Derived McKay correspondence via pure-sheaf transforms

Timothy Logvinenko

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Abstract

In most cases where it had been shown to exist the derived McKay correspondence $D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n)$ can be written as a Fourier-Mukai transform which sends point sheaves of the crepant resolution Y to pure sheaves in $D^G(\mathbb{C}^n)$. We give a sufficient (and necessary) condition for an object of $D^G(Y \times \mathbb{C}^n)$ to be the defining object of such a transform. We then use it to construct first example of the derived McKay correspondence for a non-projective crepant resolution of $\mathbb{C}^3/G$. Along the way we extract some more geometric sense out of the Intersection Theorem and learn to explicitly compute θ-stable families of G-constellations and their direct transforms.

1 Introduction

It had been observed by McKay in [McK80] that the representation graph (better known now as *McKay quiver*) of a finite subgroup G of $\mathrm{SL}_2(\mathbb{C})$ is one of the extended Dynkin graphs of affine Lie algebras of type ADE and that the configuration of irreducible exceptional divisors on the crepant resolution of $\mathbb{C}^2/G$ is dual to the regular version of the same Dynkin graph. The arising bijective correspondence between nontrivial irreducible representations of G and irreducible exceptional divisors on Y became known as *the (classical) McKay correspondence*. It has enjoyed a number of restatements and generalisations: [GSV83], [IN00],[Rei97], [IR96] to name but a few. One of the most far-reaching of them has been *the derived McKay correspondence* conjecture, which had appeared in its initial form in M. Reid's famous preprint [Rei97]. It can be stated as follows:

Conjecture 1.1. *Let G be a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$ and let Y be a crepant resolution of $\mathbb{C}^n/G$, if one exists. Then*

$$D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n) \tag{1.1}$$

where $D(Y)$ and $D^G(\mathbb{C}^n)$ are bounded derived categories of coherent sheaves on Y and of G-equivariant coherent sheaves on $\mathbb{C}^n$, respectively.

To date and to the extent of our knowledge this conjecture has been settled for the following situations:

1. $G \subset \mathrm{SL}_{2,3}(\mathbb{C})$; Y the distinguished crepant resolution G-Hilb; ([BKR01], Theorem 1.1).

2. $G \subset \mathrm{SL}_3(\mathbb{C})$ abelian; Y any projective crepant resolution; ([CI04], Theorem 1.1).

3. $G \subset \mathrm{SL}_n(\mathbb{C})$ abelian; Y any projective crepant resolution; ([Kaw05], special case of Theorem 4.2).

4. $G \subset \mathrm{Sp}_{2n}(\mathbb{C})$; Y any symplectic (crepant) resolution; ([BK04], Theorem 1.1).

In the case 3 the construction is not direct and it isn't clear what form does equivalence (1.1) take, but in each of the cases 1, 2 and 4, where (1.1) is constructed directly, we observe that the constructed functor possesses the following rather special property: it sends the point sheaves $\mathcal{O}_y$ of Y to pure sheaves (i.e. complexes with cohomologies concentrated in degree zero) in $D^G(\mathbb{C}^n)$. Another though less special (compare to [Orl97], Theorem 2.18) property shared by the functors constructed in these three cases is that each can be written as a Fourier-Mukai transform $\Phi_E(- \otimes \rho_0)$ (see Definition 3.3) for some object $E \in D^G(Y \times \mathbb{C}^n)$.

A trivial application (Proposition 3.6) of established machinery of Fourier-Mukai transforms shows that if an equivalence (1.1) is a Fourier-Mukai transform $\Phi_E(- \otimes \rho_0)$ which sends point sheaves to pure sheaves, the defining object $E \in D^G(Y \times \mathbb{C}^n)$ is necessarily a pure sheaf which is a flat family of *G-constellations* (certain finite-length coherent G-sheaves on $\mathbb{C}^n$, cf. Section 3.1) over Y. Moreover, the fibers of E have be simple ($G\text{-}\mathrm{End}_{\mathbb{C}^n}(E_{|y}) = \mathbb{C}$ for all $y \in Y$), orthogonal in all degrees ($G\text{-}\mathrm{Ext}^i_{\mathbb{C}^n}(E_{|y_1}, E_{|y_2}) = 0$ if $y_1 \neq y_2$) and the Kodaira-Spencer map has to be injective.

In this paper we give a converse result: given an arbitrary flat family $\mathcal{F}$ of G-constellations over Y, we give a condition on $\mathcal{F}$ sufficient for the functor $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ to be an equivalence (1.1). Somewhat of a surprise, in view of the above-listed properties that $\mathcal{F}$ would *a posteriori* have to possess, is that this condition only asks for the non-orthogonality locus of $\mathcal{F}$ to be of high enough codimension. Simultaneously, we show that any scheme which supports a sufficiently large and yet sufficiently orthogonal family of G-constellations has to be a crepant resolution of $\mathbb{C}^n/G$. The precise statement is:

Theorem 1.1. *Let G be a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$. Let Y be an irreducible separated scheme of finite type over $\mathbb{C}$ and let $\mathcal{F}$ be a family of G-constellations on Y such that:*

1. *Forgetful map $\pi_{\mathcal{F}} : Y \to \mathbb{C}^n/G$, which sends each G-constellation to its support, is well-defined and is a birational proper morphism.*

2. *For every* $0 \leq k < (n+1)/2$, *the codimension of the subset*

$$\mathrm{N}_k = \overline{\{(y_1, y_2) \in Y \times Y \setminus \Delta \mid G\text{-}\mathrm{Ext}^k_{\mathbb{C}^n}(\mathcal{F}_{|y_1}, \mathcal{F}_{|y_2}) \neq 0\}} \tag{1.2}$$

in $Y \times Y$ *is at least* $n + 1 - 2k$. *It is convinient to think of* N_k *as the locus of the degree* k *non-orthogonality in* $\mathcal{F}$.

Then Y *is smooth,* $\pi_{\mathcal{F}}$ *crepant and the functor* $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ *is an equivalence of categories* $D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n)$.

The proof is based on the ideas introduced in [BO95] and [BKR01], particularly on the Intersection Theorem trick introduced in the latter. But we take a different approach to Grothendieck duality issues arising when constructing the left adjoint of $\Phi_{\mathcal{F}}(- \otimes \rho_0)$, we squeeze yet more mileage out of the Intersection Theorem as the proof progresses and in the end we appeal to [Log06], Proposition 1.5 which states that outside the exceptional set of Y any family of G-constellations has to be locally isomorphic to the universal family of G-clusters. Then the locus of non-simplicity of the objects of $\mathcal{F}$ and non-injectivity of its Kodaira-Spencer map turns out to have too high a codimension to exist at all and the result quickly follows.

Thus the question of an existence of derived McKay correspondence (1.1) which sends point sheaves to pure sheaves is reduced to the question of an existence of a flat family of G-constellations satisfying nonorthogonality condition of Theorem 1.1. This is particularly important for the case of G being abelian, where all the flat families of G-constellations on a given resolution Y of $\mathbb{C}^n/G$ had been classified and their number (up to a twist by a line bundle) shown to be finite and non-zero ([Log06], Theorem 4.1).

When $n = 3$ the conditions of Theorem 1.1 reduce to only involve readily-computable (cf. Section 4.5) assumptions on orthogonality in degree 0:

Corollary 1.2. *Let* G *be a finite subgroup of* $\mathrm{SL}_3(\mathbb{C})$, *let* $\pi : Y \to \mathbb{C}^n/G$ *be a crepant resolution. Denote by* $E_1, \ldots, E_n$ *the irreducible exceptional surfaces of* π. *A point of an intersection* $E_{i_1} \cap \cdots \cap E_{i_k}$ *is said to be general if it belongs to no* E_i *other than* $E_{i_1}, \ldots, E_{i_k}$.

Let $\mathcal{F}$ *be a family of* G*-constellations on* Y *such that its forgetful map* $\pi_{\mathcal{F}}$ *agrees with* π. *If the fiber of* $\mathcal{F}$ *at a general point of any surface* E_i *is orthogonal in degree* 0 *to the fibers of* $\mathcal{F}$ *at general points of any* E_j *(including case* $j = i$*) and of any curve* $E_k \cap E_l$, *then* $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ *is an equivalence of categories* $D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n)$.

We then compute (Section 4) the following example: G is set to be the abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$ known as $\frac{1}{6}(1,1,4) \oplus \frac{1}{2}(1,0,1)$ (see Section 4.1) and Y to be a certain non-projective crepant resolution of $\mathbb{C}^3/G$ (see Section 4.2). We then construct a family of G-constellations over Y which we demonstrate to satisfy the assumptions of Corollary 1.2. This gives, as far as we know, the first example of the derived McKay correspondence for a non-projective crepant resolution of $\mathbb{C}^3/G$.

It also answers the following important question. Above mentioned properties of simplicity, orthogonality and injectivity of the Kodaira-Spencer map, which a family $\mathcal{F}$ must possess if $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is an equivalence, are common of the families obtained through a moduli construction. At present the only such construction known for G-constellations comes to us from the notion of θ-stability (see [CI04]). In dimension 3 the methods of [BKR01] apply out of the box to show that for any universal family $\mathcal{M}_\theta$ of θ-stable G-constellations $\Phi_{\mathcal{M}_\theta}(- \otimes \rho_0)$ is an equivalence. It is natural to ask whether there is any family $\mathcal{F}$ for which $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is an equivalence and which isn't one of $\mathcal{M}_\theta$. The computation in Section 4 answers it affirmatively, for due to the nature of the GIT construction involved any fine module space of θ-stable G-constellations is projective over $\mathbb{C}^n/G$. As the family constructed in Section 4 is constructed on a non-projective resolution it can't be one of $\mathcal{M}_\theta$. Thus there exists a class of families which exhibit the properties usual of a moduli family without being one, which suggests there may be a more general notion of stability on G-constellations waiting to be discovered.

The paper is organised as follows: Section 2 is abstract derived category theory on a locally noetherian scheme X. We propose a generalisation of the concept of the *homological dimension* of $E \in D^b_{\mathrm{coh}}(X)$ which we call Tor-*amplitude* and use it to show that inequality

$$\text{hom. dim. } E \geq \operatorname{codim}_X \operatorname{Supp}(E)$$

of [BM02], Corollary 5.5 refines to

$$\text{Tor-amp}\, E - \operatorname{codim}_X \operatorname{Supp} E \geq \text{coh-amp}\, E$$

Section 3 contains the proofs of Theorem 1.1 and of Corollary 1.2. In Section 4 we explicitly construct the derived McKay correspondence for a particular non-projective crepant resolution of $\mathbb{C}^3/G$. In the Appendix we prove a technical result which allows to explicitly compute the universal families $\mathcal{M}_\theta$ of θ-stable G-constellations and their direct transforms. It is used to drastically reduce the amount of computations necessary in Section 4.

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2 Cohomological and Tor amplitudes

We clarify terminology and introduce some persistent notation: by a point of a scheme we shall mean both closed and non-closed points unless specifically mentioned otherwise. Given a point x on a scheme X we shall write $(\mathcal{O}_x, \mathfrak{m}_x)$ for the local ring of x, $\mathbf{k}(x)$ for the residue field $\mathcal{O}_x/\mathfrak{m}_x$ and ι_x for the point-scheme inclusion Spec $\mathbf{k}(x) \hookrightarrow X$. Given an irreducible closed set $C \subset X$, we shall write x_C for the generic point of C and shall frequently abuse the above notation by writing $(\mathcal{O}_C, \mathfrak{m}_C)$ for the local ring of x_C.

All complexes are cochain complexes. Given a right (resp. left) exact functor F between two abelian categories $\mathcal{A}$ and $\mathcal{B}$, we denote by $\mathbf{L}\, F$ (resp. $\mathbf{R}\, F$) the left (resp. right) derived functor between the appropriate derived categories, if it exists, and by $\mathbf{L}^i\, F(\bullet)$ (resp. $\mathbf{R}^i\, F(\bullet)$) the $-i$-th cohomology of $\mathbf{L}\, F(\bullet)$ (resp. the i-th cohomology of $\mathbf{R}\, F(\bullet)$).

Lemma 2.1. *Let X be a locally noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let C be an irreducible component of the support of $\mathcal{F}$, then for every point $x \in C$, we have*

$$\mathbf{L}^i\, \iota_x^* \mathcal{F} \neq 0 \quad \textit{for } 0 \leq i \leq \operatorname{codim}_X(C) \tag{2.1}$$

Proof. Recall (cf. [Mat86], §19) that if a minimal free resolution $L_\bullet$ of a finitely generated module M for a local ring $(R, \mathfrak{m}, k)$ exists, then

$$\dim_k \operatorname{Tor}^i(M, k) = \operatorname{rk} L_i$$

Since X is locally noetherian, minimal free resolutions of $\mathcal{F}$ exist in all local rings. Therefore, as the localisation functor is exact, it suffices to prove that the length of the minimal free resolution of $\mathcal{F}$ in the local ring $(\mathcal{O}_C, \mathfrak{m}_C, k_C)$ of C is at least $\operatorname{codim}_X(C)$.

Write F_C for the stalk of $\mathcal{F}$ at the generic point of C. It is a finite-length $\mathcal{O}_C$-module: consider the standard filtration ([Ser00], I, §7, Theorem 1) of F_C by submodules $0 = M_0 \subset \cdots \subset M_n = F_C$ with each M_i/M_{i-1} isomorphic to $\mathcal{O}_C/\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Supp}_{\mathcal{O}_C}(F_C)$. As the defining ideal of C is minimal in $\operatorname{Supp}_X(\mathcal{F})$, $\operatorname{Supp}_{\mathcal{O}_C}(F_C)$ consists of just $\mathfrak{m}_C$. So each M_i/M_{i-1} is isomorphic to k_C and (M_i) is a finite composition series for F_C.

