambda) \right)^{\pm\theta_1/2}$:

$$\psi_1 = C_1 \, e^{-i\frac{\pi}{2}\theta_1} \lambda^{\frac{\theta_1-\theta_\infty}{2}} \left(1 + \frac{\theta_1}{2}\lambda + O(\lambda^2) \right) + C_2 \, e^{i\frac{\pi}{2}\theta_1} \lambda^{\frac{\theta_\infty-\theta_1}{2}} \left[-\frac{\rho}{1+\theta_1-\theta_\infty} \frac{1}{\lambda} + \right.$$

$$\left. + \frac{\rho \, \theta_1(\theta_1-\theta_\infty+2) - 2r_1 \, (\theta_1-\theta_\infty+1)}{2(\theta_1-\theta_\infty)(\theta_1-\theta_\infty+1)} + O(\lambda) \right].$$

$$\psi_2 = C_2 \, e^{i\frac{\pi}{2}\theta_1} \left(1 - \frac{\theta_1}{2}\lambda + O(\lambda^2) \right) \, \lambda^{\frac{\theta_\infty-\theta_1}{2}}.$$

Therefore, we can take as a fundamental solution the following matrix (choose $C_1 = e^{i\frac{\pi}{2}\theta_1}$, $C_2 = e^{-i\frac{\pi}{2}\theta_1}$):

$$\Psi_{OUT} = \begin{bmatrix} \left(1 + \frac{\theta_1}{2}\lambda + ... \right) \lambda^{\frac{\theta_1-\theta_\infty}{2}} & \left[-\frac{\rho}{1+\theta_1-\theta_\infty}\frac{1}{\lambda} + \frac{\rho \, \theta_1(\theta_1-\theta_\infty+2)-2r_1 \, (\theta_1-\theta_\infty+1)}{2(\theta_1-\theta_\infty)(\theta_1-\theta_\infty+1)} + ... \right] \lambda^{\frac{\theta_\infty-\theta_1}{2}} \\ 0 & \left(1 - \frac{\theta_1}{2}\lambda + ... \right) \lambda^{\frac{\theta_\infty-\theta_1}{2}} \end{bmatrix}$$

$$= \left\{ \begin{pmatrix} 0 & -\frac{\rho}{1+\theta_1-\theta_\infty} \\ 0 & 0 \end{pmatrix} \frac{1}{\lambda} + \begin{pmatrix} 1 & \frac{\rho \, \theta_1(\theta_1-\theta_\infty+2)-2r_1 \, (\theta_1-\theta_\infty+1)}{2(\theta_1-\theta_\infty)(\theta_1-\theta_\infty+1)} \\ 0 & 1 \end{pmatrix} + O(\lambda) \right\} \lambda^{\frac{\theta_1-\theta_\infty}{2} \sigma_3}$$

.

6.1 Matching

The above solution must be matched with the solution of system (39), with $\sigma = \theta_1 - \theta_\infty$.

$$\Psi_0 \left(\frac{\lambda}{x} \right) = \left[I + K_1 \frac{x}{\lambda} + O\left(\frac{x^2}{\lambda^2} \right) \right] \begin{pmatrix} \left(\frac{\lambda}{x}\right)^{\frac{\theta_1-\theta_\infty}{2}} & 0 \\ 0 & \left(\frac{\lambda}{x}\right)^{\frac{\theta_\infty-\theta_1}{2}} \end{pmatrix}.$$

From a standard computation we find:

$$K_1 = \begin{pmatrix} \frac{\theta_0^2-(\theta_x-\sigma)^2}{4\sigma} & \frac{r}{\sigma+1} \\ \frac{(\theta_0^2-(\theta_x-\sigma)^2)(\theta_0^2-(\theta_x+\sigma)^2)}{16\sigma^2(\sigma-1)r} & \frac{(\theta_x+\sigma)^2-\theta_0^2}{4\sigma(\sigma+1)} \end{pmatrix}, \quad r \neq 0.$$

Note that r in (41) is constant, because the monodromy of (39) depends on r.

The matching relation $\Psi_{OUT}(\lambda) \sim K_0(x)\Psi_0(\lambda/x)$, reads:

$$\left\{ \begin{pmatrix} 0 & -\frac{\rho}{1+\theta_1-\theta_\infty} \\ 0 & 0 \end{pmatrix} \frac{1}{\lambda} + \begin{pmatrix} 1 & \frac{\rho\,\theta_1(\theta_1-\theta_\infty+2)-2r_1\,(\theta_1-\theta_\infty+1)}{2(\theta_1-\theta_\infty)(\theta_1-\theta_\infty+1)} \\ 0 & 1 \end{pmatrix} + O(\lambda) \right\} \sim$$

$$\sim\ K_0(x)\ \left[I + K_1\frac{x}{\lambda} + ...\right] \begin{pmatrix} x^{\frac{\theta_\infty-\theta_1}{2}} & 0 \\ 0 & x^{\frac{\theta_1-\theta_\infty}{2}} \end{pmatrix}.$$

Namely:

$$\begin{cases} \begin{pmatrix} 1 & \frac{\rho\,\theta_1(\theta_1-\theta_\infty+2)-2r_1\,(\theta_1-\theta_\infty+1)}{2(\theta_1-\theta_\infty)(\theta_1-\theta_\infty+1)} \\ 0 & 1 \end{pmatrix} \sim K_0(x) \begin{pmatrix} x^{\frac{\theta_\infty-\theta_1}{2}} & 0 \\ 0 & x^{\frac{\theta_1-\theta_\infty}{2}} \end{pmatrix} \\[20pt] \begin{pmatrix} 0 & -\frac{\rho}{1+\theta_1-\theta_\infty} \\ 0 & 0 \end{pmatrix} \sim\ x\,K_0(x)\,F_1^{(0)} \begin{pmatrix} x^{\frac{\theta_\infty-\theta_1}{2}} & 0 \\ 0 & x^{\frac{\theta_1-\theta_\infty}{2}} \end{pmatrix} \end{cases}$$

The first equation above is:

$$K_0(x) \sim \begin{pmatrix} 1 & \frac{\rho\,\theta_1(\theta_1-\theta_\infty+2)-2r_1\,(\theta_1-\theta_\infty+1)}{2(\theta_1-\theta_\infty)(\theta_1-\theta_\infty+1)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\frac{\theta_1-\theta_\infty}{2}} & 0 \\ 0 & x^{\frac{\theta_\infty-\theta_1}{2}} \end{pmatrix}.$$

We substitute this in the second equation. For simplicity, denote K_{ij} the matrix elements of K_1. We obtain:

$$\begin{pmatrix} 0 & -\frac{\rho}{1+\theta_1-\theta_\infty} \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} x\,K_{11} + (*)\,K_{21}\,x^{\theta_\infty-\theta_1+1} & K_{12}\,x^{\theta_1-\theta_\infty+1} + (*)\,x\,K_{22} \\ K_{21}\,x^{\theta_\infty-\theta_1+1} & x\,K_{22} \end{pmatrix},$$

where:

$$(*) := \frac{\rho\,\theta_1(\theta_1-\theta_\infty+2)-2r_1\,(\theta_1-\theta_\infty+1)}{2(\theta_1-\theta_\infty)(\theta_1-\theta_\infty+1)}.$$

The element $(2,1)$ must vanish. This occurs iff:

$$x^{\theta_\infty-\theta_1+1} \to 0, \quad \text{for } x \to 0; \iff \Re(\theta_\infty - \theta_1) > -1. \tag{84}$$

We substitute this result in the element $(1,1)$ and then we impose that it vanishes:

$$\frac{\rho\,\theta_1(\theta_1-\theta_\infty+2)-2r_1\,(\theta_1-\theta_\infty+1)}{2(\theta_1-\theta_\infty)(\theta_1-\theta_\infty+1)}\,K_{21}\,x^{\theta_\infty-\theta_1+1} \to 0.$$

This implies that $\rho\,\theta_1(\theta_1-\theta_\infty+2)-2r_1\,(\theta_1-\theta_\infty+1) = o(x^{\theta_1-\theta_\infty-1})$.
From the element $(1,2)$ we have:

$$-\frac{\rho}{1+\theta_1-\theta_\infty} \sim K_{12}\,x^{\theta_1-\theta_\infty+1} + o(x^{\theta_1-\theta_\infty})\,K_{22}. \tag{85}$$

This relation may be satisfied in two ways: the first is that $\rho = \rho(x) = o(x^{\theta_1-\theta_\infty})$. The second is:

$$\rho = \rho(x) \sim -(1+\theta_1-\theta_\infty)K_{12}x^{\theta_1-\theta_\infty+1} \quad = -rx^{\theta_1-\theta_\infty+1}. \tag{86}$$

In both cases ρ is a function of x. We are going to prove that the monodromy of Ψ_{OUT} is independent of $\rho(x)$ (namely, of x) if and only if:

$$\rho = r_1\,\frac{\theta_1-\theta_\infty+1}{\theta_1}. \tag{87}$$

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This fact rules out the first possibility, because (85) becomes:

$$\frac{\rho(x)}{\theta_\infty - \theta_1 - 1} \sim K_{12}\, x^{\theta_1 - \theta_\infty + 1} + \text{constant} \times \rho(x)\, x,$$

so the last term in the r.h.s. is a higher order correction and ρ is given by (86). Before proving (87), we complete the matching procedure. Using (87), we find:

$$K_0(x) \sim \begin{pmatrix} 1 & g\, x^{\theta_1 - \theta_\infty + 1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\frac{\theta_1 - \theta_\infty}{2}} & 0 \\ 0 & x^{\frac{\theta_\infty - \theta_1}{2}} \end{pmatrix}, \quad g := -\frac{r}{\theta_1 - \theta_\infty + 1}$$

We are ready to compute $\hat{A}_0 = K_0 \hat{\hat{A}}_0 K_0^{-1}$, where $\hat{\hat{A}}_0$ is (40), for $\sigma = \theta_1 - \theta_\infty$:

$$\hat{A}_0 = \begin{pmatrix} (\hat{\hat{A}}_0)_{11} + g(\hat{\hat{A}}_0)_{21}\, x & \left(\frac{(\hat{\hat{A}}_0)_{12}}{x} - 2g(\hat{\hat{A}}_0)_{11} - g^2(\hat{\hat{A}}_0)_{21}\, x \right) x^{\theta_1 - \theta_\infty + 1} \\ \left((\hat{\hat{A}}_0)_{21}\, x \right) \frac{1}{x^{\theta_1 - \theta_\infty + 1}} & -(\hat{\hat{A}}_0)_{11} - g(\hat{\hat{A}}_0)_{21}\, x \end{pmatrix}. \tag{88}$$

The first term of each matrix element certainly contains no error.

We can now substitute $(\hat{A}_0)_{12} = \left(-\frac{r}{x} + O(1)\right) x^{\theta_1 - \theta_\infty + 1}$ and $\hat{A}_1 = -r_1(x) \sim \frac{\theta_1}{\theta_1 - \theta_\infty + 1} r x^{\theta_1 - \theta_\infty + 1}$ into (3), and find:

$$y(x) \sim \frac{\theta_1 - \theta_\infty + 1}{1 - \theta_\infty}, \qquad x \to 0.$$

Proof of (87). We compute the monodromy of the solution Ψ_{OUT}. At $\lambda = 0$ this is given by the matrix $M_0^{OUT} = \exp\{i\pi(\theta_1 - \theta_\infty)\sigma_3\}$, obtained by the expansion of Ψ_{OUT} at $\lambda = 0$ as we did above, with the choice $C_1 = e^{i\frac{\pi}{2}\theta_1} = 1/C_2$.

Let us study the monodromy at $\lambda = 1$. We need to expand (82) and (83) at $\lambda = 1$. First of all, we use the contiguity relation:

$$F(1 - \theta_1, \theta_\infty - \theta_1, \theta_\infty - \theta_1 + 1; \lambda) = \frac{\theta_1 - \theta_\infty}{\theta_\infty - 1} \frac{1}{\lambda} \left[(1-\lambda)^{\theta_1} - F(1 - \theta_1, \theta_\infty - \theta_1 - 1, \theta_\infty - \theta_1; \lambda) \right].$$

This is used to rewrite ψ_1:

$$\psi_1 = C_1 \lambda^{\frac{\theta_1 - \theta_\infty}{2}} (\lambda - 1)^{-\frac{\theta_1}{2}} + C_2 e^{i\pi\theta_1} \lambda^{\frac{\theta_\infty - \theta_1}{2}} (\lambda - 1)^{-\frac{\theta_1}{2}} \left[\frac{\rho - r_1}{\theta_\infty - 1} \frac{(1-\lambda)^{\theta_1}}{\lambda} + \right.$$

$$\left. + \left(\frac{\rho}{\theta_\infty - \theta_1 - 1} + \frac{r_1 - \rho}{\theta_\infty - 1} \right) \frac{1}{\lambda} F(1 - \theta_1, \theta_\infty - \theta_1 - 1, \theta_\infty - \theta_1; \lambda) \right]$$

Then, we substitute in ψ_1 the following connection formula:

$$F(1 - \theta_1, \theta_\infty - \theta_1 - 1, \theta_\infty - \theta_1; \lambda) = \frac{\Gamma(\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(\theta_\infty - 1)} F(1 - \theta_1, \theta_\infty - \theta_1 - 1, 1 - \theta_1; 1 - \lambda) +$$

$$+ \frac{\Gamma(-\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(1 - \theta_1)\Gamma(\theta_\infty - \theta_1 - 1)} (1-\lambda)^{\theta_1} F(1, \theta_\infty - 1, 1 + \theta_1 : 1 - \lambda), \qquad \theta_1, \theta_\infty \notin \mathbf{Z}.$$

