Instanton-type formal solutions for the first Painlevé hierarchy*

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Summary. Instanton-type formal solutions, which will play an important role in the description of Stokes phenomena, are discussed for the first Painlevé hierarchy. We construct instanton-type solutions by using singular-perturbative reduction of a Hamiltonian system to its Birkhoff normal form. The construction of singular-perturbative reduction to the Birkhoff normal form is also outlined.

Key words: Painlevé hierarchy, Instanton-type solutions, Hamiltonian system, Birkhoff normal form

1 Introduction

Collaborating with Kawai and partly with Aoki, I developed the exact WKB analysis for traditional (i.e., second order) Painlevé equations in the 1990's. (Cf. [5], [1], [6], [16], [17]. See also [9].) To enlarge the scope of its applicability we now try to extend the exact WKB analysis to some hierarchies, particularly the first Painlevé hierarchy $(P_{\mathrm{I}})_m$, of higher order Painlevé equations ("Toulouse Project"). To be more concrete, Toulouse Project is our project to understand the analytic structure of solutions of higher order Painlevé equations, say $(P_{\mathrm{I}})_m$, from the viewpoint of the exact WKB analysis with the following procedure:

Part 1 : Stokes geometry of $(P_{\mathrm{I}})_m$ and its relationship with that of the underlying Lax pair of $(P_{\mathrm{I}})_m$.
Part 2 : Reduction of 0-parameter solutions of $(P_{\mathrm{I}})_m$ to those of the traditional first Painlevé equation $(P_{\mathrm{I}})_1$ near a turning point of the first kind.
Part 3 : Study of the structure of 0-parameter solutions of $(P_{\mathrm{I}})_m$ near a turning point of the second kind.
Part 4 : Construction of instanton-type formal solutions of $(P_{\mathrm{I}})_m$, i.e., $(2m)$-parameter solutions of $(P_{\mathrm{I}})_m$.
Part 5 : Study of the structure of instanton-type solutions of $(P_{\mathrm{I}})_m$ near turning points.
Part 6 : Connection formulas for instanton-type solutions near turning

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points.
Part 7 : Study of the structure of instanton-type solutions near a crossing point of Stokes curves.

Among the above table of the procedure, Part 1 has already been investigated in detail in [3, 4], Part 2 is established in [7, 8] (which is a generalization of the previous result [5] for traditional Painlevé equations), and Part 3 is also well analyzed (though the results have not yet been published anywhere). Now, the purpose of this paper is to discuss the Toulouse Project Part 4, that is, to discuss the construction of formal solutions of $(P_{\mathrm{I}})_m$ containing sufficiently many (i.e., $2m$) free parameters called "instanton-type solutions".

In the case of traditional Painlevé equations there were two methods for constructing 2-parameter instanton-type formal solutions; the one is to employ the multiple-scale analysis ([1]) and the other is to use reduction of Hamiltonian systems equivalent to Painlevé equations to their Birkhoff normal form ([15]). Here, to construct $(2m)$-parameter instanton-type solutions of the first Painlevé hierarchy $(P_{\mathrm{I}})_m$, we generalize the second method so that it may be applied to $(P_{\mathrm{I}})_m$: After expressing $(P_{\mathrm{I}})_m$ in the form of a Hamiltonian system and localizing it around a 0-parameter solution (where a "0-parameter solution" means a formal solution that is algebraically constructed in a singular-perturbative manner, cf. Section 2 below), we reduce it to its Birkhoff normal form. Instanton-type formal solutions of $(P_{\mathrm{I}})_m$ are then constructed by solving explicitly the Birkhoff normal form thus obtained.

The plan of the paper is as follows: In Section 2 we first recall the explicit form of the first Painlevé hierarchy $(P_{\mathrm{I}})_m$ and state the main result to give the reader a clear image about $(2m)$-parameter instanton-type formal solutions. Next we describe an outline of the proof of the main result, i.e., an outline of the construction of instanton-type solutions of $(P_{\mathrm{I}})_m$ in Section 3. Finally in Section 4 we sketch out the proof of the existence of reduction of a Hamiltonian system in question to its Birkhoff normal form.

In ending this Introduction I would like to express my sincerest gratitude to Prof. T. Kawai for his valuable advice, kind encouragement and really stimulating discussions with him. I am very much pleased to dedicate this paper to him on the occasion of his sixtieth birthday. I also would like to thank many collaborators, especially Prof. T. Aoki and Dr. T. Koike, for stimulating and interesting discussions with them.