As $\operatorname{codim}_X(C) = \dim(\mathcal{O}_C)$, the result now follows immediately by taking a minimal free resolution of F_C and applying to it the New Intersection Theorem (see, for instance, [Rob98], Theorem 6.2.2) which states that for any local ring R the length of any nonexact complex of free R-modules with finite-length homologies is at least $\dim R$. $\square$

Lemma 2.2. *Let X be a locally noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X of finite* Tor*-dimension. For any $p \in \mathbb{Z}$ define*

$$D_p = \{x \in X \mid \mathbf{L}^i\, \iota_x^* \mathcal{F} \neq 0 \textit{ for some } i \geq p\} \tag{2.2}$$

Then each D_p is closed and $\operatorname{codim}_X(D_p) \geq p$.

Proof. It suffices to prove both claims for the case $X = \operatorname{Spec} R$ with R noetherian. Write F for $\Gamma(\mathcal{F})$. We have $\mathbf{L}^p\, \iota_x^* \mathcal{F} = \operatorname{Tor}^p(F, \mathbf{k}(x))$. The first claim follows from the upper semicontinuity theorem ([GD63], *Théorème* **7.6.9**) or directly from [GD63] *Proposition* **7.4.4** which implies that for any $i \in \mathbb{Z}$ the points x where any flat resolution of F after being localised to the local ring $\mathcal{O}_x$ can be truncated after i-th term form an open set in $\operatorname{Spec} R$.

For the second claim let C be any irreducible component of D_p, x_C be a generic point of C and let F_C be the localisation of F to the local ring $\mathcal{O}_C$. Then $\operatorname{Tor}^p_{\mathcal{O}_C}(F_C, \mathbf{k}(x_C)) \neq 0$ by the defining property of D_p. We have quite generally ([Mat86], §19, Lemma 1)

$$\operatorname{proj\,dim}_{\mathcal{O}_C} F_C = \sup\{i \in \mathbb{Z} \mid \operatorname{Tor}^i_{\mathcal{O}_C}(F_C, \mathbf{k}(x_C))\}$$

thus $\operatorname{proj\,dim}_{\mathcal{O}_C} F_C \geq p$. By Auslander-Buchsbaum equality we have

$$\operatorname{depth}_{\mathcal{O}_C} \mathcal{O}_C = \operatorname{proj\,dim}_{\mathcal{O}_C} F_C + \operatorname{depth}_{\mathcal{O}_C} F_C$$

and so we obtain $\operatorname{codim}_X C = \dim \mathcal{O}_C \geq \operatorname{depth}_{\mathcal{O}_C} \mathcal{O}_C \geq p$ as required. ☐

The main idea behind the proof of the following proposition we owe to Bondal and Orlov in [BO95], Proposition 1.5.

Proposition 2.3. *Let X be a locally noetherian scheme and $F \in D^b_{coh}(X)$ an object of finite* Tor*-dimension. Denote by $\mathcal{H}^i$ the sheaf $H^i(F)$.*

Then for any point $x \in X$, we have

$$-\sup\{i \in \mathbb{Z} \mid x \in \operatorname{Supp} \mathcal{H}^i\} = \inf\{j \in \mathbb{Z} \mid \mathbf{L}^j\, \iota_x^* F \neq 0\} \tag{2.3}$$

Moreover, for any irreducible component C of $\operatorname{Supp} F$*, we have:*

$$\operatorname{codim}_X C - \inf\{i \in \mathbb{Z} \mid C \subseteq \operatorname{Supp} \mathcal{H}^i\} = \sup\{j \in \mathbb{Z} \mid \mathbf{L}^j\, \iota_{x_C}^* F \neq 0\} \tag{2.4}$$

Proof. Fix a point $x \in X$. The main ingridient of the proof is the calculation of higher pullbacks $\mathbf{L}^j\, \iota_x^* F$ via the standard spectral sequence (eg. [GM03], Proposition III.7.10) associated to the filtration of $\mathbf{L}\, \iota_x^* F$ by the rows of the Cartan-Eilenberg resolution of F:

$$E_2^{-p,q} = \mathbf{L}^p\, \iota_x^*(\mathcal{H}^q) \Rightarrow E_\infty^{q-p} = \mathbf{L}^{p-q}\, \iota_x^*(F) \tag{2.5}$$

Denote by h and l the highest and the lowest non-zero rows of $E_2^{\bullet\bullet}$. As all rows above row h and all columns to the right of column 0 in $E_2^{\bullet\bullet}$ consist entirely of zeroes we conclude by inspection of the complex that $E_\infty^n = 0$ for all $n > h$ and $E_\infty^h = E_2^{0,h} = \mathcal{H}^h|_x$. This gives (2.3).

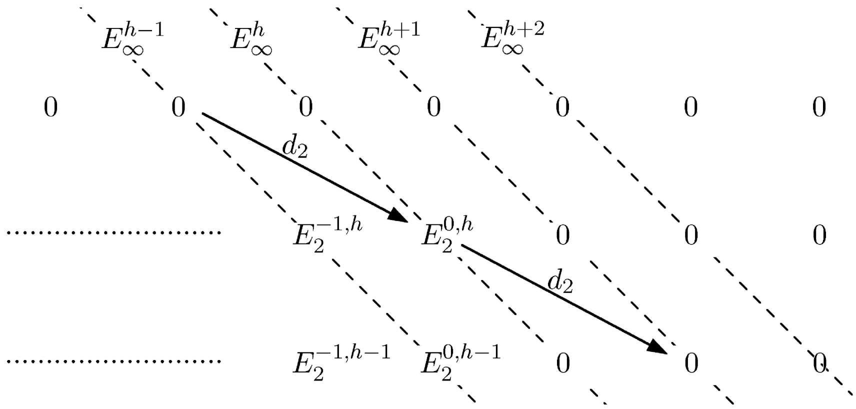


Figure 1

To obtain (2.4) let x be the generic point of C. Observe that C is an irreducible component of $\mathcal{H}^l$ by the defining property of l. Denote by d the codimension of C. By Lemma 2.2, the set of points $y \in X$, such that there is a non-zero $\mathbf{L}^p\, \iota_y^*(\mathcal{H}^q)$ with $p > d$, is closed and of codimension at least $d+1$. Then this set can not contain x for the closure of x is C whose codimension is d. Hence all columns to the left of column d in $E_2^{\bullet\bullet}$ consist entirely of zeroes. As all the rows below l consist of zeroes by the defining property of l we conclude that $E_\infty^n = 0$ for all $n > d - l$ and $E_\infty^{l-d} = E_2^{-d,l} = \mathbf{L}^d\, \iota_x^* \mathcal{H}^l$. And by Lemma 2.1, $\mathbf{L}^d\, \iota_x^* \mathcal{H}^l \neq 0$. This gives (2.4). $\square$

Definition 2.4. Let $E^\bullet$ be a cochain complex of abelian groups. Define its *cohomological amplitude,* denoted by coh-amp$(E^\bullet)$, to be the length of the minimal interval in $\mathbb{Z}$ containing the set

$$\{i \in \mathbb{Z} \mid H^i(E^\bullet) \neq 0\} \tag{2.6}$$

If no such interval exists we say that coh-amp$(E) = \infty$.

Trivially coh-amp$(E^\bullet)$ is the minimal length of a bounded complex quasi-isomorphic to $E^\bullet$, if any exist, and infinity, if none do.

Definition 2.5. Let R be a ring and $E^\bullet$ be a cochain complex of R-modules. Define its Tor-*amplitude,* denoted by Tor-amp$(E^\bullet)$, to be the length of the minimal interval in $\mathbb{Z}$ containing the set

$$\{i \in \mathbb{Z} \mid \exists\, A \in R\text{-}\mathbf{Mod} \text{ such that } \operatorname{Tor}^i_R(E^\bullet, A) \neq 0\} \tag{2.7}$$

If no such interval exists we say that Tor-amp$(E) = \infty$.

If X is a quasi-projective scheme and E an object of $D^b_{\mathrm{coh}}(X)$, Tor-amp(E) is the same as *homological dimension* of E introduced in [BM02]. It can be seen with the following lemma, an analogue of [BM02], Proposition 5.4:

Lemma 2.6. *Let X be a locally noetherian scheme and E an element of $D^b_{coh}(X)$. For any $k \in \mathbb{Z}$ the following are equivalent:*

1. *E is quasi-isomorphic to a complex of flat sheaves of length k.*

2. *$\operatorname{Tor-amp}(E) \leq k$.*

3. *There exists an interval of length k in $\mathbb{Z}$ containing the set*

$$\{i \in \mathbb{Z} \mid \exists\, x \in X \text{ such that } \mathbf{L}^i\, \iota_x^*(E) \neq 0\} \tag{2.8}$$

Proof. Implications 1 ⇒ 2 ⇒ 3 are trivial, it suffices to prove 3 ⇒ 1. Let $n, k \in \mathbb{Z}$ be such that the interval $[-n-k, -n]$ contains the set (2.8). Then (2.3) and (2.4) of Proposition 2.3 show that $H^i(F) = 0$ unless $i \in [n, n+k]$. As $\mathcal{O}_X$-**Mod** has resolutions by flat modules there exists a complex $F^\bullet$ of flat sheaves quasi-isomorphic to E and with $F_i = 0$ for all $i > n+k$. It remains to show that we can truncate $F^\bullet$ at degree n and keep it flat, i.e. that the sheaf $F^n / \operatorname{Im} F^{n-1}$ is flat. But as $H^i(F^\bullet) = 0$ for $i < n$, the complex

$$\cdots \to F^{n-2} \to F^{n-1} \to F^n \to 0 \to \dots$$

is a flat resolution of $F^n / \operatorname{Im} F^{n-1}$. Thus $\mathbf{L}^1\, \iota_x^*(F^n / \operatorname{Im} F^{n-1}) = \mathbf{L}^{-n+1}\, \iota_x^*(E)$ and so vanishes for all $x \in X$ by assumption. The result follows. □

Proposition 2.7. *Let X be a locally noetherian scheme, $E \in D^b_{coh}(X)$ an object of finite* Tor*-dimension and C an irreducible component of the support of E. Denote by E_C the localisation of E to the local ring $\mathcal{O}_C$. Then*

$$\operatorname{Tor-amp} E_C - \operatorname{codim} C = \operatorname{coh-amp} E_C$$

NB: In other words for a sufficiently general point x on C we have $\operatorname{Tor-amp} E_x = \operatorname{coh-amp} E_C + \operatorname{codim} C$.

Compare also to the inequality

$$\text{hom.dim.} E \geq \operatorname{codim} C$$

in [BM02] and [BKR01].

Proof. We apply Proposition 2.3 to $\operatorname{Spec} \mathcal{O}_C$ and F_C. Denote by H^i the i-th cohomology of F_C. In $\operatorname{Spec} \mathcal{O}_C$ the support of F_C is just $\{\mathfrak{m}_C\}$. So applying (2.3) and (2.4) to $\mathfrak{m}_C$ and subtracting the former equality from the latter we get:

$$\begin{aligned}\operatorname{codim}_{\operatorname{Spec} \mathcal{O}_C} \mathfrak{m}_C &- \inf\{i \in \mathbb{Z} \mid \mathfrak{m}_C \in \operatorname{Supp} H^i\} + \sup\{k \in \mathbb{Z} \mid \mathfrak{m}_C \in \operatorname{Supp} H^k\} \\ &= \sup\{j \in \mathbb{Z} \mid \operatorname{Tor}^j(F_C, \mathbf{k}(\mathfrak{m}_C)) \neq 0\} - \inf\{l \in \mathbb{Z} \mid \operatorname{Tor}^l(F_C, \mathbf{k}(\mathfrak{m}_C)) \neq 0\}\end{aligned}$$

The right-hand side is precisely $\operatorname{Tor-amp} F_C$ (Lemma 2.6) and the left-hand side, noting that $\mathfrak{m}_C \in \operatorname{Supp} H^k(F_C)$ if and only if $H^k(F_C) \neq 0$, equals to $\operatorname{codim}_X C - \operatorname{coh-amp} F_C$. The result follows. □

Proposition 2.8. *Let X be a locally noetherian scheme and $E \in D^b_{coh}(X)$ an object of finite* Tor*-dimension. Then*

$$\operatorname{Tor-amp} E - \operatorname{codim}_X \operatorname{Supp} E \geq \operatorname{coh-amp} E$$

NB: For a quick example of this inequality being strict, consider $X = \mathbb{A}^1$ and F being a pure sheaf formed by taking a direct sum of $\mathcal{O}_X$ with the skyscraper sheaf of any closed point.

Proof. Write l and h for the infimum and the supremum, respectively, of $\{i \in \mathbb{Z} \mid H^i(E) \neq 0\}$. Let $F^\bullet$ be a minimal length flat sheaf complex quasi-isomorphic to E and write n and m for the infimum and the supremum, respectively, of $\{i \in \mathbb{Z} \mid F^i \neq 0\}$. Observe that for any $x \in X$

$$n \leq \inf\{i \in \mathbb{Z} \mid \mathbf{L}^i \iota_x^* E \neq 0\} \leq \sup\{i \in \mathbb{Z} \mid \mathbf{L}^i \iota_x^* E \neq 0\} \leq m$$

With this in mind, if we apply (2.3) of Proposition 2.3 to any point of $\operatorname{Supp} H^h(E)$, we obtain

$$-h \geq n$$

Now let C be any irreducible component of $\operatorname{Supp} H^l$. Even though C might not be an irreducible component of $\operatorname{Supp} E$, the argument of Proposition 2.3 applies to it unchanged to yield (2.4). Therefore

$$m \geq \operatorname{codim}_X C - l$$

Adding the two inequalities, rearranging and noting that $\operatorname{codim}_X C \geq \operatorname{codim} \operatorname{Supp} E$, we obtain

$$(m - n) - \operatorname{codim} \operatorname{Supp} E \geq (h - l)$$

as required. $\square$

3 Derived McKay correspondence

Due to the index space becoming of essence we shall adopt the following shorthand throughout this section : given a scheme X by simply $D(X)$ we shall mean $D^b_{\mathrm{coh}}(X)$, the full subcategory of (unbounded) derived category of $\mathcal{O}_X$-**Mod** consisting of complexes with bounded and coherent cohomology.