Thus, ψ_1 has the following structure, when $\lambda \to 1$:

$$\psi_1 = C_1 e^{-i\frac{\pi}{2}\theta_1} \lambda^{\frac{\theta_1 - \theta_\infty}{2}} (1-\lambda)^{-\frac{\theta_1}{2}} + C_2 e^{i\frac{\pi}{2}\theta_1} \lambda^{\frac{\theta_\infty - \theta_1}{2}} \left[(1-\lambda)^{\frac{\theta_1}{2}} \sum_{n=0}^{\infty} a_n (1-\lambda)^n + \right.$$

$$\left. + \left(\frac{\rho}{\theta_\infty - \theta_1 - 1} + \frac{r_1 - \rho}{\theta_\infty - 1} \right) (1-\lambda)^{-\frac{\theta_1}{2}} \sum_{n=0}^{\infty} b_n (1-\lambda)^n \right],$$

where $a_n, b_n \neq 0$ are the coefficients that follow from the expansion $1/\lambda$ and the hyper-geometric functions at $\lambda = 1$. When $\lambda^{\pm\frac{\theta_\infty - \theta_1}{2}}$ is also expanded at $\lambda = 1$, we find that Ψ_{OUT} has the following structure:

$$\Psi_{OUT} = \begin{pmatrix} (\text{series}_1) \, (1-\lambda)^{-\frac{\theta_1}{2}} & (\text{series}_2) \, (1-\lambda)^{\frac{\theta_1}{2}} + \left(\frac{\rho}{\theta_\infty - \theta_1 - 1} + \frac{r_1 - \rho}{\theta_\infty - 1}\right) (\text{series}_3) \, (1-\lambda)^{-\frac{\theta_1}{2}} \\ 0 & (\text{series}_4) \, (1-\lambda)^{\frac{\theta_1}{2}} \end{pmatrix}.$$

Here "series" means a series of the form $\sum_{n=0}^{\infty} c_n (1-\lambda)^n$, with $c_0 \neq 0$. We just give the dominant term, with the choice $C_1 = e^{i\frac{\pi}{2}\theta_1} = 1/C_2$:

$$\Psi_{OUT} = [I + O(1-\lambda)] \times$$

$$\times \begin{pmatrix} (1-\lambda)^{-\frac{\theta_1}{2}} & \left[\frac{\rho - r_1}{\theta_\infty - 1} + (**)\frac{\Gamma(-\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(1-\theta_1)\Gamma(\theta_\infty - \theta_1 - 1)}\right] (1-\lambda)^{\frac{\theta_1}{2}} + (**)\frac{\Gamma(\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(\theta_\infty - 1)}(1-\lambda)^{-\frac{\theta_1}{2}} \\ 0 & (1-\lambda)^{\frac{\theta_1}{2}} \end{pmatrix},$$

where:

$$(**) = \frac{\rho}{\theta_\infty - \theta_1 - 1} + \frac{r_1 - \rho}{\theta_\infty - 1}.$$

Let v, w be non zero arbitrary numbers. We conclude that:

$$\Psi_{OUT} = \begin{pmatrix} v & w \\ 0 & w\left[\frac{\rho - r_1}{\theta_\infty - 1} + (**)\frac{\Gamma(-\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(1-\theta_1)\Gamma(\theta_\infty - \theta_1 - 1)}\right]^{-1} \end{pmatrix} \, [I + O(1-\lambda)] \times$$

$$\times \begin{pmatrix} (1-\lambda)^{-\frac{\theta_1}{2}} & 0 \\ 0 & (1-\lambda)^{\frac{\theta_1}{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{v} & \frac{1}{v}(**)\frac{\Gamma(\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(\theta_\infty - 1)} \\ 0 & \frac{1}{w}\left[\frac{\rho - r_1}{\theta_\infty - 1} + (**)\frac{\Gamma(-\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(1-\theta_1)\Gamma(\theta_\infty - \theta_1 - 1)}\right] \end{pmatrix}$$

The monodromy matrix is then:

$$M_1^{OUT} = Q^{-1}e^{-i\pi\theta_1\sigma_3}Q, \quad Q := \begin{pmatrix} \frac{1}{v} & \frac{1}{v}(**)\frac{\Gamma(\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(\theta_\infty - 1)} \\ 0 & \frac{1}{w}\left[\frac{\rho - r_1}{\theta_\infty - 1} + (**)\frac{\Gamma(-\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(1-\theta_1)\Gamma(\theta_\infty - \theta_1 - 1)}\right] \end{pmatrix}.$$

This is independent of $\rho(x)$ if and only if $(**) = 0$. This proves (87). $\square$

(87) simplifies considerably the structure of (83):

$$\psi_1 = C_1 \, \lambda^{\frac{\theta_1 - \theta_\infty}{2}} \, (\lambda - 1)^{-\frac{\theta_1}{2}} - C_2 \, \frac{r_1}{\theta_1}\lambda^{\frac{\theta_\infty - \theta_1}{2} - 1} \, (\lambda - 1)^{\frac{\theta_1}{2}}.$$

With the choice $C_1 = \exp\{i\frac{\pi}{2}\theta_1\} = 1/C_2$, we finally have:

$$\Psi_{OUT} = \begin{pmatrix} \lambda^{\frac{\theta_1 - \theta_\infty}{2}} \, (1-\lambda)^{-\frac{\theta_1}{2}} & -\frac{r_1(x)}{\theta_1} \, \lambda^{\frac{\theta_\infty - \theta_1}{2} - 1} \, (1-\lambda)^{\frac{\theta_1}{2}} \\ 0 & \lambda^{\frac{\theta_\infty - \theta_1}{2}}(1-\lambda)^{\frac{\theta_1}{2}} \end{pmatrix}.$$

Note: The monodromy at $\lambda = 0, 1$ is diagonal: $M_0^{OUT} = \exp\{i\pi(\theta_1 - \theta_\infty)\sigma_3\}$, $M_1^{OUT} = \exp\{-i\pi\theta_1\sigma_3\}$. It is *independent of* r_1. In case **(I)** – namely, r_1 is a non-zero constant and the system for Ψ_1 is fuchsian – if M_0^{OUT} is in diagonal form, then M_1^{OUT} is upper triangular and depends on r_1.

6.2 Higher Orders Terms

We may repeat the same procedure of section 5.6. We write a Taylor expansion $y(x) = (\theta_1 - \theta_\infty + 1)/(1 - \theta_\infty) + \sum_{n \geq 1} b_n x^n$, substitute it into (PVI) and determine recursively the b_n's. Then, we may substitute the result in the matrix elements of A_i, $i = 0, x, 1$, according to the formulae of [16], and find the higher orders of the matrix elements. We just give the result.

$$(A_x)_{11} = -(A_x)_{22} = \frac{(\theta_\infty - \theta_1)^2 + \theta_x^2 - \theta_0^2}{4(\theta_1 - \theta_\infty)} +$$

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$$+ \frac{\theta_1}{8} \frac{[(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{(\theta_1 - \theta_\infty)^2[(\theta_1 - \theta_\infty)^2 - 1]} \; x + O(x^2).$$

$$(A_x)_{12} = -r \left\{ \frac{1}{x} - \frac{\theta_1[(\theta_1 - \theta_\infty + 2)(\theta_1 - \theta_\infty) + \theta_0^2 - \theta_x^2]}{2(\theta_1 - \theta_\infty)(\theta_\infty - \theta_1 - 1)} + O(x) \right\} x^{\theta_1 - \theta_\infty + 1}.$$

$$(A_x)_{21} = \frac{1}{r} \frac{[(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{16(\theta_\infty - \theta_1)^2} \Big\{ x +$$

$$- \frac{\theta_1[(\theta_\infty - \theta_1)(\theta_\infty - \theta_1 + 2) + \theta_0^2 - \theta_x^2]}{2(\theta_\infty - \theta_1)^3(\theta_\infty - \theta_1 + 1)} \; x^2 + O(x^3) \Big\} \; \frac{1}{x^{\theta_1 - \theta_\infty + 1}}.$$

$$(A_0)_{11} = -(A_0)_{22} = \frac{(\theta_\infty - \theta_1)^2 + \theta_0^2 - \theta_x^2}{4(\theta_1 - \theta_\infty)} +$$

$$- \frac{\theta_1}{8} \frac{[(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{(\theta_1 - \theta_\infty)^2[(\theta_1 - \theta_\infty)^2 - 1]} \; x + O(x^2).$$

$$(A_0)_{12} = r \left\{ \frac{1}{x} - \frac{\theta_1[(\theta_1 - \theta_\infty)^2 + \theta_0^2 - \theta_x^2]}{2(\theta_1 - \theta_\infty)(\theta_\infty - \theta_1 - 1)} + O(x) \right\} x^{\theta_1 - \theta_\infty + 1}.$$

$$(A_0)_{21} = -\frac{1}{r} \frac{[(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{16(\theta_\infty - \theta_1)^2} \Big\{ x +$$

$$- \frac{\theta_1[(\theta_1 - \theta_\infty)^2 + \theta_0^2 - \theta_x^2]}{2(\theta_\infty - \theta_1)^3(\theta_\infty - \theta_1 + 1)} \; x^2 + O(x^3) \Big\} \; \frac{1}{x^{\theta_1 - \theta_\infty + 1}}.$$

$$(A_1)_{11} = -(A_1)_{22} = -\frac{\theta_1}{2} - \frac{\theta_1[(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{16[(\theta_\infty - \theta_1)^2 - 1](\theta_\infty - \theta_1)^2} \; x^2 + O(x^3).$$

$$(A_1)_{12} = -r \frac{\theta_1}{\theta_\infty - \theta_1 - 1} \left\{ 1 + \frac{(\theta_1 + 1)\left[(\theta_\infty - \theta_1)(\theta_\infty - \theta_1 - 2) + \theta_0^2 - \theta_x^2\right]}{2(\theta_\infty - \theta_1)(\theta_\infty - \theta_1 - 2)} \; x + O(x^2) \right\} x^{\theta_1 - \theta_\infty + 1}.$$

$$(A_1)_{21} = \left\{ \frac{\theta_1[(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{16r(\theta_\infty - \theta_1 + 1)(\theta_\infty - \theta_1)^2} \; x^2 \right\} \; \frac{1}{x^{\theta_1 - \theta_\infty + 1}}.$$

The above leading terms of $y(x)$ are related to the above formulae through (3). The truncation of $(A_0)_{12}$ and $(A_1)_{12}$ above is enough to reproduce the first two terms of solution (15):

$$y(x) = \frac{\theta_1 - \theta_\infty + 1}{1 - \theta_\infty} + \frac{\theta_1[(\theta_1 - \theta_\infty)^2 + \theta_x^2 - \theta_0^2 + 2\theta_1 - 2\theta_\infty]}{2(1 - \theta_\infty)(\theta_\infty - \theta_1)(\theta_1 - \theta_\infty + 2)} \; x + O(x^2).$$

6.3   Transpose Case and General Result

In the above computations, the condition $x^{\theta_\infty - \theta_1 + 1} \to 0$ was necessary to do the matching. We can repeat the matching procedure starting from the transpose matrices:

$$\hat{A}_1 = \begin{pmatrix} -\frac{\theta_1}{2} & 0 \\ -r_1 & \frac{\theta_1}{2} \end{pmatrix}, \quad \hat{A}_0 + \hat{A}_x = \begin{pmatrix} \frac{\theta_1 - \theta_\infty}{2} & 0 \\ r_1 & -\frac{\theta_1 - \theta_\infty}{2} \end{pmatrix}, \quad x\,A_x = \begin{pmatrix} 0 & 0 \\ \rho(x) & 0 \end{pmatrix}.$$

The procedure is exactly the same, with the necessary condition:

$$x^{\theta_1 - \theta_\infty + 1} \to 0, \quad \text{when } x \to 0.$$

As a result we obtain again exactly the matrices A_0, A_x, A_1 above. The reader can convince himself without doing any computation, simply looking at the structure of the first terms of above matrices.

For example, let us have a look at A_x. Denote the constant terms with letters c_1, c_2, ..., etc. We have:

$$A_x = \begin{pmatrix} c_1 + ... & \left\{\frac{c_2}{x} + ...\right\} x^{\theta_1 - \theta_\infty + 1} \\ \left\{c_3\, x\, + ...\right\} \frac{1}{x^{\theta_1 - \theta_\infty + 1}} & -c_1 + ... \end{pmatrix} =$$

$$= \begin{pmatrix} c_1 + ... & \left\{c_2\, x\, + ...\right\} \frac{1}{x^{\theta_\infty - \theta_1 + 1}} \\ \left\{\frac{c_3}{x} + ...\right\} x^{\theta_\infty - \theta_1 + 1} & -c_1 + ... \end{pmatrix}.$$

The role of $\theta_\infty - \theta_1 + 1$ and $\theta_1 - \theta_\infty + 1$ is just exchanged. By continuity, the matrices A_0, A_x, A_1 computed above hold *for any value of $\theta_1 - \theta_\infty \notin \mathbf{Z}$, $\theta_\infty \neq 1$.*

The matching procedure can be repeated in the same way in case $\sigma = -(\theta_\infty - \theta_1)$ and in the case $\sigma = \pm(\theta_\infty + \theta_1)$, which yields (18). For this last cases the results are obtained just by the substitution $\theta_1 \mapsto -\theta_1$.

6.4   Monodromy Data

We compute the monodromy data for the case $\sigma = \theta_1 - \theta_\infty$, all the other cases being analogous. In this case, the matching has been realized by:

$$\Psi_{OUT}(\lambda, x) := \begin{pmatrix} \lambda^{\frac{\theta_1 - \theta_\infty}{2}}(1 - \lambda)^{-\frac{\theta_1}{2}} & -\frac{r_1(x)}{\theta_1}\lambda^{\frac{\theta_\infty - \theta_1}{2} - 1}(1 - \lambda)^{\frac{\theta_1}{2}} \\ 0 & \lambda^{\frac{\theta_\infty - \theta_1}{2}}(1 - \lambda)^{\frac{\theta_1}{2}} \end{pmatrix}.$$

$$\Psi_{IN}(\lambda, x) = K_0(x)\Psi_0\left(\frac{\lambda}{x}\right), \qquad \Psi_0(\mu) = \left[I + O\left(\frac{1}{\mu}\right)\right] \mu^{\frac{\theta_1 - \theta_\infty}{2}\sigma_3}, \quad \mu \to \infty.$$

Let $\Psi(\lambda)$ denote the solution of the system (1) of (PVI), such that: $\Psi(\lambda) = [I + O(\lambda^{-1})]\,\lambda^{-\frac{\theta_\infty}{2}\sigma_3}$, $\lambda \to \infty$, $\theta_\infty \notin \mathbf{Z}$.