2 Main result — The first Painlevé hierarchy $(P_{\mathrm{I}})_m$ and its instanton-type solutions

First of all, let us recall the explicit form of the first Painlevé hierarchy $(P_{\mathrm{I}})_m$ $(m = 1, 2, \ldots)$ with a large parameter η (> 0):

$$\begin{cases} \dfrac{du_j}{dt} = 2\eta v_j \\ \dfrac{dv_j}{dt} = 2\eta(u_{j+1} + u_1 u_j + w_j) \end{cases} \qquad (P_{\mathrm{I}})_m$$

$(j = 1, \ldots, m)$, where u_j and v_j are unknown functions (u_{m+1} is conventionally assumed to be equal to 0) and w_j is a polynomial of u_k and v_l $(1 \leq k, l \leq j)$

determined by the following recursive relation:

$$w_j = \frac{1}{2}\left(\sum_{k=1}^{j} u_k u_{j+1-k}\right) + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2}\left(\sum_{k=1}^{j-1} v_k v_{j-k}\right) + c_j + \delta_{jm} t \tag{1}$$

$(j = 1, \ldots, m)$. Here c_j is a constant and δ_{jm} stands for Kronecker's delta.

The above expression of $(P_{\mathrm{I}})_m$ is a slight modification of that of Shimomura, who introduced the hierarchy in his study of the most degenerate Garnier system ([13, 14]). It is essentially the same as the P_{I} hierarchy proposed by Gordoa and Pickering ([2]). See also [11, 12]. Note that the first member of the hierarchy, i.e., $(P_{\mathrm{I}})_1$ is equivalent to (P_{I}), the traditional first Painlevé equation with a large parameter η. This is the reason why this hierarchy is called "the first Painlevé hierarchy" or "the P_{I}-hierarchy".

As is shown in [3], $(P_{\mathrm{I}})_m$ admits the following formal solution $(\hat{u}_j, \hat{v}_j)$ called a "0-parameter solution":

$$\hat{u}_j(t,\eta) = \hat{u}_{j,0}(t) + \eta^{-1}\hat{u}_{j,1}(t) + \cdots, \quad \hat{v}_j(t,\eta) = \hat{v}_{j,0}(t) + \eta^{-1}\hat{v}_{j,1}(t) + \cdots. \tag{2}$$

The 0-parameter solution is algebraically constructed in a singular-perturbative manner; $\hat{u}_{j,0}$ and $\hat{v}_{j,0}$ $(1 \le j \le m)$ are first algebraically determined (in particular, $\hat{v}_{j,0} \equiv 0$ holds) and then the other $\hat{u}_{j,k}$'s and $\hat{v}_{j,k}$'s $(k \ge 1)$ are uniquely determined in a recursive manner once (the branch of) $\hat{u}_{j,0}$ is fixed. See [3, Section 2.1] for the details. In [3] the 0-parameter solution is introduced to define the Stokes geometry (i.e., turning points and Stokes curves) of $(P_{\mathrm{I}})_m$.

The construction of 0-parameter solutions is simple and straightforward. In compensation for its simplicity 0-parameter solutions do not contain any free parameters. Thus it is impossible to discuss the Stokes phenomenon, which is observed on a Stokes curve, solely in terms of 0-parameter solutions. As a matter of fact, in the case of the traditional first Painlevé equation (P_{I}), we needed instanton-type formal solutions to describe the connection formula, the concrete expression of the Stokes phenomenon, even for a 0-parameter solution ([16]). The aim of this paper is to construct such instanton-type formal solutions with free parameters also for a higher order Painlevé equation $(P_{\mathrm{I}})_m$.

To state our main theorem, we prepare some notations. Let $(\Delta P_{\mathrm{I}})_m$ denote the linearized equation of $(P_{\mathrm{I}})_m$ at its 0-parameter solution $(\hat{u}_j, \hat{v}_j)$ (sometimes called "Fréchet derivative" for short), that is, the linear part in $(\Delta u_j, \Delta v_j)$ after the substitution $u_j = \hat{u}_j + \Delta u_j$ and $v_j = \hat{v}_j + \Delta v_j$ in $(P_{\mathrm{I}})_m$. Then $(\Delta P_{\mathrm{I}})_m$ becomes a system of linear ordinary differential equations for $(\Delta u_j, \Delta v_j)$ of the following form:

$$\frac{d}{dt}\begin{pmatrix} \Delta u_1 \\ \Delta v_1 \\ \vdots \\ \Delta v_m \end{pmatrix} = \eta C(t,\eta) \begin{pmatrix} \Delta u_1 \\ \Delta v_1 \\ \vdots \\ \Delta v_m \end{pmatrix}, \tag{3}$$

where $C(t,\eta)$ is a formal power series (in η^{-1}) with coefficients of $(2m) \times (2m)$ matrices whose entries are analytic functions of t. Note that, as is verified in [3, Section 2.1], the characteristic equation $\det(\lambda - C_0(t)) = 0$ of the top order part (i.e., the part of order 0 in η) $C_0(t)$ of $C(t,\eta)$ is an m-th degree polynomial of λ^2 (see also Section 3 and Lemma 1 below). In what follows we denote the roots of the characteristic equation $\det(\lambda - C_0(t)) = 0$ by $\pm\lambda_j(t)$ $(j = 1, \ldots, m)$. The turning points of $(P_{\mathrm{I}})_m$ are then defined in terms of $\lambda_j(t)$. There are two kinds of turning points; a turning point of the first kind is a point where λ_j vanishes for some j, and a turning point of the second kind is a point where $\lambda_j - \lambda_k$ or $\lambda_j + \lambda_k$ vanishes for some $j \neq k$.