3.1 G-constellations

For S a scheme of finite type over $\mathbb{C}$ and H a finite group acting on S on the left by automorphisms an H*-sheaf* is a sheaf $\mathcal{E}$ of $\mathcal{O}_S$-modules equipped with a lift of the H-action to $\mathcal{E}$. For technical details see the exposition in [BKR01], Section 4. We shall denote by $\mathcal{O}_S$-$\mathbf{Mod}^H$ (resp. $\operatorname{QCoh}^H S$, $\operatorname{Coh}^H S$) the abelian category of H-sheaves (resp. quasi-coherent, coherent H-sheaves) on S and by $D^G(S)$ the category $D^b_{\mathrm{coh}}(\mathcal{O}_S\text{-}\mathbf{Mod}^H)$.

Definition 3.1. Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. A *$G$-constellation* is a coherent G-sheaf $\mathcal{V}$ on $\mathbb{C}^n$ whose global sections $\Gamma(\mathcal{V})$ form a regular representation V_{reg} of G.

If $\mathcal{W}$ is another G-constellation we say that $\mathcal{V}$ and $\mathcal{W}$ are *orthogonal in degree k* if $G\text{-}\mathrm{Ext}^k_{\mathbb{C}^n}(\mathcal{V},\mathcal{W}) = G\text{-}\mathrm{Ext}^k_{\mathbb{C}^n}(\mathcal{W},\mathcal{V}) = 0$.

Let now Y be a scheme of finite type over $\mathbb{C}$. We endow Y with the trivial G-action, thus we can speak of G-sheaves on Y and on $Y \times \mathbb{C}^n$.

Definition 3.2. By a *family of G-constellations parametrised by* Y we shall mean an object $\mathcal{F}$ of $\mathrm{Coh}^G(Y \times \mathbb{C}^n)$ which is flat over Y and whose fiber at every closed point of Y is a G-constellation.

Given two subsets C and C' of Y we say that they are *orthogonal in degree k in $\mathcal{F}$*if for every closed $y \in C$ and $y' \in C'$, G-constellations $\mathcal{F}_{|y}$ and $\mathcal{F}_{|y'}$ are orthogonal in degree k. We shall say that family $\mathcal{F}$ is *orthogonal in degree k*, if Y is orthogonal to Y in degree k in $\mathcal{F}$.

The support of any G-constellation is a finite union of G-orbits in $\mathbb{C}^n$. If for every closed point $y \in Y$ the support of the fiber $\mathcal{F}_{|y}$ is a single G-orbit, then we have a well-defined 'forgetful' map $\pi_{\mathcal{F}} : \ Y \to \mathbb{C}^n/G$ which sends each $y \in Y$ to the support of $\mathcal{F}_{|y}$. If well-defined $\pi_{\mathcal{F}}$ is a morphism. This can be seen as follows: an invariant part of the pushdown $\pi_{Y*}(\mathcal{F})$ is a line bundle on Y which inherits an $\mathcal{O}_Y$-linear $\mathbb{C}[x_1,\ldots,x_n]^G$-module structure from $\mathcal{F}$. This structure defines a homomorphism $\mathbb{C}[x_1,\ldots,x_n]^G \to \mathcal{O}_Y$ and the corresponding scheme morphism $Y \to \mathbb{C}^n/G$ is precisely $\pi_{\mathcal{F}}$. We call $\pi_{\mathcal{F}}$ forgetful because it 'forgets' the G-constellation structure on $\mathcal{F}_{|y}$ and sends it to just its set-theoretical support.

3.2 Integral transforms

Definition 3.3. Let S_1 and S_2 be schemes of finite type over $\mathbb{C}$. Let E be an object of $D_{qc}(S_1 \times S_2)$ of finite Tor-dimension. By an `integral transform` Φ_E we shall mean a functor from $D_{qc}(S_1)$ to $D_{qc}(S_2)$ defined by

$$\Phi_E(-) = \mathbf{R}\,\pi_{S_2*}(E \overset{\mathbf{L}}{\otimes} \pi^*_{S_1}(-)) \tag{3.1}$$

The object E is called *the kernel* of the transform. If Φ_E is an equivalence of categories, it is further called *a Fourier-Mukai transform*.

If a group G acts on S_1 and S_2, the definition of an integral transform $D^G(S_1) \to D^G(S_2)$ is identical. If the group action on S_1 is trivial there is a functor $(- \otimes \rho_0) : \ D(S_1) \to D^G(S_1)$ which gives any sheaf the trivial G-equivariant structure. In such a case we shall also use terms *integral* and *Fourier-Mukai transform* for the functors $D(S_1) \to D^G(S_2)$ of form $\Phi_E(- \otimes \rho_0)$ where Φ_E is some integral transform $D^G(S_1) \to D^G(S_2)$.

Let now G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$, Y a scheme finite over $\mathbb{C}$ equipped with the trivial G-action and $\mathcal{F}$ a flat family of G-constellations on Y. Then $\mathcal{F}$ being flat over Y and $\mathbb{C}^n$ being non-singular imply that $\mathcal{F}$ is of finite Tor-dimension in $D^G_{qc}(Y \times \mathbb{C}^n)$.

Lemma 3.4. *Let $\mathcal{F}$ be a flat family of G-constelations on Y. Then integral transform $\Phi_{\mathcal{F}} : D^G_{qc}(Y) \to D^G_{qc}(\mathbb{C}^n)$ restricts to a functor $D^G(Y) \to D^G(\mathbb{C}^n)$.*

Proof. Since $\pi_{\mathbb{C}^n}$ is flat, the pullback $\pi^*_{\mathbb{C}^n}$ is exact and trivially takes $D^G(\mathbb{C}^n)$ to $D^G(\mathbb{C}^n)$. Since $\mathcal{F}$ is of finite Tor-dimension, $\mathcal{F} \overset{\mathbf{L}}{\otimes} -$ takes $D^G(Y \times \mathbb{C}^n)$ to $D^G(Y \times \mathbb{C}^n)$. Moreover the image $\mathrm{Im}(\mathcal{F} \overset{\mathbf{L}}{\otimes} -)$ lies in the full subcategory of $D^G(Y \times \mathbb{C}^n)$ consisting of the objects whose support lies within $\operatorname{Supp} \mathcal{F}$. Finally, since $\operatorname{Supp} \mathcal{F}$ is finite over Y, $\mathbf{R}\,\pi_{Y*}$ takes $\mathrm{Im}(\mathcal{F} \overset{\mathbf{L}}{\otimes} -)$ to $D^G(Y)$ ([GD61], *Corollaire 3.2.4*). $\square$

We shall also need the following general fact on integral transforms:

Lemma 3.5. *Let S_1, S_2 and E be as in Definition 3.3. Let p be a closed point of $S_1 \times S_2$. Denote by p_1 and p_2 the corresponding points in S_1 and S_2. Denote by $\iota_{p_1 \times S_2}$ the inclusion $S_2 \hookrightarrow S_1 \times S_2$ of the fiber over p_1. Then*

$$\Phi_E(\mathcal{O}_{p_1}) = \mathbf{L}\,\iota^*_{p_1 \times S_2} E \tag{3.2}$$

and consequently we have, in $D(\mathbb{C}\text{-}\mathbf{Mod})$, an isomorphism:

$$\mathbf{R}\operatorname{Hom}_{S_2}(\Phi_E(\mathcal{O}_{p_1}), \mathcal{O}_{p_2}) \simeq \mathbf{L}\,\iota^*_p(E)^* \tag{3.3}$$

Proof. We have a commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec}\ \mathbb{C} \arrow[r, "\iota_{p_2}"] & S_2 \arrow[r, "\iota_{p_1 \times S_2}"] \arrow[d] & S_1 \times S_2 \arrow[d, "\pi_{S_1}"] \arrow[dr, "\pi_{S_2}"] & \\
 & \operatorname{Spec}\ \mathbb{C} \arrow[r, "\iota_{p_1}"] & S_1 & S_2
\end{tikzcd}$$

By flat base change we have $\pi^*_{S_1} O_{p_1} = \iota_{p_1 \times S_2 *} \mathcal{O}_{S_2}$. By projection formula $E \overset{\mathbf{L}}{\otimes} \iota_{p_1 \times S_2 *} \mathcal{O}_{S_2} = \iota_{p_1 \times S_2 *} \mathbf{L}\,\iota^*_{p_1 \times S_2} E$. We conclude that

$$\Phi_E(\mathcal{O}_{p_1}) = \mathbf{R}\,\pi_{S_2 *}(E \overset{\mathbf{L}}{\otimes} \pi^*_{S_1} O_{p_1}) = \mathbf{R}\,\pi_{S_2 *} \iota_{p_1 \times S_2 *} \mathbf{L}\,\iota^*_{p_1 \times S_2} E$$

and the first assertion follows as $\pi_{S_2 *} \circ \iota_{p_1 \times S_2 *}$ is the identity map.

For the second assertion we use the adjunction of $\mathbf{L}\,\iota^*_{p_2}$ and $\iota_{p_2 *}$

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}_{S_2}(\Phi_E(\mathcal{O}_{p_1}), \mathcal{O}_{p_2}) &= \mathbf{R}\operatorname{Hom}_{S_2}(\mathbf{L}\,\iota^*_{p_1 \times S_2}(E), \iota_{p_2 *}\mathbb{C}) = \\
&= \mathbf{R}\operatorname{Hom}_{\mathbb{C}\text{-}\mathbf{Mod}}(\mathbf{L}\,\iota^*_p(E), \mathbb{C})
\end{aligned}$$

noting that $\iota_{p_1 \times S_2} \circ \iota_{p_2} = \iota_p$. $\square$

3.3 Main results

Proof of Theorem 1.1. We divide the proof into five steps:

Step 1: We claim that $\Phi_{\mathcal{F}}(-\otimes\rho_0)$ has a left adjoint $(\Psi_{\mathcal{F}})^G$, where $\Psi_{\mathcal{F}}$ is a certain integral transform $D^G(\mathbb{C}^n)\to D^G(Y)$ and $(-)^G : D^G(Y)\to D(Y)$ is the functor which takes a G-sheaf to its G-invariant subsheaf.

Recall that $\Phi_{\mathcal{F}} = \mathbf{R}\,\pi_{\mathbb{C}^n*}(\mathcal{F}\overset{\mathbf{L}}{\otimes}\pi_Y^*(-))$. The main issue is the left adjoint of $\pi_Y^*(-)$ as π_Y, though smooth, is manifestly non-proper. But the support of $\mathcal{F}$ is proper, which we now show to imply $\pi_Y^*(-)\overset{\mathbf{L}}{\otimes}\mathcal{F}$ having a left adjoint $D^G(Y\times\mathbb{C}^n)\to D^G(Y)$ of form $\mathbf{R}\,\pi_{Y*}(-\otimes E)$ for some $E\in D^G(Y\times\mathbb{C}^n)$. The claim then follows as $\mathbf{R}\,\pi_{\mathbb{C}^n*}$ has left adjoint $\pi_{\mathbb{C}^n}^*$ and $(-)^G$ is the left (and right) adjoint of $-\otimes\rho_0$ if G acts trivially on Y ([BKR01], Section 4.2).

To calculate the left adjoint of $\pi_Y^*(-)\overset{\mathbf{L}}{\otimes}\mathcal{F}$ we use the methods of Verdier-Deligne, as per exposition of Deligne in the note titled *"Cohomologie a support propre et construction du foncteur $f^!$"* at the end of [Har66]. The argument is quite general, so instead of $\mathbb{C}^n$ we give it for any smooth separable noetherian scheme M of finite Krull dimension. First we compactify M: we choose an open immersion $M\hookrightarrow\bar{M}$ with $\bar{M}$ smooth and proper [Nag]. Then π_Y decomposes as an open immersion $\iota : Y\times M\hookrightarrow Y\times\bar{M}$ followed by the projection $\bar{\pi}_Y : Y\times\bar{M}\to Y$. As $\bar{\pi}_Y$ is smooth and proper it is well known (e.g. [Har66], VII4.3) that the functor $\mathbf{R}\,\bar{\pi}_{Y*} : D(Y\times\bar{M})\to D(Y)$ has a right adjoint $\bar{\pi}_Y^!$ and that

$$\bar{\pi}_Y^!(-) = \bar{\pi}_{Y*}^!(-)\otimes\bar{\pi}_M^*\omega_{\bar{M}}[n]$$

where $\bar{\pi}_M : Y\times\bar{M}\to\bar{M}$ is the projection onto the second component.

On the other hand, for any noetherian scheme S and any open immersion $\iota : U\to S$ it is shown ([Har66], Deligne's note, Proposition 4) that the left adjoint to the (exact) functor $\iota^*(-)$ exists as an (exact) functor $\iota_!$ from $\mathrm{Coh}(U)$ to a category pro-$\mathrm{Coh}(S)$ which (roughly) consists of filtered inverse limits of objects in $\mathrm{Coh}(S)$. The functor $\iota_!$ is defined as follows: given $\mathcal{A}\in\mathrm{Coh}(U)$ take any $\bar{\mathcal{A}}\in\mathrm{Coh}(S)$ which restricts to $\mathcal{A}$ on U and set

$$\iota_!(\mathcal{A}) = \varprojlim \mathcal{I}^n\bar{\mathcal{A}} \tag{3.4}$$

where $\mathcal{I}$ is the ideal sheaf defining the complement $S\setminus U$. Observe that on sheaves whose support is proper $\iota_!$ restricts to the ordinary direct image functor $\iota_* : \mathrm{Coh}(U)\to\mathrm{Coh}(S)$. Indeed if the support of $\mathcal{A}$ is proper then we can take $\bar{\mathcal{A}} = \iota_*(\mathcal{A})$. Then $\mathrm{Supp}(\bar{\mathcal{A}})$ is $\iota(\mathrm{Supp}\,\mathcal{A})$ and hence disjoint from $S\setminus U$ which implies that $\mathcal{I}\bar{\mathcal{A}} = \bar{\mathcal{A}}$ and so the limit in (3.4) is just $\iota_*(\mathcal{A})$.