MATCHING $\Psi \leftrightarrow \Psi_{OUT}$.

With the choice $1 - \lambda = e^{-i\pi}(\lambda - 1)$ ($1 - \lambda > 0$ for $0 < \lambda < 1$) we have:

$$\Psi_{OUT} = \left[I + O\left(\frac{1}{\lambda}\right)\right] \lambda^{-\frac{\theta_\infty}{2}\sigma_3}\, e^{i\frac{\pi}{2}\theta_1\sigma_3}, \quad \lambda \to \infty$$

Therefore, the correct choice for Ψ_{OUT}, which matches with Ψ is:

$$\Psi_{OUT}^{Match} := \Psi_{OUT}C_{OUT}, \qquad C_{OUT} := e^{-i\frac{\pi}{2}\theta_1\sigma_3}.$$

As a consequence we obtain:

i) the monodromy of Ψ at $\lambda = 1, \infty$, which is equal to the monodromy of Ψ_{OUT}^{Match}:

$$M_1 = e^{-i\pi\theta_1\sigma_3}, \quad M_\infty = e^{-i\pi\theta_\infty\sigma_3};$$

ii) the correct choice for $\Psi_{IN}^{Match} := K_0(x)\,\Psi_0\,C_{OUT}$.

MATCHING $\Psi \leftrightarrow \Psi_{IN}^{Match}$

This is realized by construction. A consequence, we can compute the monodromy of Ψ at $\lambda = 0, x$. In order to do this, we need the local behavior of $\Psi_{IN}^{Match}(\lambda, x)$ at $\lambda = 0, x$. We start with Ψ_0, recalling that:

$$\Psi_0(\mu) = \mu^{\frac{\theta_0}{2}}(\mu - 1)^{\frac{\theta_x}{2}}\Phi_0(\mu), \qquad \Phi_0(\mu) = \begin{pmatrix} \varphi_1(\mu) & \varphi_2(\mu) \\ \xi_1(\mu) & \xi_2(\mu) \end{pmatrix}.$$

Here, φ_1, φ_2 are two independent solutions of a Gauss hyper-geometric equation (see (105) in Appendix 1):

$$\mu(1-\mu)\frac{d^2\varphi}{d\mu^2} + \left(1 + c - (a + [b+1] + 1)\,\mu\right)\frac{d\varphi}{d\mu} - a(b+1)\,\varphi = 0,$$

where $a := \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}$, $b+1 := \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1$, $c+1 := \theta_0 + 1$. The functions ξ_1 and ξ_2 are obtained from φ_1 and φ_2 by:

$$\xi = \frac{1}{r}\left[\mu(1-\mu)\frac{d\varphi}{d\mu} - a\left(\mu + \frac{b-c}{a-b}\right)\varphi\right], \tag{89}$$

In order to have generic solutions (i.e. non-logarithmic solutions) of the hyper-geometric equation, we must require:

$$\theta_1 - \theta_\infty,\ \theta_0,\ \theta_x \notin \mathbf{Z}.$$

Then, we have the following sets of independent solutions at $\mu = 0, 1, \infty$ respectively (we denote by F the Gauss hyper-geometric function):

$$\left\{\begin{array}{l} \varphi_1^{(0)} = F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2},\ \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1,\ 1 + \theta_0;\ \mu\right), \\[2mm] \varphi_2^{(0)} = \mu^{-\theta_0}\,F\left(\frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2},\ \frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1,\ 1 - \theta_0;\ \mu\right). \end{array}\right.$$

$$\left\{\begin{array}{l} \varphi_1^{(1)} = F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2},\ \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1,\ 1 + \theta_x;\ 1 - \mu\right), \\[2mm] \varphi_2^{(1)} = (1-\mu)^{-\theta_x}\,F\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2},\ \frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1,\ 1 - \theta_x;\ 1 - \mu\right). \end{array}\right.$$

$$\left\{\begin{array}{l} \varphi_1^{(\infty)} = \mu^{-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{\theta_\infty}{2} + \frac{\theta_1}{2}}\,F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2},\ \frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2},\ \theta_\infty - \theta_1;\ \frac{1}{\mu}\right), \\[2mm] \varphi_2^{(\infty)} = \mu^{-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{\theta_1}{2} + \frac{\theta_\infty}{2} - 1}\,F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1,\ \frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1,\ 2 + \theta_1 - \theta_\infty;\ \frac{1}{\mu}\right). \end{array}\right.$$

The connection formulae can be found in any book on special functions:

$$\left[\varphi_1^{(0)}, \varphi_2^{(0)}\right] = \left[\varphi_1^{(1)}, \varphi_2^{(1)}\right] C_{01}, \quad -\pi < \arg(1-\mu) < \pi.$$

$$\left[\varphi_1^{(0)}, \varphi_2^{(0)}\right] = \left[\varphi_1^{(\infty)}, \varphi_2^{(\infty)}\right] C_{0\infty}, \quad 0 < \arg\mu < 2\pi.$$

$$C_{01} := \begin{bmatrix} \dfrac{\Gamma(-\theta_x)\Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1\right)\Gamma\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}\right)} & \dfrac{\Gamma(-\theta_x)\Gamma(1-\theta_0)}{\Gamma\left(-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{\theta_\infty}{2} + \frac{\theta_1}{2} + 1\right)\Gamma\left(-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{\theta_1}{2} + \frac{\theta_\infty}{2}\right)} \\[4mm] \dfrac{\Gamma(\theta_x)\Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}\right)\Gamma\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1\right)} & \dfrac{\Gamma(\theta_x)\Gamma(1-\theta_0)}{\Gamma\left(\frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}\right)\Gamma\left(\frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1\right)} \end{bmatrix}, \tag{90}$$

$$C_{0\infty} := \begin{bmatrix} \dfrac{\Gamma\left(1 + \frac{\theta_1}{2} - \frac{\theta_\infty}{2}\right)\Gamma(1+\theta_0)e^{i\frac{\pi}{2}[\theta_0 + \theta_x + \theta_\infty - \theta_1]}}{\Gamma\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1\right)\Gamma\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1\right)} & \dfrac{\Gamma\left(1 + \frac{\theta_1}{2} - \frac{\theta_\infty}{2}\right)\Gamma(1-\theta_0)e^{i\frac{\pi}{2}[\theta_x - \theta_0 + \theta_\infty - \theta_1]}}{\Gamma\left(-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{\theta_\infty}{2} + \frac{\theta_1}{2} + 1\right)\Gamma\left(\frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1\right)} \\[4mm] -\dfrac{\Gamma\left(\frac{\theta_\infty}{2} - \frac{\theta_1}{2} - 1\right)\Gamma(1+\theta_0)e^{i\frac{\pi}{2}[\theta_0 + \theta_x + \theta_1 - \theta_\infty]}}{\Gamma\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}\right)\Gamma\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}\right)} & -\dfrac{\Gamma\left(\frac{\theta_\infty}{2} - \frac{\theta_1}{2} - 1\right)\Gamma(1-\theta_0)e^{i\frac{\pi}{2}[\theta_x - \theta_0 + \theta_1 - \theta_\infty]}}{\Gamma\left(-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{\theta_1}{2} + \frac{\theta_\infty}{2}\right)\Gamma\left(\frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}\right)} \end{bmatrix}, \tag{91}$$

From the Taylor expansion of the hyper-geometric functions in $\varphi_i^{(\infty)}$ and (89) we compute:

$$\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_1^{(\infty)} = \left[1 + O\left(\frac{1}{\mu}\right)\right]\mu^{\frac{\theta_1 - \theta_\infty}{2}}, \quad \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_2^{(\infty)} = \frac{1}{\mu}\left[1 + O\left(\frac{1}{\mu}\right)\right]\mu^{\frac{\theta_\infty - \theta_1}{2}},$$

$$\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\xi_1^{(\infty)} = \left[1 + O\left(\frac{1}{\mu}\right)\right]\mu^{\frac{\theta_1 - \theta_\infty}{2}}, \quad \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\xi_2^{(\infty)} = \frac{\theta_1 - \theta_\infty + 1}{r_0}\left[1 + O\left(\frac{1}{\mu}\right)\right]\mu^{\frac{\theta_\infty - \theta_1}{2}},$$

It follows that the matrix $\Psi_0(\mu) = \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\Phi_0(\mu)$ with the prescribed behavior $\left[I + O\left(\frac{1}{\mu}\right)\right]\mu^{\frac{\theta_1-\theta_\infty}{2}\sigma_3}$ at $\mu = \infty$ is:

$$\Psi_0(\mu) = \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(\infty)} & \varphi_2^{(\infty)} \\ \xi_1^{(\infty)} & \xi_2^{(\infty)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{r}{\theta_1-\theta_\infty+1} \end{pmatrix}$$

Let:

$$C := \begin{pmatrix} 1 & 0 \\ 0 & \frac{r}{\theta_1-\theta_\infty+1} \end{pmatrix} C_{OUT} \equiv \begin{pmatrix} 1 & 0 \\ 0 & \frac{r}{\theta_1-\theta_\infty+1} \end{pmatrix} e^{-i\frac{\pi}{2}\theta_1\sigma_3}.$$

We conclude that:

$$\Psi_{IN}^{Match}(\lambda, x) = K_0(x) \left(\frac{\lambda}{x}\right)^{\frac{\theta_0}{2}} \left(\frac{\lambda}{x} - 1\right)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(\infty)}\left(\frac{\lambda}{x}\right) & \varphi_2^{(\infty)}\left(\frac{\lambda}{x}\right) \\ \xi_1^{(\infty)}\left(\frac{\lambda}{x}\right) & \xi_2^{(\infty)}\left(\frac{\lambda}{x}\right) \end{pmatrix} C,$$

$$= K_0(x) \left(\frac{\lambda}{x}\right)^{\frac{\theta_0}{2}} \left(\frac{\lambda}{x} - 1\right)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(0)}\left(\frac{\lambda}{x}\right) & \varphi_2^{(0)}\left(\frac{\lambda}{x}\right) \\ \xi_1^{(0)}\left(\frac{\lambda}{x}\right) & \xi_2^{(0)}\left(\frac{\lambda}{x}\right) \end{pmatrix} C_{0\infty}^{-1} C,$$

$$= K_0(x) \left(\frac{\lambda}{x}\right)^{\frac{\theta_0}{2}} \left(\frac{\lambda}{x} - 1\right)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(1)}\left(\frac{\lambda}{x}\right) & \varphi_2^{(1)}\left(\frac{\lambda}{x}\right) \\ \xi_1^{(1)}\left(\frac{\lambda}{x}\right) & \xi_2^{(1)}\left(\frac{\lambda}{x}\right) \end{pmatrix} C_{01} C_{0\infty}^{-1} C.$$

The behaviors of the above matrix at $\lambda = x$, 0 is easily computed from:

$$\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_1^{(1)} =$$
$$= \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}F(...;1-\mu) = x^{-\frac{\theta_0}{2}-\frac{\theta_x}{2}}F\left(...;\frac{x-\lambda}{x}\right)\lambda^{\frac{\theta_0}{2}}(\lambda-x)^{\frac{\theta_x}{2}},$$

$$\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_2^{(1)} =$$
$$= \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}(1-\mu)^{-\frac{\theta_x}{2}} = (-1)^{\theta_x}x^{-\frac{\theta_0}{2}+\frac{\theta_x}{2}}F\left(...;\frac{x-\lambda}{x}\right)\lambda^{\frac{\theta_0}{2}}(\lambda-x)^{-\frac{\theta_x}{2}};$$

and:

$$\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_1^{(0)} =$$
$$= \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}F(...;\mu) = x^{-\frac{\theta_0}{2}-\frac{\theta_x}{2}} F\left(...;\frac{\lambda}{x}\right)(\lambda-x)^{\frac{\theta_x}{2}}\lambda^{\frac{\theta_0}{2}},$$

$$\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_2^{(0)} =$$
$$= \mu^{-\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}F(...;\mu) = x^{\frac{\theta_0}{2}-\frac{\theta_x}{2}} F\left(...;\frac{\lambda}{x}\right)(\lambda-x)^{\frac{\theta_x}{2}}\lambda^{-\frac{\theta_0}{2}}.$$

From these we compute:

$$K_0(x) \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(0)} & \varphi_2^{(0)} \\ \xi_1^{(0)} & \xi_2^{(0)} \end{pmatrix} = \psi_0^{IN}(x)(I + O(\lambda))\,\lambda^{\frac{\theta_0}{2}\sigma_3}, \quad \lambda \to 0$$

$$K_0(x) \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} \\ \xi_1^{(1)} & \xi_2^{(1)} \end{pmatrix} = \psi_x^{IN}(x)(I + O(\lambda-x))\,(\lambda-x)^{\frac{\theta_x}{2}\sigma_3}, \quad \lambda \to x.$$

Here, we don't need to explicitly give the invertible matrices $\psi_0^{IN}(x)$ and $\psi_x^{IN}(x)$. From the above procedure, we find:

$$M_0 = C^{-1}\left(C_{0\infty} \exp\{i\pi\theta_0\sigma_3\} C_{0\infty}^{-1}\right) C, \quad M_x = C^{-1}\left[C_{0\infty}\left(C_{01}^{-1} \exp\{i\pi\theta_x\sigma_3\} C_{01}\right) C_{0\infty}^{-1}\right] C.$$

We finally observe that M_1 and M_∞ are diagonal, so they are invariant for the conjugation $M \mapsto CMC^{-1}$. With this in mind, we get the result of theorem 3, point a).

We stress that $\hat{A}_0 + \hat{A}_x$ and $\hat{A}_1$ are upper (or lower) triangular matrices, and *the group generated by $M_x M_0$ and M_1 is reducible.*

7 Case $\theta_\infty = 1$, $\theta_1 = 0$, $\sigma = \pm 1$. Solution (17).

In section 6 we imposed that $\theta_\infty \neq 1$ and $\theta_\infty - \theta_1 \notin \mathbf{Z}$. Here we consider the case $\theta_\infty = 1$, $\theta_\infty - \theta_1 = 1$. We have:

$$\hat{A}_0 + \hat{A}_x = \begin{pmatrix} -\frac{1}{2} & r_1 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad \hat{A}_1 = \begin{pmatrix} 0 & -r_1 \\ 0 & 0 \end{pmatrix}.$$

Also the transpose matrices are possible. For $|\lambda| \leq |x|^{\delta_{IN}}$, we use the reduction (39) (namely, (11)). The matrices $\hat{A}_0$, $\hat{A}_x$ are given by (40) and (41) with the substitution $\sigma = 1$ or -1. For definiteness, let us take $\sigma = 1$ in the following.