Now the main result of this paper is the following:

Theorem 1. *Assume that t_0 is not a turning point of $(P_{\mathrm{I}})_m$. Suppose further that*

$$\sum_{j=1}^{m} n_j \lambda_j(t) \ \text{does not identically vanish for any} \quad (n_1, \ldots, n_m) \in \mathbb{Z}^m \setminus \{0\}. \tag{4}$$

Then, in a neighborhood of $t = t_0$, there exists a formal solution of $(P_{\mathrm{I}})_m$ of the following form:

$$\begin{aligned} u_j(t, \eta; \alpha, \beta) &= u_{j,0}(t) + \eta^{-1/2} u_{j,1/2}(t, \Psi, \Phi) + \eta^{-1} u_{j,1}(t, \Psi, \Phi) + \cdots, \\ v_j(t, \eta; \alpha, \beta) &= v_{j,0}(t) + \eta^{-1/2} v_{j,1/2}(t, \Psi, \Phi) + \eta^{-1} v_{j,1}(t, \Psi, \Phi) + \cdots, \end{aligned} \tag{5}$$

$(j = 1, \ldots, m)$. Here $u_{j,l/2}(t, \Psi, \Phi)$ and $v_{j,l/2}(t, \Psi, \Phi)$ $(l = 1, 2, \ldots)$ are polynomials in (Ψ, Φ) of degree at most l with analytic (in t) coefficients (in particular, $u_{j,1/2}$ and $v_{j,1/2}$ are linear combinations of (Ψ, Φ) with analytic coefficients), and $\Psi = (\Psi_1, \ldots, \Psi_m)$ and $\Phi = (\Phi_1, \ldots, \Phi_m)$ are "instantons", that is, formal series of exponential type of the form

$$\begin{aligned} \Psi_j &= \alpha_j \exp\left\{ \eta \int^t \left(\sum_{k=0}^{\infty} \eta^{-k} \sum_{|\mu|=k} (\mu_j + 1) g_{\mu+e_j}(t, \eta) \gamma^{\mu} \right) dt \right\}, \\ \Phi_j &= \beta_j \exp\left\{ -\eta \int^t \left(\sum_{k=0}^{\infty} \eta^{-k} \sum_{|\mu|=k} (\mu_j + 1) g_{\mu+e_j}(t, \eta) \gamma^{\mu} \right) dt \right\} \end{aligned} \tag{6}$$

$(j = 1, \ldots, m)$, where α_j and β_j are free complex constants, γ denotes $\gamma = (\gamma_1, \ldots, \gamma_m) = (\alpha_1\beta_1, \ldots, \alpha_m\beta_m)$, $\mu = (\mu_1, \ldots, \mu_m)$ $(\mu_j \in \mathbb{Z}, \mu_j \geq 0)$ and $e_j = (0, \ldots, 1, \ldots, 0)$ (i.e., only the j-th component is equal to 1 while the others are all 0) are multi-indices, and for each multi-index $\nu = (\nu_1, \ldots, \nu_m)$ $g_\nu(t, \eta)$ is a formal power series of $\eta^{-1/2}$ with analytic coefficients of the following form:

$$g_\nu(t, \eta) = \sum_{l=0}^{\infty} \eta^{-l/2} g_{\nu, l/2}(t). \tag{7}$$

We call the formal solution $(u_j(t, \eta; \alpha, \beta), v_j(t, \eta; \alpha, \beta))$ given in this theorem an "instanton-type solution" of $(P_{\mathrm{I}})_m$.

Remark 1. The top order part $(u_{j,0}(t), v_{j,0}(t))$ of $(u_j(t, \eta; \alpha, \beta), v_j(t, \eta; \alpha, \beta))$ is the same as that of the 0-parameter solution $(\hat{u}_j(t, \eta), \hat{v}_j(t, \eta))$. More important is the top order part of the instantons (Ψ_j, Φ_j); it is described by $g_{e_j,0}(t)$, which coincides with the characteristic root $\lambda_j(t)$ of the Fréchet derivative $(\Delta P_{\mathrm{I}})_m$. This fact shows the relevance of the instanton-type solutions to the Stokes phenomenon and, at the same time, validates the definition of the Stokes geometry of $(P_{\mathrm{I}})_m$ given in [3].

Remark 2. Each coefficient of $u_{j,l/2}$ (resp. $v_{j,l/2}$) may have some singularity in addition to turning points: By the construction of solutions explained below we see that the singular points of $u_{j,l/2}$ (resp. $v_{j,l/2}$) are contained at most in the union of zeros of $\sum_j n_j\lambda_j(t)$ with $|n_1|+\cdots+|n_m|\le l+1$. Similarly the singular points of a coefficient of $g_\nu(t,\eta)$ are contained in the union of zeros of $\sum_j n_j\lambda_j(t)$ with $|n_1|+\cdots+|n_m|\le 2|\nu|-1$.

