ADDENDUM TO "NUMERICALLY TRIVIAL INVOLUTIONS OF ENRIQUES SURFACES"

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In [1] we made a correction to [2]. Here we add a new one overlooked then.

The following is the earliest and simplest example of a numerically trivial involution of an Enriques surface.

Example 1. *(*[2, Example 1], [1, Example 1]*) Let μ be the involution of a Kummer surface $Km(E' \times E'')$ of product type induced by $(id_{E'}, -id_{E''})$, and β that induced from the translation by a 2-torsion point a with $a \notin E' \times 0 \cup 0 \times E''$. Then $\mu\beta$ has no fixed points and μ induces a numerically trivial involution of the Enriques surface $Km(E' \times E'')/\mu\beta$.*

Contrary to the erroneous Proposition (4.8) of [2] (see (4) in the errata below), this involution is *not* cohomologically trivial. Corollary 4 of [2] should read

Corollary 4. *A numerically trivial involution of Kummer type is not cohomologically trivial.*

Proof. We prove our assertion by constructing an elliptic fibration.

Let $\{p'_1, \dots, p'_4\}$ and $\{p''_1, \dots, p''_4\}$ be the branch of the double coverings $E' \to \mathbb{P}^1 \simeq E'/(-id)$ and $E'' \to \mathbb{P}^1 \simeq E''/(-id)$, respectively. The Kummer surface $Km(E' \times E'')$ is the minimal resolution of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with branch

$$(p'_1 \times \mathbb{P}^1 \cup \dots \cup p'_4 \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times p''_1 \cup \dots \cup \mathbb{P}^1 \times p''_4).$$

More precisely, it is the double cover of the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the 16 points (p'_i, p''_j), $i, j = 1, \dots, 4$, with branch the strict transform of these eight rational curves.

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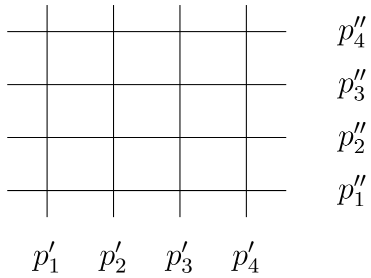

The fixed locus of μ is the inverse images of these strict transform. We denote them by

$$(A_1 \sqcup \cdots \sqcup A_4) \sqcup (B_1 \sqcup \cdots \sqcup B_4).$$

The involution $\varepsilon := \mu\beta$ of Example 1 acts on this disjoint union. Renumbering $A_1, \ldots, A_4$ and $B_1, \ldots, B_4$, we may assume that

$$\varepsilon(A_i) = A_{i+1} \quad \text{and} \quad \varepsilon(B_i) = B_{i+1}$$

for $i = 1, 3$. Then ε interchanges two divisors $A_1 + A_3 + B_2 + B_4$ and $A_2 + A_4 + B_1 + B_3$. Let Λ be the linear pencil spanned by their images

$$H_1 := p'_1 \times \mathbb{P}^1 + p'_3 \times \mathbb{P}^1 + \mathbb{P}^1 \times p''_2 + \mathbb{P}^1 \times p''_4$$

and

$$H_2 := p'_2 \times \mathbb{P}^1 + p'_4 \times \mathbb{P}^1 + \mathbb{P}^1 \times p''_1 + \mathbb{P}^1 \times p''_3$$

on $\mathbb{P}^1 \times \mathbb{P}^1$. Then Λ induces elliptic fibrations

$$\Phi_\Lambda : Km(E' \times E'')/\mu \to \Lambda(\simeq \mathbb{P}^1)$$

of the rational surface and

$$Km(E' \times E'') \to \tilde{\Lambda}(\simeq \mathbb{P}^1)$$

of the Kummer surface. The latter is the base change of the former by the double covering $\tilde{\Lambda} \to \Lambda$ with branch $[H_1]$ and $[H_2]$, and descends to an elliptic fibration f of the Enriques surface $Km(E' \times E'')/\varepsilon$ over $\tilde{\Lambda}/\bar{\varepsilon}(\simeq \mathbb{P}^1)$, where $\bar{\varepsilon}$ is the involution of $\tilde{\Lambda}$ induced by ε. Let G_1 and G_2 be the reduced part of the two multiple fibers of f. Since $\bar{\varepsilon}$ interchanges $[H_1]$ and $[H_2]$, the numerically trivial involution of Example 1 interchanges G_1 and G_2. Since the linear equivalent classes of G_1 and G_2 differ by the canonical class, it is not cohomologically trivial.

For [1, Example 2], we have $\varepsilon(A_i) = B_i$ for every $i = 1, \ldots, 4$, since a Cremona involution interchanges $p'_i \times \mathbb{P}^1$ and $\mathbb{P}^1 \times p''_i$ for every $i = 1, \ldots, 4$. The above argument works literally in this case too. (This gives a simpler proof of [1, Proposition 8].)

Now our assertion follows from [1, Theorem 3]. $\square$

Errata to [2]

(1) $\Delta(S)$ at the first line in p. 384 should read $\Delta_0(S)$.

(2) Lemma (1.2) in should be replaced by

Lemma. *Let τ be an automorphism of order 2^n of $\tilde{S}$. If some power of τ is equal to ε, then τ itself is equal to ε.*

The proof in [2] is valid under this additional assumption $\mathrm{ord}(\tau) = 2^n$ and this change does not effect the proof of Proposition (1.1)

(3) *Proof of Proposition (4.5).* The 10 rational curves in the diagram (4.4) generate a rank 10 lattice E' with $|\det E'| = 16$. The rest should be replaced by the proof of Proposition (4.8).

(4) Page 396.

Figure: $\overline{P}_0$ at the bottom should read $\overline{P}_1$.

$\ell.5$: $D_4 \perp D_4 \perp U(2)$ should be replaced by $D_4 \perp D_4 \perp (-2) \perp (2)$.

$\ell\ell.6, 7$: The class of $\overline{P}_1$ belongs to E''. Hence the 12 rational curves in the diagram (4.7) generate E'' with $|\det E''| = 64$. Therefore, the 'proof' collapses. In fact (4.8) is false as is shown above.

The second and third were pointed out by Hisanori Ohashi, to whom the author is very grateful.

References

[1] Mukai, S.: Numerically trivial involutions of Enriques surfaces, preprint RIMS-1544, Kyoto University, May, 2006.

[2] Mukai, S. and Namikawa, Y.: Automorphisms of Enriques surfaces which act trivially on the cohomology groups, Invent. math., **77**(1984), 383–397.

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