For $|\lambda| \geq |x|^{\delta_{OUT}}$, we approximate (1) with: $\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{xA_x}{\lambda^2} + \frac{A_0 + A_x}{\lambda} + \frac{A_1}{\lambda - 1} \right] \Psi_{OUT}$. For definiteness, let us consider the case when $\hat{A}_1$ and $\hat{A}_0 + \hat{A}_x$ are upper triangular. Again, we make the hypothesis that the leading terms in xA_x define an upper triangular matrix:

$$xA_x =: \begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix} + \text{ higher orders.}$$

Therefore, we will study:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{1}{\lambda^2} \begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} -\frac{1}{2} & r_1 \\ 0 & \frac{1}{2} \end{pmatrix} + \frac{1}{\lambda - 1} \begin{pmatrix} 0 & -r_1 \\ 0 & 0 \end{pmatrix} \right] \Psi_{OUT}$$

This is a reducible system. To solve it, we write Ψ_{OUT} in vector notation $\Psi_{OUT} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. The system becomes:

$$\frac{d\psi_1}{d\lambda} = -\frac{1}{2\lambda}\psi_1 + \left(\frac{\rho}{\lambda^2} + \frac{r_1}{\lambda} - \frac{r_1}{\lambda - 1} \right) \psi_2, \qquad \frac{d\psi_2}{d\lambda} = \frac{1}{2\lambda}\psi_2.$$

The solution obtained by variation of parameters is:

$$\psi_1 = C_1 \lambda^{-\frac{1}{2}} + C_2 \lambda^{-\frac{1}{2}}(\rho \ln \lambda - r_1 \ln(\lambda - 1)), \qquad \psi_2 = C_2 \lambda^{\frac{1}{2}}, \qquad C_1, C_2 \in \mathbf{C}.$$

We can choose the following fundamental matrix:

$$\Psi_{OUT} = \begin{pmatrix} \lambda^{-\frac{1}{2}} & \lambda^{-\frac{1}{2}}\left(\rho \ln \lambda - r_1 \ln(\lambda - 1)\right) \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -r_1 \ln(\lambda - 1) \\ 0 & 1 \end{pmatrix}.$$

Its monodromy relative to the loops $\lambda \mapsto \lambda e^{2\pi i}$ and $(\lambda - 1) \mapsto (\lambda - 1)e^{2\pi i}$ is respectively:

$$\Psi_{OUT} \mapsto \Psi_{OUT} \begin{pmatrix} -1 & -2\pi i\rho \\ 0 & -1 \end{pmatrix}. \qquad \Psi_{OUT} \mapsto \Psi_{OUT} \begin{pmatrix} 1 & -2\pi i r_1 \\ 0 & 1 \end{pmatrix}.$$

Therefore, ρ and r_1 *must be independent of* (the monodromy preserving deformation) x. We observe that any fundamental matrix solution of the form:

$$\Psi_{OUT} \begin{pmatrix} 1 & f(x) \\ 0 & 1 \end{pmatrix},$$

has the same monodromy of Ψ_{OUT}, for any arbitrary function of x. This fact will be used soon, with the choice $f(x) = -\rho \ln x$.

With the choice of the branch of $\ln(\lambda - 1) = \ln(1 - \lambda) + i\pi$ (i.e. $(\lambda - 1) = e^{i\pi}(1 - \lambda)$, $1 - \lambda > 0$ for $0 < \lambda < 1$), it is convenient to redefine Ψ_{OUT} by:

$$\Psi_{OUT} := \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -r_1 \ln(\lambda - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i\pi r_1 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -\frac{r_1}{\lambda} \ln(1 - \lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix}.$$

Therefore:

$$\Psi_{OUT} = \left[\begin{pmatrix} 1 & r_1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & i\pi r_1 \\ 0 & 0 \end{pmatrix} \sum_{n=1}^{\infty} \frac{\lambda^n}{n+1} \right] \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix}, \quad \lambda \to 0$$

7.1 Matching

The solution $\Psi_0\left(\frac{\lambda}{x}\right)$ has been introduced in section 4.7.

$$\Psi_0\left(\frac{\lambda}{x}\right) = \left[I + O\left(\frac{x}{\lambda}\right)\right] \begin{pmatrix} \left(\frac{\lambda}{x}\right)^{\frac{1}{2}} & 0 \\ 0 & \left(\frac{\lambda}{x}\right)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R \ln\left(\frac{\lambda}{x}\right) & 1 \end{pmatrix}, \quad R := (\hat{A}_x)_{21}.$$

Some adjustments are necessary. Let us consider the permutation matrix $P := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and redefine:

$$\Psi_0 \mapsto \Psi_0 \, P \begin{pmatrix} 1 & 0 \\ 0 & R^{-1} \end{pmatrix}, \qquad \Psi_{OUT} \mapsto \Psi_{OUT} \begin{pmatrix} 1 & -\rho\ln x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \rho^{-1} \end{pmatrix}.$$

As we have already observed, this re-definition does not affect the monodromy of Ψ_{OUT}, which is independent of x. The matching relation $\Psi_{OUT}(\lambda) \sim K_0(x)\Psi_0\left(\lambda/x\right)$ becomes:

$$\begin{pmatrix} 1 & r_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \ln\frac{\lambda}{x} \\ 0 & 1 \end{pmatrix} \sim$$

$$\sim K_0(x) \, P \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{R} \end{pmatrix} \begin{pmatrix} x^{\frac{1}{2}} & 0 \\ 0 & x^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \ln\frac{\lambda}{x} \\ 0 & 1 \end{pmatrix}.$$

The matching is thus realized, with the choice:

$$K_0(x) \sim \begin{pmatrix} 1 & \frac{r_1}{\rho} \\ 0 & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} x^{-\frac{1}{2}} & 0 \\ 0 & R\, x^{\frac{1}{2}} \end{pmatrix} P \equiv \begin{pmatrix} \frac{r_1}{\rho} & 1 \\ \frac{1}{\rho} & 0 \end{pmatrix} \begin{pmatrix} R\, x^{\frac{1}{2}} & 0 \\ 0 & x^{-\frac{1}{2}} \end{pmatrix}.$$

It follows that:

$$\hat{A}_0 = \begin{pmatrix} \frac{\theta_x^2 - \theta_0^2 - 1}{4} + \frac{r_1}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{16} x & -\frac{\rho}{x} + r_1 \frac{\theta_0^2 - \theta_x^2 + 1}{2} - \frac{r_1^2}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{16} x \\ \frac{1}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{16} x & -\frac{\theta_x^2 - \theta_0^2 - 1}{4} - \frac{r_1}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{16} x \end{pmatrix},$$

$$\hat{A}_x = \begin{pmatrix} \frac{\theta_0^2 - \theta_x^2 - 1}{4} - \frac{r_1}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{16} x & \frac{\rho}{x} + r_1 \frac{\theta_x^2 - \theta_0^2 + 1}{2} + \frac{r_1^2}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{16} x \\ -\frac{1}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{16} x & -\frac{\theta_0^2 - \theta_x^2 - 1}{4} + \frac{r_1}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{16} x \end{pmatrix}.$$

The leading term of each matrix element certainly contains no error. Note that r has simplified. Actually, the constant ρ plays the role of r.

We will not repeat again the discussion for the higher order terms. The matrix elements are Taylor expansions, corresponding to a Taylor expanded $y(x)$, the convergence of which is proved by a Briot-Bouquet argument. The first two leading terms of A_0 and A_x above are actually correct. In particular, we need:

$$(A_0)_{12} = -\frac{\rho}{x} + r_1 \frac{\theta_0^2 - \theta_x^2 + 1}{2} + O(x).$$

The leading terms of A_1 are:

$$A_1 = \begin{pmatrix} \frac{r_1}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{32} x^2 + O(x^3) & -r_1 + r_1 \frac{\theta_0^2 - \theta_x^2 - 1}{2} x + O(x^2) \\ \frac{r_1}{\rho^2} \frac{[1-(\theta_0-\theta_x)^2]^2[1-(\theta_0+\theta_x)^2]^2}{1024} x^4 + O(x^5) & -\frac{r_1}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{32} x^2 + O(x^3) \end{pmatrix}.$$

The above truncations of $(A_1)_{12}$ and $(A_0)_{12}$ corresponding to the first two terms of the Taylor expansion of the solution (17), through (3):

$$y(x) = a + \frac{1-a}{2}(1 + \theta_0^2 - \theta_x^2)\, x + O(x^2), \qquad \text{where } a := \left(1 - \frac{r_1}{\rho}\right)^{-1}. \tag{92}$$

Observe that a depends on the monodromy datum $\frac{r_1}{\rho}$.

7.2 Monodromy

The matching $\Psi_{IN} \leftrightarrow \Psi_{OUT}$ has been realized by:

$$\Psi_{OUT} = \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -r_1 \ln(\lambda - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\ln x + i\pi\frac{r_1}{\rho} \\ 0 & \frac{1}{\rho} \end{pmatrix},$$

and:

$$\Psi_{IN}(\mu) = K_0(x)\Psi_0(\mu) \begin{pmatrix} 0 & R^{-1} \\ 1 & 0 \end{pmatrix}, \qquad R = \frac{[(\theta_0 + \theta_x)^2 - 1][(\theta_0 - \theta_x)^2 - 1]}{16r}.$$

$$\Psi_0(\mu) = \left[I + O\left(\frac{1}{\mu}\right)\right] \begin{pmatrix} \mu^{\frac{1}{2}} & 0 \\ 0 & \mu^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R \ln \mu & 1 \end{pmatrix}, \qquad \mu \to \infty.$$

Namely:

$$\Psi_{IN} \sim K_0(x) P \begin{pmatrix} x^{\frac{1}{2}} & 0 \\ 0 & R^{-1} x^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \ln\frac{\lambda}{x} \\ 0 & 1 \end{pmatrix},$$

$$= \begin{pmatrix} 1 & \frac{r_1}{\rho} \\ 0 & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \ln\frac{\lambda}{x} \\ 0 & 1 \end{pmatrix},$$

for $\mu = \lambda/x \to \infty$, $|\lambda| < |x|^{\delta_{IN}}$, $x \to 0$.

MATCHING $\Psi \leftrightarrow \Psi_{OUT}$.

The correct choice of Ψ_{OUT} must match with:

$$\Psi = \left[I + O\left(\frac{1}{\lambda}\right)\right] \lambda^{-\frac{\theta_\infty}{2}\sigma_3} \lambda^L, \quad \lambda \to \infty; \qquad \theta_\infty = 1;$$

where: $L = \begin{pmatrix} 0 & \rho - r_1 \\ 0 & 0 \end{pmatrix}$. This form of L follows from the standard theory of Fuchsian systems, and from the expansion of the system (1) at $\nu := \frac{1}{\lambda} \to 0$:

$$\frac{d\Psi}{d\nu} = \left[\frac{\sigma_3}{2}\frac{1}{\nu} - (A_1 + xA_x) + O(\nu)\right] \Psi, \qquad \nu \to \infty;$$

Thus $L_{12} = -(A_1 + xA_x)_{1,2}|_{x=0} \equiv r_1 - \rho$ (this is computed from the expansions of A_1 and A_x at $x = 0$ obtained before). We expand Ψ_{OUT} at $\lambda = \infty$. We easily get:

$$\Psi_{OUT} = [I + O(\lambda^{-1})] \lambda^{-\frac{1}{2}\sigma_3} \begin{pmatrix} 1 & (\rho - r_1) \ln \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i\pi\frac{r_1}{\rho} - \ln x \\ 0 & \frac{1}{\rho} \end{pmatrix}.$$

Therefore, the correct choice is:

$$\Psi_{OUT}^{Match} := \Psi_{OUT} \begin{pmatrix} 1 & i\pi\frac{r_1}{\rho} - \ln x \\ 0 & \frac{1}{\rho} \end{pmatrix}^{-1} = \lambda^{-\frac{1}{2}\sigma_3} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -r_1 \ln(\lambda - 1) \\ 0 & 1 \end{pmatrix}.$$

As a consequence, we obtain:

i) the monodromy of Ψ at $\lambda = 1, \infty$. This coincides with that of Ψ_{OUT}^{Match}, which is easily computed from the local behavior:

$$\Psi_{OUT}^{Match}(\lambda) = [I + O(\lambda - 1)] \begin{pmatrix} 1 & -r_1 \ln(\lambda - 1) \\ 0 & 1 \end{pmatrix}, \qquad \lambda \to 1.$$

Thus, for $(\lambda - 1) \mapsto (\lambda - 1) \exp\{2\pi i\}$ and $\lambda \mapsto \lambda \exp\{2\pi i\}$ the monodromy is:

$$M_1 = \begin{pmatrix} 1 & -2\pi i r_1 \\ 0 & 1 \end{pmatrix}, \qquad M_\infty = \begin{pmatrix} -1 & 2\pi i(r_1 - \rho) \\ 0 & -1 \end{pmatrix}.$$

ii) the correct choice of Ψ_{IN}, which matches with Ψ. This is:

$$\Psi_{IN}^{Match}(\lambda, x) = \Psi_{IN}(\lambda, x) \begin{pmatrix} 1 & i\pi\frac{r_1}{\rho} - \ln x \\ 0 & \frac{1}{\rho} \end{pmatrix}^{-1} = \Psi_{IN}(\lambda, x) \begin{pmatrix} 1 & \rho \ln x - i\pi r_1 \\ 0 & \rho \end{pmatrix} =$$

$$= K_0(x)\Psi_0(\mu) \, C_{IN}, \qquad C_{IN} = \begin{pmatrix} 0 & \rho/R \\ 1 & \rho \ln x - i\pi r_1 \end{pmatrix}.$$

Important Remark: C_{IN} depends on x. But the monodromy must be independent of x. For this reason, we will have to consider the substitution

$$\Psi_{IN}^{Match} \mapsto \Psi_{IN}^{Match}\left(C_{IN}\right)^{-1} = K_0(x)\Psi_0(\mu).$$

This makes the monodromy at $\lambda = 0, x$ independent of x. The corresponding transformation:

$$\Psi_{OUT}^{Match} \mapsto \Psi_{OUT}^{Match} C_{IN}^{-1},$$

changes M_1, M_∞, but does not introduce a dependence on x. Namely:

$$M_1 \mapsto C_{IN} M_1 C_{IN}^{-1} = \begin{pmatrix} 1 & 0 \\ -2\pi i \frac{r_1}{\rho} R & 1 \end{pmatrix}, \quad M_\infty \mapsto C_{IN} M_\infty C_{IN}^{-1} = \begin{pmatrix} -1 & 0 \\ 2\pi i \left(\frac{r_1}{\rho} - 1\right) R & -1 \end{pmatrix}.$$

MATCHING $\Psi \leftrightarrow \Psi_{IN}$.