3 Outline of the construction of instanton-type solutions

As was mentioned in Introduction, we construct instanton-type solutions by using reduction of a Hamiltonian system to its Birkhoff normal form. The concrete procedure of construction consists of the following four steps.

Step 1. First we express $(P_{\mathrm{I}})_m$ in the form of a Hamiltonian system.

As is discussed in [13, 14], the first Pianlevé hierarchy $(P_{\mathrm{I}})_m$ is obtained by restricting the most degenerate Garnier system onto a one-dimensional complex curve. Since the (degenerate) Garnier system possesses a Hamiltonian structure, the first Pianlevé hierarchy also inherits such a Hamitonian structure. To be more specific, $(P_{\mathrm{I}})_m$ can be expressed in the form of a Hamiltonian system by using the canonical variable (σ_j,τ_j) defined as follows:

$$u_j=(-1)^{j-1}\sum_{k_1<\cdots<k_j}\sigma_{k_1}\cdots\sigma_{k_j}, \tag{8}$$

$$\tau_j=\frac{1}{2}\left(v_1\sigma_j^{m-1}+\cdots+v_m\right). \tag{9}$$

(Cf. [13], [10]; u_j is the j-th order fundamental symmetric polynomial of $(\sigma_1,\ldots,\sigma_m)$ (up to the sign) and τ_j is defined as the residue of coefficients of the second order linear differential equation associated with $(P_{\mathrm{I}})_m$ through isomonodromic deformations.) In what follows we use another canonical variable which is more closely attached to the original variable (u_j,v_j): Take q_j as

$$q_j=(-1)^{j-1}u_j\left(=\sum_{k_1<\cdots<k_j}\sigma_{k_1}\cdots\sigma_{k_j}\right). \tag{10}$$

As the conjugate variable of q_j we choose p_j so that it may satisfy $\sum dq_j\wedge dp_j=\sum d\sigma_j\wedge d\tau_j$ or $\sum p_j dq_j=\sum\tau_j d\sigma_j$. That is, we define p_j in such a way that

$$\tau_j=\sum_k\frac{\partial q_k}{\partial\sigma_j}p_k \tag{11}$$

may be satisfied. More explicitly, p_j is given by the following relation:

$$v_j=2(-1)^{m-j}(p_{m-j+1}+p_{m-j+2}q_1+\cdots+p_m q_{j-1}). \tag{12}$$

Remark 3. The explicit relation (12) follows from (9) and (11) in the following way: For $k=0,1,\ldots,m-1$ let $s^{(k)}$ and $\tilde{s}_j^{(k)}$ denote the k-th order fundamental symmetric polynomial of $(\sigma_1,\ldots,\sigma_m)$ and that of $(\sigma_1,\ldots,\sigma_{j-1},\sigma_{j+1},\ldots,\sigma_m)$, respectively. Then, if we define a k-th degree polynomial $F^{(k)}(z)$ of z by

$$F^{(k)}(z) = s^{(k)} - s^{(k-1)}z + s^{(k-2)}z^2 - \cdots + (-1)^k z^k, \tag{13}$$

the following relation holds for $j = 1, \ldots, m$:

$$F^{(k)}(\sigma_j) = \tilde{s}_j^{(k)}. \tag{14}$$

Taking the relation $\partial q_k/\partial\sigma_j = \partial s^{(k)}/\partial\sigma_j = \tilde{s}_j^{(k-1)}$ into account, we find that (9) and (11) together with (14) entail

$$\frac{1}{2}\left(v_1 z^{m-1} + \cdots + v_{m-1}z + v_m\right) = F^{(m-1)}(z)p_m + \cdots + F^{(1)}(z)p_2 + p_1. \tag{15}$$

Relation (12) immediately follows from comparison of like powers (in z) of (15).

Thus, in the variable (q_j, p_j), $(P_{\mathrm{I}})_m$ can be expressed in the form of the following Hamiltonian system:

$$\frac{dq_j}{dt} = \eta\frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\eta\frac{\partial H}{\partial q_j}. \tag{16}$$

For example, the Hamiltonian is explicitly given by

$$H = -\frac{1}{2}q_1^4 + \frac{3}{2}q_1^2 q_2 - \frac{1}{2}q_2^2 - 2q_1 p_2^2 - 4p_1 p_2 + c_1(-q_1^2 + q_2) - tq_1 \tag{17}$$

for $m = 2$ and by

$$\begin{aligned} H = &-\frac{1}{2}q_1^5 + 2q_1^3 q_2 - \frac{3}{2}q_1^2 q_3 - \frac{3}{2}q_1 q_2^2 + q_2 q_3 + 4q_1 p_2 p_3 + 2q_2 p_3^2 \\ &+ 4p_1 p_3 + 2p_2^2 + c_1(-q_1^3 + 2q_1 q_2 - q_3) + c_2(-q_1^2 + q_2) - tq_1 \end{aligned} \tag{18}$$

for $m = 3$.