The two adjunctions described imply together that the functor

$$\pi_Y^! = \iota^*\circ\bar{\pi}_Y^! : \quad D(Y)\to D(Y\times M)$$

has a left adjoint

$$\pi_{Y!} = \mathbf{R}\,\bar{\pi}_{Y*}\circ\iota_! : \quad \text{pro-}D(Y\times M)\to\text{pro-}D(Y)$$

and it's worth noting that, as is Deligne's point, these two functors are independent of the choice of compactification. Now, first we observe that the above extends straightforwardly to G-sheaves. Then we note that $\bar{\pi}_Y \circ \iota$ is just π_Y and $\bar{\pi}_M \circ \iota$ is the composition of π_M with $M \hookrightarrow \bar{M}$. Hence

$$\pi_Y^!(-) = \pi_Y^*(-) \otimes \pi_M^*(\omega_M)[n]$$

Thus the left adjoint of $\pi_Y^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{F}$ is

$$\mathbf{R}\, \bar{\pi}_{Y*}\iota_!(- \otimes \pi_M^*(\omega_M)[n] \overset{\mathbf{L}}{\otimes} \mathcal{F}^\vee) : \text{pro-}D^G(Y \times \mathbb{C}^n) \to \text{pro-}D^G(Y) \tag{3.5}$$

If, as in our case, the support of $\mathcal{F}^\vee$ is proper in Y then $\iota_!(- \overset{\mathbf{L}}{\otimes} \mathcal{F}^\vee) = \iota_*(- \overset{\mathbf{L}}{\otimes} \mathcal{F}^\vee)$. Thus (3.5) restricts to $D^G(Y \times M) \to D^G(Y)$ as the functor $\mathbf{R}\, \pi_{Y*}(- \otimes \pi_M^*(\omega_M)[n] \overset{\mathbf{L}}{\otimes} \mathcal{F}^\vee)$. The claim follows.

Step 2: We claim that the composition $(\Psi_{\mathcal{F}})^G \circ \Phi_{\mathcal{F}}(- \otimes \rho_0)$ is an integral transform Φ_Q for some $Q \in D(Y \times Y)$ and that for any closed point (y_1, y_2) in $Y \times Y$ and any $k \in \mathbb{Z}$ we have

$$\mathbf{L}^k\, \iota_{y_1,y_2}^* Q = G\text{-}\mathrm{Ext}^k(\mathcal{F}_{|y_1}, \mathcal{F}_{|y_2})^* \tag{3.6}$$

That a composition of two integral transforms is itself an integral transform is a standard result first proved by Mukai in [Muk81], Proposition 1.3. And since G acts on Y trivially, it follows that for any $E \in D^G(Y \times Y)$ of finite Tor-dimension

$$(-)^G \circ \Phi_E \circ (- \otimes \rho_0) = \Phi_{E^G}$$

where E^G is the G-invariant part of E. This gives the first assertion.

For the second assertion observe that by (3.2) of Lemma 3.5 for any $y \in Y$ we have $\mathcal{F}_y = \Phi_{\mathcal{F}}(\mathcal{O}_y \otimes \rho_0)$. Moreover

$$\begin{aligned} \mathbf{L}\, \iota_{y_1,y_2}^* Q &= \mathbf{R}\,\mathrm{Hom}_{D(Y)}(\Phi_Q(\mathcal{O}_{y_1}), \mathcal{O}_{y_2})^* = \\ &= \mathbf{R}\,\mathrm{Hom}_{D^G(\mathbb{C}^n)}(\Phi_{\mathcal{F}}(\mathcal{O}_{y_1} \otimes \rho_0), \Phi_{\mathcal{F}}(\mathcal{O}_{y_2} \otimes \rho_0))^* \end{aligned}$$

where the first equality is by (3.6) and the second by adjunction of $(\Psi_{\mathcal{F}})^G$ and $\Phi_{\mathcal{F}}(- \otimes \rho_0)$. The assertion follows.

Step 3: We claim that Q is a pure sheaf and that its support lies within the diagonal $Y \xrightarrow{\Delta} Y \times Y$.

First note that since $Y \times Y$ is of finite type over $\mathbb{C}$, it is certainly Jacobson (see [GD66], §10.3) and so any closed set of $Y \times Y$ is uniquely identified by its set of closed points. We shall implicitly use this property at several points of the argument below.

Recall the closed set N_k of (1.2). As the support of any G-constellation is proper and as $\omega_{\mathbb{C}^n} = \mathcal{O}_{\mathbb{C}^n} \otimes \rho_0$ as a G-sheaf since $G \subseteq \mathrm{SL}_n(\mathbb{C})$, Serre duality applies to yield

$$G\text{-}\operatorname{Ext}^k_{\mathbb{C}^n}(\mathcal{F}_{|y_1}, \mathcal{F}_{|y_2}) = G\text{-}\operatorname{Ext}^{n-k}_{\mathbb{C}^n}(\mathcal{F}_{|y_2}, \mathcal{F}_{|y_1})^*$$

and so $\operatorname{codim} N_k = \operatorname{codim} N_{n-k}$ for all k.

Let C be an irreducible component of $\operatorname{Supp} Q$. Denote by y_C its generic point, by $\mathcal{O}_C$ the local ring of y_C and by Q_C the localisation of Q to $\mathcal{O}_C$. For any k denote by M_k the set $\{y \in Y \times Y \mid \mathbf{L}^k\, \iota_y^* Q \neq 0\}$ and let l and m be the infimum and the supremum of the set $\{k \in \mathbb{Z} \mid y_C \in M_k\}$, thus $\text{Tor-amp}_{\mathcal{O}_C} Q_C = m - l$ (Lemma 2.6). On the other hand, by (3.6) the closure of $M_k \setminus \Delta$ is the set N_k, so either $y_C \in \Delta$ or $y_C \in N_l \cap N_m$. By assumption of the theorem, the latter would imply that

$$\begin{aligned} \operatorname{codim} C &\geq \operatorname{codim} N_l \geq n - 2l + 1 \\ \operatorname{codim} C &\geq \operatorname{codim} N_m = \operatorname{codim} N_{n-m} \geq 2m - n + 1 \end{aligned}$$

and therefore $\operatorname{codim} C \geq m - l + 1$. But that would make $\operatorname{codim} C$ strictly greater than $\text{Tor-amp}_{\mathcal{O}_C} Q_C$ and contradict Proposition 2.7. Thus y_C lies within Δ and, since $Y \times Y$ is separated, so does all of C.

We have now shown that $\operatorname{Supp} Q \subseteq \Delta$, so $\operatorname{codim} \operatorname{Supp} Q \geq n$. But as $\mathbb{C}^n$ is smooth and n-dimensional, (3.6) implies

$$\mathbf{L}^k\, \iota_y^* Q = 0 \qquad\qquad \forall y \in Y,\ k \notin 0, \dots, n \tag{3.7}$$

so $\text{Tor-amp}\, Q \leq n$. By Proposition 2.8 $\text{Tor-amp}\, Q = n$ and $\text{coh-amp}\, Q = 0$. Together with (3.7) this implies that Q is a pure sheaf.

Step 4: We claim that Q is the structure sheaf $\mathcal{O}_\Delta$ of the diagonal Δ and therefore $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is fully faithful.

The adjunction co-unit $\Phi_Q \to \mathrm{Id}_{D(Y)}$ induces a surjective $\mathcal{O}_{Y\times Y}$-module morphism $Q \xrightarrow{\epsilon} \mathcal{O}_\Delta$. Let K be its kernel, we then have a short exact sequence

$$0 \to K \to Q \xrightarrow{\epsilon} \mathcal{O}_\Delta \to 0 \tag{3.8}$$

Choosing some closed point $(y, y) \in \Delta$ and applying functor $\mathbf{L}\, \iota_{y,y}^*(-)$ to (3.8) we obtain a long exact sequence of $\mathbb{C}$-modules

$$\cdots \to G\text{-}\operatorname{Ext}^1_{\mathbb{C}^n}(\mathcal{F}_{|y}, \mathcal{F}_{|y})^* \xrightarrow{\alpha_y} \Omega^1_{Y,y} \to K_{y,y} \to G\text{-}\operatorname{End}_{\mathbb{C}^n}(\mathcal{F}_{|y})^* \xrightarrow{\epsilon_y} \mathbb{C} \to 0 \to \dots$$

The map ϵ_y is surjective due to the trivial automorphisms of any G-constellation. It is an isomorphism whenever $\mathcal{F}_{|y}$ is simple, i.e. trivial automorphisms are all we get. The map α_y is the dual of the Kodaira-Spencer map of $\mathcal{F}$ at $y \in Y$, which takes a tangent vector at y to the infinitesimal deformation in that direction in the family $\mathcal{F}$. Hence for any $y \in Y$, such that $\mathcal{F}_{|y}$ is simple and such that the Kodaira-Spencer map of $\mathcal{F}$ is injective at y, the long exact sequence above shows that $K|_{y,y} = 0$.

Having proved that $\operatorname{Supp} Q \subseteq \Delta$ we have proved by (3.6) that any two G-constellations in $\mathcal{F}$ are orthogonal. Denoting by q the quotient map $\mathbb{C}^n \to \mathbb{C}^n/G$ we claim that for any closed point $x \in \mathbb{C}^n/G$, such that $q^{-1}(x)$ is a free orbit of G, the fiber $\pi_{\mathcal{F}}^{-1}(x)$ consists of at most a single point. This is because, by definition of $\pi_{\mathcal{F}}$, all the G-constellations parametrised by $\pi_{\mathcal{F}}^{-1}(x)$ are supported on $q^{-1}(x)$ - and any two G-constellations supported at the same free orbit are easily seen to be isomorphic. Thus $\pi_{\mathcal{F}}$ is an isomorphism on the smooth locus X_0 of $\mathbb{C}^n/G$. By [Log06], Proposition 1.5 the family $\mathcal{F}$ on X_0 (identified with an open subset of Y via $\pi_{\mathcal{F}}$) is locally isomorphic to the canonical G-cluster family $q_*\mathcal{O}_{\mathbb{C}^n}|_{X_0}$. As any G-cluster is simple and as the Kodaira-Spencer map of $q_*\mathcal{O}_{\mathbb{C}^n}|_{X_0}$ is trivially injective $K|_{y,y} = 0$ for any $y \in X_0$. Therefore $\operatorname{codim}_{Y\times Y} \operatorname{Supp} K \geq n+1$, as X_0 is open in Δ.

On the other hand, since $\text{Tor-amp}\, Q = \text{Tor-amp}\, \mathcal{O}_\Delta = n$, the short exact sequence (3.8) implies that $\text{Tor-amp}\, K \leq n$. As that is smaller than the codimension of its support, K is 0 by Proposition 2.8. Thus $Q \simeq \mathcal{O}_\Delta$, the adjunction co-unit is an isomorphism and $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is fully faithful.

Step 5: We claim that Y is smooth, that $\pi_{\mathcal{F}} : Y \to \mathbb{C}^n/G$ is crepant and that $\Phi_{\mathcal{F}}(- \otimes \rho_0)$is an equivalence of categories.

The argument below is modeled on the one introduced in [BKR01], §6, Steps 5-7. Observe that for all $y \in Y$ the complex $\mathbf{R}\operatorname{Hom}_Y(\mathcal{O}_y, \mathcal{O}_y)$ is bounded as we have $\operatorname{Hom}_{D(Y)}(\mathcal{O}_y, \mathcal{O}_y) = \operatorname{Hom}_{D^G(\mathbb{C}^n)}(\mathcal{F}_{|y}, \mathcal{F}_{|y})$. This implies ([BKR01], Corollary 5.3) that Y is nonsingular.

Let x be any closed point of $\mathbb{C}^n/G$. By definition of $\pi_{\mathcal{F}}$ the functor $\Phi_{\mathcal{F}}(-\otimes\rho_0)$ embeds the subcategory $D_x(Y)$ of $D(Y)$ consisting of the objects supported at $\pi_{\mathcal{F}}^{-1}(x)$ into the subcategory $D^G_x(\mathbb{C}^n)$ of $D^G(\mathbb{C}^n)$ consisting of the objects supported at the orbit $q^{-1}(x)$. Thus each $D_x(Y)$ has a trivial Serre functor, which implies ([BKR01], Lemma 3.1) the crepancy of $\pi_{\mathcal{F}}$.

Finally, once we know that Y is smooth and that ω_Y is trivial, an argument identical to the one in *Step 1* shows that $(\Psi_{\mathcal{F}})^G$ is right adjoint to $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ as well as left adjoint and then [Bri99], Theorem 3.3 shows $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ to be an equivalence of categories. □

Proof of Corollary 1.2. It suffices to demonstrate that $\mathcal{F}$ satisfies the condition 2 in Theorem 1.1. Thus we have to show that $\operatorname{codim} N_0 \geq 4$ and $\operatorname{codim} N_1 \geq 2$. But as seen in proof of Theorem 1.1 N_k lies within the fibre product $Y \times_{\mathbb{C}^3/G} Y$ for all k. As $\pi_{\mathcal{F}}$ is birational its fibres are at most divisors and so the codimension of $Y \times_{\mathbb{C}^3/G} Y$ is at least 2.

It remains to show that $N_0 \geq 4$. The assumptions of the Corollary ensure that N_0 is contained in the union of all closed sets of form $(E_i \cap E_j) \times (E_k \cap E_l)$ and of form $E_i \times (E_i \cap E_j \cap E_k)$, and we note that the codimension of each one of these sets in $Y \times Y$ is 4. □

Proposition 3.6. *Let G be a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$, Y a crepant resolution of $\mathbb{C}^n/G$ and $E \in D^G(Y \times \mathbb{C}^n)$ be such that the functor $\Phi_E(- \otimes \rho_0)$ is an equivalence $D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n)$ which sends point sheaves on Y to pure sheaves. Then E is a flat family of G-constellations over Y and*

$$G\text{-}\operatorname{Ext}^i(E_{|y_1}, E_{|y_2}) = \begin{cases} \mathbb{C} & \text{if } y_1 = y_2, i = 0 \\ 0 & \text{if } y_1 \neq y_2 \end{cases} \tag{3.9}$$

and for any $y \in Y$ the (Kodaira-Spencer) map $\operatorname{Ext}^1(\mathcal{O}_y, \mathcal{O}_y) \to G\text{-}\operatorname{Ext}^1(E_{|y}, E_{|y})$ is injective.