Ψ and $\Psi_{IN}^{Match}(\lambda, x)$ are matching by construction. The monodromy of Ψ at $\lambda = 0, x$ coincides with that of $\Psi_{IN}^{Match}(\lambda, x)$. In order to compute it, we write:

$$\Psi_0(\mu) = \mu^{\frac{\theta_0}{2}}(\mu - 1)^{\frac{\theta_x}{2}}\Phi_0(\mu), \qquad \Phi_0(\mu) = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \xi_1 & \xi_2 \end{pmatrix}.$$

Φ_0 satisfies (42) with $\sigma = 1$. It is expressed in terms of two independent solutions φ_1, φ_2 of the Gauss hyper-geometric equation (see Appendix 1):

$$\mu(1-\mu)\frac{d^2\varphi}{d\mu^2} + \left(1 + c - (a + [b+1] + 1)\,\mu\right)\frac{d\varphi}{d\mu} - a(b+1)\,\varphi = 0,$$

where,

$$a = \frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2}, \quad b + 1 = \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, \quad c + 1 = \theta_0 + 1.$$

From φ_1 and φ_2 we compute:

$$\xi_i = \frac{1}{r}\left[\mu(1-\mu)\frac{d\varphi_i}{d\mu} - a\left(\mu + \frac{b-c}{a-b}\right)\varphi_i\right], \quad i = 1, 2.$$

We need a compete set of solutions at $\mu = 0, 1, \infty$. Since $a - b$ is an integer, we are in a logarithmic case. We briefly explain some preliminary facts. Let us consider a Gauss Hyper-geometric equation in standard form:

$$\mu\,(1-\mu)\frac{d^2\varphi}{d\mu^2} + \left[\gamma - (\alpha + \beta + 1)\,\mu\right]\frac{d\varphi}{d\mu} - \alpha\beta\,\varphi = 0$$

$(\alpha, \beta, \gamma$ here are not the coefficients of (PVI)! We are just using the same symbols only here). Logarithmic solutions at $\mu = 0$ may occur only if $\gamma \in \mathbf{Z}$. Logarithmic solutions at $\mu = 1$ may occur only if $\alpha + \beta - \gamma \in \mathbf{Z}$. Logarithmic solutions at $\mu = \infty$ may occur only if $\alpha - \beta \in \mathbf{Z}$. Several sub-case must be distinguished, and this is not the place to discuss them.

In our case $\alpha = a$, $\beta = b + 1$ and $\gamma = c + 1$. Therefore $\alpha - \beta = -2$. This case is logarithmic. Two independent solutions are:

$$\varphi_1^{(\infty)} = \mu^{-\beta}g_1\left(\beta, \beta - \gamma + 1, 1 + \beta - \alpha; \frac{1}{\mu}\right) = [\text{ in our case }] \ \mu^{-\alpha-2}g_1\left(\beta, \beta - \gamma + 1, 3; \frac{1}{\mu}\right),$$

$$\varphi_2^{(\infty)} = \mu^{-\beta} F\left(\beta, \beta - \gamma + 1, 1 + \beta - \alpha; \frac{1}{\mu}\right) = [\text{ in our case }] \ \mu^{-\beta} F\left(\beta, \beta - \gamma + 1, 3; \frac{1}{\mu}\right).$$

Here F is the Gauss hyper-geometric function and g_1 is a logarithmic solution introduced in Norlund's paper [21], page 7:

$$g_1(u, v, w; z) = \sum_{n=1}^{w-1} (-1)^{n-1}(n-1)! \ \frac{(u)_{-n}(v)_{-n}}{(w)_{-n}} \frac{1}{z^n} + F(u, v, w; \ z) \ \ln(-z) + $$

$$+ \sum_{n=0}^{\infty} \frac{(u)_n (v)_n}{n!(w)_n} \big[\psi(1 - u - n) + \psi(v + n) - \psi(w + n) - \psi(1 + n)\big] z^n,$$

where

$$\psi(x) := \frac{d}{dx}\Gamma(x), \qquad u \neq 1, 2, 3, \dots ; \quad |z| < 1, \quad \ln(-z) < 0 \ \text{for} \ -1 < z < 0.$$

Note that the first sum $\sum_{n=1}^{w-1}(-1)^{n-1}(n-1)! \ \frac{(u)_{-n}(v)_{-n}}{(w)_{-n}} \frac{1}{z^n}$ is a polynomial in $1/z$.

If $\theta_0, \theta_x \notin \mathbf{Z}$, we fall in the non-logarithmic cases at $\mu = 0, 1$. Thus:

$$\varphi_1^{(0)} = F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2}, \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, 1 + \theta_0; \mu\right),$$

$$\varphi_2^{(0)} = \mu^{-\theta_0} F\left(-\frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2}, -\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, 1 - \theta_0; \mu\right).$$

$$\varphi_1^{(1)} = F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2}, \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, 1 + \theta_x; 1 - \mu\right),$$

$$\varphi_2^{(1)} = (1 - \mu)^{-\theta_x} F\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{1}{2}, \frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}, 1 - \theta_x; 1 - \mu\right).$$

$$\varphi_1^{(\infty)} = \mu^{-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{3}{2}} \ g_1\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, -\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, 3; \frac{1}{\mu}\right),$$

$$\varphi_2^{(\infty)} = \mu^{-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{3}{2}} \ F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, -\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, 3; \frac{1}{\mu}\right).$$

The connection matrix between $[\varphi_1^{(0)}, \varphi_2^{(0)}]$ and $[\varphi_1^{(1)}, \varphi_2^{(1)}]$ is a standard one:

$$[\varphi_1^{(0)}, \varphi_2^{(0)}] = [\varphi_1^{(1)}, \varphi_2^{(1)}]C_{01}, \qquad 0 < \ \arg(1 - \mu) < 2\pi;$$

$$C_{01} = \begin{pmatrix} \frac{\Gamma(-\theta_x)\Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}\right)\Gamma\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{1}{2}\right)} & \frac{\Gamma(-\theta_x)\Gamma(1-\theta_0)}{\Gamma\left(-\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}\right)\Gamma\left(-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{1}{2}\right)} \\ \frac{\Gamma(\theta_x)\Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}\right)\Gamma\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{1}{2}\right)} & \frac{\Gamma(\theta_x)\Gamma(1-\theta_0)}{\Gamma\left(-\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}\right)\Gamma\left(-\frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2}\right)} \end{pmatrix} \qquad (93)$$

On the other hand, the connection matrix between $[\varphi_1^{(0)}, \varphi_2^{(0)}]$ and $[\varphi_1^{(\infty)}, \varphi_2^{(\infty)}]$ is computed in [21]. Our specific case falls in the case $\alpha - \beta = -p$, $p \geq 0$ integer. From [21], page 27, case 11, formulae (1) and (2), we deduce the connection formulae:

$$(-\mu)^{\alpha-\beta} F\left(\beta - \gamma + 1, \beta, 1 + \beta - \alpha; \frac{1}{\mu}\right) = \frac{\Gamma(\beta - \alpha + 1)\Gamma(1 - \gamma)}{\Gamma(1 - \alpha)\Gamma(\beta - \gamma + 1)}(-\mu)^{\alpha} F(\alpha, \beta, \gamma; \mu) + $$

$$+ \frac{\Gamma(\beta - \alpha + 1)\Gamma(\gamma - 1)}{\Gamma(\gamma - \alpha)\Gamma(\beta)}(-\mu)^{\alpha - \gamma + 1} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; \mu),$$

$$(-\mu)^{\alpha-\beta} g_1\left(\beta - \gamma + 1, \beta, 1 + \beta - \alpha; \frac{1}{\mu}\right) = $$

$$(-1)^{\alpha-\beta} \frac{\Gamma(1+\beta-\alpha)\Gamma(1-\beta)\Gamma(\alpha-\gamma+1)}{\Gamma(2-\gamma)} \, (-\mu)^{\alpha-\gamma+1} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; \mu).$$

Here $|\arg(-\mu)| < \pi$, $\ln(-\mu) > 0$ for $-\infty < \mu < -1$. The branch cut in the μ-plane is $[0, +\infty)$. As for the minus signs, we choose $-\mu = e^{-i\pi}\mu$, then $0 < \arg\mu < 2\pi$. Moreover, in our case $(-1)^{\alpha-\beta} = 1$. With this preparation, we can write the connection matrix:

$$[\varphi_1^{(\infty)}, \varphi_2^{(\infty)}] = [\varphi_1^{(0)}, \varphi_2^{(0)}] \, C_{\infty 0}, \qquad 0 < \arg\mu < 2\pi;$$

$$C_{\infty 0} = 2 \begin{pmatrix} 0 & \dfrac{\Gamma(-\theta_0) \, e^{-i\pi\{\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{3}{2}\}}}{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{3}{2}\right)\Gamma\left(-\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{3}{2}\right)} \\[4mm] \dfrac{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{1}{2}\right)\Gamma\left(-\frac{\theta_0}{2}+\frac{\theta_x}{2}-\frac{1}{2}\right)}{\Gamma(1-\theta_0)e^{-i\pi\{\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{3}{2}\}}} & \dfrac{\Gamma(\theta_0) \, e^{-i\pi\{-\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{3}{2}\}}}{\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{3}{2}\right)\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{3}{2}\right)} \end{pmatrix}. \qquad (94)$$

In order to write Ψ_{IN}^{Match} in terms of the $\varphi_i^{(\infty)}$, let us compute the behavior for $\mu \to \infty$ of $\mu^{\theta_0/2}(\mu-1)^{\theta_x/2} \begin{pmatrix} \varphi_1^{(\infty)} & \varphi_2^{(\infty)} \\ \xi_1^{(\infty)} & \xi_2^{(\infty)} \end{pmatrix}$. We have:

$$\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \varphi_1^{(\infty)} = -\frac{32}{[(\theta_0+\theta_x)^2-1][(\theta_0-\theta_x)^2-1]} \, \mu^{\frac{1}{2}}\left[1+O\left(\frac{1}{\mu}\right)\right] + \mu^{-\frac{3}{2}}\ln\mu\left[1+O\left(\frac{1}{\mu}\right)\right].$$

$$\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \varphi_2^{(\infty)} = \mu^{-\frac{3}{2}}\left[1+O\left(\frac{1}{\mu}\right)\right].$$

$$\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \xi_1^{(\infty)} = -\frac{2}{r} \, \mu^{-\frac{1}{2}}\ln\mu\left[1+O\left(\frac{1}{\mu}\right)\right] + O(\mu^{-\frac{1}{2}}).$$

$$\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \xi_2^{(\infty)} = \frac{2}{r} \, \mu^{-\frac{1}{2}}\left[1+O\left(\frac{1}{\mu}\right)\right].$$

Therefore:

$$\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(\infty)} & \varphi_2^{(\infty)} \\ \xi_1^{(\infty)} & \xi_2^{(\infty)} \end{pmatrix} = \left[I+O\left(\frac{1}{\mu}\right)\right] \mu^{\frac{1}{2}\sigma_3} \begin{pmatrix} 1 & 0 \\ R \ln\mu & 1 \end{pmatrix} \begin{pmatrix} -\frac{2}{rR} & 0 \\ 0 & \frac{2}{r} \end{pmatrix}.$$

We conclude that the matrix Ψ_0 used in the $\Psi_{OUT} \leftrightarrow \Psi_{IN}$ matching is:

$$\Psi_0 = \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(\infty)} & \varphi_2^{(\infty)} \\ \xi_1^{(\infty)} & \xi_2^{(\infty)} \end{pmatrix} C, \qquad C := \begin{pmatrix} -R\frac{r}{2} & 0 \\ 0 & \frac{r}{2} \end{pmatrix} = \begin{pmatrix} \frac{[1-(\theta_0+\theta_x)^2][(\theta_0-\theta_x)^2-1]}{32} & 0 \\ 0 & \frac{r}{2} \end{pmatrix}.$$

As we have already remarked, we do the transformation $\Psi_{IN}^{Match} \mapsto \Psi_{IN}^{Match} C_{IN}^{-1}$, to compute the x-independent monodromy at $\lambda = 0, x$. This means that we have to compute the monodromy of:

$$\Psi_{IN}^{Match} C_{IN}^{-1} = K_0(x)\Psi_0(\lambda/x) =$$

$$= K_0(x) \left(\frac{\lambda}{x}\right)^{\frac{\theta_0}{2}} \left(\frac{\lambda}{x}-1\right)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(\infty)} & \varphi_2^{(\infty)} \\ \xi_1^{(\infty)} & \xi_2^{(\infty)} \end{pmatrix} C,$$

$$= K_0(x) \left(\frac{\lambda}{x}\right)^{\frac{\theta_0}{2}} \left(\frac{\lambda}{x}-1\right)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(0)} & \varphi_2^{(0)} \\ \xi_1^{(0)} & \xi_2^{(0)} \end{pmatrix} C_{\infty 0} \, C,$$

$$= K_0(x) \left(\frac{\lambda}{x}\right)^{\frac{\theta_0}{2}} \left(\frac{\lambda}{x}-1\right)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} \\ \xi_1^{(1)} & \xi_2^{(1)} \end{pmatrix} C_{01} \, C_{\infty 0} \, C.$$

In order to do this, we observe that from the definition of $\varphi_1^{(0)}, \varphi_2^{(0)}$ if follows that:

$$K_0(x) \left(\frac{\lambda}{x}\right)^{\frac{\theta_0}{2}} \left(\frac{\lambda}{x}-1\right)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(0)} & \varphi_2^{(0)} \\ \xi_1^{(0)} & \xi_2^{(0)} \end{pmatrix} = \psi_0^{IN}(x)(I+O(\lambda))\lambda^{\frac{\theta_0}{2}\sigma_3},$$

for $\lambda \to 0$. From the definition of $\varphi_1^{(1)}, \varphi_2^{(1)}$, it follows that:

$$K_0(x) \left(\frac{\lambda}{x}\right)^{\frac{\theta_0}{2}} \left(\frac{\lambda}{x}-1\right)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} \\ \xi_1^{(1)} & \xi_2^{(1)} \end{pmatrix} = \psi_x^{IN}(x)(I + O(\lambda - x))(\lambda - x)^{\frac{\theta_x}{2}\sigma_3},$$

for $\lambda \to x$. It is not necessary here to write explicitly the invertible matrices $\psi_0^{IN}(x)$ and $\psi_x^{IN} x$. The above construction implies that $\Psi_{IN}^{Match} C_{IN}^{-1}$ has monodromy matrices at $\lambda = 0, x$ respectively given by:

$$M_0 = C^{-1}(C_{\infty 0})^{-1} \exp\{i\pi\theta_0\sigma_3\} C_{\infty 0}C,$$

$$M_x = C^{-1}(C_{\infty 0})^{-1}(C_{01})^{-1} \exp\{i\pi\theta_x\sigma_3\} C_{01}C_{\infty 0}C.$$

These matrices coincide with the monodromy matrices of $\Psi \ C_{IN}^{-1}$.