Step 2. In the canonical variable (q_j, p_j) there exists the following 0-parameter solution of (16), which corresponds to (2):

$$\hat{q}_j(t,\eta) = \hat{q}_{j,0}(t) + \eta^{-1}\hat{q}_{j,1}(t) + \cdots, \quad \hat{p}_j(t,\eta) = \hat{p}_{j,0}(t) + \eta^{-1}\hat{p}_{j,1}(t) + \cdots. \tag{19}$$

Then we next consider the "localization at the 0-parameter solution" of (16), that is, we introduce a new (formal) variable (ψ_j, φ_j) defined as follows:

$$q_j = \hat{q}_j + \eta^{-1/2}\psi_j, \quad p_j = \hat{p}_j + \eta^{-1/2}\varphi_j. \tag{20}$$

Since (ψ_j, φ_j) is also canonical, in the variable (ψ_j, φ_j) (16) can be expressed again in the Hamiltonian form as

$$\frac{d\psi_j}{dt} = \eta\frac{\partial K}{\partial\varphi_j}, \quad \frac{d\varphi_j}{dt} = -\eta\frac{\partial K}{\partial\psi_j}, \tag{21}$$

where

$$K = \sum_{|\mu+\nu|\geq 2} \frac{1}{\mu!\nu!}\eta^{-(|\mu+\nu|-2)/2}\frac{\partial^{|\mu+\nu|}H}{\partial q^\mu \partial p^\nu}(t,\hat{q},\hat{p})\psi^\mu\varphi^\nu. \tag{22}$$

Step 3. This is the most important step; we consider the reduction of (21) to its Birkhoff normal form.

As the localization at the 0-parameter solution is done in *Step 2*, the leading part of (21) consequently becomes linear. For example, the coefficient matrix of the top order part (in $\eta^{-1/2}$) of (21) is given by

$$
\begin{array}{cc}
 & \begin{array}{cc} \overset{k}{\vee} \qquad\qquad & \overset{k+m}{\vee} \end{array} \\
\begin{array}{c} j> \\[4em] j+m> \end{array} &
\left.\left(\begin{array}{c|c}
\dfrac{\partial^2 H}{\partial p_j \partial q_k} & \dfrac{\partial^2 H}{\partial p_j \partial p_k} \\[1em] \hline
-\dfrac{\partial^2 H}{\partial q_j \partial q_k} & -\dfrac{\partial^2 H}{\partial q_j \partial p_k}
\end{array}\right)\right|_{\substack{q_l=\hat{q}_{l,0} \\ p_l=\hat{p}_{l,0}}}
\end{array}
. \tag{23}
$$

Note that the eigenvalues of the matrix (23) exactly coincide with $\pm\lambda_j(t)$, i.e., the characteristic roots of the Fréchet derivative $(\varDelta P_{\mathrm{I}})_m$. Therefore they are distinct and non-zero outside the set of turning points.

Making use of this structure peculiar to (21), we can reduce (21) to its Birkhoff normal form, that is, we have

Theorem 2. *We assume that t_0 is not a turning point of $(P_{\mathrm{I}})_m$. We further assume* (4)*. Then, in a neighborhood of $t = t_0$, we can find a canonical transform*

$$
\psi_j = \sum_{k=0}^{\infty} \eta^{-k/2} \psi_j^{(k)}(t, \tilde{\psi}, \tilde{\varphi}, \eta^{-1/2}), \quad \varphi_j = \sum_{k=0}^{\infty} \eta^{-k/2} \varphi_j^{(k)}(t, \tilde{\psi}, \tilde{\varphi}, \eta^{-1/2}), \tag{24}
$$

where $\psi_j^{(k)}$ and $\varphi_j^{(k)}$ are homogeneous polynomials of degree $(k+1)$ in $(\tilde{\psi}, \tilde{\varphi})$, that transforms (21) *into the Birkhoff normal form*

$$
\frac{d\tilde{\psi}_j}{dt} = \eta \frac{\partial \tilde{K}}{\partial \tilde{\varphi}_j}, \quad \frac{d\tilde{\varphi}_j}{dt} = -\eta \frac{\partial \tilde{K}}{\partial \tilde{\psi}_j}, \tag{25}
$$

where

$$
\tilde{K} = \tilde{K}(t, \theta_1, \ldots, \theta_m, \eta^{-1/2}) \quad \textit{with} \quad \theta_j = \tilde{\psi}_j \tilde{\varphi}_j. \tag{26}
$$

A sketch of the proof of Theorem 2 will be given in Section 4.