Proof. That E is a pure sheaf flat over Y follows from [Bri99], Lemma 4.3. The assertion (3.9) and that of injectivity of its Kodaira-Spencer map follow immediately from $\Phi_E(-\otimes\rho_0)$ being an equivalence, noting that $E_{|y} = \Phi_E(\mathcal{O}_y \otimes \rho_0)$ by Lemma 3.5. Finally as the support of E has to be proper in order for the image of $\Phi_E(- \otimes \rho_0)$ to restrict to $D^G(\mathbb{C}^n)$ the support of each fiber $E_{|y}$ in $\mathbb{C}^n$ is proper and thus a finite union of G-orbits. The simplicity of $E_{|y}$ further implies that this support has to be a single G-orbit. A coherent G-sheaf supported on a single free orbit is isomorphic to ${V_{\mathrm{reg}}}^{\oplus k}$ and if it is simple then $k = 1$. Therefore the fibers of E supported on free orbits are G-constellations and by flatness so are all the remaining ones. ☐

4 Non-projective example

In this section we give an example of an application for the Theorem 1.1 whereby we construct, explicitly, a derived McKay correspondence for a choice of an abelian $G \subset \mathrm{SL}_3(\mathbb{C})$ and of a non-projective crepant resolution Y of $\mathbb{C}^3/G$.

4.1 The group

Our $\mathbb{C}^3$ shall be the scheme Spec R, where R is a polynomial algebra $\mathbb{C}[x_1, x_2, x_3]$ with a fixed choice of generators x_i. We give this scheme a vector space structure by identifying it with the dual of the vector space generated by x_i. Denote this vector space by V_{giv} and denote by $\check{x}_i$ the basis of V_{giv} dual to x_i. This choice of basis identifies the group $\mathrm{GL}_3(\mathbb{C}) = \mathrm{GL}(V_{\mathrm{giv}})$ with the group of invertible 3×3 complex matrices.

We then set the group G to be what is known as $\frac{1}{6}(1, 1, 4) \oplus \frac{1}{2}(1, 0, 1)$. That is, the image in $\mathrm{SL}_3(\mathbb{C})$ of the product $\mu_6 \times \mu_2$ of groups of 6th and 2nd roots of unity, respectively, under the embedding:

$$(\xi_1, \xi_2) \mapsto \begin{pmatrix} \xi_1\xi_2 & & \\ & \xi_1 & \\ & & \xi_1^4\xi_2 \end{pmatrix} \tag{4.1}$$

As G is evidently abelian all its representations are one-dimensional i.e. characters. We denote by $G^\vee$ the character group $\mathrm{Hom}(G, \mathbb{C}^*)$ of G. By $\chi_{i,j}$ we denote the character on G induced from $(\xi_1, \xi_2) \mapsto \xi_1^i \xi_2^j$ on $\mu_6 \times \mu_2$.

The (left) action of G on $\mathbb{C}^n$ induces a right action of G on R which we make into a left action by setting:

$$g \cdot f(v) = f(g^{-1} \cdot v) \qquad \text{for all } v \in \mathbb{C}^n,\ f \in R,\ g \in G \tag{4.2}$$

We then say that a rational function $f \in K(\mathbb{C}^3)$ is G-homogeneous of weight $\rho \in G^\vee$ if we have $f(g.v) = \rho(g)\ f(v)$ for all $v \in \mathbb{C}^3$ where f is defined. Beware of the confusion: the weight is the inverse of the character G acts on f with! E.g. by (4.1) we have for all $v \in \mathbb{C}^n$ and $(\xi_1, \xi_2) \in G$

$$((\xi_1, \xi_2) \cdot x_1)(v) = x_1((\xi_1, \xi_2)^{-1} \cdot v) = (\xi_1^5 \xi_2\, x_1)(v)$$

whence $(\xi_1, \xi_2) \cdot x_1 = \xi_1^5 \xi_2 x_1$, i.e. G acts on x_1 by character $\chi_{5,1}$. Rewriting the last equality as $x_1((\xi_1, \xi_2) \cdot v) = \xi_1 \xi_2 x_1(v)$ we see that the weight of x_1 is $\chi_{5,1}^{-1} = \chi_{1,1}$. Similarly the weight of x_2 is $\chi_{1,0}$ and that of x_3 is $\chi_{4,1}$.

4.2 The resolution

We define the crepant resolution Y of $\mathbb{C}^3/G$ using methods of toric geometry. For a general reading on toric geometry see [Dan78] or [Ful93]. For a detailed account of the specifics related to G-constellations see [Log03], Section 3.

We give a brief rundown to define the notation involved. By definition G is a subgroup of the maximal torus $(\mathbb{C}^*)^3 \subset \mathrm{GL}_3(\mathbb{C})$ corresponding to the maps diagonal with respect to $\check{x}_i$. This gives an exact sequence

$$0 \longrightarrow G \longrightarrow (\mathbb{C}^*)^n \longrightarrow T \longrightarrow 0 \tag{4.3}$$

Applying $\mathrm{Hom}(\bullet, \mathbb{C}^*)$ to (4.3) we obtain an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^n \xrightarrow{\rho} G^\vee \longrightarrow 0 \tag{4.4}$$

We identify each element $m = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ of the lattice $\mathbb{Z}^n$ of characters of $(\mathbb{C}^*)^n$ with a Laurent monomial $x^m = x_1^{k_1} \ldots x_n^{k_n}$ in R of that weight with respect to the action of $(\mathbb{C}^*)^n$. This identifies the map ρ in (4.4) with the weight map of the action of G on Laurent monomials and M with the sublattice in $\mathbb{Z}^n$ of (exponents of) G-invariant Laurent monomials.

Applying $\mathrm{Hom}(\bullet, \mathbb{Z})$ to (4.4) we obtain

$$0 \longrightarrow (\mathbb{Z}^n)^\vee \longrightarrow L \longrightarrow \mathrm{Ext}^1(G^\vee, \mathbb{Z}) \longrightarrow 0$$

where we write $(\mathbb{Z}^n)^\vee$ for the dual lattice of $\mathbb{Z}^n$ and L for the dual of M. Tautologically we have a $\mathbb{Z}$-valued pairing between M and L. This pairing extends uniquely to a $\mathbb{Q}$-valued pairing between $\mathbb{Z}^n$ and L. Henceforth, given $l \in L$ and $m \in \mathbb{Z}^n$, we write $l(m)$ to denote this pairing.

Write $e_i \in L$ for the elements dual to $x_i \in M$. Then $\mathbb{C}^3/G$ is the toric variety given by the fan consisting of a single cone $L_{\geq 0} = \sum \mathbb{R}_{\geq 0} e_i$. The fan of any toric resolution of $\mathbb{C}^3/G$ is a subdivision of $L_{\geq 0}$ into basic cones. The resolution morphism π is then the toric morphism corresponding to this subdivision.

The quotient torus T acts on the resolution and to each k-dimensional cone σ in the fan corresponds a $(3-k)$-dimensional orbit of T. We denote this orbit by S_σ, it is the orbit which contains limit points of the suborbits of the unique open orbit of T which are defined by those 1-parameter subgroups of T whose corresponding points of L lie in the interior of σ. We further denote by E_σ the closure of S_σ, it is the union of all orbits $S_{\sigma'}$ with $\sigma \subseteq \sigma'$.

Each exceptional divisor of π is of form E_σ for some 1-dimensional σ in the fan of the resolution and it is crepant if and only if the generatior of σ lies in the junior simplex $\Delta = \{(k_1, k_2, k_3) \in L \otimes \mathbb{R} \mid k_i > 0 \text{ and } \sum k_i = 1\}$, for details on crepancy of divisors in toric context see [Rei87], Prop. 4.8.

Calculating the elements of L contained in Δ we obtain:

$$\begin{array}{lll} e_1 = (1,0,0) & e_2 = (0,1,0) & e_3 = (0,0,1) \\ e_4 = \frac{1}{6}(1,1,4) & e_5 = \frac{1}{3}(1,1,1) & e_6 = \frac{1}{2}(1,1,0) \\ e_7 = \frac{1}{6}(1,4,1) & e_8 = \frac{1}{2}(1,0,1) & e_9 = \frac{1}{6}(4,1,1) \\ e_{10} = \frac{1}{2}(0,1,1) & & \end{array} \tag{4.5}$$

We define the resolution Y by specifying its fan $\mathfrak{F}$ to consist of the 3-dimensional cones which triangulate Δ as depicted below and all their faces.

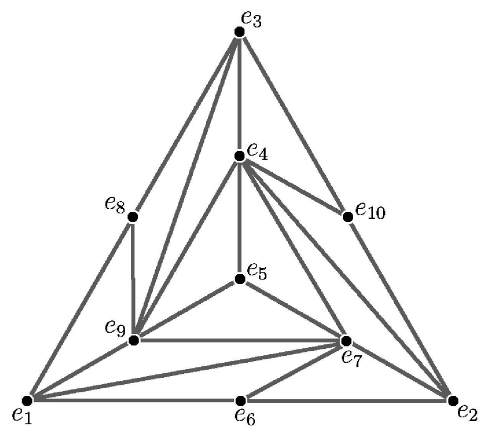


Figure 2

By an argument entirely identical to that of [KKMSD73], Chapter III, §2E, Example 2 it can be seen that the toric morphism $\pi : Y \to \mathbb{C}^3/G$ corresponding to this subdivision is non-projective.

For each cone $\langle e_i \rangle$ in the fan $\mathfrak{F}$, we denote by S_i the codimension 1 orbit $S_{\langle e_i \rangle}$ and by E_i the divisor $E_{\langle e_i \rangle}$. Similarly we use $S_{i,j}$ and $E_{i,j}$ for the codimension 2 orbit $S_{\langle e_i, e_j \rangle}$ and the surface $E_{\langle e_i, e_j \rangle}$ and we use $E_{i,j,k}$ for the toric fixed point $E_{\langle e_i, e_j, e_k \rangle}$.

4.3 The family

We define the family $\mathcal{F}$ of G-constellations on Y using the classification of G-constellation families established in [Log06]. From now on we shall operate freely by the concepts introduced in that paper and the reader is referred to it for all the definitions and other technical details.

We should note that [Log06] employs the following alternative view of G-constellations: the global section functor $\Gamma(\bullet)$ is an equivalence between the categories of quasi-coherent G-equivariant sheaves on $\mathbb{C}^3$ and of modules for the cross-product algebra $R \rtimes G$. Under this equivalence G-constellations correspond to $R \rtimes G$-modules isomorphic as G-modules to the regular representation V_{reg}. Similarly, the pushforward along $Y \times \mathbb{C}^n \to Y$ gives an equivalence of the category $\text{Coh}^G(Y \times \mathbb{C}^n)$ to the category of coherent sheaves of $\mathcal{O}_Y \otimes R \rtimes G$-modules on Y. A flat family of G-constellations on $Y \times \mathbb{C}^n$ becomes a $\mathcal{O}_Y \otimes R \rtimes G$-module on Y which is locally free of rank $|G|$ as an $\mathcal{O}_Y$-module and whose fiber at every closed point of Y is a G-constellation in the sense of $R \rtimes G$-modules. In this section *a flat family of G-constellations* shall mostly refer to a sheaf of $\mathcal{O}_Y \otimes R \rtimes G$-modules on Y.

As explained in [Log06], any proper birational map $Y \to \mathbb{C}^3/G$, such as the toric morphism π of the previous section, defines the notion of G-Cartier and G-Weil divisors on Y. By [Log06], Theorem 4.1 any flat family of G-constellations on Y whose 'forgetful' map $\pi_{\mathcal{F}}$ agrees with π (such families are called *gnat*-families for π) is of form $\oplus_{\chi \in G^\vee} \mathcal{L}(-D_\chi)$ where each D_χ is a G-Weil divisor on Y. Up to an equivalence of families we may assume that $D_{\chi_{0,0}} = 0$. Moreover, there exists ([Log06], Section 3.5) the maximal shift family $\oplus \mathcal{L}(-M_\chi)$ and for any other family $\oplus \mathcal{L}(-D_\chi)$ we have $M_\chi \geq D_\chi$ for all $\chi \in G^\vee$. In our toric context each divisor M_χ is of form $\sum q_{\chi,i} E_i$ and the coefficients $q_{\chi,i}$ can be calculated (see [Log03], Example 4.21) via formula

$$q_{\chi,i} = \inf\{e_i(m) \mid m \in M_{\geq 0} \cap \rho^{-1}(\chi)\} \tag{4.6}$$

We set the family $\mathcal{F}$ that we shall endeavor to prove to satisfy the assumptions of Theorem 1.1 to be this maximal shift family $\bigoplus \mathcal{L}(-M_\chi)$. Calculating all $q_{\chi,i}$ using formula (4.6) we obtain:

	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
$M_{\chi 0,0}$	0	0	0	0	0	0	0
$M_{\chi 2,0}$	$\frac{2}{6}$	$\frac{4}{6}$	0	$\frac{2}{6}$	0	$\frac{2}{6}$	0
$M_{\chi 4,0}$	$\frac{4}{6}$	$1\frac{2}{6}$	0	$\frac{4}{6}$	0	$\frac{4}{6}$	0
$M_{\chi 1,1}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	0
$M_{\chi 1,0}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	0	$\frac{1}{6}$	$\frac{3}{6}$
$M_{\chi 4,1}$	$\frac{4}{6}$	$\frac{2}{6}$	0	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{1}{6}$	$\frac{3}{6}$
$M_{\chi 3,1}$	$\frac{3}{6}$	1	$\frac{3}{6}$	$\frac{3}{6}$	$\frac{3}{6}$	1	0
$M_{\chi 3,0}$	$\frac{3}{6}$	1	$\frac{3}{6}$	1	0	$\frac{3}{6}$	$\frac{3}{6}$
$M_{\chi 0,1}$	1	1	0	$\frac{3}{6}$	$\frac{3}{6}$	$\frac{3}{6}$	$\frac{3}{6}$
$M_{\chi 5,1}$	$\frac{5}{6}$	$\frac{4}{6}$	$\frac{3}{6}$	$\frac{5}{6}$	$\frac{3}{6}$	$\frac{2}{6}$	0
$M_{\chi 5,0}$	$\frac{5}{6}$	$\frac{4}{6}$	$\frac{3}{6}$	$\frac{2}{6}$	0	$\frac{5}{6}$	$\frac{3}{6}$
$M_{\chi 2,1}$	$\frac{2}{6}$	$\frac{4}{6}$	0	$\frac{5}{6}$	$\frac{3}{6}$	$\frac{5}{6}$	$\frac{3}{6}$

(4.7)

4.4 Generalities on the McKay quiver of G

By a *quiver* we shall mean a vertex set Q_0, an arrow set Q_1 and a pair of maps $h\colon Q_1 \to Q_0$ and $t\colon Q_1 \to Q_0$ giving the head $hq \in Q_0$ and the tail $tq \in Q_0$ of each arrow $q \in Q_1$. By a *representation of a quiver* we shall mean a graded vector space $\bigoplus_{i \in Q_0} V_i$ and a collection $\{\alpha_q\colon V_{tq} \to V_{hq}\}_{q \in Q_1}$ of linear maps indexed by the arrow set of the quiver.