As a last simplification, we consider the transformation $M \mapsto CMC^{-1}$. We obtain the result of theorem 3, case c). Namely:

$$M_0 = (C_{\infty 0})^{-1} \exp\{i\pi\theta_0\sigma_3\} C_{\infty 0}, \qquad M_\infty = \begin{pmatrix} -1 & 0 \\ 2\pi i\left(1 - \frac{r_1}{\rho}\right) & -1 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 1 & 0 \\ 2\pi i\frac{r_1}{\rho} & 1 \end{pmatrix}, \qquad M_x = (C_{\infty 0})^{-1}(C_{01})^{-1} \exp\{i\pi\theta_x\sigma_3\} C_{01}C_{\infty 0}.$$

We observe that r does not appear in the monodromy matrices.

8   Appendix 1

Proposition 2 *Let B_0, B_1 be 2×2 matrices such that*

$$Eigenvalues \ (B_0) = 0, -c, \quad Eigenvalues \ (B_1) = 0, c - a - b.$$

and $B_0 + B_1$ is either diagonalizable:

$$B_0 + B_1 = \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \quad (it \ may \ happen \ that \ a = b),$$

or it is a Jordan form:

$$B_0 + B_1 = \begin{pmatrix} -a & 1 \\ 0 & -a \end{pmatrix}.$$

Then, B_0 and B_1 can be computed as in the following cases. Let r, s be any complex numbers.

1) Diagonalizable case.
Case $a \neq b$:

$$B_0 := \begin{pmatrix} \frac{a(b-c)}{a-b} & r \\ \frac{ab(a-c)(c-b)}{r(a-b)^2} & \frac{b(c-a)}{a-b} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{a(c-a)}{a-b} & -r \\ -(B_0)_{21} & \frac{b(b-c)}{a-b} \end{pmatrix}, \quad r \neq 0 \tag{95}$$

$$if \ a = 0: \quad B_0 = \begin{pmatrix} 0 & r \\ 0 & -c \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -r \\ 0 & c-b \end{pmatrix}. \tag{96}$$

$$If \ b = 0: \quad B_0 = \begin{pmatrix} -c & r \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} c-a & -r \\ 0 & 0 \end{pmatrix}. \tag{97}$$

$$If \ a = c \neq b: \quad B_0 = \begin{pmatrix} -a & r \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -r \\ 0 & -b \end{pmatrix}. \tag{98}$$

$$If \ b = c \neq a: \quad B_0 = \begin{pmatrix} 0 & r \\ 0 & -b \end{pmatrix}, \quad B_1 = \begin{pmatrix} -a & -r \\ 0 & 0 \end{pmatrix}. \tag{99}$$

Cases (96)–(99) are already included in (95).

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Case $a = b$. We have two sub-cases:

$$\text{If } a = b = c: \quad B_0 = \begin{pmatrix} -c - s & r \\ -\frac{s(c+s)}{r} & s \end{pmatrix}, \quad B_1 = \begin{pmatrix} s & -r \\ \frac{s(c+s)}{r} & -c - s \end{pmatrix}. \tag{100}$$

$$\text{If } a = b = 0: \quad B_0 = \begin{pmatrix} -c - s & r \\ -\frac{s(c+s)}{r} & s \end{pmatrix}, \quad B_1 = -B_0. \tag{101}$$

The transpose matrices of all the above cases are also possible.

2) Jordan case.
For $a \neq 0$ and $a \neq c$ we have:

$$B_0 = \begin{pmatrix} r & \frac{r(r+c)}{a(a-c)} \\ a(c-a) & -c - r \end{pmatrix}, \quad B_1 = \begin{pmatrix} -a - r & 1 - \frac{r(r+c)}{a(a-c)} \\ a(a-c) & c - a + r \end{pmatrix}. \tag{102}$$

For $a = 0$, or $a = c$, we have two possibilities:

$$B_0 = \begin{pmatrix} 0 & r \\ 0 & -c \end{pmatrix}, \quad B_1 = \begin{pmatrix} -a & 1 - r \\ 0 & -a + c \end{pmatrix}; \tag{103}$$

or

$$B_0 = \begin{pmatrix} -c & r \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} c - a & 1 - r \\ 0 & -a \end{pmatrix} \tag{104}$$

Proposition 3 *Let B_0 and B_1 be as in Proposition 2. The linear system:*

$$\frac{d}{dz} \begin{pmatrix} \varphi \\ \xi \end{pmatrix} = \left[\frac{B_0}{z} + \frac{B_1}{z - 1} \right] \begin{pmatrix} \varphi \\ \xi \end{pmatrix}$$

may be reduced to a Gauss hyper-geometric equation, in the following cases.

Diagonalizable case (i.e. from (95) to (101)):

$$z(1 - z) \frac{d^2\varphi}{dz^2} + \left(1 + c - (a + [b + 1] + 1) z \right) \frac{d\varphi}{dz} - a(b + 1) \varphi = 0. \tag{105}$$

The component ξ is obtained by the following equalities, according to the different cases of Proposition 2.

Cases (95) (96) (97) (98) (99):

$$\xi = \frac{1}{r} \left[z(1 - z) \frac{d\varphi}{dz} - a \left(z + \frac{b - c}{a - b} \right) \varphi \right] \tag{106}$$

Case (100):

$$\xi = \frac{1}{r} \left[z(1 - z) \frac{d\varphi}{dz} + (c + s - c\,z) \varphi \right]$$

Case (101):

$$\xi = \frac{1}{r} \left[z(1 - z) \frac{d\varphi}{dz} + (c + s) \varphi \right]$$

Jordan case (102): The equation for φ is in Gauss hypergeometic form only when $r = -a$. In this case, the matrices (102) are:

$$B_0 = \begin{pmatrix} -a & 1 \\ a(c - a) & a - c \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ a(a - c) & c - 2a \end{pmatrix}.$$

The equation is:

$$z(1 - z) \frac{d^2\varphi}{dz^2} + [1 + c - (2a + 1) z] \frac{d\varphi}{dz} - a^2 \varphi = 0,$$

$$\xi = z \frac{d\varphi}{dz} + a \varphi.$$

Jordan case (103): for $r = 1$ we get:

$$\text{For } a = 0, \qquad z(1-z)\frac{d^2\varphi}{dz^2} + (1+c-z)\frac{d\varphi}{dz} = 0.$$

$$\text{For } c = a, \quad z(1-z)\frac{d^2\varphi}{dz^2} + (1+a-(2a+1)z)\frac{d\varphi}{dz} + \left(-a^2 - \frac{a}{1-z}\right)\varphi = 0.$$

The Gauss form appears only when $a = 0$.

Jordan case (104): for $r = 1$ we get:

$$\text{For } a = 0, \quad z(1-z)\frac{d^2\varphi}{dz^2} + (1+c-z)\frac{d\varphi}{dz} + \frac{c}{1-z}\,\varphi = 0.$$

$$\text{For } c = a, \quad z(1-z)\frac{d^2\varphi}{dz^2} + (1+a-(2a+1)z)\frac{d\varphi}{dz} - a^2\varphi = 0.$$

For $r \neq 1$, we don't get a Gauss hyper-geometric form for the equation of φ in both cases (103) and (104). Nevertheless, the matrices are in upper triangular form, so the equation for ξ is solvable by elementary integration.

As a compendium to the above proposition, we recall that any irreducible representation of $\pi_1(\mathbf{CP}^1 \backslash \{0, 1, \infty\}) \mapsto GL(2, \mathbf{C})$ can be realized as the monodromy group of a Riemann (or Gauss) equation. A reducible representation (namely, M_0, M_1, M_∞ are in upper triangular form) can be realized by the monodromy of a 2×2 Fuchsian system

$$\frac{dY}{dz} = \left[\frac{B_0}{z} + \frac{B_1}{z-1}\right] Y$$

where B_0, B_1 are 2×2 upper triangular matrices. We also state the following:

Lemma 12 *Consider a 2×2 linear system:*

$$\frac{dY(z)}{dz} = A(z)\, Y, \qquad A(z) = \begin{pmatrix} a(z) & b(z) \\ 0 & c(z) \end{pmatrix}$$

such that $A(z)$ is meromorphic, with poles $a_1, a_2, ..., a_N, \infty$. The monodromy group is generated by N upper triangular monodromy matrices:

$$M_i = \begin{pmatrix} \lambda_1^{(i)} & R^{(i)} \\ 0 & \lambda_2^{(i)} \end{pmatrix}, \quad i = 1, 2, ..., N;$$

where $\lambda_1^{(i)}, \lambda_2^{(i)}, R^{(i)}$ are constants (i.e. they are independent of z) given by:

$$\lambda_1^{(i)} = \exp\{2\pi i \text{ Res } a(z)|_{a_i}\}, \quad \lambda_2^{(i)} = \exp\{2\pi i \text{ Res } c(z)|_{a_i}\},$$

$$R^{(i)} = \int_z^{z\exp\{2\pi i\}} ds\, b(s)\,\frac{u_2(s)}{u_1(s)} + \left(\lambda_1^{(i)} - \lambda_2^{(i)}\right)\int_{z_0}^z ds\, b(s)\,\frac{u_2(s)}{u_1(s)}.$$

$R^{(i)}$ depends on a chosen non-singular base point z_0, but not on z. One of the non zero $R^{(i)}$ can be put equal to 1, by re-defining $Y \mapsto YC$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1/R^{(i)} \end{pmatrix}$.

Proof: Let us write $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and the equation in the form:

$$\frac{dy_1}{dz} = a(z)y_1 + b(z)y_2, \qquad \frac{dy_2}{dz} = c(z)y_2.$$

Let $z_0 \neq a_i, \infty$, $i = 1, ..., N$. The second equation has solution:

$$y_2(z) = C_2 \, u_2, \qquad \text{where:} \quad u_2 := \exp\{\int_{z_0}^z ds \, c(s)\}, \quad C_2 \in \mathbf{C}.$$

The first equation becomes:

$$\frac{dy_1}{dz} = a(z)y_1 \; + \; C_2 \, b(z) \, u_2(z). \tag{107}$$

We solve the first equation by variation of parameters. Let:

$$u_1(z) = \exp\{\int_{z_0}^z ds \, a(s)\}$$

be a fundamental solution of the homogeneous equation:

$$\frac{dy_1}{dz} = a(z)y_1.$$

We look for a solution of (107) of the form $y_2(z) = w(z) \, u_1(z)$. Substitution gives:

$$\frac{dw}{dz} = C_2 \, b(z) \, \frac{u_2(z)}{u_1(z)} \quad \Longrightarrow \quad w(z) = C_2 \, v(z) \; + \; C_1, \qquad C_1 \in \mathbf{C}.$$

where:

$$v(z) = \int_{z_0}^z ds \, b(s) \, \frac{u_2(s)}{u_1(s)}$$

The general solution of (107) is:

$$y_1(z) = C_1 \, u_1(z) \; + C_2 \, v(z) \, u_1(z).$$

Then, a fundamental solution for the initial system can be chosen to be:

$$Y(z) = \begin{pmatrix} u_1(z) & v(z) \, u_1(z) \\ 0 & u_2(z) \end{pmatrix}$$

We compute the monodromy for $(z - a_i) \mapsto (z - a_i)e^{2\pi i}$. We have:

$$u_1(z) \mapsto \lambda_1^{(i)} \, u_1(z), \quad \lambda_1^{(i)} = \exp\{2\pi i \, \mathrm{Res} \, a(z)|_{a_i}\},$$

$$u_2(z) \mapsto \lambda_2^{(i)} \, u_1(z), \quad \lambda_2^{(i)} = \exp\{2\pi i \, \mathrm{Res} \, c(z)|_{a_i}\}. \tag{108}$$

By linearity, the vector solution $\begin{pmatrix} v(z) \, u_1(z) \\ u_2(z) \end{pmatrix}$ is transformed into a linear combination of two independent vector solutions: $R^{(i)} \begin{pmatrix} u_1(z) \\ 0 \end{pmatrix} \; + S^{(i)} \begin{pmatrix} v(z) \, u_1(z) \\ u_2(z) \end{pmatrix}$. Moreover, $S^{(i)}$ must coincide with $\lambda_2^{(i)}$, because – by virtue of (108) – $u_2(z) \mapsto \lambda_2^{(i)} u_2(z)$. Namely:

$$\begin{pmatrix} v(z) \, u_1(z) \\ u_2(z) \end{pmatrix} \mapsto R^{(i)} \begin{pmatrix} u_1(z) \\ 0 \end{pmatrix} \; + \lambda_2^{(i)} \begin{pmatrix} v(z) \, u_1(z) \\ u_2(z) \end{pmatrix}, \qquad R^{(i)} \in \mathbf{C}.$$