Step 4. In view of (26) we find that (25) can be written as

$$
\frac{d\tilde{\psi}_j}{dt} = \eta \left.\frac{\partial \tilde{K}}{\partial \theta_j}\right|_{\theta_l = \tilde{\psi}_l \tilde{\varphi}_l} \tilde{\psi}_j, \quad \frac{d\tilde{\varphi}_j}{dt} = -\eta \left.\frac{\partial \tilde{K}}{\partial \theta_j}\right|_{\theta_l = \tilde{\psi}_l \tilde{\varphi}_l} \tilde{\varphi}_j. \tag{27}
$$

In particular, this entails that

$$
\frac{d}{dt}(\tilde{\psi}_j \tilde{\varphi}_j) = \frac{d\tilde{\psi}_j}{dt} \tilde{\varphi}_j + \tilde{\psi}_j \frac{d\tilde{\varphi}_j}{dt} = 0, \tag{28}
$$

that is,

$$
\gamma_j := \tilde{\psi}_j \tilde{\varphi}_j \quad \text{does not depend on } t. \tag{29}
$$

By substituting (29) into (27) we can explicitly solve (27) to obtain

$$
\tilde{\psi}_j = \alpha_j \exp\left(\eta \int^t \left.\frac{\partial \tilde{K}}{\partial \theta_j}\right|_{\theta_l = \gamma_l} dt \right), \quad \tilde{\varphi}_j = \beta_j \exp\left(-\eta \int^t \left.\frac{\partial \tilde{K}}{\partial \theta_j}\right|_{\theta_l = \gamma_l} dt \right), \tag{30}
$$

where α_j and β_j are free complex constants of integration. Note that (29) and (30) imply

$$\gamma_j = \alpha_j \beta_j. \tag{31}$$

In this way the Birkhoff normal form (25) has been solved explicitly. If we denote the explicit solution $(\tilde{\psi}_j, \tilde{\varphi}_j)$ of (25) thus obtained by (Ψ_j, Φ_j) and substitute it into the canonical transform (24), we can obtain also a (formal) solution of (21) and consequently an instanton-type solution of $(P_{\mathrm{I}})_m$ with $(2m)$ free parameters $(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m)$. (The solution (Ψ_j, Φ_j) of (25) or (27) gives "instantons".) We have thus finished the construction of instanton-type solutions of $(P_{\mathrm{I}})_m$.

4 A sketch of the proof of Theorem 2

In this section we sketch out the proof of Theorem 2.

Let us denote $\eta^{-1/2}$ by ϵ. We want to construct a canonical transform

$$\psi_j = \sum_{k=0}^{\infty} \epsilon^k \psi_j^{(k)}(t, \tilde{\psi}, \tilde{\varphi}, \epsilon), \quad \varphi_j = \sum_{k=0}^{\infty} \epsilon^k \varphi_j^{(k)}(t, \tilde{\psi}, \tilde{\varphi}, \epsilon) \tag{32}$$

which transforms the Hamiltonian system (21) in question into its Birkhoff normal form. Here $\psi_j^{(k)}$ and $\varphi_j^{(k)}$ are assumed to be of the following form:

$$\psi_j^{(k)} = \sum_{|\mu+\nu|=k+1} \psi_j^{\mu,\nu}(t,\epsilon) \tilde{\psi}^{\mu} \tilde{\varphi}^{\nu}, \quad \varphi_j^{(k)} = \sum_{|\mu+\nu|=k+1} \varphi_j^{\mu,\nu}(t,\epsilon) \tilde{\psi}^{\mu} \tilde{\varphi}^{\nu}. \tag{33}$$

As the construction of $\psi_j^{(0)}$ and $\varphi_j^{(0)}$, i.e., the linear part with respect to $(\tilde{\psi}, \tilde{\varphi})$, is quite different from that of the nonlinear part, we discuss these two parts separately in what follows.

4.1 Construction of the linear part $\psi_j^{(0)}$ and $\varphi_j^{(0)}$

Let us write the quadratic part of the Hamiltonian (22) as

$$K = \frac{1}{2}{}^t\psi M_1 \psi + \frac{1}{2}{}^t\varphi M_2 \varphi + {}^t\varphi M_3 \psi + \cdots, \tag{34}$$

where M_j is a formal power series of ϵ whose coefficients are $m \times m$ matrices of analytic functions of t. Then (the linear part of) the Hamiltonian system (21) can be expressed as

$$\frac{d}{dt}\begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \eta \left(\begin{array}{c|c} M_3 & M_2 \\ \hline -M_1 & -{}^tM_3 \end{array}\right) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} + \cdots. \tag{35}$$

We now want to construct the linear part of a canonical transform

$$\begin{pmatrix} \psi^{(0)} \\ \varphi^{(0)} \end{pmatrix} = A \begin{pmatrix} \tilde{\psi}^{(0)} \\ \tilde{\varphi}^{(0)} \end{pmatrix} \quad \text{with} \quad A = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \tag{36}$$

(where a, b, c and d are also formal power series of ϵ with $m \times m$ matrix coefficients) in such a way that the following two conditions may be satisfied.

(A1) (36) is symplectic,

(A2) (36) diagonalizes the linear part of (35).