Definition 4.1. Let G be a finite subgroup of $\mathrm{GL}(V_{\mathrm{giv}})$. Then the *McKay quiver of* G is the quiver whose vertex set Q_0 are the irreducible representations ρ of G and whose arrow set Q_1 has precisely $\dim \operatorname{Hom}_G(\rho_i, \rho_j \otimes V_{\text{giv}})$ arrows going from the vertex ρ_i to the vertex ρ_j.

In our case G is abelian and ${V_{\text{giv}}}^\vee$ decomposes into irreducible subrepresentations as $\bigoplus \mathbb{C}x_i$. If we write U_χ for the 1-dimensional representation that G acts on by $\chi \in G^\vee$ we have by Schur's lemma

$$G\text{-}\operatorname{Hom}(U_{\chi_i} \otimes {V_{\text{giv}}}^\vee, U_{\chi_j}) = \begin{cases} \mathbb{C} & \text{if } \chi_j = \chi_i \rho(x_k)^{-1} \quad k \in \{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

Thus each vertex χ of the McKay quiver of G has three arrows emerging from it and going to vertices $\chi\rho(x_k)^{-1}$ for $k = 1, 2, 3$. We denote the arrow from χ to $\chi\rho(x_k)^{-1}$ by (χ, x_k). Calculating the whole quiver, we obtain:

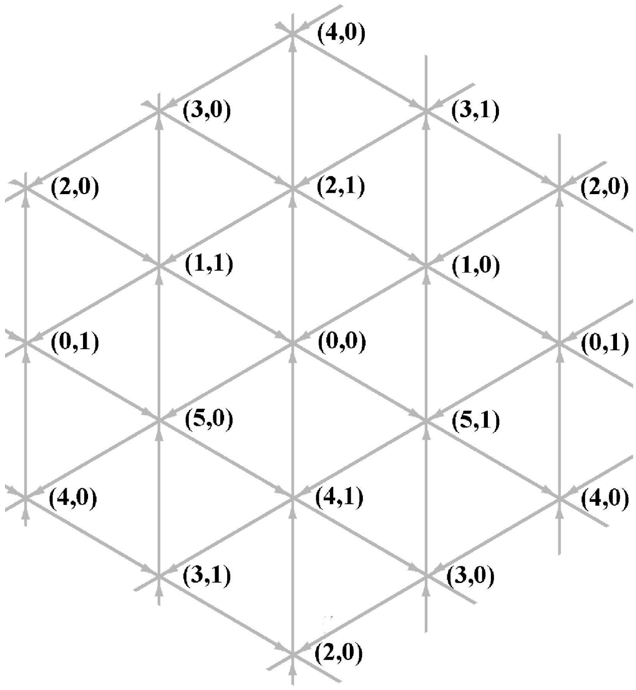


Figure 3

The way we've chosen to depict the McKay quiver reflects the fact that it has a universal cover quiver naturally embedded into $\mathbb{R}^2$. This point of view will not be essential for our argument but a curious reader should consult [CI04], Section 10.2 and [Log04], Section 6.4.

Let now A be a G-constellation in a sense of $R \rtimes G$-modules and let $\oplus A_\chi$ be its decomposition into irreducible representations of G. Then $R \rtimes G$-module structure on A defines a natural representation of the McKay quiver into the graded vector space $\oplus A_\chi$ in which the map α_{χ,x_k} corresponding to an arrow (χ, x_k) is just the multiplication by x_k, i.e.

$$\alpha_{\chi,x_k} : \; A_\chi \to A_{\chi\rho(x_k)^{-1}}, \; v \mapsto x_k \cdot v \tag{4.8}$$

4.5 Degree 0 orthogonality of G-constellations

Let A and A' be two G-constellations and ϕ be a $R \rtimes G$-module morphism $A \to A'$. Let $\bigoplus_{G^\vee} A_\chi$ and $\bigoplus_{G^\vee} A'_\chi$ be decompositions of A and A' into one-dimensional representations of G. By G-equivariance ϕ decomposes into linear maps $\phi_\chi : A_\chi \to A'_\chi$.

Let $\{\alpha_q\}$ and $\{\alpha'_q\}$ be the coresponding representations of the McKay quiver into graded vector spaces $\oplus A_\chi$ and $\oplus A'_\chi$, as per (4.8). Each α_q is a linear map between one-dimensional vector spaces A_{tq} and A_{hq} and so is either a zero-map or an isomorphism, and similarly for the maps α'_q. So for each arrow of McKay quiver we distinguish the following four possibilities:

Definition 4.2. Let q be an arrow of McKay quiver of G. With the notation above we say that with respect to an ordered pair (A, A') of G-constellations the arrow q is:

1. a type $[1,1]$ arrow, if both α_q and α'_q are isomorphisms.
2. a type $[1,0]$ arrow, if α_q is an isomorphism and α'_q is a zero map.
3. a type $[0,1]$ arrow, if α_q is a zero map and α'_q is an isomorphism.
4. a type $[0,0]$ arrow, if both α_q and α'_q are zero maps.

Proposition 4.3. *Let q and (A, A') be as in Definition 4.2 and let ϕ be any $R \rtimes G$-module morphism $A \to A'$. Then:*

1. *If q is a $[1,0]$ arrow, then $A_{hq} \subseteq \ker \phi$.*
2. *If q is a $[0,1]$ arrow, then $A_{tq} \subseteq \ker \phi$.*
3. *If q is a $[1,1]$ arrow, then A_{tq} and A_{hq} either both lie in $\ker \phi$ or both don't.*

Proof. Write $q = (\chi, i)$ where $\chi \in G^\vee$ and $i \in \{1,2,3\}$. Recall that α_q is the map $A_{tq} \to A_{hq}$ corresponding to the action of x_i on A_{tq}. Then R-equivariance of the morphism ϕ implies the commutative square

$$\begin{tikzcd} A_{hq} \arrow[r, "\phi_{hq}"] & A'_{hq} \\ A_{tq} \arrow[u, "\alpha_q"] \arrow[r, "\phi_{tq}"'] & A'_{tq} \arrow[u, "\alpha'_q"'] \end{tikzcd}$$

from which all three claims immediately follow. ∎

Corollary 4.4. *Let (A, A') be an ordered pair of G-constellations. If every component of the McKay quiver path-connected by $[1,1]$-arrows has either a $[0,1]$-arrow emerging from it or a $[1,0]$-arrow entering it, then*

$$\mathrm{Hom}_{R \rtimes G}(A, A') = 0$$

If, also, every component has either a $[0,1]$-arrow entering it or a $(1,0)$-arrow emerging from it, then we further have

$$\mathrm{Hom}_{R \rtimes G}(A', A) = 0$$

and therefore A and A' are orthogonal in degree 0.

4.6 Divisors of zeroes

Definition 4.5. Let $\mathcal{V} = \bigoplus \mathcal{L}(-D_\chi)$ be a family of G-constellations on Y and $q = (\chi, x_k)$ be an arrow in the McKay quiver of G. We define the *divisor of zeroes* B_q of q in $\mathcal{V}$ to be the Weil divisor

$$D_{\chi^{-1}} + (x_i) - D_{\chi^{-1}\rho(x_i)} \tag{4.9}$$

NB: B_q is always an ordinary, integral Weil divisor on Y.

Proposition 4.6. *Let $\mathcal{V} = \bigoplus \mathcal{L}(-D_\chi)$ be a family of G-constellations on Y, (χ, x_k) an arrow in the McKay quiver of G and B_{χ,x_k} its divisor of zeroes in $\mathcal{V}$. Let y be a closed point of Y and G-constellation A be the fiber $\mathcal{V}_{|y}$.*

Then in the corresponding representation $\{\alpha_q\}_{q \in Q_1}$ of the McKay quiver the map $\alpha_(\chi, x_k)$ is a zero map if and only if $y \in B_{\chi,x_k}$.

Proof. By its definition (4.8) the map $\alpha_{\chi,x_k} : A_\chi \to A_{\chi\rho(x_k)^{-1}}$ is the action of x_k on the χ-eigenspace A_χ. This map is the restriction to the point y of the global section β of the $\mathcal{O}_Y$-module

$$\mathcal{H}om_{G,\mathcal{O}_Y}(\mathcal{O}_Y x_k \otimes \mathcal{V}_\chi, \mathcal{V}_{\chi\rho^{-1}(x_k)}) \tag{4.10}$$

defined by $x_k \otimes s \mapsto x_k \cdot s$ for any section s of the χ-eigensheaf $\mathcal{V}_\chi$.

As G acts on a monomial of weight χ by χ^{-1} the χ-eigensheaf of $\mathcal{V}$ is $\mathcal{L}(-D_{\chi^{-1}})$. Hence (4.10) is canonically isomorphic to the following sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^3)$:

$$\mathcal{L}(D_{\chi^{-1}} + (x_k) - D_{\chi^{-1}\rho(x_k)}) \tag{4.11}$$

and the isomorphism maps β to the global section $1 \in K(\mathbb{C}^3)$ of (4.11). Which vanishes precisely on the Weil divisor $B_{\chi,x_k} = D_{\chi^{-1}} + (x_k) - D_{\chi^{-1}\rho(x_k)}$. ∎

In toric context the principal G-Weil divisors (x_k) can be calculated with a formula given in [Log03], Proposition 3.2. In our case we obtain:

$$\begin{aligned}(x_1) &= E_1 + \frac{1}{6}E_4 + \frac{1}{3}E_5 + \frac{1}{2}E_6 + \frac{1}{6}E_7 + \frac{1}{2}E_8 + \frac{4}{6}E_9 \\ (x_2) &= E_2 + \frac{1}{6}E_4 + \frac{1}{3}E_5 + \frac{1}{2}E_6 + \frac{4}{6}E_7 + \frac{1}{6}E_9 + \frac{1}{2}E_{10} \\ (x_3) &= E_3 + \frac{4}{6}E_4 + \frac{1}{3}E_5 + \frac{1}{6}E_7 + \frac{1}{2}E_8 + \frac{1}{6}E_9 + \frac{1}{2}E_{10}\end{aligned} \tag{4.12}$$

With these values and the expressions (4.7) for the divisors M_χ we use the formula (4.9) and for every arrow of the McKay quiver calculate its divisor of zeroes in the family $\mathcal{F} = \bigoplus \mathcal{L}(-M_\chi)$:

$$\begin{array}{ll}
B_{\chi_{0,0},1} = E_1 & B_{\chi_{1,1},1} = E_1 + E_4 + E_5 + E_6 + E_7 + E_8 + E_9 \\
B_{\chi_{0,0},2} = E_2 & B_{\chi_{1,1},2} = E_2 + E_6 + E_7 \\
B_{\chi_{0,0},3} = E_3 & B_{\chi_{1,1},3} = E_3 + E_4 + E_8 \\
B_{\chi_{4,0},1} = E_1 & B_{\chi_{1,0},1} = E_1 + E_6 + E_9 \\
B_{\chi_{4,0},2} = E_2 & B_{\chi_{1,0},2} = E_2 + E_4 + E_5 + E_6 + E_7 + E_9 + E_{10} \\
B_{\chi_{4,0},3} = E_3 & B_{\chi_{1,0},3} = E_3 + E_4 + E_{10} \\
B_{\chi_{2,0},1} = E_1 + E_5 + E_9 & B_{\chi_{4,1},1} = E_1 + E_8 + E_9 \\
B_{\chi_{2,0},2} = E_2 + E_5 + E_7 & B_{\chi_{4,1},2} = E_2 + E_7 + E_{10} \\
B_{\chi_{2,0},3} = E_3 + E_4 + E_5 & B_{\chi_{4,1},3} = E_3 + E_4 + E_5 + E_7 + E_8 + E_9 + E_{10} \\
B_{\chi_{5,1},1} = E_1 + E_6 + E_8 + E_9 & B_{\chi_{3,1},1} = E_1 + E_6 + E_8 + E_9 \\
B_{\chi_{5,1},2} = E_2 + E_6 & B_{\chi_{3,1},2} = E_2 + E_5 + E_6 + E_7 + E_9 \\
B_{\chi_{5,1},3} = E_3 + E_8 & B_{\chi_{3,1},3} = E_3 + E_4 + E_5 + E_8 + E_9 \\
B_{\chi_{5,0},1} = E_1 + E_6 & B_{\chi_{3,0},1} = E_1 + E_5 + E_6 + E_7 + E_9 \\
B_{\chi_{5,0},2} = E_2 + E_6 + E_7 + E_{10} & B_{\chi_{3,0},2} = E_2 + E_6 + E_7 + E_{10} \\
B_{\chi_{5,0},3} = E_3 + E_{10} & B_{\chi_{3,0},3} = E_3 + E_4 + E_5 + E_7 + E_{10} \\
B_{\chi_{2,1},1} = E_1 + E_8 & B_{\chi_{0,1},1} = E_1 + E_4 + E_5 + E_8 + E_9 \\
B_{\chi_{2,1},2} = E_2 + E_{10} & B_{\chi_{0,1},2} = E_2 + E_4 + E_5 + E_7 + E_{10} \\
B_{\chi_{2,1},3} = E_3 + E_4 + E_8 + E_{10} & B_{\chi_{0,1},3} = E_3 + E_4 + E_8 + E_{10}
\end{array} \tag{4.13}$$

4.7 A sample calculation

The orthogonality criterion in Corollary 4.4 and the data in the table (4.13) of divisors of zeroes are all that we need to check whether any two G-constellations in the family $\mathcal{F}$ are orthogonal in degree 0. In this subsection we are going to go step by step through such a calculation and verify that any point on the two-dimensional torus orbit S_8 and any point on the one-dimensional torus orbit $S_{1,7}$ are orthogonal in degree 0 in $\mathcal{F}$.