Thus:

$$Y(z) = \begin{pmatrix} u_1(z) & v(z) \, u_1(z) \\ 0 & u_2(z) \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1^{(i)} \, u_1(z) & \lambda_2^{(i)} \, v(z) \, u_1(z) \; + \; R^{(i)} \, u_1(z) \\ 0 & \lambda_2^{(i)} \, u_2(z) \end{pmatrix} =$$

$$= Y(z) \begin{pmatrix} \lambda_1^{(i)} & R^{(i)} \\ 0 & \lambda_2^{(i)} \end{pmatrix}.$$

Let C_i be a small loop around a_i. To find $R^{(i)}$ let us observe that:

$$u_1(z) \, v(z) \mapsto \lambda_1^{(i)} u_1(z) \, \tilde{v}(z)$$

where:

$$\tilde{v}(z) = v(z) + K_i(z), \qquad K_i(z) := \int_z^{z\exp\{2\pi i\}} ds\, b(s)\, \frac{u_2(s)}{u_1(s)}.$$

Thus:

$$Y(z) = \begin{pmatrix} u_1(z) & v(z)\, u_1(z) \\ & \\ 0 & u_2(z) \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1^{(i)}\, u_1(z) & \lambda_1^{(i)}\, v(z)\, u_1(z) + K_i(z)u_1(z) \\ & \\ 0 & \lambda_2^{(i)}\, u_2(z) \end{pmatrix}.$$

We must have: $\lambda_1^{(i)} v(z) u_1(z) + K_i(z) u_1(z) \equiv \lambda_2^{(i)} v(z) u_1(z) + R^{(i)} u_1(z)$; namely:

$$R^{(i)} = K_i(z) + \left(\lambda_1^{(i)} - \lambda_2^{(i)}\right) v(z).$$

$\square$

9   Appendix 2: Formal Asymptotic Expansion

1) We consider systems (55):

$$\frac{dY}{dz} = \left[\Omega + \frac{D_1}{z} + \sum_{n=2}^{\infty} \frac{D_n}{z^n}\right] Y := D(z)\, Y, \qquad \Omega = \mathrm{diag}(\omega_1, \omega_2, ..., \omega_n),$$

with $\omega_i \neq \omega_j$, for $i \neq j$. We introduce a gauge transformation $Y = G(z)\tilde{Y}$, such that:

$$\frac{\tilde{Y}}{dz} = \left[G^{-1}(z)D(z)G(z) - G^{-1}(z)\frac{dG(z)}{dz}\right] \tilde{Y},$$

be in the simple form:

$$\frac{d\tilde{Y}}{dz} = \left[\Omega + \frac{\Omega_1}{z}\right] \tilde{Y}, \qquad \Omega,\ \Omega_1\ \text{diagonal}.$$

Formally, we write $G(z)$ as:

$$G(z) = I + \frac{G_1}{z} + \frac{G_2}{z^2} + ..., \qquad z \to \infty.$$

If the G_n's can be determined, we get the formal solution:

$$Y(z) \sim \left[I + \sum_{n=1}^{\infty} \frac{G_n}{z^n}\right] \exp\left\{z\,\Omega + \Omega_1 \ln(z)\right\}, \qquad z \to \infty$$

For a sector of angular with $\pi + \epsilon$, $\epsilon > 0$ sufficiently small (but finite and non zero), there exists a unique solution $Y(z)$ with the above asymptotic expansion [2].

In order to determine G_n and Ω_1, we solve $D(z)G(z) - \partial_z G(z) = G(z)(\Omega_0 + \Omega_1 z^{-1})$:

$$\left(\Omega + \sum_{n=1}^{\infty} \frac{D_n}{z^n}\right)\left(I + \sum_{n=1}^{\infty} \frac{G_n}{z^n}\right) + \sum_{n=2}^{\infty} \frac{(n-1)G_{n-1}}{z^n} = \left(I + \sum_{n=1}^{\infty} \frac{G_n}{z^n}\right)\left(\Omega + \frac{\Omega_1}{z}\right).$$

We identify equal powers of z^{-1}. From the power $1/z$ we get:

$$\Omega_1 = \mathrm{diag}(\omega_1^{(1)}, ..., \omega_n^{(1)}), \quad \omega_i^{(1)} := (\Omega_1)_{ii} = (D_1)_{ii}, \qquad (G_1)_{ij} = -\frac{(D_1)_{ij}}{\omega_i - \omega_j}.$$

From the power $1/z^2$, for $i \neq j$, we compute $(G_2)_{ij}$, and for $i = j$ we compute:

$$(G_1)_{ii} = -(D_2)_{ii} - \sum_{k \neq i} (D_1)_{ik}(G_1)_{ki}.$$

From the power $1/z^n$ we get:

$$(G_{n-1})_{ii} = \frac{1}{n-1}\left\{-\left(D_n + D_{n-1}G_1 + ... + D_2G_{n-2}\right)_{ii} - \sum_{k\neq i}(D_1)_{ik}(G_{n-1})_{ki}\right\},$$

$$(G_n)_{ij} = \frac{1}{\omega_i - \omega_j}\left\{[\omega_j^{(1)} - \omega_i^{(1)} - (n-1)](G_{n-1})_{ij} - \sum_{k\neq i}(D_1)_{ik}(G_{n-1})_{kj} - \right.$$

$$\left. - (D_n + D_{n-1}G_1 + ... + D_2G_{n-2})_{ij}\right\}, \qquad i \neq j$$

2) We consider the system (56):

$$\frac{dY}{dz} = \left[x^2\Lambda\ z + x\Lambda + \frac{E_1}{z} + \sum_{n=2}^{\infty}\frac{E_n}{z^n}\right]Y \ := E(z)\ Y, \qquad \Lambda = \ \text{diag}\ (\lambda_1, \lambda_2, ..., \lambda_n).$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$. We introduce a gauge transformation:

$$Y(z) = K(z)\ \tilde{Y}(z)$$

in order to reach the simple form:

$$\frac{d\tilde{Y}}{dz} = \left[K^{-1}(z)E(z)K(z) - K^{-1}(z)\frac{dK(z)}{dz}\right]\ \tilde{Y}$$

$$\equiv \left[x^2\Lambda\ z + x\Lambda + \frac{\Lambda_1}{z}\right]\ \tilde{Y}, \qquad \Lambda_1\ \text{diagonal.}$$

Formally, we write:

$$K(z) \sim I + \frac{K_1}{z} + \frac{K_2}{z} + ... = I + \sum_{n=1}^{\infty}\frac{K_n}{z}.$$

Provided that we can determine the matrices K_n, we obtain the formal solution:

$$Y(z) \sim \left[I + \sum_{n=1}^{\infty}\frac{K_n}{z^n}\right]\ \exp\left\{\frac{x^2}{2}\Lambda\ z^2\ + x\Lambda\ z\ \Lambda_1\ln\ x\right\}$$

For a sector of angular with $\frac{\pi}{2} + \epsilon$, $\epsilon > 0$ sufficiently small (but finite and non zero), there exists a unique solution $Y(z)$ with the above asymptotic expansion [2].

In order to determine K_n and Λ_1, we solve $E(z)K(z) - \partial_z K(z) = K(z)(x^2\Lambda z^2 + \Lambda z + \Omega_1 z^{-1})$:

$$\left(x^2\Lambda z^2 + \Lambda z + \sum_{n=1}^{\infty}\frac{E_n}{z^n}\right)\left(I + \sum_{n=1}^{\infty}\frac{K_n}{z^n}\right) + \sum_{n=2}^{\infty}\frac{(n-1)K_{n-1}}{z^n} =$$

$$= \left(I + \sum_{n=1}^{\infty}\frac{K_n}{z^n}\right)\left(x^2\Lambda z^2 + \Lambda z + \frac{\Omega_1}{z}\right).$$

We identify equal powers of z^{-1}.

Power z. It is an identity: $x^2\Lambda = x^2\Lambda$.

Power z^0.

$$x\Lambda + x^2\Lambda K_1 = x\Lambda + x^2 K_1\Lambda \quad \implies \quad [\Lambda, K_1] = 0$$

This means that K_1 is a diagonal matrix.

Power $1/z$. We obtain the equation:

$$x^2\ [\Lambda, K_2] = \Lambda_1 - E_1 \quad \implies \quad (\Lambda_1)_{ii} = (E_1)_{ii}, \quad (G_2)_{ij} = -\frac{(E_1)_{ij}}{x^2(\lambda_i - \lambda_j)}, \quad i \neq j.$$

Power $1/z^2$. We obtain the equation:

$$x^2[\Lambda, K_3] = x[K_2, \Lambda] + K_1\Lambda_1 - E_1K_1 - K_1 - E_2 \equiv \frac{E_1 - \Lambda_1}{x} + K_1\Lambda_1 - E_1K_1 - K_1 - E_2$$

Thus:

$$(K_1)_{ii} = -(E_2)_{ii}$$

$$(K_3)_{ij} = \frac{1}{x^2(\lambda_i - \lambda_j)}\left[(E_1)_{ij}\left(\frac{1}{x} + (E_2)_{jj}\right) - (E_2)_{ij}\right], \quad i \neq j.$$

Power $1/z^2$. We obtain the equation:

$$x^2[\Lambda, K_4] = x[K_3, \Lambda] + K_2\Lambda_1 - E_1K_2 - E_2K_1 - 2K_2 - E_3$$

The diagonal part gives:

$$2(K_2)_{ij} = \left[(E_2)_{ii}\right]^2 - (E_3)_{ii} - \sum_{k \neq i}(E_1)_{ik}(K_2)_{ki}.$$

The non diagonal part gives $(K_4)_{ij}$, $i \neq j$.

We content ourselves with these results, namely the determination of K_1 and K_2. With the same procedure, we can determine all the K_n's.

10   Appendix 3: Birational Transformations

All the solutions of (PVI) of the form:

$$y(x) = b_0 + b_1 x + b_2 x^2 + ... + b_N x^N + ... \tag{109}$$

are obtained from the matching procedure of sections 5, 6, 7. By this we mean that solutions of type (109) are the solutions given by the matching procedure, or they can be obtained from solutions given by the matching procedure via one of the birational transformations of [22] and the transformation (19).

Birational transformations are symmetries of (PVI), namely invertible transformations:

$$y'(x) = \frac{P(x, y(x))}{Q(x, y(x))}, \quad x' = \frac{p(x)}{q(x)}, \qquad (\theta_0, \theta_x, \theta_1, \theta_\infty) \mapsto (\theta_0', \theta_x', \theta_1', \theta_\infty')$$

such that $y(x)$ satisfies (PVI) with coefficients $\theta_0, \theta_x, \theta_1, \theta_\infty$ and variable x, if and only if $y'(x')$ satisfies (PVI) with coefficients $\theta_0', \theta_x', \theta_1', \theta_\infty'$ and variable x'. The functions P, Q are polynomials; p, q are linear; the transformation of the θ_μ's is an element of a linear representation of one of the following groups. Permutation group; the Weyl group of the root system D_4; the group of translations $v := (v_1, v_2, v_3, v_4) \mapsto v + e_j$, j=1,2,3,4 (where $e_1 = (1, 0, 0, 0)$, ..., $e_4 = (0, 0, 0, 1)$). [7]

* Permutation group:

$$x^1: \quad \theta_1' = \theta_0, \quad \theta_0' = \theta_1; \qquad \theta_x' = \theta_x, \quad \theta_\infty' = \theta_\infty; \qquad y'(x) = y(x) - 1, \quad x = 1 - x'.$$

$$x^2: \quad \theta_0' = \theta_\infty - 1, \quad \theta_\infty' = \theta_0 + 1; \qquad \theta_1' = \theta_1, \quad \theta_x' = \theta_x; \qquad y'(x) = \frac{1}{y(x)}, \quad x = \frac{1}{x'}.$$

$$x^3: \quad \theta_0' = \theta_x, \quad \theta_x' = \theta_0; \qquad \theta_1' = \theta_1, \quad \theta_\infty' = \theta_\infty; \qquad y'(x) = \frac{x - y(x)}{x - 1}, \quad x = \frac{x'}{x' - 1}.$$

* Weyl Group:

w_1:

$$\theta_1' = -\theta_1; \qquad \theta_0' = \theta_0, \quad \theta_x' = \theta_x, \quad \theta_\infty' = \theta_\infty.$$

---

[7]We note that θ_1, θ_x, θ_0 are defined up to the sign. Moreover, θ_∞ is defined up to $\theta_\infty \mapsto 2 - \theta_\infty$, and for this reason symmetries are described in [22] in terms of $\chi_\infty := \theta_\infty - 1$.

w_2:

$$\theta_0' = \frac{\theta_0 + \theta_1 + \theta_x + \theta_\infty}{2} - 1, \quad \theta_1' = \frac{\theta_0 + \theta_1 - \theta_x - \theta_\infty}{2} + 1,$$

$$\theta_x' = \frac{\theta_0 - \theta_1 + \theta_x - \theta_\infty}{2} + 1, \quad \theta_\infty' = \frac{\theta_0 - \theta_1 - \theta_x + \theta_\infty}{2} + 1$$

w_3:

$$\theta_\infty' = 2 - \theta_\infty; \quad \theta_0' = \theta_0, \quad \theta_x' = \theta_x, \quad \theta_1' = \theta_1.$$

w_4:

$$\theta_\infty' = 2 - \theta_\infty; \quad \theta_x' = 2 - \theta_x; \quad \theta_0' = \theta_0, \quad \theta_1' = \theta_1.$$

The variable $x' = x$, but $y'(x)$ is quite complicated and will not be given here (see [22]).