First, the top order term (with respect to ϵ) of (36) can be constructed by applying

Lemma 1. *Assume that the top order term of the coefficient of* (35)

$$\left(\begin{array}{c|c} M_{3,0} & M_{2,0} \\ \hline -M_{1,0} & -{}^tM_{3,0} \end{array}\right) = \left.\left(\begin{array}{c|c} M_3 & M_2 \\ \hline -M_1 & -{}^tM_3 \end{array}\right)\right|_{\epsilon=0}, \tag{37}$$

which coincides with (23)*, has distinct eigenvalues. Then we can find a symplectic matrix T that satisfies*

$$T^{-1}\left(\begin{array}{c|c} M_{3,0} & M_{2,0} \\ \hline -M_{1,0} & -{}^tM_{3,0} \end{array}\right)T = \left(\begin{array}{ccc|ccc} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_m & & & \\ \hline & & & -\lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & -\lambda_m \end{array}\right), \tag{38}$$

where $\pm\lambda_j(t)$ are eigenvalues of (37)*, i.e., of* (23)*.*

As the proof of Lemma 1 is an exercise of the linear algebra, we omit it here. Since the assumption of Lemma 1 is satisfied outside the set of turning points, the existence of the top order term of (36) is guaranteed by this lemma.

Once the top order term is constructed, higher order terms (with respect to ϵ) of (36) are determined in the following manner: If we let X, Y and Z denote bd^{-1}, ca^{-1} and $1-{}^tXY$ $(=1-{}^td^{-1}{}^tbca^{-1})$, respectively, we find that the conditions (A1) and (A2) are equivalent to

$${}^tX = X, \quad M_3X + {}^tX{}^tM_3 + {}^tXM_1X + M_2 - \epsilon^2\frac{\partial X}{\partial t} = 0, \tag{39}$$

$${}^tY = Y, \quad {}^tM_3Y + {}^tYM_3 + M_1 + {}^tYM_2Y + \epsilon^2\frac{\partial Y}{\partial t} = 0, \tag{40}$$

$$d = {}^t(Za)^{-1}, \tag{41}$$

$$\begin{aligned} &a^{-1}Z^{-1}\left[M_3 + {}^tXM_1 + M_2Y + {}^tX{}^tM_3Y + \epsilon^{2t}X\frac{\partial Y}{\partial t}\right]a \\ &\quad - \epsilon^2a^{-1}\frac{\partial a}{\partial t} : \text{diagonal}. \end{aligned} \tag{42}$$

Since the top order term has already been constructed, we may assume that

$$a = 1 + O(\epsilon^2),\ \ b = O(\epsilon^2),\ \ c = O(\epsilon^2),\ \ d = 1 + O(\epsilon^2), \tag{43}$$

$$M_1 = O(\epsilon^2),\ \ M_2 = O(\epsilon^2),\ \ M_3 = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix} + O(\epsilon^2). \tag{44}$$

Equations (39) and (40) then uniquely determine the formal power series X and Y, respectively. Consequently $Z = 1-{}^tXY$ is also fixed. Furthermore, substituting X, Y and Z thus determined into (42), we may as well determine $a = 1+\epsilon^2a_2+\epsilon^4a_4+\cdots$ so that (42) is satisfied. In this way, by using (41) in addition, we can construct higher order terms of a, b, c and d, that is, the higher order terms of (36).

4.2 Construction of the nonlinear part

To construct the nonlinear part of the canonical transform (32), we make use of a generating function of the following form:

$$W(t,\tilde{\psi},\varphi)=\sum_{|\mu+\nu|\geq 2}\epsilon^{|\mu+\nu|-2}w^{\mu,\nu}\tilde{\psi}^{\mu}\varphi^{\nu}. \tag{45}$$

The canonical transform

$$\psi=\psi(t,\tilde{\psi},\tilde{\varphi},\epsilon),\quad \varphi=\varphi(t,\tilde{\psi},\tilde{\varphi},\epsilon) \tag{46}$$

induced by the generating function W is determined by

$$\psi_j=-\frac{\partial W}{\partial\varphi_j},\quad \tilde{\varphi}_j=-\frac{\partial W}{\partial\tilde{\psi}_j}, \tag{47}$$

and the new Hamiltonian $\tilde{K}$ for $(\tilde{\psi},\tilde{\varphi})$ is described in terms of the original Hamiltonian K and the generating function W as follows:

$$\tilde{K}=K(t,\psi(t,\tilde{\psi},\tilde{\varphi},\epsilon),\varphi(t,\tilde{\psi},\tilde{\varphi},\epsilon),\epsilon)+\epsilon^2\frac{\partial W}{\partial t}(t,\tilde{\psi},\varphi(t,\tilde{\psi},\tilde{\varphi},\epsilon),\epsilon). \tag{48}$$

Thus, for the construction of a canonical transform that reduces (21) to its Birkhoff normal form, it suffices to fix each coefficient $w^{\mu,\nu}$ of the generating function W so that

$$\text{any term of the form } \tilde{\psi}^{\mu}\tilde{\varphi}^{\nu} \text{ with } \mu\neq\nu \text{ may not appear in } \tilde{K}. \tag{49}$$