Let a be any point of S_8. Then a lies on no divisor E_i other than E_8. Observe that as all divisors of zeroes B_q consist entirely of divisors E_i we have $a \in B_q$ if and only if $E_8 \subset B_q$. Let G-constellation A be the fiber of $\mathcal{F}$ at a and $\{\alpha_q\}$ be the corresponding representation of the McKay quiver. By Proposition 4.6 for any arrow q the map α_q is a zero map if and only if $E_8 \in B_q$. From the table (4.13) we mark all the zero-maps on the diagram of the McKay quiver by drawing a line through the coresponding arrow:

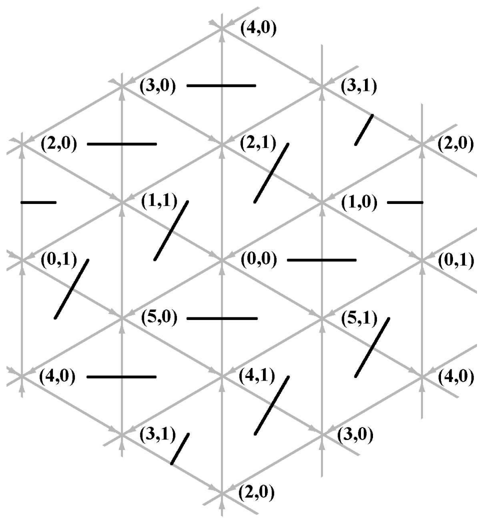


Figure 4

Similarly if b is a point of $S_{1,7}$ then b lies on no E_i other than E_1 and E_7. Let B be the fiber of $\mathcal{F}$ at b and $\{\beta_q\}$ be the corresponding representation of the McKay quiver. As above β_q is a zero-map if and only if either E_1 or E_7 belongs to B_q. Marking all the zero-maps on the McKay quiver we obtain:

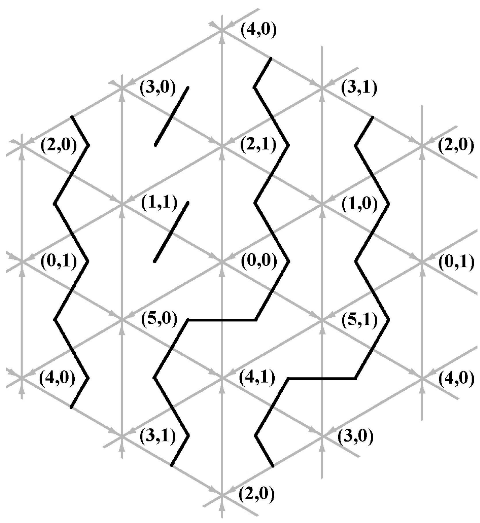


Figure 5

If we combine Figure 4 and Figure 5 the only arrows left unmarked would be the arrows of type $[1,1]$ (Definition 4.2) with respect to the pair (A,B):

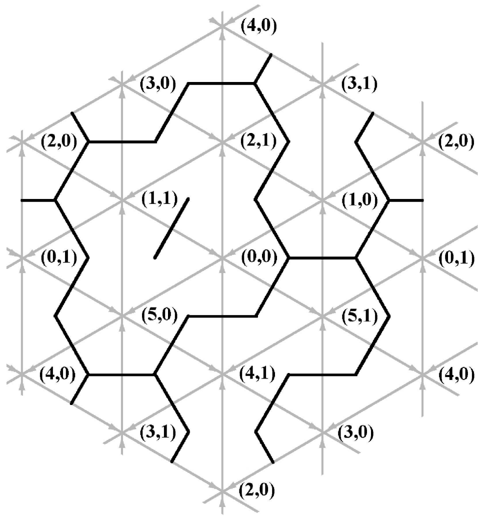


Figure 6

Figure 6 makes clear what the components path-connected by $[1,1]$-arrows are: $\{\chi_{0,0},\chi_{2,1},\chi_{5,0},\chi_{1,1}\}$, $\{\chi_{5,1},\chi_{4,1},\chi_{2,0}\}$, $\{\chi_{1,0},\chi_{3,1}\}$ and $\{\chi_{0,1},\chi_{4,0},\chi_{3,0}\}$. Now, with Corollary 4.4 in mind, we search the borders of these four regions for the $[1,0]$ and $[0,1]$-arrows. The $[1,0]$-arrows are unmarked on Figure 4 but marked on Figure 5 and vice versa for $[0,1]$. On Figure 7 we've marked on the border of each region an incoming and an outgoing $[0,1]$-arrow:

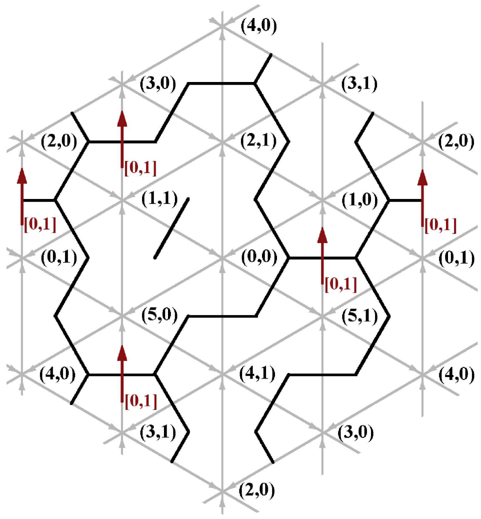


Figure 7

By the Corollary 4.4 we see that A and B are orthogonal in degree 0.

4.8 Direct transforms

By construction of the family $\mathcal{F}$ its forgetful map $\pi_{\mathcal{F}} : Y \to \mathbb{C}^n/G$ coincides with the toric morphism π of the section 4.2. As π is certainly proper and birational to show that the integral transform $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is an equivalence of categories it now suffices by Corollary 1.2 to succesfully carry out the calculation of Section 4.7 for every pair $(S_i, S_{j,k})$ and every pair (S_i, S_j) on Y thus showing their degree 0 orthonogality in $\mathcal{F}$. However we are going to significantly reduce the amount of calculations necessary by pointing out that our family $\mathcal{F}$ is a *direct transform* of the universal family $\mathcal{M}_{\theta_+}$ of G-clusters on G-Hilb$(\mathbb{C}^3)$, the fine moduli space of G-clusters and another toric crepant resolution of $\mathbb{C}^3/G$ ([Nak00]).

By the *direct transform* we mean the following: let Y' and Y'' be two toric crepant resolutions of $\mathbb{C}^3/G$. Then their toric fans in L have the same set of 1-dimensional cones and therefore the unions of torus orbits of codimension 1 or less in Y' and in Y'' can be naturally identified. We denote this common open set by U. Since the compliment of U is of codimension 2 in Y' (resp. Y'') any line bundle or divisor on U extends uniquely to a line bundle or a divisor on Y' (resp. Y''). The same is true of a family of G-constellations as for G abelian any such family is a direct sum of line bundles. Thus for any family $\mathcal{V}'$ of G-constellations on Y' we define its direct transform $\mathcal{V}''$ to Y'' to be the unique extension to Y'' of the restriction of $\mathcal{V}'$ to U. Observe that if $\mathcal{V}'$ is of form $\bigoplus_\chi \mathcal{L}(-D'_\chi)$ for some G-Weil divisors D'_χ on Y' then $\mathcal{V}''$ is the family $\bigoplus \mathcal{L}(-D''_\chi)$ where each D'' is the direct transform of D'.

Our claim that $\mathcal{F}$ is the direct transform of the universal family $\mathcal{M}_{\theta_+}$ follows from a more general statement which on a given Y explicitly identifies the direct transform of the universal family $\mathcal{M}_\theta$ for any θ. In particular, the maximal shift family is always the transform of $\mathcal{M}_{\theta_+}$. The proof is somewhat technical and we defer it to the Appendix (Section 5.1).

Consider now the set of all cones common to both the fan of Y and the fan of G-Hilb$(\mathbb{C}^3)$. We can identify Y and G-Hilb$(\mathbb{C}^3)$ along the open set U' which consists of all the corresponding torus orbits. As $\mathcal{F}$ is the direct transform of $\mathcal{M}_{\theta_+}$ the restriction of $\mathcal{F}$ to $U' \subset Y$ is isomorphic to the restriction of $\mathcal{M}_{\theta_+}$ to $U' \subset G$-Hilb$(\mathbb{C}^3)$. The family $\mathcal{M}_{\theta_+}$ is everywhere orthogonal in all degrees (an immediate consequence of [BKR01], Theorem 1.1) so on $U' \subset Y$ the family $\mathcal{F}$ is also orthogonal in all degrees. Thus any pair of torus orbits in Y whose corresponding cones are also contained in the fan of G-Hilb$(\mathbb{C}^3)$ are orthogonal in $\mathcal{F}$ in all degrees.

4.9 Final calculations

A detailed description of an algorythm which allows one to calculate the toric fan of G-Hilb$(\mathbb{C}^3)$ can be found in [CR02]. Applying it to our group G we obtain:

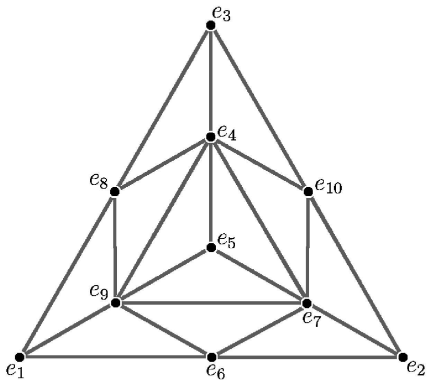
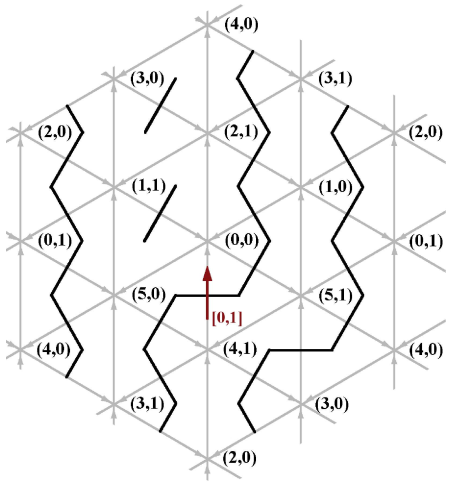


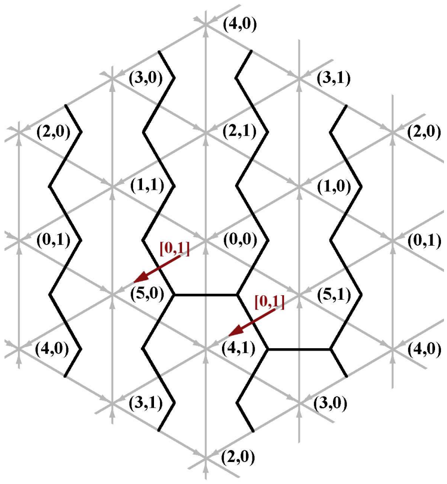
Figure 8

Comparing Figure 8 with the fan of Y on Figure 2 we see that only codimension 2 torus orbits of Y whose corresponding cones aren't also contained in the fan of G-Hilb$(\mathbb{C}^3)$ are $S_{1,7}$, $S_{2,4}$ and $S_{3,9}$. By the argument in Section 4.8 to establish that $\mathcal{F}$ satisfies the assumptions of Theorem 1.1, it now remains to demonstrate that each of these three orbits is orthogonal in degree 0 in $\mathcal{F}$ to every torus orbit of codimension 1. But as the fan of Y and the numerical data 4.7 defining $\mathcal{F}$ are invariant with respect to the rotations of L which correspond to the cyclic permutations of e_1, e_2 and e_3, it suffices to treat any single one of $S_{1,7}$, $S_{2,4}$ and $S_{1,7}$ - for the three corresponding cones in L get permuted by these rotations.

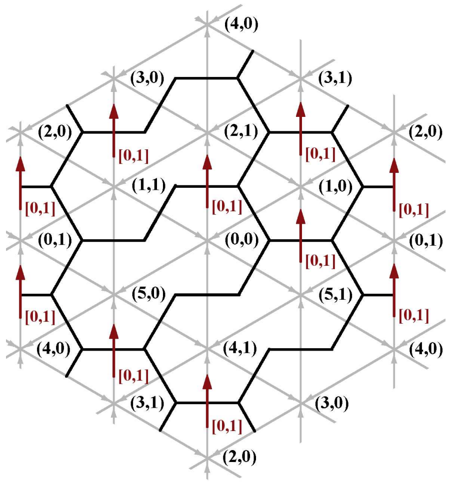
We choose to treat $S_{1,7}$. In Section 4.7 we have established the orthogonality of $S_{1,7}$ and S_8. We repeat that calculation for $S_{1,7}$ and every other orbit S_i and list below the analogues of Figure 7. From them, as elaborated in Section 4.7, the reader could readily ascertain that Corollary 4.4 applies in each case to show the orthogonality in $\mathcal{F}$ of the torus orbits involved.