* Shift $l_j: \ v \mapsto v + e_j$:

$$l_1: \qquad \theta_0' = \theta_0 + 1, \ \theta_1' = \theta_1 + 1; \qquad \theta_x' = \theta_x, \ \theta_\infty' = \theta_\infty.$$

$$l_2: \qquad \theta_0' = \theta_0 + 1, \ \theta_1' = \theta_1 - 1; \qquad \theta_x' = \theta_x, \ \theta_\infty' = \theta_\infty.$$

$$l_3: \qquad \theta_x' = \theta_x + 1, \ \theta_\infty' = \theta_\infty + 1; \qquad \theta_0' = \theta_0, \ \theta_1' = \theta_1.$$

$$l_4: \qquad \theta_x' = \theta_x + 1, \ \theta_\infty' = \theta_\infty - 1; \qquad \theta_0' = \theta_0, \ \theta_1' = \theta_1.$$

The variable $x' = x$, but $y'(x)$ is quite complicated and will not be given here (see [22]).

For the Taylor solutions, we have $\sigma = \theta_1 - \theta_\infty$. Denote $\sigma' := \theta_1' - \theta_\infty'$. If we start from a Taylor solution constructed in sections 5, 6, 7 by means of the matching procedure, developed for $\sigma = \theta_1 - \theta_\infty$, then the birational transformations allow to obtain the solutions defined for $\sigma = \pm(\theta_1 \pm \setminus \mp \theta_\infty) + n$, $n \in \mathbf{Z}$. This is a consequence of the following actions:

$$l_1 \text{ and } l_4: \ \sigma \mapsto \sigma' = \sigma + 1. \qquad l_2 \text{ and } l_3: \ \sigma \mapsto \sigma' = \sigma - 1$$

$$w_3 \cdot l_1 \cdot l_1: \ \sigma \mapsto \sigma' = \theta_1 + \theta_\infty. \qquad w_1 \cdot w_3 \cdot l_1 \cdot l_1: \ \sigma \mapsto \sigma' = -\sigma.$$

Note also that (PVI) is invariant for $\theta_1 \mapsto -\theta_1$. This maps $\sigma \mapsto -\sigma$. Other actions are:

$$w_1: \quad \sigma' = -(\theta_1 + \theta_\infty); \qquad w_2: \quad \sigma' = \sigma; \qquad w_3: \quad \sigma' = \theta_1 + \theta_\infty - 2; \qquad w_4: \quad \sigma' = \theta_1 + \theta_\infty - 2.$$

$$x^1: \quad \sigma' = \theta_0 - \theta_\infty; \qquad x^2: \quad \sigma' = \theta_1 - \theta_0; \qquad x^3: \quad \sigma' = \theta_1 - \theta_\infty = \sigma.$$

11   Appendix 4: Examples of Taylor Expansions

We give the solutions of (PVI) of the form $y(x) = \sum_{n=0}^\infty b_n \, x^n$, $b_0 \neq 0$, depending on the value of the coefficients $\theta_0, \theta_1, \theta_x, \theta_\infty$. Similar examples can be constructed for solutions $y(x) = \sum_{n=1}^\infty b_n \, x^n$, $b_1 \neq 0$, by the symmetry (19).

We always denote b_n the coefficients, though they change case by case.

* Example 1: (PVI) always has one solution (15) when $\theta_\infty - \theta_1 \notin \mathbf{Z}$, and one solution (18) when $\theta_\infty + \theta_1 \notin \mathbf{Z}$.

* Example 2: Case $\theta_1 + \theta_\infty = 0$.

i) There are solutions (16), if $\theta_x = \pm\theta_0$, $\theta_\infty \neq 1$.

ii) There is a solution (15), defined for $\theta_\infty \neq 1$, $\frac{2n+1}{2}$, where $n \in \mathbf{Z}$. The condition $\theta_\infty \neq \frac{2n+1}{2}$ follows from the condition $\theta_1 - \theta_\infty \neq n$ in (15), when $\theta_1 = -\theta_\infty$. The denominators of the coefficients b_n vanish for half-integer θ_∞, and for $\theta_\infty = 1$.

iii) For $\theta_\infty = \frac{2n+1}{2}$, solutions $y(x) = \sum_{n=0}^\infty b_n x^n$ in ii) are not defined. On the other hand, solutions (16) are defined, provided that $\theta_x = \pm\theta_0$. Moreover, for a given $\theta_\infty = \frac{2n+1}{2}$, there may be solutions

equivalent to (16), provided that $\theta_x \pm \theta_0$ assumes some integer values. For example, consider $\theta_\infty = \frac{3}{2}$. We have (16):

$$y(x) = -2 \; + a\; x \; + \left(\theta_0^2 - 1 + \frac{3}{2}\; a - \frac{1}{2}\; a^3\right)\; x^2 \; + \; \sum_{n=3}^{\infty} b_n(a;\theta_0,\frac{3}{2})\; x^n, \quad \theta_x = \pm\theta_0;$$

and a solution equivalent to (16):

$$y(x) = 4 - (2 \pm \theta_0)\; x \; + a\; x^2 \; + \sum_{n=3}^{\infty} b_n(a;\theta_0)\; x^n, \quad \theta_x = \pm(1 + \theta_0), \quad \pm(1 - \theta_0).$$

iv) If $\theta_\infty = 0, 1$, we do not have any solution of the desired form, except for the singular solutions $y = 1$.

* **Example 3:** Case $\theta_\infty - \theta_1 = 0$.

i) We have solutions (16)

ii) We have the solution (18), with the substitution: $\theta_1 = \theta_\infty$. This is defined for $\theta_\infty \neq 1, \frac{2n+1}{2}$.

iii) For $\theta_\infty = \frac{2n+1}{2}$ we find solutions of exactly the form (16), when $\theta_x = \pm\theta_0$. Moreover, for any given $\theta_\infty = \frac{2n+1}{2}$, there may be solutions equivalent to (16), provided that $\theta_x \pm \theta_0$ has some integer value. For example, consider $\theta_\infty = -\frac{1}{2}$. There are solutions (16):

$$y(x) = \frac{2}{3} \; + a\; x \; + \sum_{n=2}^{\infty} b_n(a;\theta_0, -\frac{1}{2})\; x^n, \quad \theta_x = \pm\theta_0;$$

and the equivalent solutions:

$$y(x) = \frac{4}{3} - \frac{2 \pm \theta_0}{9}\; x \; + a\; x^2 \; + \sum_{n=3}^{\infty} b_n(a;\theta_0)x^n, \quad \theta_x = \pm(1 + \theta_0) \; \text{ or } \; \pm(1 - \theta_0).$$

Another example: consider $\theta_\infty = \frac{1}{2}$. In this case we just have (16), or the singular solution $y = 1$.

iv) For $\theta_\infty = 0, 1$ we don't have solutions of the desired form, except for the singular solutions $y = 0, 1$.

* **Example 4:** Case $\theta_1 - \theta_\infty = -1$.

i) We have the solution (18), defined for $\theta_\infty \neq 1, \frac{2n+1}{2}$, where $n \in \mathbf{Z}$.

ii) For any $\theta_\infty = \frac{2n+1}{2}$, solutions (16) are defined, when $\theta_0 \pm \theta_x = 0$. Moreover, solutions may exist equivalent to (16) by symmetry, provided that $\theta_0 \pm \theta_x$ is some integer.

iii) For $\theta_\infty = 1$ we are exactly in the case (17).

iv) For $\theta_\infty = 0$ we have solutions equivalent to (17):

$$y(x) = 2 + \frac{\theta_0^2 - \theta_x^2 - 3}{6}\; x + a\; x^2 + \left\{\frac{[(\theta_0 - \theta_x)^2 - 9][(\theta_0 + \theta_x)^2 - 9](\theta_0^2 - \theta_x^2)}{4320} \; + \; a\; \frac{\theta_0^2 - \theta_x^2 + 3}{6}\right\}\; x^3 + ...$$

* **Example 5:** Case $\theta_1 - \theta_\infty = 1$.

i) We have solution (18), with the substitution of $\theta_1 = \theta_\infty + 1$, and defined for $\theta_\infty \neq 1, \frac{2n+1}{2}, n \in \mathbf{Z}$.

ii) Equivalent to (16) by symmetry, we have the solution:

$$y(x) = \frac{2}{1 - \theta_\infty} \; + \frac{(\theta_\infty + 1)(2 \pm \theta_0)}{3(\theta_\infty - 1)}\; x \; + ax^2 + \sum_{n=3}^{\infty} b_n(a, \theta_\infty, \theta_0)x^n,$$

$$\theta_\infty \neq 0, 1; \quad \theta_x = \pm(1 - \theta_0) \text{ or } \pm(1 + \theta_0)$$

The two signs in the coefficient of x depend on the choice $\theta_x = \pm(1 - \theta_0)$ or $\pm(1 + \theta_0)$ respectively. Similar change of signs occur in all the coefficients b_n.

iii) If $\theta_\infty = 0$, the solution in case ii) is not defined (denominators in the coefficients b_n vanish). We have anyway a solution equivalent to (17):

$$y(x) = 2 \ + \frac{\theta_0^2 - \theta_x^2 - 3}{6} \ x \ + a \ x^2 \ + \sum_{n=3}^{\infty} b_n(a; \theta_0, \theta_\infty) x^n.$$

iv) If $\theta_\infty = \frac{2n+1}{2}$ the solution in i) is not defined. Solutions exist equivalent to (16), provided that $\theta_x \pm \theta_0$ is some integer.

v) For $\theta_\infty = -1$ we have a solution equivalent to (17):

$$y(x) = 1 \ + a \ x^2 \ + \ \frac{a(\theta_x^2 - \theta_0^2 + 3)}{6} \ x^3 \ + \sum_{n=4}^{\infty} b_n(a; \theta_0, \theta_x) \ x^n.$$

vi) For $\theta_\infty = 1$, solutions of the desired form do not exist, except for $y = 1$.

We could proceed at our pleasure, choosing any value $\theta_\infty \pm \theta_1$ integer. We would always have solutions of three kinds. 1) *One* out of the two solutions (15) and (18). 2) Solutions equivalent to (16) – at least when θ_∞ is half integer – provide that $\theta_x \pm \theta_0$ is some integer. 3) Solutions equivalent to (17), for θ_∞ equal to some integer.

References

[1] A.V. Andreev, A.V. Kitaev: *Connection formulae for asymptotics of the fifth Painlevé transcendent on the real axis.* Nonlinearity, **13**, (2000), 1801-1840.

[2] W.Balser, W.B.Jurkat, D.A.Lutz: *Birkhoff Invariants and Stokes' Multipliers for Meromorphic Linear Differential Equations*, Journal Math. Analysis and Applications, **71**, (1979), 48-94.

[3] W.Balser, W.B.Jurkat, D.A.Lutz: *On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities*, SIAM J. Math. Anal., **12**, (1981), 691-721.

[4] P.Boalch: *From Klein to Painlevé, via Fourier, Laplace and Jimbo*, math.AG/0308221 (2004)

[5] A.D.Bruno, I.V. Goryuchkina: *Expansions of Solutions of the Sixth Painlevé Equation.* Doklady Mathematics, **69**, (2004), 733-737

[6] B.Dubrovin: *Painlevé trascendents in two-dimensional topological field theory.* The Painlevé Property, One Century later, R.Conte, Springer 1999.

[7] B.Dubrovin-M.Mazzocco: *Monodromy of Certain Painlevé-VI trascendents and Reflection Groups.* Invent. math., **141**, (2000), 55-147.

[8] D.Guzzetti: *On the Critical Behavior, the Connection Problem and the Elliptic Representation of a Painlevé 6 Equation .* Mathematical Physics, Analysis and Geometry, **4**, (2001), 293-377.

[9] D.Guzzetti: *Inverse Problem and Monodromy Data for 3-dimensional Frobenius Manifolds.* Mathematical Physics, Analysis and Geometry, **4**, (2001), 254-291.

[10] D.Guzzetti: *The Elliptic Representation of the General Painlevé 6 Equation.* Communications in Pure and Applied Mathematics, **55**, (2002), 1280-1363.

[11] D.Guzzetti: *The Elliptic Representation of the Sixth Painlevé Equation.* Publications of the SMF, (2006) to appear.

[12] N.Hitchin: *Hypercomplex Manifolds and the Space of Framings*, The Geometric Universe, Oxford Univ. Press, (1998), 9-30.

[13] E.L.Ince: *Ordinary Differential Equations*. Dover Publications.

[14] K.Iwasaki, H.Kimura, S.Shimomura, M.Yoshida: *From Gauss to Painleve'*. Aspects of Mathematics **16**, (1991).

[15] M.Jimbo: *Monodromy Problem and the Boundary Condition for Some Painlevé Trascendents.* Publ. RIMS, Kyoto Univ.,**18** (1982), 1137-1161.

[16] M.Jimbo, T.Miwa, K.Ueno: *Monodromy Preserving Deformations of Linear Ordinary Differential Equations with Rational Coefficients (I).* Physica **D 2** , (1981), 306

[17] K.Kaneko: *Sixth Painlevé transcendents which are analytic at the fixed singularitie.* Preprint (2006).

[18] K.Kaneko, Y.Ohyama: *Fifth Painlevé transcendents which are analytic at the origin.* To appear.

[19] M.Mazzocco: *Picard and Chazy Solutions to the Painlevé VI Equation.* Math. Ann., **321**, no.1, (2001), 157-195.

[20] M.Mazzocco: *Rational Solutions of the Painlevé VI Equation,* J. Phys. A: Math. Gen. **34**, (2001), 2281–2294.

[21] N.E.Norlund: *The Logaritmic Solutions of the Hypergeometric Equation.* Mat.Fys.Skr.Dan.Vid.Selsk., **2**, (1963), no. 5, 1-58

[22] K.Okamoto: *Studies on the Painlevé Equation. I – Sixth Painlevé Equation P_{VI}*, Ann. Mat. Pura Appl. **148**, (1987), 337-381.

[23] S.Shimomura: *Painlevé Trascendents in the Neighbourhood of Fixed Singular Points.* Funkcial.Ekvac., **25**, (1982), 163-184.

Series Expansions of Painlevé Trascendents in the Neighbourhood of a Fixed Singular Point. Funkcial.Ekvac.,**25**, (1982), 185-197.

Supplement to "Series Expansions of Painlevé Trascendents in the Neighborhood of a Fixed Singular Point". Funkcial.Ekvac.,**25**, (1982), 363-371.

A Family of Solutions of a Nonlinear Ordinary Differntial Equation and its Application to Painlevé Equations (III), (V), (VI). J.Math. Soc. Japan, **39**, (1987), 649-662.