Note that the construction of the linear part of the canonical transform has been already finished in Section 4.1. Hence we may assume that the quadratic part of the original Hamiltonian K has the form (34) where $M_1=M_2=0$ and M_3 is a diagonal matrix whose top order term is given by the right-hand side of (38), and further that

$$w^{\mu,\nu}=-1\ \ (\text{for } \mu=\nu),\quad w^{\mu,\nu}=0\ \ (\text{for } \mu\neq\nu) \tag{50}$$

in case $|\mu+\nu|=2$. Using this "induction hypothesis" and the expression (48) of $\tilde{K}$, we can verify the following Lemma 2 through explicit computations similar to those of [15, Section 2.2].

Lemma 2. *For $|\mu+\nu|\geq 3$ the requirement (49) is equivalent to an equation of the following form:*

$$\left(\sum_{j=1}^{m}(\mu_j-\nu_j)\lambda_j+O(\epsilon^2)\right)w^{\mu,\nu}+\epsilon^2\frac{\partial}{\partial t}w^{\mu,\nu}=R(t,w^{\mu',\nu'},\epsilon^2), \tag{51}$$

where the indices (μ',ν') that appear in $R(t,w^{\mu',\nu'},\epsilon)$ of the right-hand side run in the set $\{(\mu',\nu')\,;\,|\mu'+\nu'|\leq|\mu+\nu|-1\}$.

Thus the terms $w^{\mu,\nu}$ with $\mu\neq\nu$ can be recursively determined.

This completes the proof of Theorem 2.

References

[1] T. Aoki, T. Kawai and Y. Takei, WKB analysis of Painlevé transcendents with a large parameter. II, *Structure of Solutions of Differential Equations*, World Scientific, 1996, pp.1–49.

[2] P.R. Gordoa and A. Pickering, Nonisospectral scattering problems: A key to integrable hierarchies, *J. Math. Phys.*, **40**(1999), 5749–5786.

[3] T. Kawai, T. Koike, Y. Nishikawa and Y. Takei, On the Stokes geometry of higher order Painlevé equations, *Astérisque*, Vol. 297, 2004, pp. 117–166.

[4] T. Kawai, T. Koike, Y. Nishikawa and Y. Takei, On the complete description of the Stokes geometry for the first Painlevé hierarchy, *RIMS Kôkyûroku*, Vol. 1397, 2004, pp. 74–101.

[5] T. Kawai and Y. Takei, WKB analysis of Painlevé transcendents with a large parameter. I, *Adv. Math.*, **118**(1996), 1–33.

[6] T. Kawai and Y. Takei, WKB analysis of Painlevé transcendents with a large parameter. III, *Adv. Math.*, **134**(1998), 178–218.

[7] T. Kawai and Y. Takei, On WKB analysis of higher order Painlevé equations with a large parameter, *Proc. Japan Acad., Ser. A*, **80**(2004), 53–56.

[8] T. Kawai and Y. Takei, WKB analysis of higher order Painlevé equations with a large parameter — Local reduction of 0-parameter solutions for Painlevé hierarchies (P_J) ($J =$ I, II-1 or II-2), *Adv. Math.*, in press.

[9] T. Kawai and Y. Takei, *Algebraic Analysis of Singular Perturbation Theory*, Translations of Mathematical Monographs, Vol. 227, Amer. Math. Soc., 2005. (Originally published in Japanese by Iwanami, Tokyo in 1998.)

[10] H. Kimura, The degeneration of the two dimensional Garnier system and the polynomial Hamiltonian structure, *Ann. Mat. Pura Appl.*, **155**(1989), 25–74.

[11] N. A. Kudryashov, The first and second Painlevé equations of higher order and some relations between them, *Phys. Lett. A*, **224**(1997), 353–360.

[12] N. A. Kudryashov and M. B. Soukharev, Uniformization and transcendence of solutions for the first and second Painlevé hierarchies, *Phys. Lett. A*, **237**(1998), 206–216.

[13] S. Shimomura, Painlevé property of a degenerate Garnier system of (9/2)-type and of a certain fourth order non-linear ordinary differential equation, *Ann. Scuola Norm. Sup. Pisa*, **29**(2000), 1–17.

[14] S. Shimomura, A certain expression of the first Painlevé hierarchy, *Proc. Japan Acad., Ser. A*, **80**(2004), 105–109.

[15] Y. Takei, Singular-perturbative reduction to Birkhoff normal form and instanton-type formal solutions of Hamiltonian systems, *Publ. RIMS, Kyoto Univ.*, **34**(1998), 601–627.

[16] Y. Takei, An explicit description of the connection formula for the first Painlevé equation, *Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear*, Kyoto Univ. Press, 2000, pp. 271–296.

[17] Y. Takei, On an exact WKB approach to Ablowitz-Segur's connection problem for the second Painlevé equation, *ANZIAM J.*, **44**(2002), 111–119.