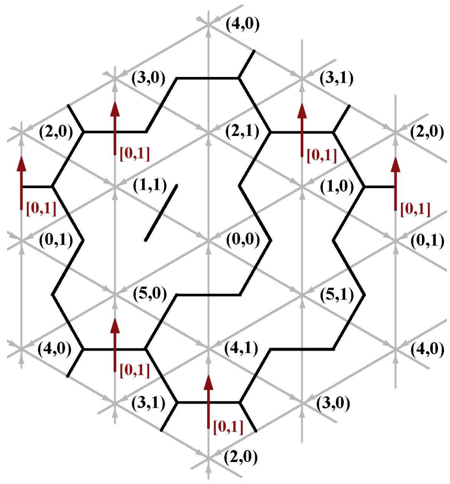
$(S_1, S_{1,7})$ and $(S_7, S_{1,7})$



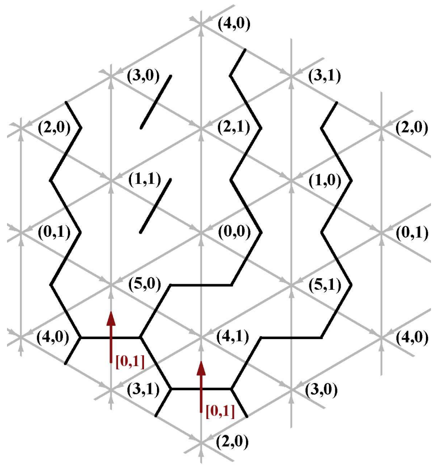
$(S_2, S_{1,7})$



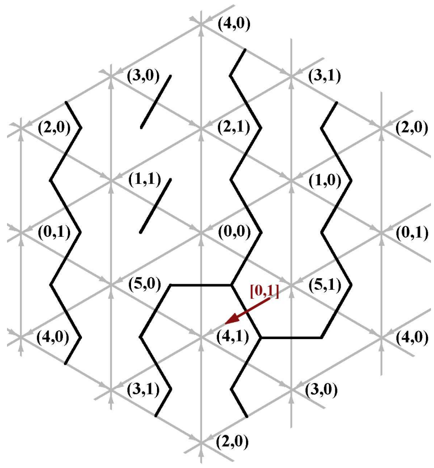
$(S_3, S_{1,7})$



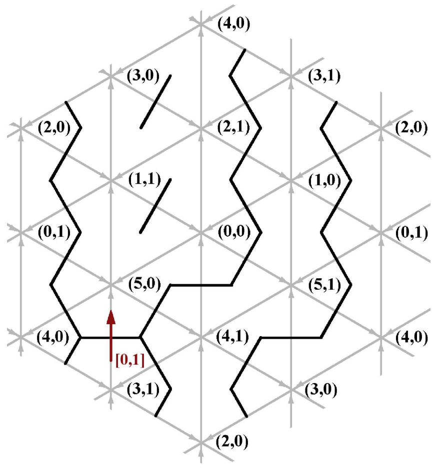
$(S_4, S_{1,7})$



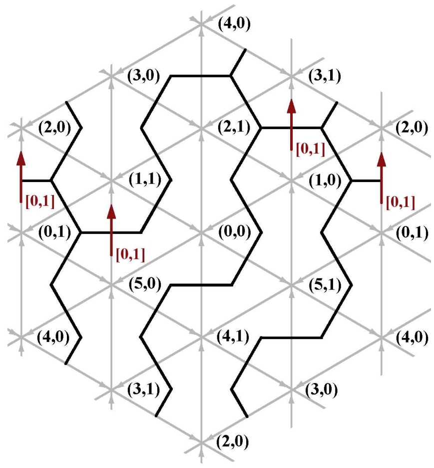
$(S_5, S_{1,7})$



$(S_6, S_{1,7})$



$(S_9, S_{1,7})$



$(S_{10}, S_{1,7})$

We conclude that the integral transform $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is an equivalence of categories $D(Y) \to D^G(\mathbb{C}^3)$ and that *a posteriori* the family $\mathcal{F}$ is everywhere orthogonal in all degrees.

5 Appendix

5.1 Theta stability and G-constellations

Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. We recall basic facts about θ-stability for G-constellations, for more detail the reader should consult [CI04], Section 2.1. Let $\mathbb{Z}(G) = \bigoplus_{\rho \in \operatorname{Irr} G} \mathbb{Z}\rho$ be the representation ring of G and set

$$\Theta = \{\theta \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}(G), \mathbb{Q}) \mid \theta(V_{\mathrm{reg}}) = 0\}$$

For any $\theta \in \Theta$ we say that a G-constellation A is θ-stable (resp. θ-semistable) if for every sub-$R \rtimes G$-module B of A we have $\theta(B) > 0$ (resp. $\theta(B) \geq 0$). We say that θ is generic if every θ-semistable G-constellation is θ-stable. This is equivalent to θ being non-zero on any proper subrepresentation of V_{reg}. The fine moduli space M_θ of θ-stable G-constellations can be constructed via GIT theory and for $G \subseteq \mathrm{SL}_3(\mathbb{C})$ and θ generic M_θ is a projective crepant resolution of $\mathbb{C}^3/G$. Any two θ-stable G-constellations are either orthogonal in degree 0 or isomorphic so the universal family $\mathcal{M}_\theta$, which M_θ comes equipped with, is everywhere orthogonal in degree 0.

Assume from now on that G is abelian. Given any toric resolution Y of $\mathbb{C}^n/G$ and any generic $\theta \in \Theta$ the following allows us to explicitly compute, in terms of the classification in [Log06], the family that is the direct transform to Y of the universal family $\mathcal{M}_\theta$:

Proposition 5.1. *Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, θ be an element of Θ and let $Y \xrightarrow{\pi} \mathbb{C}^n/G$ be a resolution. Denote by $\mathfrak{E}$ the set of all irreducible exceptional divisors of π and all divisors $(x_i^{|G|})$ for $i \in \{1, \dots, n\}$ and by U the open subset of Y consisting of points lying on at most one divisor in $\mathfrak{E}$. We define a map*

$$w_\theta : \left\{ \text{normalized gnat-families on } Y \xrightarrow{\pi} \mathbb{C}^n/G \right\} \to \mathbb{Q} \tag{5.1}$$

as follows: given a normalized family $\mathcal{V}$ write it as $\bigoplus \mathcal{L}(-D_\chi)$. By [Log06], Proposition 3.15 each G-Weil divisor D_χ is of form $\sum_{E \in \mathfrak{E}} q_{\chi,E} E$. We set

$$w_\theta(\mathcal{V}) = \sum_{E \in \mathfrak{E}} \sum_{\chi \in G^\vee} \theta(\chi) q_{\chi,E} \tag{5.2}$$

Let $\mathcal{M}$ be a family which maximizes $w_\theta(\mathcal{M})$, such exists as the domain of definition of w_θ is finite ([Log06], Corollary 3.16). Then for any point $y \in U$ the fiber of $\mathcal{M}$ at y is a θ-semistable G-constelation. If, moreover, θ is generic then a normalized family θ-semistable over U is unique.

Proof. Write $\mathcal{M}$ as $\bigoplus \mathcal{L}(-M_\chi)$. Suppose that the fiber of $\mathcal{M}$ is not θ-semistable at some $y \in U$. Denote this fiber by A, its decomposition into irreducible representations by $\bigoplus_{\chi \in G^\vee} A_\chi$ and the corresponding representation of the McKay quiver by $\{\alpha_q\}$. As A isn't θ-semistable there exists a non-empty proper subset I of $G^\vee$ such that $A' = \bigoplus_{\chi \in I} A_\chi$ is a sub-$R \rtimes G$-module of A and $\theta(A') < 0$. Denote by J the compliment $G^\vee \setminus I$. Denote by $Q_{I \to J}$ the subset $\{q \in Q_1 \mid tq \in I, hq \in J\}$ of the arrow set Q_1 of the McKay quiver and similarly for $Q_{J \to I}$, $Q_{I \to I}$, $Q_{J \to J}$. Then A' being closed under the action of R implies that for any $q \in Q_{I \to J}$ the map α_q is a zero map. Which by Proposition 4.6 implies $y \in B_q$.

As the divisors M_χ consist entirely of prime Weil divisors in $\mathfrak{E}$ so do all the divisors of zeroes B_q. As y lies on all B_q with $q \in Q_{I \to J}$, y lies on at least one divisor in $\mathfrak{E}$. But as $y \in U$ y also lies on at most one divisor in $\mathfrak{E}$. Denote this unique divisor by E then we have

$$q \in Q_{I \to J} \Rightarrow E \subset B_q \tag{5.3}$$

Define a new G-Weil divisor set $\{M'_\chi\}$ by setting M'_χ to be M_χ if $\chi \in I$ and $M_\chi + E$ if $\chi \in J$. Then divisors $\{B'_q\}$ defined from $\{M'_\chi\}$ by equations (4.9) can be expressed as

$$B'_q = \begin{cases} B_q & \text{if } q \in Q_{I \to I}, Q_{J \to J} \\ B_q + E & \text{if } q \in Q_{J \to I} \\ B_q - E & \text{if } q \in Q_{I \to J} \end{cases} \tag{5.4}$$

Since $\{B_q\}$ are all effective (5.4) and (5.3) imply that $\{B'_q\}$ are also all effective. Therefore $\bigoplus \mathcal{L}(-M'_\chi)$ is a normalized *gnat*-family. But

$$w_\theta(\mathcal{M}') = w_\theta(\mathcal{M}) + \sum_{\chi \in J} \theta(\chi) \tag{5.5}$$

which contradicts the maximality of $w_\theta(\mathcal{M})$ since $\sum_{\chi \in J} \theta(\chi) = -\theta(A') > 0$.

For the second claim let $\mathcal{N} = \bigoplus \mathcal{L}(-N_\chi)$ be another normalized family θ-semistable over U. Let B'_q be divisors of zeroes of $\mathcal{N}$. Then

$$B_q - B'_q = (M_{tq} - N_{tq}) - (M_{hq} - N_{hq}) \tag{5.6}$$

Take any $E' \in \mathfrak{E}$ such that the sets $\{m_{\chi,E'}\}$ and $\{n_{\chi,E'}\}$ of the coefficients of E' in $\{M_\chi\}$ and $\{N_\chi\}$ are distinct. Then $J' = \{\chi \in G^\vee \mid n_{\chi,E'} > m_{\chi,E'}\}$ is a non-empty proper subset of $G^\vee$. Denote by I' its compliment. For any $q \in Q_{I' \to J'}$ the coefficient of E' in the RHS of (5.6) is strictly positive. As B'_q is effective we conclude that $q \in Q_{I' \to J'}$ implies $E' \subset B_q$. So for any $y \in E'$ the restriction $(\bigoplus_{\chi \in I'} \mathcal{L}(M_\chi))|_y$ is a sub-$R \rtimes G$-module of $\mathcal{M}_{|y}$. But as $\mathcal{M}$ is θ-semistable on U and as $U \cap E' \neq \emptyset$ we must have $\sum_{\chi \in I'} \theta(\chi) \geq 0$. Similarly if $q \in Q_{J' \to I'}$, then the RHS of (5.6) is strictly negative, so $E' \subset B'_q$ and θ-semistability of $\mathcal{N}$ implies $\sum_{\chi \in J'} \theta(\chi) = -\sum_{\chi \in I'} \theta(\chi) \geq 0$. Therefore $\sum_{\chi \in I'} \theta(\chi) = 0$ and θ is not generic. $\square$

Corollary 5.2. *Let G be the finite subgroup of $\mathrm{SL}_3(\mathbb{C})$, $Y \xrightarrow{\pi} \mathbb{C}^n/G$ be a crepant toric resolution and $\theta \in \Theta$ be generic.*

Denote by $\mathcal{M}$ the unique normalized gnat-family on Y which maximizes the map w_θ. Its existence is warrantied by the Proposition 5.1. Then $\mathcal{M}$ is the direct transform of the universal family $\mathcal{M}_\theta$ from the fine module space M_θ of θ-stable G-constellations.

Proof. By the first claim of Proposition 5.1, $\mathcal{M}$ is θ-stable on the open subset U of Y consisting of all torus orbits of codimension 1 or less. So, by its definition, is the direct transform of $\mathcal{M}_{\theta_+}$ to Y. So by the second claim of Proposition 5.1 $\mathcal{M}$ and the direct transform of $\mathcal{M}_{\theta_+}$ must be isomorphic. □

Define $\theta_+ \in \Theta$ by $\theta_+(\chi_0) = 1 - |G|$ and $\theta_+(\chi) = 1$ for $\chi \neq \chi_0$. Evidently θ_+ is generic. As follows from the original observation by Ito and Nakajima in [IN00], §3 G-clusters can be identified with θ_+-stable G-constellations, thus identifying $G\text{-Hilb}(\mathbb{C}^3)$ with the fine moduli space M_{θ_+}.

In Section 4.3 we defined the family $\mathcal{F} = \bigoplus \mathcal{L}(-M_\chi)$ to be the maximal shift family on Y (see [Log06], Section 3.5) so for any other normalized *gnat*-family $\oplus \mathcal{L}(-D_\chi)$ we have

$$M_\chi \geq D_\chi$$

for all $\chi \in G^\vee$. Therefore $\mathcal{F}$ is the family which maximizes w_{θ_+} and thus Proposition 5.2 verifies our claim that $\mathcal{F}$ is the direct transform of the universal family $\mathcal{M}_{\theta_+}$.

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